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Representability of Cohomology Theories on CW-complexes

Bachelor's Project Mathematics

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Abstract

In topology, it is in general very difficult to decide whether or not two spaces are homotopy equivalent. In algebraic topology, therefore, algebraic invariants are associated with each space, reflecting some topological data. Examples of this are homology and cohomology, that originated from measuring the number and size of holes in topological spaces. They turn out to be versatile and related to much that is cared about in algebraic topology. In this thesis, we will show that, as long as we restrict ourselves to a certain subclass of topological spaces, namely that of CW-complexes, all cohomology theories are representable by a sequence of CW-complexes. To do so, we first explore singular homology and cohomology, define what generalised cohomology theories are, and use the homotopy theory of CW-complexes to prove this main result.

Contents

Introduction

A central challenge in topology is to show if two given topological space X and Y are homeomorphic or homotopy equivalent. In practice, there is often not a natural candidate for a homeomorphism or homotopy equivalence, and to directly show no such map exists is neigh impossible. Therefore, algebraic topology seeks to assign to each topological space X some algebraic structure that is invariant under homeomorphism or homotopy equivalence. If two spaces X and Y have non-isomorphic invariants associated to them, consequently they cannot be homeomorphic or homotopy equivalent, respectively. It is in practice doable to check whether or not groups or modules (or whatever the algebraic structure is) are isomorphic, so this is a viable approach.

The fundamental group $\pi_1(X, x)$ is one example of such an invariant, and measures the amount of different loops there are in X on a certain base point x (up to homotopy). As such, it is also a measure of the amount of holes in the space, although not a very reliable one in that. For instance, we know that $\pi_1(S^2) = 0$, although we would certainly say the 2-sphere has a two-dimensional hole (since the boundary of this hole is a surface). Apparently, the fundamental group cannot recognise this, so it does not capture all the information we are after.

If we want to define an algebraic invariant on an arbitrary space, it is natural to want to measure the amount and dimension of the holes of such a space. They are namely not only visibly homotopy invariants on the space, but are themselves visible: they are clear geometric aspects of the space, one of the few we have left once we make the step from more rigid geometry like differential geometry to topology.

This leads us to singular homology and cohomology, which we will be covering in the first two chapters of this thesis. These assign for each $n \geq 0$ a module over a ring to a given space. The most important properties of singular cohomology can be used as a starting point to define what we wish a cohomology theory in general to be. We then do not have an explicit description of the inner workings of the cohomology theory anymore, as we do have with singular cohomology. What we will however be concerned with showing is that despite this, all cohomology theories that map to abelian groups (which are $\mathbb{Z}$-modules) actually arise in a similar manner, as long as we restrict ourselves to CW-complexes only: for any cohomology theory, there exist fixed CW-complexes such that the algebraic invariants that the theory assigns to a CW-complex X is the set of homotopy classes of maps from X into those fixed spaces (and we will also see that this set carries a group structure). This is of huge theoretical and practical importance, we will encounter some beautiful consequences of it along the way.

This thesis is meant as a Bachelor's level treatment of cohomology, CW-complexes, homotopy theory and representability of said cohomology theories. As such, not much prior knowledge is required. A solid understanding of point-set topology and of group and module theory are needed, and a basic understanding the fundamental group and homotopy are recommended. Although we do include some category theory, module theory and homological algebra in the appendices, we omit most of the proofs of the statements there, and it certainly helps if the reader is already a bit familiar with those topics.

Conventions

We quickly write down a few conventions that we adopt in this thesis, that are not entirely standard.

- Every ring is understood to have a unit.
- The empty set $\varnothing$ is also considered a topological space because of category theoretical considerations. However, arbitrary topological spaces are often implicitly assumed to be nonempty in the text. This is because arguments with maps to and from empty spaces require some care, although it is always simple to check if a certain passage still holds for the empty space, and to check that results hold for empty spaces as well. It is therefore not worth the effort to always write down a separate passage for the empty space.
- A space X is quasi-compact if every open cover of X admits a finite subcover. A space X is compact if it is both quasi-compact and Hausdorff.
- A space X is locally compact if for every point $x \in X$ and open neighbourhood $U \subseteq X$, there is a compact (in the sense of the above convention) neighbourhood $K \subseteq X$ of x with $K \subseteq U$. K being a neighbourhood of x means that there also exists an open $O \ni x$ with $O \subseteq K$.
- If $X' \subseteq X$ are two topological spaces, a map $f : X \to X'$ is a deformation retraction if $f|_{X'} = \mathrm{id}_{X'}$ and if there exists a homotopy $H : X \times [0, 1] \to X$ from id_X to $\iota \circ f$, where $\iota : X' \hookrightarrow X$ is the inclusion, such that $H(\,\cdot\,, t)|_{X'} = \mathrm{id}_{X'}$ for all t.

Notation

Ab	The category of abelian groups.	Example A.4.
Cf,C_uf	The reduced and unreduced mapping cone of a map.	Definition 3.45.
CX, C_uX	The reduced and unreduced cone on X.	Definition 3.42.
C_*	The pointed version of a category C.	
$\mathsf{C}(2)$, $\mathsf{C}(3)$	The category of pairs or triplets of objects in C.	
$\mathsf{C}^{\mathrm{opp}}$	The opposite category of a category C.	Definition A.8.
${}_R\mathsf{Chain}$, ${}_R\mathsf{cChain}$	The category of left R-module chain or cochain complexes.	Notation B.20.
$C_\bullet(X;M)$	The singular chain complex with coefficients in M.	Definition 1.15.
$C^\bullet(X;M)$	The singular cochain complex with coefficients in M.	Definition 2.2.
$C_\bullet(X,X';M)$	The relative chain complex with coefficients in M.	Definition 1.27.
$C^\bullet(X,X';M)$	The relative cochain complex with coefficients in M.	Above Proposition 2.12.
CMon	The category of commutative monoids.	Definition 4.25.
$\mathsf{CohomTh}$	The category of generalised cohomology theories on CW-complexes.	Beginning of Section 5.2.
CW	The category of CW-complexes.	Notation 3.22.
cCW	The category of path-connected CW-complexes.	Notation 6.27.
$\mathrm{H}_n(X;M)$	The n-th singular homology module with coefficients in M.	Definition 1.16.
$\mathrm{H}^n(X;M)$	The n-th singular cohomology module with coefficients in M.	Definition 2.6.
$\mathrm{H}_n(X,X';M)$	The n-th relative homology module with coefficients in M.	Definition 1.27.
$\mathrm{H}^n(X,X';M)$	The n-th relative cohomology module with coefficients in M.	Definition 2.16.
hC	The homotopy version of a category C.	
$\mathrm{Hom}_R(\,\cdot\,,\,\cdot\,)$	The Hom-functor in the category of R-modules.	Notation B.3.
$K(A,n)$	The (an) Eilenberg-MacLane space of type $K(A,n)$.	Notation 6.50.
m_f, M_f	The reduced and unreduced mapping cylinder of a map.	Definition 3.44.
$\mathrm{Map}(X,Y)$	The set of continuous maps between topological spaces.	Notation 1.9.
$\mathrm{Map}^\bullet(X,Y)$	The set of pointed continuous maps between pointed topological spaces.	Notation 6.6.
$M[S]$	The linearisation of a set S over a module M.	Definition B.4.
$\mathsf{rCohomTh}$	The category of reduced cohomology theories on pointed CW-complexes.	Beginning of Section 5.2.
SX	The unreduced suspension of a space X.	Definition 3.43.
$\mathcal{S}_n(X)$	The set of singular n-simplices on a topological space.	Notation 1.4.
Set	The category of sets.	Example A.4.
Top	The category of topological spaces.	Example A.4.
X_+	The addition of a base point to a topological space.	Remark 1.26.
Δ^n	The standard n-simplex in $\mathbb{R}^{n+1}$.	Definition 1.1.
ΣX	The reduced suspension of a space X.	Definition 3.43.
ΩX	The loop space of X.	Definition 6.8.
$\Omega{-}\mathsf{spec}$	The category of Ω-spectra.	Definition 6.24.
$[\,\cdot\,,\,\cdot\,]$	The set of homotopy classes of maps between spaces, pairs or triplets.	Notation 4.2.
$[\,\cdot\,,\,\cdot\,]^\bullet$	The set of homotopy classes of pointed maps between spaces or pairs.	Notation 4.2.
$\int_{\mathsf{C}} F$	The category of elements of a functor F.	Definition 5.19.
$\vee$	The wedge sum of topological spaces.	Definition 3.37.
$\wedge$	The smash product of topological spaces.	Definition 3.40.

Notation of references

This thesis consists of literature research, and consequently contains plenty of references in the text. Sometimes, we mention during a section that a part that follows is taken from a certain source; other moments, we add references at the most important theorems only, especially when consequences of it are fairly straightforward. When we add the reference not in the theorem statement but at the beginning of the proof, it is understood that both the theorem and its proof are taken from that source. Remarks taken from sources also get their references, and definitions that are either very important or that appear in multiple different forms in the literature do as well.

When we add a part of a proof or make entirely our own, state an interesting result not taken from the literature, or have interesting contributions to make in the text that are our own, we will either make that clear in the text, or we will denote it with the symbol (†) for clarity. None of these additions on our side are however new results: it only means we did not need sources to come up with them.

There is one exception to this notation, namely our usage of Theorem A.51. This theorem states that left adjoint functors commute with colimits and right adjoint functors with limits, and we will use it multiple times througout this thesis. Each time we do so, it has been an addition on our side, as none of the sources that we have used for arguments about algebraic topology have used it. However, since sometimes it is only briefly mentioned by us within a proof taken from a reference, we will never explicitly use our above defined symbol when we use that theorem, as that would often be inappropriate.

Chapter 1

Preliminaries

We begin this thesis by recalling the theory of singular homology. This provides a natural setup for singular cohomology and its generalisations, and is moreover better accessible for a reader which has not yet encountered more advanced algebraic topology. As stated in the introduction, the motivation for singular homology is that we want to determine for a given topological space X whether or not X has any holes, and if so, if we can say anything about the "size" or "dimension" of that hole.

Singular homology approximately does the following. If we let Δ^n be a regular n-simplex in $\mathbb{R}^n$, then the image of any continuous map $f : \Delta^n \to X$ can be considered to be a "continuous n-simplex in X". We can use these n-simplices in X to build something that looks like it could be the boundary of a continuous $(n+1)$-simplex in X, given by a continuous map $\Delta^{n+1} \to X$ (where "looks like it could" will be captured in precise mathematics). If it actually is, then there are no holes within that construction, since the image of a continuous map $\Delta^{n+1} \to X$ is contractible. However, if it is not, then that indicates that there is some sort of n-dimensional hole there. Singular homology provides a precise and algebraic method to carry out this idea. It turns out that singular homology is reasonably well-computable since it satisfies a few pleasant properties, which are of great theoretical interest. We will mainly focus on the theoretical aspects, and not on the practical and computational ones.

Although the title of the chapter suggests we are going to quickly go through the material, we are going to go in depth at some points, not in the least because this allows us to better understand singular cohomology in the next chapter. However, on one hand it is not meant to learn for the first time about singular homology, and on the other hand some of the more technical proofs will be skipped.

This chapter, and for that matter all subsequent ones, relies heavily on concepts from abstract algebra and category theory, such as functors, modules, and chain complexes. A brief overview of these topics is given in Appendices A and B.

1.1 Singular homology

In this section, we will define singular homology and derive a few basic properties of it. We follow the route that [23] takes, and generalise using [7] the construction to modules over a ring R. We will tend use more category theory jargon than in [23], but this does not generalise further the arguments he gives. Rather, it serves to keep an overview, a sort of "bird-eye view", since there will be many induced maps appearing on the next pages. How and why they arise is best captured using the language of functors.

As stated above, we have to begin with explaining what n-dimensional simplices on X exactly are. Our starting point is at the Euclidian simplices.

Definition 1.1. For $n \geq 0$, the *standard n-simplex* is the subset $\Delta^n = \{(t_0, \ldots, t_n) \mid t_i \geq 0 \text{ for all } i, \sum_{i=0}^{n} = 1\}$ of $\mathbb{R}^{n+1}$, equipped with the subspace topology. $\Diamond$

The reader should visualise these sets for $n = 0, 1, 2$ and convince himself or herself that these indeed define simplices: Δ^0 is a singleton, Δ^1 is a line segment, Δ^2 is a triangle with its interior, and so forth. Moreover, we see that there are inclusions $\Delta^0 \subset \Delta^1 \subset \Delta^2 \subset \ldots$, which is partly the reason of this particular definition. In fact, the boundary of the standard $(n+1)$-simplex consists of $n+2$ copies of the standard n-simplex. Motivated by this observation, we define for any $n \geq 1$ and $i \in \{0, \ldots n\}$ the so-called *face-maps*

$$\delta_i^n : \Delta^{n-1} \to \Delta^n : (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}). \tag{1.1}$$

In terms of the standard basis (e_i) of $\mathbb{R}^{n+1}$ (which are the vertices of Δ^n), δ_i^n sends Δ^{n-1} to its copy in Δ^n opposite to the vertex e_i. We will often omit the superscript and simply write δ_i, leaving the dimensions implicit. Note that the face maps are continuous, and in fact topological embeddings.

The next step is to define simplices on arbitrary topological spaces.

Convention 1.2. For the rest of this chapter, let R be a commutative ring and M be a left R-module. ⊙

Definition 1.3. Let X be a topological space and n be a nonnegative integer. A *singular n-simplex in X* is a continuous map $\sigma : \Delta^n \to X$. ◊

Notation 1.4. Let X be a topological space and $n \geq 0$. We write $\mathcal{S}_n(X)$ for the set of singular n-simplices on X. ○

Recall the definition of the M-linearisation of a set (see Definition B.4).

Definition 1.5. Let X be a topological space and $n \geq 0$. We define $C_n(X; M) \coloneqq M[\mathcal{S}_n(X)]$. We shorten $C_n(X; \mathbb{Z})$ to $C_n(X)$ (where $\mathbb{Z}$ is understood to be a $\mathbb{Z}$-module). ◊

Remark 1.6. For us, it is convenient to also set $C_n(X; M) = 0$ for negative n, because of the technicality in Definition 1.46 of a generalised homology theory that we wish them to be defined for all integers n, and not just for nonnegative ones. Note that setting $C_n(X; M) = 0$ for integers $n < 0$ is in line with the definition of $C_n(X; M)$ for $n \geq 0$. Indeed, if $n < 0$, then it is acceptable to say that $\{\sigma : \Delta^n \to X\} = \varnothing$, and hence $C_n(X; M) = M[\varnothing] = 0$. ▽

Remark 1.7. We must stress that, in general, a singular n-simplex in X *does not* need to be an element of $C_n(X; M)$, because there does not need to be an element $1 \in M$. Only M-linear combinations of such simplices are contained in $C_n(X; M)$. ▽

Convention 1.8. We write the elements of $C_n(X; M)$ as $\sum_{\sigma \in \mathcal{S}_n(X)} m_\sigma \sigma$ and do not mention that, of course, m_σ must be zero for all but finitely many σ. This is implicitly assumed specifically in this notation. ⊙

We are going to build a chain complex out of our above defined modules, and therefore need to define differentials. Let a topological space X be given. For $n \geq 1$, a face map $\delta_i : \Delta^{n-1} \to \Delta^n$ induces an R-linear map

$$(\delta_i)_* : C_n(X; M) \to C_{n-1}(X; M), \sum_{\sigma \in \mathcal{S}_n(X)} m_\sigma \sigma \mapsto \sum_{\sigma \in \mathcal{S}_n(X)} m_\sigma (\sigma \circ \delta_i). \tag{1.2}$$

This can be captured in the language of category theory as follows.

Notation 1.9. We write $\mathrm{Map}(\,\cdot\,, \,\cdot\,)$ for the Hom-functor $\mathsf{Top}^{\mathrm{opp}} \times \mathsf{Top} \to \mathsf{Set}$. For any two topological spaces X and Y, we call $\mathrm{Map}(X, Y)$ a *mapping space.* ○

Remark 1.10. We will later, in Definition 6.1 to be precise, equip these mapping spaces with a topology, which explains why we call them spaces. However, until then, these spaces are only sets. ▽

We see that $\mathcal{S}_n(X) = \mathrm{Map}(\Delta^n, X)$, so the face-maps induce a map of sets $\mathrm{Map}(\delta_i, X) : \mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X)$. Applying the functor $M[\,\cdot\,]$ to this map yields an R-linear map $C_n(X; M) = M[\mathcal{S}_n(X)] \to M[\mathcal{S}_{n-1}(X))] = C_{n-1}(X; M)$, which is of the form (1.2). We also obtain the following proposition.

Proposition 1.11. *For each $n \geq 0$, $\mathcal{S}_n$ is a functor* $\mathsf{Top} \to \mathsf{Set}$. *Consequently, $C_n(\,\cdot\,; M) = M[\mathcal{S}_n(\,\cdot\,)]$ is a functor* $\mathsf{Top} \to {}_R\mathsf{Mod}$ *for all $n \in \mathbb{Z}$.* □

Remark 1.12. [23] In fact, even more is true: the face-maps themselves are actually induced by a more abstract morphism between totally ordered sets of nonnegative integers: if we let Δ be the category of ordered sets $[n] = (0 < 1 < \ldots < n)$ and order preserving maps between them (in the weakly monotone sense, so these maps do not need to preserve the strict order $<$, but only the order $\leq$), then the collection of standard n-simplices is actually a functor $\Delta \to \mathsf{Set}$ which sends $[n]$ to Δ^n and an order preserving map $\alpha : [m] \to [n]$ to the map $\alpha_* : \Delta^m \to \Delta^n$ determined by sending a basis vector e_i of Δ^m to $e_{\alpha(i)}$ of Δ^n. The face-maps are induced by the unique order preserving map $[n-1] \to [n]$ which maps an i to $i+1$ and $i-1$ to $i-1$, as is now clear from (1.1).

A *simplicial set* is a functor $\Delta^{\mathrm{opp}} \to \mathsf{Set}$, and the category of all simplicial sets is denoted by sSet (which is a functor category, see Definition A.22). We see that for each topological space X, $\mathcal{S}(X)$ is actually a simplicial set, which sends a map $\alpha : [m] \to [n]$ in Δ to the map $\mathcal{S}_n(X) \to \mathcal{S}_m(X), \sigma \mapsto \sigma \circ \alpha_*$. Furthermore, $\mathcal{S}$ itself is a functor $\mathsf{Top} \to \mathsf{sSet}$, which sends a continuous map $f : X \to Y$ to the natural transformation $\mathcal{S}(X) \to \mathcal{S}(Y)$ consisting of maps $\mathcal{S}_n(X) \to \mathcal{S}_n(Y) : \sigma \mapsto f \circ \sigma$.

The diagrams below summarise the situation: from left to right, it shows the functor $\Delta \to \mathsf{Set}$, the simplicial functor $\mathcal{S} : \mathsf{Top} \to \mathsf{sSet}$, and the action of the simplicial set $\mathcal{S}(X)$ and a morphism $\mathcal{S}(f) : \mathcal{S}(X) \to \mathcal{S}(Y)$ of simplicial sets.

$$\begin{tikzcd}
\Delta \arrow[r] & \mathsf{Set} \\
{[m]} \arrow[r, mapsto] \arrow[d, "\alpha"] & \Delta^m \arrow[d, "\alpha_*"] \\
{[n]} \arrow[r, mapsto] & \Delta^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Top} \arrow[r, "\mathcal{S}"] & \mathsf{sSet} \\
X \arrow[r, mapsto] \arrow[d, "f"] & \mathcal{S}(X) \arrow[d, "\mathcal{S}(f)"] \\
Y \arrow[r, mapsto] & \mathcal{S}(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^{\mathrm{opp}} \arrow[r, "\mathcal{S}(X)"] & \mathsf{Set} \\
{[n]} \arrow[r, mapsto] \arrow[d, "\alpha^{\mathrm{opp}}"] & \mathcal{S}_n(f) \arrow[d, "-\circ\alpha_*"] \\
{[m]} \arrow[r, mapsto] & \mathcal{S}_m(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S}(X) \arrow[r, "\mathcal{S}(f)"] & \mathcal{S}(Y) \\
\mathcal{S}_n(X) \arrow[r, "-\circ\alpha_*"] \arrow[d, "f\circ-"] & \mathcal{S}_m(X) \arrow[d, "f\circ-"] \\
\mathcal{S}_n(Y) \arrow[r, "-\circ\alpha_*"] & \mathcal{S}_m(Y)
\end{tikzcd}$$

(†) M-linearising the right-most diagram, we see that in particular for any continuous map $f : X \to Y$ and any face map δ_i, the diagram

$$\begin{tikzcd}
C_n(X; M) \arrow[r, "(\delta_i)_*"] \arrow[d, "f_*"] & C_{n-1}(X; M) \arrow[d, "f_*"] \\
C_n(Y; M) \arrow[r, "(\delta_i)_*"] & C_{n-1}(Y; M)
\end{tikzcd}$$

is commutative. ▽

Now we define our differentials.

Definition 1.13. Let X be a topological space. For $n \geq 1$, we define the map $\partial_n : C_n(X; M) \to C_{n-1}(X; M)$ by

$$\partial_n = \sum_{i=0}^{n} (-1)^i (\delta_i)_*,$$

and we let $\partial_0 : C_0(X; M) \to 0$ be the zero map. ♢

Explicitly, these maps are given by

$$\partial_n\Bigg(\sum_{\sigma \in \mathcal{S}_n(X)} m_\sigma \sigma \Bigg) = \sum_{\sigma \in \mathcal{S}_n(X)} \sum_{i=0}^{n} (-1)^i m_\sigma (\sigma \circ \delta_i).$$

The geometric motivation for this definition lies in the observation that the differential sends a singular n-simplex to its boundary. Intuitively, for such a simplex σ, the $(n-1)$-simplex $\sigma \circ \delta_i$ is the part of the boundary of σ opposite to $\sigma(e_i)$. The sum in the expression of the differential is alternating to make the orientations of the simplices on the boundary match.

The maps ∂_n are called the *singular boundary operators* for the above reason. They are also clearly R-linear. The following lemma is the last ingredient we need to define the singular chain complex.

Lemma 1.14. *For each n, it holds that $\partial_{n-1} \circ \partial_n = 0$.*

Proof. The proof consists of elementary algebra and is rather straightforward, and hence omitted. It is given in [23], however. □

Definition 1.15. For a topological space X, the chain complex $(C_\bullet(X; M), \partial_\bullet)$ is called the *singular chain complex (with coefficients in M).* ♢

For a topological space X, the associated singular chain complex looks like this:

$$\begin{tikzcd} \dots \arrow[r, "\partial_3"] & C_2(X;M) \arrow[r, "\partial_2"] & C_1(X;M) \arrow[r, "\partial_1"] & C_0(X;M) \arrow[r, "\partial_0"] & 0 \arrow[r] & \dots \end{tikzcd}$$

Definition 1.16. Let X be a topological space. The *n-th singular homology module (with coefficients in M)* $\mathrm{H}_n(X; M)$ is defined as the n-th homology module of the singular chain complex $C_\bullet(X; M)$, that is,

$$\mathrm{H}_n(X;M) = \mathrm{H}_n(C_\bullet(X;M)) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

We shorten $\mathrm{H}_n(X; \mathbb{Z})$ to $\mathrm{H}_n(X)$. ♢

Remark 1.17. We are now in a position to verify that singular homology does what we promised. Indeed, the kernel of a differential consists of singular n-chains with zero boundary. For 1-chains, this simply means that the 1-chain is a loop consisting of singular 1-simplices on X, and for 2-chains, this means that the singular 2-simplices of which a chain consists form a "closed" shape, analoguous to the boundary of a tetrahedron.

The differential maps an n-chain to its boundary chain. If therefore every cycle (element of the kernel of a differential) lies in the image of the differential, there is for every "closed" shape in X a singular simplex which has this shape as boundary. In other words, we can then "fill in every n-simplex on X".

Singular homology measures as a quotient module the extent to which this fails to be the case, and therefore measures if the space X has holes in it, and measures a bit more abstractly also their size. ▽

As we saw in Remark 1.12, for any continuous map $f : X \to Y$ between topological spaces, the induced R-linear maps $f_* : C_n(X; M) \to C_n(Y; M)$ commute with the induced maps $(\delta_i)_*$. By R-linearity of both maps, and since the differential is an R-linear combination of the maps $(\delta_i)_*$, it follows that $\partial \circ f_* = f_* \circ \partial$, and the diagram

$$\begin{tikzcd} \dots \arrow[r, "\partial"] & C_{n+1}(X;M) \arrow[r, "\partial"] \arrow[d, "f_*"] & C_n(X;M) \arrow[r, "\partial"] \arrow[d, "f_*"] & C_{n-1}(X;M) \arrow[r, "\partial"] \arrow[d, "f_*"] & \dots \\ \dots \arrow[r, "\partial"] & C_{n+1}(Y;M) \arrow[r, "\partial"] & C_n(Y;M) \arrow[r, "\partial"] & C_{n-1}(Y;M) \arrow[r, "\partial"] & \dots \end{tikzcd}$$

commutes.

It is clear from the definition that $(\mathrm{id}_X)_*$ equals the identity map on the chain complex for each topological space X, and there is an equality of induced chain maps $(gf)_* = g_* f_*$ for all composable continuous maps f and g. Indeed, $C_n(\,\cdot\,; M)$ is a functor for each n (as it is trivially also a functor in negative degrees), so in each degree we have $((gf)_*)_n = (g_*)_n\,(f_*)_n$. Hence taking the singular chain complex is a functor $C_\bullet(\,\cdot\,; M) : \mathsf{Top} \to {}_R\mathsf{Chain}$.

Proposition 1.18. *The n-th singular homology $\mathrm{H}_n(\,\cdot\,; M) = \mathrm{H}_n \circ C_\bullet(\,\cdot\,; M)$ is a functor $\mathsf{Top} \to {}_R\mathsf{Mod}$.* □

Corollary 1.19. *Let X and Y be topological spaces. If X and Y are homeomorphic, then $\mathrm{H}_n(X; M) \cong \mathrm{H}_n(Y; M)$ for all n.*

Proof. This follows directly from Corollary A.15. □

Lemma 1.20. *Let X be a topological space. Then $\mathrm{H}_n(X;M) = 0$ for all $n < 0$.* ☐

Therefore, we usually implicitly assume we are dealing with nonnegative n when talking about singular homology. There is nothing interesting happening when n is negative.

So what is the situation? For any continuous map $f : X \to Y$ of topological spaces, there are induced maps $f_* = \mathrm{H}_n(f;M)$ on singular homology by virtue of the composition of functors $\mathrm{H}_n \circ C_\bullet(\,\cdot\,;M) : \mathsf{Top} \to {}_R\mathsf{Mod}$. Explicitly, for nonnegative n they are given by

$$f_* : \mathrm{H}_n(X;M) \to \mathrm{H}_n(Y;M), \left[\sum_{\sigma\in\mathcal{S}_n(X)} m_\sigma\sigma\right] \mapsto \left[f_*\left(\sum_{\sigma\in\mathcal{S}_n(X)} m_\sigma\sigma\right)\right] = \left[\sum_{\sigma:\Delta^n\to X} m_\sigma(f\circ\sigma)\right] \tag{1.3}$$

for $\sum_{\sigma\in\mathcal{S}_n(X)} m_\sigma\sigma \in \ker\partial_n$. It is maybe a good time to start computing some singular homology modules. The following two examples are admittedly the easiest ones, but they are important nonetheless.

Example 1.21. The empty space $\varnothing$ has a trivial singular chain complex by definition (see Definition B.4), and therefore $\mathrm{H}_n(\varnothing;M) = 0$ for all n. △

Example 1.22. [23] Consider the one-point space $*$. For each $n \geq 0$, there is precisely one continuous map $\sigma : \Delta^n \to *$. If $S = \{s\}$ is a set with only one element, there is a canonical isomorphism $M[S] \to M, ms \mapsto m$, so $C_n(*;M) \cong M$ for $n \geq 0$ (and zero in negative degrees). Under these isomorphisms, for $n > 1$ the differential ∂_n is associated to

$$d_n : M \to M : m \mapsto \sum_{i=0}^{n}(-1)^i m = \begin{cases} m, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

The singular chain complex $C_\bullet(*;M)$ thus is isomorphic to the chain complex

$$\dots \xrightarrow{d_3} M \xrightarrow{d_2} M \xrightarrow{d_1} M \xrightarrow{d_0} 0 \longrightarrow \dots$$

Since isomorphic chain complexes have isomorphic homology modules, for even $n > 0$ we find $\mathrm{H}_n(*;M) \cong \ker d_n/\operatorname{im} d_{n+1} = 0/0 = 0$, for odd $n > 0$ we find $\mathrm{H}_n(*;M) \cong \ker d_n/\operatorname{im} d_{n+1} = M/M = 0$, and at $n = 0$, it holds that $\mathrm{H}_0(*;M) \cong \ker d_0/\operatorname{im} d_1 = M/0 \cong M$. All in all,

$$\mathrm{H}_n(*;M) \cong \begin{cases} M, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$ △

Proposition 1.23. *If X is a (nonempty) path connected topological space, then $\mathrm{H}_0(X;M) \cong M$.* ☐

Proof. [23] There is a canonical bijection $\mathcal{S}_0(X) \to X, \sigma \mapsto \sigma(1)$, which induces an isomorphism $C_0(X;M) \cong M[X]$. Let two points $x, y \in X$ be given, and let $\gamma : I \to X$ be a path from x to y (which exists, as X is path connected). A homeomorphism $\Delta^1 \xrightarrow{\sim} I$ allows us to consider γ as an element of $\mathcal{S}_1(X)$. From the definition of the singular boundary operator, we see that for all continuous $\sigma : \Delta^1 \to X$ and all $m \in M$ it holds that $\partial(m\sigma) = m\sigma(1) - m\sigma(0)$ (multiplication by m is needed, as there does not need to be an element $1 \in M$, and hence σ cannot automatically be seen as an element of $C_1(X;M)$). Hence $my = mx + \partial(m\gamma)$ for all $m \in M$. However, this implies that $[mx] = [my]$ in $\mathrm{H}_0(X;M)$, so each homology class in $\mathrm{H}_0(X;M)$ is represented by an element of the form $[mx_0]$, where x_0 is a fixed point in X and $m \in M$.

This means that the homomorphism $\psi : M[X] \cong C_0(X;M) \to \mathrm{H}_0(X;M)$ factors as the precomposition of the homomorphism $M[X] \to M, \sum_{x\in X} m_x x \mapsto \sum_{x\in X} m_x$ with the homomorphism $\varphi : M \to \mathrm{H}_0(X;M), m \mapsto [mx_0]$. On the other hand, the unique map $X \to *$ to the one-point space induces the so-called *augmentation map* $\varepsilon : \mathrm{H}_0(X;M) \to \mathrm{H}_0(*;M) \cong M, [mx_0] \mapsto m$ (by (1.3), and where the isomorphism was shown in Example 1.22). We see that ε and φ are each other's inverses, and therefore they are both isomorphisms. This shows that $\mathrm{H}_0(X;M) \cong M$. ☐

1.2 Relative homology

Singular homology also comes with a relative variant. Heuristically, it measures the extent to which the homology of a topological space X agrees with the homology of a subspace of X. It will be very convenient to consider a category consisting of such pairs.

Definition 1.24. A *pair of topological spaces* (X, X') consists of a topological space X and a subspace $X' \subseteq X$. A *morphism of pairs* $f : (X, X') \to (Y, Y')$ is a continuous map $f : X \to Y$ such that $f(X') \subseteq Y'$, that is, such that f restricts to a map $X' \to Y'$. $\Diamond$

Notation 1.25. The category of topological pairs is denoted by $\mathsf{Top}(2)$. $\bigcirc$

Remark 1.26. (†) There are a few standard functors between Top, Top_* and $\mathsf{Top}(2)$ that we will encounter later on. Therefore we list them here.

Firstly, there is a functor $J : \mathsf{Top} \to \mathsf{Top}(2), X \mapsto (X, \varnothing)$, which sends a map $f : X \to Y$ to $If = f : (X, \varnothing) \to (Y, \varnothing)$. We see that I is injective on objects and morphisms, and fully faithful, so Top can be considered as a full subcategory of $\mathsf{Top}(2)$. Moreover, it is left adjoint to the forgetful functor $\mathsf{Top}(2) \to \mathsf{Top}$ that forgets the subspace, which is not difficult to show.

Secondly, there is a fully faithful inclusion $\mathsf{Top}_* \to \mathsf{Top}(2)$, which is right adjoint to the quotient functor $Q : \mathsf{Top}(2) \to \mathsf{Top}_*, (X, X') \to (X/X', *)$, where the base point of X/X' is the equivalence class of the set X'. This adjunction is basically a restatement of the universal property of the quotient topology.

Thirdly, note that the forgetful functor $\mathsf{Top}_* \to \mathsf{Top}$ is right adjoint to the functor $\cdot_+ : \mathsf{Top} \to \mathsf{Top}_*, X \mapsto X_+ \coloneqq (X \sqcup \{*\}, *)$, which adds a discrete base point to a space. This is actually a consequence of the adjoint functors of the previous paragraphs, since $X_+ = X/\varnothing$ for any topological space X. $\triangledown$

Let (X, X') be a pair of a topological spaces. We will again follow [23] and generalise the concepts according to [7]. The inclusion $\iota : X' \hookrightarrow X$ induces a chain map $\iota_* : C_\bullet(X'; M) \to C_\bullet(X; M)$, and from its definition, it is clear that it consists of inclusions $\iota_n : C_n(X'; M) \hookrightarrow C_n(X; M)$ in each degree: this follows from the observation that any singular n-simplex on X' is also a singular n-simplex on X. A consequence of this observation is that the singular boundary operator belonging to $C_\bullet(X'; M)$ is simply the one belonging to $C_\bullet(X; M)$ restricted to the respective submodules. Example B.28 allows us to construct a short exact sequence of chain complexes

$$0 \longrightarrow C'_\bullet(X; M) \xrightarrow{\iota_*} C_\bullet(X; M) \xrightarrow{\pi_\bullet} C_\bullet(X; M)/C'_\bullet(X; M) \longrightarrow 0. \tag{1.4}$$

Definition 1.27. The *relative chain complex* of a pair of spaces (X, X') is defined as $C_\bullet(X, X'; M) \coloneqq C_\bullet(X; M)/C'_\bullet(X; M)$. The *$n$-th relative homology module* is the n-th homology of the relative chain complex, that is,

$$\mathrm{H}_n(X, X'; M) \coloneqq \mathrm{H}_n(C_\bullet(X, X'; M)).$$ $\Diamond$

Proposition 1.28. *The relative chain complex is a functor $C_\bullet(\,\cdot\,, \cdot\,; M) : \mathsf{Top}(2) \to {}_R\mathsf{Chain}$, and for each n, relative homology is a functor $\mathrm{H}_n(\,\cdot\,, \cdot\,; M) : \mathsf{Top}(2) \to {}_R\mathsf{Mod}$.*

Proof. As shown in Example B.32, a morphism of pairs $f : (X, X') \to (Y, Y')$ induces a chain map between relative chain complexes, and a R-module homomorphism between the relative homology modules, from the explicit description of which both can easily seen to be functorial. $\square$

Theorem 1.29. *Let (X, X') be a pair of topological spaces, and use the notation of (1.4). Then there is a long exact sequence*

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_2(X; M) \arrow[r, "\mathrm{H}_2(\pi)"] & \mathrm{H}_2(X, X'; M) \arrow[dll, "\alpha_2"] & \\
\mathrm{H}_1(X'; M) \arrow[r, "\mathrm{H}_1(\iota)"] & \mathrm{H}_1(X; M) \arrow[r, "\mathrm{H}_1(\pi)"] & \mathrm{H}_1(X, X'; M) \arrow[dll, "\alpha_1"] & \\
\mathrm{H}_0(X'; M) \arrow[r, "\mathrm{H}_0(\iota)"] & \mathrm{H}_0(X; M) \arrow[r, "\mathrm{H}_0(\pi)"] & \mathrm{H}_0(X, X'; M) \arrow[r] & 0
\end{tikzcd}$$

Moreover, this sequence is natural in (X, X'): if $f : (X, X') \to (Y, Y')$ is a morphism of pairs, then there is a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_1(X;M) \arrow[r, "\mathrm{H}_1(\pi)"] \arrow[d, "\mathrm{H}_1(f)"] & \mathrm{H}_1(X,X';M) \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{H}_1(\bar{f})"] & \mathrm{H}_0(X';M) \arrow[r, "\mathrm{H}_0(\iota)"] \arrow[d, "\mathrm{H}_0(f)"] & \mathrm{H}_0(X;M) \arrow[r, "\mathrm{H}_0(\pi)"] \arrow[d, "\mathrm{H}_0(f)"] & \mathrm{H}_0(X,X';M) \arrow[r] \arrow[d, "\mathrm{H}_0(\bar{f})"] & 0 \\
\dots \arrow[r] & \mathrm{H}_1(Y;M) \arrow[r, "\mathrm{H}_1(\pi)"] & \mathrm{H}_1(Y,Y';M) \arrow[r, "\alpha_1"] & \mathrm{H}_0(Y';M) \arrow[r, "\mathrm{H}_0(\iota)"] & \mathrm{H}_0(Y;M) \arrow[r, "\mathrm{H}_0(\pi)"] & \mathrm{H}_0(Y,Y';M) \arrow[r] & 0
\end{tikzcd}$$

Figure 1.1

Proof. This follows directly from Theorem B.29, and Example B.32. ☐

Remark 1.30. The relative homology modules allow us to measure in a way how the homology of a topological space X relates to the homology of a subspace X'. Note that it does not generally hold that $\mathrm{H}_n(X'; M)$ is a submodule of $\mathrm{H}_n(X; M)$. We will see later in Corollary 1.35 for instance that the zeroth homology of a space X with n path connected components is isomorphic to M^n, and a subspace can have more path connected components than the original space. ▽

Lemma 1.31. *(i) The map $\mathrm{H}_n(X'; M) \to \mathrm{H}_n(X; M)$ induced by the inclusion is an isomorphism for all $n \geq 0$ if and only if $\mathrm{H}_n(X, X'; M) = 0$ for all $n \geq 0$.*

(ii) Let $f : (X, X') \to (Y, Y')$ be a morphism of topological pairs. If two of the maps $\mathrm{H}_n f : \mathrm{H}_n(X'; M) \to \mathrm{H}_n(Y'; M)$, $\mathrm{H}_n f : \mathrm{H}_n(X; M) \to \mathrm{H}_n(Y; M)$ and $\mathrm{H}_n \bar{f} : \mathrm{H}_n(X, X'; M) \to \mathrm{H}_n(Y, Y'; M)$ are isomorphisms for all $n \geq 0$, then so is the other.

Proof. The first statement follows from Lemma B.10(v), and the second from Figure 1.1 and the Five Lemma B.13. ☐

Proposition 1.32. (†) *Let $J : \mathsf{Top} \to \mathsf{Top}(2)$ be the inclusion functor of Remark 1.26. Then there is an isomorphism of functors $\mathrm{H}_n(\,\cdot\,, \cdot\,; M) \circ J \cong \mathrm{H}_n(\,\cdot\,; M)$ for all n. In other words, for any topological space X, there is for all n an isomorphism $\mathrm{H}_n(X, \varnothing; M) \cong \mathrm{H}_n(X; M)$, which is natural in X.*

Proof. This is immediate from Theorem 1.29, Example 1.21 and Lemma A.25. ☐

1.3 Properties of singular homology

Now we are ready to delve into some larger properties of singular homology and relative homology. This section culminates in Definition 1.46 and Theorem 1.47, which summarise the main results, which turn out to be somewhat characteristic of singular homology.

Definition 1.33. Let X be a topological space. The set of path connected components of X is denoted by $\pi_0(X)$. ◊

It is no coincidence that the notation is so similar to that of the fundamental group of X (with a base point). In Definition 4.23 the exact relation is explained.

Proposition 1.34. *Let X be a topological space. Index the path connected components of X as $\pi_0(X) = \{X_\alpha \mid \alpha \in A\}$. Then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism $\bigoplus_{\alpha \in A} \mathrm{H}_n(X_\alpha; M) \xrightarrow{\sim} \mathrm{H}_n(X; M)$ for all n.*

Similarly, Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a disjoint union of spaces (not necessarily the path connected components). Then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism $\bigoplus_{\alpha \in A} \mathrm{H}_n(X_\alpha; M) \xrightarrow{\sim} \mathrm{H}_n(X; M)$ for all n.

Proof. [23] We only prove the first statement, since the second one is proved entirely analogously. Since each n-simplex Δ^n is path connected, there is for each singular n-simplex $\sigma : \Delta^n \to X$ an $\alpha \in A$ such that $\operatorname{im} \sigma \subseteq X_\alpha$. Therefore, we see that $\mathcal{S}_n(X) = \bigsqcup_{\alpha \in A} \mathcal{S}_n(X_\alpha)$. An application of Lemma B.7 now yields an isomorphism $\bigoplus_{\alpha\in A} C_n(X_\alpha\ \ M) \cong C_n(X;M)$, induced by the inclusions $\iota_\alpha : X_\alpha \hookrightarrow X$. As mentioned in Example B.28, each map ι_α induce a chain map $C_\bullet(X_\alpha;M) \to C_\bullet(X;M)$, and together they induce a chain map $\bigoplus_{\alpha\in A} C_\bullet(X_\alpha;M) \to C_\bullet(X;M)$, which is an isomorphism since it is an isomorphism in each degree (see Lemma B.23). One can demonstrate that $\mathrm{H}_n(\bigoplus_{\alpha\in A} C_\bullet(X_\alpha;M)) \cong \bigoplus_{\alpha\in A} \mathrm{H}_n(C_\bullet(X_\alpha;M))$, so now Proposition B.26 yields the desired isomorphism. □

Corollary 1.35. *Let X be a topological space. Then $\mathrm{H}_0(X;M) \cong \bigoplus_{\pi_0(X)} M$.*

Proof. This is immediate from Proposition 1.34, Proposition 1.23, and Example 1.21 in case X is empty. □

We now move on to the most important property of singular homology, for our purposes at least, namely the homotopy invariance of it. We first need to know what we understand under a homotopy of a map between pairs of spaces.

Convention 1.36. Unless explicitly stated otherwise, we write I for the unit interval $[0,1]$ (with inherited Euclidean topology). ⊙

Definition 1.37. Let $f, g : (X, X') \to (Y, Y')$ be two maps of pairs of topological spaces. Then f and g are *homotopic* if there exists a homotopy $H : X \times I \to Y$ between f and g (that is, $H(\,\cdot\,,0) = f$ and $H(\,\cdot\,,1) = g$) such that $H(X' \times I) \subseteq Y'$. ◊

In other words, $f : (X,X') \to (Y,Y')$ and $g : (X,X') \to (Y,Y')$ are homotopic if there exists a homotopy H between f and g that is also a map $(X,X') \to (Y,Y')$ at each time $t \in I$.

Being homotopic is an equivalence relation between maps of pairs of spaces that respects composition, and therefore we have also have a homotopy category of pairs of topological spaces.

Notation 1.38. The category of pairs of topological spaces and equivalence classes of homotopic maps between them is denoted by $\mathsf{hTop}(2)$, the *homotopy category of topological pairs.* ○

Theorem 1.39. (Homotopy invariance of singular homology) *Let (X,X') and (Y,Y') be two pairs of topological spaces, and suppose $f, g : (X,X') \to (Y,Y')$ are homotopic maps. Then $\mathrm{H}_n f = \mathrm{H}_n g$ as maps $\mathrm{H}_n(X,X';M) \to \mathrm{H}_n(Y,Y';M)$ for all n.*

Proof. See [20] for a direct proof, and [23] for a more category theoretical proof (which has the preference of the author). Both proofs can be generalised to work for R-modules as well. The main idea of the latter proof is that $\mathrm{H}(\,\cdot\,;M)$ is a composition

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "\mathcal{S}"] & \mathsf{sSet} \arrow[r] & {}_R\mathsf{Chain} \arrow[r, "\mathrm{H}_n"] & {}_R\mathsf{Mod} \end{tikzcd}$$

of functors, and in the first three of the above categories there is a notion of homotopy. In case of ${}_R\mathsf{Chain}$, it is a chain homotopy as defined in Definition B.33. By Proposition B.37, it suffices to show that the first two functors above preserve the particular notion of homotopy in each category, and this is exactly what that proof does. □

We deduce, just like in [20] or [23], the following consequences.

Corollary 1.40. *Let X and Y be two topological spaces, and suppose $f, g : X \to Y$ are homotopic maps. Then $\mathrm{H}_n f = \mathrm{H}_n g$ as maps $\mathrm{H}_n(X;M) \to \mathrm{H}_n(Y;M)$ for all n.*

Proof. A map $X \to Y$ is the same as a map $(X,\varnothing) \to (Y,\varnothing)$ (cf. Remark 1.26), and a homotopy between two maps $X \to Y$ is the same thing as a homotopy between two maps $(X,\varnothing) \to (Y,\varnothing)$. Theorem 1.39 and Proposition 1.32 then yield the result. □

Corollary 1.41. *Let $f : X \to Y$ be homotopy equivalence of topological spaces. Then $\mathrm{H}_n f : \mathrm{H}_n(X;M) \to \mathrm{H}_n(Y;M)$ is an isomorphism for each n.*

Moreover, if X' and Y' are subspaces of X and Y respectively such that f restricts to a homotopy equivalence $X' \to Y'$, then the commutative diagram of long exact sequences of homology in Figure 1.1 has all isomorphisms as vertical arrows.

Proof. This is immediate from the previous corollary and Lemma 1.31. ☐

Remark 1.42. The statement of the theorem and corollaries above is equivalent to saying that the singular homology functors $\mathsf{Top} \to {}_R\mathsf{Mod}$ and the relative singular homology functors $\mathsf{Top}(2) \to {}_R\mathsf{Mod}$ factor through the categories hTop (see Example A.4(vi)) and $\mathsf{hTop}(2)$, respectively. This means that singular homology cannot distinguish between objects on the level of homotopy equivalence: it only "sees information up to homotopy and homotopy equivalence". ▽

Corollary 1.43. *If X is a contractible space, then*

$$\mathrm{H}_n(X;M) \cong \begin{cases} M, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$ ☐

One of the most computationally important properties of singular homology is the following theorem. It allows us to remove a suitable part of our space without affecting the relative homology modules. As a result, it can considerably simplify certain computations of homology, when used in combination with e.g. the long exact sequence of singular homology.

Theorem 1.44. (Excision Theorem) *Let (X, X') be a pair of topological spaces, and suppose $Y \subseteq X'$ is a subspace such that $\overline{Y} \subseteq (X')^\circ$. Then the inclusion $X \setminus Y \hookrightarrow X$ induces for each n an isomorphism $\mathrm{H}_n(X \setminus Y, X' \setminus Y; M) \xrightarrow{\sim} \mathrm{H}_n(X, X'; M)$.*

Proof. The proof is rather involved and technical. In [23] and [11] it is given. The basic idea is that singular simplices on X can be subdivided into smaller simplices in such a way that the relative homology remains unchanged. Therefore, we can replace simplices by smaller ones for any finite amount of times. Since a simplex is quasi-compact in X (as the image of a quasi-compact space), at some point the smaller simplex will either lie completely in $X \setminus Y$ or in $(X')^\circ$. Therefore, each cycle is equivalent to one that does not intersect Y, and hence the isomorphism in the Excision Theorem follows. ☐

Remark 1.45. [11] Note that the Excision Theorem is equivalent to the following statement:

> Let X be a topological space with subspaces A and B such that $X = A^\circ \cup B^\circ$. Then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $\mathrm{H}_n(A, A \cap B; M) \xrightarrow{\sim} \mathrm{H}_n(X, B; M)$ for each n.

Namely, given the Excision Theorem, then for X, A and B as above, we can take $Y = X \setminus A$, which satisfies $\overline{Y} = X \setminus A^\circ \subseteq B^\circ$, so we can use the Excision Theorem. Noting that $X \setminus Y = A$ and $B \setminus Y = A \cap B$, it tells us that the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $\mathrm{H}_n(A, A \cap B; M) \xrightarrow{\sim} \mathrm{H}_n(X, B; M)$ for each n.

Conversely, if $\mathrm{H}_n(\,\cdot\,, \cdot\,; M)$ satisfies the statement above, then for a pair (X, X') of topological spaces a subspace $Y \subseteq X'$ satisfying $\overline{Y} \subseteq (X')^\circ$, we define $A = X \setminus Y$ and $B = X'$. Then $A^\circ = X \setminus \overline{Y} \supseteq X \setminus (X')^\circ = X \setminus B^\circ$, so $A^\circ \cup B^\circ = X$. The given statement then yields the Excision Theorem. ▽

The key results about singular homology turn out to more or less define singular homology up to natural isomorphism, or at least on a suitable subcategory of Top, which we will define in Chapter 3. Therefore, they are known as the *Eilenberg-Steenrod axioms for homology.* We will come back to this uniqueness statement in Remark 6.64. For now, let us state these axioms

Definition 1.46. (Eilenberg-Steenrod axioms for homology) [13] A sequence of functors $\mathrm{h}_n(\,\cdot\,, \cdot\,) : \mathsf{Top}(2) \to {}_R\mathsf{Mod}$ with $n \in \mathbb{Z}$, together with homomorphisms $\alpha_{n,(X,X')} : \mathrm{h}_n(X, X') \to \mathrm{h}_{n-1}(X', \varnothing)$ that are natural in the topological pair (X, X'), is called a *generalised homology theory* if it satisfies the following four properties (where it is understood that $\mathrm{h}_n(X)$ is to be interpreted as $\mathrm{h}_n(X, \varnothing)$):

(i) *(Homotopy invariance)* Let (X, X') and (Y, Y') be two pairs of topological spaces, and suppose $f, g : (X, X') \to (Y, Y')$ are homotopic maps. Then $\mathrm{h}_n f = \mathrm{h}_n g$ as maps $\mathrm{h}_n(X, X') \to \mathrm{h}_n(Y, Y')$ for all n.

(ii) *(Excision)* Let X be a topological space with subspaces A and B such that $X = A^\circ \cup B^\circ$. Then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $\mathrm{h}_n(A, A \cap B) \xrightarrow{\sim} \mathrm{h}_n(X, B)$ for each n.

(iii) *(Long exact sequence)* Let (X, X') be a pair of topological spaces. Then the inclusions $X' \hookrightarrow X$ and $(X, \varnothing) \to (X, X')$ induce a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{h}_{n+1}(X) \arrow[r] & \mathrm{h}_{n+1}(X, X') \arrow[dll, "\alpha_{n+1,(X,X')}" description] \\
\mathrm{h}_n(X') \arrow[r] & \mathrm{h}_n(X) \arrow[r] & \mathrm{h}_n(X, X') \arrow[dll, "\alpha_{n,(X,X')}" description] \\
\mathrm{h}_{n-1}(X') \arrow[r] & \mathrm{h}_{n-1}(X) \arrow[r] & \dots
\end{tikzcd}$$

of homology, which is natural in the pair (X, X').

(iv) *(Sums)* Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a disjoint union of spaces. Then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism $\bigoplus_{\alpha \in A} \mathrm{h}_n(X_\alpha) \xrightarrow{\sim} \mathrm{h}_n(X)$ for all n.

The functors h_n are called an *ordinary homology theory* if moreover they satisfy

(v) *(Dimension)* The homology modules of the one-point space satisfy $\mathrm{h}_n(*) = 0$ for all $n \neq 0$. ♢

Theorem 1.47. *For each R-module M, the sequence of relative singular homology functors $\mathrm{H}_n(\,\cdot\,, \,\cdot\,; M)$ is an ordinary homology theory.* □

This concludes our introduction to singular homology. Of course, with these results we are able to compute many homology modules, define new concepts and derive more results, but that is not the purpose of this thesis. In the next chapters, we will introduce singular cohomology and focus on cohomology in general.

Chapter 2

Singular cohomology

In many ways, the theory of singular cohomology is dual to the theory of singular homology. The basic motivation is still the same, namely that we are algebraically measuring the amount and size of holes in a topological space, and our approach is in some sense dual to the one taken with singular homology.

In fact, singular cohomology does not capture more information about a space X than singular homology, which is shown in Theorem 2.20. It only captures the same information slightly differently. For a start, singular cohomology is a contravariant functor, which in practice means that many properties of singular homology have reversed arrows in the case of singular cohomology. Furthermore, singular cohomology actually allows a multiplicative structure, which gives us more ways to reason with it, and in some cases it allows easier or more natural arguments. We will however not need this extra structure, but refer to [13] for an exposition.

It are these slightly different and dualised properties of singular cohomology that make the question of representability a bit more convenient to answer than the case of singular homology.

2.1 Dualising singular homology

Convention 2.1. In this chapter, let R be a *principal ideal domain*, and M a left R-module. ⊙

The construction of singular cohomology follows the construction of the dual cochain complex in Definition B.40. We generalise using [7] the construction of singular cohomology given in [13] for abelian groups to R-modules.

Definition 2.2. Let X be a topological space. The *singular cochain complex (with coefficients in M)* is defined as $C^\bullet(X; M) \coloneqq \mathrm{Hom}_R(C_\bullet(X; R), M)$. In the case of $R = \mathbb{Z}$, We shorten $C^\bullet(X; \mathbb{Z})$ to $C^\bullet(X)$. ♢

Notation 2.3. We write ∂^* for the codifferentials $\mathrm{Hom}_R(\partial, M)$ of the singular cochain complex. We also slightly adapt the indexing for consistency as follows: $(\partial^*)^n = \mathrm{Hom}_R(\partial_{n+1}, M)$. ○

Remark 2.4. There is an isomorphism $C^\bullet(X; M) \cong \mathrm{Hom}_{\mathsf{Set}}(\mathcal{S}_n(X), M)$ of R-modules (where the right-hand side carries the structure of an R-module inherited from M). Therefore, we can also see the elements of $C^\bullet(X; M)$ as maps of sets from singular n-simplices to M, rather than as R-linear maps from singular cochains with coefficients in R to M. The reader might prefer this to get an idea of the meaning of the singular cochain complex. ▽

Remark 2.5. Since the singular cochain complex $C^\bullet(\,\cdot\,; M)$ is the precomposition of the functor $C_\bullet(\,\cdot\,; R) : \mathsf{Top} \to {}_R\mathsf{Chain}$ with the contravariant Hom-functor $\mathrm{Hom}_R(\,\cdot\,; M) : {}_R\mathsf{Chain}^{\mathrm{opp}} \to {}_R\mathsf{cChain}$ (see Lemma B.39), it itself is a functor $\mathsf{Top}^{\mathrm{opp}} \to {}_R\mathsf{cChain}$. ▽

Explicitly, the singular cochain complex with coefficients in M of a topological space X is the cochain complex

$$0 \longrightarrow \mathrm{Hom}_R(C_0(X; R), M) \xrightarrow{(\partial^*)^0} \mathrm{Hom}_R(C_1(X; R), M) \xrightarrow{(\partial^*)^1} \mathrm{Hom}_R(C_2(X; R), M) \xrightarrow{(\partial^*)^2} \dots$$

Any continuous map $f : X \to Y$ of topological spaces induces a map $f^* : C^\bullet(Y; M) \to C^\bullet(X; M)$ by functoriality of the singular cochain complex. Explicitly, it is given by

$$(f^*)^n : C^n(Y; M) \to C^n(X; M), \varphi \mapsto \varphi \circ (f_*)_n,$$

where $f_* : C_\bullet(X; R) \to C_\bullet(Y; R)$ is the induced map on the singular chain complexes.

