university of groningen
faculty of science and engineering
mathematics and applied mathematics

Mitigation of queue formations on the motorway using temporal reduction in speed limits

Bachelor's Project Mathematics

July 2021

Student: Jennifer Towers

First supervisor: Dr. B. Besselink

Second assessor: Dr. H. Jardón-Kojakhmetov

Abstract

The study of traffic flow has become increasingly popular since its conception in the 1950s. One of the main focuses in this field is on the ability to accurately simulate traffic networks at different levels of detail. These simulations can then be tailored to a specific situation and be used to optimise the flow of traffic on a given stretch of road. This study looks into the mitigation of queue formation as a result of an active bottleneck along a simple stretch of motorway, with no entrances/exits, via the introduction of a temporarily reduced speed limit. Through the use of shock wave analysis, it was found that a reduced speed limit can greatly reduce the queue that forms immediately upstream of the bottleneck, but depending on it's characteristics could also disrupt traffic to an almost equal degree the moment it is taken out of effect. Three characteristics are found to be of importance; the duration, the reach and finally the magnitude of the temporary reduced speed limit. The optimum duration is given by the total queuing time, which is the length of time it takes for the queue at the bottleneck to fully dissipate and traffic flow to return to normal. From this the optimum distance from the bottleneck where the reduced speed limit is put into effect can be determined such that the forward propagating shock wave, as a result of the change in speed, does not catch up with the queue. Finally, a lower bound is determined for the magnitude of the reduced speed limit so that traffic remains in a free-flow state when it is lifted, and further congestion is avoided. It is concluded that the use of a temporary reduction in speed limit can be effective for the mitigation of traffic jams caused by an active bottleneck, provided these three characteristics are chosen in a certain way.

Contents

1 Introduction

Mathematical modelling in motorway traffic control has been used over the past decades to optimise traffic flow, increase safety and is becoming increasingly more important in sustainable development. The concept of modelling motorway traffic control has recently become revitalised due to technological developments allowing for a more efficient and accurate transfer of data from speculation into practice. Macroscopic models are used to represent motorway traffic as a dynamical system, where the flow of vehicles is seen as a unique stream with variables such as density, mean speed, and flow. These models are advantageous when used for control purposes, due to their relatively low computational effort and minimal parameters [1]. Alongside this, macroscopic models can be classified as either continuous or discrete, based on the independent variables of space and time, where continuous models represent the dynamical system using differential equations, and discrete models represent the system through difference equations.

These models can be used in practice to determine mitigation techniques for congestion formed for various reasons, such as an accident or roadworks, which due to their effect on traffic flow are referred to as active bottlenecks. Because of the increasing number of vehicle's on the roads, and the inability to instantly change the already existing infrastructure to accommodate this increase in demand, the mitigation of queue formation is a primary focus in traffic flow theory [8]. One of the easiest to apply methods is the introduction of a temporary reduction in speed limit on a given stretch of road, where the infrastructure required to implement it depends on the nature of the active bottleneck. For example, roadworks which will last for an extended period of time may only require physical speed limit signs for the stretch of road under construction, where an accident that occurs and lasts for a shorter period of time would require electronic signs that can be activated to the chosen speed limit along the necessary stretch of road. As the latter case requires more investment to use in practice, it is important to determine how effective this method is at reducing queue formation. This project then looks into how a temporary reduction in the speed limit for a simple stretch of motorway, with no exits/entrances nor overtaking, affects the flow of traffic and how it can used to reduce the queue formation caused by an active bottleneck, such as an accident, which will be resolved after a relatively short period of time.

The following research question will be explored:

What effect does a temporal reduction in the speed limit have on queuing dynamics on a simple stretch of motorway?

In order to answer this question, shock wave analysis is used to determine the expected effect of a sudden reduction in the speed limit on queuing characteristics before a stretch of simple motorway is simulated, where a base case with an active bottleneck is initially considered and the queuing dynamics determined. Here, the traffic is assumed to flow as in a triangular fundamental diagram and the discrete macroscopic cell transmission model (CTM) is chosen to be used for the simulation due to its lower computational effort and equivalence to the continuous macroscopic Lighthill, Whitham and Richards model (LWR). The simulation is then run with three different temporary speeds for a given amount of time, all with magnitude less than the base case. The queuing characteristics are then calculated to determine which one allows for the quickest dissipation of the queue, the shortest maximum queue and the shortest reach of the queue, as well as any adverse effects that occur at the moment the reduced speed limit is lifted.

The first section introduces the basics of traffic theory and explains how models are classified, namely microscopic versus macroscopic and continuous versus discrete. Then the report looks into the continuous Lighthill, Whitham and Richards model before showing its discrete equivalent the cell transmission model. Both these models are widely accepted in traffic flow theory and serve as a basis for later models to be developed. After this shock wave theory is explored for traffic that flows on the triangular fundamental diagram where initially the effect alone of a reduction in speed limit on the evolution of traffic is determined before its relation to queuing characteristics is established. Finally, an example is discussed using the CTM to determine the flow of traffic and the effect on the mitigation of a queue formed by an active bottleneck, as in the case of an accident, that a temporary reduction in the speed limit has.

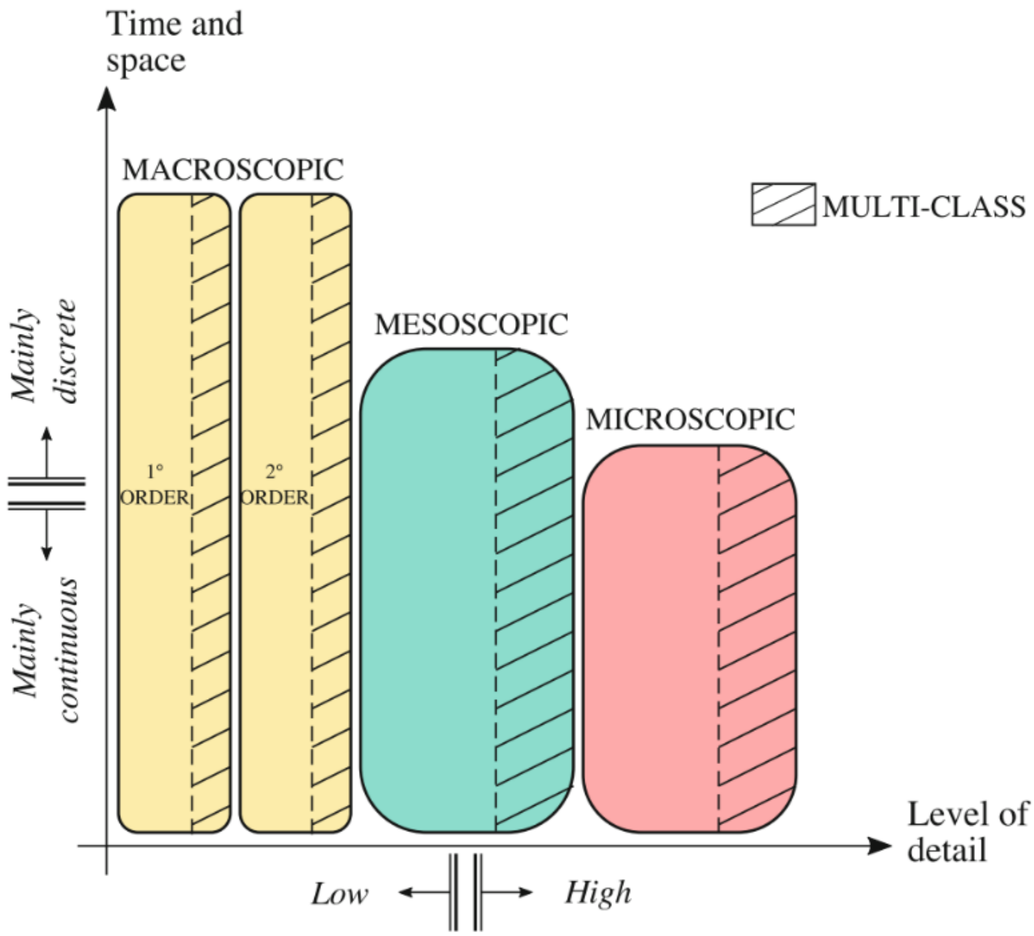

Figure 1: Classification of traffic models according to their time and space representation and their level of detail [1].

2 Traffic Flow Theory

In order to be able to predict the evolution of traffic in the presence of an active bottleneck, traffic flow modelling can be used to depict an accurate account of traffic behaviour. For a specific scenario, different models can be chosen based on the level of detail that is required. To this effect, this section will focus on traffic flow modelling and how it can be utilised to determine the evolution of traffic in a given situation.

2.1 Modelling Traffic Flow

Traffic flow theory is a relatively new contribution to the scientific community and has laid the basis for various traffic models to emerge. Ever since the first traffic model was introduced in 1955 by Lighthill, Whitham and Richards [10] [13], there have been many other traffic models proposed, all with specific characteristics which allow them to be classified differently [5]. One of the main ways of classifying traffic models is based on their level of detail. The main classifications are *microscopic*: a high level of detail, modelling cars separately; *macroscopic*: a lower level of detail which models a stretch of road as one unity; and finally *mesoscopic*: a level of detail in between the other two mentioned, which often includes a mixture of microscopic and macroscopic variables.

Alongside these, models can also be classified by other variables such as a model which accounts for multiple classes of vehicle (car, lorry, bike etc.) as well as being classified as either continuous or discrete. Figure 1 visualises how these classifications intertwine with each other respective to time and space. This section will explore the relationship between microscopic and macroscopic models before delving deeper into the traffic flow theory corresponding to both continuous and discrete macroscopic models.

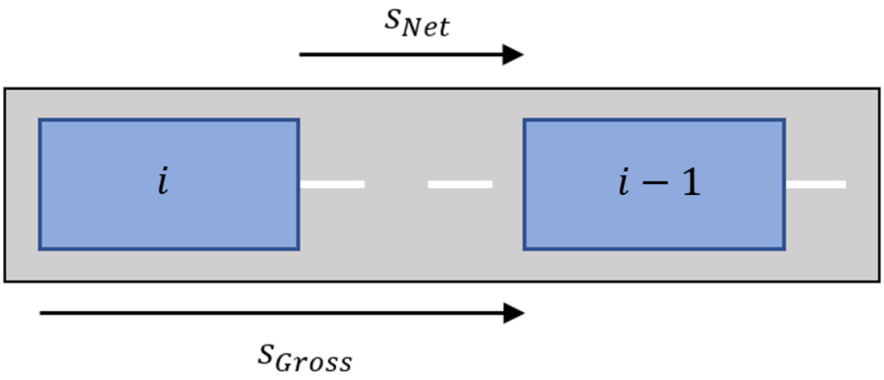

Figure 2: The difference between net and gross space headway

Microscopic models

As the name would suggest, microscopic models look at a road system in a high level of detail. Often the models are comprised of several individually described driver-vehicle combinations, taking into account each vehicle's position at all times. Each vehicle is numbered, where vehicle i follows directly behind vehicle $i-1$ and the variables that are often used to describe the movements of vehicle i for a given position x and time t in such systems are:

- v_i : Speed [distance/time]
- s_i : Space headway [distance]
- h_i : Time headway [time]

Here, the Space headway is the distance between vehicle i and $i-1$. This can be calculated as either a net value (not including the length of the vehicles themselves) or as a gross value (including the length of the following vehicle) as shown in Figure 2. The time headway is then the time it takes for the following vehicle to travel the distance of the space headway. Similarly, this can be measured as either net or gross. Unless otherwise stated it is assumed that these variables are measured as gross values [6].