Definition 2.6. Let X be a topological space. The *n-th singular cohomology module (with coefficients in M)* $\mathrm{H}^n(X; M)$ is defined as the n-th cohomology module of the singular cochain complex $C^\bullet(X; M)$, that is,

$$\mathrm{H}^n(X; M) \coloneqq \mathrm{H}^n(C^\bullet(X; M)) = \ker{(\partial^*)^n} / \mathrm{im}\,{(\partial^*)^{n-1}}.$$

We shorten $\mathrm{H}^n(X; \mathbb{Z})$ to $\mathrm{H}^n(X)$. ◊

Proposition 2.7. *Singular cohomology is a functor* $\mathrm{H}^n(\,\cdot\,; M) : \mathsf{Top}^{\mathrm{opp}} \mapsto {}_R\mathsf{Mod}$. ☐

Corollary 2.8. *Let X and Y be topological spaces. If X and Y are homeomorphic, then for each n there is an isomorphism* $\mathrm{H}^n(X; M) \cong \mathrm{H}^n(Y; M)$. ☐

Lemma 2.9. *Let X be a topological space. Then* $\mathrm{H}^n(X; M) = 0$ *for all* $n < 0$. ☐

Remark 2.10. By this functoriality, each continuous map $f : X \to Y$ of topological spaces induces a map

$$f^* : \mathrm{H}^n(Y; M) \cong \mathrm{H}^n(X; M), [\varphi] \mapsto [\varphi \circ (f_*)_n]$$

on cohomology modules. ▽

Example 2.11. As we saw in Example 1.22, the singular chain complex $C_\bullet(*; R)$ is isomorphic to the chain complex

$$\ldots \xrightarrow{\mathrm{id}} R \xrightarrow{0} R \xrightarrow{\mathrm{id}} R \xrightarrow{0} R \longrightarrow 0.$$

Applying the $\mathrm{Hom}_R(\,\cdot\,, M)$-functor and using the isomorphism $\mathrm{Hom}_R(R^{(A)}, M) \cong M^A$ for all sets A, we find that $C^\bullet(*; M)$ is isomorphic to the cochain complex

$$0 \longrightarrow M \xrightarrow{0} M \xrightarrow{\mathrm{id}} M \xrightarrow{0} M \xrightarrow{\mathrm{id}} \ldots$$

The cohomology of this complex is easy to take and gives us $\mathrm{H}^n(*; M) = 0$ if $n \geq 1$ and $\mathrm{H}^0(*; M) \cong M$.

The empty space $\varnothing$ had a trivial singular chain complex $C_\bullet(\varnothing; R)$, and hence $C^\bullet(\varnothing; M)$ is trivial as well. Therefore, $\mathrm{H}^n(\varnothing; M) = 0$ for all n. △

For a pair of spaces (X, X') we can also define relative singular cohomology, by dualising the relative chain complex $C_\bullet(X, X'; M)$: we let the *relative singular cochain complex* be the cochain complex $C^\bullet(X, X'; M) \coloneqq \mathrm{Hom}_R(C_\bullet(X, X'; R), M)$. Of course, we wish to get a long exact sequence of cohomology, so we need the following proposition.

Proposition 2.12. *Let (X, X') be a pair of spaces. With the notation as in Equation (1.4), there is a short exact sequence*

$$0 \longrightarrow C^\bullet(X, X'; M) \xrightarrow{(\pi^*)^\bullet} C^\bullet(X; M) \xrightarrow{\iota^*} C^\bullet(X'; M) \longrightarrow 0 \tag{2.1}$$

of cochain complexes.

Proof. This follows from exactness of (1.4) and Corollary B.43. ☐

Remark 2.13. Note that this proposition also tells us that it would not have made a difference if we had mimicked the construction of the relative chain complex and defined the relative cochain complex as the kernel of the map $C^\bullet(X; M) \to C^\bullet(X'; M)$ induced by the inclusion $X' \hookrightarrow X$, instead of dualising the relative chain complex. We can therefore confidently define relative cohomology below. ▽

Remark 2.14. The reason why we take R to be a PID in this chapter (see Convention 2.1) is to be able to apply the Algebraic Universal Coefficient Theorem B.44. Corollary B.43 is a corollary of it, so the result of the above proposition is not generally true if R is not a PID. Likewise, the Universal Coefficient Theorem 2.20 below requires R to be a PID, for the same reason. ▽

Remark 2.15. As a continuation of Remark 2.4, note that there is an R-linear isomorphism between $C^\bullet(X, X'; M)$ and the set of maps of sets $\mathcal{S}_n(X) \to M$ whose restriction to X' is trivial. ▽

Definition 2.16. Let (X, X') be a pair of topological spaces. Their *n-th relative cohomology module* is defined as the cohomology of the relative cochain complex. In other words,

$$\mathrm{H}^n(X, X'; M) := \mathrm{H}^n(C^\bullet(X, X'; M)). \qquad \diamondsuit$$

Proposition 2.17. *For each n, relative cohomology is a functor $\mathrm{H}^n(\,\cdot\,,\,\cdot\,; M) : \mathsf{Top}(2)^{\mathrm{opp}} \to {}_R\mathsf{Mod}$.* □

Theorem 2.18. *Let (X, X') be a pair of topological spaces, and use the notation of (2.1). Then there is a long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(X, X'; M) \arrow[r, "\mathrm{H}^0(\pi)"] & \mathrm{H}^0(X; M) \arrow[r, "\mathrm{H}^0(\iota)"] & \mathrm{H}^0(X'; M) \arrow[dll, "\alpha_1^*" description] \\
 & \mathrm{H}^1(X, X'; M) \arrow[r, "\mathrm{H}^1(\pi)"] & \mathrm{H}^1(X; M) \arrow[r, "\mathrm{H}^1(\iota)"] & \mathrm{H}^1(X'; M) \arrow[dll, "\alpha_2^*" description] \\
 & \mathrm{H}^2(X, X'; M) \arrow[r, "\mathrm{H}^2(\pi)"] & \mathrm{H}^2(X; M) \arrow[r] & \dots
\end{tikzcd}$$

Moreover, this sequence is natural in the pair (X, X').

Proof. This follows from Proposition 2.12, Theorem B.30, and the statement of Corollary B.31 in case of cochains and cohomology. □

Corollary 2.19. (†) *Let $J : \mathsf{Top} \to \mathsf{Top}(2), X \mapsto (X, \varnothing)$ be the inclusion functor of Remark 1.26. Then there is a natural isomorphism $\mathrm{H}^n(\,\cdot\,,\,\cdot\,; M) \circ J \cong \mathrm{H}^n(\,\cdot\,; M)$ for all n.* □

The Eilenberg-Steenrod axioms for singular homology carry over with appropriate dualising to the case of singular cohomology. However, this dualising is not trivial, and we need some auxiliary results. The following theorem is a direct consequence of the Algebraic Universal Coefficient Theorem B.44.

Theorem 2.20. (Universal Coefficient Theorem) [7] *There is a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_R(\mathrm{H}_{n-1}(X, X'; R), M) \longrightarrow \mathrm{H}^n(X, X'; M) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_n(X, X'; R), M) \longrightarrow 0,$$

which is natural in the pair (X, X'). The sequence is split as well, although not naturally in the pair (X, X'). □

Remark 2.21. The Universal Coefficient Theorem tells us two things at the same time: first, the singular cohomology of a pair (X, X') of spaces can be computed from its singular homology, and hence singular cohomology does not capture more information about the pair than singular homology does (as already mentioned in the beginning of this chapter) [11]. Second, using different coefficients does also not capture more information about the pair: the singular cohomology with coefficients in M depends on the pair (X, X') only via the singular homology with coefficients in R. Hence anything that singular homology with coefficients in R cannot distinguish between is also indistinguishable for singular cohomology with coefficients in M. ▽

Remark 2.22. On a historical side note, we would like to mention that the Universal Coefficient Theorem is directly related to the birth of category theory, as its founders, Saunders MacLane and Samuel Eilenberg, needed a new framework to capture the ideas needed to state and prove the Universal Coefficient Theorem after, thanks to some good fortune, discovering it [22]. ▽

Definition 2.23. (Eilenberg-Steenrod axioms for cohomology) [13] A sequence of functors $\mathrm{h}^n(\,\cdot\,,\,\cdot\,) : \mathsf{Top(2)}^{\mathrm{opp}} \to {}_R\mathsf{Mod}$ with $n \in \mathbb{Z}$, together with homomorphisms $\alpha_{n,(X,X')} : \mathrm{h}^n(X', \varnothing) \to \mathrm{h}^{n+1}(X, X')$ that are natural in the topological pair (X, X'), is called a *generalised cohomology theory* if it satisfies the following four properties (where it is understood that $\mathrm{h}^n(X)$ is to be interpreted as $\mathrm{h}^n(X, \varnothing)$):

(i) *(Homotopy invariance)* Let (X, X') and (Y, Y') be two pairs of topological spaces, and suppose $f, g : (X, X') \to (Y, Y')$ are homotopic maps. Then $\mathrm{h}^n f = \mathrm{h}^n g$ as maps $\mathrm{h}^n(Y, Y') \to \mathrm{h}^n(X, X')$ for all n.

(ii) *(Excision)* Let X be a topological space with subspaces A and B such that $X = A^\circ \cup B^\circ$. Then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $\mathrm{h}^n(X, B) \xrightarrow{\sim} \mathrm{h}^n(A, A \cap B)$ for each n.

(iii) *(Long exact sequence)* Let (X, X') be a pair of topological spaces. Then the inclusions $X' \hookrightarrow X$ and $(X, \varnothing) \to (X, X')$ induce a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{h}^{n-1}(X) \arrow[r] & \mathrm{h}^{n-1}(X') \arrow[dll, "\alpha_{n-1,(X,X')}" description] \\
 & \mathrm{h}^n(X, X') \arrow[r] & \mathrm{h}^n(X) \arrow[r] & \mathrm{h}^n(X') \arrow[dll, "\alpha_{n,(X,X')}" description] \\
 & \mathrm{h}^{n+1}(X, X') \arrow[r] & \mathrm{h}^{n+1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

of cohomology, which is natural in the pair (X, X').

(iv) *(Products)* Let $X = \coprod_{\alpha \in A} X_\alpha$ be a disjoint union of spaces. Then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism $\mathrm{h}_n(X) \xrightarrow{\sim} \prod_{\alpha \in A} \mathrm{h}_n(X_\alpha)$ for all n.

The functors h^n are called an *ordinary cohomology theory* if moreover they satisfy

(v) *(Dimension)* The cohomology modules of the one-point space satisfy $\mathrm{h}^n(*) = 0$ for all $n \neq 0$. ◊

Remark 2.24. [11] Instead of the excision axiom above, we could have taken the dual statement of the Excision Theorem 1.44:

> *(Excision Theorem for singular cohomology)* Let (X, X') be a pair of topological spaces, and suppose $Y \subseteq X'$ is a subspace such that $\overline{Y} \subseteq (X')^\circ$. Then the inclusion $X \setminus Y \hookrightarrow X$ induces for each n an isomorphism $\mathrm{h}^n(X, X') \xrightarrow{\sim} \mathrm{h}^n(X \setminus Y, X' \setminus Y)$.

The equivalence between these two axioms is shown exactly as in Remark 1.45. ▽

Theorem 2.25. *For any R-module M, relative singular cohomology is an ordinary cohomology theory.*

Proof. [13] The natural isomorphism $\mathrm{H}^n(X, \varnothing; M) \cong \mathrm{H}^n(X; M)$ is the content of Corollary 2.19. We will prove the remaining statements in order for $n \geq 0$ (the case of $n < 0$ is trivial).

(i) As mentioned in the "proof" of Theorem 1.39, the two homotopic maps f and g induce chain homotopic maps $f_*, g_* : C_\bullet(X, X'; M) \to C_\bullet(Y, Y'; M)$. Let $(h_n)_{n \in \mathbb{Z}}$ be a homotopy from f_* to g_*. It is straightforward to check that the induced maps between dual cochain modules $(h_n^*)_{n \in \mathbb{Z}}$ form a cochain homotopy between f^* and g^*. Indeed, the chain homotopy identity

$$(f_*)_n - (g_*)_n = \partial_{n+1} h_n + h_{n-1} \partial_n$$

implies

$$(f^*)^n + (g^*)^n = (h_*)^n (\partial^*)^n + (\partial^*)^{n-1} (h_{n-1})^*$$

(with the adapted indexing of the codifferentials as in Notation 2.3). Proposition B.37 applies to the cochain case as well and implies that $\mathrm{H}^n f = \mathrm{H}^n g$.

(ii) The chain complexes $C_\bullet(X, X')$ and $C_\bullet(X \setminus Y, X' \setminus Y)$ are free modules on the sets $\mathcal{S}_n(X) \setminus \mathcal{S}_n(X')$ and $\mathcal{S}_n(X \setminus Y) \setminus \mathcal{S}_n(X' \setminus Y)$, and the Excision Theorem 1.44 implies that the inclusion $X \setminus Y \hookrightarrow X$ induces isomorphisms on all homology modules. Then Corollary B.45 yields induced isomorphisms $\mathrm{H}^n(X, X'; M) \xrightarrow{\sim} \mathrm{H}^n(X \setminus Y, X' \setminus Y; M)$, and by Remark 2.24, this shows that singular cohomology satisfies the excision axiom.

(iii) This is Theorem 2.18.

(iv) By Remark 2.4 and Propositions A.49 and A.50, we have isomorphims of chain complexes

$$C^\bullet(X;M) = C^\bullet(\bigsqcup_{\alpha\in A} X_\alpha; M) \cong \mathrm{Hom}_{\mathsf{Set}}(\bigsqcup_{\alpha\in A} \mathcal{S}_\bullet(X_\alpha), M) \cong \prod_{\alpha\in A} \mathrm{Hom}_{\mathsf{Set}}(\mathcal{S}_\bullet(X_\alpha), M) \cong \prod_{\alpha\in A} C^\bullet(X_\alpha; M),$$

and hence $\mathrm{H}^n(X;M) \cong \prod_{\alpha\in A} \mathrm{H}^n(X_\alpha;M)$.

(v) This is Example 2.11. □

Corollary 2.26. *If X is a contractible space, then*

$$\mathrm{H}^n(X;M) \cong \begin{cases} M, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$ □

2.2 Reduced singular cohomology

The previous corollary shows that the zeroth singular cohomology module with coefficients in M of any contractible space is isomorphic to M (it measures the amount of path-connected components). Since we like to think of these spaces as having no holes, it would be nice to be able to adjust the definition of singular cohomology slightly in order to make the cohomology of any contractible space trivial in any degree. It also seems to be attractive for computational reasons (for instance in long exact sequences). There is another important reason to do this, namely that this allows us to prove representability of singular cohomology more easily. Details of this will be presented in due time, namely in Sections 6.3 and 6.4. We will devote this section to the adjusting the construction of singular cohomology slightly to obtain this new cohomology theory. We dualise the construction of reduced singular homology as given in [23], and generalise again using [7] the construction to R-modules.

Let X be a topological space. The unique continuous map $X \to *$[1] induces maps $\varepsilon_n : C^n(*;M) \to C^n(X;M)$. Since these form a chain map, we obtain the identity $\partial^0\varepsilon_0 = \varepsilon_1\partial^0 = 0$, since $\partial^0 : C_0(*,M) \to C_1(*;M)$ is the zero map (see Example 2.11). Hence we have a chain complex

$$\begin{tikzcd} 0 \arrow[r] & C^0(*;M) \arrow[r, "\varepsilon"] & C^0(X;M) \arrow[r, "\partial^0"] & C^1(X;M) \arrow[r, "\partial^1"] & C^2(X;M) \arrow[r] & \dots \end{tikzcd}$$

where $\varepsilon := \varepsilon_0$ is the *augmentation map*. This chain complex (with $C^n(X;M)$ appearing in degree n) is called the *reduced singular cochain complex*, and we denote it by $\widetilde{C}^\bullet(X;M)$.

We have $C^0(X;M) = \mathrm{Hom}_R(C_0(X;R), M) \cong \mathrm{Hom}_R(R^{(X)}, M) \cong M^X$, and likewise $C^0(*;M) \cong M$, and the augmentation map ε can be seen to correspond to the diagonal $M \to M^X : m \mapsto (m)_{x\in X}$ under these isomorphisms. Therefore, it is injective.

Definition 2.27. Let X be a topological space. Its *n-th reduced singular cohomology module (with coefficients in M* is defined as $\widetilde{\mathrm{H}}^n(X;M) := \mathrm{H}^n(\widetilde{C}^\bullet(X;M))$. ◊

Remark 2.28. For any continuous map $f : X \to Y$ of topological spaces, there is an identity $(X \to *) = (Y \to *) \circ f$. Therefore, there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^0(*;M) \arrow[r, "\varepsilon"] & C^0(X;M) \arrow[r, "\partial^0"] & C^1(X;M) \arrow[r, "\partial^1"] & C^2(X;M) \arrow[r] & \dots \\ 0 \arrow[r] & C^0(*;M) \arrow[r, "\varepsilon"] \arrow[u, "\mathrm{id}"] & C^0(Y;M) \arrow[r, "\partial^0"] \arrow[u, "f^*"'] & C^1(Y;M) \arrow[r, "\partial^1"] \arrow[u, "f^*"] & C^2(Y;M) \arrow[r] \arrow[u, "f^*"] & \dots \end{tikzcd}$$

(where commutativity of all squares but the left one above follows from f already inducing a cochain map on the singular cochain complexes) which shows that $f : X \to Y$ induces a chain map $\widetilde{f}^* : \widetilde{C}^\bullet(Y;M) \to \widetilde{C}_\bullet(X;M)$. Since taking cohomology of a cochain complex is a functor as well, this shows reduced cohomology is a functor $\mathsf{Top}^{\mathrm{opp}} \to {}_R\mathsf{Mod}$. ▽

---

[1] If X is empty, there is a set-theoretical map $\varnothing \to *$, which is vacuously continuous. Although certain statements here will not make sense for an empty space, the conclusions do.

Example 2.29. The reduced singular cohomology of the empty space is from its description easily seen to be trivial for all n except for $n = -1$, when $\widetilde{\mathrm{H}}^{-1}(\varnothing; M) \cong M$. On the other hand, $\widetilde{\mathrm{H}}^{n}(*; M) = 0$ for all n, by a similar computation as in Example 2.11. △

Convention 2.30. Reduced cohomology is, as this example show, more suited for nonempty spaces. To avoid technicalities and pathetic cases for the empty space, when dealing with reduced singular cohomology we will always assume the space we plug in is nonempty. ⊙

For any space X, the map $X \to *$ to the one-point space also induces a map $\mathrm{H}^0(*; M) \to \mathrm{H}^0(X; M)$ by functoriality of singular cohomology. This gives us functors $\operatorname{coker}(\mathrm{H}^n(*; M) \to \mathrm{H}^n(\,\cdot\,; M)) : \mathsf{Top}^{\mathrm{opp}} \to {}_R\mathsf{Mod}, X \mapsto \operatorname{coker}(\mathrm{H}^n(*; M) \to \mathrm{H}^n(X; M))$. Indeed, for any continuous map $f : X \to Y$, it holds true that $(X \to *) = (Y \to *) \circ f$, so

$$(\mathrm{H}^n(*; M) \to \mathrm{H}^n(X; M)) = \mathrm{H}^n f \circ (\mathrm{H}^n(*; M) \to \mathrm{H}^n(Y; M)).$$

This shows that $\mathrm{H}^n f$ passes through the quotient $\operatorname{coker}(\mathrm{H}^n(*; M) \to \mathrm{H}^n(Y; M)) \to \operatorname{coker}(\mathrm{H}^n(*; M) \to \mathrm{H}^n(X; M))$, which establishes $\operatorname{coker}(\mathrm{H}^n(*; M) \to \mathrm{H}_0(\,\cdot\,; M))$ as a functor. Of course, for $n \geq 1$ this equals singular cohomology, since $\mathrm{H}^n(*; M) = 0$ then by Example 2.11. The connection with reduced singular cohomology is now as given in the following proposition.

Proposition 2.31. *There is for each nonzero n a natural isomorphism $\widetilde{\mathrm{H}}^n(\,\cdot\,; M) \cong \mathrm{H}^n(\,\cdot\,; M)$ of functors $\mathsf{Top}\backslash\{\varnothing\}^{\mathrm{opp}} \to {}_R\mathsf{Mod}$. Moreover, there is also a natural isomorphism $\widetilde{\mathrm{H}}^0(\,\cdot\,; M) \cong \operatorname{coker}\big(\mathrm{H}^0(*; M) \to \mathrm{H}^0(\,\cdot\,; M)\big)$ of functors $\mathsf{Top} \setminus \{\varnothing\}^{\mathrm{opp}} \to {}_R\mathsf{Mod}$.*

Proof. The only slightly nontrivial case of the first statement is if $n = -1$. However, the augmentation map ε is injective as we saw above, so $\widetilde{\mathrm{H}}^{-1}(\,\cdot\,; M) \cong 0 \cong \mathrm{H}_{-1}(\,\cdot\,; M)$, where 0 denotes the zero functor $\mathsf{Top}^{\mathrm{opp}} \to {}_R\mathsf{Mod}, X \mapsto 0$.

For the second statement, note that $C^0(*; M) = \mathrm{H}^0(*; M)$ by Example 2.11, and consequently that the diagram

$$\begin{tikzcd}
\mathrm{H}^0(*; M) \arrow[r] \arrow[d, equal] & \mathrm{H}^0(X; M) = \ker \partial^0 \arrow[d, hook] \\
C^0(*; M) \arrow[r, "\varepsilon"] & C^0(X; M)
\end{tikzcd}$$

commutes. This means that $\operatorname{im} \varepsilon = \operatorname{im}\big(\mathrm{H}^0(*; M) \to \mathrm{H}^0(X; M)\big)$ as submodules of $\ker \partial^0 \subseteq C^0(X; M)$. Finally, we find

$$\widetilde{\mathrm{H}}^0(X; M) = \ker \partial^0 / \operatorname{im} \varepsilon = \mathrm{H}^0(X; M)/\operatorname{im}\big(\mathrm{H}^0(*; M) \to \mathrm{H}^0(X; M)\big) = \operatorname{coker}\big(\mathrm{H}^0(*; M) \to \mathrm{H}^0(X; M)\big),$$

showing the claim, as naturality is obvious now. □

Remark 2.32. Just like in the case of ordinary singular cohomology, for a pair of nonempty topological spaces (X, X') the inclusion induces a cochain map $\widetilde{C}^\bullet(X; M) \to \widetilde{C}^\bullet(X'; M)$. Consequently we can set $\widetilde{C}^\bullet(X, X'; M) := \ker\Big(\widetilde{C}^\bullet(X; M) \to \widetilde{C}^\bullet(X'; M)\Big)$, which can be seen to be isomorphic to $C^\bullet(X, X'; M)$, since the module in degree -1 is trivial (here we use Proposition 2.12 to see $C^\bullet(X, X'; M)$ as the kernel of the map $C^\bullet(X; M) \to C^\bullet(X'; M)$). We write the cohomology of this reduced cochain complex as $\widetilde{\mathrm{H}}^n(X, X'; M)$, but note that $\widetilde{\mathrm{H}}^n(X, X'; M) = \mathrm{H}^n(X, X'; M)$ for all n, so this is more for notational consistency. By Theorem B.30, therefore, there is a long exact sequence

$$0 \longrightarrow \widetilde{\mathrm{H}}^0(X, X'; M) \longrightarrow \widetilde{\mathrm{H}}^0(X; M) \longrightarrow \widetilde{\mathrm{H}}^0(X'; M) \longrightarrow \widetilde{\mathrm{H}}^1(X, X'; M) \longrightarrow \widetilde{\mathrm{H}}^1(X; M) \longrightarrow \dots$$

of reduced singular cohomology (like there is also one for reduced singular homology [23]), which is natural in the pair (X, X'). We again should stress that this sequence fails in case X' is the empty space: in that case, we must add the nontrivial cohomology at degree -1 at the start of the sequence as first nontrivial module. In particular, by Example 2.11, there is for any pointed space $(X,*)$ an isomorphism $\mathrm{H}^n(X,*;M) \cong \widetilde{\mathrm{H}}^n(X;M)$, which is natural in the pointed space $(X,*)$. This gives us another description of what reduced singular cohomology does on (nonempty) spaces. ▽

Lemma 2.33. *(i) Let $f,g : X \to Y$ be two homotopic maps between nonempty topological spaces. Then $\widetilde{\mathrm{H}}^n f = \widetilde{\mathrm{H}}^n g$ for all n.*

(ii) Let $f : X \to Y$ be homotopy equivalence of nonempty topological spaces. Then $\widetilde{\mathrm{H}}^n f : \widetilde{\mathrm{H}}^n(Y;M) \to \widetilde{\mathrm{H}}^n(X;M)$ is an isomorphism for each n.

Moreover, if X' and Y' are nonempty subspaces of X and Y respectively such that f restricts to a homotopy equivalence $X' \to Y'$, then the induced map between the natural long exact sequences of reduced singular cohomology of (X,X') and (Y,Y') consists entirely of isomorphisms.

(iii) If X is a contractible space, then $\widetilde{\mathrm{H}}^n(X;M) = 0$ for all n.

Proof. This follows from Proposition 2.31 and Theorem 2.25. □

The natural isomorphism $\mathrm{H}^n(X,*;M) \cong \widetilde{\mathrm{H}}^n(X;M)$ derived above has an interesting consequence: for any pair (X,X') of pointed spaces with the same base point (which are consequently both nonempty), it implies the existence of a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X,X';M) \longrightarrow \mathrm{H}^0(X,*;M) \longrightarrow \mathrm{H}^0(X',*;M) \longrightarrow \mathrm{H}^1(X,X';M) \longrightarrow \mathrm{H}^1(X,*;M) \longrightarrow \dots$$

which is also natural in this pair of pointed spaces. This turns out not to be a coincidence, and can even be generalised a bit: we end this chapter with the so-called *long exact sequence of a triplet of spaces*. It is a property of all generalised cohomology theory that they have one, and the proof will be a first example of how to use the Eilenberg-Steenrod axioms. Of course, we first need to define what a triplet of spaces is, but it will not be a surprise.

Definition 2.34. A *triplet of topological spaces* (X,X',X'') consists of a topological space X and subspaces $X'' \subseteq X' \subseteq X$. A *morphism of triplets* $f : (X,X',X'') \to (Y,Y',Y'')$ is a continuous map $f : X \to Y$ such that $f(X') \subseteq Y'$ and $f(X'') \subseteq Y''$. The category of topological triplets is denoted by $\mathsf{Top}(3)$. ◊

Note that any map $f : (X,X',X'') \to (Y,Y',Y'')$ of triplets can also be seen as three maps of topological pairs.

Theorem 2.35. (Long exact sequence of cohomology of a triplet) [20] *Let (X,X',X'') be a triplet of topological spaces and h^* a generalised cohomology theory. Then the inclusions $(X',X'') \hookrightarrow (X,X'') \hookrightarrow (X,X')$ induce a long exact sequence*

$$\dots \longrightarrow \mathrm{h}^n(X,X') \longrightarrow \mathrm{h}^n(X,X'') \longrightarrow \mathrm{h}^n(X',X'') \longrightarrow \mathrm{h}^{n+1}(X,X') \longrightarrow \dots$$

which is natural in the triplet (X,X',X'').

Proof. (†) We provide the proof that Munkres [20] omitted (as it was an exercise in his book). Let $i : X' \hookrightarrow X$, $i' : X'' \hookrightarrow X$ and $i'' : X'' \hookrightarrow X'$ be the inclusions, let $\iota : (X,\varnothing) \to (X,X')$, $\iota' : (X,\varnothing) \to (X,X'')$ and $\iota'' : (X',\varnothing) \to (X',X'')$ be the inclusions of pairs, and let $j'' : (X,X'') \to (X,X')$ be the identity on X and $j : (X',X'') \to (X,X'')$ the inclusion. Denote by α, α' and α'' respectively the snake map in the long exact sequences of cohomology of the pairs (X,X'), (X,X'') and (X',X''). By naturality of the long exact sequence of cohomology, j and j'' induce a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{h}^n(X,X') \arrow[r, "\iota^*"] \arrow[d, "j''^*"] & \mathrm{h}^n(X) \arrow[r, "i^*"] \arrow[d, "\mathrm{id}"] & \mathrm{h}^n(X') \arrow[r, "\alpha"] \arrow[d, "i''^*"] & \mathrm{h}^{n+1}(X,X') \arrow[r] \arrow[d, "j''^*"] & \dots \\
\dots \arrow[r] & \mathrm{h}^n(X,X'') \arrow[r, "\iota'^*"] \arrow[d, "j^*"] & \mathrm{h}^n(X) \arrow[r, "i'^*"] \arrow[d, "i^*"] & \mathrm{h}^n(X'') \arrow[r, "\alpha'"] \arrow[d, "\mathrm{id}"] & \mathrm{h}^{n+1}(X,X'') \arrow[r] \arrow[d, "j^*"] & \dots \\
\dots \arrow[r] & \mathrm{h}^n(X',X'') \arrow[r, "\iota''^*"] & \mathrm{h}^n(X') \arrow[r, "i''^*"] & \mathrm{h}^n(X'') \arrow[r, "\alpha''"] & \mathrm{h}^{n+1}(X',X'') \arrow[r] & \dots
\end{tikzcd}$$

Hence, the diagram

$$\begin{tikzcd}
\cdots \arrow[dr] \arrow[rr, bend left] & & \mathrm{h}^{n-1}(X'') \arrow[dr, "\alpha'"] \arrow[rr, bend left, "\alpha''"] & & \mathrm{h}^n(X',X'') \arrow[dr, "\iota''^*"] \arrow[rr, dashed, bend left, "\alpha\circ\iota''^*"] & & \mathrm{h}^{n+1}(X,X') \arrow[dr, dashed, "j''^*"] \arrow[rr, bend left, "\iota^*"] & & \mathrm{h}^{n+1}(X) \arrow[dr] \arrow[rr, bend left] & & \cdots \\
& \mathrm{h}^{n-1}(X') \arrow[ur, "i''^*"] \arrow[dr, "\alpha"] & & \mathrm{h}^n(X,X'') \arrow[ur, dashed, "j^*"] \arrow[dr, "\iota'^*"] & & \mathrm{h}^n(X') \arrow[ur, "\alpha"] \arrow[dr, "i''^*"] & & \mathrm{h}^{n+1}(X,X'') \arrow[ur, "\iota'^*"] \arrow[dr, dashed] & & \cdots & \\
\cdots \arrow[ur] \arrow[rr, dashed, bend right] & & \mathrm{h}^n(X,X') \arrow[ur, dashed, "j''^*"] \arrow[rr, bend right, "\iota^*"'] & & \mathrm{h}^n(X) \arrow[ur, "i^*"] \arrow[rr, bend right, "i'^*"'] & & \mathrm{h}^n(X'') \arrow[ur, "\alpha'"] \arrow[rr, bend right] & & \cdots & &
\end{tikzcd}$$

is also commutative. Moreover, since the map $j'' \circ j : (X', X'') \to (X, X')$ equals the composite $[(X', X') \to (X, X')] \circ [(X', X'') \to (X', X')]$ and $\mathrm{h}^n(X', X') = 0$ for all n (by the long exact sequence of the pair (X', X')), it follows that $j^* \circ j''^* = 0$ as well. Now, repeated application of the Braid Lemma implies that the dashed sequence is exact, as we wanted to show. Naturality of this sequence follows from commutativity of the diagram

$$\begin{tikzcd}
(X', X'') \arrow[r, hook] \arrow[d, "f"] & (X, X'') \arrow[r, hook] \arrow[d, "f"] & (X, X') \arrow[d, "f"] \\
(Y', Y'') \arrow[r, hook] & (Y, Y'') \arrow[r, hook] & (Y, Y')
\end{tikzcd}$$

for any map $f : (X, X', X'') \to (Y, Y', Y'')$ of triplets, and from the naturality of the snake maps in the long exact sequences of pairs. □

Chapter 3

CW-complexes

In this chapter, we will finally introduce the class of topological spaces to which we will mostly restrict our attention for the rest of this thesis (as promised before Definition 1.46), namely that of the *CW-complexes*. These are spaces that are build by taking a few points, and then (in this order) "glue" lines segments, discs, cubes, etc. to the already existing space. This gluing can be done in all kinds of ways, and it turns out that many spaces we encounter in practice are in fact CW-complexes, although we will not give many examples of CW-complexes. The construction of a CW-complex allows us to generalise arguments about (hyper)spheres to those spaces, and allows inductive arguments over the structure of a space, both of which are especially helpful in homotopy theory. Together with the fact that most spaces we care about are CW-complexes, this justifies a separate study of these spaces.

3.1 Pushout diagrams

Recall the definition of a pushout of a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & X \\ Y & \end{tikzcd}$$

Figure 3.1

as given in Example A.46, namely as the colimit of the above diagram in Top. We state the definition again for convenience.

Definition 3.1. Consider the diagram in Figure 3.1. The *pushout* $X \cup_A Y$ is the topological space $(X \sqcup Y)/\sim$, where $\sim$ is the equivalence relation generated by $f(a) \sim g(a)$ for all $a \in A$. It comes with continuous maps $\iota_X : X \to X \sqcup Y \to X \cup_A Y$ and $\iota_Y : Y \to X \sqcup Y \to X \cup_A Y$, where the last arrow is the projection belonging to the quotient space. ◊

Remark 3.2. Although we do not notate the maps f and g in the pushout, it of course does depend on these maps as well, and not only on the spaces A, X and Y. ▽

Remark 3.3. The construction of a pushout can be visualised as follows: given two spaces X and Y, we take their disjoint union, and then glue two points $x \in X$ and $y \in Y$ together if there is an $a \in A$ such that $f(a) = x$ and $g(a) = y$. If we picture A as its image in X and Y, we glue those respective images together (of course, if f and g are not injective, we cannot make such an identification, but this image might help to understand what we are doing). ▽

The fact that the pushout is a colimit of course means it satisfies and is characterised by a universal property, and looking at Figure 3.1, we see that it is the following.

Proposition 3.4. [22] *For every commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "\iota_X"] \arrow[ddr, bend left] & \\
Y \arrow[r, "\iota_Y"] \arrow[drr, bend right] & X \cup_A Y \arrow[dr, dashed, "\exists! h"] & \\
& & Z
\end{tikzcd}$$

of topological spaces and continuous maps, there exists a unique map $h : X \cup_A Y \to Z$ making the above diagram commutative. We sometimes write this map as $f \cup_A g$. ☐

If there is a space Z such that there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

satisfying this universal property, we say the above diagram is a *pushout square* [22]. It then immediately follows that there is a homeomorphism from Z to $X \cup_A Y$ which identifies the maps $X \to Z$ and $Y \to Z$ above with ι_X and ι_Y, respectively.

Remark 3.5. For a diagram as in Figure 3.1 but now of *pointed* topological spaces, the pushout still exists and is formed by equipping the non-pointed pushout $X \cup_A Y$ with the canonical base point $[*]$, which equals the images $\iota_X(*)$ and $\iota_Y(*)$ of the base points of X and Y, since all the maps in the pushout diagram are pointed now.

The basic theory of pushouts of pointed topological spaces is therefore entirely similar to the non-pointed case, and we only need to add a base point and the word "pointed" on appropriate places. ▽

Lemma 3.6. *Let X be a topological space, and suppose A, B are two closed or two open subspaces of X. Then $A \cup_{A\cap B} B \cong A \cup B$ via a homeomorphism that identifies $\iota_A : A \to A \cup_{A\cap B} B$ and $\iota_B : B \to A \cup_{A\cap B} B$ with the inclusions $A \hookrightarrow A \cup B$ and $B \hookrightarrow A \cup B$.*

Proof. [23] Let Z be an arbitrary topological space, and suppose we are given two continuous maps $f : A \to Z$ and $g : B \to Z$ such that $f|_{A\cap B} = g|_{A\cap B}$. The pasting or gluing lemma from point-set topology tells us that the unique map $A \cup B \to Z$ that restricts to f and g on A and B, respectively, is continuous as well. Therefore, $A \cup B$ satisfies the universal property of the pushout, and is hence homeomorphic to $A \cup_{A\cap B} B$ via a homeomorphism that identifies ι_A and ι_B with the inclusions $A \hookrightarrow A \cup B$ and $B \hookrightarrow A \cup B$. ☐

Remark 3.7. By the universal property of the pushout, any two continuous maps $X \to Z$ and $Y \to Z$ that agree on (the images of) A can be glued uniquely to a map $X \cup_A Y \to Z$. A natural question is if the same holds for homotopies. The answer turns out to be affirmative, as explain in [23]. We present a more category theoretical argument here.

(†) Given continuous maps $X \xleftarrow{f} A \xrightarrow{g} Y$, we ask if for every two homotopies $F : X \times I \to Z$ and $G : Y \times I \to Z$ fitting in the commutative diagram

$$\begin{tikzcd}
A \times I \arrow[r, "f\times\mathrm{id}"] \arrow[d, "g\times\mathrm{id}"] & X \times I \arrow[d, "\iota_X\times\mathrm{id}"] \arrow[ddr, bend left, "F"] & \\
Y \times I \arrow[r, "\iota_Y\times\mathrm{id}"] \arrow[drr, bend right, "G"'] & (X \cup_A Y) \times I \arrow[dr, dashed] & \\
 & & Z
\end{tikzcd}$$

Figure 3.2

there is a unique map $(X \cup_A Y) \times I \to Z$ making the diagram commute still. We will use the result of Proposition 6.4 (whose proof is independent of our usage of it here) that $- \times I$ is a left adjoint functor $\mathsf{Top} \to \mathsf{Top}$, since I is a locally compact space. By Theorem A.51 then, it commutes with colimits, so

$$\begin{tikzcd}
A \times I \arrow[r, "f\times\mathrm{id}"] \arrow[d, "g\times\mathrm{id}"] & X \times I \arrow[d, "\iota_X\times\mathrm{id}"] \\
Y \times I \arrow[r, "\iota_Y\times\mathrm{id}"] & (X \cup_A Y) \times I
\end{tikzcd}$$

is a pushout square (note that the morphisms are also the ones induced by the functor $- \times I$). Therefore, $(X \cup_A Y) \times I$ satisfies the universal property of the pushout and there is a homeomorphism $(X \cup_A Y) \times I \cong X \times I \cup_{A \times I} Y \times I$. Therefore, the dashed line morphism in Figure 3.2 does indeed exist, and is unique. In other words, the pushout also glues homotopies that agree on $A \times I$. ▽

Lemma 3.8. *Consider the pushouts*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "\iota_X"] \\
Y \arrow[r, "\iota_Y"] & X \cup_A Y
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
B \arrow[r, "k"] \arrow[d, "h"] & Z \arrow[d, "i_Z"] \\
Y \arrow[r, "i_Y"] & Y \cup_B Z
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_Y \circ g"] & X \arrow[d, "j_X"] \\
Y \cup_B Z \arrow[r, "j_Y"] & X \cup_A (Y \cup_B Z)
\end{tikzcd}$$

of topological spaces. Then

$$\begin{tikzcd}
B \arrow[r, "\iota_Y \circ h"] \arrow[d, "k"] & X \cup_A Y \arrow[d, "j_X \cup_A (j_Y \circ i_Y)"] \\
Z \arrow[r, "j_Y \circ i_Z"] & X \cup_A (Y \cup_B Z)
\end{tikzcd}$$

Figure 3.3

is a pushout square. In particular, there is a homeomorphism $(X \cup_A Y) \cup_B Z \cong X \cup_A (Y \cup_B Z)$.

Proof. (Proposed by drs. J. Becerra) This will follow from a more general assertion in category theory that if the left and right square in a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E \arrow[d] \\
C \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

are pushout squares in a certain category C, then so is the outer rectangle. In fact, this is fairly obvious: given a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drrr, bend right] & D \arrow[r] \arrow[drr, dashed] & F \arrow[dr, dotted] & \\
 & & & G
\end{tikzcd}$$

of solid arrows, then the dashed morphism $D \to G$ exists since the left square is a pushout square, and then the dotted morphism $F \to G$ exists since the right square is a pushout square. This shows that the outer rectangle is also a pushout square.

Now, returning to the statement of the lemma, first note that $j_X \circ f = j_Y \circ i_Y \circ g$ by the third pushout square, so the map $j_X \cup_A (j_Y \circ i_Y)$ does indeed exist. Therefore $(j_X \cup_A (j_Y \circ i_Y)) \circ \iota_Y \circ h = j_Y \circ i_Y \circ h = j_Y \circ i_Z \circ k$, so Figure 3.3 does commute. We obtain a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[d, "k"'] & Y \arrow[r, "\iota_Y"] \arrow[d, "i_Y"'] & X \cup_A Y \arrow[d, "j_X \cup_A (j_Y \circ i_Y)"'] \\
Z \arrow[r, "i_Z"] & Y \cup_B Z \arrow[r, "j_Y"] & X \cup_A (Y \cup_B Z))
\end{tikzcd}$$

that consists of a left and right pushout squares. By our observation above, Figure 3.3 is a pushout square. □

3.2 Construction of CW-complexes

What we are interested in is gluing n-dimensional disks $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ to a given space along their boundaries $\partial D^n = S^{n-1}$. If we let $i : \partial D^n \to D^n$ denote the inclusion, we can for any continuous map $f : \partial D^n \to Y$ form the pushout $X \cup_{\partial D^n} D^n$. More generally, we can glue any number of n-dimensional discs to X as we want.

Definition 3.9. [19] Suppose $X \to Y$ is a continuous map, J is a set (which we equip with the discrete topology) and $f : J \times \partial D^n \to X$ is a continuous map. If there is a pushout square

$$\begin{tikzcd}
J \times \partial D^n \arrow[r, "f"] \arrow[d, "\mathrm{id}_J \times i"] & X \arrow[d] \\
J \times D^n \arrow[r] & Y
\end{tikzcd}$$

we say that Y *arises from* X *by attaching* n*-cells along* f. We call f the *attaching map.* ◊

Note that by putting the discrete topology on J, we have guaranteed that a map $f : J \times \partial D^n \to X$ is continuous if and only if for each j the restriction $f|_{\{j\} \times \partial D^n}$ is continuous. Together with Lemma 3.8, this implies that there is no difference between attaching a set of n-cells to X all at once or one at a time.

Remark 3.10. A discrete topological space J of course canonically satisfies $J \times \partial D^n \cong \bigsqcup_{j \in J} \partial D^n$. This identification with the disjoint union might help with visualising the process of attaching n-cells. ▽

Definition 3.11. [19] Let a topological space A be given. A *CW-complex relative to* A consists of a topological space X together with subspaces $A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots$ satisfying:

(i) For each $n \geq 0$, X_n arises from X_{n-1} by attaching (possibly zero) n-cells.

(ii) X is the colimit of the diagram $A = X_{-1} \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ ◊

Notation 3.12. We often omit the subspaces X_n in our notation, and simply call X a CW-complex relative to A. We also tend to call this "a relative CW-complex (X, A)". ○

Remark 3.13. The fact that $X = \operatorname{colim}_n X_n$ can be translated as saying that $X = \bigcup_{n=-1}^{\infty} X_n$ and that X carries the so-called *weak topology*: a set $S \subseteq X$ is open if and only if $S \cap X_n$ is open for all $n \geq -1$. Indeed, as we saw in Example A.53, the colimit of the above diagram has $\bigcup_n X_n$ as underlying set by Proposition A.48, and is equipped with the final topology with respect to the maps of sets $\iota_n : X_n \hookrightarrow X$. The condition that a set $S \subseteq X$ is open if and only if $S \cap X_n$ is open for all $n \geq -1$ is just a different way of saying that X carries this very same final topology with respect to the inclusions $\iota_n : X_n \hookrightarrow X$. ▽

Corollary 3.14. *Let X be a CW-complex relative to A. Then a map $f : X \to Y$ of topological spaces is continuous if and only if each restriction $f|_{X_n} : X_n \to Y$ is continuous.* □

Definition 3.15. Let X be a CW-complex relative to A. Then:
(i) We call the subspace X_n the *n-skeleton of X*.
(ii) (X, A) is *finite-dimensional* if there is an n such that $X = X_n$.
(iii) (X, A) is *finite* if it is finite dimensional and at each step, we only attach finitely many n-cells.
(iv) X is called an *absolute CW-complex* if $A = \varnothing$. ◇

Convention 3.16. When we talk about a CW-complex X without mentioning any space to which it is relative, it is understood that X is an absolute CW-complex. ⊙

Remark 3.17. Milnor [18] showed that any topological manifold is homotopy equivalent to a CW-complex. Any compact smooth manifold allows even a CW-structure itself, as shown in [21]. Any topological graph is also a CW-complex. Therefore, almost all spaces we encounter in practice are at least homotopy equivalent to a CW-complex, with many actually being CW-complexes. ▽

Definition 3.18. [20] Let (X, A) be a relative CW-complex. A *subcomplex* of (X, A) is a relative CW-complex (X', A), where $X' \subseteq X$ is a closed subspace of X that is a union of cells of X. ◇

Notice that we do not require a subcomplex to contain n-cells of any degree except -1. Note also that it depends on the cell structure whether or not a space is a subcomplex of another space. For instance, S^0 is a CW-complex consisting of two 0-cells, and we can realise S^1 as a CW-complex consisting of one 0-cell with one 1-cell attached to it. In this case, S^0 is a subspace of S^1, but not a subcomplex.

Remark 3.19. There are two standard choices of CW-structure on the n-spheres. Firstly, we can realise S^0 as two 0-cells, and S^n as a single n-cell attached to a single 0-cell. Secondly, we can let S^0 consist of two 0-cells, and if we have defined the structure on S^n, then we can form S^{n+1} by attaching two $(n+1)$-cells to S^n using the identity $\partial D^{n+1} \to S^n$ as attaching maps. In this case S^n is a subcomplex of S^m for all $n < m$. It depends on the situation which structure is more convenient to use. ▽

Definition 3.20. A *CW-pair* (X, X') is a topological pair with X a CW-complex and $X' \subseteq X$ a subcomplex of X. ◇

A CW-pair is also a relative CW-complex, since a CW-complex X can be build from a subcomplex X' by attaching all the cells which are not yet in X'.

Convention 3.21. We will always assume that a base point of a CW-complex is a 0-cell of it. ⊙

We can actually always refine the CW-structure of a CW-complex X in such a way that an arbitrary point $x \in X$ becomes a 0-cell.

Notation 3.22. The category of CW-complexes and continuous maps between them is written as CW. The category of pointed CW-complexes is denoted by CW_*, and the category of CW-pairs by $\mathsf{CW}(2)$, seen as full subcategories of Top_* and $\mathsf{Top}(2)$, respectively. ○

Definition 3.23. Let (X, A) and (Y, B) be two relative CW-complexes. A continuous map $f : X \to Y$ is *cellular* if $f(X_n) \subseteq Y_n$ for all $n \geq -1$. ◇

Remark 3.24. Instead of letting the category of CW-complexes be the full subcategory of Top, it seems it would have been more natural to take only the cellular maps as morphisms, since we tend to pick structure preserving maps as morphisms in categories. However, this would require us include for each topological space the multiple copies of it, but with different CW-structures, as objects in the category. We do not want this: objects in CW are just topological spaces that admit a CW-structure (and in arguments we will often implicitly take a particular CW-structure). However, we will see later that for our purposes the choice does not matter that much: by the Cellular Approximation Theorem 4.40 any continuous map $f : X \to Y$ between CW-complexes is homotopic to a cellular map. We are in this thesis only concerned with properties that are conserved under homotopy, so we can almost always immediately take a map between CW-complexes to be cellular. $\triangledown$

Now we turn to some point-set topological properties of CW-complexes and cells attachments. Since our main interest lies with cohomology and homotopy theory, we omit the proofs of most statements, and instead refer to [19], or equivalently [23] for them.

Proposition 3.25. *Let X be an arbitrary topological space, and J a set. Let $q : X \sqcup (J \times D^n) \to X \cup_{J\times\partial^D_n} J \times D^n$ be the quotient map belonging to the attachment of n-cells.*

- *(i) The image $q(X)$ is closed in $X \cup_{J\times\partial^D_n} J \times D^n$, and q gives an embedding $X \to X \cup_{J\times\partial^D_n} J \times D^n$.*
- *(ii) The image $q(J \times (D^n)^\circ)$ is open in $X \cup_{J\times\partial^D_n} J \times D^n$, and q gives an embedding $J \times (D^n)^\circ \to X \cup_{J\times\partial^D_n} J \times D^n$.*
- *(iii) Suppose for every $j \in J$ we are given an open subset $V_j \subseteq \{j\} \times D^n$ with $\{j\} \times \partial D^n \subseteq V_j$. Then $X \cup \bigcup_{j\in J} q(V_j)$ is open in $X \cup_{J\times\partial^D_n} 1J \times D^n$.*

Proposition 3.26. *If X is Hausdorff, so is $X \cup_{J\times\partial^D_n} J \times D^n$.*

Corollary 3.27. *Let (X, A) be a relative CW-complex. If A is Hausdorff, then X is as well.*

Corollary 3.28. *If X is compact and J is finite, then $X \cup_{J\times\partial^D_n} J \times D^n$ is compact as well.*

Corollary 3.29. *If (X, A) is a finite relative CW-complex and A is compact, then X is compact.*

Lemma 3.30. *Any CW-complex X is locally path-connected. Consequently, X is path-connected if and only if it is connected.*

There is no guarantee that standard operations on topological spaces behave nicely when we apply them to relative CW-complexes, in the sense that the result will in that case again be a relative CW-complex. One of the most important operations for which a complication arises is the product of two spaces: naively, the product of two relative CW-complexes (X, A) and (Y, B) seems to be the space $(X \times Y, A \times B)$, which has a natural cell structure with the cells being the products of single cells of X and Y. However, the topologies do not match: for $(X \times Y, A \times B)$ to be a relative CW-complex, we need a stronger topology than the product topology.

Write $\widehat{X} = A \bigsqcup_{n=1}^{\infty} J_n \times D^n$ and $\widehat{Y} = B \bigsqcup_{n=1}^{\infty} J'_n \times D^n$, where we shortened the notation of the disjoint unions a bit. Then there are quotient maps $q : \widehat{X} \to X$ and $q' : \widehat{Y} \to Y$.

Definition 3.31. The *CW-product* of relative CW-complexes (X, A) and (Y, B) is the pair of sets $(X \times Y, A \times B)$ with n-skeleton $(X \times Y)_n = \bigcup_{i+j=n} X_i \times Y_j$, and equipped with the final topology with respect to the product map $q \times q' : \widehat{X} \times \widehat{Y} \to X \times Y$. $\Diamond$

The last part of the definition guarantees that the CW-product actually arises from $A \times B$ by attaching the cells in the mentioned n-skeletons. This is not a trivial statement and of course needs a proof, but we will not give that here. The reader is instead referred to [23].

Convention 3.32. Whenever we consider the product of two (relative) CW-complexes, it is understood that we are taking the CW-product of these spaces. $\odot$

It does happen that this finer topology coincides with the product topology.

Proposition 3.33. *Let (X, A) and (Y, B) be two relative CW-complexes, and suppose Y is locally compact. Then the topology on the CW-product $(X \times Y, A \times B)$ coincides with the product topology on it.*

Proof. This is shown in [23]. ☐

Corollary 3.34. *If (X, A) and (Y, B) are relative CW-complexes, (Y, B) is finite and B is compact, then the topology on the CW-product $(X \times Y, A \times B)$ coincides with the product topology on it.*

Proof. This follows from Corollary 3.29 and the previous corollary. ☐

However, many other constructions do behave nice with respect to (absolute) CW-copmlexes. We have the following important result about pushouts of CW-complexes, which has numerous useful applications.

Proposition 3.35. *Suppose X and Y are CW-complexes, and A is a subcomplex of X. If $f : A \to Y$ is a cellular map, then the pushout $X \cup_A Y$ is a CW-complex as well. Moreover, Y is a subcomplex of $X \cup_A Y$.*

Proof. The proof can be found in [9]. ☐

Corollary 3.36. *Let X be a CW-complex and A be a subcomplex. Then X/A is also a CW-complex.*

Proof. X/A is the pushout of the diagram $X \hookleftarrow A \to *$, and the map $A \to *$ is of course cellular. ☐

Note that the preferred base point of a quotient CW-complex X/A is a 0-cell of it. Therefore, it is a pointed CW-complex, in line with Convention 3.21.

3.3 Cones, suspensions, mapping cones and mapping cylinders

There are a few useful constructions of topological spaces which we will often use in the remainder of this thesis. Then it will be clear why these constructions are interesting with respect to our topic. In short, they either behave well in homotopy theory or reflect certain homotopy information of a certain space or map between spaces. We follow [11] and [13] in all the definitions to come in this section.

Definition 3.37. Let $(X_\alpha, *_\alpha)_{\alpha \in A}$ be a family of pointed topological spaces. Their *wedge sum* is the pointed topological space $\bigvee_{\alpha \in A} X_\alpha := \coprod_{\alpha \in A} X_\alpha / \{*_\alpha \mid \alpha \in A\}$. ◊

Remark 3.38. The wedge sum comes with canonical inclusions $\iota_\alpha : X_\alpha \hookrightarrow \bigvee_{\alpha \in A} X_\alpha$, which establish the wedge sum as the coproduct in Top_*. ▽

Remark 3.39. In Remark 3.5, we described the pointed pushout of a given diagram $X \xleftarrow{f} A \xrightarrow{g} Y$ of pointed spaces and maps. We can see this pushout equals $(X \vee Y)/\sim$, with $\sim$ again the equivalence relation generated by $f(a) \sim g(a)$ for all $a \in A$. ▽

We can identify $X \vee Y$ with $(X \times \{*\}) \cup (\{*\} \times Y)$ and consider $X \vee Y$ as a subspace of $X \times Y$. We can then make the following definition.

Definition 3.40. Let $(X, *)$ and $(Y, *)$ be two pointed topological spaces. Their *smash product* is the pointed topological space $X \wedge Y := X \times Y / X \vee Y$. ◊

The smash product fits in a pushout square

$$\begin{tikzcd}
X \vee Y \arrow[r, hook] \arrow[d] & X \times Y \arrow[d] \\
* \arrow[r] & X \wedge Y
\end{tikzcd}$$

in Top_*, and the smash product is a functorial construction $\mathsf{Top}_* \times \mathsf{Top}_* \to \mathsf{Top}_*$: if $f : X \to X'$ and $g : Y \to Y'$ are pointed maps of spaces, then the induced map $X \vee Y \to X' \vee Y'$ equals the restriction of the induced map $X \times Y \to X' \times Y'$, so there is an induced map $X \wedge Y = X \times Y / X \vee Y \to X' \times Y' / X' \vee Y' = X' \wedge Y'$.

Lemma 3.41. *(i) Let X and Y be pointed topological spaces. Then $X \wedge Y \cong Y \wedge X$ naturally in both X and Y.*

(ii) Let X, Y and Z be pointed topological spaces. Then $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ naturally in all three spaces.

Proof. The first statement is obvious from the definition. For the second, the interested reader may consider it an exercise in quotient spaces to show that

$$(X \wedge Y) \wedge Z \cong X \times Y \times Z/(X \times Y \cup X \times Z \cup Y \times Z),$$

naturally in all three spaces, where we identify $X \times Y$ with $X \times Y \times \{*\}$, and similarly for the other two subspaces appearing on the right. This yields the second statement. ☐

Definition 3.42. Let X be a topological space. Then the *unreduced cone over* X is defined as the space $C_uX = X \times I/(X \times \{1\})$. We consider X as a subspace of C_uX by identifying it with (the image of) $X \times \{0\}$ in the quotient space.

Now suppose $(X, *)$ is a pointed space. The *(reduced) cone over* X is the pointed space $CX = C_uX/(\{*\} \times I)$ with the equivalence class of $*$ as base point. We again consider $(X, *)$ as a pointed subspace of CX by identifying it with the image of $X \times \{0\}$ in the quotient space. ◊

Definition 3.43. Let X be a topological space. Then the *unreduced suspension over* X is the space SX obtained from the space $X \times [-1, 1]$ by collapsing separately the subspaces $X \times \{-1\}$ and $X \times \{1\}$ to a point. We consider X as a subspace of SX by identifying it with (the image of) $X \times \{0\}$ in the quotient space.

Now suppose $(X, *)$ is a pointed space. The *(reduced) suspension over* X is the pointed space $\Sigma X = SX/(\{*\} \times [-1, 1])$ with the equivalence class of $*$ as base point. We again consider $(X, *)$ as a pointed subspace of ΣX by identifying it with the image of $X \times \{0\}$ in the quotient space. ◊

Definition 3.44. Let $f : X \to Y$ be a continuous map. The *mapping cylinder* is defined as the pushout $M_f = X \times I \cup_{X \times \{1\}} Y$, in other words, the space making

$$\begin{tikzcd}
X \times \{1\} \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X \times I \arrow[r] & M_f
\end{tikzcd}$$

a pushout diagram. If f, X and Y are pointed, then the *reduced mapping cylinder* is the quotient $m_f = M_f/(\{*\} \times I)$, where $*$ denotes the base point of X.

We consider X and Y as subspaces of M_f and m_f by identifying them with the images of $X \times \{0\}$ and Y in the quotients M_f and m_f, respectively. ◊

Definition 3.45. Let $f : X \to Y$ be a pointed continuous map. The *mapping cone* is the pointed pushout $Cf = CX \cup_X Y$, in other words, the space making

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x \mapsto {[(x,0)]}"] & Y \arrow[d] \\
CX \arrow[r] & Cf
\end{tikzcd}$$

a pushout diagram. The *unreduced mapping cone* is the pushout $C_uf = C_uX \cup_X Y$. ◊

Remark 3.46. Let X be a pointed topological space. Then there are homeomorphisms $C_uX \cong X_+ \wedge I$, $CX \cong X \wedge I$, and $\Sigma X \cong X \wedge S^1$, where $\cdot_+$ is the base point adding functor of Remark 1.26 and where we take 1 as the base point of I. ▽

Lemma 3.47. [11] *Let X be a pointed topological space, and $f : X \to Y$ a continuous map. Then*

(i) C_uX *and* CX *are contractible.*
(ii) $SX \cong C_uX \cup_X C_uX = C_u i_u$ *where* $i_u : X \hookrightarrow C_uX$ *is the inclusion, and* $\Sigma X \cong CX \cup_X CX = Ci$*, where* $i : X \to CX$ *is the inclusion.*
(iii) $C_uf \cong M_f/(X \times \{0\})$*, and* $Cf \cong m_f/(X \times \{0\})$*.*
(iv) Y *is a deformation retract of both* M_f *and* m_f*.*

Proof. (i) is obvious, and (ii) follows from Lemma 3.6. (iii) is obvious as soon as we realise we could also have defined $M_f = X \times I \cup_{X\times\{0\}} Y$ and the reduced statement follows similarly. For (iv), note that the homotopies $(X \times I) \times I \to X \times \{1\}, ((x, s), t) \mapsto (x, (1 - t)s + t)$ and $Y \times I \to Y, (y, t) \mapsto y$ induce by inclusion in the disjoint union $(X \times I) \sqcup Y$ and taking the quotient homotopies $F : (X \times I) \times I \to M_f$ and $G : Y \times I \to M_f$. They satisfy $G(f(x), t) = [f(x)] = [(x, 1)] = F((x, 1), t)$ for all $x \in X$ and $t \in I$, so by Remark 3.7 there is a homotopy $H : M_f \times I \to M_f$ that glues F and G. By the description of F and G, H is seen to be a homotopy between the identity on M_f and a retract $M_f \to Y \subseteq M_f$ that is stationary on Y at all times. Therefore, Y is a deformation retract of M_f. The same homotopies F and G also induce similarly the deformation of m_f on Y. □

Remark 3.48. The last statement of this lemma implies that any continuous map $f : X \to Y$ factors through a closed inclusion $X \to M_f$ and a homotopy equivalence $M_f \to Y$, and a similar story holds for pointed continuous maps. Therefore, when working up to homotopy we can safely assume that every pointed or non-pointed continuous map is a pointed or non-pointed closed inclusion. ▽

Remark 3.49. (†) All the above constructions are functorial:
(i) The unreduced cone is a functor $C_u : \mathsf{Top} \to \mathsf{Top}$ that sends a morphism $f : X \to Y$ to the morphism $C_uf : C_uX \to C_uY, [(x, t)] \mapsto [(f(x), t)]$.
(ii) The (reduced) cone is a functor $C : \mathsf{Top}_* \to \mathsf{Top}_*$ which acts the same as the above functor on morphisms. Continuity of this induced map follows from the pointedness assumption the morphisms now have and the universal property of the quotient topology.
(iii) The unreduced and reduced suspensions are functors $S : \mathsf{Top} \to \mathsf{Top}$ and $\Sigma : \mathsf{Top}_* \to \mathsf{Top}_*$, respectively, that act similarly to the cone functors above on morphisms. In fact, the homeomorphisms in Lemma 3.47(ii) can even be seen to be natural isomorphisms of functors (the pushout is also a functor from a category of diagrams (see Example A.23)).
(iv) The mapping cylinder is a functor from the category of diagrams of shape $\bullet \to \bullet$ of topological spaces to the category Top. It sends a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

to the morphism $M_f \to M_{f'}$ induced by the maps $\varphi \times \mathrm{id} : X \times I \to X' \times I$ and $\psi : Y \to Y'$. Similarly the functoriality of the reduced mapping cylinder is established.
(v) The mapping cone is a functor from the category of diagrams $\bullet \to \bullet$ of pointed topological spaces to the category Top_*. It sends a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

of pointed spaces and morphisms to the pointed morphism $C_f \to C_{f'}$ induced by the maps $C(\varphi) : CX \to CX'$ and $\psi : Y \to Y'$. The unreduced mapping cone is a functor in the same way by replacing pointed spaces and morphisms by regular ones.

All the homeomorphisms in Remark 3.46 are natural, and the right hand sides could have been taken as definition instead (and will be used when convenient). ▽

When working with CW-complexes, the definitions of the smash product, cones, suspensions, mapping cylinders and mapping cones need to be modified slightly: the continuous maps f mentioned in the definition of the mapping cylinder and mapping cone must be cellular, and every time a product of two spaces occurs, it has to be mentioned that it is the CW-product, and the resulting space therefore does not necessarily carry the product topology. Now, by Corollary 3.34, the latter actually does not change the definitions of anything other than the smash product, and even the smash product remains unchanged if one of its terms is a locally compact CW-complex. Using Proposition 3.35, Corollary 3.36 and 3.46, we obtain the following results:

Lemma 3.50. *(i) If $(X_\alpha)_{\alpha\in A}$ is a family of pointed CW-complexes, then $\bigvee_{\alpha\in A} X_\alpha$ is a pointed CW-complex, which contains all the X_α as subcomplexes.*

(ii) If X and Y are pointed CW-complexes, then $X \wedge Y$ is a pointed CW-complex as well.

(iii) If X is an non-pointed resp. pointed CW-complex, then C_uX and SX resp. CX and ΣX are non-pointed resp. pointed CW-complexes, which contain X as subcomplex.

(iv) If $f : X \to Y$ is a cellular map between non-pointed resp. pointed CW-complexes, then C_uf and M_f resp. Cf and m_f are non-pointed resp. pointed CW-complexes as well. Moreover, Y is a subcomplex of all of them, and X is a subcomplex of M_f and m_f (whichever is appropriate). ☐

Chapter 4

Homotopy theory of CW-complexes

After having defined what CW-complexes are in the previous chapter, we will now study these spaces from a homotopy theoretical perspective. We will encounter various examples in which the CW-structure of a space allows inductive arguments for results that were first shown to hold for spheres, illustrating the usefulness of the definition of a CW-complex. Next to the important theorems that will be key in later chapters, we will also introduce the higher homotopy groups of a topological space, that are not only needed to state and prove the former results, but are also a central element in algebraic topology and homotopy theory: loosely stated, they describe how many different ways (up to homotopy) there are to map spheres of a certain dimension into your space, and are as such higher-dimensional analogues of the fundamental group.

4.1 The homotopy extension property

We begin by introducing some notation. Recall Definition 1.37 of a homotopy between maps of pairs of topological spaces. Since a pointed space is also a pair of spaces, we also know what it means for two pointed maps to be *pointedly* homotopic: for two pointed maps $f, g : (X, *) \to (Y, *))$, a pointed homotopy from f to g is a homotopy $H : X \times I \to Y$ such that $H(*, t) = *$ for all $t \in I$. Note that this is the same as giving a map $\widetilde{H} : X \wedge I_+ \to Y$ (with $\cdot_+$ the base-point adding functor of Remark 1.26) such that $\widetilde{H}([x, 0]) = f(x)$ and $\widetilde{H}([x, 1]) = g(x)$.

All in all, we get a homotopy category of pointed topological spaces, which is a full subcategory of $\mathsf{hTop}(2)$.

Notation 4.1. The homotopy category of pointed spaces is denoted by hTop_*. ○

Notation 4.2. For the rest of this thesis, we let $[\,\cdot\,,\,\cdot\,]$ denote the Hom-functor $\mathsf{hTop}^{\mathrm{opp}} \times \mathsf{hTop} \to \mathsf{Set}$, and $[\,\cdot\,,\,\cdot\,]^\bullet$ the Hom-functor $\mathsf{hTop}_*^{\mathrm{opp}} \times \mathsf{hTop}_* \to \mathsf{Set}$. ○

In other words, for any two topological spaces X and Y, $[X, Y]$ denotes the set of equivalence classes of homotopic maps $X \to Y$, and for two pointed spaces $(X, *)$ and $(Y, *)$, $[X, Y]^\bullet$ (we often omit the base point in this notation) denotes the set of equivalence classes of homotopic pointed maps $(X, *) \to (Y, *)$. The latter set has a natural choice of base point, namely the constant map that maps X entirely to the base point of Y. It is immediate that any induced map between these Hom-sets preserves this preferred element. Hence $[\,\cdot\,,\,\cdot\,]^\bullet$ is a functor to the category Set_* of pointed sets, rather than just ordinary ones.

The reader might already be familiar with homotopies relative to a certain subspace. It is indeed likely to have been covered in any introductory course in topology, but since the amount of different types of homotopies make it pleasant to be able to reread definitions, we present it again.

Definition 4.3. Let $f, g : (X, X') \to (Y, Y')$ be two maps of pairs of spaces that agree on X'. A *homotopy between f and g relative to X'* is a homotopy $H : X \times I \to Y$ from f to g such that $H(x', t) = f(x')(= g(x'))$ for all $x' \in X'$ and $t \in I$. ◊

As such, a homotopy relative to a subspace is a special case of a homotopy of a map of pairs, but clearly a much stronger requirement on a homotopy of maps. A pointed homotopy is one example of both a homotopy of pairs and a homotopy relative to the base point.

Related to this is a special type of homotopy equivalences that are of interest to us, namely the ones the ones that also restrict to the identity on a certain subspace. A deformation retract is an example of this. It will be convenient to have a notation for this.

Notation 4.4. Let (X, A) and (Y, B) be two topological pairs. If there is a homotopy equivalence $X \simeq Y$ via maps $f : X \to Y$ and $g : Y \to X$ that restrict to mutually inverse homeomorphisms $A \xrightarrow{\sim} B$ and $B \xrightarrow{\sim} A$, respectively, and via homotopies $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$ that are constant on A and B, respectively, then we write $X \simeq Y \operatorname{rel} A$, and say that *$X$ and Y are homotopy equivalent relative to A* (and then they are of course also homotopy equivalent relative to B). ○

With these notational matters out of the way, we can start with the actual content of this section, which all revolves around the following idea.

Definition 4.5. [23] Let (X, X') be a pair of topological spaces. It is said to have the *homotopy extension property* (HEP) if for any topological space Z and any continuous map $f : X \to Z$, any homotopy $H : X' \times I \to Z$ from the restriction $f|_{X'}$ of f can be extended to a homotopy $\tilde{H} : Y \times I \to Z$ from f. ◇

In other words, a pair (X, X') has the HEP if for every commutative diagram

$$\begin{tikzcd}
X' \arrow[r, hook, "\iota"] \arrow[d, "x' \mapsto (x',0)"'] & X \arrow[d, "x \mapsto (x,0)"'] \arrow[ddr, bend left, "f"] & \\
X' \times I \arrow[r, hook, "\iota \times \mathrm{id}"] \arrow[drr, bend right, "H"'] & X \times I \arrow[dr, dashed, "\tilde{H}"'] & \\
 & & Z
\end{tikzcd}$$

there is a homotopy $\tilde{H} : X \times I \to Z$ such that the resulting diagram is also commutative.

Remark 4.6. [19] We can rephrase this definition for X' closed in X once again by saying that the pair (X, X') has the HEP if every map $\mathcal{H} : (X' \times I) \cup_{X'} X \to Z$ factors through the map $(\iota \times \mathrm{id}) \cup_{X'} (x \mapsto (x, 0)) : (X' \times I) \cup_{X'} X \to X \times I$, as illustrated in the following diagram:

$$\begin{tikzcd}
(X' \times I) \cup_{X'} X \arrow[r, "\mathcal{H}"] \arrow[d, "(\iota \times \mathrm{id}) \cup_{X'} (x \mapsto (x,0))"'] & Z \\
X \times I \arrow[ur, dashed] &
\end{tikzcd}$$

By Lemma 3.6, $(X' \times I) \cup_{X'} X \cong (X' \times I) \cup (X \times \{0\}) \subseteq X \times I$ (and this uses that X' is assumed to be closed in X now), so we finally end up with a pair (X, X') having the HEP if and only if for every diagram

$$\begin{tikzcd}
(X' \times I) \cup (X \times \{0\}) \arrow[r, "\mathcal{H}"] \arrow[d, hook] & Z \\
X \times I \arrow[ur, dashed] &
\end{tikzcd}$$

there is an arrow making the diagram commutative. ▽

The importance of a pair of spaces having the HEP lies for instance in the following results.

Proposition 4.7. *Let (X, A) and (Y, B) be two topological pairs with the HEP, and suppose $f : X \to Y$ is a homotopy equivalence that restricts to a homeomorphism $A \xrightarrow{\sim} B$. Then $X \simeq Y \operatorname{rel} A$ via f.*

Proof. The proof is a bit long, and can be found in [11]. ☐

Corollary 4.8. *Suppose X and Y are two pointed spaces with the HEP with respect to that base point. If $f : X \to Y$ is a pointed map that is also a (not necessarily pointed) homotopy equivalence, then f is a pointed homotopy equivalence.* ☐

Proposition 4.9. *Suppose the topological pair (X, X') satisfies the HEP and X' is contractible. Then the quotient map $q : X \to X/X'$ is a homotopy equivalence. In particular, if $(X, X', *)$ is a topological triplet and X' is contractible, then the quotient map $q : X \to X/X'$ is a pointed homotopy equivalence.*

Proof. [11] Let $H : X' \times I \to X$ be a contraction of X' onto a point in X', and consider the identity $\mathrm{id}_X : X \to X$. Since (X, X') has the HEP, there exists a homotopy $\widetilde{H} : X \times I \to X$ extending both id_X and H. Write $\widetilde{H}_t$ for the continuous map $X \to X : x \mapsto \widetilde{H}(x, t)$. Because $\widetilde{H}_t(X') \subseteq X'$ for each $t \in I$, there exist maps $\widetilde{h}_t : X/X' \to X/X'$ such that $q\widetilde{H}_t = \widetilde{h}_t q$, and these maps $\widetilde{h}_t$ also constitute to a homotopy $\widetilde{h} : X/X' \times I \to X/X'$. Moreover, $\widetilde{H}_1(X')$ equals a single point, so there is a map $f : X/X' \to X$ such that $fq = \widetilde{H}_1 \simeq \widetilde{H}_0 = \mathrm{id}_X$. Therefore, $qfq = q\widetilde{H}_1 = \widetilde{h}_1 q$, and surjectivity of q now implies $qf = \widetilde{h}_1 \simeq \widetilde{h}_0 = \mathrm{id}_{X/X'}$. Therefore, $q : X \to X/A$ is a homotopy equivalence. The last statement follows from the fact that the quotient map is pointed and Corollary 4.8. ☐

We will prove that any CW-pair has the HEP, and, more generally, any relative CW-complex has it with respect to the space which it is relative to. To do so, we need a convenient necessary and sufficient condition for a pair of spaces to have the HEP. The following lemma is an easy observation.

Lemma 4.10. *Let (X, X') be a pair of topological spaces. Then X' is a retract of X if and only if any continuous map $f : X' \to Z$ can be extended to a continuous map $\tilde{f} : X \to Z$.*

Proof. For necessity, let $r : X \to X'$ be a retract. Then any continuous map $f : X' \to Z$ can be extended by precomposition with r to a continuous map $f \circ r : X \to Z$. For sufficiency, suppose any continuous map $f : X' \to Z$ can be extended to a continuous map $\tilde{f} : X \to Z$. Then the identity $\mathrm{id}_{X'}$ can be extended to a continuous map $X \to X'$, which therefore establishes X' as a retract of X. ☐

Corollary 4.11. [23] *A pair (X, X') of topological spaces with X' closed in X has the HEP if and only if $(X' \times I) \cup (X \times \{0\})$ is a retract of $X \times I$.*

Proof. Remark 4.6 and the previous lemma yield this statement. ☐

Lemma 4.12. *If $n \geq 0$, then the CW-pair $(D^n, \partial D^n)$ has the HEP.*

Proof. [23] We will construct a retraction $D^n \times I \to \partial(D^n \times I) \cup (D^n \times \{0\})$, which is sufficient by Corollary 4.11. We consider $D^n \times I$ as a subspace of $\mathbb{R}^{n+1}$. Let $P = (0, \ldots, 0, 2)$. For any $(x, t) \in D^n \times I$, we set $r(x, t)$ to be the intersection of the line in $\mathbb{R}^{n+1}$ through P and (x, t) with $(\partial D^n \times I) \cup (D^n \times \{0\})$. It is not difficult to see that this is indeed a retraction. ☐

Lemma 4.13. *Suppose (X, X') is a topological pair such that X arises from X' by attaching n-cells. Then (X, X') has the HEP.*

Proof. [23] First note that X' is closed in X, so it again suffices to construct a suitable retraction. Let $X = X' \cup_{J \times \partial D^n} J \times D^n$, where we as usual consider X' to be a closed subspace of this. Using the last lemma, we obtain a retraction

$$r : J \times D^n \times I \to (J \times \partial D^n \times I) \cup (J \times D^n \times \{0\}).$$

We obtain, using Lemmata 3.8 and 3.6 and the fact that $-\times I$ and $-\times\{0\}$ commute with pushouts (as they are left adjoints, see Proposition 6.4), the homeomorphisms

$$\begin{aligned}
X' \times I \cup X \times \{0\} &\cong X' \times I \cup ((X' \cup_{J\times\partial D^n} J \times D^n) \times \{0\}) \\
&\cong X' \times I \cup (X' \times \{0\} \cup_{J\times\partial D^n\times\{0\}} J \times D^n \times \{0\}) \\
&\cong X' \times I \cup_{X'\times\{0\}} (X' \times \{0\} \cup_{J\times\partial D^n\times\{0\}} J \times D^n \times \{0\}) \\
&\cong (X' \times I \cup_{X'\times\{0\}} X' \times \{0\}) \cup_{J\times\partial D^n\times\{0\}} J \times D^n \times \{0\} \\
&\cong X' \times I \cup_{J\times\partial D^n\times\{0\}} J \times D^n \times \{0\} \\
&\cong (X' \times I \cup_{J\times\partial D^n\times I} J \times \partial D^n \times I) \cup_{J\times\partial D^n\times\{0\}} J \times D^n \times \{0\} \\
&\cong X' \times I \cup_{J\times\partial D^n\times I} (J \times \partial D^n \times I \cup_{J\times\partial D^n\times\{0\}} J \times D^n \times \{0\}) \\
&\cong X' \times I \cup_{J\times\partial D^n\times I} (J \times \partial D^n \times I \cup J \times D^n \times \{0\}),
\end{aligned}$$

and a homeomorphism $X \times I \cong X' \times I \cup_{J\times\partial D^n\times I} J \times D^n \times I$. Now, the retraction r induces a retraction

$$X \times I \cong X' \times I \cup_{J\times\partial D^n\times I} J \times D^n \times I \xrightarrow{\mathrm{id}\cup r} X' \times I \cup_{J\times\partial D^n\times I} (J \times \partial D^n \times I \cup J \times D^n \times \{0\}) \cong X' \times I \cup X \times \{0\}.$$

This completes the proof. ☐

Theorem 4.14. *Let (X, A) be a relative CW-complex. Then (X, A) has the HEP.*

Proof. [23] A is closed in X, and hence we will inductively construct a retraction $r : X \times I \to A \times I \cup X \times \{0\}$. Let $r_{-1} : A \times I \to A \times I$ be the identity, and once a retraction $r_n : X_n \times I \to A \times I \cup X_n \times \{0\}$ is defined, we use the retraction $r'_{n+1} : X_{n+1} \times I \to X_n \times I \cup X_{n+1} \times \{0\}$ from the last proposition and let r_{n+1} be the map

$$X_{n+1} \times I \xrightarrow{r'_{n+1}} X_n \times I \cup X_{n+1} \times \{0\} \xrightarrow{r_n\cup\mathrm{id}} A \times I \cup X_{n+1} \times \{0\},$$

which can be seen to be both well-defined and continuous. Moreover, it is a retraction as a composition of retractions. Then $r : X \times I \to A \times I \cup X \times \{0\}$ is defined as $r(x, t) = r_n(x, t)$ for $x \in X_n$. This is well-defined, since $r_n|_{X_{n-1}\times I} = r_{n-1}$ by construction. Moreover, it is continuous for the following reason: by definition, X is the colimit of the diagram

$$A \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots$$

Since $-\times I$ commutes with colimits, $X \times I$ is the colimit of

$$A \times I \hookrightarrow X_0 \times I \hookrightarrow X_1 \times I \hookrightarrow X_2 \times I \hookrightarrow \dots$$

Therefore, $r : X \times I \to A \times I \cup X \times \{0\}$ is continuous iff each map $X_n \times I \xrightarrow{r_n} A \times I \cup X_n \times \{0\} \hookrightarrow A \times I \cup X \times \{0\}$ is continuous, which holds true. Lastly, r is a retraction as well, essentially by definition. ☐

We have shown that $A \times I \cup X \times \{0\}$ is a retract of $X \times I$. Even more is true: it is a deformation retract.

Lemma 4.15. [11] *Let (X, A) be a relative CW-complex. Then $A \times I \cup X \times \{0\}$ is a deformation retract of $X \times I$.* ☐

Proof. (This proof was proposed by drs. J. Becerra) Note that $X \times \{0\}$ is a deformation retract of both $X \times I$ and $A \times I \cup X \times \{0\}$. Therefore, the unnamed inclusions in the commutative diagram

$$\begin{tikzcd}
X \times \{0\} \arrow[rr, hook] \arrow[dr, hook] & & X \times I \\
& A \times I \cup X \times \{0\} \arrow[ur, hook, "\iota"] &
\end{tikzcd}$$

are homotopy equivalences. Consequently, so is $\iota : A \times I \cup X \times \{0\} \hookrightarrow X \times I$. Also note that $(X \times I, A \times I \cup X \times \{0\})$ is a relative CW-complex, and has therefore the HEP by the previous theorem. Now, Proposition 4.7 implies that $\iota : A \times I \cup X \times \{0\} \hookrightarrow X \times I$ establishes $A \times I \cup X \times \{0\}$ as a deformation retract of $X \times I$. ☐

The fact that each relative CW-complex has the HEP is key in proving some other important results about CW-complexes in relation to homotopy theory. We present a few of them.

Lemma 4.16. *Let (X, X') be a CW-pair and suppose Y is another CW-complex. If we are given two homotopic maps $f, g : X' \to Y$, then the pushouts $X \cup_f Y$ and $X \cup_g Y$ of the diagrams*

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f"] & X \arrow[d] \\ Y \arrow[r] & X \cup_f Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, hook] \arrow[d, "g"] & X \arrow[d] \\ Y \arrow[r] & X \cup_g Y \end{tikzcd}$$

respectively, satisfy $X \cup_f Y \simeq X \cup_g Y \operatorname{rel} Y$.

Proof. [11] Let $H : X' \times I \to Y$ be a homotopy from f to g, and consider the pushout $(X \times I) \cup_{X' \times I} Y$, which contains both $X \cup_f Y$ and $X \cup_g Y$ as subspaces. By Lemma 4.15, there is a deformation retraction $X \times I \to X' \times I \cup X \times \{0\}$, and this gives us a deformation retraction $(X \times I) \cup_{X' \times I} Y \to X \cup_f Y$, which the identity on Y. A similar argument gives us a a deformation retraction $(X \times I) \cup_{X' \times I} Y \to X \cup_g Y$ which the identity on Y, and combining the two a homotopy equivalence $X \cup_f Y \simeq X \cup_g Y \operatorname{rel} Y$. ☐

Proposition 4.17. *(i) Let X be a pointed CW-complex. Then the quotient maps $C_u X \to CX$ and $SX \to \Sigma X$ are pointed homotopy equivalences.*

(ii) Let $f : X \to Y$ be a pointed cellular map between pointed CW-complexes. Then the quotient maps $M_f \to m_f$ and $C_u f \to Cf$ are pointed homotopy equivalences.

Proof. All statements follow from Theorem 4.14 and Proposition 4.9, since the subspace we collapse is always a subcomplex. ☐

Remark 4.18. The statement of the lemma holds in more generality for pointed spaces $(X, *)$ that satisfy the HEP [13]. Such a space is called *well-pointed*. ▽

We will later need the following construction, namely that of the mapping telescope. We will not need it in all its generality, but only in the case where the maps involved are inclusions. From its description, it should be clear why we call it a "telescope".

Definition 4.19. [11] Let $(Y_n)_{n \geq 1}$ be a sequence of CW-complexes such that Y_n is a subcomplex of Y_{n+1} for all n (not necessarily the n-skeleton: do not be confused by the sequence notation), and set $Y = \operatorname{colim}_n Y_n$ with respect to the inclusions $Y_n \hookrightarrow Y_{n+1}$. The *mapping telescope of* Y is now defined as $T_Y = \cup_{n=1}^{\infty} Y_n \times [n, n+1]$ as a subcomplex of $Y \times [1, \infty)$. If all the Y_n are pointed, then the *reduced mapping telescope* is defined as $t_Y = \cup_{n=1}^{\infty} Y_n \wedge [n, n+1]_+$ as a subcomplex of $Y \wedge [1, \infty)_+$. ◊

Note that the union in the definition of the reduced mapping telescope identifies all base points of the complexes considered, so the reduced mapping telescope comes with a canonical base point.

Lemma 4.20. *Let (Y_n) and Y be as above. Then Y is a deformation retract of both T_Y and t_Y.*

Proof. [11] It is clear that the projection $p' : Y \times [1, \infty) \to Y$ is a deformation retraction, so to show that T_Y deformation retracts to Y, we only need to show that T_Y is a deformation retract of $Y \times [1, \infty)$. Indeed, let $i : T_Y \hookrightarrow Y \times [1, \infty)$, $\iota' : Y \hookrightarrow Y \times [1, \infty)$ and $\iota : Y \hookrightarrow T_Y$ be the inclusions, set $p = p' \circ i$, and suppose we have found a deformation retract $\pi : Y \times [1, \infty) \to T_Y$. If $H' : Y \times [1, \infty) \times I \to Y \times [1, \infty)$ is a homotopy from $\iota' \circ p'$ to $\mathrm{id}_{Y \times [1,\infty)}$ relative to Y, then one may check that $H = \pi \circ H' \circ (i \times \mathrm{id}_I) : T_Y \times I \to T_Y$ defines a homotopy from $\iota \circ p$ to id_{T_Y} relative to Y, so that Y indeed would be a deformation retract of T_Y.

To show that this deformation retraction $Y \times [1, \infty) \to T_Y$ exists, define for all $n \in \mathbb{N}$ the subspace $Z_n = T_Y \cup Y \times [n, \infty)$ of $Y \times [1, \infty)$. Since the CW-pair (Y, Y_n) has the homotopy extension property, there is a deformation retraction $r'_n : Y \times [n, n+1] \to Y_n \times [n, n+1] \cup Y \times \{n+1\}$ induced by the one in Lemma 4.15. The identity on Z_{n+1} glues with r'_n to form a deformation retraction $r_n : Z_n \to Z_{n+1}$.

Let $p' : T_Y \to Y$ be the restriction of the projection $p : Y \times [1, \infty) \to Y$ to T_Y. If $\iota_{n+1} : Z_{n+1} \hookrightarrow Z_n$ and $\iota : T_Y \hookrightarrow Z_1$ are the inclusions, and if we let $H_n : Z_n \times I \to Z_n$ be the homotopy from $\iota_n \circ r_n$ to id_{Z_n} relative to Z_{n+1}, then we can perform H_n during the time interval $[1/2^{n-1}, 1/2^n]$, and glue these maps together to get a homotopy from $\iota \circ r$ (by its definition) to the identity on Z_1 which is stationary on T_Y and is performed during the time interval $[0, 1]$. This map is continuous and well-defined, which shows that T_Y is a deformation retract of $Y \times [1, \infty)$.

Now, in the reduced case (when Y is also pointed) we simply write $R = \{*\} \times [1, \infty)$ and note that $t_Y \cong T_Y/R$ and $Y \wedge [1, \infty)_+ \cong Y \times [1, \infty)/R$. Since the deformation retraction $Y \times [1, \infty) \to T_Y$ is stationary on R, we obtain a deformation retraction $Y \wedge [1, \infty)_+ \to t_Y$, and since the projections $p : T_Y \to Y$ and $p' : Y \times [1, \infty) \to Y$ and factor through R as well, they define induced projections $t_Y \to Y$ and $Y \wedge [1, \infty)_+ \to Y$. These establish Y as a deformation retract of t_Y. ☐

We end this section with obtaining the so-called Puppe sequence, an important result about the relation between a map and its mapping cone in homotopy theory. To get there, we first need the following lemma, which already contains part of the Puppe sequence.

Lemma 4.21. *Let $f : X \to Y$ be a pointed map between pointed topological spaces, and let another pointed space Z be given. Then the sequence $X \xrightarrow{f} Y \overset{\iota}{\hookrightarrow} Cf$ induces an exact sequence*

$$[Cf, Z]^\bullet \longrightarrow [Y, Z]^\bullet \longrightarrow [X, Z]^\bullet,$$

of pointed sets, which is natural in Z and in the map $f : X \to Y$.

Proof. To establish the exact sequence, we follow [13]. Suppose a pointed map $g : Y \to Z$ satisfies that $g \circ f$ is homotopic to the constant pointed map $c : X \to Z$. Then there is a homotopy $H : X \times I \to Z$ such that $H(X \times \{1\}) = *$, which means H factors through a continuous map $\overline{H} : CX \to Z$. By definition, it fits in a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook, "\iota"] \arrow[ddr, bend left, "g"] & \\
CX \arrow[r, hook, "\iota_{CX}"] \arrow[drr, bend right, "\overline{H}"'] & Cf \arrow[dr, dashed, "h"] & \\
 & & Z
\end{tikzcd}$$

so the universal property of the pushout gives us a map $h : Cf \to Z$ such that $g = h \circ \iota$.

On the other hand, for any map $h : Cf \to Z$ the map $h \circ \iota \circ f$ is null-homotopic, because the map $h \circ \iota_{CX} : CX \to Z$, where $\iota_{CX} : CX \to Cf$ is the map in the pushout square above, induces a homotopy $X \times I \to Z$ from $h \circ \iota \circ f$ to c.

(†) Now, for any commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "h"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

between two maps $f : X \to Y$ and $f' : X' \to Y'$, the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] & Y \arrow[dr, "h"] & \\
CX \arrow[dr, "C(g)"] & & Y' \arrow[d] \\
 & CX' \arrow[r] & Cf'
\end{tikzcd}$$

commutes, so the universal property of the pushout gives an induced map $Cf \to Cf'$ which fits in a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] & Y \arrow[d] \arrow[dr, "h"] & \\
CX \arrow[r] \arrow[dr, "C(g)"] & Cf \arrow[dr] & Y' \arrow[d] \\
 & CX' \arrow[r] & Cf'
\end{tikzcd}$$

This implies that there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[r, hook] \arrow[d, "h"] & Cf \arrow[d] \\
X' \arrow[r, "f'"] & Y' \arrow[r, hook] & Cf'
\end{tikzcd}$$

induced by the morphism between f and f'. Naturality of the exact sequence in Z and $f : X \to Y$ follows then from functoriality of $[\,\cdot\,,\,\cdot\,]^\bullet$. ☐

Proposition 4.22. (Puppe sequence) *Let $f : X \to Y$ be a pointed map between pointed topological spaces, and let another pointed space Z be given. Then the sequence $X \xrightarrow{f} Y \overset{\iota}{\hookrightarrow} Cf$ induces a long exact sequence*

$$\ldots \longrightarrow [\Sigma Cf, Z]^\bullet \longrightarrow [\Sigma Y, Z]^\bullet \longrightarrow [\Sigma X, Z]^\bullet \longrightarrow [Cf, Z]^\bullet \longrightarrow [Y, Z]^\bullet \longrightarrow [X, Z]^\bullet,$$

which is natural in Z and in the map $f : X \to Y$.

Proof. We will only show this statement for pointed CW-complexes X and Y, and with f a pointed cellular map, since we will not need the statement in its full generality. For the full proof, see [13], which we will modify for our special case. The details about naturality were added by us.

We only need to show that the part of the sequence that is written out in the statement of the lemma is exact and natural in Z, as inductively it will then follow that the whole sequence is exact and natural in Z. By Lemma 3.50, Y is a subcomplex of Cf. Now, $C\iota = Cf \cup_Y CY$ is a CW-complex as well by Proposition 3.35, and since ι is an inclusion of a subcomplex, CY can be regarded as a contractible subcomplex of $C\iota$. This implies that $C\iota \simeq C\iota/CY \cong Cf/Y \cong \Sigma X$ pointedly by Proposition 4.9. This homotopy equivalence is also natural in f: indeed, a morphism between $f : X \to Y$ and $f' : X' \to Y'$ induces as in the previous lemma a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[r, hook, "\iota"] \arrow[d, "h"] & Cf \arrow[r, hook] \arrow[d] & C\iota \arrow[d] \\
X' \arrow[r, "f'"] & Y' \arrow[r, hook, "\iota'"] & Cf' \arrow[r, hook] & C\iota'
\end{tikzcd}$$

and the map $C\iota \to C\iota'$ sends the subspace CY into CY'. Since taking the quotient is a functor, we finally end up with a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[r, hook, "\iota"] \arrow[d, "h"] & Cf \arrow[r, hook] \arrow[d] & C\iota \arrow[r, "\simeq"] \arrow[d] & C\iota/CY \arrow[r, "\sim"] \arrow[d] & \Sigma X \arrow[d, "\Sigma g"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, hook, "\iota'"] & Cf' \arrow[r, hook] & C\iota' \arrow[r, "\simeq"] & C\iota'/CY' \arrow[r, "\sim"] & \Sigma X'
\end{tikzcd}$$

This shows that the homotopy equivalence $C\iota \simeq \Sigma X$ is natural in f. Hence $[C\iota, Z]^\bullet \cong [\Sigma X, Z]^\bullet$ as pointed sets, naturally in both Z and f.

Furthermore, the inclusion $i : Cf \hookrightarrow C\iota$ induces a pointed homotopy equivalence $Ci \simeq Ci/C(Cf) \cong Ci/Cf \cong \Sigma Y$, again natural in f. Lastly, we will later (independently of our current work) show in Corollary 6.9 that Σ is a left adjoint functor, and hence commutes with pushouts, which implies there is a pointed homeomorphism $C(\Sigma f) \cong \Sigma Cf$, also natural in f.

Putting all this together, and using the preceding lemma repeatedly, we find that the sequence

$$[\Sigma Cf, Z]^\bullet \longrightarrow [\Sigma Y, Z]^\bullet \longrightarrow [\Sigma X, Z]^\bullet \longrightarrow [Cf, Z]^\bullet \longrightarrow [Y, Z]^\bullet \longrightarrow [X, Z]^\bullet$$

is exact and natural in Z and f by functoriality of $[\,\cdot\,,\,\cdot\,]^\bullet$, and as we remarked earlier, this is all we need to show. □

For fixed Z, the Puppe sequence tells us that the functor $[\,\cdot\,, Z]^\bullet$ satisfies some sort of long exact sequence condition. When we define a generalised reduced cohomology theory in Definition 5.1, the nature of this long exact sequence will become more clear, and it is a clear hint that we might be able to use these sets of homotopy classes to construct a generalised reduced cohomology theory.

4.2 Higher homotopy groups

We can recover and generalise the definition of the fundamental group (or rather, its underlying set) by choosing a base point $* \in S^n$ (and fixing it for the rest of this thesis), and defining the following pointed set:

Definition 4.23. [23] Let $(X, *)$ be a pointed topological space. Then for $n \geq 0$ we define the *n-th homotopy group*

$$\pi_n(X, *) \coloneqq [S^n, X]^\bullet.$$

It has the equivalence class of the constant map $c_X : (S, *) \to (X, *), s \mapsto *$ as preferred element. We tend to omit the base point in our notation and simply write $\pi_n(X)$. ◊

Lemma 4.24. *π_n is a functor $\mathsf{hTop}_* \to \mathsf{Set}_*$ for each $n \geq 0$.* □

Given the importance of the fundamental group, we can expect these homotopy groups contain much and important information about our spaces, and this turns out to be absolutely true. A few examples of this can be found in the rest of this thesis.

We will show later that for positive n, this pointed set indeed carries a natural group structure. However, for $n = 0$, it most certainly does not. In fact, it is not difficult to see that the above definition of $\pi_0(X, *)$ agrees with Definition 1.33, but now we also choose the path connected component of the base point $*$ of X as preferred element.

There is a homeomorphism $I^n/\partial I^n \cong S^n$, which means that a pointed map $S^n \to X$ is the same as a map $(I^n, \partial I^n) \to (X, *)$ of topological pairs. Similar to the group structure on the fundamental group, we can then, as in [19] define for any pointed topological space X, and $1 \leq i \leq n$ the binary operation $+_i$ on $\mathrm{Map}(S^n, X)$, which sends two pointed maps $\alpha, \beta : S^n \to X$ to

$$\alpha +_i \beta : S^n \to X, (t_1, \ldots, t_i, \ldots, t_n) \mapsto \begin{cases} \alpha(t_1, \ldots, 2t_i, \ldots, t_n) & \text{if } 0 \leq t_i < \frac{1}{2}, \\ \beta(t_1, \ldots, 2t_1 - 1, \ldots, t_n) & \text{if } \frac{1}{2} \leq t_i \leq 1, \end{cases}$$

which is continuous by the pasting lemma, and seen to be well-defined (in that ∂I^n is mapped on the base point of X). This operation factors through homotopy: if F and G are pointed homotopies $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$, respectively, then $\alpha +_i \beta \simeq \alpha' +_i \beta'$ via the pointed homotopy

$$H : S^n \times I \to X, ((t_1, \ldots, t_i, \ldots, t_n), s) \mapsto \begin{cases} F((t_1, \ldots, 2t_i, \ldots, t_n), s) & \text{if } 0 \leq t_i < \frac{1}{2}, \\ G((t_1, \ldots, 2t_1 - 1, \ldots, t_n), s) & \text{if } \frac{1}{2} \leq t_i \leq 1. \end{cases}$$

Therefore, we have n binary operations on $\pi_n(X)$, denoted by the same notation $+_i$, which all have the same twosided unit $[c_X]$. This might seem to turn out a bit complicated, but luckily we are saved by the lemma below.

Definition 4.25. [22] Let a set M together with a map $\cdot : M \times M \to M$ and a particular element $e \in M$ be given. Then $(M, \cdot, e)$ is a *monoid* if $\cdot$ is associative and e is a two-sided identity element of $\cdot$. It is a *commutative monoid* if moreover $\cdot$ is commutative.

A *homomorphism of monoids* $f : M \to N$ is a map that satisfies $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in M$ and maps the identity element of M to the identity element of N. The category of monoids is denoted by Mon, and of commutative monoids by CMon. ◊

Lemma 4.26. (Eckmann-Hilton argument) *Let A be a set, and suppose we are given two binary operations $\bullet, * : A \times A \to A$ with twosided unital elements $1_\bullet$ and 1_*. If*

$$(a \bullet b) * (c \bullet d) = (a * c) \bullet (b * d)$$

for all $a, b, c, d \in A$, then the two operations and their units coincide, and the operations are associative and commutative. Therefore, they define the same commutative monoid structure on M. In particular, if $\bullet$ and $*$ allow inverse elements, they define the same abelian group structure on A.

Proof. We slightly generalise the argument given in [19]. First note that

$$1_\bullet = 1_\bullet \bullet 1_\bullet = (1_\bullet * 1_*) \bullet (1_* * 1_\bullet) = (1_\bullet \bullet 1_*) * (1_* \bullet 1_\bullet) = 1_* * 1_* = 1_*,$$

and therefore we see that

$$a * d = (a \bullet 1_\bullet) * (1_\bullet \bullet d) = (a * 1_*) \bullet (1_* * d) = a \bullet d,$$

that is, $\bullet = *$. Hence

$$b \bullet c = b * c = (e \bullet b) * (c \bullet e) = (e * c) \bullet (b * e) = c \bullet b,$$

and

$$a \bullet (b \bullet d) = (a * 1_*) \bullet (b * d) = (a \bullet b) \bullet (1_\bullet \bullet d) = (a \bullet b) \bullet d.$$

Therefore $\bullet$ and $*$ are commutative and associative. If they allow inverse elements, they hence define the same commutative group structure, i.e. abelian group structure, on A. ☐

Remark 4.27. The condition that $(a \bullet b) * (c \bullet d) = (a * c) \bullet (b * d)$ for all $a, b, c, d \in A$ in the lemma above can be rephrased in a more insightful way. A *magma* is an algebraic structure consisting of a set M and a binary operation $\bullet : M \times M \to M$ (without any further conditions on this operation). We can define for two magmas $(M, \bullet)$ and $(N, *)$ a *product magma* $M \times N$, which has $M \times N$ as the underlying set, and the binary operation $\cdot$ given by $(m_1, n_1) \cdot (m_2, n_2) = (m_1 \bullet m_2, n_1 * n_2)$ for all $m_1, m_2 \in M$ and $n_1, n_2 \in N$. A *unital magma* is a magma M which contains a twosided unit element 1. Finally, a *homomorphism of magmas* $f : (M, \bullet) \to (N, *)$ is a map of sets $f : M \to N$ such that $f(m_1 \bullet m_2) = f(m_1) * f(m_2)$ for all $m_1, m_2 \in M$. Now, the Eckmann-Hilton argument says that if $(M, \bullet, 1_\bullet)$ and $(M, *, 1_*)$ are unital magmas, and if $* : (M, \bullet) \times (M, \bullet) \to (M, \bullet)$ is a homomorphism of magmas (not assumed to be unital), then both unital magma structures coincide and define a commutative monoid structure on M. ▽

Corollary 4.28. *For any pointed topological space X and $n \geq 1$, $\pi_n(X)$ carries naturally the structure of a group, which is even abelian if $n \geq 2$.*

Remark 4.29. Naturality means here that a continuous map $f : X \to Y$ induces a group homomorphism $\pi_n(X) \to \pi_n(Y)$. In other words, the corollary states that for $n \geq 1$, π_n is a functor $\mathsf{hTop}_* \to \mathsf{Grp}$, or even $\mathsf{hTop}_* \to \mathsf{Ab}$ in case $n \geq 2$. Note that we will often treat the π_n also as functors from Top_* rather than hTop_*. This of course should not cause confusion. ▽

Proof. [19] We already know that the fundamental group is a functor, so we only need to consider the case $n \geq 2$. Just like in the case of the fundamental group, it is clear that the operations $+_i$ allow inverse elements.

Now, if pointed maps $\alpha, \beta, \gamma, \delta : S^n \to X$ and $1 \leq i, j \leq n$ are given, then $(\alpha +_i \beta) +_j (\gamma +_i \delta)$ is the map

$$S^n \to X, (t_1, \dots, t_i, \dots, t_j, \dots t_n) \mapsto \begin{cases} \alpha(t_1, \dots, 2t_i, \dots, 2t_j, \dots t_n) & \text{if } 0 \leq t_i, t_j \leq \frac{1}{2}, \\ \beta(t_1, \dots, 2t_i - 1, \dots, t_j, \dots t_n) & \text{if } \frac{1}{2} \leq t_i \leq 1 \text{ and } 0 \leq t_j \leq \frac{1}{2}, \\ \gamma(t_1, \dots, t_i, \dots, 2t_j - 1, \dots t_n) & \text{if } 0 \leq t_i \leq \frac{1}{2} \text{ and } \frac{1}{2} \leq t_j \leq 1, \\ \delta(t_1, \dots, 2t_i - 1, \dots, 2t_j - 1, \dots t_n) & \text{if } \frac{1}{2} \leq t_i, t_j \leq 1, \end{cases}$$

which similarly equals $(\alpha +_j \gamma) +_i (\beta +_j \delta)$. Therefore $([\alpha] +_i [\beta]) +_j ([\gamma] +_i [\delta]) = ([\alpha] +_j [\gamma]) +_i ([\beta] +_j [\delta])$, and the Eckmann-Hilton argument applies and gives $\pi_n(X)$ unambiguously the structure of an abelian group.

A continuous map $f : X \to Y$ induces the map $f_* : \pi_n(X) \to \pi_n(Y) : [\alpha] \mapsto [f \circ \alpha]$, which respects $+_1$ (and hence all $+_i$ and the group structure on $\pi_n(X)$ and $\pi_n(Y)$) by the same argument as in the fundamental group. Therefore f_* is a group homomorphism. It is also clear that f_* only depends on the homotopy class of f, so π_n is a functor $\mathsf{hTop}_* \to \mathsf{Ab}$. □

Just like we wanted to consider homology and cohomology of pairs of spaces, we also want to consider homotopy groups of such pairs. It will turn out that a pair of spaces then also induces a long exact sequence of homotopy groups, and this allows us to relate the homotopy groups of a subspace to that of the space itself. First recall from Definition 2.34 what a map between triplets of topological spaces is.

Definition 4.30. Two maps $f, g : (X, X', X'') \to (Y, Y', Y'')$ of triplets of topological spaces are *homotopic* if there exists a homotopy $H : X \times I \to Y$ such that $H(X' \times I) \subseteq Y'$ and $H(X'' \times I) \subseteq Y''$. ♢

Any pair of pointed spaces is also a triplet of spaces, and as such, we let for any two pointed pairs (X, X') and (Y, Y') of topological spaces the set $[(X, X'), (Y, Y')]^\bullet$ denote the set of homotopy classes of maps of triplets $(X, X', *) \to (Y, Y', *)$. This is a pointed set, with the homotopy class of the constant map $(X, X', *) \to (Y, Y', *)$ on the base point as preferred element. More generally, for two triplets (X, X', X'') and (Y, Y', Y'') of spaces, we let $[(X, X', X''), (Y, Y', Y'')]$ denote the set of homotopy classes of maps between these triplets.

Definition 4.31. [23] Let (X, X') be a pair of pointed spaces. For each $n \geq 1$. Then the *n-th relative homotopy group* is defined as the pointed set $\pi_n(X, X', *) = [(D^n, S^{n-1}), (X, X')]^\bullet$. Again, we often omit the base point in notation and write $\pi_n(X, X')$. ♢

Lemma 4.32. *For each $n \geq 1$, the relative homotopy group is a functor $\pi_n : \mathsf{hTop}(2)_* \to \mathsf{Set}_*$.* □

For $n \geq 2$, the relative homotopy groups also allow a natural group structure. Let (X, X') be a pointed pair of spaces, and let $\alpha, \beta : (D^n, S^{n-1}, *) \to (X, X', *)$ represent two homotopy classes of maps. Consider I^{n-1} as the subset $\{(t_1, \dots, t_n \in I^n \mid t_n = 0\}$ of I^n, and set $J^{n-1} = \{(t_1, \dots, t_n \in I^n \mid t_n = 1, \text{or}\, t_i = 0, 1\, \text{for some}\ 1 \leq i \leq n-1\}$. There is a homeomorphism $(D^n, S^{n-1}, *) \cong (I^n / J^{n-1}, \partial I^{n1} / J^{n-1}, J^{n-1})$, which allows us to see α and β as maps $(I^n, \partial I^n, J^{n-1}) \to (X, X', *)$. Now, for $1 \leq i \leq n-1$ the maps $+_i$ defined above give us maps $\alpha +_i \beta : (I^n, \partial I^n, J^{n-1}) \to (X, X', *)$, and once more these maps respects homotopy. Similarly as the case for absolute homotopy groups, for $n \geq 2$ there is a group structure on $\pi_n(X, X')$ defined by each of these maps [23], and by the Eckmann-Hilton argument 4.26, this group structure is abelian when $n \geq 3$.

Lemma 4.33. *The relative homotopy group π_2 is a functor $\mathsf{hTop}(2)_* \to \mathsf{Grp}$, and for each $n \geq 3$, the relative homotopy group is a functor $\pi_n : \mathsf{hTop}(2)_* \to \mathsf{Ab}$.* □

To describe the long exact sequence of the homotopy groups, we need the following lemma.

Lemma 4.34. *For all $n \geq 1$, there for any pointed space $(X, *)$ a natural bijection $\pi_n(X, *) \cong \pi_n(X, *, *)$.*

Proof. This follows from $\pi_n(X, *, *) \cong [(I^n, \partial I^n, J^{n-1}), (X, *, *)] \cong [(I^n, \partial I^n), (X, *)] \cong \pi_n(X, *)$. □

For a pointed pair (X, X'), let $i : X' \hookrightarrow X$ be the inclusion. This defines maps $i_n \coloneqq \pi_n(i) : \pi_n(X') \to \pi_n(X)$ for all $n \geq 0$. Now, the isomorphism $\pi_n(X, *) \cong \pi_n(X, *, *)$ and the map $(X, *, *) \to (X, X', *)$ that acts as the identity on X gives us for each $n \geq 1$ a map $j_n : \pi_n(X, *) \to \pi_n(X, X', *)$. Lastly, for each $n \geq 1$ we define

$p_n : \pi_n(X, X', *) \to \pi_{n-1}(X', *), [\alpha] \mapsto [\alpha|_{S^{n-1}}]$, which works because for $[\alpha] \in \pi_n(X, X', *)$ a representative α is a map $(D^n, S^{n-1}, *) \to (X, X', *)$, so $\alpha|_{S^{n-1}}$ is a pointed map $S^{n-1} \to X'$. Moreover, a homotopy between two maps $(D^n, S^{n-1}, *) \to (X, X', *)$ restricts to a homotopy between their restrictions $(S^{n-1}, *) \to (X', *)$ by definition of a homotopy of maps of triplets. Also note that it is a group homomorphism for $n \geq 2$. Namely, for two maps $\alpha, \beta : (I^n, \partial I^n, J^{n-1}) \to (X, X', *)$, the maps $(\alpha +_i \beta)|_{\partial I^n}$ and $\alpha|_{\partial I^n} +_i \beta|_{\partial I^n}$ coincide.

Proposition 4.35. *Let (X, X') be a pointed pair of spaces. Then the sequence*

$$\begin{tikzcd} \ldots \arrow[r] & \pi_1(X') \arrow[r, "i_1"] & \pi_1(X) \arrow[r, "j_1"] & \pi_1(X, X') \arrow[r, "p_1"] & \pi_0(X') \arrow[r, "i_0"] & \pi_0(X) \end{tikzcd}$$

of groups and pointed sets is exact, and moreover natural in the pointed pair (X, X').

Proof. The proof can be found in [23]. ☐

We end this section with addressing the way the homotopy groups depend on the base point. Since π_0 does not depend on the choice of base point at all, we are only interested in the higher homotopy groups. We know from introductory topology courses that the fundamental group is basically independent of the choice of base points within a path-connected component. This turns out to hold as well for higher homotopy groups. In fact, they are very well-behaved with respect to paths between points in the space. We present three useful results making the situation precise.