It can be easily seen that the variables are related as

$$s_i = v_i h_i. \tag{1}$$

for all vehicles i. Whereas (1) provides a simple relationship between the three variables, the intricacy of microscopic traffic models comes from determining the behavior of each vehicle i in relation to the directly preceding vehicle $i-1$. To illustrate this, the acceleration of vehicle i is determined by knowing the velocity of vehicle i and $i-1$ as well as the space headway between the two vehicles. To this effect the acceleration is given by

$$\dot{v_i} = f(v_i, v_{i-1}, s_{i-1}) \tag{2}$$

for some function f, which itself can also be a complex function [12]. Due to this intricate level of detail, tracking the behaviour of each individual vehicle, microscopic models can lead to very large-scale systems which require a high computational effort, especially when used to model road networks.

Macroscopic models

As opposed to microscopic models, macroscopic models do not focus on driver-vehicle combinations individually. Instead, traffic is viewed as a unique stream in analogy with fluid and gas flow theory. The models are described by the following variables:

- u : Mean speed [distance/time]
- ρ : Density [vehicles/distance]
- ϕ : Flow [vehicles/time]

for a given position x at time t. Each variable is derived from its microscopic counterpart via a simple conversion of the dimensions, so that they match those mentioned above. In particular,

$$\phi = \frac{1}{\langle h \rangle}, \quad \rho = \frac{1}{\langle s \rangle}, \quad u = \langle v \rangle.$$

where the angled brackets indicate the mean. Then the desired measures of distance and time can be applied as needed, for example if $\langle h \rangle = 5$ seconds for a given stretch of motorway, then $\phi = 1/5$ (s/veh) $=$ 3600 (s/h)/5 (s/veh) $=$ 720 (veh/h).

As in (1) the variables are related as

$$\phi = u\rho. \tag{3}$$

Because of its relatively lower level of detail, macroscopic modelling is far less computationally expensive than microscopic models and is therefore often favoured in application for control purposes. This is also due to its variables which can be easily calibrated to suit a specific situation as well as its ability to model large-scale networks. Macroscopic models also have the further classification based on the number of state-variables and can be classed as first-order, second-order or even higher-order models [1].

2.2 Continuous vs. Discrete modelling

As previously mentioned, traffic models can be classified as either continuous or discrete. Continuous models use variables as continuous functions of space x and time t, whereas discrete models increment the variables using uniform space and time steps i and k, respectively.

2.2.1 LWR model

The original first-order continuous model to be proposed was in 1956 by Lighthill, Whitham and Richards and acts as a basis for all traffic models, both continuous and discrete. In [9] they proposed a one-dimensional wave motion applicable to both fluid motion as well as motorway traffic flow [11]. The main variables considered in continuous traffic flow theory are

- $\phi(x,t)$: Traffic flow [veh/hr]
- $\rho(x,t)$: Traffic density [veh/km]
- $u(x,t)$: Mean speed [km/hr]
- $A(x,t)$: Cumulative number of vehicles at some location x at time t

for some location x and time t. Hence, the *hydrodynamic equation* is formulated similarly to (1) and (3) as

$$\phi(x,t) = u(x,t)\rho(x,t) \tag{4}$$

which states that the flow of traffic is attributed to the density times the mean speed as functions of space x and time t.

Secondly, the conservation law of vehicle flows for a homogeneous stretch of road with no entrances or exits can be derived via the relationship between $\phi(x,t)$, $\rho(x,t)$ and $A(x,t)$. It should be noted that $A(x,t)$ is a step function which discretely counts the number of vehicles for a given position x and time t, but can be approximated by the smooth function $\tilde{A}(x,t)$ where $A(x,t_j) = \tilde{A}(x,t_j)$ for times t_j at a cross section x. From Figure 3 it is clear that the traffic flow at a certain point x during the period t_1 to t_2 is given by

$$\phi(x, t_1 \ to \ t_2) = \frac{\tilde{A}(x,t_2) - \tilde{A}(x,t_1)}{t_2 - t_1}. \tag{5}$$

Similarly, the density between two points x_1 and x_2 at a given time t is given by

$$\rho(x_1 \ to \ x_2, t) = \frac{-(\tilde{A}(x_2,t) - \tilde{A}(x_1,t))}{x_2 - x_1}. \tag{6}$$

Note that the negative of the numerator is taken to allow for positive density. This is because of the assumption that cars may not pass each other, meaning $\tilde{A}(x_2,t) > \tilde{A}(x_1,t)$ for all t. Taking the limit of (5) and (6) as $(t_2 - t_1) \to 0$ and $(x_2 - x_1) \to 0$ respectively, leads to

$$\phi(x,t) = \frac{\partial \tilde{A}(x,t)}{\partial t} \tag{7}$$

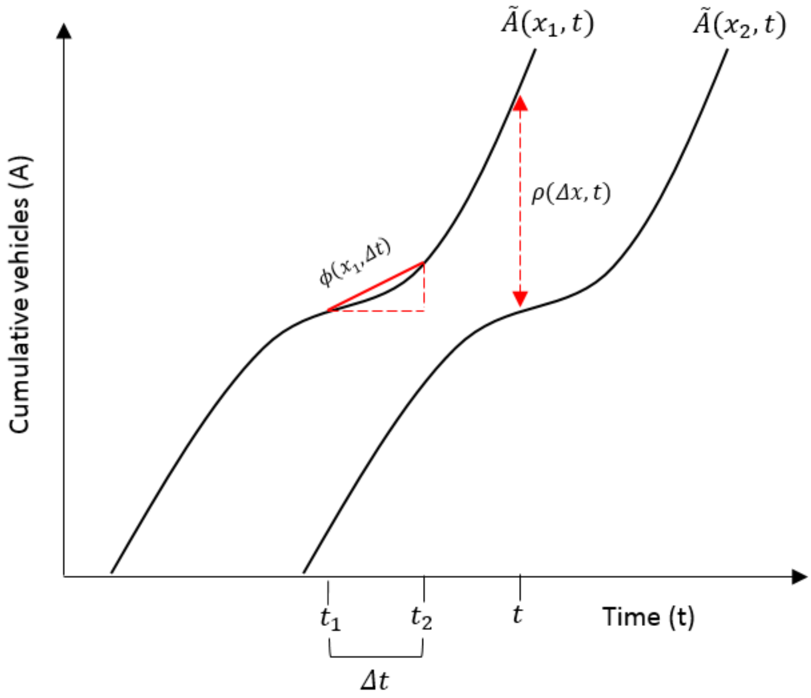

Figure 3: Graph of cumulative flow at two locations (such that $x_2 - x_1 = 1$) showing relationship between $\phi(x, t)$, $\rho(x, t)$ and $\tilde{A}(x, t)$.

$$\rho(x,t) = -\frac{\partial \tilde{A}(x,t)}{\partial x}. \tag{8}$$

Assume $\tilde{A}(x, t)$ is twice continuously differentiable, then the identity is

$$\frac{\partial^2 \tilde{A}(x,t)}{\partial t \partial x} = \frac{\partial^2 \tilde{A}(x,t)}{\partial x \partial t}.$$

Substituting this result in (7) and (8) gives

$$\frac{\partial \phi(x,t)}{\partial x} = -\frac{\partial \rho(x,t)}{\partial t}.$$

Hence, we obtain

$$\frac{\partial \phi(x,t)}{\partial x} + \frac{\partial \rho(x,t)}{\partial t} = 0, \tag{9}$$

which gives the *conservation equation* [4].

The hydrodynamic and conservation equations, as in (4) and (9), provide the building blocks for all continuous traffic flow models alongside the use of steady-state relations which form the relationship between two of the variables $\phi(x, t)$, $\rho(x, t)$ or $u(x, t)$. The most common steady-state relationship is one that relates the traffic flow and the density in a *fundamental diagram*, with some function Q that satisfies the following conditions

$$Q(0) = 0, \quad Q(\rho^j) = 0, \quad \frac{\partial Q(\rho)}{\partial \rho}\bigg|_{\rho=\rho^c} = 0, \tag{10}$$

where ρ^j is the jam density (the moment when $u = 0$ due to a traffic jam) and ρ^c is the critical density (the moment at which the road has reached maximum flow capacity ϕ^c). The LWR model is then comprised of (4) and (9) along with the assumption that

$$\phi(x,t) = Q(\rho(x,t)). \tag{11}$$

The conservation equation can then be re-written as

$$\frac{\partial Q(\rho(x,t))}{\partial x} + \frac{\partial \rho(x,t)}{\partial t} = 0 \tag{12}$$