The fact that any pointed map induces a homomorphism on homotopy groups can also be rephrased as saying that any non-pointed map $f : X \to Y$ yields for any choice of $x_0 \in X$ a homomorphism $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$. In what follows, we denote such a homomorphism by f_*.

Lemma 4.36. *Let X be a topological space, take $n \geq 1$ and let $\gamma : I \to X$ be a path in X. Then γ induces an isomorphism $\gamma_* : \pi_n(X, \gamma(1)) \to \pi_n(X, \gamma(0))$ satisfying the following properties:*

(i) If $\gamma, \gamma' : I \to X$ are homotopic relative to $\{0, 1\}$, then $\gamma_* = \gamma'_*$.
(ii) If γ is constant, then γ_* is the identity.
(iii) If $\gamma, \delta : I \to X$ satisfy $\gamma(1) = \delta(0)$, then $(\gamma \star \delta)_* = \gamma_* \circ \delta_*$, where $\star$ denotes the composition of paths.
(iv) If $f : X \to Y$ is continuous, then the square

$$\begin{tikzcd} \pi_n(X, \gamma(1)) \arrow[r, "f_*"] \arrow[d, "\gamma_*"] & \pi_n(Y, f \circ \gamma(1)) \arrow[d, "(f\circ\gamma)_*"] \\ \pi_n(X, \gamma(0)) \arrow[r, "f_*"] & \pi_n(Y, f \circ \gamma(0)) \end{tikzcd}$$

commutes.

Proof. The proof is very similar to the proof that the fundamental group is a group, and therefore we only describe the nature of the induced homomorphism γ_*. Full details can be found in [23].

Let $\alpha : (D^n, S^{n-1}) \to (X, \gamma(1))$ represent an element $[\alpha] \in \pi_n(X, \gamma(1))$. Consider for now $\mathbb{R}^n$ as the subspace $\mathbb{R}^n \times \{1\}$ of $\mathbb{R}^{n+1}$. Let $P = (0, -1) \in \mathbb{R}^n \times \mathbb{R}$ and let $p : D^n \cup_{S^{n-1} \times \{1\}} S^{n-1} \times I \to \mathbb{R}^n$ be the map that sends a point $x \in D^n \cup_{S^{n-1} \times \{1\}} S^{n-1} \times I$ to the intersection of the line through P and x with $\mathbb{R}^n$ (let us stress that D^n and $\mathbb{R}^n$ are seen as subspaces $D^n \times \{1\}$ and $\mathbb{R}^n \times \{1\}$ of $\mathbb{R}^{n+1}$). p defines a homeomorphism from $D^n \cup_{S^{n-1} \times \{1\}} S^{n-1} \times I$ onto its image in $\mathbb{R}^n$, which is another disc centered at the origin. Using the standard homeomorphism from such a disc to D^n, we get a homeomorphism $p : D^n \cup_{S^{n-1} \times \{1\}} S^{n-1} \times I \xrightarrow{\sim} D^n$ (which we indeed also will call p).

Now, γ induces a map $\gamma \circ \mathrm{pr}_1 : S^{n-1} \times I \to X$, and using the fact that $\alpha(S^{n-1}) = \gamma(1)$, the universal property of the pushout gives us a map $(D^n, S^{n-1}) \cong (D^n \cup_{S^{n-1} \times \{1\}} S^{n-1}, S^{n-1} \times \{0\}) \xrightarrow{\alpha \cup \gamma \circ \mathrm{pr}} (X, x_0)$. We let $\gamma_*([\alpha])$ be the homotopy class represented by this last map.

As said, verifying that γ_* is a group homomorphism and that is satisfies properties (i) to (iv) is certainly not trivial, but it is intuitive and straightforward, and the first three of these properties furthermore imply that γ_* is an isomorphism. ☐

Corollary 4.37. *Let $f : X \to Y$ be continuous, take $n \geq 1$ and pick two points $x_0, x_1 \in X$ in the same path component. Then $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is injective, surjective or bijective if and only if $f_* : \pi_n(X, x_1) \to \pi_n(Y, f(x_1))$ is injective, surjective or bijective, respectively.*

Proof. [23] Let $\gamma : I \to X$ be a path from x_0 to x_1. Then the diagram

$$\begin{tikzcd}
\pi_n(X, x_1) \arrow[r, "f_*"] \arrow[d, "\gamma_*"] & \pi_n(Y, f(x_1)) \arrow[d, "(f\circ\gamma)_*"] \\
\pi_n(X, x_0) \arrow[r, "f_*"] & \pi_n(Y, f(x_0))
\end{tikzcd}$$

commutes, and γ_* and $(f \circ \gamma)_*$ are isomorphisms. ☐

Lemma 4.38. *Let $f, g : X \to Y$ be two homotopic continuous maps between topological spaces, and let $H : X \times I \to Y$ be a homotopy from f to g. Pick a point $x_0 \in X$. Let $\gamma = H|_{\{x_0\}\times I} : I \to Y$ be the path from $f(x_0)$ to $g(x_0)$ in Y that H gives. Then the diagram*

$$\begin{tikzcd}
 & \pi_n(X, x_0) \arrow[dl, "g_*"] \arrow[dr, "f_*"] & \\
\pi_n(Y, g(x_0)) \arrow[rr, "\gamma_*"] & & \pi_n(Y, f(x_0))
\end{tikzcd}$$

commutes for all $n \geq 1$.

Proof. [23] Let $\alpha : (D^n, S^{n-1}) \to (X, x_0)$ represent an element $[\alpha] \in \pi_n(X, x_0)$, and define $F = H \circ (\alpha \times \mathrm{id}_I) : D^n \times I \to Y$, which is a homotopy from $f \circ \alpha$ to $g \circ \alpha$ satisfying $F(x, t) = \gamma(t)$ for all $x \in S^{n-1}$. We will show that we can produce a homotopy between $f \circ \alpha$ and $g \circ \alpha$ relative to S^{n-1}. To do so, consider once more the cylinder $D^n \times I$, and consider for each $x \in D^n$ the line L_x through $P = (0, 2) \in D^n \times \mathbb{R}$ and $(x, 1)$. Let $l_x : I \to D^n \times I$ be the affine linear map parametrising the part of L_x in $D^n \times I$, with $l_x(1) = (x, 1)$ and $l_x(0) \in D^n \cup_{S^{n-1}\times\{0\}} S^{n-1} \times I$. It is clear that $l_x(t)$ is continuous in both x and t, so we can define a homotopy $G : D^n \times I \to Y, (x, t) \mapsto F(l_x(t))$, which can (after a bit of care) seen to be a homotopy relative to S^{n-1} between $f \circ \alpha$ and a representative of $\gamma_*([g \circ \alpha])$. This shows that $f_*([\alpha]) = [f \circ \alpha] = \gamma_*[g \circ \alpha] = \gamma_* \circ g_*([\alpha])$ and hence that the diagram above commutes. ☐

4.3 Approximation theorems and the Whitehead Theorem

We have yet to cover three main results from the homotopy theory of CW-complexes, namely the Cellular Approximation Theorem, the Whitehead Theorem (and the important related Proposition 4.54) and the CW-approximation Theorem. The first one is for us more a technical yet helpful tool, and we will not go into the details of its proof. The Whitehead Theorem and the results that lead to it do however contain key concepts for us, and we will give a full proof. CW-approximation is more than just a technical tool, but mainly because it just so happens that we can later give a proof of it using the Brown Representability Theorem. We will therefore skip a more direct proof.

We will begin with stating the Cellular Approximation Theorem and a related result.

Lemma 4.39. *Let (X, A) be a CW-pair and let (Y, B) be a topological pair with B nonempty. Assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$ and all $n > 0$ such that $X \setminus A$ has (open) cells of dimension n, and also assume in case $X \setminus A$ has 0-cells, that then the inclusion $B \hookrightarrow Y$ induces a surjection $\pi_0(B) \to \pi_0(Y)$. Then every map $f : (X, A) \to (Y, B)$ is homotopic relative to A to a map $X \to B$.*

Proof. The proof can be found in [11]. ☐

Theorem 4.40. (Cellular Approximation Theorem) *Let $f : (X, A) \to (Y, B)$ be a continuous map between relative CW-complexes. Then f is homotopic relative to A to a cellular map $(X, A) \to (Y, B)$.*

Proof. The proof can be found in [11], or in [19] and [23] in slightly more detail. ☐

Corollary 4.41. *For all $n < m$, it holds that $\pi_n(S^m) = 0$.*

Proof. Consider S^m and S^n as CW-complexes with one 0-cell as base point and one m-cell or n-cell respectively attached. Since $n < m$, the Cellular Approximation Theorem implies that any map $S^n \to S^m$ is pointedly homotopic to the constant map on the base point, which means that $\pi_n(S^m) = [S^n, S^m]^\bullet = 0$. ☐

Now we turn our attention to the Whitehead Theorem and the notion of a weak homotopy equivalence.

Definition 4.42. [23] A map $f : X \to Y$ of topological spaces is a *weak homotopy equivalence* if f induces isomorphisms $\pi_n(X, x_0) \xrightarrow{\sim} \pi_n(Y, f(x_0))$ for all $n \geq 0$ and all choices of base point $x_0 \in X$ (and the induced map is understood to be a pointed bijection in case of $n = 0$). ♢

A large part of algebraic topology is devoted to the study of homotopy groups. A weak homotopy equivalence is then comparable with an isomorphism: it shows that from the point of view of homotopy groups alone, two spaces cannot be distinguished from each other. It is natural to consider especially on CW-complexes, given the importance of spheres in general and given the fact that CW-complexes are more or less built from spheres. However, we should note that a weak homotopy equivalence is not required to have a weak homotopy equivalence as “inverse”: the inverses of the isomorphisms such a map induces on homotopy groups need not be induced by a continuous map. This leads us to the following definition:

Definition 4.43. Let X and Y be two topological spaces. They are *weakly homotopy equivalent* if there exist a finite sequence of topological spaces $X_0 = X, X_1, X_2, \dots, X_n = Y$ with the property that there exists for each $0 \leq i \leq n-1$ a weak homotopy equivalence $X_i \to X_{i+1}$ or a weak homotopy equivalence $X_{i+1} \to X_i$. ♢

To weakly homotopy equivalent spaces clearly have isomorphic homotopy groups. Now, the Whitehead Theorem states that any weak homotopy equivalence between CW-complexes must be a homotopy equivalence (and hence that CW-complexes being weakly homotopy equivalent means they are homotopy equivalent). This justifies the idea that the behaviour of CW-complexes in homotopy theory is determined by their behaviour with respect to the spheres. Before we can get there, we show a bunch of result about weak homotopy equivalences and related concepts that we will use later.

Lemma 4.44. *If $f : X \to Y$ and $g : Y \to Z$ are weak homotopy equivalences, then so is $g \circ f$.* ☐

Lemma 4.45. *Suppose $f : X \to Y$ is a homotopy equivalence. Then f is also a weak homotopy equivalence.*

Proof. Let us begin by noting the proof is not entirely trivial, since f is not required to be a pointed homotopy equivalence in any way for arbitrary base points. If it was, then functoriality of π_n would immediately yield the statement, but now we need to work a bit, and we follow [23].

It is clear that f induces an isomorphism $\pi_0(X) \to \pi_0(Y)$. Let $g : Y \to X$ be a map such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, let x_0 be an arbitrary point in X and let γ be the path from $g \circ f(x_0)$ to x_0 given by the homotopy $g \circ f \simeq \mathrm{id}_X$. Since $(g \circ f)_* = g_* \circ f_*$, via Lemma 4.38 we obtain for $n \geq 1$ a commutative diagram

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "f_*"] \arrow[dr, "\gamma_*"'] & \pi_n(Y, f(x_0)) \arrow[d, "g_*"] \\
 & \pi_n(X, g \circ f(x_0))
\end{tikzcd}$$

Since γ_* is an isomorphism by Lemma 4.36, f_* must be injective. A similar argument using the homotopy $f \circ g \simeq \mathrm{id}_Y$ implies that f_* is surjective, and therefore $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $n \geq 1$. Since x_0 was arbitrary, we are done. ☐

Lemma 4.46. *If $f, g : X \to Y$ are homotopic maps and f is a weak homotopy equivalence, then so is g.*

Proof. Once more it is clear that g must induce a bijection $\pi_0(X) \to \pi_0(Y)$ if we are given that f does, since the homotopy between the two gives us paths from $f(x)$ to $g(x)$ for any $x \in X$. For $n \geq 1$, we can use Lemma 4.38, and the fact that both f_* and γ_* are isomorhisms in this case to conclude that g_* is also an isomorphism. Since x_0 is arbitrary, we are done. ☐

Remark 4.47. By Corollary 4.37, if a continuous map $f : X \to Y$ induces isomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $n \geq 0$, and for a particular $x_0 \in X$, then it also induces isomorphisms $f_* : \pi_n(X, x_1) \to \pi_n(Y, f(x_1))$ for any other choice of x_1 within the same path component as x_0. Therefore, for a path-connected space X we only need to check a single point to see whether or not f is a weak homotopy equivalence. ▽

Lemma 4.48. [23] *Let (X, X') be a topological pair, and assume the inclusion $\iota : X' \hookrightarrow X$ induces a bijection $\pi_0(X') \to \pi_0(X)$. Then ι is a weak homotopy equivalence if and only if $\pi_n(X, X', x_0) = 0$ for all $n \geq 1$ and all $x_0 \in X'$.*

Proof. This follows from exactness of the long exact sequence of homotopy groups of the triplet (X, X', x_0) in Proposition 4.35, where x_0 ranges over X'. ☐

Definition 4.49. A pair (X, X') of topological spaces is *n-connected* if for every $m \leq n$ and every map $f : (D^m, S^{m-1}) \to (X, X')$ there is a homotopy relative to S^{m-1} from f to a map with image in X'. ♢

Remark 4.50. In the special case of $n = 0$, we adopt the convention that $(D^0, S^{-1}) = (*, \varnothing)$, so that a pair (X, X') is 0-connected if the inclusion $X' \hookrightarrow X$ induces a surjective map $\pi_0(X') \to \pi_0(X)$. ▽

Lemma 4.51. *Let (X, X') be a pair of spaces and $n \geq 1$. Then (X, X') is n-connected if and only if the inclusion $X' \hookrightarrow X$ induces a surjection $\pi_0(X') \to \pi_0(X)$ and $\pi_i(X, X', x_0) = 0$ for all $1 \leq i \leq n$ and all choices of $x_0 \in X'$.*

Proof. Sufficiency follows from the definition of n-connectedness and Lemma 4.39. For necessity, let $x_0 \in X'$ be arbitrary, and note that any map $\alpha : (D^i, S^{i-1}, *) \to (X, X', x_0)$ now is homotopic relative to S^{i-1} and $*$ to a map with image in X'. By contracting D^i to a point, we obtain a pointed homotopy (now trivially relative to X') from the latter map to the constant map onto x_0. This shows that $[\alpha] = 0 \in \pi_i(X, X', x_0)$, the latter group (or pointed set if $n = 1$) therefore is trivial. ☐

Lemma 4.52. [23] *Let (X, X') be a pair of spaces, and suppose the inclusion $\iota : X' \hookrightarrow X$ is a weak homotopy equivalence. Then (X, X') is n-connected for every n.*

Proof. The inclusion ι induces a bijection $\pi_0(X') \to \pi_0(X)$ by being a weak homotopy equivalence, so (X, X') is 0-connected. Moreover, this allows us to use Lemma 4.48, which implies that $\pi_n(X, X') = 0$ for any choice of base point and all $n \geq 1$. By the previous lemma, (X, X') is n-connected for these n as well. ☐

Lemma 4.53. *Let (X, X') be a CW-pair.*

(i) If all the cells in $X \setminus X'$ have dimension greater than n, then (X, X') is n-connected.

(ii) The pair (X, X_n) is n-connected, and the inclusion $X_n \hookrightarrow X$ induces isomorphisms $\pi_i(X_n, x_0) \to \pi_i(X, x_0)$ for $i < n$ (which is understood to be a pointed bijection in case $i = 0$) and a surjection $\pi_n(X_n, x_0) \to \pi_n(X, x_0)$, for all choices of base point $x_0 \in X'$.

Proof. [11] For the first statement, let $m \leq n$ and $f : (D^m, S^{m-1}) \to (X, X')$ be a map of pairs. Using the Cellular Approximation Theorem on f and the given fact that $X \setminus X'$ has no cells of dimension less than or equal to n, there must be a map $g : (D^m, S^{m-1}) \to (X, X')$ with image in X' to which f is homotopic relative to S^{m-1}. Therefore, (X, X') is n-connected.

For the second statement, first note that (X, X_n) is n-connected by the first part, and therefore the inclusion $X' \to X$ induces a surjection $\pi_0(X') \to \pi_0(X)$. In particular, if $n = 0$, the pair (X, X_n) is n-connected. If on the other hand $n \geq 1$, Lemma 4.51 and the long exact sequence of homotopy groups of the pair (X, X') in Proposition 4.35 together imply that the inclusion also induces a surjection $\pi_n(X') \to \pi_n(X)$, isomorphisms $\pi_i(X') \to \pi_i(X)$ for $1 \leq i \leq n - 1$ and an injection $\pi_0(X') \to \pi_0(X)$. Since the map $\pi_0(X') \to \pi_0(X)$ is apparently both injective and surjective, it is a bijection. ☐

After all these rather short and technical statements, let us turn towards proving the Whitehead Theorem. It will in fact follow from a more general statement, which is in itself of great importance to us.

Proposition 4.54. *Suppose $f : Y \to Z$ is a weak homotopy equivalence between topological spaces or pointed topological spaces, respectively. Then f induces natural isomorphisms $[\,\cdot\,, Y] \cong [\,\cdot\,, Z]$ of functors $\mathsf{hCW}^{\mathrm{opp}} \to \mathsf{Set}$ or $[\,\cdot\,, Y]^\bullet \cong [\,\cdot\,, Z]^\bullet$ of functors $\mathsf{hCW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$, respectively.*

Proof. [13] By Lemma A.25, we only need to show that for every non-pointed or pointed CW-complex X the map

$$[X, Y] \to [X, Z], \text{ or } [X, Y]^\bullet \to [X, Z]^\bullet,$$

induced by f, respectively, is a bijection. It is namely also clear that the map induced by f on the pointed sets of homotopy classes is pointed. In the proof below we will include between parentheses what needs to be adjusted in the pointed, rather than the non-pointed case.

Note that Z is (pointedly) homotopy equivalent to M_f via a homotopy equivalence that identifies f with the inclusion $\iota : Y \hookrightarrow M_f$ (see Remark 3.48). Therefore, it suffices to show that ι induces a bijection $[X, Y] \to [X, M_f]$. Since f was assumed to be a weak homotopy equivalence, so is ι, and therefore the pair (M_f, Y) satisfies by Lemma 4.48 the conditions of Lemma 4.39 (we assume that Y is nonempty, as the proposition is trivial in case Y (and hence Z) is empty). We can therefore apply said lemma to a map $(X, \varnothing) \to (M_f, Y)$ (or in the pointed case, a map $(X, *) \to (M_f, Y)$), and find that it is homotopic to a map $X \to Y$ (relative to the base point of X), which implies that $\iota \circ - : [X, Y] \to [X, M_f]$ is surjective.

For injectivity, assume that two maps $g, h : X \to Y$ induce homotopic maps $\iota \circ g \simeq \iota \circ h$. If we apply the lemma to the homotopy $H : (X \times I, X \times \partial I) \to (M_f, Y)$ between these maps, we find that H is homotopic relative to $X \times \partial I$ to a homotopy $G : X \times I \to Y$. This means that $G(\,\cdot\,, 0) = g$ and $G(\,\cdot\,, 1) = h$, so g and h are also homotopic. (In the pointed case, we must take apply the lemma to the homotopy $(X \times I, X \times \partial I \cup \{*\} \times I) \to (M_f, Y)$.) This shows that ι indeed induces a bijection, which is all we needed to show. ☐

Remark 4.55. (†) Note that there is a partial converse to the above proposition: if Y and Z are pointed and path-connected, and f induces a natural isomorphism $[\,\cdot\,, Y]^\bullet \cong [\,\cdot\,, Z]^\bullet$ of functors $\mathsf{hCW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$, then f is a pointed weak homotopy equivalence. To see this, fill in the spheres S^n in the natural isomorphism to show that $f_* : \pi_n(Y, *) \to \pi_n(Z, *)$ is an isomorphism for all $n \geq 0$. Since both Y and Z are assumed to be path-connected, Remark 4.47 implies that f is indeed a pointed weak homotopy equivalence. ▽

Theorem 4.56. (The Whitehead Theorem) [23] *Let $f : X \to Y$ be a weak homotopy equivalence between two CW-complexes. Then f is a homotopy equivalence.*

Proof. (†) By the previous proposition, f induces a natural isomorphism $h^f : h^X \xrightarrow{\sim} h^Y$, where $h : \mathsf{hCW} \to \mathsf{Fun}(\mathsf{hCW}^{\mathrm{opp}}, \mathsf{Set})$ is the Yoneda functor of Remark A.31. Now that X and Y are CW-complexes, we can apply the Yoneda Lemma A.34 and use that h is fully faithful to conclude that f must be an isomorphism in the category hCW, in other words, a homotopy equivalence. ☐

The same argument can of course be applied in the pointed case, so we also have the following pointed verson of the Whitehead Theorem.

Theorem 4.57. (Pointed Whitehead Theorem) *Let $f : X \to Y$ be a pointed weak homotopy equivalence between two pointed CW-complexes. Then f is a pointed homotopy equivalence.* ☐

The last result we need is the CW-approximation Theorem (not to be confused with the Cellular Approximation Theorem), and its uniqueness.

Theorem 4.58. (CW-approximation) *Let X be a topological space. Then there exists a CW-complex Y and a weak homotopy equivalence $f : Y \to X$. If X is pointed, both Y and f may be chosen to be pointed as well.*

Proof. A direct proof which constructs an explicit CW-pair is given in [11]. We will later also give a more abstract proof of a slightly stronger version of the theorem as Theorem 6.40 via the Brown Representability Theorem 6.31, the proof of which is independent of CW-approximation. ☐

Proposition 4.59. *Let* X *be a topological space, and suppose* Y *and* Y' *are both CW-complexes with weak homotopy equivalences* $f : Y \to X$ *and* $f' : Y' \to X$*. Then there exists a homotopy equivalence* $g : Y \to Y'$ *such that the diagram*

$$\begin{tikzcd} Y \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes up to homotopy. The statement with pointed spaces, maps and homotopies also holds. In other words, either non-pointed or pointed CW-approximation is unique up to homotopy equivalence.

Proof. (†) By Proposition 4.54 there is a natural isomorphism $[\,\cdot\,, Y] \cong [\,\cdot\,, X] \cong [\,\cdot\,, Y']$ of functors $\mathsf{hCW}^{\text{opp}} \to \mathsf{Set}$. The Yoneda Lemma tells us that this isomorphism is induced by an isomorphism $g : Y \to Y'$ in hCW (which means g is a homotopy equivalence). Moreover, by definition g fits in a commutative diagram

$$\begin{tikzcd} {[\,\cdot\,, Y]} \arrow[rr, "g\circ -"] \arrow[dr, "f\circ -"'] & & {[\,\cdot\,, Y']} \arrow[dl, "f'\circ -"] \\ & {[\,\cdot\,, X]} & \end{tikzcd}$$

and by filling in the space Y in the first argument and considering the identity id_Y, we find that $f \simeq f' \circ g$, which means that the diagram in the statement indeed commutes up to homotopy. The pointed case is treated entirely similar. ☐

Chapter 5

Cohomology on CW-complexes

It is time to turn our attention towards cohomology again, although this time immediately restricted to CW-complexes. We will define generalised reduced cohomology theories and generalised cohomology theories on CW-complexes, and derive a few useful results about them. The reader might wonder while reading the first section why we suddenly consider a new kind of cohomology theory, but there are two good reasons for it. Firstly, generalised reduced cohomology theories are "basically the same" as generalised cohomology theories, as long as we restrict ourselves to CW-complexes. This is made precise in Theorem 5.11 in the next section. Secondly, they are however in some situations easier to study, just like reduced singular cohomology is sometimes easier to use than ordinary singular cohomology.

The last section of this chapter deals with a few technical properties of reduced cohomology functors. We present a more abstract treaty on particular functors from the category of pointed CW-complexes to the category of pointed sets, since this will also be especially useful when proving the Brown Representability Theorem.

5.1 Reduced and generalised cohomology theories on CW-complexes

Definition 5.1. **(Generalised reduced cohomology theory)** [13] A sequence of functors $\widetilde{\mathrm{h}}^n : \mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Ab}$ with $n \in \mathbb{Z}$, together with natural isomorphisms $\varsigma_n : \widetilde{\mathrm{h}}^{n+1} \circ \Sigma \cong \widetilde{\mathrm{h}}^n$ of functors $\mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Ab}$, is called a *generalised reduced cohomology theory* if it satisfies the following properties:

(i) *(Homotopy invariance)* If $f, g : X \to Y$ are pointedly homotopic maps, then $\widetilde{\mathrm{h}}^n f = \widetilde{\mathrm{h}}^n g$ for all n.

(ii) *(Exact sequence)* Any pointed cellular map $f : X \to Y$ induces together with the inclusion $Y \hookrightarrow Cf$ an exact sequence

$$\widetilde{\mathrm{h}}^n(Cf) \to \widetilde{\mathrm{h}}^n(Y) \to \widetilde{\mathrm{h}}^n(X).$$

(iii) *(Wedges)* Let $\bigvee_{\alpha \in A} X_\alpha$ be a wedge sum of pointed spaces. Then the inclusions $X_\alpha \hookrightarrow \bigvee_{\alpha \in A} X_\alpha$ induce an isomorphism $\widetilde{\mathrm{h}}^n(\bigvee_{\alpha \in A} X_\alpha) \xrightarrow{\sim} \prod_{\alpha \in A} \widetilde{\mathrm{h}}^n(X_\alpha)$.

We often refer to a generalised reduced cohomology theory simply as a reduced cohomology theory. ◊

Remark 5.2. We require the map $f : X \to Y$ in part (ii) of the above definition to be cellular in order for Cf to be a CW-complex (see Lemma 3.50) and hence for $\widetilde{\mathrm{h}}^n(Cf)$ to be well-defined. For any pointed continuous map $g : X \to Y$, there is a pointed homotopy equivalence from g to a pointed cellular map $f : X \to Y$ by the Cellular Approximation Theorem, and Cg and Cf are homotopy equivalent relative to Y by Lemma 4.16, so by homotopy invariance of $\widetilde{\mathrm{h}}^*$, our restriction to cellular maps should not feel too restrictive. ▽

We will not give any examples of reduces cohomology theories on CW_* (except the trivial one sending all pointed CW-complexes to the trivial group) until Example 5.13, where we show that reduced singular cohomology of pointed CW-complexes is one such theory. In Section 6.5 we will give two more examples of reduced cohomology theories, however not always defined strictly on CW_*.

Proposition 5.3. [13] *Let $f : X \to Y$ be a pointed cellular map of CW-complexes. Then there is a long exact sequence*

$$\ldots \longrightarrow \widetilde{\mathrm{h}}^{n-1}(X) \longrightarrow \widetilde{\mathrm{h}}^{n}(Cf) \longrightarrow \widetilde{\mathrm{h}}^{n}(Y) \longrightarrow \widetilde{\mathrm{h}}^{n}(X) \longrightarrow \widetilde{\mathrm{h}}^{n+1}(Cf) \longrightarrow \widetilde{\mathrm{h}}^{n+1}(Y) \longrightarrow \ldots$$

induced by f, which is natural in the map $f : X \to Y$.

Proof. (†) We essentially saw in the proof of exactness and naturality of the Puppe sequence (Proposition 4.22) that there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[r, hook, "\iota"] \arrow[d, "h"] & Cf \arrow[r, hook] \arrow[d] & C\iota \arrow[r, "\simeq"] \arrow[d] & C\iota/CY \arrow[r, "\sim"] \arrow[d] & \Sigma X \arrow[d, "\Sigma g"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, hook, "\iota'"] & Cf' \arrow[r, hook] & C\iota' \arrow[r, "\simeq"] & C\iota'/CY' \arrow[r, "\sim"] & \Sigma X'
\end{tikzcd}$$

By naturality of the suspension isomorphism and the exact sequence in Definition 5.1(ii), this gives us for each $n \in \mathbb{Z}$ a sequence

$$\widetilde{\mathrm{h}}^{n-1}(X) \xrightarrow[\sim]{\varsigma_{n-1}^{-1}} \widetilde{\mathrm{h}}^{n}(\Sigma X) \xrightarrow{\sim} \widetilde{\mathrm{h}}^{n}(C\iota) \longrightarrow \widetilde{\mathrm{h}}^{n}(Cf) \xrightarrow{\widetilde{\mathrm{h}}^{n}\iota} \widetilde{\mathrm{h}}^{n}(Y),$$

which is natural in f and reduces to a short exact sequence

$$\widetilde{\mathrm{h}}^{n-1}(X) \longrightarrow \widetilde{\mathrm{h}}^{n}(Cf) \longrightarrow \widetilde{\mathrm{h}}^{n}(Y).$$

Similarly, we obtain a short exact sequence

$$\widetilde{\mathrm{h}}^{n}(Y) \longrightarrow \widetilde{\mathrm{h}}^{n}(X) \longrightarrow \widetilde{\mathrm{h}}^{n+1}(Cf),$$

which is also natural in f. Together with exactness of the sequence in Definition 5.1(ii), this yields exactness and naturality of the sequence in the statement. ☐

Let us return to generalised cohomology theories and take as ground ring $R = \mathbb{Z}$ (so such a theory maps pairs of spaces to abelian groups now). We wish to restrict such a generalised cohomology theory to CW-pairs to make a comparison with reduced cohomology theories possible. To make sure it is clear what we mean by this, and to be able to check what we mean by it is well-defined, we will restate the axioms of such a theory for CW-pairs.

Definition 5.4. (Eilenberg-Steenrod axioms for cohomology on CW-pairs) A sequence of functors $\mathrm{h}^n(\,\cdot\,,\,\cdot\,) : \mathsf{CW(2)}^{\mathrm{opp}} \to \mathsf{Ab}$ with $n \in \mathbb{Z}$, together with homomorphisms $\alpha_{n,(X,X')} : \mathrm{h}^n(X', \varnothing) \to \mathrm{h}^{n+1}(X, X')$ that are natural in the CW-pair (X, X'), is called a *generalised cohomology theory on* $\mathsf{CW(2)}$ if it satisfies the following four properties (where it is understood that $\mathrm{h}^n(X)$ is to be interpreted as $\mathrm{h}^n(X, \varnothing)$):

(i) *(Homotopy invariance)* Let (X, X') and (Y, Y') be two CW-pairs, and suppose $f, g : (X, X') \to (Y, Y')$ are homotopic maps. Then $\mathrm{h}^n f = \mathrm{h}^n g$ as maps $\mathrm{h}^n(Y, Y') \to \mathrm{h}^n(X, X')$ for all n.

(ii) *(Excision)* Let X be a CW-complex with subcomplexes A and B such that $X = A^\circ \cup B^\circ$. Then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $\mathrm{h}^n(X, B) \xrightarrow{\sim} \mathrm{h}^n(A, A \cap B)$ for each n.

(iii) *(Long exact sequence)* Let (X, X') be a CW-pair. Then the inclusions $X' \hookrightarrow X$ and $(X, \varnothing) \to (X, X')$ induce a long exact sequence

$$\begin{tikzcd}
 & \ldots \arrow[r] & \mathrm{h}^{n-1}(X) \arrow[r] & \mathrm{h}^{n-1}(X') \arrow[dll, "\alpha_{n-1,(X,X'))}"] \\
 & \mathrm{h}^{n}(X, X') \arrow[r] & \mathrm{h}^{n}(X) \arrow[r] & \mathrm{h}^{n}(X') \arrow[dll, "\alpha_{n,(X,X')}"] \\
 & \mathrm{h}^{n+1}(X, X') \arrow[r] & \mathrm{h}^{n+1}(X) \arrow[r] & \ldots
\end{tikzcd}$$

of cohomology, which is natural in the pair (X, X').

(iv) *(Products)* Let $X = \coprod_{\alpha \in A} X_\alpha$ be a disjoint union CW-complexes. Then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism $\mathrm{h}_n(X) \xrightarrow{\sim} \prod_{\alpha \in A} \mathrm{h}_n(X_\alpha)$ for all n.

The functors h^n are called an *ordinary cohomology theory on* $\mathsf{CW}(2)$ if moreover they satisfy

(v) *(Dimension)* The cohomology groups of the one-point space satisfy $\mathrm{h}^n(*) = 0$ for all $n \neq 0$. ◊

For all the axioms (and in particular the excision axiom) all the pairs of spaces that are plugged in the cohomology functors are CW-pairs, so the definition above makes sense.

Proposition 5.5. [11] *Let* h^* *be a generalised cohomology theory on* $\mathsf{CW}(2)$*. Then for each CW-pair* (X, X') *and each* $n \in \mathbb{Z}$*, the quotient map* $(X, X') \to (X/X', *)$ *induces an isomorphism* $\mathrm{h}^n(X/X', *) \xrightarrow{\sim} \mathrm{h}^n(X, X')$*, which is natural in* (X, X')*.*

Remark 5.6. Let Q be the quotient functor $\mathsf{Top}(2) \to \mathsf{Top}_*, (X, X') \mapsto (X/X', *)$ of Remark 1.26. Then the above proposition says that $\mathrm{h}^n \cong \mathrm{h}^n \circ Q$ as functors $\mathsf{CW}(2) \to \mathsf{Ab}$ for all n. ▽

Proof. (†) Let $\iota : X' \hookrightarrow X$ denote the inclusion, and consider the CW-complex $Y = M_\iota \cup_{X'} C_u X'$ (which is X, with a cylinder on X' glued to it, and on top of that cylinder an unreduced cone glued to it). Let $A = M_\iota$ and $B = X' \times I \cup_{X'} C_u X'$ be two subcomplexes of Y (consisting of X and the cylinder, and the cylinder and the cone, respectively). Note that $A \cap B \cong X' \times I$, and that there is a deformation retraction $(A, A \cap B) \to (X, X')$ (so in particular an isomorphism in the category $\mathsf{hTop}(2)$), a canonical homeomorphism $(Y, B) \to (C_u\iota, C_u X')$ of pairs obtained by seeing the cylinder in Y as the lower half of a cone on X', and a quotient map $(C_u\iota, C_u X') \to (C\iota/C_u X', *) \cong (X/X', *)$ which consists of a homotopy equivalences $C_u\iota \to X/X'$ and $C_u X' \to *$ by the fact that $C_u X'$ is contractible and Proposition 4.9.

The inclusion $(X, X') \hookrightarrow (A, A \cap B)$ is the homotopy inverse of the deformation retraction of pairs above, so it induces for each n an isomorphism $\mathrm{h}^n(A, A \cap B) \xrightarrow{\sim} \mathrm{h}^n(X, X')$. By the excision axiom, there is an isomorphism $\mathrm{h}^n(Y, B) \xrightarrow{\sim} \mathrm{h}^n(A, A \cap B)$ induced by the inclusion $(A, A \cap B) \hookrightarrow (Y, B)$. By the above homeomorphism of pairs, there is an isomorphism $\mathrm{h}^n(C_u\iota, C_u X') \xrightarrow{\sim} \mathrm{h}^n(Y, B)$. By the long exact sequence of the pairs $(C_u\iota, C_u X')$ and $(X/X', *)$ and the Five Lemma, the map $(C_u\iota, C_u X') \to (X/X', *)$ induces an isomorphism $\mathrm{h}^n(X/X', *) \xrightarrow{\sim} \mathrm{h}^n(C_u\iota, C_u X')$. Since the composition $(X, X') \hookrightarrow (A, A \cap B) \hookrightarrow (Y, B) \xrightarrow{\sim} (C_u\iota, C_u X') \to (X/X', *)$ equals the quotient map $(X, X') \to (X/X', *)$, this shows that the latter quotient map induces an isomorphism $\mathrm{h}^n(X/X', *) \xrightarrow{\sim} \mathrm{h}^n(X, X')$. Since quotienting is a functor, this isomorphism is also natural. □

We are now also in a position to show that the requirement in the excision axiom of a generalised cohomology theory can be weakened somewhat: it is not necessary that the *interiors* of the subcomplexes cover the CW-complex, only that the complexes themselves do.

Corollary 5.7. (Excision) [11] *Let* h^* *be a generalised cohomology theory on* $\mathsf{CW}(2)$*. Suppose* X *is a CW-complex with subcomplexes* A *and* B *such that* $X = A \cup B$*. Then the inclusion* $(A, A \cap B) \hookrightarrow (X, B)$ *induces an isomorphism* $\mathrm{h}^n(X, B) \xrightarrow{\sim} \mathrm{h}^n(A, A \cap B)$ *for each* n*.*

Proof. (†) There is a homeomorphism $(A/(A \cap B), *) \xrightarrow{\sim} (X/B, *)$ which fits in a commutative diagram

$$\begin{tikzcd}
(A, A \cap B) \arrow[r, hook] \arrow[d] & (X, B) \arrow[d] \\
(A/(A \cap B), *) \arrow[r, "\sim"] & (X/B, *)
\end{tikzcd}$$

where the vertical maps are the quotient maps, so Proposition 5.5 tells us that the induced diagram

$$\begin{tikzcd}
\mathrm{h}^n(A, A \cap B) & \mathrm{h}^n(X, B) \arrow[l] \\
\mathrm{h}^n(A/(A \cap B), *) \arrow[u, "\wr"] & \mathrm{h}^n(X/B, *) \arrow[l, "\sim"'] \arrow[u, "\wr"]
\end{tikzcd}$$

has isomorphisms as vertical maps (and as lower horizontal map, since it is induced by a homeomorphism). Therefore the top map is an isomorphism as well, which is what we wanted to show. ☐

Corollary 5.8. (Mayer-Vietoris sequence) *Let* h^* *be a generalised cohomology theory on* $\mathsf{CW}(2)$*. Suppose* X *is a CW-complex with subcomplexes* A *and* B *such that* $X = A \cup B$*, and write* $j_A : A \cap B \hookrightarrow A$*,* $j_B : A \cap B \hookrightarrow B$*,* $i_A : A \hookrightarrow X$ *and* $i_B : B \hookrightarrow X$ *for the respective inclusions. Then there is an exact sequence*

$$\ldots \longrightarrow \mathrm{h}^{n-1}(A \cap B) \longrightarrow \mathrm{h}^n(X) \xrightarrow{(\mathrm{h}^n i_A, \mathrm{h}^n i_B)} \mathrm{h}^n(A) \oplus \mathrm{h}^n(B) \xrightarrow{\mathrm{h}^n j_B - \mathrm{h}^n j_A} \mathrm{h}^n(A \cap B) \longrightarrow \ldots$$

Proof. The inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces a morphism between the long exact sequences of the pairs (X, B) and $(A, A \cap B)$, and induces isomorphisms $\mathrm{h}^n(X, B) \xrightarrow{\sim} \mathrm{h}^n(A, A \cap B)$ by the previous corollary. Therefore, Lemma B.14 yields the statement. ☐

5.2 The equivalence of categories

The collection of all generalised cohomology theories on $\mathsf{CW}(2)$ (mapping to abelian groups) forms a category $\mathsf{CohomTh}$: a morphism η between two generalised cohomology theories h^* and k^* consists of level-wise natural transformations $\eta_n : \mathrm{h}^n \to \mathrm{k}^n$ that respect the snake map of the long exact sequence: if we write for a CW-pair (X, X') the snake maps that h^* and k^* come equipped with as $\alpha_{n,(X,X')} : \mathrm{h}^n(X') \to \mathrm{h}^{n+1}(X, X')$ and $\alpha'_{n,(X,X')} : \mathrm{k}^n(X') \to \mathrm{k}^{n+1}(X, X')$, respectively, then the diagram

$$\begin{tikzcd}
\mathrm{h}^n(X') \arrow[r, "\alpha_{n,(X,X')}"] \arrow[d, "\eta_{X'}"] & \mathrm{h}^{n+1}(X, X') \arrow[d, "\eta_{n+1,(X,X')}"] \\
\mathrm{k}^n(X') \arrow[r, "\alpha'_{n,(X,X')}"] & \mathrm{k}^{n+1}(X, X')
\end{tikzcd}$$

must be commutative for all CW-pairs (X, X'). In particular, η induces for each CW-pair (X, X') a morphism between the long exact sequences of (X, X') with respect to h^* and k^*.

Likewise, the collection of all reduced cohomology theories on CW_* (which are by our definition required to map to abelian groups) forms a category $\mathsf{rCohomTh}$, this time with a morphism η between two reduced cohomology theories $\widetilde{\mathrm{h}}^*$ and $\widetilde{\mathrm{k}}^*$ consisting of level-wise natural transformations $\eta_n : \widetilde{\mathrm{h}}^n \to \widetilde{\mathrm{k}}^n$ that respect the natural suspension isomorphism: if we write $\eta_{n+1}\Sigma$ for the natural transformation $\widetilde{\mathrm{h}}^{n+1} \circ \Sigma \to \widetilde{\mathrm{k}}^{n+1} \circ \Sigma$ determined by $(\eta_{n+1}\Sigma)_X = \eta_{n+1,\Sigma X}$ for all $X \in \mathsf{CW}_*$, and if we write ς_n and ς'_n for the suspension isomorphisms that $\widetilde{\mathrm{h}}^*$ resp. $\widetilde{\mathrm{k}}^*$ come equipped with, then the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{h}}^{n+1} \circ \Sigma \arrow[r, "\varsigma_n", "\sim"'] \arrow[d, "\eta_{n+1}\Sigma"] & \widetilde{\mathrm{h}}^n \arrow[d, "\eta_n"] \\
\widetilde{\mathrm{k}}^{n+1} \circ \Sigma \arrow[r, "\varsigma'_n", "\sim"'] & \widetilde{\mathrm{k}}^n
\end{tikzcd}$$

of functors and natural transformations must be commutative. We will spend this section on showing that these two categories of cohomology theories are equivalent.

Proposition 5.9. *There is a functor* $F : \mathsf{CohomTh} \to \mathsf{rCohomTh}$ *which sends a generalised cohomology theory* h^* *to the reduced cohomology theory* $F\mathrm{h}^*$ *determined by* $F\mathrm{h}^n(X) = \mathrm{h}^n(X, *)$ *for any pointed CW-complex* X *and* $n \in \mathbb{Z}$*.*

Proof. We roughly follow the proof in [13], with drs. J. Becerra providing a general strategy of the proof, with the details provided by us. For a generalised cohomology theory h^* and $n \in \mathbb{Z}$, the object $F\mathrm{h}^n : \mathsf{CW}_* \to \mathsf{Ab}, X \mapsto \mathrm{h}^n(X, *)$ is a functor since h^n is and a pointed continuous map is also a morphism of pairs of spaces.

We will now construct the suspension isomorphism. By Lemma 3.47(ii), there is for any $X \in \mathsf{CW}_*$ a natural pointed homeomorphism $r_1 : Ci \xrightarrow{\sim} \Sigma X$, where $i : X \hookrightarrow CX$ is the inclusion. This gives us an isomorphism $p_1 \coloneqq \mathrm{h}^{n+1} r_1 : \mathrm{h}^{n+1}(\Sigma X, *) \cong \mathrm{h}^{n+1}(Ci, *)$ for all n, which is natural in X since r_1 is.

The long exact sequence of the triplet $(Ci, CX, *)$ of Theorem 2.35 and the fact that CX is contractible on its base point (so that the long exact sequence of the pair $(CX, *)$ implies that $\mathrm{h}^n(CX, *) = 0$ for all n) give us an isomorphism $p_2 : \mathrm{h}^{n+1}(Ci, *) \xrightarrow{\sim} \mathrm{h}^{n+1}(Ci, CX)$, induced by the inclusion $(Ci, *) \hookrightarrow (Ci, CX)$. This isomorphism is natural in X since any pointed map $f : X \to Y$ between two pointed CW-complexes induces a map $(C(X \hookrightarrow CX), CX, *) \to (C(Y \hookrightarrow CY), CY, Y, *)$ which therefore commutes with the inclusion which induces the isomorphism. The maps on cohomology induced by f will therefore commute with these isomorphisms.

If we let $A = CX$ be the "original" reduced cone in Ci, and $B = CX$ the one added in the construction of Ci, then both are subcomplexes of Ci that satisfy $Ci = A \cup B$ and $A \cap B = X$. By the excision property (Corollary 5.7), the inclusion $(A, X) \hookrightarrow (Ci, B)$ induces an isomorphism $p_3 : \mathrm{h}^{n+1}(Ci, CX) \xrightarrow{\sim} \mathrm{h}^{n+1}(CX, X)$, which is also natural in X: any pointed map $f : X \to Y$ between two pointed CW-complexes also induces like above a map $(C(X \hookrightarrow CX), CX, X, *) \to (C(Y \hookrightarrow CY), CY, Y, *)$, which consequently commutes with the inclusions considered in the excision argument. Since these inclusions induce the isomorphism, the maps induced by f commute with the respective isomorphisms.

Lastly, there is an isomorphism $p_4 : \mathrm{h}^n(X, *) \xrightarrow{\sim} \mathrm{h}^{n+1}(CX, X)$, which follows from the long exact sequence of the triplet $(CX, X, *)$ and the fact that CX is contractible on its base point. Explicitly, if we write $\iota : (X, \varnothing) \hookrightarrow (X, *)$ for the inclusion and $\alpha_{n,(CX,X)}$ for the snake map $\mathrm{h}^n(X) \to \mathrm{h}^{n+1}(CX, X)$ in the long exact sequence of the pair (CX, X), then $p_4 = \alpha_{n,(CX,X)} \circ \mathrm{h}^n \iota$. This isomorphism is also natural in X, since the assignment $X \mapsto (CX, X, *)$ is functorial and the long exact sequence of the triplet is natural. Putting everything together, we get a natural isomorphism

$$\mathrm{h}^{n+1}(\Sigma X, *) \xrightarrow[\sim]{p_1} \mathrm{h}^{n+1}(Ci, *) \xrightarrow[\sim]{p_2} \mathrm{h}^{n+1}(Ci, CX) \xrightarrow[\sim]{p_3} \mathrm{h}^{n+1}(CX, X) \xrightarrow[\sim]{p_4^{-1}} \mathrm{h}^n(X, *),$$

and we define $\varsigma_{n,X} \coloneqq p_4^{-1} \circ p_3 \circ p_2 \circ p_1$. This defines the suspension isomorphism ς_n for all n.

Now we verify $F\mathrm{h}^*$ satisfies the axioms of a reduced cohomology theory. Firstly, homotopy invariance is trivial. Secondly, for any pointed cellular map $f : X \to Y$ between CW-complexes there is a pointed homeomorphism $Cf \cong m_f/X$, and by Proposition 5.5 this gives us an isomorphism $\mathrm{h}^n(Cf, *) \cong \mathrm{h}^n(m_f, X)$ induced by the quotient map $q : (m_f, X) \to (Cf, *)$. If we denote by $p : m_f \to Y$ the standard pointed deformation retraction, then there is a diagram

$$\begin{tikzcd}
(X,*) \arrow[r, hook] \arrow[d, "\mathrm{id}_X"] & (m_f,*) \arrow[r, hook] \arrow[d, "p"] & (m_f,X) \arrow[d, "q"] \\
(X,*) \arrow[r, "f"] & (Y,*) \arrow[r, hook] & (Cf,*)
\end{tikzcd}$$

which commutes up to homotopy. By the long exact sequence of the triplet $(m_f, X, *)$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{h}^n(m_f,X) \arrow[r] & \mathrm{h}^n(m_f,*) \arrow[r] & \mathrm{h}^n(X,*) \\
\mathrm{h}^n(Cf,*) \arrow[r] \arrow[u, "\mathrm{h}^n q"] & \mathrm{h}^n(Y,*) \arrow[r, "f"] \arrow[u, "\mathrm{h}^n p"] & \mathrm{h}^n(X,*) \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

in which the vertical maps are all isomorphism and the upper row is exact. This is the exact sequence $\mathrm{h}^n(Cf, *) \to \mathrm{h}^n(Y, *) \to \mathrm{h}^n(X, *)$ induced by f which we were after.

Now, thirdly, if $\bigvee_{\alpha \in A} X_\alpha$ is a wedge sum of pointed CW-complexes $(X_\alpha, *_\alpha)$, there is a homeomorphism $\bigvee_{\alpha \in A} X_\alpha \cong \bigsqcup_{\alpha \in A} X_\alpha / \bigsqcup_{\alpha \in A} \{*_\alpha\}$ (this is also a consequence of the fact that taking the quotient is a left adjoint, whereby it commutes with colimits). The inclusions $\iota_\alpha : (X_\alpha, *_\alpha) \hookrightarrow (X, *)$ give us inclusions $\overline{\iota_\alpha}(X_\alpha, *_\alpha) \to$

$(\bigsqcup_{\alpha\in A} X_\alpha, \bigsqcup_{\alpha\in A}\{*_\alpha\})$ between pairs of CW-complexes, and the diagram

$$\begin{tikzcd}
(X_\alpha, *_\alpha) \arrow[r, "\overline{\iota_\alpha}"] \arrow[dr, "\iota_\alpha"] & (\bigsqcup_{\alpha\in A} X_\alpha, \bigsqcup_{\alpha\in A}\{*_\alpha\}) \arrow[d] \\
& \bigvee_{\alpha\in A} X_\alpha \cong \bigsqcup_{\alpha\in A} X_\alpha / \bigsqcup_{\alpha\in A}\{*_\alpha\}
\end{tikzcd}$$

commutes. Taking cohomology, the inclusions ι_α induce by Proposition 5.5 an isomorphism $F\mathrm{h}^n(\bigvee_{\alpha\in A} X_\alpha) = \mathrm{h}^n(\bigvee_{\alpha\in A} X_\alpha, *) \cong \mathrm{h}^n(\bigsqcup_{\alpha\in A} X_\alpha, \bigsqcup_{\alpha\in A}\{*_\alpha\})$. The long exact sequence of the pair $(\bigsqcup_{\alpha\in A} X_\alpha, \bigsqcup_{\alpha\in A}\{*_\alpha\})$ and the product axiom of cohomology gives a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{h}^n(\bigsqcup_{\alpha\in A} X_\alpha, \bigsqcup_{\alpha\in A}\{*_\alpha\}) \arrow[r] \arrow[d] & \mathrm{h}^n(\bigsqcup_{\alpha\in A} X_\alpha) \arrow[r] \arrow[d, "\wr"] & \mathrm{h}^n(\bigsqcup_{\alpha\in A}\{*_\alpha\}) \arrow[r] \arrow[d, "\wr"] & \dots \\
\dots \arrow[r] & \prod_{\alpha\in A} \mathrm{h}^n(X_\alpha, *_\alpha) \arrow[r] & \prod_{\alpha\in A} \mathrm{h}^n(X_\alpha) \arrow[r] & \prod_{\alpha\in A} \mathrm{h}^n(*_\alpha) \arrow[r] & \dots
\end{tikzcd}$$

where the first vertical map is given by the product of the maps $\mathrm{h}^n(\bigsqcup_{\alpha\in A} X_\alpha, \bigsqcup_{\alpha\in A}\{*_\alpha\}) \to \mathrm{h}^n(X_\alpha, *_\alpha)$ induced by the inclusions $\overline{\iota_\alpha}$. By the Five Lemma, there is thus an isomorphism $\mathrm{h}^n(\bigsqcup_{\alpha\in A} X_\alpha, \bigsqcup_{\alpha\in A}\{*_\alpha\}) \cong \prod_{\alpha\in A} \mathrm{h}^n(X_\alpha, *_\alpha)$, which implies $F\mathrm{h}^n(\bigvee_{\alpha\in A} X_\alpha) \cong \prod_{\alpha\in A} F\mathrm{h}^n(X_\alpha)$, induced by the ι_α. Therefore $F\mathrm{h}^*$ is a reduced cohomology theory.

Let $\eta : \mathrm{h}^* \to \mathrm{k}^*$ be a morphism of generalised cohomology theories in $\mathsf{CohomTh}$. Then for $n \in \mathbb{Z}$ and each pointed CW-complex X, $\eta_{n,(X,*)}$ is a map $F\mathrm{h}^n(X) = \mathrm{h}^n(X,*) \to \mathrm{k}^n(X,*) = F\mathrm{k}^n(X)$, which obviously constitutes to a natural transformation $F\mathrm{h}^n \to F\mathrm{k}^n$. We have to show that these maps commute with the suspension isomorphism. To see this, note that since η_n and η_{n+1} are natural transformations between cohomology functors, they are assumed to commute with maps induced by morphisms on CW-pairs, and are assumed to commute with the snake maps of the long exact sequences of pairs. By the above definition of the suspension isomorphism, this implies that the diagram

$$\begin{tikzcd}
\mathrm{h}^{n+1}(\Sigma X, *) \arrow[r, "\varsigma_n", "\sim"'] \arrow[d, "\eta_{n+1,(\Sigma X,*)}"] & \mathrm{h}^n(X,*) \arrow[d, "\eta_{n,(X,*)}"] \\
\mathrm{k}^{n+1}(\Sigma X, *) \arrow[r, "\varsigma_n'", "\sim"'] & \mathrm{k}^n(X,*)
\end{tikzcd}$$

must commute, where ς_n and ς_n' are the suspension isomorphisms of h^* and k^* in degree n, respectively. Therefore, the maps $\eta_{n,(X,*)}$ constitute for all $X \in \mathsf{CW}_*$ and $n \in \mathbb{Z}$ to a morphism between $F\mathrm{h}^*$ and $F\mathrm{k}^*$. We let $F\eta$ be this morphism. It is clear that this assignment respects composition and the identity. This shows F is a well-defined functor. ☐

Proposition 5.10. [11] *There is a functor* $G : \mathsf{rCohomTh} \to \mathsf{CohomTh}$ *which sends a reduced cohomology theory* $\widetilde{\mathrm{h}}^*$ *to the generalised cohomology theory* $G\widetilde{\mathrm{h}}^*$ *determined by* $G\widetilde{\mathrm{h}}^n(X,X') = \widetilde{\mathrm{h}}^n(X/X')$ *for any CW-pair* (X,X') *and* $n \in \mathbb{Z}$.

Proof. (†) Let a reduced cohomology theory $\widetilde{\mathrm{h}}^*$ be given. $G\widetilde{\mathrm{h}}^n$ is for each $n \in \mathbb{Z}$ a functor, since $\widetilde{\mathrm{h}}^n$ and taking the quotient is.