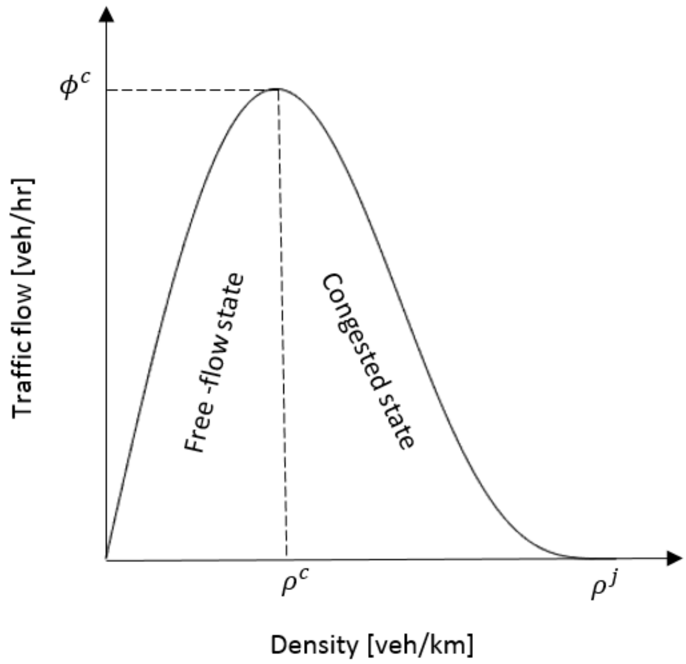

Figure 4: Example of a fundamental diagram

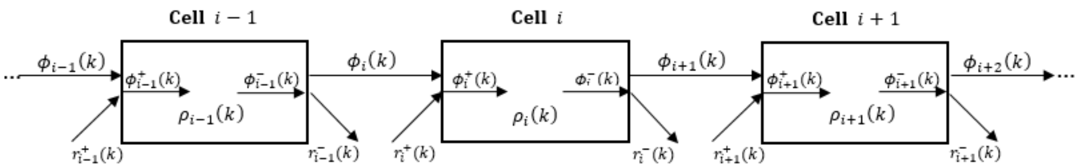

Figure 5: CTM visualised for a road stretch with on/off ramps

for a strictly concave C^2 function $Q(\rho(x,t))$. This ensures (12) belongs to the class of hyperbolic conservation laws. A common function for Q is given by

$$Q(\rho) = u^f \rho(x,t)\left(1 - \left(\frac{\rho(x,t)}{\rho^j}\right)^l\right)^m \tag{13}$$

where $m < l < 0$ are parameters and u^f is the free flow speed. As seen in Figure 4, the left side of the diagram shows traffic in its free flow state where the free flow speed u^f is given by the slope of the curve at the origin. This is where the traffic is able to flow undisturbed as the road is not yet at full capacity allowing the traffic flow to increase with the density. The corresponding density at the point that the road meets maximum flow capacity is known as the critical density, and marks the turning point for the curve and the transition into the congested state. In this state the traffic flow then begins to decrease as the density increases, where the slope of the curve gives the wave congestion speed $w(x,t)$. Finally the traffic flow reaches 0 as the density reaches the jam density and the traffic comes to a standstill.

Although the model is sound and has paved the way for many other models to be developed, there are some limitations which cause unrealistic outputs when modelling real life scenarios. The main one is due to the assumption that vehicles adjust their speed instantaneously, leading to no inertial effects being represented in the model. This can result in unrealistic acceleration and deceleration of vehicles.

2.2.2 Cell Transmission Model

Whereas the LWR model is continuous, in reality it is often more practical to simulate traffic theory through a discrete model,due to its lower computation effort. The CTM, which was first introduced by Daganzo in [2], accounts for this via breaking up a homogeneous section of road of length L km into N sections, known as cells

$i = 1, ..., N$, and calculating the in and outflows of each cell for a given time step $k = 1, ..., K$ of duration T hours. The following variables then describe the flow of traffic, for some cell i at time step k:

- $u_i(k)$: Mean speed at time kT [km/hr]
- $\rho_i(k)$: Density at time kT [veh/km]
- $\phi_i(k)$: Traffic Flow in time interval $[kT, (k+1)T)$ [veh/hr]
- $\phi_i^+(k)$: Inflow of cell i in time interval $[kT, (k+1)T)$ [veh/hr]
- $\phi_i^-(k)$: Outflow of cell i in time interval $[kT, (k+1)T)$ [veh/hr]
- $r_i^+(k)$: On-ramp inflow of cell i in time interval $[kT, (k+1)T)$ [veh/hr]
- $r_i^-(k)$: Off-ramp outflow of cell i in time interval $[kT, (k+1)T)$ [veh/hr]
- $\lambda_i(k) \in [0, 1)$: Split ratio of cell i in time interval $[kT, (k+1)T)$

Then the *hydrodynamic equation* can be given in discrete form as

$$\phi_i(k) = u_i(k)\rho_i(k) \tag{14}$$

and the *conservation equation* is transformed from the partial differential equation (12) of the LWR model, into a difference equation for a cell i at a given time step k given by

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{L}(\phi_i^+(k) - \phi_i^-(k)) \tag{15}$$

for some $i = 1, ..., N$ and $k = 1, ..., K$.

From Figure 5, it is clear that the inflow and outflow of each cell respectively are computed as

$$\phi_i^+(k) = \phi_i(k) + r_i^+(k) \tag{16}$$

$$\phi_i^-(k) = \phi_{i+1}(k) + r_i^-(k). \tag{17}$$

The flow of traffic exiting cell i via an off ramp is determined by the split ratio λ_i, which denotes the percentage of vehicles which take the off-ramp at cell i, and the outflow of cell i such that

$$r_i^-(k) = \lambda_i(k)\phi_i^-(k) = \lambda_i(k)(\phi_{i+1}(k) + r_i^-(k)),$$

which can be re-written as

$$r_i^-(k) = \frac{\lambda_i(k)}{1 - \lambda_i(k)}\phi_{i+1}(k). \tag{18}$$

A simplified version of the motorway model can be constructed with the assumption of no on/off ramps, as shown in Figure 6, where

$$\lambda_i(k) = 0.$$

Therefore

$$r_i^+(k) = r_i^-(k) = 0$$

and hence, we find

$$\phi_i(k) = \phi_i^+(k) = \phi_{i-1}^-(k).$$

This allows (15) to be re-written as

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{L}(\phi_i(k) - \phi_{i+1}(k)) \tag{19}$$

for all $i = 1, ..., N$ and $k = 1, ..., K$.

It should be noted that in order for the CTM to be equivalent to the LWR, the step size modelling constraint must also be satisfied

$$u_i(k)\Delta T \leqslant \Delta x_i \tag{20}$$

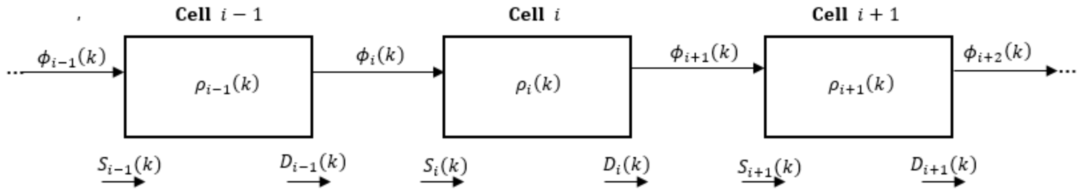

Figure 6: CTM visualised for a road stretch without on/off ramps

for all cells i of length Δx_i and time step ΔT. This ensures that any vehicle traveling maximum speed cannot travel more than one cell in one time step. It is shown in [7] that the CTM is equivalent to the LWR through the use of the Godunov scheme described in [14].

Whereas (14) - (19) can be viewed as a discritized version of (4) and (12), A vital addition to the cell transmission model are the supply and demand functions for each cell, which provide the characteristics to the discrete model that allow the traffic to flow as in the fundamental diagram, as seen in Figure 6. They are defined as follows

- $D_i(k) :=$ The demand of cell i, i.e. the flow that can be sent from cell i to cell $i+1$ in the time interval $[kT,(k+1)T)$
- $S_i(k) :=$ The supply of cell i, which is the flow that can be received by cell i from cell $i-1$ in the time interval $[kT,(k+1)T)$

and are given by

$$D_{i-1}(k) = \min\{(1-\lambda_{i-1}(k))u_{i-1}\rho_{i-1}(k), q_{i-1}^{max}\} \tag{21}$$

$$S_i(k) = \min\{w_i(\rho_i^j - \rho_i(k)), q_i^{max}\} \tag{22}$$

where q_i^{max} is the maximum capacity of cell i. Note for the simplified case (21) becomes

$$D_{i-1}(k) = \min\{u_{i-1}\rho_{i-1}(k), q_{i-1}^{max}\}. \tag{23}$$

Hence, the traffic flow for each cell i at time step k can be calculated by

$$\phi_i(k) = \min\{D_{i-1}(k), S_i(k)\}. \tag{24}$$

The demand and supply functions can be plotted as a function of the density for cell $i-1$, as shown in Figure 7, where the demand function assures that traffic is in a free-flow state for $\rho_{i-1} < \rho_{i-1}^c$ and the supply function assures that congestion begins to form in cell i, and the immediately preceding cells, if $\rho_i \geq \rho_i^c$. To this effect, it is clear that they both assure that the maximum capacity of cell $i-1$ and i cannot be exceeded and hence ensure that the traffic flow follows a steady-state relationship with the traffic density. Just as in the LWR model, the steady-state relationship Q is defined such that the conditions in (10) are satisfied. In the CTM model this is often represented by a triangular fundamental diagram, as in Figure 8, which by (24), can be viewed as a combination of the free-flow and congestion branches of the demand and supply functions respectively.