For any two homotopic maps $f, g : (X,X') \to (Y,Y')$ of pairs, there is an induced pointed homotopy between the induced maps $\bar{f}, \bar{g} : X/X' \to Y/Y'$, so homotopy invariance of $\widetilde{\mathrm{h}}^n$ implies homotopy invariance of $G\widetilde{\mathrm{h}}^n$.

Now let X be a CW-complex with two subcomplexes A and B that satisfy $X = A^\circ \cup B^\circ$. We proceed just like in the proof of Corollary 5.7: there is a homeomorphism $(A/(A\cap B), *) \xrightarrow{\sim} (X/B, *)$ which fits in a commutative diagram

$$\begin{tikzcd}
(A, A\cap B) \arrow[r, hook] \arrow[d] & (X,B) \arrow[d] \\
(A/(A\cap B), *) \arrow[r, "\sim"] & (X/B, *)
\end{tikzcd}$$

where the vertical maps are the quotient maps, so the definition of $G\widetilde{\mathrm{h}}^n$ immediately tells us that the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $G\widetilde{\mathrm{h}}^n(X, B) \xrightarrow{\sim} G\widetilde{\mathrm{h}}^n(A, A \cap B)$.

Next, for a pair (X, X') of non-pointed CW-complexes the inclusion $\iota : X' \hookrightarrow X$ induces a pointed inclusion $\iota_+ : X'_+ = X'/\varnothing \to X/\varnothing = X_+$, and the quotient map $q' : C\iota_+ \cong C_u\iota \to X/X'$ is a pointed homotopy equivalence, as we by now know all too well. If we let $q : (X_+, X'_+) \mapsto (X_+/X'_+, *) \cong (X/X', *)$ be the other quotient map, then there is a commutative diagram

$$\begin{tikzcd}
(X'_+, *) \arrow[r, hook, "\iota_+"] & (X_+, *) \arrow[r, hook] \arrow[dr, "q"] & (C\iota_+, *) \arrow[d, "q'"] \\
& & (X/X', *)
\end{tikzcd}$$

and therefore by Proposition 5.3 also a commutative diagram

$$\begin{tikzcd}
\dots & \widetilde{\mathrm{h}}^{n+1}(X_+) \arrow[l] & \widetilde{\mathrm{h}}^{n+1}(C\iota_+)) \arrow[l] & \widetilde{\mathrm{h}}^n(X'_+) \arrow[l] & \widetilde{\mathrm{h}}^n(X_+) \arrow[l] & \widetilde{\mathrm{h}}^n(C\iota_+) \arrow[l] & \widetilde{\mathrm{h}}^{n-1}(X'_+) \arrow[l] & \dots \arrow[l] \\
& & \widetilde{\mathrm{h}}^{n+1}(X/X') \arrow[ul, "\widetilde{\mathrm{h}}^n q"] \arrow[u, "\wr"] & & & \widetilde{\mathrm{h}}^n(X/X') \arrow[ul, "\widetilde{\mathrm{h}}^n q"] \arrow[u, "\wr"] & &
\end{tikzcd}$$

where the top row is exact and induced by the inclusions $X'_+ \hookrightarrow X_+ \hookrightarrow C\iota_+$. The diagram

$$\begin{tikzcd}
(X', \varnothing) \arrow[r, hook] \arrow[d] & (X, \varnothing) \arrow[r, hook] \arrow[d] & (X, X') \arrow[d] \\
(X'_+, *) \arrow[r, hook] & (X_+, *) \arrow[r, "q"] & (X/X', *)
\end{tikzcd}$$

where the vertical maps are quotient maps, also commutes, and $G\widetilde{\mathrm{h}}^n(X)$ is short for $G\widetilde{\mathrm{h}}^n(X, \varnothing) = \widetilde{\mathrm{h}}^n(X/\varnothing) = \widetilde{\mathrm{h}}^n(X_+)$. This implies that the inclusions $(X', \varnothing) \hookrightarrow (X, \varnothing) \hookrightarrow (X, X')$ induce a commutative diagram

$$\begin{tikzcd}
\dots & G\widetilde{\mathrm{h}}^{n+1}(X) \arrow[l] & G\widetilde{\mathrm{h}}^{n+1}(C\iota) \arrow[l] & G\widetilde{\mathrm{h}}^n(X') \arrow[l] & G\widetilde{\mathrm{h}}^n(X) \arrow[l] & G\widetilde{\mathrm{h}}^n(C\iota) \arrow[l] & G\widetilde{\mathrm{h}}^{n-1}(X') \arrow[l] & \dots \arrow[l] \\
& & G\widetilde{\mathrm{h}}^{n+1}(X, X') \arrow[ul] \arrow[u, "\wr"] & & & G\widetilde{\mathrm{h}}^n(X, X') \arrow[ul] \arrow[u, "\wr"] & &
\end{tikzcd}$$

where the top row is exact. If we write p_{n+1} for the isomorphism $G\widetilde{\mathrm{h}}^{n+1}(X, X') \xrightarrow{\sim} G\widetilde{\mathrm{h}}^{n+1}(C\iota)$ in the diagram above and $r_n : G\widetilde{\mathrm{h}}^n(X') \to G\widetilde{\mathrm{h}}^{n+1}(C\iota)$ for the map appearing in the diagram above, then we define the snake maps $\alpha_{n,(X,X')} \coloneqq p'_{n+1} \circ r_n$. This gives us the long exact sequence of the pair (X, X'). Naturality of this sequence (and hence of the snake maps in particular) follows from a map $(X, X') \to (Y, Y')$ of CW-pairs inducing a map $(C\iota_{X,+}, X_+, X'_+) \to (X\iota_{Y,+}, Y_+, Y'_+)$ and naturality of the long exact sequence in Proposition 5.3.

Finally, for a disjoint union $\bigsqcup_{\alpha \in A} X_\alpha$ of non-pointed CW-complexes, there is a pointed homeomorphism $\left(\bigsqcup_{\alpha \in A} X_\alpha\right)_+ \cong \bigvee_{\alpha \in A} (X_\alpha)_+$ (this is also a consequence of the functor $X \to X_+$ being a left adjoint, as remarked in Remark 1.26), which associates the inclusions $(\iota_\alpha)_+ : (X_\alpha)_+ \hookrightarrow (\bigsqcup_{\alpha \in A} X_\alpha)_+$ with $i_\alpha : (X_\alpha)_+ \hookrightarrow \bigvee_{\alpha \in A} (X_\alpha)_+$. Therefore the inclusions induce an isomorphism $G\widetilde{\mathrm{h}}^n(\bigsqcup_{\alpha \in A} X_\alpha) \cong \widetilde{\mathrm{h}}^n(\bigvee_{\alpha \in A} (X_\alpha)_+) \cong \prod_{\alpha \in A} \widetilde{\mathrm{h}}^n((X_\alpha)_+) = \prod_{\alpha \in A} G\widetilde{\mathrm{h}}^n(X_\alpha)$. Therefore $G\widetilde{\mathrm{h}}^*$ is a generalised cohomology theory.

For a morphism $\eta : \widetilde{\mathrm{h}}^* \to \widetilde{\mathrm{k}}^*$ and a CW-pair (X, X'), for each n the map $\eta_{n,X/X'} : G\widetilde{\mathrm{h}}^n(X, X') \to G\widetilde{\mathrm{k}}^n(X, X')$ is a homomorphism, and since η_n is a natural transformation, these maps define a natural transformation $G\widetilde{\mathrm{h}}^n \to G\widetilde{\mathrm{k}}^n$. We need to show that these transformations commute with the snake maps $G\widetilde{\mathrm{h}}^n(X') \to G\widetilde{\mathrm{h}}^{n+1}(X, X')$ of the long exact sequence of a pair (X, X'). We saw above that these snake maps are induced by the pointed homotopy equivalence $C\iota_+ \simeq X/X'$, with $\iota_+ : X'_+ \hookrightarrow X_+$ the inclusion, by the homotopy equivalence $Ci \simeq \Sigma X'$, with $i : X_+ \hookrightarrow C\iota_+$ the inclusion, and by the suspension isomorphism of $\widetilde{\mathrm{h}}^*$. Since the η_n are assumed to commute with homomorphisms induced by maps of pointed CW-complexes and with the suspension isomorphism, they will therefore also commute with the snake maps, and therefore they induce a morphism $G\widetilde{\mathrm{h}}^* \to G\widetilde{\mathrm{k}}^*$. We let $G\eta$ be this morphism, and it is not difficult to see that this assignment respects composition and the identity. Therefore, G is a well-defined functor as well. ☐

Theorem 5.11. *The functors F and G as in the previous two propositions give an equivalence of categories (see Definition A.26) between* $\mathsf{CohomTh}$ *and* $\mathsf{rCohomTh}$*.*

Proof. (†) On the one hand, the functor GF sends a generalised cohomology theory h^* to the generalised homology theory $GF\mathrm{h}^*$ determined by $GF\mathrm{h}^n(X, X') = \mathrm{h}^n(X/X', *)$ for all CW-pairs (X, X') and $n \in \mathbb{Z}$. Proposition 5.5 then gives us a natural isomorphism $GF \cong \mathrm{id}_{\mathsf{CohomTh}}$.

On the other hand, for a reduced cohomology theory $\widetilde{\mathrm{h}}^*$, the reduced cohomology theory $FG\widetilde{\mathrm{h}}^*$ is determined by $FG\widetilde{\mathrm{h}}^n(X) = \widetilde{\mathrm{h}}^n(X/*) \cong \widetilde{\mathrm{h}}^n(X)$ for all pointed CW-complexes X and $n \in \mathbb{Z}$. Since the homeomorphism $X/* \cong X$ is natural in the pointed CW-complex X, we find $FG \cong \mathrm{id}_{\mathsf{rCohomTh}}$ as well. This shows that F and G are equivalences of categories. ☐

Corollary 5.12. [11] *Let h^* be a generalised cohomology theory on* $\mathsf{CW}(2)$*. Then there exists a reduced cohomology theory $\widetilde{\mathrm{h}}^*$ on* CW_* *such that $\mathrm{h}^n(X, X') \cong \widetilde{\mathrm{h}}^n(X/X')$ naturally for all CW-pairs (X, X') and $n \in \mathbb{Z}$. Conversely, given a reduced cohomology theory $\widetilde{\mathrm{h}}^*$ on* CW_**, there exists a generalised cohomology theory h^* on* $\mathsf{CW}(2)$ *such that $\widetilde{\mathrm{h}}^n(X) \cong \mathrm{h}^n(X, *)$ naturally for all pointed CW-complexes X and $n \in \mathbb{Z}$.* ☐

Example 5.13. In Remark 2.32, we deduced that there is a natural isomorphism $\mathrm{H}^n(\,\cdot\,, \cdot\,; A) \cong \widetilde{\mathrm{H}}^n(\,\cdot\,; A) \circ U$ of functors $\mathsf{Top}_*^{\mathrm{opp}} \to \mathsf{Ab}$ for any abelian group A and any integer n, where $U : \mathsf{Top}_*^{\mathrm{opp}} \to \mathsf{Top}^{\mathrm{opp}}$ is the forgetful functor. Restricting to CW_*, we see that singular cohomology and reduced singular cohomology are related to each other just as in the above corollary. In particular, this shows formally that reduced singular cohomology is a reduced cohomology theory. △

Corollary 5.14. (Mayer-Vietoris sequence for reduced cohomology) *Let $\widetilde{\mathrm{h}}^*$ be a reduced cohomology theory on* CW_**. Suppose X is a pointed CW-complex with pointed subcomplexes A and B such that $X = A \cup B$, and write $j_A : A \cap B \hookrightarrow A$, $j_B : A \cap B \hookrightarrow B$, $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ for the respective inclusions. Then there is an exact sequence*

$$\dots \longrightarrow \widetilde{\mathrm{h}}^{n+1}(A \cap B) \longrightarrow \widetilde{\mathrm{h}}^n(X) \xrightarrow{(\widetilde{\mathrm{h}}^n i_A, \widetilde{\mathrm{h}}^n i_B)} \widetilde{\mathrm{h}}^n(A) \oplus \widetilde{\mathrm{h}}^n(B) \xrightarrow{\widetilde{\mathrm{h}}^n j_B - \widetilde{\mathrm{h}}^n j_A} \widetilde{\mathrm{h}}^n(A \cap B) \longrightarrow \dots$$

Proof. Use the previous corollary to find a generalised cohomology theory h^* on $\mathsf{CW}(2)$ that satisfies $\widetilde{\mathrm{h}}^n(X) = \mathrm{h}^n(X, *)$ naturally for all pointed CW-complexes X and $n \in \mathbb{Z}$. The inclusion $(A, A \cap B, *) \hookrightarrow (X, B, *)$ induces a morphism between the long exact sequences of the triplets $(X, B, *)$ and $(A, A \cap B, *)$ (in h^*), and induces isomorphisms $\mathrm{h}^n(X, B) \xrightarrow{\sim} \mathrm{h}^n(A, A \cap B)$ by Corollary 5.7. Therefore, Lemma B.14 yields the statement. ☐

Corollary 5.15. *Let $\widetilde{\mathrm{h}}^*$ be a reduced cohomology theory on* CW_**, and write h^* for its associated generalised cohomology theory obtained from Corollary 5.12. Then for each $m \geq 0$ and $n \in \mathbb{Z}$, we have $\widetilde{\mathrm{h}}^n(S^m) = \mathrm{h}^{n-m}(*)$.*

Proof. For $m = 0$, this follows from $\widetilde{\mathrm{h}}^n(S^0) = \widetilde{\mathrm{h}}^n(*/\varnothing) \cong \mathrm{h}^n(*, \varnothing) = \mathrm{h}^n(*)$. To show the claim for all m, we use the suspension isomorphism and the pointed homotopy equivalence $S^{m+1} \cong SS^m \simeq \Sigma S^m$ to obtain $\widetilde{\mathrm{h}}^{n+1}(S^{m+1}) \cong \widetilde{\mathrm{h}}^n(S^m)$, which inductively shows the claim for all n and m. ☐

5.3 Interlude on semi-Brown functors

Two of the axioms of a reduced cohomology theory on CW_*, the homotopy invariance axiom and the wedge axiom, in themselves already allow us to deduce a few interesting results about functors satisfying them. Since we will use these in the proof of the Brown Representability Theorem 6.31, it is worth to take the time to deduce them formally in this section.

Notation 5.16. Let F be a contravariant functor from a subcategory of topological spaces to the category of sets, pointed sets or abelian groups, respectively (it will be clear from context). If $X' \subseteq X$ is a subspace of a topological space X and $\iota : X' \hookrightarrow X$ is the inclusion, such that all three are contained in the subcategory, then for any $a \in FX$ we denote by $a|_{X'}$ the element $F\iota(a) \in FX'$, and we call $a|_{X'}$ the *restriction of a to X'*. ○

Definition 5.17. A functor $F : \mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$ is called a *semi-Brown functor* if it satisfies the following two properties:

- (i) *(Homotopy invariance)* If $f, g : X \to Y$ are two pointedly homotopic maps, then $Ff = Fg$.
- (ii) *(Wedges)* If $(X_\alpha)_{\alpha \in A}$ is a family of spaces in CW_*, then the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism $F(\bigvee_{\alpha \in A} X_\alpha) \xrightarrow{\sim} \prod_{\alpha \in A} F(X_\alpha)$. ◊

Note that we only require F to map to pointed sets, and not to abelian groups. The name is chosen because, by lack of a better name, we hint at Definition 6.29 of a Brown functor, which is required to satisfy one further axiom and be defined only on the path-connected CW-complexes.

Lemma 5.18. *Let Y be an arbitrary pointed topological space. Then $[\,\cdot\,, Y]^\bullet$ is a semi-Brown functor.*

Proof. Homotopy invariance is obvious, and the wedge axiom is a restatement of the universal property of the wedge sum (cf. Proposition A.50). □

Definition 5.19. Let F be a contravariant functor from a subcategory of topological spaces to the category of sets, pointed sets or abelian groups, respectively (again it will be clear from context). Let (X, x) and (Y, y) be two pairs of spaces $X, Y \in \mathsf{C}$ and $x \in FX$, $y \in FY$. A *morphism* $f : (X, x) \to (Y, y)$ is a morphism $f : X \to Y$ in D such that $Ff(y) = x$. By functoriality of F, we obtain a category $\int_{\mathsf{C}} F$ of such pairs and morphisms between them, called the *category of elements of F* [22]. ◊

Convention 5.20. In all that follows here, we assume that $F : \mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$ is a semi-Brown functor. ⊙

Lemma 5.21. (†)

(i) Suppose $X, Y, Z \in \mathsf{CW}_*$, and let $f : X \to Z$ and $g : Y \to Z$ be two pointed maps. Then there is a commutative diagram

$$\begin{tikzcd}
FZ \arrow[r, "F(f\vee g)"] \arrow[d, equal] & F(X \vee Y) \arrow[d, "\wr"] \\
FZ \arrow[r, "(Ff, Fg)"] & FX \times FY
\end{tikzcd}$$

where the inclusions $X, Y \hookrightarrow X \vee Y$ induce the vertical pointed bijection on the right.

(ii) Suppose $X, Y, Z, W \in \mathsf{CW}_*$, and let $f : X \to W$ and $g : Y \to Z$ be two pointed maps. If we denote by h the induced map $X \vee Y \to W \vee Z$, then there is a commutative diagram

$$\begin{tikzcd}
F(W \vee Z) \arrow[r, "Fh"] \arrow[d, "\wr"] & F(X \vee Y) \arrow[d, "\wr"] \\
FW \times FZ \arrow[r, "(Ff, Fg)"] & FX \times FY
\end{tikzcd}$$

where the vertical isomorphisms are induced by the inclusions.

Proof. Let $\iota_X : X \hookrightarrow X \vee Y$ and $\iota_Y : Y \hookrightarrow X \vee Y$ be the inclusions, and note that the wedge axiom of F gives us a commutative diagram

$$\begin{tikzcd}
FX & & & \\
& FX \times FY \arrow[ul, "\mathrm{pr}_{FX}"'] \arrow[dl, "\mathrm{pr}_{FY}"] & F(X \vee Y) \arrow[l, "\sim"'] \arrow[ull, bend right, "F\iota_X"'] \arrow[dll, bend left, "F\iota_Y"] & FZ \arrow[l, "F(f\vee g)"'] \\
FY & & &
\end{tikzcd}$$

Since $F\iota_X \circ F(f \vee g) = F((f \vee g) \circ \iota_X) = Ff$ and similarly $F\iota_Y \circ F(f \vee g) = Fg$, it follows that the map $FZ \to FX \times FY$ associated with $f \vee g$ is (Ff, Fg).

For the second statement, let $\iota_W : W \hookrightarrow W \vee Z$ and $\iota_Z : Z \hookrightarrow W \vee Z$ be the inclusions, and note that the induced map h equals $(\iota_W \circ f) \vee (\iota_Z \circ g)$, and hence the bijection $F(X \vee Y) \cong FX \times FY$ given by the wedge axiom associates Fh with the map $F(W \vee Z) \to FX \times FY, u \mapsto (Ff \circ F\iota_W(u), Fg \circ F\iota_Z(u)) = (Ff(u|_W), Fg(u|_Z))$ by the first part. Now, under the bijection $F(W \vee Z) \cong FW \times FZ$ also given by the wedge axiom, this is associated with the map $FW \times FZ \to FX \times FY, (w, z) \mapsto (Ff(w), Fg(z))$. ☐

Corollary 5.22. *$F(*)$ equals the one-point set $\{*\}$.*

Proof. [11] For any $X \in \mathsf{CW}_*$, let $\iota : * \hookrightarrow X$ be the inclusion. Then the wedge $\mathrm{id}_X \vee \iota : X \vee * \to X$ is a homeomorphism, so the previous lemma implies that the map $(\mathrm{id}_{FX}, F\iota) : FX \to FX \times F(*)$ is a bijection. This is only possible if $F(*)$ equals the one-point set. ☐

Notation 5.23. In what follows, we will often by slight abuse of notation write both the map $X \vee Y \to W \vee Z$ induced by two maps $f : X \to W$ and $g : Y \to Z$, and also the map $X \vee Y \to Z$ induced by two maps $f : X \to Z$ and $g : Y \to Z$ as $f \vee g$. ○

Lemma 5.24. (†) *Let $X, Y \in \mathsf{CW}_*$, and let $\iota_X : X \hookrightarrow X \vee Y$ and $\iota_Y : Y \hookrightarrow X \vee Y$ be the inclusions. Let $p_Y : Y \to *$ be the unique map of this kind, and write e_Y for the base point of FY. If $f : X \vee * \xrightarrow{\sim} X$ is the canonical homeomorphism, then the composition*

$$FX \xrightarrow[\sim]{Ff} F(X \vee *) \xrightarrow{F(\mathrm{id} \vee p_Y)} F(X \vee Y) \xrightarrow[\sim]{(F\iota_X, F\iota_Y)} FX \times FY$$

equals the map $FX \to FX \times FY : x \mapsto (x, e_Y)$.

Proof. The map $f \circ (\mathrm{id} \vee p_Y) \circ \iota_X : X \to X$ equals the identity, while $f \circ (\mathrm{id} \vee p_Y) \circ \iota_Y : Y \to X$ is pointedly null-homotopic. Homotopy invariance of F and the fact that F sends continuous maps to pointed maps of sets therefore imply that the composition above does send any $x \in FX$ to $(x, e_Y) \in FX \times FY$. ☐

Lemma 5.25. *For any $X \in \mathsf{CW}_*$, $F(\Sigma X)$ carries naturally a group structure.*

Proof. We fill in the details of the short proof in [11]. Consider the quotient map $q : \Sigma X \to \Sigma X / X \cong \Sigma X \vee \Sigma X$, and let i_1 and i_2 denote the inclusions $\Sigma X \hookrightarrow \Sigma X \vee \Sigma X$ on the first and second copy, respectively. In what follows, we will write S_1 for this first copy and S_2 for the second copy. By the wedge axiom, there is an isomorphism $F(S_1 \vee S_2) \xrightarrow{\sim} F(S_1) \times F(S_2)$ induced by i_1 and i_2, so we obtain a map $m : F(S_1) \times F(S_2) \xrightarrow{\sim} F(S_1 \vee S_2) \xrightarrow{Fq} F(\Sigma X)$, which we claim defines the structure of a group. Explicitly, if $(a_1, a_2) \in F(S_1) \times F(S_2)$, and $\alpha \in F(S_1 \vee S_2)$ is the unique element such that α restricts to a_n on S_n for $n = 1, 2$, then $m(a_1, a_2) = Fq(\alpha)$.

For associativity, let S_3 be a third copy of ΣX, and let $\iota_n : \Sigma X \hookrightarrow S_1 \vee S_2 \vee S_3$ be inclusions on the n-th copy, and let $j_1 : \Sigma X \hookrightarrow \Sigma X \vee S_3$ and $j_2 : S_3 \hookrightarrow \Sigma X \vee S_3$ be two further inclusions.

The operation $F(S_1) \times F(S_2) \times F(S_3) \to F(\Sigma X), (a_1, a_2, a_3) \mapsto m(m(a_1, a_2), a_3)$ factors as

$$\begin{tikzcd}
F(S_1) \times F(S_2) \times F(S_3) & F(S_1 \vee S_2) \times F(S_3) \arrow[l, "(Fi_1, Fi_2) \times \mathrm{id}"', "\sim"] \arrow[r, "Fq \times \mathrm{id}"] & F(\Sigma X) \times F(S_3) & F(\Sigma X \vee S_3) \arrow[l, "(Fj_1, Fj_2)"', "\sim"] \arrow[r, "Fq"] & F(\Sigma X) \\
(a_1, a_2, a_3) \arrow[r, mapsto] & (\alpha, a_3) \arrow[r, mapsto] & (Fq(\alpha), a_3) \arrow[r, mapsto] & \beta \arrow[r, mapsto] & Fq(\beta)
\end{tikzcd}$$

with α such that $Fi_n(\alpha) = a_n$ for $n = 1, 2$, and β such that $Fj_1(\beta) = Fq(\alpha)$ and $Fj_2(\beta) = a_3$. Now consider the composite

$$\begin{tikzcd}
F(S_1) \times F(S_2) \times F(S_3) & F(S_1 \vee S_2 \vee S_3) \arrow[l, "(F\iota_1, F\iota_2, F\iota_3)"', "\sim"] \arrow[r, "F(q \vee \mathrm{id})"] & F(\Sigma X \vee S_3) \\
(a_1, a_2, a_3) \arrow[r, mapsto] & a \arrow[r, mapsto] & b
\end{tikzcd}$$

with a such that $F\iota_n(a) = a_n$ for $n = 1, 2, 3$ (which implies that $F(\iota_1 \vee \iota_2)(a) = \alpha$ by Lemma 5.21 and the wedge axiom) and b such that $Fj_1(b) = Fq(F(\iota_1 \vee \iota_2)(a)) = Fq(\alpha)$ and $Fj_2(b) = F\iota_3(a) = a_3$. This implies that $b = \beta$, and therefore $m(m(a_1, a_2), a_3) = F((q \vee \mathrm{id}) \circ q)(a_1, a_2, a_3)$. Likewise,$m(a_1, m(a_2, a_3)) = F((\mathrm{id} \vee q) \circ q$, so by homotopy invariance of F, associativity of m would follow as soon as we show that $(q \vee \mathrm{id}) \circ q \simeq (\mathrm{id} \vee q) \circ q$ pointedly.

To do so, consider the maps

$$f : [-1, 1] \to [-1, 5], t \mapsto \begin{cases} 2t + 1, & \text{if } -1 \le t \le 0, \\ 4t + 1, & \text{if } 0 \le t \le 1 \end{cases}, \text{ and } g : [-1, 1] \to [-1, 5], t \mapsto \begin{cases} 4t + 3, & \text{if } -1 \le t \le 0, \\ 2t + 3, & \text{if } 0 \le t \le 1 \end{cases},$$

which are clearly homotopic relative to $\{-1, 1\}$ (the reader should try to visualise what these maps are doing). Using that we can consider $S_1 \vee S_2 \vee S_3$ to be the space $X \times [-1, 5]/(X \times \{-1, 1, 3, 5\} \cup \{*\} \times [-1, 5])$, the maps f and g induce maps $\widetilde{f}, \widetilde{g} : \Sigma X \to S_1 \vee S_2 \vee S_3$ (whose well-definedness can be checked), which are then also homotopic, and even pointedly homotopic. However, $\widetilde{f} = (q \vee \mathrm{id}) \circ q$ and $\widetilde{g} = (\mathrm{id} \vee q) \circ q$, which means these right-hand maps are also pointedly homotopic, which is what we wanted to show. Therefore, m is associative.

Let $e \in F(\Sigma X)$ be the base point, and let $\iota : * \hookrightarrow \Sigma X$ be the inclusion and consider $p : \Sigma X \to *$. These maps fit in a diagram

$$\begin{tikzcd}
\Sigma X \arrow[r, "\mathrm{id}"] \arrow[d, "q"'] & \Sigma X \\
\Sigma X \vee \Sigma X \arrow[r, "\mathrm{id} \vee p"] & \Sigma X \vee * \arrow[u, "\mathrm{id} \vee \iota"']
\end{tikzcd}$$

that commutes up to homotopy. Applying F to this diagram, we obtain by homotopy invariance of F a commutative diagram

$$\begin{tikzcd}
F(\Sigma X) & F(\Sigma X) \arrow[l, "\mathrm{id}"'] \arrow[d, "F(\mathrm{id} \vee \iota)"'] \\
F(\Sigma X \vee \Sigma X) \arrow[u, "Fq"] & F(\Sigma X \vee *) \arrow[l, "F(\mathrm{id} \vee p)"']
\end{tikzcd}$$

Under the bijections $F(\Sigma X \vee \Sigma X) \cong F(\Sigma X) \times F(\Sigma X)$ and $F(\Sigma X \vee *) \cong F(\Sigma X) \times F(*)$, we see using Lemmata 5.21 and 5.24 that an element $a \in F(\Sigma X)$ in the upper right corner is sent by the lower three maps first to $(a, *)$, then to (a, e) and then to $m(a, e)$. By commutativity of the above diagram, $m(a, e) = a$. Similarly, we find that $m(e, a) = a$. Since $a \in F(\Sigma X)$ was arbitrary, we have checked that e is the unit element of $F(\Sigma X)$.

Lastly, let $r : \Sigma X \to \Sigma X : [(x, t)] \mapsto [(x, -t)]$. We claim that $Fr : F(\Sigma X) \to F(\Sigma X)$ is the inversion map. Since the map

$$f : [-1, 1] \to [-1, 1], t \mapsto \begin{cases} 2t + 1, & \text{if } -1 \le t \le 0, \\ 1 - 2t, & \text{if } 0 \le t \le 1 \end{cases}$$

is null-homotopic relative to $\{0, 1\}$, the induced map $\widetilde{f} : \Sigma X \to \Sigma X$ is also pointedly null-homotopic. By homotopy invariance, $F\widetilde{f} : F(\Sigma X) \to F(\Sigma X)$ is the constant map on the base point e. It is not difficult to see that $\widetilde{f} = (\mathrm{id} \vee r) \circ q$, which implies that $m(a, Fr(a)) = F((\mathrm{id} \vee r) \circ q)(a) = e$ for all $a \in F(\Sigma X)$. Similary, we find $m(Fr(a), a) = e$, also for all such a, and with that, we have found an inverse operation. All in all, we have shown that $F(\Sigma X)$ allows a group structure.

For naturality, let $f : X \to Y$ be a continuous pointed map between pointed CW-complexes, and write $q_X : \Sigma X \to \Sigma X \vee \Sigma X$ and $q_Y : \Sigma Y \to \Sigma Y \vee \Sigma Y$ for the respective quotient maps that define the groups structures on $F(\Sigma X)$ and $F(\Sigma Y)$ (with multiplication maps m_X and m_Y). Then there is a commutative diagram

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma f"] \arrow[d, "q_X"] & \Sigma Y \arrow[d, "q_Y"] \\
\Sigma X \vee \Sigma X \arrow[r, "\Sigma f \vee \Sigma f"] & \Sigma Y \vee \Sigma Y
\end{tikzcd}$$

With Lemma 5.21, this implies that for each $(a_1, a_2) \in F(\Sigma Y) \times F(\Sigma Y)$, it holds that $F(\Sigma f)(m_Y(a_1, a_2)) = m_X(F(\Sigma f)(a_1), F(\Sigma f)(a_2))$, and therefore $F(\Sigma f)$ is a group homomorphism. This shows that the group structure is natural. ☐

In particular, for all $n \geq 1$ the set $F(S^n)$ carries the structure of a group, which follows from $S^n \cong SS^{n-1} \simeq \Sigma S^{n-1}$ and homotopy invariance of F. Note also that Lemma 5.18 implies that $[\Sigma X, Y]^\bullet$ carries a group structure for all pointed CW-complexes X and Y, which is natural in X.

Remark 5.26. (†) We now have a rather explicit description of the group structure on reduced cohomology groups for reduced suspensions: let $\widetilde{\mathrm{h}}^*$ be a reduced cohomology theory on CW_*, and let X be an arbitrary pointed CW-complex. Then $\widetilde{\mathrm{h}}^n(\Sigma X)$ does not only carry its original natural (since $\widetilde{\mathrm{h}}^n \circ \Sigma$ is a functor) group structure, but by the above lemma also another natural group structure, and we claim these coincide. Indeed, let $\iota_1, \iota_2 : \Sigma X \hookrightarrow \Sigma X \vee \Sigma X$ be the inclusions on the first and second copies, and let $q : \Sigma X \to \Sigma X / X \cong \Sigma X \vee \Sigma X$ be the quotient map. The second group structure is given by the composite $\widetilde{\mathrm{h}}^n q \circ \widetilde{\mathrm{h}}^n(\iota_1, \widetilde{\mathrm{h}}^n \iota_2)^{-1} : \widetilde{\mathrm{h}}^n(\Sigma X) \times \widetilde{\mathrm{h}}^n(\Sigma X) \to \widetilde{\mathrm{h}}^n(\Sigma X)$, which is a group homomorphism in the first group structure since $\widetilde{\mathrm{h}}^*$ is a functor $\mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Ab}$. The Eckmann-Hilton argument (and Remark 4.27) now implies that both group structures indeed coincide. This gives us a good picture of the group law on $\widetilde{\mathrm{h}}^n(\Sigma X)$, despite not knowing anything of $\widetilde{\mathrm{h}}^*$ except the axioms it satisfies. ▽

In the following lemma, F will no longer be a functor $\mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$, but a co- or contravariant functor $F : \mathsf{CW}_* \to \mathsf{Ab}$. For F to be a semi-Brown functor now means it basically satisfies the same axioms as before, properly adjusted in the covariant case.

Lemma 5.27. *Let $F : \mathsf{CW}_* \to \mathsf{Ab}$ be a co- or contravariant semi-Brown functor. Given two pointed CW-complexes X and Y and two pointed maps $f, g : \Sigma X \to Y$, let $[f] \star [g]$ be the group product of homotopy classes of f and g in the group $[\Sigma X, Y]^\bullet$. Then $F(f \star g) = Ff + Fg$ as homomorphisms of abelian groups.*

Proof. [11] We only will cover the contravariant case, since the covariant case is treated analogously. Let $q : \Sigma X \to \Sigma X \vee \Sigma X$ be the quotient map belonging to collapsing $X \subseteq \Sigma X$ to a point, let $i_1, i_2 : \Sigma X \hookrightarrow \Sigma X \vee \Sigma X$ be the inclusions in the first and second copy, and let $p : \Sigma X \to *$ be the unique map between these spaces.

The composite $FK \xrightarrow{f \vee g} F(\Sigma X \vee \Sigma X) \xrightarrow{\sim} F(\Sigma X) \times F(\Sigma X)$ sends an arbitrary $u \in FK$ to $(Ff(u), Fg(u))$ by Lemma 5.21. On the other hand, the composite $F(\Sigma X) \times F(\Sigma X) \xrightarrow{\sim} F(\Sigma X \vee \Sigma X) \xrightarrow{Fq} F(\Sigma X)$ sends any pair $(a_1, 0)$ to a_1, by the same argument as in the previous lemma, when we showed the base point of $F(\Sigma X)$ was the unit element of the group law on the latter set. Similarly, the composite considered sends $(0, a_2)$ to a_2 for any $a_2 \in F(\Sigma X)$. Since $F(\Sigma X)$ is an abelian groups and all induced maps are homomorphisms of abelian groups (since F is a functor to Ab), this implies that (a_1, a_2) is sent to $a_1 + a_2$.

All in all, we find $F(f \star g) = Fq \circ F(f \vee g)$ equals the composite

$$\begin{tikzcd} FK \arrow[r, "F(f \vee g)"] & F(\Sigma X \vee \Sigma X) \arrow[r, "\sim"] & F(\Sigma X) \times F(\Sigma X) \arrow[r, "\sim"] & F(\Sigma X \vee \Sigma X) \arrow[r, "Fq"] & F(\Sigma X) \end{tikzcd}$$

and hence sends any $u \in FK$ to $Ff(u) + Fg(u)$. This shows that $F(f \star g) = Ff + Fg$. ☐

Chapter 6

Ω-spectra and representability of cohomology

We have now (almost) all the tools we need to show that generalised and reduced cohomology theories on CW-complexes are representable, as we will show in this chapter. The first two sections are devoted to introducing the kind of objects that will later turn out to be the representing objects, whereas the third and most important section deals with the Brown Representability Theorem and the representability of cohomology. In the fourth section, we apply these results to singular cohomology and give a short treaty of Eilenberg-MacLane spaces and the uniqueness of cohomology. In the last section, we give, without going into too much detail, two other examples of cohomology theories, as a teaser of what one can do with cohomology theories other than singular cohomology.

6.1 Mapping spaces and the suspension-loop adjunction

We have used before that for commutative rings R, the Hom-sets of ${}_R\mathsf{Mod}$ carry naturally the structure of an R-module. Something similar is the case with the Hom-sets of Top: for any two topological spaces X and Y, we can equip $\mathrm{Map}(X,Y)$ with a convenient topology.

Definition 6.1. Let X and Y be two topological spaces. For any compact set $K \subseteq X$ and open set $O \subseteq Y$, we let $N(K,O) = \{f \in \mathrm{Map}(X,Y) \mid f(K) \subseteq O\}$. The *compact-open* topology on $\mathrm{Map}(X,Y)$ is the topology generated by the subbase $\{N(K,O) \mid K \subseteq X \,\text{compact}, O \subseteq Y \,\text{open}\}$. ◊

Convention 6.2. From this point forwards, we will always assume that a mapping space carries the compact-open topology. ⊙

Proposition 6.3. $\mathrm{Map}(\,\cdot\,,\,\cdot\,)$ *is a functor* $\mathsf{Top}^{\mathrm{opp}} \times \mathsf{Top} \to \mathsf{Top}$.

Proof. We only need to check that the induced maps between mapping spaces are continuous. Let X, Y and Z be topological spaces, and suppose $f : X \to Y$ is a continuous map. For a compact set $K \subseteq X$ and open $O \subseteq Z$, we know that $f(K) \subseteq Y$ is also compact. The induced map $- \circ f : \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z)$ therefore satisfies $(- \circ f)^{-1}(N(K,O)) = \{g \in \mathrm{Map}(Y,Z) \mid gf(K) \subseteq O\} = N(f(K),O)$. Since the compact-open topology has sets $N(K,O)$ as subbase, $- \circ f$ is continuous. Moreover, for a continuous map $g : Y \to Z$, we have $(g \circ -)^{-1}(N(K,O)) = \{f \in \mathrm{Map}(X,Y) \mid gf(K) \subseteq O\} = N(K, g^{-1}(O))$. Therefore, $g \circ - : \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Z)$ is also continuous. □

As we saw in Example A.38(i), there is for any given set Y a natural isomorphism $\mathrm{Hom}_{\mathsf{Set}}(- \times Y, \,\cdot\,) \cong \mathrm{Hom}_{\mathsf{Set}}(-, \mathrm{Hom}_{\mathsf{Set}}(Y, \,\cdot\,))$. We could ask whether or not such an adjunction also holds in Top. It turns out it does −as long as we assume that Y is locally compact. What follows is a key result, and hence deserves a thorough proof.

Proposition 6.4. *Let Y be a locally compact space. Then there is an adjunction $- \times Y \dashv \operatorname{Map}(Y, \cdot\,)$ of functors* $\mathsf{Top} \to \mathsf{Top}$.

Proof. (Roughly following [13]) In other words, we need to show there is a natural bijection of sets[1] $\operatorname{Map}(- \times Y, \cdot\,) \cong \operatorname{Map}(-, \operatorname{Map}(Y, \cdot\,))$. We use the same natural bijection as in Example A.38(i) (or rather its restriction the given mapping spaces). We thus send a continuous map $f : X \times Y \to Z$ to $\tilde{f} : X \to \operatorname{Map}(Y, Z), x \mapsto f(x, \cdot\,)$, and a continuous map $g : X \to \operatorname{Map}(Y, Z)$ to $\tilde{g} : X \times Y \to Z, (x, y) \mapsto g(x)(y)$. We then only need to show that these assignments send continuous maps to continuous maps, for this shows the bijection in the mentioned example indeed restricts to a bijection on mapping spaces.

Let us first check the first assignment sends a continuous map $f : X \times Y \to Z$ to a continuous map. Let $K \subseteq Y$ be compact, and $O \subseteq Z$ be open. Then $\tilde{f}^{-1}(N(K, O)) = \{x \in X \mid f(x, K) \subseteq O\} = \{x \in X \mid (x, K) \subseteq f^{-1}(O)\}$, which is open by Lemma 6.5 below. This shows $\tilde{f}$ is continuous.

Now, let a continuous map $g : X \to \operatorname{Map}(Y, Z)$, and an open set $O \subseteq Z$ be given. Then $\tilde{g}^{-1}(O) = \{(x, y) \in X \times Y \mid g(x)(y) \in O\} = \{(x, y) \in X \times Y \mid y \in g(x)^{-1}(O)\}$. Note that $g(x) \in \operatorname{Map}(Y, Z)$ is continuous, so $g(x)^{-1}(O)$ is open in Y. Let $(x_0, y_0) \in \tilde{g}^{-1}(O) = \{(x, y) \in X \times Y \mid y \in g(x)^{-1}(O)\}$ be fixed. Since Y is locally compact, there is a compact set $K_{x_0} \subseteq Y$ and an open set $U_{x_0} \subseteq Y$ such that $y_0 \in U_{x_0} \subseteq K_{x_0} \subseteq g(x_0)^{-1}(O)$. The set $N(K_{x_0}, O)$ is open in $\operatorname{Map}(Y, Z)$, so $O_{x_0} = g^{-1}(N(K_{x_0}, O) = \{x \in X \mid g(x)(K_{x_0}) \subseteq O\}$ is open in X, and contains x_0. Now, for every $(x, y) \in O_{x_0} \times U_{x_0}$ it holds that $g(x)(y) \in g(x)(U_{x_0}) \subseteq g(x)(K_{x_0}) \subseteq O$, since $x \in O_{x_0}$. This shows that $(x_0, y_0) \in O_{x_0} \times U_{x_0} \subseteq \tilde{g}^{-1}(O)$, and therefore $\tilde{g}$ is also continuous. □

Lemma 6.5. *Let X and Y be topological spaces, and let $K \subseteq Y$ be compact and $O \subseteq X \times Y$ be open. Then the set $U = \{x \in X \mid (x, K) \subseteq O\}$ is open.*

Proof. Let $x_0 \in U$. Then there are for each $y \in K$ open $O_X^y \in X$ and $O_Y^y \in Y$ such that $(x_0, y) \in O_X^y \times O_Y^y \subseteq O$. It is clear that the sets $O_X^y \times O_Y^y$ cover the compact set (x_0, K), so there is a finite number of element $y_1, \ldots, y_n \in K$ such that the sets $O_X^{y_i} \times O_Y^{y_i}$ cover (x_0, K). It now holds that $O_X = \bigcap_{i=1}^n O_X^{y_i}$ is open in X and contains x_0, and it is not difficult to see that if $x \in O_X$, then $x \in U$. This shows U is open in X. □

There is a similar adjunction when we are considering pointed spaces and homotopy classes of continuous pointed maps. To prove it, we first restrict the above adjunction to pointed spaces and pointed continuous maps.

Notation 6.6. We let $\operatorname{Map}^\bullet(\cdot\,, \cdot\,)$ denote the Hom-functor $\mathsf{Top}_*^{\mathrm{opp}} \times \mathsf{Top}_* \to \mathsf{Top}_*$. For two pointed spaces X and Y, the constant map on the base point of Y is preferred element of $\operatorname{Map}^\bullet(X, Y)$, and $\operatorname{Map}^\bullet(X, Y)$ is equipped with the subspace topology inherited from $\operatorname{Map}(X, Y)$. ○

Proposition 6.7. *Let Y be a pointed locally compact space. Then there is an adjunction $- \wedge Y \dashv \operatorname{Map}^\bullet(Y, \cdot\,)$ of functors* $\mathsf{Top}_* \to \mathsf{Top}_*$.

Proof. [13] Let X and Z be pointed spaces. The adjunction of Proposition 6.4 sends a pointed map $g : X \to \operatorname{Map}^\bullet(Y, Z)$ to the continuous map $\tilde{g} : X \times Y \to Z, (x, y) \mapsto g(x)(y)$, which is pointed as well. Even more is true: for $(x, y) \in X \vee Y \subseteq X \times Y$, we have $\tilde{g}(x, y) = *$, since $g(*) : Y \to Z$ is the constant map to the base point, and $g(x)$ is pointed for every $x \in X$. Therefore, the characteristic property of the quotient gives us a continuous map $\overline{g} : X \wedge Y \to Z$.

Conversely, if $f : X \wedge Y \to Z$ is pointedly continuous, then it induces a pointed continuous map $\hat{f} : X \times Y \to Z$, and our adjunction of Proposition 6.4 then produces a continuous map $\tilde{f} : X \to \operatorname{Map}(Y, Z), x \mapsto \hat{f}(x, \cdot\,)$. Since $\hat{f}$ factors through $X \wedge Y$, $\tilde{f}$ maps actually to the subspace $\operatorname{Map}^\bullet(Y, Z)$ and is pointed as such a map.

It is not difficult to show that these two assignments between $\operatorname{Map}^\bullet(X \wedge Y, Z)$ and $\operatorname{Map}^\bullet(X, \operatorname{Map}^\bullet(Y, Z))$ are bijections, which shows the claim. □

Definition 6.8. Let X be a pointed topological space. The *loop space* ΩX is defined as $\operatorname{Map}^\bullet(S^1, X)$. ◊

From its definition, it is immediately clear that taking the loop space is a functor $\Omega : \mathsf{Top}_* \to \mathsf{Top}_*$. Furthermore, Remark 3.46 lets us translate the previous adjunction to the following result:

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[1] This bijection does not need to be continuous, even though it maps between mapping spaces.

Corollary 6.9. (suspension-loop adjunction) *There is an adjunction* $\Sigma \dashv \Omega$ *of functors* $\mathsf{Top}_* \to \mathsf{Top}_*$. ☐

Lemma 6.10. *Let* X *and* Y *be two non-pointed resp. pointed spaces. Then* $\pi_0(\mathrm{Map}(X,Y)) \cong [X,Y]$ *resp.* $\pi_0(\mathrm{Map}^\bullet(X,Y)) \cong [X,Y]^\bullet$, *naturally in both* X *and* Y.

Proof. (Proposed by drs. J. Becerra) We only show the statement in the pointed case, as the non-pointed case is treated similarly. Any path in $\mathrm{Map}^\bullet(X,Y)$ corresponds one-to-one and naturally to a map $I_+ \to \mathrm{Map}^\bullet(X,Y)$. Via the adjunction of Proposition 6.7 it therefore corresponds via a natural bijection to a map $X \wedge I_+ \to Y$, which is a pointed homotopy $X \to Y$. ☐

Corollary 6.11. *Let* $X \in \mathsf{Top}$ *and* $Y \in \mathsf{Top}_*$. *Then there is a bijection* $[X_+,Y]^\bullet \cong [X,Y]$, *which is natural in both* X *and* Y.

Proof. (Also proposed by drs. J. Becerra) The previous lemma and the adjunction between $\cdot_+$ and the forgetful functor $\mathsf{Top}_* \to \mathsf{Top}$ from Remark 1.26 give bijections $[X_+,Y]^\bullet \cong \pi_0(\mathrm{Map}^\bullet(X_+,Y)) \cong \pi_0(\mathrm{Map}(X,Y)) \cong [X,Y]$ which are natural in both spaces. ☐

Proposition 6.12. *Let* Y *be a pointed locally compact space. Then there is a natural bijection* $[- \wedge Y, \cdot\,]^\bullet \cong [-, \mathrm{Map}^\bullet(Y, \cdot\,)]^\bullet$.

Proof. By Lemma 6.10 and Proposition 6.7, we have

$$[- \wedge Y, \cdot\,]^\bullet \cong \pi_0(\mathrm{Map}^\bullet(- \wedge Y, \cdot\,)) \cong \pi_0(\mathrm{Map}^\bullet(-, \mathrm{Map}^\bullet(\,\cdot\,,\,\cdot\,))) \cong [-, \mathrm{Map}^\bullet(Y, \cdot\,)]^\bullet$$

naturally. ☐

Corollary 6.13. (suspension-loop adjunction in the homotopy category) *There is a natural bijection* $[\Sigma(\,\cdot\,), \cdot\,]^\bullet \cong [\,\cdot\,, \Omega(\,\cdot\,)]^\bullet$. ☐

6.2 Ω-spectra

Loop spaces are key in systematically producing generalised reduced cohomology theories, and a cleverly defined sequence together with some extra structure regarding loop spaces does the trick. These are the Ω-spectra. We will not consider nontrivial examples of such spectra in this section, but leave those for later parts, such as Lemma 6.51 and Section 6.5. Rather, we will discuss their properties in relation to cohomology here. First we will explore the natural structure loop spaces carry to make this possible.

Recall that the loop space of a given pointed topological space X is the topological space $\Omega X = \mathrm{Map}^\bullet(S^1, X)$ of pointed loops on X (hence of course the name). As such, it carries the same operations that descend to the group law and inversion on the fundamental group, namely concatenation and reversion of loops. It is therefore a "group up to homotopy", which we call an H-group.

Definition 6.14. [13] A topological space $(X,*)$ with two continuous maps $m : X \times X \to X$ and $i : X \to X$ and a fixed element $e \in X$ is called an *H-group* if it satisfies the following conditions:

(i) $m(m(\,\cdot\,,\,\cdot\,),\,\cdot\,) \simeq m(\,\cdot\,, m(\,\cdot\,,\,\cdot\,))$ as maps $X \times X \times X \to X$.

(ii) $m(e,\,\cdot\,) \simeq \mathrm{id}_X \simeq m(\,\cdot\,, e)$ as maps $X \to X$.

(iii) $m(i(-),-) \simeq c_e \simeq m(-,i(-)))$ as maps $X \to X$ (so all entries must be the same element), where c_e is the constant map that maps every element of X to e.

X is a *pointed H-group* if it has base point e and m and i are pointed maps. ◊

Lemma 6.15. *Let* X *be a pointed topological space. Let* $c_X : S^1 \to X$ *be the constant loop on* $*$ *in* X. *Then* ΩX *is a pointed H-group with respect to concatenation and reversion of loops, and with* c_X *as unit element.* ☐

Lemma 6.16. [13] *Let* Z *a pointed H-group. Then* $[X,Z]^\bullet$ *carries naturally the structure of a group for all* X.

Remark 6.17. Here, naturality means that for every homotopy class $[f]$ of continuous map $X \to Y$, each representative $f : X \to Y$ induces a group homomorphism $- \circ f : [Y, Z]^\bullet \to [X, Z]^\bullet$, which does not depend on the chosen representative. $\triangledown$

Proof. (†) Let m and i denote the multiplication and inversion map, and $e \in Z$ the unit element belonging to the pointed H-group structure on Z. For two pointed continuous maps $g, h : Y \to Z$, we let $f \star g := m \circ (f, g) : Y \to Z, y \mapsto m(f(y), g(y))$, which is clearly a pointed map, since f, g and m are. If $f \simeq f'$ and $g \simeq g'$ pointedly, then $(f, g) \simeq (f', g')$ pointedly as maps $Y \to Z \times Z$, and hence $f \star g \simeq f' \star g'$ pointedly. Therefore we get an induced binary operation $\star$ on $[Y, Z]^\bullet$. We also set $\mathrm{inv}(f) = i \circ f : Y \to Z$. This also passes to a map $\mathrm{inv} : [Y, Z]^\bullet \to [Y, Z]^\bullet$.

If $c_Y : Y \to Z, y \mapsto e$ is the constant pointed map, then the properties of m and i as in the definition of a pointed H-group above imply that $\star$ is associative, has c_Y as unit element and inv is the inversion operation on $[Y, Z]^\bullet$. This shows that $[Y, Z]^\bullet$ carries a group structure. Now suppose $f : X \to Y$ and $g, h : Y \to Z$ are pointed maps. Then $(g \star h) \circ f = (g \circ f) \star (h \circ f)$ is clear from the definition of $\star$, so passing to homotopy classes of maps, f induces a group homomorphism $[Y, Z]^\bullet \to [X, Z]^\bullet$. If $f \simeq f'$ pointedly, then $(g \star h) \circ f \simeq (g \star h) \circ f'$ pointedly, so this group homomorphism depends only on the homotopy class $[f]$. $\square$

Note that the natural unit element of $[X, Z]^\bullet$ coincides with the preferred base point.

Remark 6.18. Using Lemmata 5.18 and 5.25, we defined a group structure on $[\Sigma X, Z]^\bullet$ which was natural in X. For two pointed maps $f, g : \Sigma X \to Z$, the group law was $[f] \star [g] = [(f \vee g) \circ q]$, where $q : \Sigma X \to \Sigma X/X \cong \Sigma X \vee \Sigma X$ is the quotient map. We can also define a natural group structure on $[\Sigma X, Z]^\bullet$ by pulling back the natural group structure on the set $[X, \Omega Z]^\bullet$ with which it is in bijection. A careful inspection of the suspension-loop adjunction and the description of the groups law on $[X, \Omega Z]^\bullet$ given in the proof of the previous lemma will reveal that these two group structures actually coincide. In particular, this shows that the suspension-loop adjunction $[\Sigma X, Z]^\bullet \cong [X, \Omega Z]^\bullet$ is an isomorphism of groups.

Note that we cannot use the same argument as in Remark 5.26 to show these two group structures coincide, since we needed there that there is also a natural group structure when we plug in the space $\Sigma X \vee \Sigma X$. This latter space is however not generally a suspension itself, so we cannot apply our knowledge that sets of the form $[\Sigma Y, Z]^\bullet$ carry group structures. This also means that $[\Sigma X, Z]^\bullet$ need not be an abelian group. $\triangledown$

Lemma 6.19. *Let Z be a pointed topological space and let $\Omega^2 Z = \Omega(\Omega Z)$ denote its second loop space. Then $[X, \Omega^2 Z]^\bullet$ carries naturally the structure of an abelian group.*

Proof. [6] We only need to show that the natural group structure on $[X, \Omega^2 Z]^\bullet$ is abelian. Since the reduced suspension $\Sigma S^1 \cong S^1 \wedge S^1$ is actually homeomorphic to S^2, and not only pointedly homotopy equivalent, the suspension-loop adjunction allows us to see an element of $\Omega^2 Z = \mathrm{Map}^\bullet(S^1, \mathrm{Map}^\bullet(S^1, Z))$ as a map $S^2 \to Z$. We already saw in the verification that the group structure on higher homotopy groups was abelian that this means that $\Omega^2 Z$ is an abelian H-group: it is also commutative up to homotopy. The group structure on $[X, \Omega^2 Z]$ as described in the proof of Lemma 6.16 now shows that for any two maps $f, g : X \to \Omega^2 Z$ it holds that $f \star g \simeq g \star f$ pointedly, so $[f] \star [g] = [g] \star [f]$ in $[X, \Omega^2 Z]^\bullet$. $\square$

Using Lemma A.25, we suggestively summarise the results so far in the following way.