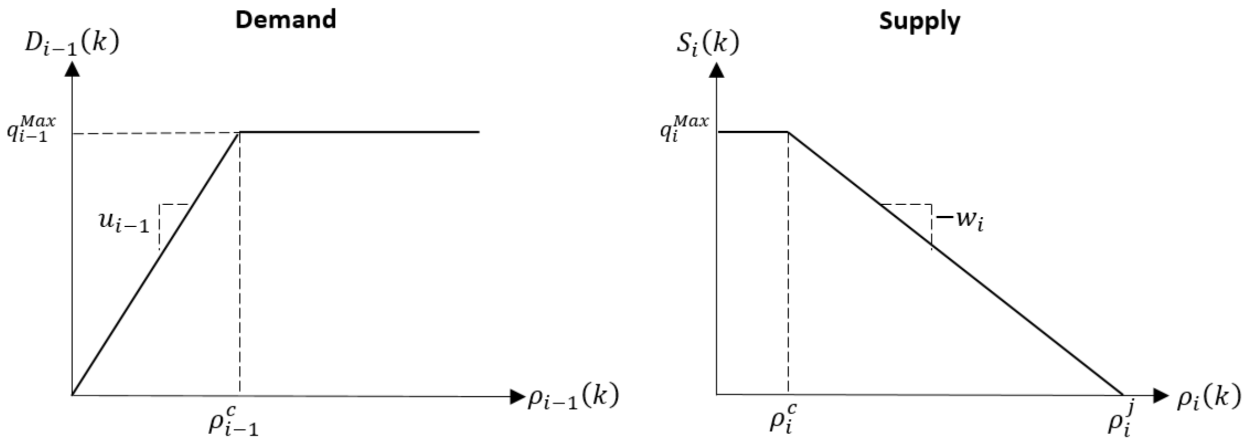

Figure 7: Demand and supply as a function of the density

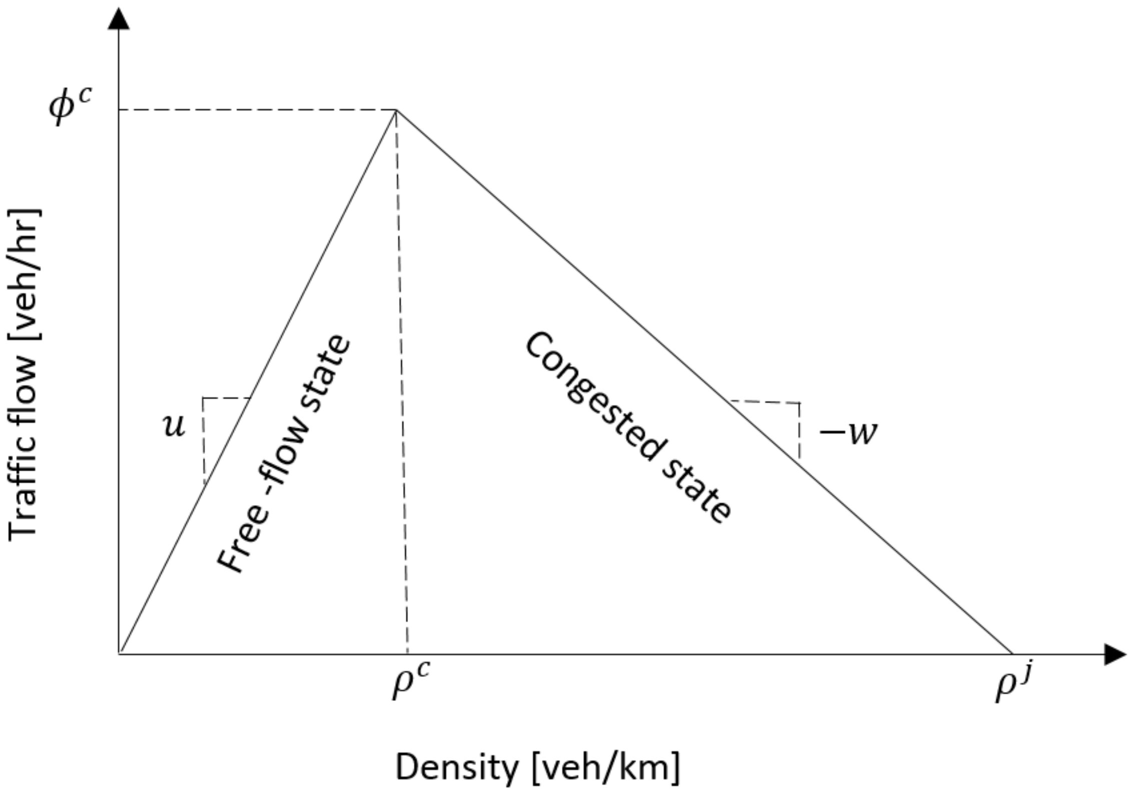

Figure 8: Triangular fundamental diagram

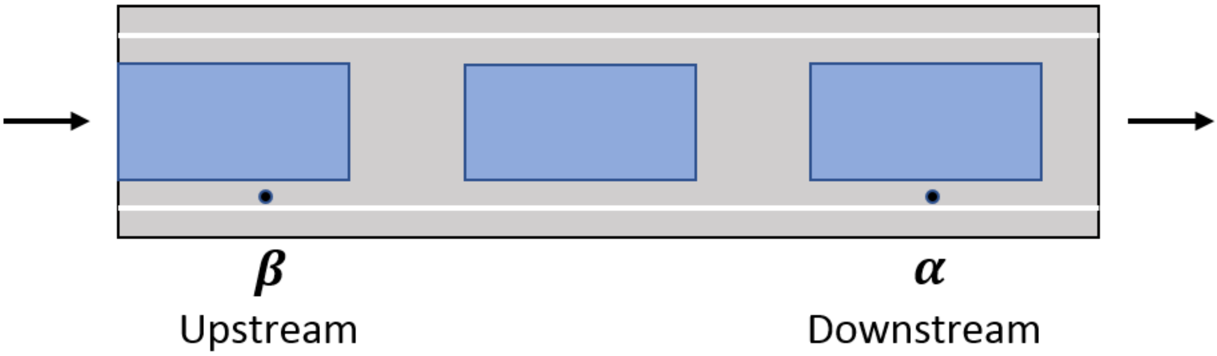

Figure 9: Position of different traffic states

3 Shock wave analysis

Another tool that can be useful when determining the evolution of traffic in the presence of an active bottleneck is shock wave analysis, where traffic is broken up into different states with different characteristics, namely the flow rate ϕ, the density ρ and the speed u. Being able to determine the effect that these changes in state have on the evolution of traffic provides a deeper understanding into the formation of congestion and exactly why it occurs, hence leading to the notion of how it can be mitigated. Throughout this section shock wave analysis will be explained for a given stretch of road, with no entrances/exits and no ability to overtake, such that the flow of traffic is assumed to follow the triangular fundamental diagram.

The triangular fundamental diagram relates the flow of traffic ϕ and the traffic density ρ by

$$\phi(\rho) = \begin{cases} u\rho & \text{if } \rho < \rho^c \\ \phi^c - \dfrac{\rho - \rho^c}{\rho^j - \rho^c}\phi^c & \text{if } \rho \geq \rho^c \end{cases} \tag{25}$$

3.1 Shock wave propagation

Now consider a stretch of motorway with two different states α and β. Take state α to be downstream of traffic with the properties ϕ_α, ρ_α and u_α and state β to be upstream of traffic with ϕ_β, ρ_β and u_β, where a point downstream of traffic is located at the right of the considered section and upstream is located at the left assuming traffic flows left to right, as shown in Figure 9.

A shock wave is the boundary at which the vehicles change speed, with the assumption that vehicles can change speed instantaneously. This means that there are no vehicles in the wave itself and hence the wave also does not have a length. Since the wave is empty the flow of vehicles entering the wave must be equivalent to the flow of vehicles exiting the wave.

Let the speed of the wave be denoted as Ω. Then the hydrodynamic equation and the equivalence relations

$$\phi = u\rho \tag{26}$$

and

$$\phi_{entrance} = \phi_{exit} \tag{27}$$

can be used to derive the the speed of the shock wave between the two different states of traffic α and β, notated as $\Omega_{\alpha\beta}$.

Assume we are viewing the wave as a moving observer, moving at the speed of the wave. Then the speed of the moving frame of reference for the downstream state α is $u_\alpha - \Omega_{\alpha\beta}$. By (26) we get

$$\phi_{entrance} = \rho_\alpha(u_\alpha - \Omega_{\alpha\beta}).$$

Similarly, the exit rate is given by

$$\phi_{exit} = \rho_\beta(u_\beta - \Omega_{\alpha\beta}).$$

From (27) we find

$$\rho_\alpha(u_\alpha - \Omega_{\alpha\beta}) = \rho_\beta(u_\beta - \Omega_{\alpha\beta})$$

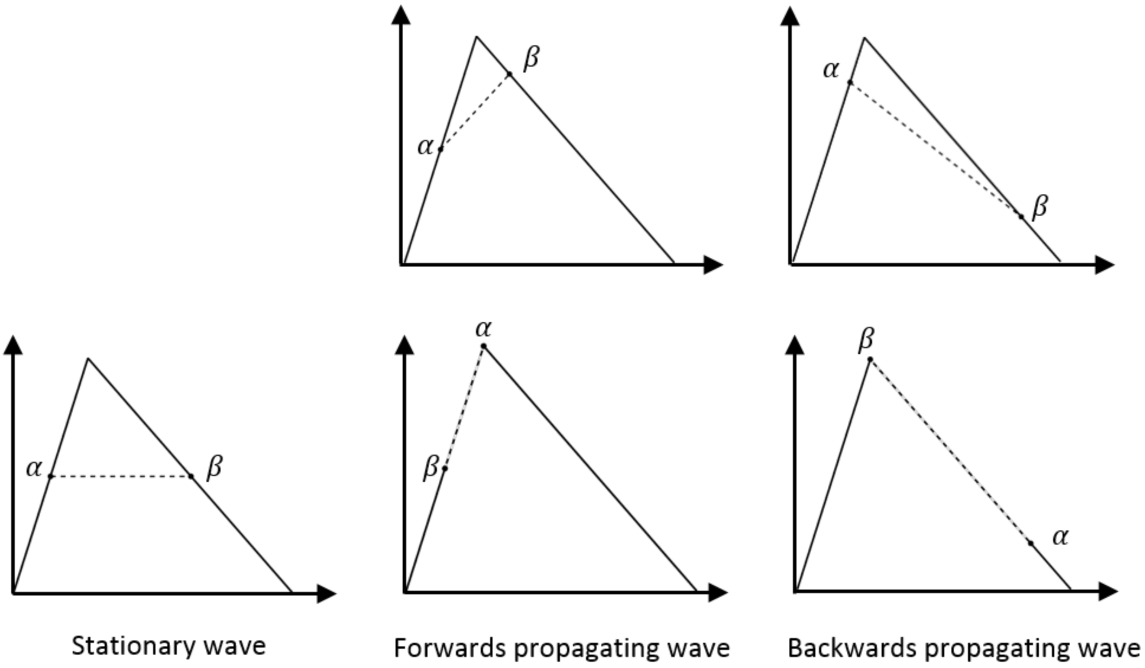

Figure 10: Graphical representation of shock waves

which can be rewritten as

$$\rho_\alpha u_\alpha - \rho_\alpha \Omega_{\alpha\beta} = \rho_\beta u_\beta - \rho_\beta \Omega_{\alpha\beta}.$$

Then using (26), we find

$$\phi_\alpha - \rho_\alpha \Omega_{\alpha\beta} = \phi_\beta - \rho_\beta \Omega_{\alpha\beta}$$

$$\phi_\alpha - \phi_\beta = (\rho_\alpha - \rho_\beta)\Omega_{\alpha\beta}$$

and hence the wave speed between state α and β is given by

$$\Omega_{\alpha\beta} = \frac{\phi_\alpha - \phi_\beta}{\rho_\alpha - \rho_\beta} = \frac{\Delta\phi}{\Delta\rho}. \tag{28}$$

Graphically this can be interpreted as the gradient of a line intersecting the two states on the fundamental diagram; α: $(\rho_\alpha, \phi_\alpha)$ and β: (ρ_β, ϕ_β). Three scenarios result based on different values of ϕ and ρ, namely a stationary shock wave, a forwards propagating shock wave and a backwards propagating shock wave.

Case 1: Stationary wave

$$\phi_\alpha = \phi_\beta \implies \Omega_{\alpha\beta} = 0.$$

In this case the shock wave remains at the same position in space and no shock wave is seen in the flow of traffic, intuitively as the respective flows are equal. Note that although the traffic flows are equal in each state that does not mean that the densities are equal, as seen in Figure 10.

An example of a stationary shock wave can be envisaged as the head of queue caused by a lane closure on a stretch of motorway. The position of the shock wave occurs where the lane closure begins and acts as the boundary between the queue forming upstream of traffic and the moment at which the cars are able to freely move through the section with the lane closure. At the instant that the car leading the queue at a given moment in time moves into the section of road with the roadworks, it moves from a congested state into a free-flow state once more, however the flow of traffic still remains the same as what it was in the queue. Hence, a change of state as described by a stationary shock wave occurs.