Lemma 6.20. *Let Z be a given pointed topological space.*

(i) $[\,\cdot\,, \Omega Z]^\bullet$ *is a functor* $\mathsf{hTop}_*^{\mathrm{opp}} \to \mathsf{Grp}$.
(ii) *For* $n \geq 2$, $[\,\cdot\,, \Omega^n Z]^\bullet$ *is a functor* $\mathsf{hTop}_*^{\mathrm{opp}} \to \mathsf{Ab}$.
(iii) *There is a natural isomorphism* $[\Sigma(\,\cdot\,), Z]^\bullet \cong [\,\cdot\,, \Omega Z]^\bullet$ *of functors* $\mathsf{hTop}_*^{\mathrm{opp}} \to \mathsf{Grp}$.
(iv) *For* $n \geq 2$, *there is a natural isomorphism* $[\Sigma(\,\cdot\,), \Omega^{n-1} Z]^\bullet \cong [\,\cdot\,, \Omega^n Z]^\bullet$ *of functors* $\mathsf{hTop}_*^{\mathrm{opp}} \to \mathsf{Ab}$. $\square$

It looks like we are close to defining for a fixed pointed space Z a reduced cohomology theory $X \mapsto [X, \Omega^* Z]^\bullet$, but this fails for three reasons, as [11] explains: firstly, we need also cohomology groups in negative degrees, secondly, the sets $[X, \Omega^n Z]^\bullet$ are not necessarily abelian groups for $n = 0, 1$, and thirdly, a careful inspection of Definition 5.1 of a generalised reduced cohomology theory and part (iii) of the above lemma will reveal that the latter statement is then not the suspension isomorphism in a reduced cohomology theory, since the indices are the wrong way around. If we want to form a reduced cohomology theory and have a suspension isomorphism, then rather than to begin with a topological space Z and define higher cohomology groups by using the higher loop spaces, we have to work the other way around and start with some sort of "infinite loop space" and as we define higher reduced cohomology groups use the "lower loop spaces", and moreover hope that this approach also turns the zeroth and first cohomology group into actual abelian groups and gives us cohomology groups in negative degrees. This is the intuition behind the definition of an Ω-spectrum.

Definition 6.21. An Ω*-spectrum* is a sequence $(K_n)_{n\geq 0}$ of pointed CW-complexes together with pointed weak homotopy equivalences $\varphi_n : K_n \to \Omega K_{n+1}$. ♢

Although the weak homotopy equivalences are part of the data of an Ω-spectrum, we often omit them in notation. We must also remark that our earlier usage of the term "infinite loop space" was somewhat correct: given an Ω-spectrum (K_n), each K_n, and especially K_0 is sometimes called an *infinite loop space*, for the clear reason that it is weakly homotopy equivalent to $\Omega^m K_{n+m}$ for all $m \geq 0$.

Remark 6.22. [11] There are three things worth mentioning about the definition of an Ω-spectrum. First of all, it is not necessary to restrict it to CW-complexes. It is only that for our purposes it is more convenient to do so. It is a consequence of pointed CW-approximation (Theorem 4.58) and Proposition 4.54 that there is for our purposes not really a difference between these two possible definitions. Namely, in Theorem 6.26 we will see that the Hom-functors in hCW_* determined by the elements of an Ω-spectrum induce a reduced cohomology theory on CW_*, and these Hom-functors are naturally isomorphic if we replace the elements of the spectrum by spaces that admit weak homotopy equivalences to our original spaces.

Secondly, Milnor [18] showed that for any pointed CW-complex Z, the loop space ΩZ is pointedly homotopy equivalent to a pointed CW-complex. By the Whitehead Theorem then, if Y is a pointed CW-complex as well, a pointed weak homotopy equivalence $Y \to \Omega Z$ is a pointed homotopy equivalence. As such, the weak homotopy equivalences in an Ω-spectrum are all pointed homotopy equivalences.

Lastly, CW-approximation shows that any element K_n of an Ω-spectrum is determined up to pointed homotopy equivalence by the loop space ΩK_{n+1}. Indeed, there exists a pointed CW-approximation K for this space (and hence also a pointed weak homotopy equivalence $K \to \Omega K_{n+1}$), and any two of these are pointedly homotopy equivalent in such a way that the weak homotopy equivalences they come with are identified up to pointed homotopy, as we argued in Proposition 4.59. Following [11], we can use this same argument to define, given an Ω-spectrum $(K_n)_{n\geq 0}$, also the spaces K_n for negative n, namely inductively as pointed CW-approximations of the loop spaces ΩK_{n+1}. The particular choice of CW-approximation does not matter for our purposes, but since the proof of the CW-approximation Theorem in [11] gives us an explicit example, for definiteness we take that one each time. ▽

Convention 6.23. From now on, when mentioning an Ω-spectrum, we will always assume it also carries these spaces with negative indices and weak homotopy equivalences with negative indices as structure maps. ⊙

Definition 6.24. [16] Given two Ω-spectra (K_n) and (K'_n), with structure maps (φ_n) and (φ'_n), respectively, a *morphism of* Ω*-spectra* (or just "map") between them is a collection $f_n : K_n \to K'_n$ of continuous maps such that each square

$$\begin{tikzcd}
K_n \arrow[r, "\varphi_n"] \arrow[d, "f_n"] & \Omega K_{n+1} \arrow[d, "\Omega f_{n+1}"] \\
K'_n \arrow[r, "\varphi'_n"] & \Omega K'_{n+1}
\end{tikzcd}$$

commutes. (This is of course a special case of morphisms in a diagram category.) In this way, the collection of Ω-spectra and maps between them forms a category, denoted by $\Omega{-}\mathsf{spec}$. ♢

Remark 6.25. It is not entirely standard to take these morphisms in the category of Ω-spectra. For instance, Adams [1] takes homotopy classes of maps as morphisms. We choose however to follow May [16] because we will not be concerned with the subtleties that come with the more advanced theory of Ω-spectra that justify some modifications. ▽

We are now in the endgame of this thesis: the following theorem is the first central one with regards to our topic of representability of cohomology theories.

Theorem 6.26. *There is a functor $C : \Omega{-}\mathsf{spec} \to \mathsf{rCohomTh}$ which sends an Ω-spectrum (K_n) to the generalised reduced cohomology theory $X \mapsto [X, K_n]^\bullet$ on pointed CW-complexes.*

Proof. We have two show two things: first, given an Ω-spectrum (K_n), we have to check that the assignment $X \mapsto [X, K_n]^\bullet$ defines a generalised reduced cohomology theory on CW-complexes, and second, we need to check that this assignment respects maps of Ω-spectra and morphisms of generalised reduced cohomology theories.

Let us start with the first statement, following [11]. For each $n \in \mathbb{Z}$, there is a weak homotopy equivalence $K_n \to \Omega^2 K_{n+2}$, so Proposition 4.54 and Lemma 6.20 imply that $X \mapsto [X, K_n]^\bullet$ is a functor $\mathsf{hCW}_*^{\mathrm{opp}} \to \mathsf{Ab}$. This is the same as saying it is a functor $\mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Ab}$ which satisfies homotopy invariance.

Now, the Puppe sequence 4.22 implies that this functor also satisfies the exact sequence condition 5.1(ii), and the universal property of the wedge product implies that it also satisfies the wedge axiom of a reduced cohomology theory: indeed, for a wedge sum $\bigvee_{\alpha \in A} X_\alpha$, we get a bijection $[\bigvee_{\alpha \in A} X_\alpha, K_n]^\bullet \xrightarrow{\sim} \prod_{\alpha \in A} [X_\alpha, K_n]^\bullet$ induced by the inclusions $X_\alpha \hookrightarrow \bigvee_{\alpha \in A} X_\alpha$. By naturality of the abelian group structure on the sets appearing above, each such inclusion induces a group homomorphism, and hence the their product induces a group homomorphism as well. Since the latter is apparently bijective, it is an isomorphism of abelian groups.

Therefore we only need to give the suspension isomorphisms. For each $n \in \mathbb{Z}$, there is a natural isomorphism $[\,\cdot\,, K_n]^\bullet \xrightarrow[\sim]{\varphi_n \circ -} [\,\cdot\,, \Omega K_{n+1}]^\bullet \xrightarrow{\sim} [\Sigma(\,\cdot\,), K_{n+1}]^\bullet$ by Proposition 4.54 and Lemma 6.20 (so this second isomorphism is induced by the suspension-loop adjunction). We define the suspension isomorphism ς_n to be the inverse of this isomorphism. This shows that the assignment C is well-defined on the level of objects.

(†) As for its functoriality, let $(f_n : K_n \to K'_n)$ be a map between two Ω-spectra (K_n) and (K'_n). Each f_n induces a natural transformation $[\,\cdot\,, K_n]^\bullet \to [\,\cdot\,, K'_n]^\bullet$, so we only need to check this transformation commutes with the suspension isomorphisms. For a pointed CW-complex X, there is a commutative diagram

$$\begin{tikzcd}
{[X, K_n]^\bullet} \arrow[r, "\varphi_n \circ -"] \arrow[d, "f_n \circ -"] & {[X, \Omega K_{n+1}]^\bullet} \arrow[d, "\Omega f_{n+1} \circ -"] \\
{[X, K'_n]^\bullet} \arrow[r, "\varphi_n \circ -"] & {[X, \Omega K'_{n+1}]^\bullet}
\end{tikzcd}$$

since $\varphi'_n \circ f_n = \Omega f_{n+1} \circ \varphi_n$. Now consider the diagram

$$\begin{tikzcd}
{[X, \Omega K_{n+1}]^\bullet} \arrow[r, "\sim"] \arrow[d, "\Omega f_{n+1} \circ -"] & {[\Sigma X, K_{n+1}]^\bullet} \arrow[d, "f_{n+1} \circ -"] \\
{[X, \Omega K'_{n+1}]^\bullet} \arrow[r, "\sim"] & {[\Sigma X, K'_{n+1}]^\bullet}
\end{tikzcd}$$

with the horizontal isomorphisms induced by the suspension-loop adjunction. It commutes by said suspension-loop adjunction, so we obtain a commutative diagram

$$\begin{tikzcd}
{[X, K_n]^\bullet} \arrow[r, "\varphi_n \circ -"] \arrow[d, "f_n \circ -"] & {[X, \Omega K_{n+1}]^\bullet} \arrow[r, "\sim"] \arrow[d, "\Omega f_{n+1} \circ -"] & {[\Sigma X, K_{n+1}]^\bullet} \arrow[d, "f_{n+1} \circ -"] \\
{[X, K'_n]^\bullet} \arrow[r, "\varphi_n \circ -"] & {[X, \Omega K'_{n+1}]^\bullet} \arrow[r, "\sim"] & {[\Sigma X, K'_{n+1}]^\bullet}
\end{tikzcd}$$

with the inverses of the suspension isomorphisms as rows. By naturality of all maps, the diagram is natural in X, and this shows (f_n) induces a morphism between the generalised reduced cohomology theories $C(K_n)$ and $C(K'_n)$. It is easy to see that this assignment respects composition and the identities, which shows C is a functor. □

6.3 The Brown Representability Theorem

We want to strengthen the result of Theorem 6.26 and prove what will be the main result of this thesis, namely that any generalised reduced cohomology theory on CW-complexes, and consequently (by Theorem 5.11) any generalised cohomology theory on CW-complexes, is representable by an Ω-spectrum. This means that each individual functor of which such a theory consists is representable by a CW-complex, and the collection of these CW-complexes forms an Ω-spectrum. Alternatively, the functor C of Theorem 6.26 is essentially surjective.

The most difficult part of the proof is finding a CW-complex that represents a single functor of a reduced cohomology theory, and this part of the proof deserves its own theorem, as its arguments apply to more than cohomology functors alone. Almost this entire section will be dedicated to this theorem, called the Brown Representability Theorem.[2] At the end of this section, we apply the theorem to give a proof of the CW-approximation Theorem and then move on to the representability of generalised and reduced cohomology theories.

Now let us get started. Recall the kind of restriction introduced in Notation 5.16, and the definition of a semi-Brown functor in Definition 5.17.

Notation 6.27. We denote by cCW the full subcategory of CW consisting of path-connected CW-complexes, and by cCW_* the full subcategory of CW_* consisting of path-connected pointed CW-complexes. ○

Remark 6.28. By Lemma 3.30, any connected pointed CW-complex is also path-connected, so any CW-complex can be written as disjoint union of its path-connected components, something which is not generally possible for arbitrary topological spaces, since the topology of the disjoint union of the path-connected components may not agree with the topology on the space itself. ▽

Definition 6.29. A functor $F : \mathsf{cCW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$ is called a *Brown functor* if it satisfies the following three properties:

(i) *(Homotopy invariance)* If $f, g : X \to Y$ are two pointedly homotopic maps, then $Ff = Fg$.

(ii) *(Mayer-Vietoris axiom)* Suppose $X \in \mathsf{cCW}_*$ can be written as $X = A \cup B$ for two path-connected pointed subcomplexes A and B of X, such that both A and B have the base point of X as their base point. Then, if $a \in FA$ and $b \in FB$ satisfy $a|_{A\cap B} = b|_{A\cap B}$, then there is an $x \in FX$ such that $x|_A = a$ and $x|_B = b$.

(iii) *(Wedges)* If $(X_\alpha)_{\alpha\in A}$ is a family of spaces in cCW_*, then the inclusions $X_\alpha \hookrightarrow \bigvee_{\alpha\in A} X_\alpha$ induce an isomorphism $F(\bigvee_{\alpha\in A} X_\alpha) \xrightarrow{\sim} \prod_{\alpha\in A} F(X_\alpha)$. ♢

Remark 6.30. (By drs. J. Becerra) We can restate the Mayer-Vietoris axiom a bit more category theoretically (that is, only in terms of objects and arrows) as follows: by the universal property of the pullback (see Example A.46(ii)) of the diagram $FA \to F(A\cap B) \leftarrow FB$ induced by the inclusions, and the fact that there is a commutative diagram

$$\begin{tikzcd}
FX \arrow[r] \arrow[d] & FA \arrow[d] \\
FB \arrow[r] & F(A\cap B)
\end{tikzcd}$$

induced by the inclusions (since the corresponding diagram before applying F is commutative as well), there is a map $FX \to FA \times_{F(A\cap B)} FB$ fitting in the diagram

$$\begin{tikzcd}
FX \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& FA \times_{F(A\cap B)} FB \arrow[r] \arrow[d] & FA \arrow[d] \\
& FB \arrow[r] & F(A\cap B)
\end{tikzcd}$$

---

[2] There are multiple related theorems deserving of the name Brown Representability Theorem. We reserve it for Theorem 6.31 which contains the representability of a single functor, although statements regarding representability of reduced or generalised cohomology theories are also known under this name.

The Mayer-Vietoris axiom now is that this map is surjective. ▽

Theorem 6.31. **(Brown Representability Theorem)** *Let $F : \mathsf{cCW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$ be a Brown functor. Then there exists a path-connected pointed CW-complex K and an element $u \in FK$ such that the natural transformation $T_u : [\,\cdot\,, K]^\bullet \to F$ of functors determined by $T_{u,X} : [X, K]^\bullet \to FX, [f] \mapsto Ff(u)$ is a natural isomorphism.*

As said above, the proof is quite long, and there is still a bit of preparatory work to be done. First we want to remark a few things.

Remark 6.32. Let F be a contravariant functor from a category C to the category of sets, pointed sets or abelian groups, respectively (it will be clear from context). A pair $(K, u) \in \int_{\mathsf{C}} F$ is called *universal* for the functor F if there is a natural isomorphism $\mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, K) \xrightarrow{\sim} F$ determined by $\mathrm{Hom}_{\mathsf{C}}(X, K) \xrightarrow{\sim} FX, f \mapsto Ff(u)$ for all $X \in \mathsf{C}$. By the Yoneda Lemma, any representable functor allows such a pair (as that lemma produces the element $u \in FK$ from any natural isomorphism that shows F is representable). Moreover, although this pair is not unique (since there may be multiple natural isomoprhisms from $\mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, K)$ to F), the object K is unique up to isomorphism, for instance by Corollary A.36. ▽

Lemma 6.33. *Let K be an arbitrary pointed topological space. Then $[\,\cdot\,, K]^\bullet$ is a Brown functor.*

Proof. [11] The homotopy and wedge axiom are clearly satisfied (and were already covered by the way in Lemma 5.18). For the Mayer-Vietoris axiom, let $X = A \cup B$ be as in its statement and let $f : A \to K$ and $g : B \to K$ be two pointed maps. Suppose the homotopy classes $[f]$ and $[g]$ restrict to the same element in $[A \cap B, K]^\bullet$. This means that $f|_{A\cap B} \simeq g|_{A\cap B}$ pointedly. Let H be a pointed homotopy between $f|_{A\cap B}$ and $g|_{A\cap B}$. The diagram

$$\begin{tikzcd}
A \cap B \arrow[r, hook] \arrow[d, "c\mapsto(c,0)"'] & A \arrow[d, "c\mapsto(c,0)"'] \arrow[ddr, bend left, "f"] & \\
(A \cap B) \times I \arrow[r, hook] \arrow[drr, bend right, "H"] & A \times I \arrow[dr, dashed] & \\
 & & K
\end{tikzcd}$$

commutes, so by the homotopy extension property, the dashed homotopy exists. It is a pointed homotopy since H is and the diagram above commutes. Therefore, f is pointedly homotopic to a map f' that satisfies $f'|_{A\cap B} = g|_{A\cap B}$. We can glue f' and g to form a map $h : X \to K$, and $[h]$ restricts to $[f'] = [f]$ on A and to $[g]$ on B. This shows that $[\,\cdot\,, K]^\bullet$ satisfies the Mayer-Vietoris axiom and hence is a Brown functor. □

This shows that being a Brown functor is not only sufficient, but also necessary in order for a functor to be representable (assuming the Brown Representability Theorem holds in the first place).

Convention 6.34. From now on, suppose $F : \mathsf{cCW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$ is a Brown functor. ⊙

We advise the reader to quickly take a look again at section 5.3 again, where we already deduced a lot of results about Brown functors, which we will use here again, most notably Lemma 5.25. Here, we will follow the proof of Theorem 6.31 as given in [4] and [11].

Lemma 6.35. *For any $X \in \mathsf{CW}_*$ and for all pairs $(K, u) \in \int_{\mathsf{cCW}_*} F$, the map $T_{u,X} : [\Sigma X, K]^\bullet \to F(\Sigma X)$ is a group homomorphism.*

Proof. Let $X, K \in \mathsf{CW}_*$ be arbitrary. We recall from Remark 6.18 that the natural group structure on $[\Sigma X, K]^\bullet$ given by the suspension-loop adjunction and the natural group structure on it given by the fact that $[\,\cdot\,, K]^\bullet$ is a semi-Brown functor (Lemma 5.18) coincide.

Write the group law on $F(\Sigma X)$ as $m(\,\cdot\,,\,\cdot\,)$ once more. Let $(K, u) \in \int_{\mathsf{cCW}_*} F$ be arbitrary, and let two homotopy classes $[f], [g] \in [\Sigma X, K]^\bullet$ be given. Then $T_{u,X}([f] \star [g]) = F([f] \star [g])(u) = F((f \vee g) \circ q)(u) = Fq \circ F(f \vee g)(u) = m(Ff(u), Fg(u)) = m(T_{u,X}[f], T_{u,X}[g])$ by Lemma 5.21, which shows that $T_{u,X}$ is a group homomorphism. Since both X and the pair (K, u) were arbitrary, we are done. □

Lemma 6.36. *The functor F satisfies the exact sequence axiom in Definition 5.1(ii).*

Proof. By considering the mapping cylinder, it suffices by homotopy invariance of F to cover only the case of a pointed inclusion of a subcomplex into a CW-complex. Therefore, let $\iota : X \hookrightarrow Y$ be such an inclusion. Then the composition $X \overset{\iota}{\hookrightarrow} Y \overset{\iota_Y}{\hookrightarrow} C\iota$ is pointedly null-homotopic (which we also showed in the proof of Lemma 4.21). Since the constant pointed map $c : X \to C\iota$ factors as $X \to * \hookrightarrow C\iota$ and F sends each maps to pointed maps of sets, Lemma 5.22 implies that $Fc : F(C\iota) \to FX$ is the constant pointed map. By homotopy invariance, $F\iota \circ F\iota_Y = F(\iota_Y \circ \iota) = Fc$, which shows that $\operatorname{im} F\iota_Y \subseteq \ker F\iota$.

On the other hand, suppose $y \in FY$ is such that $F\iota(y) = * \in FX$. For the inclusion $\iota_X : X \to CX$, $F\iota_X$ is a pointed map $F(CX) \to FX$, so the base point of $F(CX)$ restricts thus to the base point of FX. Note that $X, Y, CX, C\iota \in \mathsf{cCW}_*$, and hence the decomposition $C\iota = CX \cup Y$ satisfies the requirements of the Mayer-Vietoris axiom (where it should be noted that $CX \cap Y = X$). This yields an element $x \in F(C\iota)$ such that $F\iota_Y(x) = y$, which shows $\ker F\iota \subseteq \operatorname{im} F\iota_Y$. This shows F satisfies the exact sequence axiom. ☐

Now we can turn to the main part of the proof of the Representability Theorem. The following definition hints at the strategy of the proof.

Definition 6.37. Let (K, u) be a pair with $K \in \mathsf{cCW}_*$ and $u \in FK$. For $n \geq 1$, we say (K, u) is *n-universal* if the homomorphism $T_{u,S^i} : \pi_i(K) \to F(S^i)$ is an isomorphism for all $1 \leq i < n$ and surjective for $i = n$. The pair is *π_*-universal* if it is n-universal for all $n \geq 1$. ◊

In other words, the pair is n-universal if it satisfies the Brown Representability Theorem for spheres S^i, with $1 \leq i < n$, and satisfies it "half" for the sphere S^n (of course, we cannot consider the sphere S^0, since this not a path-connected space). It will turn out that the theorem follows once we can show that F is representable on this subcategory of spheres, which constitutes to finding a π_*-universal pair (K, u). This happens often when working with CW-complexes, as we have seen: usually, we first show results for spheres and use the CW-structure to generalise them to all CW-complexes.

The next lemma shows that we can extend any given pair $(Z, z) \in \int_{\mathsf{cCW}_*} F$ to a π_*-universal pair, which in particular shows existence of π_*-universal pairs.

Lemma 6.38. *For $(Z, z) \in \int_{\mathsf{cCW}_*} F$ there exists a π_*-universal pair (K, u) such that Z is a subcomplex of K and $u|_Z = z$.*

Put differently, the conditions on K and u are saying that the inclusion $Z \hookrightarrow K$ gives a map $(Z, z) \to (K, u)$ of pairs.

Proof. We will inductively construct n-universal pairs (K_n, u_n) such that Z is a subcomplex of K_n and $u_n|Z = z$.

To begin with, let $K_1 = Z \vee \left(\bigvee_{\alpha \in F(S^1)} S^1_\alpha\right)$, where S^1_α is just an indexed copy of S^1. This space is clearly a path-connected pointed CW-complex. Since the respective inclusions induce an isomorphism $F(K_1) \cong FZ \times \prod_{\alpha \in F(S^1)} F(S^1)$ by the wedge axiom, there must be an element $u_1 \in F(K_1)$ such that $u_1|_Z = z$ and $u_1|_{S^1_\alpha} = \alpha$ for all $\alpha \in F(S^1)$. Since for any $\alpha \in F(S^1)$ the inclusion loop $\iota_\alpha : S^1 \hookrightarrow S^1_\alpha \subseteq K_1$ satisfies (as shown above) $T_{u_1,S^1}(\iota_\alpha) = F\iota_\alpha(u_1) = u_1|_{S^1_\alpha} = \alpha \in F(S^1)$, the pair (K_1, u_1) is 1-universal.

Now for the inductive step, assume we have already constructed an n-universal pair (K_n, u_n) such that Z is a subcomplex of K_n and $u_n|_Z = z$ (an overview of the construction in this and the following paragraphs is given in Figure 6.1). The group homomorphism $T_{u_n,S^n} : \pi_n(K_n) \to F(S^n)$ is then surjective, but not necessarily injective. Let for each $\alpha \in \ker T_{u_n,S^n}$ a representative $f_\alpha : S^n \to K_n$ be given, and set $f = \bigvee_{\alpha \in \ker T_{u_n,S^n}} f_\alpha : \bigvee_{\alpha \in \ker T_{u_n,S^n}} S^n_\alpha \to K_n$. (In this whole induction step, we will when writing the indices α assume they range over $\ker T_{u_n,S^n}$.) There is a deformation retraction $m_f \to K_n$ (and this reduced mapping cylinder is also a path-connected pointed CW-complex), so there is also a pointed bijection $F(K_n) \xrightarrow{\sim} F(m_f)$ induced by this map, by homotopy invariance of F. Therefore, we can consider u_n as an element of $F(m_f)$, and by definition of f, it holds that u_n restricts to the trivial element of $F(\bigvee_\alpha S^n_\alpha)$: indeed, the inclusion $\iota : \bigvee_\alpha S^n_\alpha \hookrightarrow m_f$ is associated by the deformation retract $m_f \to K_n$ with the map f, and hence the restriction of u_n to $\bigvee_\alpha S^n_\alpha$

equals by homotopy invariance of F and Lemma 5.21 the element $Ff(u_n) = \prod_\alpha Ff_\alpha(u_n) = *$ by definition of the f_α, and this is indeed the trivial element.

By Lemma 6.36), f and the inclusion $K_n \hookrightarrow Cf$ induce an exact sequence $F(Cf) \to F(K_n) \to F(\bigvee_\alpha S^n_\alpha)$, which implies that there is an element $w \in F(Cf)$ such that $w|_{K_n} = u_n$.

Now, for each map $f_\alpha : S^n \to K_n$, note that Cf_α is obtained by attaching a single $(n+1)$-cell to K_n along f_α. This means that Cf is obtained by attaching for each α an $(n+1)$-cell to K_n along f_α, and in particular this shows Cf is also a path-connected and pointed CW-complex. We now let $K_{n+1} = Cf \vee (\bigvee_{\beta \in F(S^{n+1})}) S^{n+1}_\beta$, which is a path-connected and pointed CW-complex and clearly contains Z as a subcomplex. Like before, there must by the wedge axiom exist an element u_{n+1} that restricts to w on Cf and to β on each S^{n+1}_β, and since w restricts to u_n on K_n and u_n restricts to z on Z, u_{n+1} also restricts to z on Z.

$$\begin{tikzcd}
 & (\bigvee_\alpha S^n_\alpha, *) \arrow[d, "f=\bigvee_\alpha f_\alpha"] & & (S^{n+1}_\beta, \beta) \arrow[d, hook] \\
(Z, z) \arrow[r, hook] & (K_n, u_n) \arrow[r, hook] \arrow[d, hook] & (Cf, w) \arrow[r, hook] & (K_{n+1}, u_{n+1}) \\
 & (m_f, u_n) & &
\end{tikzcd}$$

Figure 6.1: For all $\beta \in F(S^{n+1})$, this diagram commutes in $\int_{\mathsf{cCW}_*} F$.

In particular, this implies that the inclusion $\iota_{K_n} : K_n \hookrightarrow K_{n+1}$ induces a commutative diagram

$$\begin{tikzcd}
\pi_i(K_n) \arrow[rr, "\pi_i(\iota_{K_n})"] \arrow[dr, "T_{u_n,S^i}"'] & & \pi_i(K_{n+1}) \arrow[dl, "T_{u_{n+1},S^i}"] \\
 & F(S^i) &
\end{tikzcd}$$

for any $i \geq 1$. Indeed, for any representative $g : S^i \to K$ of a homotopy class $[g] \in \pi_i(K_n)$, we have $T_{u_{n+1},S^i} \circ \pi_i(\iota_{K_n}([g]) = T_{u_{n+1},S^i}([\iota_{K_n} \circ g]) = F(\iota_{K_n} \circ g)(u_{n+1}) = Fg \circ F\iota_{K_n}(u_{n+1}) = Fg(u_n) = T_{u_n,S^i}([g])$.

Now, since K_{n+1} arises from K_n by attaching only $(n+1)$-cells, Corollary 4.53 tells us that $\pi_i(\iota_{K_n})$ is an isomorphism for $1 \leq i < n$ and a surjection for $i = n$. By the induction hypothesis, the same holds for T_{u_n,S^i} , and hence the same holds for T_{u_{n+1},S^i}. Furthermore, we have $\ker T_{u_{n+1},S^n} = \pi_n(\iota_{K_n})(\ker T_{u_n,S^n})$ by surjectiveness of all maps, and for all $\alpha \in \ker T_{u_n,S^n}$ we attached an $(n+1)$-cell to K_n along a map $f_\alpha : S^n \to K_n$ representing α. That means that each $f_\alpha \circ \iota_{K_n}$ is pointedly null-homotopic in K_{n+1}, so the kernel of T_{u_{n+1},S^n} is trivial, which means this map is also an isomorphism. Lastly, $T_{u_{n+1},S^{n+1}}$ is surjective by construction, similarly to the case of T_{u_1,S^1}. Therefore, (K_{n+1}, u_{n+1}) is $(n+1)$-universal.

We have therewith constructed for each $n \geq 1$ an n-universal pair (K_n, u_n) that contains Z as a subcomplex and such that u_n restricts to u_i on K_i for all $1 \leq i \leq n$ and to z on Z.

We now will produce the π_*-universal pair (K, u) with the help of our n-universal pairs (K_n, u_n). Namely, we let K be the colimit of the diagram

$$\begin{tikzcd}
K_1 \arrow[r, hook, "\iota_{K_1}"] & K_2 \arrow[r, "\iota_{K_2}"] & K_3 \arrow[r, hook] & \ldots
\end{tikzcd}$$

Therefore, K is a pointed CW-complex, that is path-connected (for any two $x, y \in K$, there is an n with $x, y \in K_n$, and K_n is path-connected) and contains Z as a subcomplex.

We will use the mapping telescope construction of Definition 4.19 to find an element $u \in FK$ that restricts to u_n on each K_n (we summarise the next paragraphs in Figure 6.2). We let $t_K = \cup_{n=1}^{\infty} K_n \wedge [n, n+1]_+$ be the reduced mapping telescope of K. By Lemma 4.20, the inclusion $\iota : K \to t_K$ is a homotopy equivalence, and this means that $F\iota : F(t_K) \to FK$ is a pointed bijection. Define $A = \cup_{n \geq 1 \,\mathrm{odd}} K_n \wedge [n, n+1]_+$ and

$B = \cup_{n \geq 2\,\text{even}} K_n \wedge [n, n+1]_+$. Then $A \cup B = t_K$ and $A \cap B = \bigvee_{n=1}^{\infty} K_n$, whereas $A \simeq \bigvee_{n \geq 1\,\text{odd}} K_n$ and $B \simeq \bigvee_{n \geq 1\,\text{even}} K_n$. We have already seen how this implies that there are $a \in FA$ and $b \in FB$ that restrict to u_n on K_n for $n \geq 1$ odd or even, respectively, and since $u_{n+1}|_{K_n} = u_n$, a and b both restrict to u_n on K_n for all $n \geq 1$. Since $a \in FA$ and $b \in FB$ therefore restrict to the same element of $\prod_{n \geq 1} F(K_n) \cong F(\bigvee_{n \geq 1} K_n) = F(A \cap B)$, the Mayer-Vietoris axiom implies that there is a $t \in F(t_K)$ such that t restricts to both a and b on A and B, respectively, and consequently to u_n on K_n for all n. Under the previously established bijection $F\iota : F(t_K) \to FK$, there is an $u \in FK$ that restricts to u_n on K_n for all n. This of course also means that u restricts to z on Z.

$$\begin{tikzcd}
 & & (A,a) \arrow[dr, hook] & \\
(Z,z) \arrow[r, hook] & (K_n,u_n) \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & (K,u) \arrow[r, hook, "\iota"] & (t_K,t) \\
 & & (B,b) \arrow[ur, hook] &
\end{tikzcd}$$

Figure 6.2: For all $n \geq 1$, this diagram commutes in $\int_{\mathsf{cCW}_*} F$.

Finally, we have to verify that (K,u) is π_*-universal. For this, now let $\iota_{K_n} : K_n \hookrightarrow K$ denote the inclusion of K_n into K, and consider the commutative diagram

$$\begin{tikzcd}
\pi_i(K_n) \arrow[rr, "\pi_i(\iota_{K_n})"] \arrow[dr, "T_{u_n,S^i}"'] & & \pi_i(K) \arrow[dl, "T_{u,S^i}"] \\
 & F(S^i) &
\end{tikzcd}$$

for any $i \geq 1$ (which is commutative for the same reason that the earlier diagram of this form was, namely that u restricts to u_n on K_n for all n). If we choose $n > i+1$, then both $\pi_i(\iota_{K_n})$ and T_{u_n,S^i} are isomorphisms by Corollary 4.53 and the fact that (K_n,u_n) is n-universal. Therefore, for all i, the map T_{u,S^i} must be an isomorphism. This shows (K,u) is indeed a π_*-universal pair and completes the proof. ☐

Lemma 6.39. *Let (K,u) be a π_*-universal pair, and let (X,X') be a CW-pair in cCW_*. Then for each $x \in FX$ and each map $f : X' \to K$ that satisfies $Ff(u) = x|_{X'}$, there exists a map $g : X \to K$ that extends f and satisfies $Fg(u) = x$.*

In other words, the lemma asserts that given any diagram

$$\begin{tikzcd}
(X',x') \arrow[r, "f"] \arrow[d, hook] & (K,u) \\
(X,x) \arrow[ur, dashed, "g"] &
\end{tikzcd}$$

in $\int_{\mathsf{cCW}_*} F$, the dashed arrow g always exists.

Proof. By homotopy invariance of F, we may replace K by the mapping cylinder m_f (and u by the corresponding element of $F(m_f)$). Therefore, we may to begin with already assume that $f : X' \to K$ is an inclusion of a subcomplex. Since $Ff(u) = x|_{X'}$ by assumption, this means that $u|_{X'} = x|_{X'}$. Set $Z = X \cup_{X'} K$ (we again present an overview in Figure 6.3). The Mayer-Vietoris axiom tells us that there is a $z \in FZ$ such that $z|_K = u$ and $z|_X = x$. Now we use the previous lemma to extend the pair (Z,z) to a π_*-universal pair (K',u'). Since K

is a subcomplex of Z, it is a subcomplex of K', and since u' restricts to z on Z, and the latter restricts to u on K, the inclusion $\iota : K \hookrightarrow K'$ satisfies $F\iota(u') = u$. Once again, we obtain a commutative diagram

$$\begin{tikzcd}
\pi_n(K) \arrow[rr, "\pi_n(\iota)"] \arrow[dr, "T_{u,S^n}"'] & & \pi_n(K') \arrow[dl, "T_{u',S^n}"] \\
& F(S^n) &
\end{tikzcd}$$

for all n, in which the maps T_{u,S^n} and T_{u',S^n} are isomorphisms by π_*-universalness of (K, u) and (K', u'). Therefore, the inclusion $\iota : K \hookrightarrow K'$ induces isomorphisms on all homotopy groups, and since K is path-connected, Remark 4.47 says that it is consequently a weak homotopy equivalence, and a homotopy equivalence by the Whitehead Theorem[3]. By Proposition 4.7 (applied to the map $(K, K) \to (K', K)$ induced by ι), K is a deformation retract of K', so $F\iota : FK' \to FK$ is a bijection. Since $X \subseteq Z \subseteq K'$ is a subcomplex of K', the inclusion $i : X \hookrightarrow K'$ is homotopic relative to X' to a map $g : X \to K$, which therefore extends f. Moreover, this means that $Fg : FK \to FX$ coincides with the map $FK \xrightarrow{(F\iota)^{-1}} FK' \xrightarrow{Fi} FX$. Since $u'|_K = u$ and $u'|_X = (u'|_Z)|_X = z|_X = x$, we have $Fg(u) = Fg(u'|_K) = Fi(u') = u'|_X = x$, which completes the proof.

$$\begin{tikzcd}
(X', x|_{X'}) \arrow[r, hook, "f"] \arrow[d, hook] & (K, u) \arrow[d, hook] \arrow[dr, hook, "\iota"] & \\
(X, x) \arrow[ur, "g"] \arrow[r, hook] \arrow[rr, bend right, "i"'] & (Z, z) \arrow[r, hook] & (K', u')
\end{tikzcd}$$

Figure 6.3

□

Now we can prove the Brown Representability Theorem.

Proof of Theorem 6.31. We only need to show that an arbitrary π_*-universal pair (K, u) is universal for the functor F. For this, let $X \in \mathsf{cCW}_*$ be arbitrary. Note that $F(*) = *$ implies that the unique map $\iota : * \to K$ satisfies $u|_* = F\iota(u) = *$. The previous lemma applied to the CW-pair $(X, *)$ implies then that the map $T_{u,X} : [X, K]^\bullet \to FX, [f] \mapsto Ff(u)$ is surjective.

Now for injectivity, suppose that two pointed maps $f, g : X \to K$ satisfy $Ff(u) = Fg(u)$. These maps f and g induce a continuous map $h : X \wedge \partial I_+ \to K$. Note that $X \wedge \partial I_+$ is path-connected, so if we let $p : X \wedge I_+ \to X$ be the projection (or rather, the map through which the projection $X \times I \to X$ factors in the quotient), then $Fp : FX \to F(X \wedge I_+)$ is well-defined. If we set $x \coloneqq F(f \circ p)(u) = Fp \circ Ff(u) = Fp \circ Fg(u) = F(g \circ p)(u) \in F(X \wedge I_+)$ and let $i : X \wedge \partial I_+ \hookrightarrow X \wedge I_+$ be the inclusion, then the pointed homeomorphism $X \wedge I_+ \cong X \vee X$ and Lemma 5.21 imply that $x|_{X \wedge \partial I_+} = Fi(u) = F(f \circ p \circ i)(u) = F(f \vee f)(u) = (Ff, Ff)(u) = (Ff, Fg)(u) = Fh(u)$.

We can thus apply the previous lemma to the CW-pair $(X \wedge I_+, X \wedge \partial I_+)$, the map h, and the element x, and obtain a map $H : X \wedge I_+ \to K$ which extends h, which is thus a pointed homotopy from f to g. This means that $[f] = [g]$ in $[X, K]^\bullet$, which shows injectivity of $T_{u,X}$. Therefore T_u is a natural isomorphism of functors $[\,\cdot\,, K]^\bullet \to F$. □

As promised, we can now give a proof of the CW-approximation Theorem, and even a slightly stronger statement. Note that we did not use the CW-approximation in the proof of the Brown Representability Theorem, nor in any statements that were referenced to therein.

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[3] This step right here illustrates why we have restricted ourselves to path-connected pointed CW-complexes, rather than all pointed CW-complexes.

Theorem 6.40. *Let X be a pointed topological space that contains a pointed CW-complex A as a subspace, whose intersection with each path-connected component of X is path-connected. Then there exists a pointed CW-pair (Y, A) and a pointed weak homotopy equivalence $Y \to X$ which restricts to the identity on A.*

Proof. (†) First consider the case in which X is path-connected, so that A is also path-connected. We have already seen that the functor $[\,\cdot\,, X]^\bullet : \mathsf{cCW}_* \to \mathsf{Set}_*$ is a Brown functor. The inclusion $\iota : A \hookrightarrow X$ induces the element $[\iota]$ of $[A, X]^\bullet$, so by Lemma 6.38 the pair $(A, [\iota])$ can be extended to a π_*-universal pair $(Y, [f'])$ where Y is a path-connected pointed CW-complex that contains A as a subcomplex and $[f'] \in [Y, X]^\bullet$ is such that $[\iota] = [f' \circ i]$, where $i : A \hookrightarrow Y$ is the inclusion. In other words, the restriction of a representative $f' : Y \to X$ of $[f']$ to the subcomplex A is homotopic to the inclusion ι. Since (Y, A) has the homotopy extension property, f' is homotopic to a map $f : Y \to X$ that restricts to the identity on A, and note that therefore $(Y, [f])$ is also a π_*-universal extension of $(A, [\iota])$.

We also showed that any π_*-universal pair represents the Brown functor, so there is also a natural isomorphism $[\,\cdot\,, Y]^\bullet \cong [\,\cdot\,, X]^\bullet$ determined by $[Z, Y]^\bullet \to [Z, X]^\bullet [g] \mapsto [f \circ g]$ for all $Z \in \mathsf{cCW}_*$. Now, taking $Z = S^n$ for all $n \geq 1$, we see that f induces isomorphisms $\pi_n(Y) \to \pi_n(X)$ for all $n \geq 1$. Since X and Y are also path-connected, f induces a (pointed) bijection $\pi_0(Y) \to \pi_0(X)$, and moreover the choice of base point does not matter for the higher homotopy groups by Remark 4.47. Therefore, f is the required weak homotopy equivalence, which restricts to the identity on A.

Now, if X is not path-connected, index the path-connected components of X as X_α, and the path-connected intersection of X_α with A as A_α. we can use the above procédé to come up with CW-approximations $f_\alpha : (Y_\alpha, A_\alpha) \to (X_\alpha, A_\alpha)$ that restrict to the identity on A_α for each such component. Let $Y = \coprod_\alpha Y_\alpha$ and $f : (Y, A) \to (X, A)$ be the map induced by the f_α. Since all these latter maps are weak homotopy equivalences, so is f. Indeed, for arbitrary $y \in Y_\alpha \subseteq Y$, we have $\pi_n(Y, y) \cong \pi_n(Y_\alpha, y)$ and $\pi_n(X, f(y)) = \pi_n(X_\alpha, f_\alpha(y))$, and moreover $\pi_n f$ is associated with $\pi_n f_\alpha$ under these isomorphisms. Since each of the f_α restrict to the identity on A_α, we see that f restricts to the identity on A, and hence we are done. □

We now end this section with addressing the representability of reduced cohomology theories first, and generalised cohomology theories afterwards.

Lemma 6.41. *Let $\widetilde{\mathrm{h}}^*$ be a reduced cohomology theory. Then for each $n \in \mathbb{Z}$, the functor $\widetilde{\mathrm{h}}^n$ satisfies the Mayer-Vietoris axiom, and is hence a Brown functor.*

Proof. [11] Let $X = A \cup B$ be a decomposition of a pointed CW-complex into pointed subcomplexes, and write $j_A : A \cap B \hookrightarrow A$, $j_B : A \cap B \hookrightarrow B$, $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ for the respective inclusions. Suppose there are $a \in \widetilde{\mathrm{h}}^n(A)$ and $b \in \widetilde{\mathrm{h}}^n(B)$ that restrict to the same element in $\widetilde{\mathrm{h}}^n(A \cap B)$, that is, such that $\widetilde{\mathrm{h}}^n j_A(a) = \widetilde{\mathrm{h}}^n j_B(b)$. Exactness of the Mayer-Vietoris sequence (Corollary 5.14) now implies that there exists an $x \in \widetilde{\mathrm{h}}^n(X)$ such that $\widetilde{\mathrm{h}}^n i_A(x) = a$ and $\widetilde{\mathrm{h}}^n i_B(x) = b$, which is to say that x restricts to a on A and to b on B, which shows $\widetilde{\mathrm{h}}^n$ satisfies the Mayer-Vietoris axiom. □

Theorem 6.42. *Let $\widetilde{\mathrm{h}}^*$ be a generalised reduced cohomology theory on $\mathsf{CW}(2)$, and let $C : \Omega{-}\mathsf{spec} \to \mathsf{rCohomTh}$ be the functor of Theorem 6.26. Then there exists an Ω-spectrum (K_n) and a natural isomorphism $\widetilde{\mathrm{h}}^* \cong C(K_n)$ of reduced cohomology theories. In other words, C is essentially surjective.*

Proof. We add some details to the proof given in [11]. Let $U : \mathsf{Ab} \to \mathsf{Set}_*$ be the forgetful functor. The Brown Representability Theorem (which we can apply by the previous lemma) gives us for each $n \in \mathbb{Z}$ a space $K'_n \in \mathsf{cCW}_*$ and a natural isomorphism $U \circ \widetilde{\mathrm{h}}^n \cong [\,\cdot\,, K'_n]^\bullet$ seen as functors $\mathsf{cCW}_*^{\mathrm{opp}} \to \mathsf{Set}_*$. For an arbitrary pointed CW-complex X (not necessarily path-connected), the suspension isomorphism and the suspension-loop adjunction imply that $U \circ \widetilde{\mathrm{h}}^n(X) \cong U \circ \widetilde{\mathrm{h}}^{n+1}(\Sigma X) \cong [\Sigma X, K'_{n+1}]^\bullet \cong [X, \Omega K'_{n+1}]^\bullet$ naturally as pointed sets. Now, as said earlier, the spaces $\Omega K'_{n+1}$ are pointedly homotopy equivalent to CW-complexes [18], but for self-containedness, we let K_n be a CW-approximation of $\Omega K'_{n+1}$ (which is essentially unique by Lemma 4.59) for each $n \in \mathbb{Z}$. Using Proposition 4.54, we obtain a natural isomorphism $U \circ \widetilde{\mathrm{h}}^n \cong [\,\cdot\,, K_n]^\bullet$ of functors $\mathsf{CW}_* \to \mathsf{Set}_*$.

Another application of the suspension isomorphism and the suspension-loop adjunction gives $[\,\cdot\,, K_n]^\bullet \cong U \circ \widetilde{\mathrm{h}}^n \cong U \circ \widetilde{h}^{n+1} \circ \Sigma \cong [\Sigma(\,\cdot\,), K_{n+1}]^\bullet \cong [\,\cdot\,, \Omega K_{n+1}]$, so the Yoneda Lemma implies that this isomoprhism is induced by a homotopy equivalence $K_n \to \Omega K_{n+1}$. Lemma 4.45 now implies that the (K_n) form an Ω-spectrum.

Lastly, we need to show that for each $n \in \mathbb{Z}$ and $X \in \mathsf{CW}_*$ the pointed bijection $[X, K_n]^\bullet \cong \widetilde{\mathrm{h}}^n(X)$ is a group homomorphism. If we can establish that, then $\widetilde{\mathrm{h}}^n \cong [\,\cdot\,, K_n]^\bullet$ as functors $\mathsf{CW}_*^{\mathrm{opp}} \to \mathsf{Ab}$ and the Ω-spectrum (K_n) respresents the reduced cohomology theory $\widetilde{\mathrm{h}}^*$.

Now, to show this pointed bijection is a group homomorphism, note that we have isomorphisms $[X, K_n]^\bullet \cong [X, \Omega K'_{n+1}]^\bullet \cong [\Sigma X, K'_{n+1}]^\bullet$ and $\widetilde{\mathrm{h}}^{n+1}(\Sigma X) \cong \widetilde{\mathrm{h}}^n(X)$ of abelian groups, and therefore it suffices to show that the pointed bijection $\widetilde{\mathrm{h}}^n(\Sigma X) \cong [\Sigma X, K'_n]^\bullet$ is a group homomorphism for each n. Let $u_n \in \widetilde{\mathrm{h}}^n(K'_n)$ be the object that induces the isomorphism of the Brown Representability Theorem. For two maps $f, g : \Sigma X \to K'_n$, the equation $T_{u_n,\Sigma X}([f]+[g]) = T_{u_n,\Sigma X}[f] + T_{u_n,\Sigma X}[g]$ that would show this map is a homomorphism of abelian groups translates to $\widetilde{\mathrm{h}}^n([f]+[g])(u_n) = \widetilde{\mathrm{h}}^n[f](u_n) + \widetilde{\mathrm{h}}^n[g](u_n)$, and by Lemma 5.27, this holds. Therefore the bijection in the Representability Theorem is indeed a homomorphism, which finishes the proof. ☐

Combining Corollary 6.11 with the previous theorem and the equivalence of categories $\mathsf{CohomTh} \to \mathsf{rCohomTh}$ of Theorem 5.11 (and in particular Corollary 5.12), we obtain the following equally important theorem.

Theorem 6.43. *Let* h^* *be a generalised cohomology theory on* $\mathsf{CW}(2)$*. Then there exists an* Ω*-spectrum* (K_n) *such that*

(i) for all pairs $(X, X') \in \mathsf{CW}(2)$ *and all* n *there is an isomorphism* $\mathrm{h}^n(X, X') \cong [X/X', K_n]^\bullet$*, which is natural in the pair* (X, X')*;*

(ii) for all CW-complexes X *and all* n *there is an isomorphism* $\mathrm{h}^n(X) \cong [X, K_n]$*, which is natural in* X*.* ☐

6.4 Eilenberg-MacLane spaces

Let us now look at reduced singular cohomology again. We are going to more closely investigate the Ω-spectrum that respresents it. Given an abelian group A, any space K_n that satisfies $\widetilde{\mathrm{H}}^n(\,\cdot\,; A) \cong [\,\cdot\,, K_n]^\bullet$, must, plugging in the spheres S^m and using Corollary 5.15 and Example 2.11, satisfy

$$\pi_m(K_n) \cong \widetilde{\mathrm{H}}^n(S^m; A) \cong \begin{cases} A, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

This leads us to the following definition.

Definition 6.44. [7] Let $n \geq 0$ and G be a set which is a group if $n \geq 1$ and abelian in case $n \geq 2$. A pointed CW-complex K is an *Eilenberg-MacLane space of type* $K(G, n)$ if $\pi_m(K) = 0$ for $m \neq n$ and $\pi_n(K) \cong G$. ◊

Example 6.45. From covering theory we know that any pointed space X that allows a contractible universal covering space $\widetilde{X}$ is an Eilenberg-MacLane space of type $K(\pi_1(X), 1)$. Indeed, X is path-connected since $\widetilde{X}$ is, and for $n \geq 2$, S^n is path-connected and satisfies $\pi_1(S^n) = 0$ (for instance by Corollary 4.41). Therefore, any pointed map $f : S^n \to X$ lifts via the universal covering map $p : \widetilde{X} \to X$ to a pointed map $\widetilde{f} : S^n \to \widetilde{X}$ [10], as illustrated in the diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

Since $\widetilde{X}$ was assumed to be contractible, $f = p \circ \widetilde{f}$ is pointedly null-homotopic. Therefore $\pi_n(X) = 0$ for $n \geq 2$. This shows that the only possibly nontrivial homotopy group of X is $\pi_1(X)$, so X is an Eilenberg-MacLane space of type $K(\pi_1(X), 1)$.

This means for instance that the circle S^1 is a $K(\mathbb{Z}, 1)$, as it has $\mathbb{R}$ as universal covering space. By the Seifert-Van Kampen Theorem then, $\bigvee_{i=1}^n S^1$ is a $K(F_n, 1)$, where F_n is the free group on n generators. It is also possible to show that the infinite dimensional complex projective space $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ [13]. △

Remark 6.46. (†) It is clear that the discrete set G is an Eilenberg-MacLane space of type $K(G,0)$, and that any Eilenberg-MacLane space of this type is homotopy equivalent to this space. Indeed, if K is any such space, that we have $\pi_0(K) \cong G$, and we can write $\{K_g \mid g \in G\}$ for the path-connected components of K. By definition, for each such path-connected component we have $\pi_m(K_g) = 0$ for all $m \geq 0$. The Whitehead Theorem implies that any inclusion $* \hookrightarrow K_g$ is a homotopy equivalence, and therefore $K \simeq G$. ▽

Convention 6.47. For the following passage, let A be an abelian group and write (K_n) for the Ω-spectrum representing reduced singular cohomology with coefficients in A. ⊙

By the above remark, we have $K_0 \simeq A$ as a discrete space, and obviously we have $K_n \simeq *$ for negative n. For positive n, we have seen that K_n is an Eilenberg-MacLane space of type $K(A,n)$. This is all that we can say, in the sense that this is enough: it turns out that any choice of Eilenberg-MacLane spaces forms an Ω-spectrum and represents reduced singular cohomology, as we will demonstrate below. For now, we wish to show uniqueness of Eilenberg-MacLane spaces of type $K(A,n)$ for all n, and not just $n = 0$. To do so, we need an important theorem in algebraic topology, which we introduce here without proof (and not in the most general form).

We first however need to talk a bit about singular homology again. It is not difficult to show that $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ for all $n \geq 0$ since it satisfies all the Eilenberg-Steenrod axioms in Definition 1.46. Now, this means that there is a generator $\alpha \in \mathrm{H}_n(S^n)$ of this group, which is unique up to sign. Fix a particular choice. Consider for a path-connected space X (not necessarily a CW-complex) the *Hurewicz homomorphism* $h : \pi_n(X) \to \mathrm{H}_n(X), [f] \mapsto \mathrm{H}_n f(\alpha)$ (note that path-connectedness of X implies that base points do not matter). That this is a homomorphism follows similarly as Lemma 5.27.

Theorem 6.48. (Hurewicz) *Let X be a path-connected space.*

(i) If $n = 1$, the Hurewicz map induces an isomorphism $\pi_1(X)^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{H}_1(X)$.

(ii) If $n \geq 2$ and $\pi_i(X) = 0$ for $0 \leq i \leq n-1$, then the Hurewicz map $h : \pi_n(X) \to \mathrm{H}_n(X)$ is an isomorphism.

Proof. This is shown in [11]. □

Proposition 6.49. *Let $n \geq 0$ and let K be an Eilenberg-MacLane space of type $K(A,n)$. Then $K \simeq K_n$ pointedly, where K_n is as in Convention 6.47.*

Proof. We already covered the case $n = 0$ in Remark 6.46, so let us take $n \geq 1$. Our proof, roughly following [7], will be a prime example of abstract nonsense and will use a few large results obtained over the past pages.

Let $u_n \in \mathrm{H}^n(K_n; A)$ be the element that induces the natural isomorphism $[\,\cdot\,, K_n]^\bullet \xrightarrow{\sim} \widetilde{\mathrm{H}}^n(\,\cdot\,; A) = \mathrm{H}^n(\,\cdot\,; A)$ (since $n \geq 1$). We have $\mathrm{H}_n(K) \cong \pi_n(K) \cong A$ by the Hurewicz Theorem, with the Hurewicz map h providing the first isomorphism, and moreover we have $\mathrm{H}_{n-1}(K) \cong \pi_{n-1}(K) = 0$ by the same theorem. The Universal Coefficient Theorem 2.20 now implies that the map Φ given by said theorem is an isomorphism $\mathrm{H}^n(K; A) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Ab}}(\mathrm{H}_n(K), A) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(K), A) \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathrm{H}_n(K), A) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(0, A) \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathrm{H}_n(K), A)$, by Proposition B.42(ii). All in all we obtain an isomorphism

$$\Psi_K : [K, K_n]^\bullet \xrightarrow{\quad\sim\quad} \widetilde{\mathrm{H}}^n(K; A) = \mathrm{H}^n(K; A) \xrightarrow[\sim]{\Phi} \mathrm{Hom}_{\mathsf{Ab}}(\mathrm{H}_n(K), A)) \xrightarrow[\sim]{-\circ h} \mathrm{Hom}_{\mathsf{Ab}}(\pi_n(K), A).$$

Let $f : K \to K_n$ represent an element $[f] \in [K, K_n]^\bullet$, let a group homomorphism $\varphi : C_n(K) \to A$ (where $C_n(K)$ is a group appearing in the singular chain complex, last encountered in the first two chapters) represent the element $\mathrm{H}^n f(u_n) \in \mathrm{H}^n(K; A)$ and let $\gamma : S^n \to K_n$ represent an element $[\gamma] \in \pi_n(K_n)$. If α is a generator of $\mathrm{H}_n(S^n)$ (represented by $a \in C_n(S^n)$), the description of the Hurewicz isomorphism $h : \pi_n(K) \to \mathrm{H}_n(K)$ above implies (after careful inspection) that the homomorphism $\Psi_K([f]) : \pi_n(K) \to A$ is determined by $\Psi_K([f])([\gamma]) = \varphi \circ C_n\gamma(a)$, where $C_n\gamma : C_n(S^n) \to C_n(K)$ is the induced map on the n-th singular chain group.

There is for $n \geq 1$ a preferred isomorphism $\pi_n(K_n) \xrightarrow{\sim} A$ obtained in a similar manner as the composition

$$\Psi_{S^n} : \pi_n(K_n) = [S^n, K_n]^\bullet \xrightarrow{\quad\sim\quad} \widetilde{\mathrm{H}}^n(S^n; A) = \mathrm{H}^n(S^n; A) \xrightarrow[\sim]{\Phi} \mathrm{Hom}_{\mathsf{Ab}}(\mathrm{H}_n(S^n), A)) \xrightarrow[\sim]{g \mapsto g(\alpha)} A.$$

This isomorphism can be seen (after another careful inspection) to send the element $\pi_n f([\gamma]) = [f \circ \gamma]$ to $\varphi \circ C_n\gamma(a)$ as well. All in all, we obtain an isomorphism

$$[K, K_n]^\bullet \xrightarrow[\sim]{\Psi_K} \mathrm{Hom}_{\mathsf{Ab}}(\pi_n(K), A) \xleftarrow[\sim]{\Psi_{S^n} \circ -} \mathrm{Hom}_{\mathsf{Ab}}(\pi_n(K), \pi_n(K_n)),$$

which by our previous observations sends $[f]$ to $\pi_n f$ (we will celebrate this result by deducing Lemma 6.52 below from it). Now, since $\pi_n(K) \cong A \cong \pi_n(K_n)$, there is an isomorphism $\pi_n(K) \to \pi_n(K_n)$ on the right of the isomorphism above, which is apparently induced by (a pointed homotopy class of) a pointed map $f : K \to K_n$. Since the homotopy groups of K and K_n are trivial everywhere else, this map f is a pointed weak homotopy equivalence between CW-complexes, and therefore a pointed homotopy equivalence $K \to K_n$ by the Whitehead Theorem. ☐

Notation 6.50. Since all Eilenberg-MacLane spaces of type $K(A, n)$ are apparently pointedly homotopy equivalent, we write by a slight abuse of notation simply $K(A, n)$ for any such space. ○

Lemma 6.51. *The sequence $(K(A, n))$ in n forms an Ω-spectrum (the Eilenberg-MacLane spectrum) that represents $\widetilde{\mathrm{H}}^n(\,\cdot\,; A)$.*

Proof. Since we found in (K_n) an Ω-spectrum and all Eilenberg-MacLane spaces of the same type are pointedlty homotopy equivalent, this is clear. ☐

We draw a few further conclusions from the proof of Proposition 6.49.

Lemma 6.52. *Let $n \geq 1$.*

(i) For every pointed CW-complex X, there is an isomorphism $[X, K(A, n)] \cong [X, K(A, n)]^\bullet$.

(ii) If X is a pointed CW-complex such that $\pi_i(X) = 0$ for $0 \leq i \leq n-1$, then taking the n-th homotopy group induces an isomorphism $[X, K(A, n)]^\bullet \xrightarrow[\sim]{\pi_n} \mathrm{Hom}_{\mathsf{Ab}}(\pi_n(X), \pi_n(K(A, n)))$.

(iii) If A' is another abelian group, then there is an isomorphism $\mathrm{H}^n(K(A', n); A) \cong [K(A', n), K(A, n)]^\bullet \cong \mathrm{Hom}_{\mathsf{Ab}}(A', A)$.

Proof. The first statement follows by Lemma 6.51 and Theorem 6.43 from the isomorphisms $[X, K(A, n)] \cong \mathrm{H}^n(X; A) = \widetilde{\mathrm{H}}^n(X; A) \cong [X, K(A, n)]^\bullet$, as $n \geq 1$. The second statement is shown just as the case for X an Eilenberg-MacLane space in the proof of Proposition 6.49 (as we only needed to apply the Hurewicz Theorem to this first Eilenberg-MacLane space, which only requires the restrictions imposed on X in the statement). The third statement is obvious now. ☐

Corollary 6.53. (†) *Let $n \geq 1$, and denote by $\mathsf{hEM}_*(n)$ the full subcategory of hCW_* consisting of Eilenberg-MacLane spaces of type $K(A, n)$, where A ranges over all abelian groups (and n is fixed).*

(i) The functor $\pi_n : \mathsf{hEM}_*(n) \to \mathsf{Ab}$ is an equivalence of categories.

(ii) There is a fully faithful functor $k(\,\cdot\,, n) : \mathsf{Ab} \to \mathsf{hTop}_*$ that sends an abelian group A to a particular Eilenberg-MacLane space $k(A, n)$ of type $K(A, n)$.

Proof. The functor $\pi_n : \mathsf{hEM}_*(n) \to \mathsf{Ab}$ is fully faithful by Lemma 6.52(ii), and is essentially surjective by definition of the Eilenberg-MacLane spaces (as for any abelian group A the existence of $K(A, n)$ is guaranteed by representability of reduced singular cohomology). By Proposition A.28, therefore, it is an equivalence of categories. This shows the first claim. For the second, note that there must now be a functor $k(\,\cdot\,, n) : \mathsf{Ab} \to \mathsf{hEM}_*(n)$ such that $\pi_n(k(A, n)) \cong A$ naturally in A for any abelian group A, which means that $k(A, n)$ is an Eilenberg-MacLane space of type $K(A, n)$. After considering the inclusion $\mathsf{hEM}_*(n) \to \mathsf{hTop}_*$, we see that $k(\,\cdot\,, n)$ is also a fully faithful functor $\mathsf{Ab} \to \mathsf{hTop}_*$. ☐

Remark 6.54. This is certainly a nontrivial equivalence of categories, and a particularly daring mathematician could now try to study abelian group theory by devoting his or her live to the study of Eilenberg-MacLane spaces. In this sense, one could say that most of abelian group theory is apparently hidden within the Eilenberg-MacLane spaces. However, it is clear that the more useful observation is that much of the homotopy theory of Eilenberg-MacLane spaces can be entirely captured within the language of abelian groups. ▽

Now, for another application of the representability of singular cohomology, we will talk about *cohomology operations*. As we shortly states at the beginning of Chapter 2, one of the advantages that singular cohomology has over singular homology is that the abelian groups of the former can be combined within a single ring, whereas the latter does not (at east not naturally). We could ask if there is even more algebraic structure lying around. What happens if we also allow us to vary the coefficients? This is a rough justification for the following definition.[4]

Definition 6.55. [7] Given $m, n \geq 0$ and two abelian groups A and A', a *cohomology operation of type* (n, m, A, A') is a natural transformation $\mathrm{H}^n(\,\cdot\,; A) \to \mathrm{H}^m(\,\cdot\,; A')$. ◊

Example 6.56. We will give a nontrivial example of a cohomology operation that arises quite naturally (pun intended), taken from [7]. Consider for any prime p the short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$. By definition of the singular cochain complex, it can be seen to induce for each topological space X a short exact sequence

$$0 \longrightarrow C^{\bullet}(X, \mathbb{Z})) \xrightarrow{(\cdot p)\circ -} C^{\bullet}(X, \mathbb{Z}) \longrightarrow C^{\bullet}(X, \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

of said cochain complexes. By Theorem B.30, we obtain a long exact sequence

$$\dots \longrightarrow \mathrm{H}^{n-1}(X; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X; \mathbb{Z}) \longrightarrow \mathrm{H}^n(X; \mathbb{Z}) \longrightarrow \mathrm{H}^n(X; \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X; \mathbb{Z}) \longrightarrow \dots$$

of cohomology. The *Bockstein homomorphisms* are the maps $\delta_n : \mathrm{H}^n(X; \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^{n+1}(X; \mathbb{Z})$. They are natural in X because the long exact sequence of cohomology in Theorem B.30 is. Therefore, they define natural transformations $\mathrm{H}^n(\,\cdot\,; \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^{n+1}(\,\cdot\,; \mathbb{Z})$, and hence give us for all $n \geq 0$ cohomology operations of type $(n, n+1, \mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$. △

The representability of singular cohomology allows us to determine in some cases the number of such cohomology operations of a certain type.

Lemma 6.57. *Let $m, n \geq 0$ and let A and A' be two abelian groups.*

(i) If $n > m$, there are no cohomology operations of type (n, m, A, A') except the trivial one.

(ii) The cohomology operations of type (n, n, A, A') are in bijection with $\mathrm{Hom}_{\mathsf{Ab}}(A, A')$.

Proof. By the Yoneda Lemma and the fact that Eilenberg-MacLane spaces represent singular cohomology, we have an isomorphism $\mathrm{Nat}(\mathrm{H}^n(\,\cdot\,; A), \mathrm{H}^m(\,\cdot\,; A')) \cong \mathrm{H}^m(K(A, n); A')$. If $m < n$, then the Hurewicz Theorem 6.48 implies that $\mathrm{H}^m(K(A, n); A') = 0$, which shows that in this case there are no cohomology operations of type (n, m, A, A'). If $m = n$, then Lemma 6.52(iii) implies that $\mathrm{H}^n(K(A, n); A') \cong \mathrm{Hom}_{\mathsf{Ab}}(A, A')$. ☐

We already have shown that Eilenberg-MacLane spaces $K(A, n)$ exist and are unique for abelian groups A, but have not touched on the existence nor uniqueness of *all* types. We will not need it, but state the result nonetheless.

Lemma 6.58. *Let n and G be as in the definition of an Eilenberg-MacLane space. Then there exists a CW-complex K which is an Eilenberg-MacLane space of type $K(G, n)$. Moreover, all such spaces are pointedly homotopy equivalent.*

Proof. We already covered the case in which $n = 0$ and the case in which G is abelian. The only remaining case, existence and uniqueness of a $K(G, 1)$ for G not abelian, is shown in [11]. ☐

The following corollary is a result too beautiful not to include here.

Corollary 6.59. (†) *Let $(G_n)_{n \geq 0}$ be a sequence of a set G_0 and groups G_n for $n \geq 1$, which are abelian in case $n \geq 2$. Then there exists a CW-complex K such that $\pi_n(K) \cong G_n$ for all n.*

---

[4] This definition is actually really interesting and important, but we have simply not explored enough algebraic topology here to fully justify it.

Proof. Simply let $K' = \prod_{n=0}^{\infty} K(G_n, n)$, equipped with the regular product topology (so K' is not necessarily a CW-complex). Since $[\,\cdot\,, K']^{\bullet} \cong \prod_{n=0}^{\infty}[\,\cdot\,, K(G_n, n)]^{\bullet}$ for every choice of base point in K', we have $\pi_n(K') \cong G_n$ for all K. Now we can pick for K a CW-approximation of K', and we are done. □

We end this section with two among the most important theorems about reduced and generalised cohomology theories that we can now relatively easily prove. The second could also have been proven at the end of the previous section, but it is more naturally stated along with the first one.

Theorem 6.60. (Uniqueness of ordinary cohomology)[11]

(i) Let h^* *and* k^* *be two ordinary cohomology theories on* $\mathsf{CW}(2)$*. If* $\mathrm{h}^0(*) \cong \mathrm{k}^0(*)$*, then* $\mathrm{h}^* \cong \mathrm{k}^*$ *as cohomology theories.*

(ii) Let $\widetilde{\mathrm{h}}^*$ *and* $\widetilde{\mathrm{k}}^*$ *be two reduced cohomology theories on* CW_* *that satisfy* $\widetilde{\mathrm{h}}^n(S^0) \cong 0 \cong \widetilde{\mathrm{k}}^n(S^0)$ *for* $n \neq 0$*. If* $\widetilde{\mathrm{h}}^0(S^0) \cong \widetilde{\mathrm{k}}^0(S^0)$*, then* $\widetilde{\mathrm{h}}^* \cong \widetilde{\mathrm{k}}^*$ *as reduced cohomology theories.*

Remark 6.61. In particular, this implies that any ordinary cohomology theory h^* on $\mathsf{CW}(2)$ is naturally isomorphic to the singular cohomology theory $\mathrm{H}^*(\,\cdot\,,\,\cdot\,; \mathrm{h}^0(*))$, and the reduced statement is also true (below we will see that this is actually the way we prove this theorem). Note that this also implies that $\mathrm{h}^n(X, X') = 0$ for all CW-pairs (X, X') and all $n < 0$. ▽

Proof. (†) First note that both statements are equivalent by Corollaries 5.12 and 5.15. We will therefore only prove the second. Let (K_n) an Ω-spectra that represents $\widetilde{\mathrm{h}}^*$, and write $A = \widetilde{\mathrm{h}}^0(S^0)$. By Corollary 5.15, we must have $\pi_m(K_n) = [S^m, K_n]^{\bullet} \cong \widetilde{h}^n(S^m)$, which is trivial if $m \neq n$ and isomorphic to A if $m = n$. Therefore, every K_n must be an Eilenberg-MacLane space of type $K(A, n)$, and therefore (K_n) is the Eilenberg-MacLane spectrum. This implies that $\widetilde{\mathrm{h}}^*$ is naturally isomorphic to $\widetilde{\mathrm{H}}^n(\,\cdot\,; A)$. The same holds for $\widetilde{\mathrm{k}}^*$, so $\widetilde{\mathrm{h}}^* \cong \widetilde{\mathrm{k}}^*$ as reduced cohomology theories. □

Remark 6.62. If the reader is familiar with de Rham cohomology on smooth manifolds, he or she might recall the De Rham Theorem, that states that singular cohomology with coefficients in $\mathbb{R}$ is isomorphic to de Rham cohomology on smooth manifolds [14]. This is related to the above theorem, since all smooth manifolds are homotopy equivalent to a CW-complex [18]. ▽

Theorem 6.63. (Uniqueness of generalised cohomology)[11]

(i) Let h^* *and* k^* *be two generalised cohomology theories on* $\mathsf{CW}(2)$*, and suppose there is a morphism* $\eta : \mathrm{h}^* \to \mathrm{k}^*$ *of generalised cohomology theories in* $\mathsf{CohomTh}$*. If* $\eta^n_{*,\varnothing} : \mathrm{h}^n(*, \varnothing) \to \mathrm{k}^n(*, \varnothing)$ *is an isomorphism for all* n*, then* η *is an isomorphism of cohomology theories.*

(ii) Let $\widetilde{\mathrm{h}}^*$ *and* $\widetilde{\mathrm{k}}^*$ *be two reduced cohomology theories on* CW_**, and suppose there is a morphism* $\eta : \widetilde{\mathrm{h}}^* \to \widetilde{\mathrm{k}}^*$ *of generalised cohomology theories in* $\mathsf{rCohomTh}$*. If* $\eta^n_{S^0} : \widetilde{\mathrm{h}}^n(S^0) \to \widetilde{\mathrm{k}}^n(S^0)$ *is an isomorphism for all* n*, then* η *is an isomorphism of reduced cohomology theories.*

Proof. (†) By Corollaries 5.12 and 5.15 both statements are equivalent, so we will only show the second. Let (K_n) and (K'_n) be two Ω-spectra that represent $\widetilde{\mathrm{h}}^*$ and $\widetilde{\mathrm{k}}^*$, respectively. This means that for each n, the natural transformation η^n induces a natural transformation $[\,\cdot\,, K_n]^{\bullet} \to [\,\cdot\,, K'_n]^{\bullet}$, which by the Yoneda Lemma is induced by a unique pointed homotopy class $[f_n]$ of maps $K_n \to K'_n$. Choose a particular such map for each n.

Since η^n commutes with the suspension isomorphism, we have $\eta^n_{S^m}$ to be an isomorphism for every $m \geq 0$. Therefore, f_n induces isomorphisms $[S^m, K_n]^{\bullet} \xrightarrow{\sim} [S^m, K'_n]^{\bullet}$ for each m. Write $\widehat{K}_n$ and $\widehat{K'}_n$ for the path-connected components of K_n and K'_n that contain the base point. By the Whitehead Theorem, $f_n|_{\widehat{K}_n} : \widehat{K}_n \to \widehat{K'}_n$ is a pointed homotopy equivalence. This means that $[f_n|_{\widehat{K}_n}] \circ - : [\,\cdot\,, \widehat{K}_n]^{\bullet} \xrightarrow{\sim} [\,\cdot\,, \widehat{K'}_n]^{\bullet}$ is a natural isomorphism of functors for each n. Also note that, by definition, the η^n commute with the suspension isomorphism, and therefore the maps $[f_n] \circ -$ commute with the suspension-loop adjunction.

Now let X be any pointed CW-complex. Since ΣX is path-connected, we find isomorphisms fitting in a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{h}}^n(X) \arrow[r, "\sim"] \arrow[d, "\eta^n_X"] & {[X, K_n]^\bullet} \arrow[r, "\sim"] \arrow[d, "{[f_n]\circ-}"] & {[\Sigma X, K_{n+1}]^\bullet} \arrow[r, "\sim"] \arrow[d, "{[f_{n+1}]\circ-}"] & {[\Sigma X, \widehat{K}_{n+1}]^\bullet} \arrow[d, "{[f_{n+1}|_{\widehat{K}_{n+1}}]\circ-}", "\wr"'] \\
\widetilde{\mathrm{k}}^n(X) \arrow[r, "\sim"] & {[X, K'_n]^\bullet} \arrow[r, "\sim"] & {[\Sigma X, K'_{n+1}]^\bullet} \arrow[r, "\sim"] & {[\Sigma X, \widehat{K'}_{n+1}]^\bullet}
\end{tikzcd}$$

Therefore, η^n_X is an isomorphism for all pointed CW-complexes X and all integers n, which shows that $\eta : \widetilde{\mathrm{h}}^* \to \widetilde{\mathrm{k}}^*$ is an isomorphism of reduced cohomology theories. □

Remark 6.64. The above theorems also holds when we replace ordinary, generalised or reduced cohomology theories with ordinary, generalised or reduced homology theories, as shown in [11]. (The axioms of a reduced homology theory are not important to us, but can be guessed from Definitions 1.46 and 5.1.) ▽

Remark 6.65. The above two uniqueness theorems, as well as the existence and uniqueness of Eilenberg-MacLane spaces can also be proven more constructively, without using the representability of cohomology, as is for instance done in [13] and [11]. In fact, when doing so, we can conclude that the Eilenberg-MacLane spaces using the same abelian group form an Ω-spectrum, and uniqueness of ordinary cohomology theories (and its reduced version) then allow us to conclude that it must represent singular cohomology. However, such a proof would require much work[5], while they follow relatively easily from the representability of cohomology.

Note also that if we take representability of cohomology as a given, that then the first uniqueness theorem reduces to showing that any pointed CW-complex X with homotopy groups concentrated in a single degree is unique up to pointed homotopy equivalence, while the second reduced to the Whitehead Theorem. In other words, we managed to reduce statements about cohomology to purely homotopy theoretic statements. This is exactly what the representability of cohomology says is possible. ▽

6.5 Two further examples of cohomology theories

We will end this thesis by giving two more examples of reduced cohomology theories defined on a subcategory of topological spaces, and address their representability. This section serves mainly to illustrate that there are more interesting cohomology theories in algebraic topology, which also can be representable, and give the examples of stable cohomotopy and topological K-theory. We only give a short introduction and omit proofs. We however would like to note that behind each of these examples lies a whole field of study.

Definition 6.66. Let X be a pointed topological space. The *n-th cohomotopy group* of X is the set $\pi^n(X) \coloneqq [X, S^n]^\bullet$. ◊

Generally, these cohomotopy groups actually do not carry group structures. A notable exception is of course when X is pointedly homotopy equivalent to a reduced suspension. This can be a motivation to consider the spaces $\pi^{n+k}(\Sigma^k X) = [\Sigma^k X, S^{n+k}]^\bullet$, which are groups for $k \geq 1$, and are even abelian if $k \geq 2$. The functor Σ provides us with natural maps $[Y, Z]^\bullet \to [\Sigma Y, \Sigma Z]^\bullet$ for all pointed spaces Y and Z, so we can form the colimit $\pi^n_s(X)$ of the diagram

$$\pi^n(X) \xrightarrow{\Sigma} \pi^{n+1}(\Sigma^1 X) \xrightarrow{\Sigma} \pi^{n+2}(\Sigma^2 X) \longrightarrow \dots,$$

which turns out to actually exist, to carry a natural abelian group structure, and, although we will not show it here, to constitute to a cohomology theory on CW_* [1]. We call π^*_s *stable cohomotopy theory*, and it turns out that it is represented by an Ω-spectrum QS^0, where $Q : \mathsf{Top}_* \to \Omega{-}\mathsf{spec}$ is a certain functor [1].

---

[5] While working on this thesis, I actually typed out a large part of the proof of uniqueness of cohomology, and it would have taken me at least five pages to show it completely. Existence and uniqueness of Eilenberg-MacLane spaces would require a similar amount of pages to show it directly.

Topological K-theory, on the other hand, measures in a sense how many vector bundles there exist over a given compact space. Of course, we will now need to define a vector bundle.

Definition 6.67. Let X be a topological space. A *complex vector bundle* over X consists of a topological space E, together with a continuous surjection $p : E \to X$ such that:

(i) For each $x \in X$, $E_x \coloneqq p^{-1}(x)$ carries the structure of an vector space over $\mathbb{C}$.

(ii) For each such x there exist an open neighbourhood $U \subseteq X$ of x and an integer $n \geq 0$ such that:

  (a) There is a homeomorphism φ fitting in the commutative diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "p"'] & & U \times \mathbb{C}^n \arrow[dl, "\mathrm{pr}_1"] \\ & X & \end{tikzcd}$$

  (b) For each $x' \in U$, the map $\varphi(x', \cdot\,) : p^{-1}(x') \to \{x'\} \times \mathbb{C}^n$ is a linear map (and consequently an isomorphism). ♢

If we denote by CHaus the full subcategory of Top consisting of compact spaces, then the set $\mathrm{Vect}^\bullet_{\mathbb{C}}(X)$ consisting of isomorphism classes of complex vector bundles over a compact space X actually carries a natural commutative monoid structure, and $\mathrm{Vect}^\bullet_{\mathbb{C}}$ turns out to be a functor $\mathsf{CHaus}^{\mathrm{opp}} \to \mathsf{CMon}$ [12]. Even more is true: homotopic maps induce the same homomorphisms of commutative monoids by a result shown in [2], so $\mathrm{Vect}^\bullet_{\mathbb{C}}$ factors through the homotopy category $\mathsf{hCHaus}^{\mathrm{opp}}$.

Now, the forgetful functor $U : \mathsf{Ab} \to \mathsf{CMon}$ admits a left adjoint $K : \mathsf{CMon} \to \mathsf{Ab}$, known as the *Grothendieck construction* [2], and we can then define $K(X) \coloneqq K(\mathrm{Vect}^\bullet_{\mathbb{C}}(X))$. This gives us a homotopy invariant functor $K : \mathsf{CHaus}^{\mathrm{opp}} \to \mathsf{Ab}$. Now, for a pointed compact space X, with $i : * \hookrightarrow X$ the inclusion, we set $\widetilde{K}X \coloneqq \ker(Ki)$.

We now define $\widetilde{K}^{2n} = \widetilde{K}$ and $\widetilde{K}^{2n+1} = \widetilde{K} \circ \Sigma$ for $n \in \mathbb{Z}$. This is the reduced cohomology theory on CHaus_* (with the exact sequence axiom stated slightly differently than we did it for pointed CW-complexes) which is called *(reduced) topological K-theory* [12].

We cannot apply the Brown Representability Theorem to conclude that it is representable, since it is not defined for all CW-complexes. However, it does turn out to be representable [1]. To describe the representing objects, let $BU(n) = \mathrm{G}(\mathbb{C}^\infty, n)$ be an infinite Grassmanian, if desired defined as the colimit of the diagram

$$\begin{tikzcd} \mathrm{G}(\mathbb{C}^n, n) \arrow[r] & \mathrm{G}(\mathbb{C}^{n+1}, n) \arrow[r] & \mathrm{G}(\mathbb{C}^{n+2}, n) \arrow[r] & \dots \end{tikzcd}$$

There exist natural maps $BU(n) \to BU(n+1)$ sending $V \subseteq \mathbb{C}^\infty$ to $V \oplus \mathbb{C} \subseteq \mathbb{C}^\infty \oplus \mathbb{C} \cong \mathbb{C}^\infty$, and their colimit gives us a space BU. It turns out that K^0 is represented by $BU \times \mathbb{Z}$, and therefore $\widetilde{K}^1$ is represented by $\Omega(BU \times \mathbb{Z})$ [1].

There is something called *Bott periodicity*, which states that there is a pointed weak homotopy equivalence $BU \times \mathbb{Z} \to \Omega^2(BU \times \mathbb{Z})$ [3]. This gives us an Ω-spectrum

$$\begin{tikzcd} \dots \arrow[r] & \Omega(BU \times \mathbb{Z}) \arrow[r] & BU \times \mathbb{Z} \arrow[r] & \Omega(BU \times \mathbb{Z}) \arrow[r] & BU \times \mathbb{Z} \arrow[r] & \dots \end{tikzcd}$$

that represents K-theory on CHaus_* [1].

Discussion and conclusion

In this thesis, we have given an introduction to cohomology theories on CW-complexes, and have shown their representability by an Ω-spectrum. As such, we have obtained a strong link between cohomology on the one hand, and homotopy theory of CW-complexes on the other.

Cohomology originated in a desire to measure holes in topological spaces in order to define homotopy invariants on topological spaces. These are of use in particular because they can help us determine if two given spaces are homotopy equivalent. We covered singular homology first and then dualised its construction to obtain singular cohomology as an important example of cohomology, and have shown a few of its properties. These properties were then used to define what we wanted a generalised cohomology theory to be, and as such, by studying singular cohomology we already gained some familiarity with generalised cohomology theories.

In order to show such theories are representable, we restricted ourselves to CW-complexes, which we then introduced. They turned out to satisfy quite a few pleasant properties with regards to homotopy theory, such as the homotopy extension property, but also results like the Cellular Approximation Theorem and the Whitehead Theorem. These results, and the CW-approximation Theorem, all were very important to us in later parts, and justified devoting a separate chapter to them.

Reduced cohomology theories on pointed CW-complexes, and in particular the equivalence of categories between the reduced and unreduced generalised cohomology theories on the CW-complexes, allowed us to pass the question of representability of generalised cohomology on these spaces to a question of representability of the reduced cohomology theories. This was an important step, as it allowed us to apply the Brown Representability Theorem, using which we finally showed that all the reduced cohomology theories on pointed CW-complexes are representable by an Ω-spectrum, and therefore that all unreduced cohomology theories on CW-pairs are as well.

The Brown Representability Theorem holds, as we saw, for more general functors than the ones in a cohomology theory alone, and we saw in our proof of the CW-approximation Theorem one example of that. We also studied the Eilenberg-MacLane spaces, that represent (reduced) singular cohomology, in some detail, and drew a few conclusions about them and singular cohomology. In cohomology operations, namely, lies part of the study of additional algebraic structure on singular cohomology. Lastly, we gave two other examples of representable cohomology theories on certain topological spaces, next to singular cohomology.

The proof of the Brown Representability Theorem used many abstract concepts from category theory and ideas that could be captured or generalised in that language. This was already noted by Brown himself in [5], and he defined a type of pairs of categories which he called a *homotopy category*, which admits an abstract definition of a Brown functor, which are then shown to be representable. This provides us with a generalisation of the theorem to a multitude of situations.

Also, Ω-spectra are special cases of something called *spectra*. A spectrum is a sequence $(K_n)_{n\geq 0}$ of pointed topological spaces together with maps (not necessarily weak homotopy equivalences) $\varphi_n : \Sigma K_n \to K_{n+1}$, and via the suspension loop adjunction, each Ω-spectrum defines a spectrum. The interplay between spectra and Ω-spectra in particular is a central topic in modern algebraic topology, with books like [15], [1], and [17] being devoted to it. The functor $Q : \mathsf{Top}_* \to \Omega{-}\mathsf{spec}$ which we mentioned when covering stable cohomotopy theory in Section 6.5 actually arises in that context, for instance in [17]. Spectra can be used to produce both cohomology and homology theories, and are needed in order to talk about the representability of generalised homology theories, which we did not do in this thesis [7].

Next to stable cohomotopy, there is also stable homotopy theory, and determining the way it acts on the spheres is also a central topic in modern algebraic topology, with [7] calling the computation of the stable homotopy groups of S^0 "the holy grail of homotopy theory".

Topological K-theory was one of the first generalised cohomology theories that were stuied extensively, and has a wide range of applications both within and outside of algebraic topology [12]. We only briefly mentioned its main idea, but it would certainly deserve to have a separate thesis written about it.

Lastly, we would like to note that cohomology can also be studied further by using *spectral sequences*. We will not explain what they are here, but instead refer to [13] for an exposition. It also provides ways to show certain results about homotopy theory that we already encountered in this thesis, such as the Hurewicz Theorem.

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Even though I can thank others for proofreading this text, undoubtedly there are still typos and other errors. Any mistakes appearing in this thesis are of course wholly my own.

Appendix A

Category theory

Category theory is absolutely indispensable in many areas of (theoretical) mathematics. Originally founded in the context of abstract algebra and algebraic topology [22], it has shown to capture many different mathematical ideas in a unified language, and it enables mathematicians to look straight at the fundamentals of the matter. Here, we quickly give an overview of the material that is needed in this thesis.

A.1 Categories and functors

The idea of a category originates from the observation that in (theoretical) mathematics, we often study objects that have a certain structure between them, and do that by in turn studying maps between those objects that preserve or respect that structure in some sense. To get an overview of these objects and maps, we arrive at the following definition. All the definitions and propositions in this appendix can be found in [22]. We omit many proofs, but it should be noted that many of them are actually quite doable for any reader.

Definition A.1. A *category* C is an object that consists of

(i) a collection $\operatorname{ob}\mathsf{C}$ of *objects*,
(ii) for any two objects $X, Y \in \operatorname{ob}\mathsf{C}$, a collection $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ of *morphisms* between X and Y,
(iii) for every object $X \in \operatorname{ob}\mathsf{C}$ an *identity morphism* $\operatorname{id}_X \in \operatorname{Hom}_{\mathsf{C}}(X, X)$,
(iv) for any three objects $X, Y, Z \in \operatorname{ob}\mathsf{C}$ a map

$$\circ : \operatorname{Hom}_{\mathsf{C}}(X, Y) \times \operatorname{Hom}_{\mathsf{C}}(Y, Z) \to \operatorname{Hom}_{\mathsf{C}}(X, Z), (f, g) \mapsto g \circ f,$$

called *composition* such that

(a) for all $X, Y \in \operatorname{ob}\mathsf{C}$ and every $f \in \operatorname{Hom}_{\mathsf{C}}(X, Y)$, it holds that $f \circ \operatorname{id}_X = f$ and $\operatorname{id}_Y \circ f = f$,
(b) for all $X, Y, Z, T \in \operatorname{ob}\mathsf{C}$ and every $f \in \operatorname{Hom}_{\mathsf{C}}(X, Y)$, $g \in \operatorname{Hom}_{\mathsf{C}}(Y, Z)$ and $h \in \operatorname{Hom}_{\mathsf{C}}(Z, T)$, it holds that $h \circ (g \circ f) = (h \circ g) \circ f$. ◊

Notation A.2. We will write $f : X \to Y$ for a morphism $f \in \operatorname{Hom}_{\mathsf{C}}(X, Y)$, and write $X \in \mathsf{C}$ instead of $X \in \operatorname{ob}\mathsf{C}$. We will also often shorten a composition $g \circ f$ to simply gf. ○

Remark A.3. In the above definition, the use of the word "collection" is intentional. We do not always want the size of categories or the morphisms between two objects to be limited by a set. Our definition suffices for our purposes, but the reader may ask how rigorous it is. For a proper investigation of the foundations of category theory, see [8]. ▽

Example A.4. In principle every collection of objects and arrows that satisfies the conditions in Definition A.1 is a category. However, there are some important examples (which were of course some of the reasons categories were introduced):

(i) The collection of all sets with maps of sets between them (and the identity and composition as usual) forms the category Set.

(ii) The collection of all abelian groups and homomorphisms between them forms the category Ab.

(iii) The collection of all topological spaces and continuous maps between them forms the category Top.

(iv) The collection of all pointed topological spaces and pointed continuous maps between them forms the category Top_*. Recall that a pointed topological space is a pair $(X, *)$, where X is a topological space and $* \in X$. A pointed continuous map $f : (X, *) \to (Y, *)$ is a continuous map $f : X \to Y$ such that $f(*) = *$.

(v) Similarly, the collection of all pointed sets and pointed maps between them forms the category Set_*.

(vi) The collection of all topological spaces and homotopy classes of continuous maps between them forms the *homotopy category of topological spaces* hTop[1]. That is, for two topological spaces X and Y, we set $\mathrm{Hom}_{\mathsf{hTop}}(X, Y) = \{f : X \to Y \mid f \text{ is continuous}\}/\sim$, and take the composition law $[g] \circ [f] := [gf]$ for two composable morphisms f and g in Top. We know from topology that this is well-defined.

(vii) The *empty category* $\mathsf{0}$ is the category without objects or morphisms. The *trivial category* $\mathsf{1}$ is the category with one object and one morphism, namely the identity on that object. △

Definition A.5. Let C be a category. C is *locally small* if for any two objects $X, Y \in \mathsf{C}$, $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is a set, and C is *small* if furthermore $\mathrm{ob}\,\mathsf{C}$ is a set. ◊

Definition A.6. Let C' and C be two categories. Then C' is a *subcategory* of C if $\mathrm{ob}\,\mathsf{C}'$ forms a subcollection of $\mathrm{ob}\,\mathsf{C}$, and for any two $X, Y \in \mathsf{C}'$, $\mathrm{Hom}_{\mathsf{C}'}(X, Y)$ is a subcollection of $\mathrm{Hom}_{\mathsf{C}}(X, Y)$. ◊

Definition A.7. Let C and D be two categories. The *product category* $\mathsf{C} \times \mathsf{D}$ is the category that consists of pairs (X, Y) of an object $X \in \mathsf{C}$ and an object $Y \in \mathsf{D}$, and with morphisms $(f, g) : (X_1, Y_1) \to (X_2, Y_2)$ that consist of a morphism $f : X_1 \to X_2$ in C and a morphism $g : Y_1 \to Y_2$ in D. ◊

Definition A.8. Let a category C be given. The *opposite category* $\mathsf{C}^{\mathrm{opp}}$, also called the *dual category*, is defined as $\mathrm{ob}\,\mathsf{C}^{\mathrm{opp}} = \mathrm{ob}\,\mathsf{C}$, and for any two $X, Y \in \mathsf{C}^{\mathrm{opp}}$, we set $\mathrm{Hom}_{\mathsf{C}^{\mathrm{opp}}}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X)$. Composition is defined for $X, Y, Z \in \mathrm{Hom}_{\mathsf{C}^{\mathrm{opp}}}(X, Y)$ as $\mathrm{Hom}_{\mathsf{C}^{\mathrm{opp}}}(X, Y) \times \mathrm{Hom}_{\mathsf{C}^{\mathrm{opp}}}(Y, Z) \to \mathrm{Hom}_{\mathsf{C}^{\mathrm{opp}}}(X, Z), (f, g) \mapsto f \circ g$ (where $\circ$ is composition in C). ◊

Remark A.9. We should note that this duality in the definition of a category means that each categorical construction has a "co-construction" obtained by reversing all the arrows. Similarly, every result has a co-result, which can be proved by reversing all the arrows in the proof of the original statement. This allows us to prove two results at once each time. Most of the time, we omit the dual construction, definition or result, but sometimes, when we think it is important enough, mention the dual concept. ▽

Definition A.10. Let C be a category and $f : X \to Y$ a morphism in C. Then f is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_y$. If f is an isomorphism, this map g is called its *inverse* and is often denoted by f^{-1}. Two objects X and Y in C are *isomorphic* if there exists an isomorphism $X \to Y$, and we sometimes write $X \cong Y$. ◊

Example A.11. (i) In any category C, the identity morphisms id_X are isomorphisms $X \to X$ for any $X \in \mathsf{C}$.

(ii) An isomorphism in Set is a bijection of sets, and an isomorphism in Top is a homeomorphism. In hTop, an isomorphism is a homotopy equivalence between topological spaces.

(iii) The composition of two composable isomorphisms in a category C is again an isomorphism. The inverse f^{-1} of an isomorphism f is also an isomorphism, and $(f^{-1})^{-1} = f$. △

The above definition of an isomorphism applies to all categories, so we do not need to explicitly define isomorphisms between specific objects that form a particular category anymore. However, we will still sometimes do this, just to also give a bit more classical introduction to new material for readers that are not that used to category theory.

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[1] This terminology is not entirely standard. Since I am simply a Backelor's student, I like to call it this, but others sometimes like to call it the *naive homotopy category*, since a subcategory (see Definition A.6) of it is actually more natural and convenient to study, which is then dubbed the *(true) homotopy category*. We will, as is clear, not adopt this convention.

Categories are not only used to get an overview of objects and structure respecting maps between them. Categories themselves are mathematical objects, with the structure of objects, arrows and composition. To study category theory, or describe constructions we come across in other mathematics, we would like to have a sense of a structure respecting map between categories. The following definition gives us just that.

Definition A.12. Let C and D be two categories. A *(covariant) functor* F between C and D is an object consisting of

(i) for any object $X \in \mathsf{C}$, an object $F(X) \in \mathsf{D}$,

(ii) for any morphism $f : X \to Y$ in C, a morphism $F(f) : F(X) \to F(Y)$ in D, such that
  (a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathsf{C}$,
  (b) $F(gf) = F(g)F(f)$ for all composable morphisms f and g in C.

We write this as $F : \mathsf{C} \to \mathsf{D}$.

A *contravariant functor* F between C and D is a covariant functor $F : \mathsf{C}^{\mathrm{opp}} \to \mathsf{D}$. We again write this as $F : \mathsf{C} \to \mathsf{D}$. ♢

When we refer to a functor, it is understood that we talk about a covariant functor. A contravariant functor $\mathsf{C} \to \mathsf{D}$ is either explicitly called that, or referred to as a functor $\mathsf{C}^{\mathrm{opp}} \to \mathsf{D}$. In cases where it cannot cause confusion, we shorten $F(X)$ and $F(f)$ to FX and Ff, respectively.

Example A.13. (i) Any category C has an identity functor id_{C} that sends every object X and morphism f to itself, that is, $\mathrm{id}_{\mathsf{C}}X = X$ and $\mathrm{id}_{\mathsf{C}}f = f$.

(ii) For any locally small category C, there are three functors baring the name *Hom-functor*. Firstly, there is for any $C \in \mathsf{C}$ the covariant Hom-functor $\mathrm{Hom}_{\mathsf{C}}(C, \cdot\,) : \mathsf{C} \to \mathsf{Set}, X \mapsto \mathrm{Hom}_{\mathsf{C}}(C, X)$, which sends a morphism $f : X \to Y$ in C to the morphism

$$\mathrm{Hom}_{\mathsf{C}}(C, f) = f \circ - : \mathrm{Hom}_{\mathsf{C}}(C, X) \to \mathrm{Hom}_{\mathsf{C}}(C, Y), h \mapsto f \circ h.$$

(See Figure A.1.) Secondly, there is for any $C \in \mathsf{C}$ the contravariant Hom-functor $\mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, C) : \mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}, X \mapsto \mathrm{Hom}_{\mathsf{C}}(X, C)$, which sends a morphism $f : X \to Y$ in C to the morphism

$$\mathrm{Hom}_{\mathsf{C}}(f, C) = - \circ f : \mathrm{Hom}_{\mathsf{C}}(Y, C) \to \mathrm{Hom}_{\mathsf{C}}(X, C), h \mapsto h \circ f.$$

(See Figure A.1.) Lastly, there is the bifunctor $\mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, \cdot\,) : \mathsf{C}^{\mathrm{opp}} \times \mathsf{C} \to \mathsf{Set}, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{C}}(X, Y)$, which sends morphisms $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ in C to the morphism

$$\mathrm{Hom}_{\mathsf{C}}(f, g) = g \circ - \circ f : \mathrm{Hom}_{\mathsf{C}}(X_2, Y_1) \to \mathrm{Hom}_{\mathsf{C}}(X_1, Y_2), h \mapsto ghf.$$

(See Figure A.1.) This Hom-functor is contravariant in the first argument, and covariant in the second.

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[dr, dashed] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"] & C \\ Y \arrow[ur, "h"] & \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "h"] & Y_1 \arrow[d, "g"] \\ X_1 \arrow[u, "f"] \arrow[r, dashed] & Y_2 \end{tikzcd}$$

Figure A.1: The induced maps of the Hom-functors.

(iii) There are many examples of so-called *forgetful functors*. These are functors that “forget” part of the structure of objects, essentially leaving the category unchanged: it sends objects and morphisms to themselves, but in a category with less structure in it. Examples are $\mathsf{Ab} \to \mathsf{Set}$, which sends an abelian group to the underlying set, and $\mathsf{Top}_* \to \mathsf{Top}$, which forgets the choice of base point.

(iv) Any commutative diagram of sets, topological spaces, groups, rings, etc. can be considered to be a functor. To give an example, suppose that

$$\begin{tikzcd} X_1 \arrow[r, "f_3"] \arrow[d, "f_2"] & X_2 \arrow[dl, "f_1"] \\ X_3 & \end{tikzcd}$$

is a commutative diagram in some category C. If we let S be the formal category

$$\begin{tikzcd}
1 \arrow[r, "\varphi_3"] \arrow[d, "\varphi_2"] & 2 \arrow[dl, "\varphi_1"] \\
3 &
\end{tikzcd}$$

(where we omitted the identity morphisms) such that $\varphi_1\varphi_3 = \varphi_2$, then the first diagram is a functor $X : \mathsf{S} \to \mathsf{C}, i \mapsto X_i$ which sends a morphism φ_i to f_i.

(v) The composition of two functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ is the map $GF = G \circ F : \mathsf{C} \to \mathsf{E}$, which sends an object $X \in \mathsf{C}$ to $GFX \in \mathsf{E}$, and a morphism $f : X \to Y$ in C to the morphism $GFf : GFX \to GFY$ in E. This is easily seen to be a functor, and for any $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{E}$, and $H : \mathsf{E} \to \mathsf{K}$, it satisfies $H(GF) = (HG)F$ and $F \circ \mathrm{id}_{\mathsf{C}} = \mathrm{id}_{\mathsf{D}} \circ F$. △

Proposition A.14. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between two categories, and suppose $\mathcal{C}$ is a commutative diagram of objects and arrows in C. Then its image $F(\mathcal{C})$ is a commutative diagram in D.* ☐

Corollary A.15. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor, and suppose $f : X \to Y$ is an isomorphism in C. Then $Ff : FX \to FY$ is an isomorphism in D.* ☐

The following definition allows us to state a partial converse to Proposition A.14, but it will turn out that it in itself is also an important concept.

Definition A.16. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between locally small categories. Then

(i) F is *full* if the assignment $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(FX, FY), f \mapsto Ff$ is surjective for all $X, Y \in \mathsf{C}$,

(ii) F is *faithful* if the assignment $\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(FX, FY), f \mapsto Ff$ is injective for all $X, Y \in \mathsf{C}$.

A functor that is both full and faithful is called *fully faithful*. ♢

Definition A.17. A subcategory is called *full* if the inclusion functor is full. ♢

The following lemma is not difficult to show and a good exercise for a reader that is new to category theory, and hence we omit the proof.

Lemma A.18. *Let $F : \mathsf{C} \to \mathsf{D}$ be a fully faithful functor between two locally small categories. Given a commutative diagram $\mathcal{D}$ of objects in the image $F(\mathsf{C})$, and arrows in D, there is a commutative diagram $\mathcal{C}$ in C that is mapped by F on $\mathcal{D}$.* ☐

Corollary A.19. *Let $F : \mathsf{C} \to \mathsf{D}$ be a fully faithful functor between two locally small categories, and let $X, Y \in \mathsf{C}$. Suppose $h : FX \to FY$ is an isomorphism in D. Then there is a unique isomorphism $f : X \to Y$ in C such that $Ff = h$.* ☐

Corollary A.20. *Let $F : \mathsf{C} \to \mathsf{D}$ be a fully faithful functor between two locally small categories, and let $Y \in F(\mathsf{C})$. Then there is up to isomorphism a unique $X \in \mathsf{C}$ such that $FX = Y$. In particular, given a commutative diagram $\mathcal{D}$ in D, there is a commutative diagram $\mathcal{C}$ in C such that $F(\mathcal{C}) = \mathcal{D}$, and it is unique up to isomorphism in its objects, and once the objects are chosen, unique in its morphisms.* ☐

A.2 Natural transformations and the Yoneda Lemma

Definition A.21. Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. A *natural transformation* or *morphism* η from F to G is an object that consists of the following: for each object $X \in \mathsf{C}$ a morphism $\eta_X : FX \to GX$ in D, such that for every morphism $f : X \to Y$ in C the square

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\eta_X"] & FY \arrow[d, "\eta_Y"] \\
GX \arrow[r, "Gf"] & GY
\end{tikzcd}$$

commutes in D. We write this as $\eta : F \to G$. ♢

Note that we can compose natural transformations by composing the morphisms they consist of: given two natural transformations $\eta : F \to G$ and $\varepsilon : G \to H$, their composition is the natural transformation $\varepsilon\eta : F \to H$ defined by $(\varepsilon\eta)_X = \varepsilon_X \eta_X$ (this is easily seen to be a natural transformation). There is also an identity transformation $\mathrm{id}_F : F \to F$, consisting of maps $\mathrm{id}_{FX} : FX \to FX$. In this way, we arrive at the so-called functor category.

Definition A.22. Let C and D be two categories. The *functor category* $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is the category that has functors $F : \mathsf{C} \to \mathsf{D}$ as objects, and natural transformations $\eta : F \to G$ between these functors as morphisms. For two functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{C} \to \mathsf{D}$ in $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$, we write $\mathrm{Nat}(F, G) = \mathrm{Hom}_{\mathsf{Fun}(\mathsf{C},\mathsf{D})}(F, G)$. ◊

Example A.23. For any category C and shape of a commutative diagram, there exists a category consisting of commutative diagrams in C that have that shape. As we remarked in Example A.13(vii), any commutative diagram of a fixed shape can be made a functor $X : \mathsf{S} \to \mathsf{C}$ from some small formally commutative category S, which determines the shape of the diagram. The *category of commutative diagrams in* C *of shape* S is then the functor category $\mathsf{Fun}(\mathsf{S}, \mathsf{C})$. △

In the light of Definition A.10, the following definition is obsolete, but we give it anyway.

Definition A.24. Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. A natural transformation $\eta : F \to G$ is an *isomorphism* between F and G if there exists a natural transformation $\varepsilon : G \to F$ such that $\varepsilon\eta = \mathrm{id}_F$ and $\eta\varepsilon = \mathrm{id}_G$. In this case F and G are said to be *(naturally) isomorphic*, and we sometimes write $F \cong G$. ◊

Lemma A.25. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. A natural transformation $\eta : F \to G$ is an isomorphism if and only if $\eta_X : FX \to GX$ is an isomorphism for each $X \in \mathsf{C}$.* ☐

Definition A.26. A functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence of categories* if there exists a functor $G : \mathsf{D} \to \mathsf{C}$ and two natural isomorphisms $FG \cong \mathrm{id}_{\mathsf{D}}$ and $GF \cong \mathrm{id}_{\mathsf{C}}$. If such an equivalence of categories exists, C and D are *equivalent categories.* ◊

Definition A.27. A functor $F : \mathsf{C} \to \mathsf{D}$ is *essentially surjective* if for every $D \in \mathsf{D}$, there is a $C \in \mathsf{C}$ such that $FC \cong D$. ◊

Proposition A.28. *A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. The proof can be found in [24]. ☐

Proposition A.29. *Let C, D and E be locally small categories, let $F : \mathsf{D} \to \mathsf{E}$ be a fully faithful functor, and let $G_1, G_2 : \mathsf{C} \to \mathsf{D}$ be two functors. Suppose there is a natural isomorphism $FG_1 \cong FG_2$. Then there is a natural isomorphism $G_1 \cong G_2$.*

Proof. A natural isomorphism $FG_1 \cong FG_2$ consists of commutative diagrams in $F(\mathsf{D})$ for each morphism $f : X \to Y$ in C, so Lemma A.18 gives us an essentially unique natural transformation $G_1 \to G_2$. By Lemma A.25 and Corollary A.19 this natural transformation is in fact an isomorphism. ☐

Notation A.30. Let C be a locally small category. For an object $C \in \mathsf{C}$, we write

$$h_C \coloneqq \mathrm{Hom}_{\mathsf{C}}(C, \cdot\,) : \mathsf{C} \to \mathsf{Set}, \text{ and } h^C \coloneqq \mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, C) : \mathsf{C}^{\mathrm{opp}} \to \mathsf{Set} \qquad ○$$

Remark A.31. There is a functor $h : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Set}) : X \mapsto h^X$, which sends a morphism $f : X \to Y$ to the natural transformation $h^f : h^X \to h^Y$ consisting for each $Z \in \mathsf{C}$ of the morphism $(h^f)_Z : \mathrm{Hom}_{\mathsf{C}}(Z, X) \to \mathrm{Hom}_{\mathsf{C}}(Z, Y), g \mapsto fg$.

Likewise, there is a functor $h : \mathsf{C}^{\mathrm{opp}} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set}) : X \mapsto h_X$, which sends a morphism $f : X \to Y$ to the natural transformation $h_f : h_Y \to h_X$ consisting for each $Z \in \mathsf{C}$ of the morphism $(h_f)_Z : \mathrm{Hom}_{\mathsf{C}}(Y, Z) \to \mathrm{Hom}_{\mathsf{C}}(X, Z), g \mapsto gf$. ▽

Definition A.32. Let C be a locally small category, and let $F : \mathsf{C} \to \mathsf{Set}$ and $G : \mathsf{C}^{\text{opp}} \to \mathsf{Set}$ be two functors. Then F is *representable* if there is an object $C \in \mathsf{C}$ such that there is a natural isomorphism $F \xrightarrow{\sim} h_C$, and then C is said to *represent* F. G is *co-representable* if there is an object $C \in \mathsf{C}$ such that there is a natural isomorphism $G \xrightarrow{\sim} h^C$, and then C is said to *co-represent* G. ♢

Theorem A.33. (Yoneda Lemma) *Let C be a locally small category. There is a natural isomorphism*

$$\mathrm{Nat}(h_{(\,\cdot\,)}, -) \xrightarrow{\sim} -(\,\cdot\,)$$

of functors $\mathsf{C} \times \mathsf{Fun}(\mathsf{C}, \mathsf{Set}) \to \mathsf{Set}$. Explicitly, for $X \in \mathsf{C}$ and a functor $F : \mathsf{C} \to \mathsf{Set}$, the bijection of sets is given by

$$\mathrm{Nat}(h_X, F) \xrightarrow{\sim} FX, (\eta : h_X \to F) \mapsto \eta_X(\mathrm{id}_X)$$

with inverse

$$FX \xrightarrow{\sim} \mathrm{Nat}(h_X, F), x \mapsto (\eta_Y : h_X Y \to FY, f \mapsto Ff(x))_{Y \in \mathsf{C}}.$$

In particular, the functor $h : \mathsf{C}^{\text{opp}} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set}) : X \mapsto h_X$ is fully faithful.

Proof. The proof is doable for the interested reader, but a bit long. It can be found in [22]. ☐

Theorem A.34. (Dual Yoneda Lemma) *Let C be a locally small category. There is a natural isomorphism*

$$\mathrm{Nat}(h^{(\,\cdot\,)}, -) \xrightarrow{\sim} -(\,\cdot\,)$$

of functors $\mathsf{C}^{\text{opp}} \times \mathsf{Fun}(\mathsf{C}^{\text{opp}}, \mathsf{Set}) \to \mathsf{Set}$. Explicitly, for $X \in \mathsf{C}$ and a functor $F : \mathsf{C}^{\text{opp}} \to \mathsf{Set}$, the bijection of sets is given by

$$\mathrm{Nat}(h^X, F) \xrightarrow{\sim} FX, (\eta : h^X \to F) \mapsto \eta_X(\mathrm{id}_X)$$

with inverse

$$FX \xrightarrow{\sim} \mathrm{Nat}(h^X, F), x \mapsto (\eta_Y : h^X Y \to FY, f \mapsto Ff(x))_{Y \in \mathsf{C}}.$$

In particular, the functor $h : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\text{opp}}, \mathsf{Set}) : X \mapsto h^X$ is fully faithful. ☐

Corollary A.35. *Let C be a locally small category, and suppose there is for two objects $X, Y \in \mathsf{C}$ a natural isomorphism $h_X \xrightarrow{\sim} h_Y$. Then X and Y are isomorphic.*

Proof. This follows from the fact that $h : \mathsf{C}^{\text{opp}} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set}) : X \mapsto h_X$ is fully faithful, and Corollary A.19. ☐

Corollary A.36. *Let $F : \mathsf{C} \to \mathsf{Set}$ be a representable functor. Then the object C that represents F is unique up to isomorphism.* ☐

A.3 Adjunctions, limits and colimits

Definition A.37. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be two functors between locally small categories. An *adjunction* between F and G is a natural isomorphism

$$\alpha : \mathrm{Hom}_{\mathsf{D}}(F(\,\cdot\,)), \,\cdot\,) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, G(\,\cdot\,)).$$

In this case, F is *left adjoint* to G, and G is *right adjoint* to F, and sometimes write $F \dashv G$. ♢

Unwinding the definitions (and using Lemma A.25), we see that an adjunction from F to G consists of bijections of sets

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathsf{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(X, GY)$$

for each $X \in \mathsf{C}$ and $Y \in \mathsf{D}$, which are natural in both X and Y.

Example A.38. Adjoint functors are truly all around in mathematics, but we will for reasons of length only give two examples that are of interest to us.

(i) For any set Y, the functor $- \times Y : \mathsf{Set} \to \mathsf{Set}$ is left adjoint to the Hom-functor $\mathrm{Hom}_{\mathsf{Set}}(Y, \cdot\,) : \mathsf{Set} \to \mathsf{Set}$. The bijection of Hom-sets sends a map $f : X \times Y \to Z$ to the map $X \to \mathrm{Hom}_{\mathsf{Set}}(Y, Z), x \mapsto f(x, \cdot\,)$, and inversely, a map $g : X \to \mathrm{Hom}_{\mathsf{Set}}(Y, Z)$ to a map $X \times Y \to Z, (x, y) \mapsto g(x)(y)$. This adjunction is also known as the exponential law of sets: if we let Z^Y denote $\mathrm{Hom}_{\mathsf{Set}}(Y, Z)$, then the adjunction says that there is a bijection $Z^{(X \times Y)} \cong (Z^Y)^X$, which is natural in both X and Z.