Case 2: Forwards propagating

Two scenarios can cause a shock wave to propagate downstream of traffic, namely

$$\left.\begin{array}{l}\phi_\alpha > \phi_\beta \text{ and } \rho_\alpha > \rho_\beta \\ \phi_\alpha < \phi_\beta \text{ and } \rho_\alpha < \rho_\beta\end{array}\right\} \implies \Omega_{\alpha\beta} > 0.$$

In these cases the shock wave is strictly positive, $\Omega_{\alpha\beta} > 0$, and moves **downstream** with the traffic. Intuitively this makes sense, as in the first scenario the shock wave follows the increase in the traffic flow and density. Whereas the second scenario is created by α being a free-flow state and β being a congested state. Then the shock wave also follows the respective decrease in traffic flow and density, except at a slower speed. The following remark can be deduced from Figure 10:

Remark. *A forwards propagating shock wave resulting from a change in state between free-flow and congested conditions is always slower than a forwards propagating shock wave between two free-flow states.*

Returning back to the example given in the case of a stationary wave, a forwards propagating shock wave occurs at the moment that the roadworks are first introduced. Here we can see a difference in the cars directly downstream of where the roadworks were instated, which remain unaffected, and the cars that go through the roadworks. The shock wave can be envisaged as a platoon of cars at a lower density, and hence further spread out from each other, following a platoon of cars at a higher density, with a shorter headway. In this case the forward propagating wave will travel with a speed equal to the free-flow speed, as both states remain in free-flow conditions.

Case 3: Backwards propagating

This can also be a result of two scenarios, namely

$$\left.\begin{array}{l}\phi_\alpha < \phi_\beta \text{ and } \rho_\alpha > \rho_\beta \\ \phi_\alpha > \phi_\beta \text{ and } \rho_\alpha < \rho_\beta\end{array}\right\} \implies \Omega_{\alpha\beta} < 0.$$

Here the shock wave is strictly negative and travels **upstream** against the flow of traffic. The first scenario occurs when both states are congested. In this case the shock wave moves with the speed of the congestion wave speed ω. The second scenario however occurs again when α is in a free-flow state and β is in a congested state. Similarly to the previous case this causes a shock wave, but of lower magnitude to the first scenario.

Remark. *A backwards propagating shock wave resulting from a change in state between free-flow and congested conditions is always slower than a backwards propagating shock wave between two congested states.*

Intuitively, a backwards propagating shock wave is the easiest to envisage as it is often made explicitly visible by the break lights of the vehicles. A simple example of a backwards propagating shock wave is the wave of break lights that flows towards you as a driver catching up to the end of a queue that is forming.

From all of the cases we can deduce the following:

$$|\Omega_{\alpha\beta}| \geq |\Omega_{\alpha\gamma}| \tag{29}$$

assuming $\phi^c \leq \rho^j/2$. If this condition is not satisfied the relation is mirrored and becomes $|\Omega_{\alpha\beta}| < |\Omega_{\alpha\gamma}|$. Where α and β are states in the same branch of the fundamental diagram and γ is a state in the respective different branch of the fundamental diagram. In other words, the shock waves are bounded by the speed at which the traffic flows, either in free-flow state or in congested state.

3.2 Bottlenecks

One of the most common reasons for changes in traffic states, and hence congestion forming due to a sudden change in traffic states, are known as bottlenecks. A bottleneck is the point in the road where less traffic can flow downstream than the demand of traffic flow that is upstream. This is often caused by a lane closure, traffic accidents or roadworks in practice. Due to the sudden drop in supply of the road at a given point x_0, congestion will start to form upstream of the bottleneck, whereas conditions downstream of the bottleneck remain in a free-flow state.

The presence of an active bottleneck on a simple stretch of motorway causes a backwards propagating shock wave with the following properties:

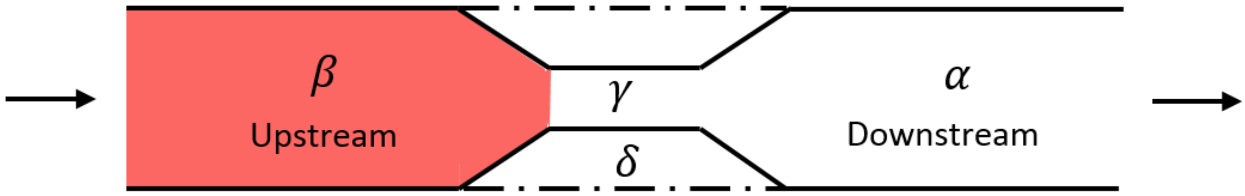

Figure 11: Bottleneck on a stretch of road

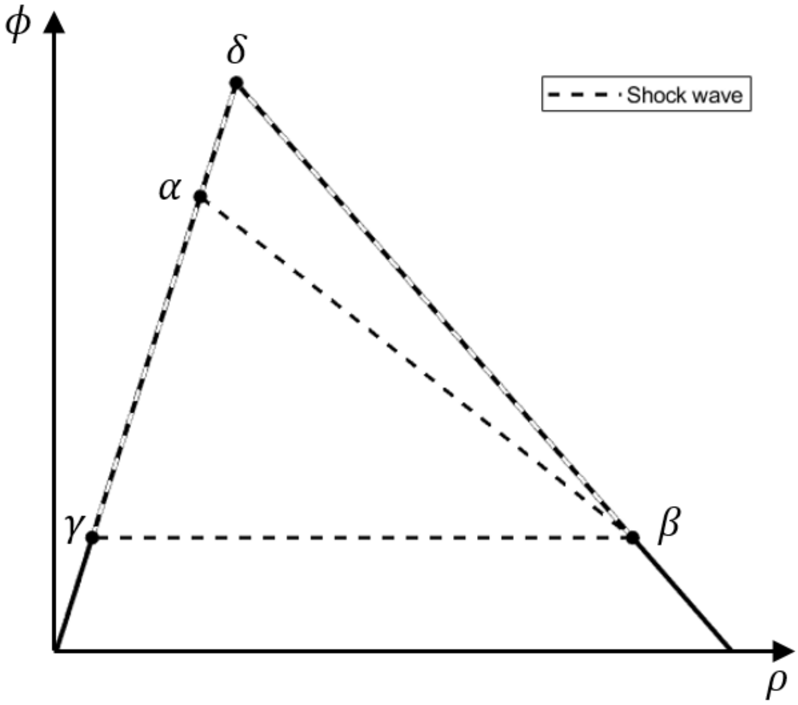

Figure 12: States of traffic during a bottleneck on the fundamental diagram

$$\phi_\alpha > \phi_\beta \text{ and } \rho_\alpha < \rho_\beta$$

such that α is a state of free-flow traffic and β is a state of congestion. The resulting congestion then forms a queue which will begin to dissipate once either the demand of traffic is below the capacity of the bottleneck or when the bottleneck becomes inactive (i.e. supply returns back to maximum road capacity). Four different traffic states can be observed in the presence of a bottleneck, as shown in Figure 11. Take α to be the free-flow state downstream of the bottleneck, where the traffic flows undisturbed as it did before the bottleneck became active. The state immediately upstream of the active bottleneck is then given by β and is a state of congestion, where ϕ_β equals the reduced supply as in the bottleneck due to the inability of the vehicle's to flow at a rate higher than the blockage directly in front of them. The state immediately downstream of the active bottleneck is then given by γ, with $\phi_\gamma = \phi_\beta$ but where the traffic is now in a free-flow state. Finally, δ is the state of traffic at the moment that the bottleneck becomes inactive, where ϕ_δ is then equal to the maximum flow capacity ϕ^c of the motorway and the vehicle's are now able move freely as before.

The shock waves that propagate from the changes of states can be visualised on the fundamental diagram, as in Figure 12. These can then be calculated as in (28). It is clear from the diagram that $\Omega_{\alpha\gamma} = \Omega_{\alpha\delta} = u$, which can be interpreted as the shock waves between the respective states moving with the speed of the traffic. It is also clear that $\Omega_{\beta\gamma} = 0$, and is hence a stationary wave. Finally, $\Omega_{\alpha\beta}$ is the backwards propagating shock wave which begins the formation of the congestion and $\Omega_{\beta\delta} = \omega$ is the backwards propagating shock wave that forms when the queue begins to dissipate. These are often referred to as the stop and go waves respectively, as they are often studied in the scenario where the bottleneck causes traffic to halt entirely. They can be envisaged as the wave of break lights travelling towards you, as a vehicle catching up to the queue, and similarly the wave of break lights going out as the vehicles in front of you remove their breaks to accelerate away as the queue begins to dissipate.

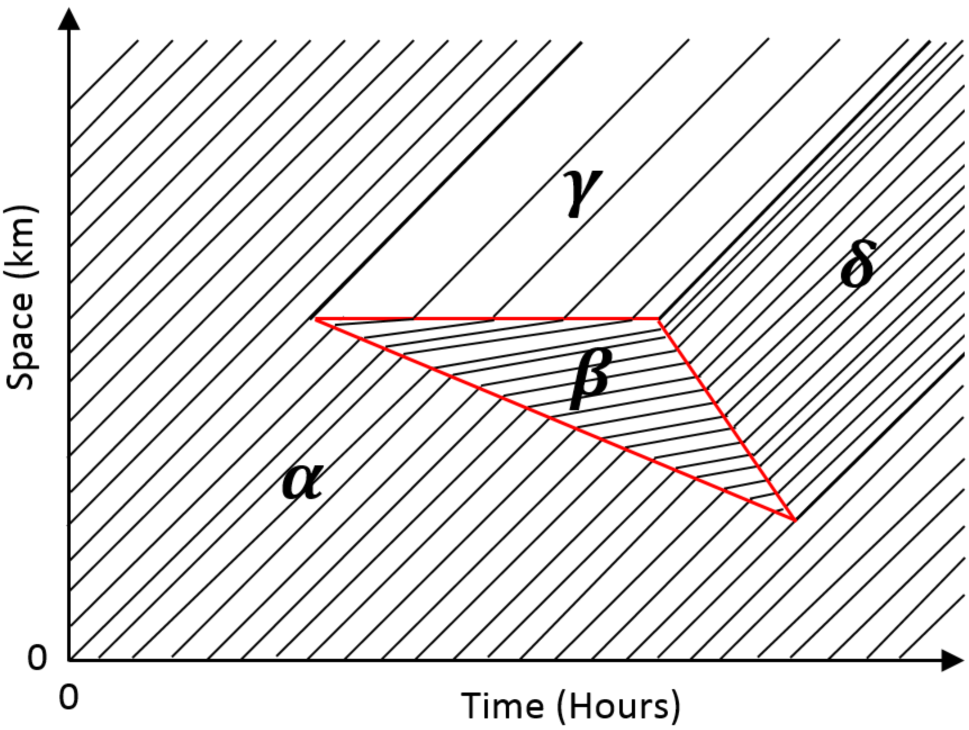

Figure 13: Space time diagram for a temporal bottleneck

3.2.1 Queuing Characteristics

By knowing the magnitude and direction of shock waves in traffic, we can predict the evolution of queue formation caused by an active bottleneck [3]. Shock wave analysis can be used to determine the following queuing characteristics:

- Q_{max}: Maximum Queue length [km]
- Q_r: The reach of the Queue [km]
- T_d: Time for queue to dissipate [hours]
- T_Q: Total queuing time [hours]

Where the reach of the queue determines the distance upstream from the bottleneck that the flow of traffic is disturbed. It can be helpful to visualise the evolution of traffic in a space-time diagram, where each line corresponds to the trajectory of one vehicle. Figure 13 shows the different traffic states, as well as the resulting shock waves. The triangle that forms around state β can be used to determine the queuing characteristics mentioned before, as seen in Figure 14. Through basic geometry it can then be deduced that

$$\frac{Q_r - Q_{max}}{|\Omega_{\alpha\beta}|} = \frac{Q_r}{|\Omega_{\beta\delta}|} \tag{30}$$

where

$$Q_{max} = T_b|\Omega_{\alpha\beta}| \tag{31}$$

for T_b being the amount of time that the bottleneck is active.

We can then be re-write (30) to determine the reach of the queue as

$$Q_r = \frac{Q_{max}}{1 - |\frac{\Omega_{\alpha\beta}}{\Omega_{\beta\delta}}|}. \tag{32}$$

The time for the queue to dissipate and the total duration of the queue can then easily be calculated by

$$T_d = \frac{Q_r - Q_{max}}{|\Omega_{\alpha\beta}|} \tag{33}$$

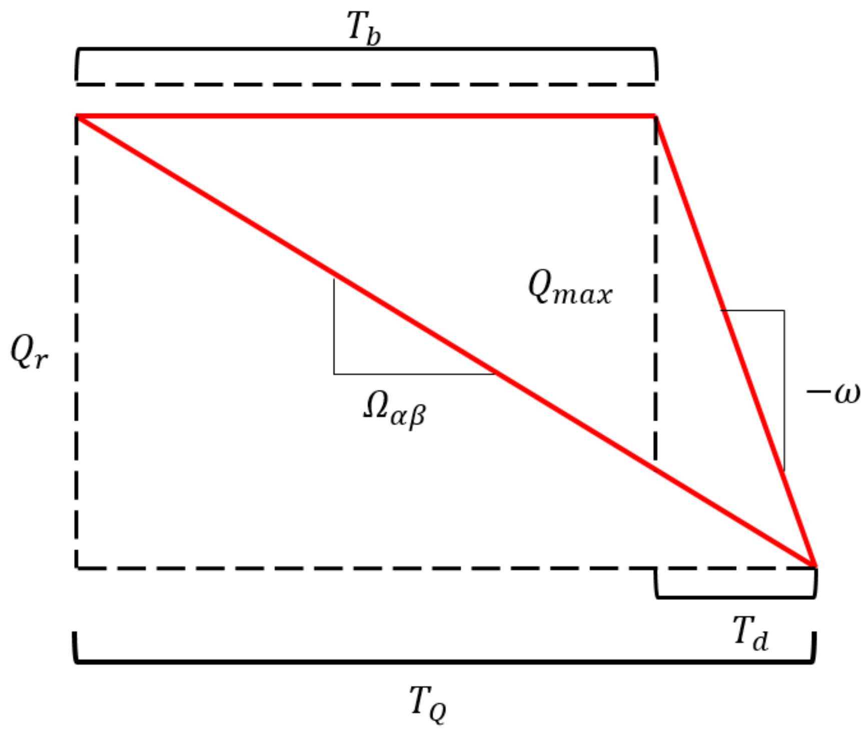

Figure 14: Queue formation on the space time diagram with temporal bottleneck

and

$$T_Q = T_b + T_d. \tag{34}$$

Note that in this scenario the bottleneck is assumed to be stationary, so (34) can also be written as

$$T_Q = \frac{Q_r}{|\Omega_{\alpha\beta}|}. \tag{35}$$

3.3 Temporal reduction of the speed limit

A traffic control technique that is often used in reality is to introduce a temporary reduction in the speed limit for a stretch of the motorway. This can be implemented through the use of electronic speed limit signs placed along the road which can be activated when necessary to the desired new speed limit. The main idea behind this technique is to increase the safety of the road situation, as vehicles may need to drastically slow down in the case of congestion. In this section, the effect of a temporal reduction in the speed limit on shock wave propagation will be analysed initially for a simple stretch of motorway with no congestion, and then for a simple stretch of motorway with a temporal bottleneck downstream. Finally, the use of this technique for the mitigation of queuing formation will be discussed.

Take u' to be a temporal reduction of the speed limit on a given stretch of motorway such that $u' < u$. Here the speed limit is assumed to be the free-flow speed, the traffic flow follows the triangular fundamental diagram and the vehicles change speed instantaneously. Assume that the remaining road characteristics remain the same, i.e:

$$\phi^{c'} = \phi^c \quad \text{and} \quad \rho^{j'} = \rho^j$$

From the hydrodynamic equation, it is clear that the critical density is increased when the reduced speed limit is in effect,

$$\rho^c < \rho^{c'} \tag{36}$$

which is an intuitive result as with a lower speed, the headway between vehicles is safely reduced. Another consequence is given by

$$|\omega| < |\omega'| \tag{37}$$

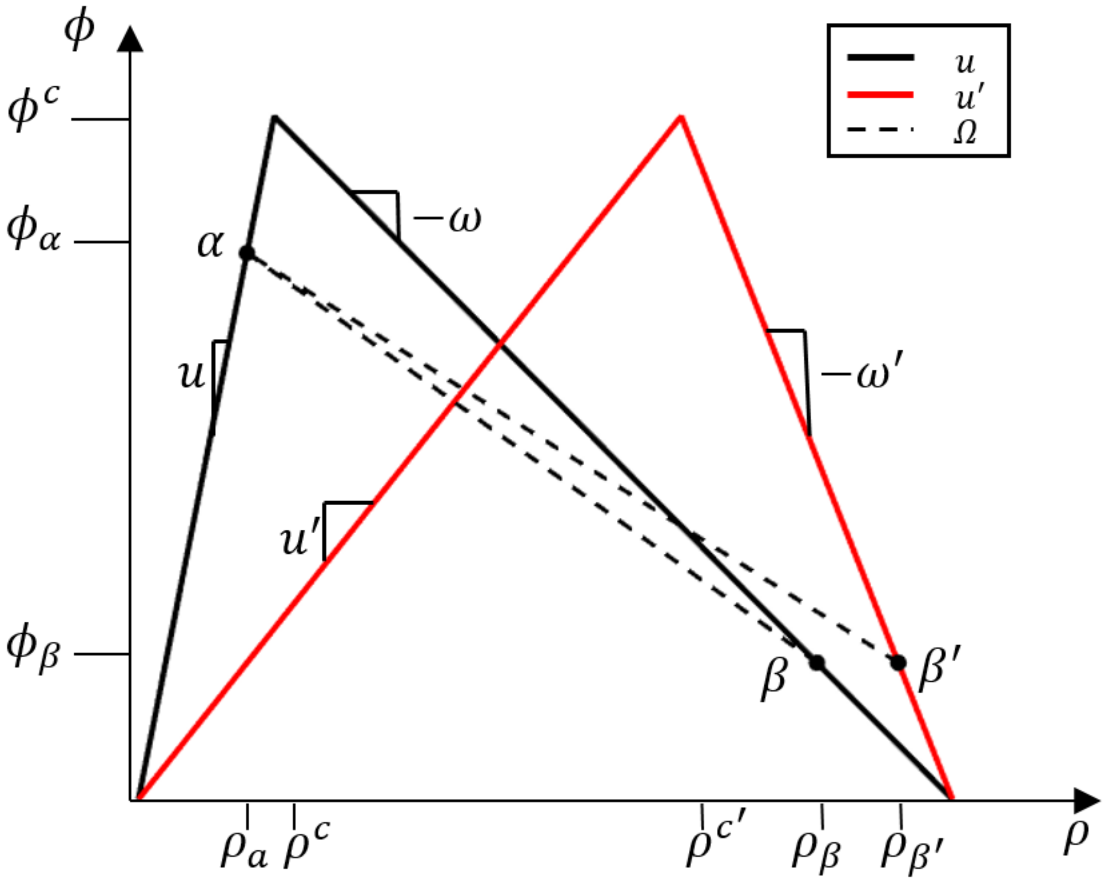

Figure 15: Comparison of fundamental diagrams with different free-flow speeds such that $u' < u$.

which states that the magnitude of the congestion wave speed is increased when a reduced speed limit is in effect. This is also expected, as with a reduced headway, the breaking distance in between vehicles is also reduced resulting in congestion forming quicker.

Now assume that there is a change of state resulting in a shock wave. We can compare the magnitude of the shock wave in a scenario where no new speed limit is introduced upstream of the traffic and where a slower speed limit u' is introduced for the state β'. As seen in 15, the free-flow and congested branches of the resultant fundamental diagram for u' have a gentler and steeper slope respectively, shifting the position of the critical density to the right and leading to (36) and (37). Alongside this, the effect on a backwards propagating shock wave is clear. Assuming that

$$\phi_{\beta'} = \phi_\beta,$$

the following relation can then be deduced

$$|\Omega_{\alpha\beta}| \geq |\Omega_{\alpha\beta'}| \tag{38}$$

indicating that the shock wave between two states with different speed limits, as described above, are lower in magnitude and hence slower.

The effect that a temporal immediate reduction in the speed limit has on the evolution of traffic depends on the density $\rho_{\alpha'}$ and the critical density ρ^c, and falls into two cases.