(ii) The forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$ is left adjoint to the functor $\cdot_{\mathrm{triv}}$ which sends a set X to the topological space X with trivial (indiscrete) topology. Furthermore, U is right adjoint to the functor $\cdot_{\mathrm{disc}}$ which sends a set X to the topological space X with discrete topology. △

Proposition A.39. *Let* C *and* D *be two locally small categories, and suppose a functor* $G : \mathsf{D} \to \mathsf{C}$ *is right adjoint to both the functors* $F_1 : \mathsf{C} \to \mathsf{D}$ *and* $F_2 : \mathsf{C} \to \mathsf{D}$. *Then* F_1 *and* F_2 *are naturally isomorphic. Dually, right adjoints are also unique up to natural isomorphism.*

Proof. There are by definition of adjointess natural isomorphisms

$$h_{(\,\cdot\,)}F_1 = \mathrm{Hom}_{\mathsf{D}}(F_1(\,\cdot\,), \,\cdot\,) \cong \mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, G(\,\cdot\,)) \cong \mathrm{Hom}_{\mathsf{D}}(F_2(\,\cdot\,), \,\cdot\,) = h_{(\,\cdot\,)}F_2$$

By the Yoneda Lemma, $h : \mathsf{C}^{\mathrm{opp}} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set}) : X \mapsto h_X$ is fully faithful, and therefore Proposition A.29 applies and immediately yields an isomorphism $F_1 \cong F_2$. The dual statement is proved similarly. ☐

Convention A.40. In what follows, S will be a small category. ⊙

Let C be a category, and $X : \mathsf{S} \to \mathsf{C}$ a functor. For an object $s \in \mathsf{S}$, we will shorten $X(s)$ to X_s.

Definition A.41. Let C be a category. The *limit* of a functor $X : \mathsf{S} \to \mathsf{C}$ consists of

(i) an object $\lim_{\mathsf{S}} X$ of C,

(ii) for every object $s \in \mathsf{S}$ a morphism $\pi_s : \lim_{\mathsf{S}} X \to X_s$,

subject to the conditions

(i) for every arrow $\varphi : s_1 \to s_2$ in S, it holds that $\pi_{s_2} = X(\varphi) \circ \pi_{s_1}$;

(ii) for every $T \in \mathsf{C}$ and for every family of arrows $t_s : T \to X_s$ such that $t_{s_2} = X(\varphi) \circ t_{s_1}$ for all $\varphi : s_1 \to s_2$ in S, there is a *unique* morphism $h : T \to \lim_{\mathsf{S}} X$ satisfying $t_s = \pi_s h$ for all $s \in \mathsf{S}$. ◊

Definition A.42. Let C be a category. The *colimit* of a functor $X : \mathsf{S} \to \mathsf{C}$ consists of

(i) an object $\mathrm{colim}_{\mathsf{S}} X$ of C,

(ii) for every object $s \in \mathsf{S}$ a morphism $\iota_s : X_s \to \mathrm{colim}_{\mathsf{S}} X$,

subject to the conditions

(i) for every arrow $\varphi : s_1 \to s_2$ in S, it holds that $\iota_{s_2} \circ X(\varphi) = \iota_{s_1}$;

(ii) for every $T \in \mathsf{C}$ and for every family of arrows $t_s : X_s \to T$ such that $t_{s_2} \circ X(\varphi) = t_{s_1}$ for all $\varphi : s_1 \to s_2$ in S, there is a *unique* morphism $h : \mathrm{colim}_{\mathsf{S}} X \to T$ satisfying $t_s = h\iota_s$ for all $s \in \mathsf{S}$. ◊

Notation A.43. We will write $\lim_{\mathsf{S}} X$, $\lim_{\mathsf{S}} X_s$ or $\lim_s X_s$ for the limit of a functor $X : \mathsf{S} \to \mathsf{C}$, and leave the arrows in C or S implicit. In the same way, we write $\mathrm{colim}_{\mathsf{S}} X$, $\mathrm{colim}_{\mathsf{S}} X_s$ or $\mathrm{colim}_s X_s$ for the colimit of a functor $X : \mathsf{S} \to \mathsf{C}$. ○

Proposition A.44. *Let* C *be a category, and* $X : \mathsf{S} \to \mathsf{C}$ *be a functor. If they exist,* $\lim_{\mathsf{S}} X$ *and* $\mathrm{colim}_{\mathsf{S}} X$ *are unique up to unique isomorphism in* C. ☐

Remark A.45. The limit or colimit of a functor need not exist. We will see later that it luckily always does in case $\mathsf{C} = \mathsf{Set}$, and in Propositions A.49 and A.50 we present a necessary and sufficient condition for a candidate for the (co)limit to actually be the (co)limit of a functor, at least in a locally small category. ▽

Example A.46. (i) Taking S to be a discrete category (that is, a category which only consists of objects and the identity morphisms on them), the limit and colimit of a functor $X : \mathsf{S} \to \mathsf{C}$ are known as the categorical *product* and *coproduct* of the objects X_s. In Set and Top, the product and coproduct are the cartesian product and disjoint union, respectively.

(ii) If S is the diagram

$$\begin{tikzcd} & 1 \arrow[d, "\varphi"] \\ 2 \arrow[r, "\psi"] & 3 \end{tikzcd}$$

then we call the limit of a functor $X : \mathsf{S} \to \mathsf{C}$ the *fibered product of* $X(\varphi)$ *and* $X(\psi)$. It is commonly denoted by $X_1 \times_{X_3} X_2$.

The colimit of a functor $X : \mathsf{S}^{\mathrm{opp}} \to \mathsf{C}$, that is, a diagram

$$\begin{tikzcd} X_3 \arrow[r, "X(\varphi)"] \arrow[d, "X(\psi)"] & X_1 \\ X_2 & \end{tikzcd}$$

in C, is called the *pushout* (or *fiber coproduct*).

(a) In Set, the fibered product of two maps $f : X \to Z$ and $g : Y \to Z$ is the set $X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$. The pushout of two maps $f : Z \to X$ and $g : Z \to Y$ is the set $(X \amalg Y)/\sim$, where $\sim$ is the equivalence relation generated by $f(z) \sim g(z)$.

(b) In Top, the fibered product of two maps $f : X \to Z$ and $g : Y \to Z$ is the set $X \times_Z \times Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ with the subspace topology inherited from the product space $X \times Y$. The pushout of two maps $f : Z \to X$ and $g : Z \to Y$ is the set $(X \amalg Y)/\sim$ with firstly the disjoint union topology, and secondly the quotient topology. Here $\sim$ is again the equivalence relation generated by $f(z) \sim g(z)$ for $z \in Z$.

(iii) If $\mathsf{S} = \mathsf{0}$, the empty category, then a limit reduces to an object $F \in \mathsf{C}$ such that for each $C \in \mathsf{C}$ there is precisely one morphism $C \mapsto F$. We call this a *final object*. The colimit is an object $I \in \mathsf{C}$ such that for each $C \in \mathsf{C}$ there is precisely one morphism $I \to C$. We call this an *initial object*, or a *co-final object*. Examples include various empty and one-point sets, spaces and algebraic objects. △

The following two propositions tell us that limits and colimits of functors $X : \mathsf{S} \to \mathsf{Set}$ always exist, and give explicit descriptions for them. Their proofs consist of simply checking this description satisfies the definition of the limit and colimit, and are hence omitted.

Proposition A.47. *Let* $X : \mathsf{S} \to \mathsf{Set}$ *be a functor. Then* $\lim_{\mathsf{S}} X$ *exists, and equals*

$$\left\{ (x_s)_{s\in\mathsf{S}} \in \prod_{s\in\mathsf{S}} X_s \mid X(\varphi)(x_{s_1}) = x_{s_2} \text{ for all arrows } \varphi : s_1 \to s_2 \right\}$$

together with the projections maps π_s *to the sets* X_s. □

Proposition A.48. *Let* $X : \mathsf{S} \to \mathsf{Set}$ *be a functor. Then* $\operatorname{colim}_{\mathsf{S}} X$ *exists, and equals on the level of objects*

$$\left(\coprod_{s\in\mathsf{S}} X_s \right) \Big/ \sim$$

where $\sim$ *is the equivalence relation generated by* $x_{s_1} \sim X(\varphi)(x_{s_1})$ *for all* $\varphi : s_1 \to s_2$ *and* $x_{s_1} \in X_{s_1}$. *Furthermore, it has as arrows the maps*

$$\iota_s : X_s \hookrightarrow \textstyle\coprod_{s\in\mathsf{S}} X_s \twoheadrightarrow \left(\coprod_{s\in\mathsf{S}} X_s\right)/\sim . \qquad \square$$

We can now give a necessary and sufficient condition for the existence of limits and colimits in locally small categories. Moreover, it also characterises limits and colimits in terms of universal properties, because of the (dual) Yoneda Lemma (and in particular Corollary A.35). Their proofs are not that difficult and consist of verifying definitions, but they can also be found in [24].

Proposition A.49. *Let* C *be a locally small category and* $X : \mathsf{S} \to \mathsf{C}$ *a functor. Let* $L \in \mathsf{C}$. *Then* $\lim_{\mathsf{S}} X$ *exists and is isomorphic to* L *if and only if there is a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, L) \cong \lim_{\mathsf{S}} \mathrm{Hom}_{\mathsf{C}}(\,\cdot\,, X_s)$$

of functors $\mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$. ☐

Proposition A.50. *Let* C *be a locally small category and* $X : \mathsf{I} \to \mathsf{C}$ *a functor. Let* $C \in \mathsf{C}$. *Then* $\operatorname{colim}_{\mathsf{S}} X$ *exists and is isomorphic to* C *if and only if there is a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{C}}(C, \,\cdot\,) \cong \lim_{\mathsf{S}^{\mathrm{opp}}} \mathrm{Hom}_{\mathsf{C}}(X_s, \,\cdot\,)$$

of functors $\mathsf{C} \to \mathsf{Set}$.

The last major result of this section and appendix is a very useful one, and illustrates the way category theory can summarise a lot of mathematical results in various fields of study.

Theorem A.51. *Let* $F : \mathsf{C} \to \mathsf{D}$ *and* $G : \mathsf{D} \to \mathsf{C}$ *be two functors between locally small categories, and suppose* F *is left adjoint to* G. *Then the following two statements hold true.*

(i) (Left adjoints commute with colimits) *Let* $X : \mathsf{S} \to \mathsf{C}$ *be a functor, and assume that* $\operatorname{colim}_{\mathsf{S}} X$ *exists in* C. *Then* $\operatorname{colim}_{\mathsf{S}}(FX)$ *exists in* D, *and there is an isomorphism*

$$\operatorname{colim}_{\mathsf{S}}(FX) \cong F(\operatorname{colim}_{\mathsf{S}} X).$$

(ii) (Right adjoints commute with limits) *Let* $X : \mathsf{S} \to \mathsf{D}$ *be a functor, and assume that* $\lim_{\mathsf{S}} X$ *exists in* D. *Then* $\lim_{\mathsf{S}}(FX)$ *exists in* C, *and there is an isomorphism*

$$\lim_{\mathsf{S}}(GX) \cong G(\lim_{\mathsf{S}} X).$$

Proof. [24] We only proof the first statement (since the second is its dual). By adjointness of F and G, and by Proposition A.50, there are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}}(F(\operatorname{colim}_{\mathsf{S}} X_s), \,\cdot\,) &\cong \mathrm{Hom}_{\mathsf{C}}(\operatorname{colim}_{\mathsf{S}} X_s, G(\,\cdot\,))\\ &\cong \lim_{\mathsf{S}^{\mathrm{opp}}} \mathrm{Hom}_{\mathsf{C}}(X_s, G(\,\cdot\,))\\ &\cong \lim_{\mathsf{S}^{\mathrm{opp}}} \mathrm{Hom}_{\mathsf{C}}(FX_s, \,\cdot\,),\end{aligned}$$

and by Proposition A.50 again, this implies the existence of $\operatorname{colim}_{\mathsf{S}}(FX)$ and the isomorphism $\operatorname{colim}_{\mathsf{S}}(FX) \cong F(\operatorname{colim}_{\mathsf{S}} X)$. ☐

Remark A.52. A functor that commutes with limits and colimits like in the theorem above is also called *continuous* and *cocontinuous*, respectively, for a clear reason [22]. Therefore, the theorem above states that left adjoints are cocontinuous and right adjoints are continuous. ▽

Example A.53. [24] The forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$ is both a left and a right adjoint. Therefore, any limit or colimit in Top can, if it exists, be constructed by giving the underlying limit or colimit of sets a suitable topology. One can check that the descriptions of the limit and colimit of sets in Propositions A.47 and A.48 can be made limits and colimits of topological spaces by equipping them with the initial topology with respect to the projection maps π_s and final topology with respect to the inclusion maps ι_s, repectively. Indeed, the underlying set ensures that the unique map from the universal property in the definition of the limit and colimit in fact exists, and it is continuous by the universal property of the initial and final topology. Therefore, *all limits and colimits exist in* Top! △

Appendix B

Homological algebra

In this appendix, we review the theory of short exact sequences of modules over rings, and use these to set up the theory of chain complexes and homological algebra. The latter is simply the study of homology from a purely algebraic perspective, which simplifies or at the least clarifies a lot of the proofs in algebraic topology, and (co)homology theory in general. In the third and last section, we state the Algebraic Universal Coefficient Theorem, which establishes a strong relation between homology and cohomology in particular cases.

We omit almost all the proofs in this appendix, but most of them can be shown to hold with rather straightforward methods, such as diagram chasing or verifying the obvious approach works.

B.1 Exact sequences of modules

Convention B.1. Throughout this section, R will be a ring (with unit). ⊙

Notation B.2. The category of left R-modules and R-linear maps is denoted by ${}_R\mathsf{Mod}$. ○

Notation B.3. Let R be a ring. Then we shorten $\operatorname{Hom}_{{}_R\mathsf{Mod}}(\,\cdot\,,\,\cdot\,)$ to $\operatorname{Hom}_R(\,\cdot\,,\,\cdot\,)$. ○

Definition B.4. [23] Let M be an R-module and S a set. The *M-linearisation of S*, denoted by $M[S]$, is defined as $M^{(S)}$. ◊

We think of such a module $M[S]$ as the set of formal sums $\sum_{s\in S} m_s s$, where $m_s \in M$ and at most finitely many of those are non-zero. This description becomes an R-module with term-wise addition and scalar multiplication.

Remark B.5. Given a map $f : S \to T$ of sets, there is an induced map $M[f] : M[S] \to M[T], \sum_{s\in S} m_s s \mapsto \sum_{s\in S} m_s f(s)$, and this is easily seen to establish M-linearisation as a functor $M[\,\cdot\,] : \mathsf{Set} \to {}_R\mathsf{Mod}$. Note that this induced map is determined exactly like how a map between vector spaces is determined by how it acts on a basis. This is no coincidence, as a vector space is a linearisation of a set of base vectors over a field. ▽

Proposition B.6. *Let M be an R-module. Then there is a natural isomorphism $M[\,\cdot\,] \cong R[\,\cdot\,] \otimes_R M$ of functors* $\mathsf{Set} \to {}_R\mathsf{Mod}$. □

Lemma B.7. (†) *Let $(S_\alpha)_{\alpha\in A}$ be a collection of disjoint sets, and set $S := \bigsqcup_{\alpha\in A} S_\alpha$. Then the inclusions $\iota_\alpha : S_\alpha \hookrightarrow S$ induce an isomorphism $\bigoplus_{\alpha\in A} M[S_\alpha] \to M[S]$.*

Proof. It is a fairly straightforward verification that $M[S]$ with the induced inclusion maps satisfies the universal property of the coproduct $\bigoplus_{\alpha\in A} M[S_\alpha]$, and hence it is isomorphic to it. □

Remark B.8. (†) Since the free module functor $\mathsf{Set} \to {}_R\mathsf{Mod}, S \mapsto R^{(S)}$ is left adjoint to the forgetful functor ${}_R\mathsf{Mod} \to \mathsf{Set}$, and the Tensor-Hom adjunction says that $- \otimes_R M$ is left adjoint to $\operatorname{Hom}_R(M, \cdot\,)$, the above Proposition is also a direct consequence of Theorem A.51. ▽

The following definitions and results, until and including Proposition B.13, are taken from [24].

Definition B.9. A sequence

$$\dots \longrightarrow M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \dots$$

of R-modules and R-linear maps is *exact* if $\ker f_n = \operatorname{im} f_{n-1}$ for all n. A *short exact sequence* is an exact sequence of the form $0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$. ◊

Lemma B.10. *(i) A sequence* $0 \longrightarrow M \xrightarrow{f} N$ *of R-modules is exact iff f is injective.*

(ii) A sequence $M \xrightarrow{f} N \longrightarrow 0$ *of R-modules is exact iff f is surjective.*

(iii) A sequence $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ *of R-modules is exact iff f is an isomorphism.*

(iv) A sequence $0 \longrightarrow M \longrightarrow 0$ *of R-modules is exact iff $M = 0$.*

(v) A sequence $M_1 \xrightarrow{f_1} M_2 \longrightarrow M_3 \longrightarrow M_4 \xrightarrow{f_4} M_5$ *of R-modules, with f_1 and f_4 isomorphisms, is exact iff $M_3 = 0$.* □

Lemma B.11. *Let* $0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$ *be a short exact sequence of R-modules. Then the following are equivalent:*

(i) There exists a retraction of f_1, that is, an R-linear map $r : M_2 \to M_1$ such that $rf_1 = \mathrm{id}_{M_1}$.

(ii) There exists a section of f_2, that is, an R-linear map $s : M_3 \to M_2$ such that $f_2 s = \mathrm{id}_{M_3}$.

(iii) There exists an isomorphism $\varphi : M_2 \to M_1 \oplus M_3$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f_1"] \arrow[d, "\mathrm{id}_{M_1}"] & M_2 \arrow[r, "f_2"] \arrow[d, "\varphi"] & M_3 \arrow[r] \arrow[d, "\mathrm{id}_{M_3}"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "\iota_{M_1}"] & M_1 \oplus M_3 \arrow[r, "\pi_{M_3}"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Moreover, is any of these conditions is met (and therewith all of them), the isomorphism φ of part (iii) identifies the retraction r with the projection $M_1 \oplus M_3 \to M_1$, and the section s with the inclusion $M_3 \hookrightarrow M_1 \oplus M_3$.

Proof. This can be found in [24]. □

Definition B.12. A short exact sequence of R-modules is *split* if it satisfies any (and hence all) of the conditions in the previous lemma. ◊

Proposition B.13. (Five Lemma) *Consider the following commutative diagram*

$$\begin{tikzcd}
M_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & M_3 \arrow[r, "g_3"] \arrow[d, "f_3"] & M_4 \arrow[r, "g_4"] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r, "h_1"] & N_2 \arrow[r, "h_2"] & N_3 \arrow[r, "h_3"] & N_4 \arrow[r, "h_4"] & N_5
\end{tikzcd}$$

of R-modules with exact rows. If f_1 is surjective, f_2 and f_4 are isomorphisms, and f_5 is injective, then f_3 is an isomorphism.

Proof. [24] We only show injectivity of f_3, since the proof of surjectivity turns out to be analogous. Suppose $m_3 \in M_3$ is such that $f_3(m_3) = 0$. Then $0 = h_3 f_3(m_3) = f_4 g_3(m_3)$, and since f_4 is injective, it must be that $m_3 \in \ker g_3 = \operatorname{im} g_2$. Let $m_2 \in M_2$ be such that $g_2(m_2) = m_3$. Then $0 = f_3 g_2(m_2) = h_2 f_2(m_2)$, so $f_2(m_2) \in \ker h_2 = \operatorname{im} h_1$. Let $n_1 \in N_1$ be such that $h_1(n_1) = f_2(m_2)$. Since f_1 is surjective, there exists an $m_1 \in M_1$ such that $f_1(m_1) = n_1$, and then it follows that $f_2(m_2) = h_1 f_1(m_1) = f_2 g_1(m_1)$. Since f_2 is injective, it holds that $m_2 = g_1(m_1)$, and then exactness of the top row gives $m_3 = g_2(m_2) = g_2 g_1(m_1) = 0$. Therefore, f_3 is injective. □

The following two results can be shown by fairly straightforward diagram chases, and hence their proofs are omitted.

Lemma B.14. [11] *Consider a commutative diagram*

$$\begin{tikzcd}
\dots \arrow[r] & E_{n-1} \arrow[r, "h_{n-1}"] \arrow[d, equal] & C_n \arrow[r, "f_n"] \arrow[d, "\gamma_n"] & D_n \arrow[r, "g_n"] \arrow[d, "\delta_n"] & E_n \arrow[r, "h_n"] \arrow[d, equal] & C_{n+1} \arrow[r] \arrow[d, "\gamma_{n+1}"] & \dots \\
\dots \arrow[r] & E_{n-1} \arrow[r, "h'_{n-1}"] & C'_n \arrow[r, "f'_n"] & D'_n \arrow[r, "g'_n"] & E_n \arrow[r, "h'_n"] & C'_{n+1} \arrow[r] & \dots
\end{tikzcd}$$

of R-modules with exact rows. Then the sequence

$$\dots \longrightarrow D'_{n-1} \xrightarrow{h_{n-1}\circ g'_{n-1}} C_n \xrightarrow{\gamma_n - f_n} C'_n \oplus D_n \xrightarrow{f'_n \oplus \delta_n} D'_n \xrightarrow{h_n \circ g'_n} C_{n+1} \longrightarrow \dots$$

is also exact.

Lemma B.15. (Braid Lemma) [20] *Consider a commutative diagram*

$$\begin{tikzcd}
& A \arrow[rr, bend left, "\eth_2"] \arrow[dr, "\delta_1"] & & B \arrow[rr, bend left, "d_3"] \arrow[dr, "\eth_3"] & & C \arrow[rr, bend left, "\partial_5"] \arrow[dr, "d_4"] & & D \\
E \arrow[ur, "\eth_1"] \arrow[dr, "\partial_1"] & & F \arrow[ur, "d_2"] \arrow[dr, "\delta_2"] & & G \arrow[ur, "\partial_4"] \arrow[dr, "\eth_4"] & & H \arrow[ur, "\delta_5"] & \\
& I \arrow[ur, "d_1"] \arrow[rr, bend right, "\partial_2"'] & & J \arrow[ur, "\partial_3"] \arrow[rr, bend right, "\delta_3"'] & & K \arrow[ur, "\delta_4"] & &
\end{tikzcd}$$

where the sequences

$$E \xrightarrow{\eth} A \xrightarrow{\eth} B \xrightarrow{\eth} G \xrightarrow{\eth} K$$

$$E \xrightarrow{\partial} I \xrightarrow{\partial} J \xrightarrow{\partial} G \xrightarrow{\partial} C \xrightarrow{\partial} D$$

$$A \xrightarrow{\delta} F \xrightarrow{\delta} J \xrightarrow{\delta} K \xrightarrow{\delta} H \xrightarrow{\delta} D$$

are exact, and the composite $I \to F \to B$ is zero. Then the sequence

$$I \xrightarrow{d} F \xrightarrow{d} B \xrightarrow{d} C \xrightarrow{d} H$$

is also an exact sequence.

B.2 Chain complexes and homological algebra

Convention B.16. Throughout this section, R will be a ring. ⊙

Definition B.17. [24] A *chain complex* $(C_\bullet, \partial_\bullet)$ *of R-modules* is a diagram

$$\dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} \dots$$

of R-modules and R-module homomorphisms, such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. Dually, a *cochain complex* $(C^\bullet, \partial^\bullet)$ *of R-modules* is a diagram

$$\dots \xrightarrow{\partial^{-3}} C^{-2} \xrightarrow{\partial^{-2}} C^{-1} \xrightarrow{\partial^{-1}} C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \xrightarrow{\partial^2} \dots$$

of R-modules and R-module homomorphisms, such that $\partial^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{Z}$. The homomorphisms are called *(co)differentials*, the elements of the R-modules of a (co)chain complex *(co)chains*, elements in the kernels of (co)differentials *(co)cycles*, and elements in the images of (co)differentials *(co)boundaries.* ◊

Notation B.18. We usually suppress notation such as indices and write for instance $\partial^2 = 0$ to denote either $\partial_n \circ \partial_{n+1} = 0$ or $\partial^{n+1} \circ \partial^n = 0$. We will from now on also only write $C_\bullet$ and $C^\bullet$ for a chain complex and a cochain complex when there can be no confusion about the (co)differentials. ○

Everything in this section can be found in [24], unless indicated otherwise.

Definition B.19. A *chain map* or *morphism of chain complexes* $f_\bullet : C_\bullet \to D_\bullet$ is a collection of R-linear maps $f_n : C_n \to D_n$ such that the diagram

$$\begin{tikzcd}
\dots \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & C_{-1} \arrow[r, "\partial_{-1}"] \arrow[d, "f_{-1}"] & C_{-2} \arrow[r, "\partial_{-2}"] \arrow[d, "f_{-2}"] & \dots \\
\dots \arrow[r, "\delta_3"] & D_2 \arrow[r, "\delta_2"] & D_1 \arrow[r, "\delta_1"] & D_0 \arrow[r, "\delta_0"] & D_{-1} \arrow[r, "\delta_{-1}"] & D_{-2} \arrow[r, "\delta_{-2}"] & \dots
\end{tikzcd}$$

commutes. A *cochain map* is defined similarly. ◊

The identity map on a chain complex is the collection of identity maps on each individual module. The composition of two chain maps $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to E_\bullet$ is defined by $(g \circ f)_n = g_n \circ f_n$. It can be seen from the definition that $g \circ f : C_\bullet \to E_\bullet$ is also a chain map. The (left) R-module chain complexes and chain maps therefore form a category:

Notation B.20. The category of R-module chain complexes and chain maps is denoted by ${}_R\mathsf{Chain}$, and the cochain complex category is denoted by ${}_R\mathsf{cChain}$. ○

Proposition B.21. (†) *The categories* ${}_R\mathsf{Chain}$ *and* ${}_R\mathsf{cChain}$ *are isomorphic.*

Proof. Consider the functor $F : {}_R\mathsf{Chain} \to {}_R\mathsf{cChain}$ which sends a chain complex $(C_\bullet, \partial_\bullet)$ to the cochain complex $(C'^\bullet, \partial'^\bullet)$ given by $C'^n = C_{-n}$ and $\partial'^n = \partial_{-n}$, and sends a chain map $f_\bullet : C_\bullet \to D_\bullet$ to the cochain map $f'^\bullet : C'^\bullet \to D'^\bullet$ given by $f'^n = f_{-n}$ (that this is a functor is clear). There is a completely analogously defined functor $G : {}_R\mathsf{cChain} \to {}_R\mathsf{Chain}$, and we see that $GF = \mathrm{id}_{{}_R\mathsf{Chain}}$ and $FG = \mathrm{id}_{{}_R\mathsf{cChain}}$. Therefore, the categories ${}_R\mathsf{Chain}$ and ${}_R\mathsf{cChain}$ are isomorphic. □

This shows that there is no intrinsic difference between chain and cochain complexes of R-modules. This isomorphism of categories also means that every purely categorical theoretical result, definition or example in ${}_R\mathsf{Chain}$ has an associated result, definition or example in ${}_R\mathsf{cChain}$, and vice versa, obtained by "mirroring the indices". Therefore, we often only state those for chain complexes, unless we deem stating both important enough. Actually, in [24], results are also only stated for chain complexes, but we now know how to dualise the results to obtain the statements about cochain complexes, so when appropriate, we will do this.

Definition B.22. Let $C_\bullet$ and $D_\bullet$ be chain complexes in ${}_R\mathsf{Chain}$, and suppose $f : C_\bullet \to D_\bullet$ is a chain map between them. Then f is an *isomorphism of chain complexes* if there exists another chain map $g : D_\bullet \to C_\bullet$ such that $gf = \mathrm{id}_{C_\bullet}$ and $fg = \mathrm{id}_{D_\bullet}$. If there exists an isomorphism between $C_\bullet$ and $D_\bullet$, these chain complexes are said to be *isomorphic.* ◊

Lemma B.23. *Let $C_\bullet$ and $D_\bullet$ be two chain complexes, and let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map. Then $f_\bullet$ is an isomorphism of chain complexes if and only if $f_n : C_n \to D_n$ is an isomorphism of modules for each $n \in \mathbb{Z}$.* □

The condition that the composition of two (composable) differentials in a chain complex yields the zero map is equivalent to $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$. Therefore, the following definition makes sense.

Definition B.24. Let $C_\bullet$ be a chain complex. For each $n \in \mathbb{Z}$, the *n-th homology module* is defined as the quotient module

$$\mathrm{H}_n(C_\bullet) = \ker \partial_n / \mathrm{im}\, \partial_{n+1}.$$

For a cochain complex $C^\bullet$, the *n-th cohomology module* is defined as the quotient module

$$\mathrm{H}^n(C^\bullet) = \ker \partial_n / \mathrm{im}\, \partial_{n-1}.$$ ◊

The module structure has the follwowing explicit description: for $c, c' \in \ker \partial_n \subseteq C_n$ and $r \in R$, we have $[c] + [c'] = [c + c']$ and $r[c] = [rc]$.

The homology of a chain complex measures how close a chain complex is to being an exact sequence. The following lemma illustrates this. Its dual statement also holds, and the proofs are trivial.

Lemma B.25. *A chain complex $C_\bullet$ is exact if and only if $\mathrm{H}_n(C_\bullet) = 0$ for all $n \in \mathbb{Z}$.* □

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, it holds by definition that $f_{n-1}\partial_n = \delta_n f_n$ for all $n \in \mathbb{Z}$. This implies $f_{n-1}(\mathrm{im}(\partial_n)) \subseteq \mathrm{im}(\delta_n)$ and $f_n(\ker(\partial_n)) \subseteq \ker(\delta_n)$. There is therefore an induced map $\mathrm{H}_n(f_\bullet) : \mathrm{H}_n(C_\bullet) \to \mathrm{H}_n(D_\bullet), [c] \mapsto [f_n(c)]$. This establishes a functor $\mathrm{H}_n : {}_R\mathsf{Chain} \to {}_R\mathsf{Mod}$. Likewise, n-th cohomology is also a functor $\mathrm{H}^n : {}_R\mathsf{cChain} \to {}_R\mathsf{Mod}$. The following proposition is then immediate (see Corollary A.15).

Proposition B.26. *Let $C_\bullet$ and $D_\bullet$ be two chain complexes of R-modules. If they are isomorphic, then $\mathrm{H}_n(C_\bullet) \cong \mathrm{H}_n(D_\bullet)$ as R-modules for all n.* □

Definition B.27. A sequence

$$\begin{tikzcd}
\dots \arrow[r] & C_\bullet \arrow[r, "f_\bullet"] & D_\bullet \arrow[r, "g_\bullet"] & E_\bullet \arrow[r] & \dots
\end{tikzcd}$$

$$\begin{tikzcd}
 & \dots \arrow[d] & \dots \arrow[d] & \dots \arrow[d] & \\
\dots \arrow[r] & C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & D_{n+1} \arrow[r, "g_{n+1}"] \arrow[d] & E_{n+1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & C_n \arrow[r, "f_n"] \arrow[d] & D_n \arrow[r, "g_n"] \arrow[d] & E_n \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & D_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & \dots \\
 & \dots & \dots & \dots &
\end{tikzcd}$$

of chain complexes and chain maps is *exact* if each row in the above diagram is an exact sequence of modules. An exact sequence $0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$ of chain complexes is called a *short exact sequence.* $\lozenge$

Example B.28. [23] Let $(C'_\bullet, \partial'_\bullet)$ and $(C_\bullet, \partial_\bullet)$ be two chain complexes such that C'_n is a submodule of C_n for every n, and such that the inclusions $\iota_n : C'_n \hookrightarrow C_n$ constitute to a chain map $\iota_\bullet : C'_\bullet \to C_\bullet$ (this happens precisely when each ∂'_n is the restriction of ∂_n to C'_n). Define the *quotient complex* $(C_\bullet/C'_\bullet, \bar{\partial}_\bullet)$, consisting of the quotient modules C_n/C'_n and with the differentials $\bar{\partial}_n : C_n/C'_n \to C_{n-1}/C'_{n-1}, c_n + C'_n \mapsto \partial_n(c_n) + C'_{n-1}$. This is clearly R-linear, and well-defined: if $c_n + C'_n = \widetilde{c_n} + C'_n$, then $c_n - \widetilde{c_n} \in C'_n$. Therefore, $\partial_n(c_n) - \partial_n(\widetilde{c_n}) = \partial_n\iota_n(c_n - \widetilde{c_n}) = \iota_{n-1}\partial'_n(c_n - \widetilde{c_n}) = \partial'_n(c_n - \widetilde{c_n}) \in C'_{n-1}$, that is, $\bar{\partial}_n(c_n + C'_n) = \partial_n(c_n) + C'_{n-1} = \partial_n(\widetilde{c_n}) + C'_{n-1} = \bar{\partial}_n(\widetilde{c_n} + C'_n)$. Moreover, $\bar{\partial}^2 = 0$, because $\partial^2 = 0$. Lastly, let $\pi_n : C_n \to C_n/C'_n$ be the canonical projections. Then $\bar{\partial}_n\pi_n = \pi_{n-1}\partial_n$ is obvious, so we have a chain map $\pi_\bullet : C_\bullet \to C_\bullet/C'_\bullet$. All in all, we obtain a short exact sequence of chain modules

$$\begin{tikzcd}
0 \arrow[r] & C'_\bullet \arrow[r, "\iota_\bullet"] & C_\bullet \arrow[r, "\pi_\bullet"] & C_\bullet/C'_\bullet \arrow[r] & 0.
\end{tikzcd}$$

△

Theorem B.29. *Let a short exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet \arrow[r, "f_\bullet"] & D_\bullet \arrow[r, "g_\bullet"] & E_\bullet \arrow[r] & 0
\end{tikzcd}$$

of chain complexes be given. Then there exists an exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_{n+1}(D_\bullet) \arrow[r, "\mathrm{H}_{n+1}(g_\bullet)"] & \mathrm{H}_{n+1}(E_\bullet) \arrow[dll, rounded corners, "\alpha_{n+1}"] \\
\mathrm{H}_n(C_\bullet) \arrow[r, "\mathrm{H}_n(f_\bullet)"] & \mathrm{H}_n(D_\bullet) \arrow[r, "\mathrm{H}_n(g_\bullet)"] & \mathrm{H}_n(E_\bullet) \arrow[dll, rounded corners, "\alpha_n"] \\
\mathrm{H}_{n-1}(C_\bullet) \arrow[r, "\mathrm{H}_{n-1}(f_\bullet)"] & \mathrm{H}_{n-1}(D_\bullet) \arrow[r] & \dots
\end{tikzcd}$$

of homology modules. This sequence is called the long exact sequence of homology*.*

Proof. [24] We will only give the construction of the diagonal "snake"[1] maps α_n, and leave it to the reader to verify well-definedness of this map, and exactness in all degrees. Those aspects of the proof consist namely mainly of diagram chasing, which is rather straightforward.

For $[e_n] \in E_n$, first use surjectivity of g_n to pick an element $x_n \in D_n$ such that $g(x_n) = e_n$. Since $e_n \in \ker d_n$, it holds that $g_{n-1}\delta_n(x_n) = d_n g_n(x_n) = 0$, so $\delta_n(x_n) \in \ker g_{n-1} = \operatorname{im} f_{n-1}$. Since f_{n-1} is injective, there exists a unique $c_{n-1} \in C_{n-1}$ with $f_{n-1}(c_{n-1}) = \delta_n(x_n)$. This c_{n-1} satisfies $f_{n-2}\partial_{n-1}(c_{n-1}) = \delta_{n-1}f_{n-1}(c_{n-1}) = \delta_{n-1}\delta_n(x_n) = 0$, so $c_{n-1} \in \ker \partial_{n-1}$ by injectivity of f_{n-2}. We can therefore set $\alpha_n : \mathrm{H}_n(E_\bullet) \to \mathrm{H}_{n-1}(C_\bullet), [e_n] \mapsto [c_{n-1}]$. ☐

Theorem B.30. *Let a short exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet \arrow[r, "f^\bullet"] & D^\bullet \arrow[r, "g^\bullet"] & E^\bullet \arrow[r] & 0
\end{tikzcd}$$

of cochain complexes be given. Then there exists an exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}^{n-1}(D^\bullet) \arrow[r, "\mathrm{H}^{n-1}(g^\bullet)"] & \mathrm{H}^{n-1}(E^\bullet) \arrow[dll, rounded corners, "\alpha_{n-1}"] \\
\mathrm{H}^n(C^\bullet) \arrow[r, "\mathrm{H}^n(f^\bullet)"] & \mathrm{H}^n(D^\bullet) \arrow[r, "\mathrm{H}^n(g^\bullet)"] & \mathrm{H}^n(E^\bullet) \arrow[dll, rounded corners, "\alpha^n"] \\
\mathrm{H}^{n+1}(C^\bullet) \arrow[r, "\mathrm{H}^{n+1}(f^\bullet)"] & \mathrm{H}^{n+1}(D^\bullet) \arrow[r] & \dots
\end{tikzcd}$$

of cohomology modules. This sequence is called the long exact sequence of cohomology*.* ☐

Corollary B.31. *Let*

$$\begin{tikzcd}
0 \arrow[r] & (C_\bullet, \partial_\bullet) \arrow[r, "f_\bullet"] \arrow[d, "p_\bullet"] & (D_\bullet, \delta_\bullet) \arrow[r, "g_\bullet"] \arrow[d, "q_\bullet"] & (E_\bullet, d_\bullet) \arrow[r] \arrow[d, "r_\bullet"] & 0 \\
0 \arrow[r] & (C'_\bullet, \partial'_\bullet) \arrow[r, "f'_\bullet"] & (D'_\bullet, \delta'_\bullet) \arrow[r, "g'_\bullet"] & (E'_\bullet, d'_\bullet) \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of chain complexes and chain maps with exact rows, and let $\alpha_n : \mathrm{H}_n(E_\bullet) \to \mathrm{H}_{n-1}(C_\bullet)$ and $\alpha'_n : \mathrm{H}_n(E'_\bullet) \to \mathrm{H}_{n-1}(C'_\bullet)$ be the maps in the associated long exact sequences of homology. Then there is a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(C_\bullet) \arrow[r, "\mathrm{H}_n(f_\bullet)"] \arrow[d, "\mathrm{H}_n(p_\bullet)"] & \mathrm{H}_n(D_\bullet) \arrow[r, "\mathrm{H}_n(g_\bullet)"] \arrow[d, "\mathrm{H}_n(q_\bullet)"] & \mathrm{H}_n(E_\bullet) \arrow[r, "\alpha_n"] \arrow[d, "\mathrm{H}_n(r_\bullet)"] & \mathrm{H}_{n-1}(C_\bullet) \arrow[r, "\mathrm{H}_{n-1}(f_\bullet)"] \arrow[d, "\mathrm{H}_n(p_\bullet)"] & \mathrm{H}_{n-1}(D_\bullet) \arrow[r] \arrow[d, "\mathrm{H}_{n-1}(q_\bullet)"] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(C'_\bullet) \arrow[r, "\mathrm{H}_n(f'_\bullet)"] & \mathrm{H}_n(D'_\bullet) \arrow[r, "\mathrm{H}_n(g'_\bullet)"] & \mathrm{H}_n(E'_\bullet) \arrow[r, "\alpha'_n"] & \mathrm{H}_{n-1}(C'_\bullet) \arrow[r, "\mathrm{H}_{n-1}(f'_\bullet)"] & \mathrm{H}_{n-1}(D'_\bullet) \arrow[r] & \dots
\end{tikzcd}$$

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[1] There is a lemma called the Snake Lemma, which is a special case of this theorem. It also features these diagonal maps, which for some people resemble snakes.

consisting of the long exact sequences of homology. In other words, the long exact sequence of homology is natural in the short exact sequence of chain complexes.

Proof. [23] The most difficult part is to prove naturality in the squares containing the snake maps α_n. The other squares can easily be seen to be commutative by the definition of the induced map on homology and the commutativity of the chain complex diagram in the statement of this corollary. Hence we will only prove the commutativity of the firstly mentioned squares.

Let $[e_n] \in \mathrm{H}_n(E_\bullet)$, and pick $x_n \in D_n$ and $c_{n-1} \in C_{n-1}$ as in the construction of α_n. Then $\alpha_n([e_n]) = [c_{n-1}]$. Now we consider the elements $r_n(e_n) \in E'_n$, $q_n(x_n) \in D'_n$ and $p_{n-1}(c_{n-1}) \in C'_{n-1}$. Then $g'_n q_n(x_n) = r_n g_n(x_n) = r_n(e_n)$, and $f'_{n-1} p_{n-1}(c_{n-1}) = q_{n-1} f_{n-1}(c_{n-1}) = q_{n-1}\delta_n(x_n) = \delta'_n q_n(x_n)$. Therefore, from the construction of the map α'_n we get $\alpha'_n([r_n(e_n)]) = [p_{n-1}(c_{n-1})]$, which shows commutativity of the considered square. ☐

Example B.32. [23] As in Example B.28, let $(C'_\bullet, \partial'_\bullet)$ and $(C_\bullet, \partial_\bullet)$ be two chain complexes such that C'_n is a submodule of C_n for every n, and such that the inclusions $\iota_n : C'_n \hookrightarrow C_n$ constitute to a chain map $\iota_\bullet : C'_\bullet \to C_\bullet$ (that is, such that ∂'_n is the restriction of ∂_n to C'_n for each n). In the aforementioned example we constructed the quotient complex $(C_\bullet/C'_\bullet, \bar{\partial}_\bullet)$, and saw that we have a short exact sequence

$$0 \longrightarrow C'_\bullet \xrightarrow{\iota_\bullet} C_\bullet \xrightarrow{\pi_\bullet} C_\bullet/C'_\bullet \longrightarrow 0.$$

Let

$$0 \longrightarrow D'_\bullet \xrightarrow{\iota_\bullet} D_\bullet \xrightarrow{\pi_\bullet} D_\bullet/D'_\bullet \longrightarrow 0$$

be another short exact sequence of chain complexes of this form, and suppose that we are given a chain map $f_\bullet : C_\bullet \to D_\bullet$ such that $f_n(C'_n) \subseteq D'_n$ for all n. Then there is an induced map $\bar{f}_n : C_n/C'_n \to D_n/D'_n, c + C'_n \mapsto f(c) + D'_n$, and we see that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C'_\bullet \arrow[r, "\iota_\bullet"] \arrow[d, "f_\bullet"] & C_\bullet \arrow[r, "\pi_\bullet"] \arrow[d, "f_\bullet"] & C_\bullet/C'_\bullet \arrow[r] \arrow[d, "\bar{f}_\bullet"] & 0. \\
0 \arrow[r] & D'_\bullet \arrow[r, "\iota_\bullet"] & D_\bullet \arrow[r, "\pi_\bullet"] & D_\bullet/D'_\bullet \arrow[r] & 0
\end{tikzcd}$$

By Corollary B.31, we obtain a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}_n(C'_\bullet) \arrow[r, "\mathrm{H}_n(\iota_\bullet)"] \arrow[d, "\mathrm{H}_n(f_\bullet)"] & \mathrm{H}_n(C_\bullet) \arrow[r, "\mathrm{H}_n(\pi_\bullet)"] \arrow[d, "\mathrm{H}_n(f_\bullet)"] & \mathrm{H}_n(C_\bullet/C'_\bullet) \arrow[r, "\alpha_n"] \arrow[d, "\mathrm{H}_n(\bar{f}_\bullet)"] & \mathrm{H}_{n-1}(C'_\bullet) \arrow[r, "\mathrm{H}_{n-1}(\iota_\bullet)"] \arrow[d, "\mathrm{H}_n(f_\bullet)"] & \mathrm{H}_{n-1}(C_\bullet) \arrow[r] \arrow[d, "\mathrm{H}_{n-1}(f_\bullet)"] & \dots \\
\dots \arrow[r] & \mathrm{H}_n(D'_\bullet) \arrow[r, "\mathrm{H}_n(\iota_\bullet)"] & \mathrm{H}_n(D_\bullet) \arrow[r, "\mathrm{H}_n(\pi_\bullet)"] & \mathrm{H}_n(D_\bullet/D'_\bullet) \arrow[r, "\alpha_n"] & \mathrm{H}_{n-1}(D'_\bullet) \arrow[r, "\mathrm{H}_{n-1}(\iota_\bullet)"] & \mathrm{H}_{n-1}(D_\bullet) \arrow[r] & \dots
\end{tikzcd}$$

of the long exact sequences of homology. △

Definition B.33. Let $f_\bullet, g_\bullet : (C_\bullet, \partial_\bullet) \to (D_\bullet, \delta_\bullet)$ be two chain maps. A *homotopy* from $f_\bullet$ to $g_\bullet$ consists of R-linear maps $h_n : C_n \to D_{n+1}$, satisfying

$$g_n - f_n = \delta_{n+1} h_n + h_{n-1} \partial_n$$

for all $n \in \mathbb{Z}$. If there exists a homotopy from $f_\bullet$ to $g_\bullet$, they are said to be *homotopic*. A homotopy between cochain maps is defined analogously. ◊

Remark B.34. We do not require any commutativity of the homotopy maps with the given chain maps or differentials in the definition above. ▽

It is easy to verify that homotopy is an equivalence relation on the chain maps $C_\bullet \to D_\bullet$. It also respects composition:

Lemma B.35. *Let $f_\bullet, g_\bullet : (C_\bullet, \partial_\bullet) \to (D_\bullet, \delta_\bullet)$ be two homotopic chain maps. For any chain map $s_\bullet : D_\bullet \to E_\bullet$ and $t_\bullet : B_\bullet \to C_\bullet$, the compositions $s_\bullet f_\bullet$ and $s_\bullet g_\bullet$ are homotopic, and the compositions $f_\bullet t_\bullet$ and $g_\bullet t_\bullet$ are homotopic.* □

Definition B.36. The *homotopy category of chain complexes of R-modules* is the category ${}_R\mathsf{hChain}$ having chain complexes of R-modules as objects and equivalence classes of chain maps modulo homotopy as morphisms. The *homotopy category of cochain complexes of R-modules* ${}_R\mathsf{hcChain}$ is defined similarly. ◊

The preceding lemma shows that composition of morphisms in these homotopy categories is indeed well-defined. Chain homology also behaves nicely with respect to the homotopy category, as the next proposition shows.

Proposition B.37. *Let $f_\bullet, g_\bullet : (C_\bullet, \partial_\bullet) \to (D_\bullet, \delta_\bullet)$ be two homotopic chain maps. Then $\mathrm{H}_n f = \mathrm{H}_n g$ for all n.*

Proof. [24] Let $(h_n)_{n\in\mathbb{Z}}$ be a homotopy from f to g, and let $c + \operatorname{im}\partial_{n+1}$ be an element of $\mathrm{H}_n(C_\bullet)$. Then we have

$$\begin{aligned}\mathrm{H}_n f(c + \operatorname{im}\partial_{n+1}) - \mathrm{H}_n g(c + \operatorname{im}\partial_{n+1}) &= (f(c) + \operatorname{im}\delta_{n+1}) - (g(c) + \operatorname{im}\delta_{n+1}) = f(c) - g(c) + \operatorname{im}\delta_{n+1} \\ &= \delta_{n+1} \circ h_n(c) + h_{n-1} \circ \partial_n(c) + \operatorname{im}\delta_{n+1} = h_{n-1}(0) + \operatorname{im}\delta_{n+1} = 0,\end{aligned}$$

since $c \in \ker\partial_n$. □

B.3 Dual cochain complexes and the Algebraic Universal Coefficient Theorem

Convention B.38. Throughout this section, R will be a *commutative* ring. ⊙

Since R is assumed to be commutative, the Hom-sets carry naturally the structure of an R-module. Moreover, the contravariant Hom-functor sends trivial maps to trivial maps, so we obtain the following result.

Lemma B.39. *For any R-module M, $\operatorname{Hom}_R(\,\cdot\,, M)$ is a functor ${}_R\mathsf{Chain}^{\mathrm{opp}} \to {}_R\mathsf{cChain}$.* □

Definition B.40. [13] Let $C_\bullet$ be a chain complex of R-modules, and suppose M is an R-module. The *dual cochain complex (with coefficients in M)* is the cochain complex $\operatorname{Hom}_R(C_\bullet, M)$. ◊

Remark B.41. Every chain map $f : C_\bullet \to D_\bullet$ induces a cochain map $(f^*)^\bullet := \operatorname{Hom}_R(f_\bullet, M) : \operatorname{Hom}_R(D_\bullet, M) \to \operatorname{Hom}_R(C_\bullet, M)$, which we call the *dual cochain map.* ▽

A natural question to ask is how the homology of a chain complex relates to the cohomology of the dual cochain complex. A first guess could be that the cohomology of the dual cochain complex is the dual of the homology of the chain complex. Put differently, given a chain complex $C_\bullet$, we could hope that there is a commutative diagram

$$\begin{tikzcd}
{}_R\mathsf{Chain} \arrow[r, "\operatorname{Hom}_R(\,\cdot\,, M))"] \arrow[d, "\mathrm{H}_n"] & {}_R\mathsf{cChain} \arrow[d, "\mathrm{H}^n"] \\
{}_R\mathsf{Mod} \arrow[r, "\operatorname{Hom}_R(\,\cdot\,, M))"] & {}_R\mathsf{Mod}
\end{tikzcd}$$

of functors, or at least up to natural isomorphism. There is even a homomorphism $\Phi : \mathrm{H}^n(\operatorname{Hom}_R(C_\bullet, M)) \to \operatorname{Hom}_R(\mathrm{H}_n(C_\bullet), M)$, which sends a cohomology class $[\varphi]$ to the map $\Phi([\varphi]) : \mathrm{H}_n(C_\bullet) \to M, [c] \mapsto \varphi(c)$ [13]. (This is well-defined, but we will not show that here. The reader is of course invited to do so.) This map is the natural candidate for an isomorphism between the cohomology of the dual cochain complex and the dual of the homology of the chain complex. However, in general this map will not be an isomorphism. In some cases, we can even measure the extent to which it fails to be an isomorphism. For our purposes, it is enough to consider the case in which R is a principal ideal domain and the chain complex consists entirely of free modules. First, we need to introduce a new functor.

Proposition B.42. *Let R be a commutative ring. There are for $n \geq 0$ functors $\operatorname{Ext}^n_R(\,\cdot\,,\,\cdot\,) : {}_R\mathsf{Mod}^{\mathrm{opp}} \times {}_R\mathsf{Mod} \to {}_R\mathsf{Mod}$ such that the following three properties hold:*

(i) There is a natural isomorphism $\operatorname{Ext}^0_R(\,\cdot\,,\,\cdot\,) \cong \operatorname{Hom}_R(\,\cdot\,,\,\cdot\,)$.

(ii) If M is a free R-module and $n > 0$, then $\operatorname{Ext}^n_R(M, N) = 0$ for any module N.

(iii) Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be a short exact sequence of R-modules, and N be an R-module. Then there is a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(M_3, N) \arrow[r] & \operatorname{Hom}_R(M_2, N) \arrow[r] & \operatorname{Hom}_R(M_1, N) \arrow[dll, out=-10, in=170] \\
& \operatorname{Ext}^1_R(M_3, N) \arrow[r] & \operatorname{Ext}^1_R(M_2, N) \arrow[r] & \operatorname{Ext}^1_R(M_1, N) \arrow[dll, out=-10, in=170] \\
& \operatorname{Ext}^2_R(M_3, N) \arrow[r] & \operatorname{Ext}^2_R(M_2, N) \arrow[r] & \dots
\end{tikzcd}$$

which is natural in the short exact sequence and in N.

Moreover, the functors Ext^n_R are determined up to natural isomorphism by these three conditions.

Proof. See [13] for the proof. ☐

Corollary B.43. [24] *If $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is a short exact sequence of R-modules, with M_3 free, and N is another R module, then the sequence*

$$0 \longrightarrow \operatorname{Hom}_R(M_3, N) \longrightarrow \operatorname{Hom}_R(M_2, N) \longrightarrow \operatorname{Hom}_R(M_1, N) \longrightarrow 0$$

is also short exact.

Proof. This follows from the long exact sequence in Proposition B.42(iii) and from property (ii) in the same proposition (applied to M_3, as it is free). ☐

We are now ready to state the main theorem of this section.

Theorem B.44. (Algebraic Universal Coefficient Theorem) *Let R be a PID and $C_\bullet$ a chain complex of* free *R-modules. For any R-module M, there is a short exact sequence*

$$0 \longrightarrow \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(C_\bullet), M) \longrightarrow \mathrm{H}^n(\operatorname{Hom}_R(C_\bullet, M)) \xrightarrow{\Phi} \operatorname{Hom}_R(\mathrm{H}_n(C_\bullet), M) \longrightarrow 0,$$

where Φ is the map defined above. This sequence is natural in $C_\bullet$, and is split, although not naturally in $C_\bullet$.

Proof. For the proof, see [11], or [7] if the reader wishes to do parts of the proof as an exercise. ☐

Corollary B.45. *Let $f_\bullet : C_\bullet \to D_\bullet$ be a chain map between chain complexes of free modules, that also induces isomorphisms on the homology modules. Then the dual map $(f^*)^\bullet : \operatorname{Hom}_R(D_\bullet, M) \to \operatorname{Hom}_R(C_\bullet, M)$ induces isomorphisms on the cohomology modules.*

Proof. [13] Naturality of the short exact sequence of the Algebraic Universal Coefficient Theorem gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(D_\bullet), M) \arrow[r] \arrow[d] & \mathrm{H}^n(\operatorname{Hom}_R(D_\bullet, M)) \arrow[r] \arrow[d, "\mathrm{H}^n(f_*)^\bullet"] & \operatorname{Hom}_R(\mathrm{H}_n(D_\bullet), M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(C_\bullet), M) \arrow[r] & \mathrm{H}^n(\operatorname{Hom}_R(C_\bullet, M)) \arrow[r] & \operatorname{Hom}_R(\mathrm{H}_n(C_\bullet), M) \arrow[r] & 0
\end{tikzcd}$$

Since $f_\bullet$ induces an isomorphism on each homology module, the left and right vertical arrow are isomorphisms (by Corollary A.15). Therefore, so is the middle (for instance by the Five Lemma). ☐

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