Case 1: free-flow state

Assume $\rho_{\alpha'} < \rho^c$. As seen in Figure 16, all of the states of traffic remain on the free-flow branches of the respective fundamental diagrams. This ensures that no congestion is formed by the shock waves that result from the sudden change in speed limit u to u'. From the space time diagram we can see how the traffic moves downstream following the shock waves in platoons of their respective density. A shock wave $|\Omega_{\epsilon'\alpha'}|$, with magnitude u', occurs at the moment the reduced speed limit is introduced. Since it is assumed here that there are no entrances/exits, the quantity of cars on the road must remain the same at the time of the speed reduction. Because of this $\rho_\epsilon = \rho_\alpha$, which in turn causes an immediate reduction in the traffic flow from ϕ_α to

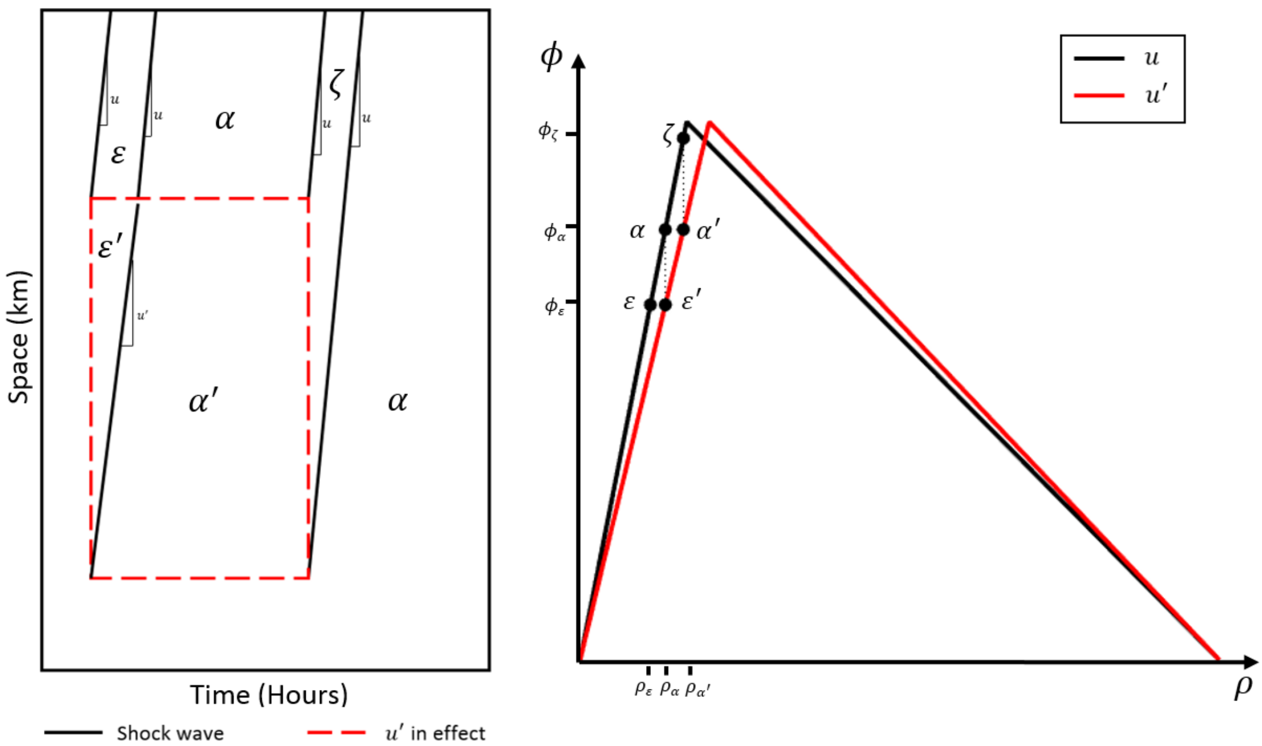

Figure 16: Evolution of traffic when $\rho_{\alpha'} \leq \rho^c$.

$\phi_{\epsilon'}$. From the hydrodynamic equation, the change in traffic flow is then directly proportional to the change in speed, such that

$$\phi_{\epsilon'} = a\phi_\alpha \tag{39}$$

where $a = \frac{u'}{u}$. It should be noted that a' is the lower limit of a given by

$$a' = \frac{\phi^c}{u\rho^j}, \tag{40}$$

which ensures that $\rho^{c'} < \rho^j$. If this constraint is not met, then the critical density would be higher than the jam density, meaning that the traffic would be in a free-flow state when it should be at a stand-still.

At the moment that the reduced speed limit is introduced, we see that the traffic immediately upstream of the new speed limit will remain at a flow rate of ϕ_α, where as the traffic that is now driving at a reduced speed in turn has a reduced flow rate of ϕ_ϵ, as the density remains at ρ_α. We see the traffic with the initial flow rate ϕ_α push into the area of the reduced speed limit and hence increase the traffic flow rate from ϕ_ϵ to $\phi_{\alpha'} = \phi_\alpha$. Then the wave of vehicles increases in density $\rho_{\alpha'} = \frac{\phi_{\alpha'}}{u'}$, which causes the shock wave $\Omega_{\epsilon\alpha'}$ as in Figure 16.

The change in density is also directly proportional to the reduction in speed, such that

$$\rho_\alpha = a\rho_{\alpha'} \tag{41}$$

and hence,

$$\Delta\phi_u = \phi_\alpha(1-a) \quad \Delta\rho_u = \rho_{\alpha'}(1-a)$$

for $\phi_\alpha > \phi_{\epsilon'}$ and $\rho_\alpha < \rho_{\alpha'} < \phi^c$.

Case 2: Congested state

Now assume that $\rho_{\alpha'} \geq \rho^c$. This causes a congestion wave $\Omega_{\delta\zeta}$ with magnitude ω at the point of transition between states δ and ζ. This, combined with the shock wave $\Omega_{\alpha\zeta}$, create the area of congestion as seen in Figure 17.

The duration of the congestion depends on $\rho_{\alpha'}$ and ρ^i, where ρ^i is the moment of intersection between the free-flow and congestion branches of the fundamental diagrams and ϕ^i is then the corresponding traffic flow.

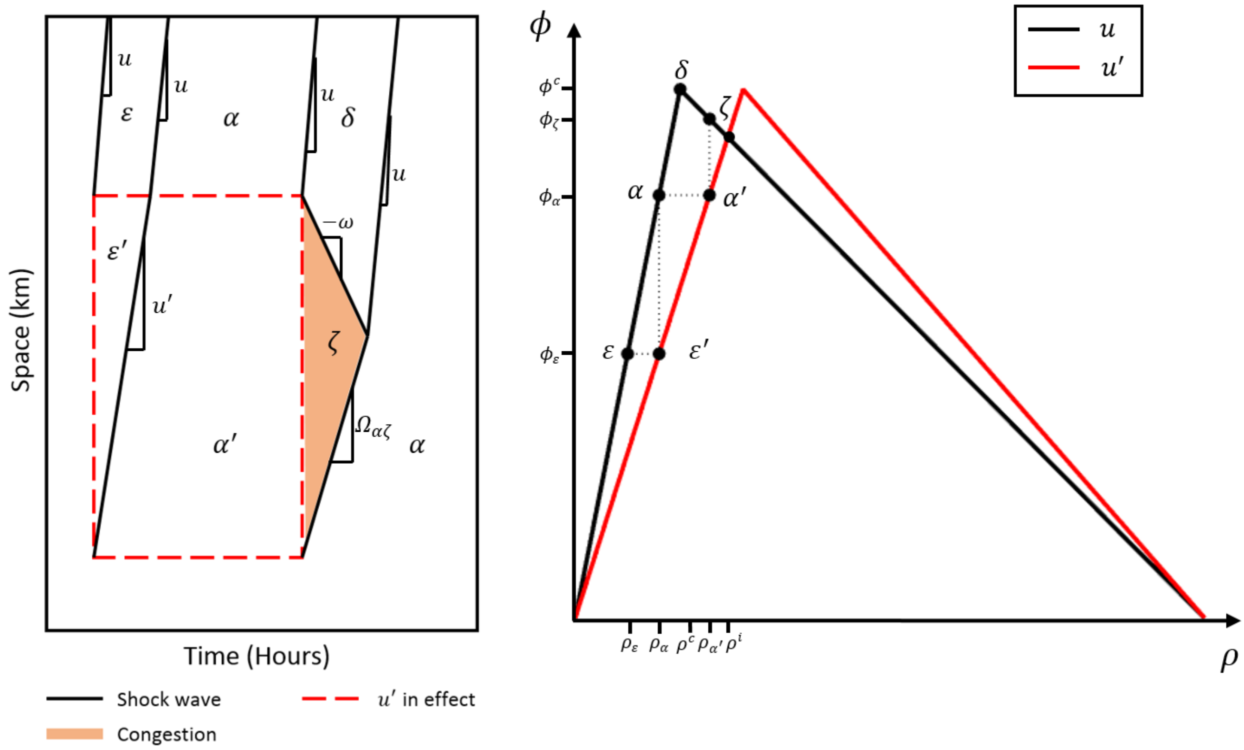

Figure 17: Evolution of traffic when $\rho_{\alpha'} > \rho^c$.

Figure 17 demonstrates the case where $\rho_{\alpha'} \leq \rho^i$, meaning $\phi_\alpha \leq \phi^i$ and $\phi_\zeta \geq \phi^i$. In this scenario $\Omega_{\alpha\zeta} \geq 0$ and is hence a forwards propagating shock wave.

Now let $\rho_{\alpha'} > \rho^i$. Similarly, the corresponding traffic flows are then $\phi_\alpha > \phi^i$ and hence $\phi_\zeta < \phi^i$. By (29), this then creates a backwards propagating shock wave $\Omega_{\alpha\zeta} < 0$ of magnitude $|\Omega_{\alpha\zeta}| \leq |\omega|$. This results in an area of congestion as shown in Figure 18. This begins the formation of a queue, the characteristics of which depend on the point of intersection of $\Omega_{\delta\zeta}$ and $\Omega_{\alpha\zeta}$.

In order to avoid congestion it is then suggested to choose u' such that the conditions of the free-flow case are met wherever possible.

3.3.1 Bottleneck with temporal reduction in speed limit

We can determine how an introduction of a reduced speed limit u' during an active bottleneck effects the queuing formation by evaluating the queuing characteristics with $u' < u$. Assume u' is introduced upstream of the bottleneck sometime between when it becomes active and when it becomes inactive. Also assume that immediately downstream of the bottleneck, traffic returns to the original speed limit u. As all the vehicles are assumed to instantaneously change speed, the state directly upstream of the bottleneck will change and effect the formation of the queue. Exactly how it is effected depends on three characteristics of the temporal speed limit u'. These are

- $|u'| :=$ The magnitude of the temporal speed limit
- $u'_r :=$ The reach of the temporal speed limit
- $u'_d :=$ The duration of the temporal speed limit

All these factors intertwine with each other to determine the formation of the queue directly upstream of the bottleneck. In order to analyse their effect, they will each be studied individually.

Magnitude of u'.

Assume that u'_r and u'_d are chosen such that the forwards propagating shock wave created by the change in speed does not intersect the queue upstream of the bottleneck at any point. From (31), in order to minimise the maximum queue we must minimise the magnitude of the stop wave. This is achieved through choosing u'

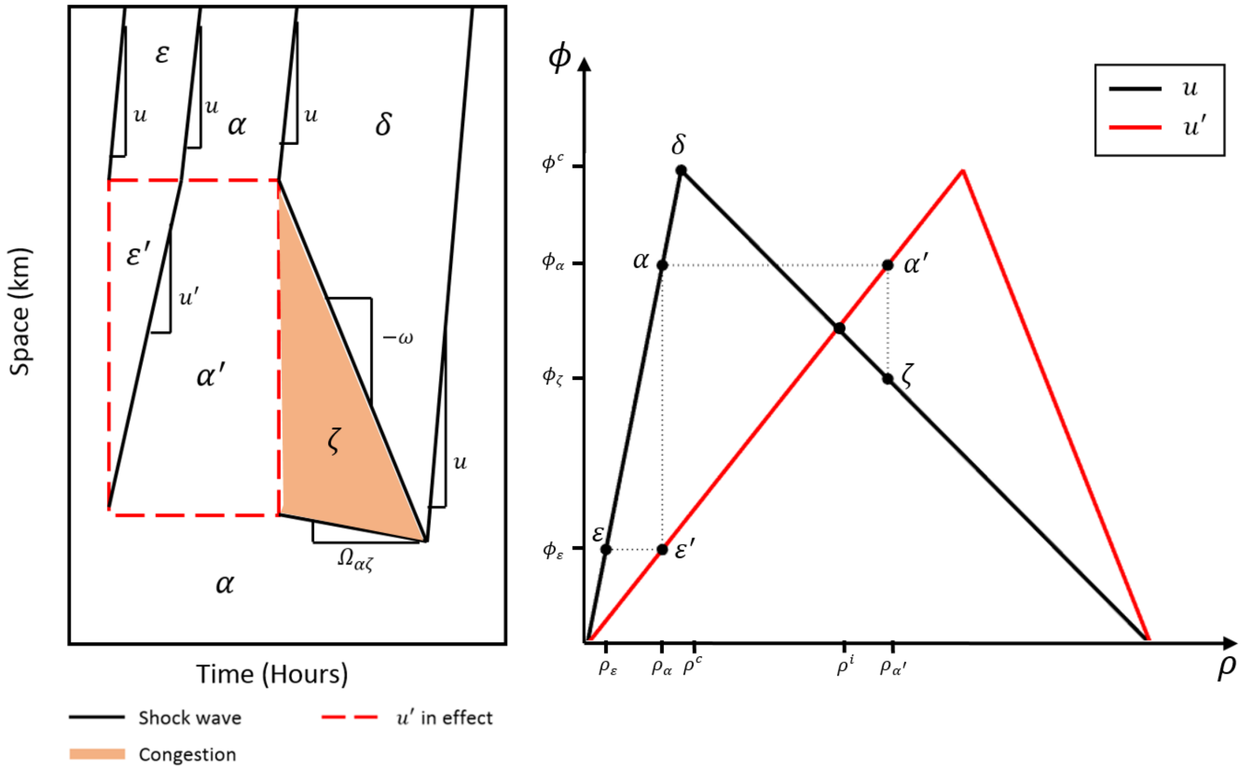

Figure 18: Evolution of traffic when $\rho_{\alpha'} > \rho^i$.

such that the traffic flow directly upstream of the bottleneck is the same, or even less, than the flow at the bottleneck itself. This mitigates the queue entirely by making $|\Omega_{\beta'\epsilon'}| \leq 0$, assuming $\phi_\gamma \neq 0$. From this the optimum reduced speed limit is given by

$$\frac{\phi^c}{\rho^j} < u' \leq \frac{\phi_\beta}{\rho_\alpha}, \tag{42}$$

where $\dfrac{\phi^c}{\rho^j} = a'u$ is the lower bound for u'. If this condition cannot be satisfied, then u' can be chosen such that $|\dfrac{\phi_\beta}{\rho_\alpha} - u'|$ is minimised.

Assume $\dfrac{\phi^c}{\rho^j} < u' = \dfrac{\phi_\beta}{\rho_\alpha}$. Then $|\Omega_{\beta'\epsilon'}| = 0$ and hence $Q_{max} = 0$. Note that this also means that $Q_r = 0$ and the queue is entirely mitigated.

Now assume $\dfrac{\phi^c}{\rho^j} < u' < \dfrac{\phi_\beta}{\rho_\alpha}$. This would change the stop wave from an backwards to a forwards propagating shock wave. In this case, the state immediately upstream of the bottleneck becomes a free-flow state instead of a congested state, and hence the effects of the bottleneck are entirely mitigated.

Although the optimum speed limit may mitigate the effects of the bottleneck, we have already shown that if u' is too low then a congested state is formed at the moment that u' is taken out of effect. Hence to ensure a free-flow state upstream of the bottleneck, the constant a should be chosen such that

$$\frac{\rho_\alpha}{a} = \rho_{\alpha'} < \rho^c \implies \frac{\rho_\alpha}{\rho^c} < a < 1$$

and hence,

$$\frac{u\rho_\alpha}{\rho^c} < u' < u. \tag{43}$$

In reality it may not be practical to introduce u' as its "optimum" magnitude as described above. In this case the other characteristics can be used to attempt to mitigate the queue formation.

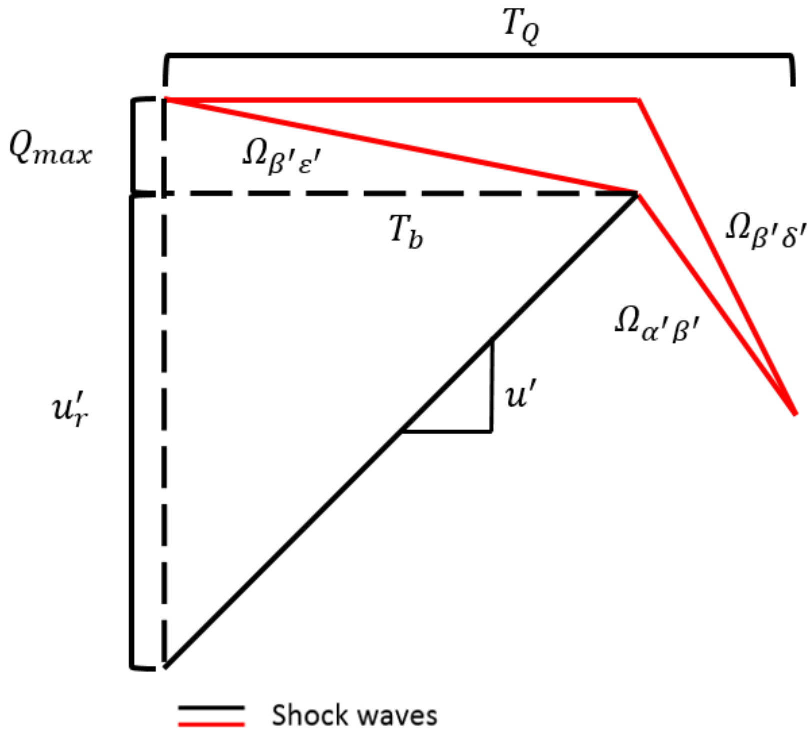

Figure 19: The reach of the temporal speed limit

The reach of u'

The reach of the temporal speed limit u'_r is defined as the distance upstream from the bottleneck that u' is put into effect. Assume that $|u'|$ is large enough so that $\Omega_{\beta'\epsilon'} < 0$ and $u'_d \geq T_Q$. The effect of this on the queue depends on where the forwards propagating shock wave $\Omega_{\alpha'\epsilon'}$ that results from the introduction of u' catches up with the stop and go waves created by the bottleneck.

In order to minimise Q_{max}, we want the flow directly upstream of the bottleneck to be as low as possible. As $\phi_\epsilon < \phi_\alpha$, by (38)

$$|\Omega_{\beta'\epsilon'}| < |\Omega_{\alpha'\beta'}| \tag{44}$$

and hence,

$$T_b|\Omega_{\beta'\epsilon'}| < T_b|\Omega_{\alpha'\beta'}|. \tag{45}$$

Therefore, u'_r should be large enough such that $\Omega_{\alpha'\epsilon'}$ catches up with the queue at the moment that the bottleneck becomes inactive. From Figure 19 and the pythagoras theorem

$$u'_r = \sqrt{(u'T_b)^2 - (T_b)^2} + Q_{max}. \tag{46}$$

Where T_b is the time moment that $|\Omega_{\epsilon'\alpha'}|$ intersects the queue formation. Note that if $\Omega_{\epsilon'\alpha'}$ intersects the stop wave then it is split in two as

$$\text{Stop wave} = \begin{cases} \Omega_{\beta'\epsilon'}, & \text{left side} \\ \Omega_{\alpha'\beta'}, & \text{right side} \end{cases} \tag{47}$$

creating an area of congestion shaped like a boomerang, as in Figure 19.

By minimising the reach of the queue, we also inadvertently minimise the dissipation time and hence the total queuing time. by (44) it stands to reason that the minimal Q_r occurs when $\Omega_{\alpha\epsilon'}$ catches up to the queue no sooner than at the tail of the queue. This is visualised in Figure 20 where

$$u'_r = \sqrt{(u'T_q)^2 - (T_q)^2} + Q_r. \tag{48}$$

Then the stop and go waves have magnitude $|\Omega_{\beta'\epsilon'}|$ and $|\Omega_{\beta'\delta'}|$, respectively. With u'_r chosen as in (48) or larger, T_d and T_Q are both minimised.

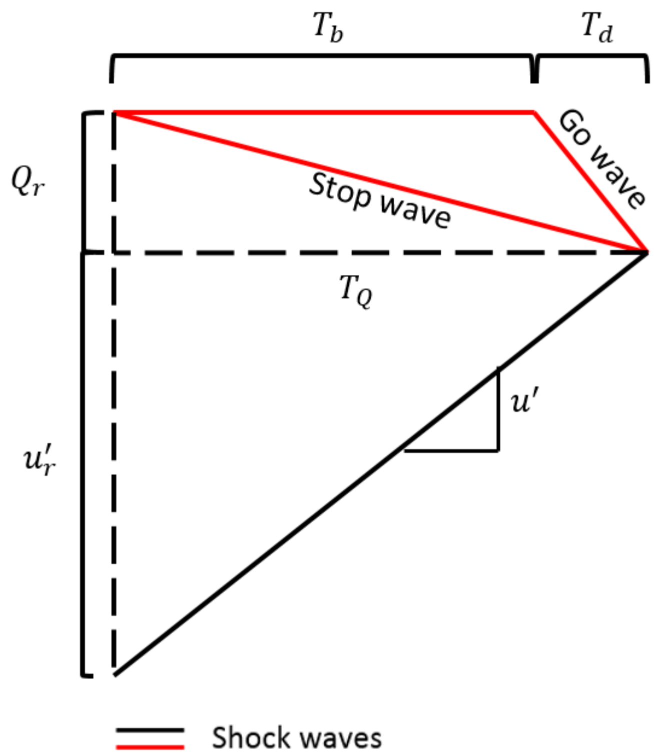

Figure 20: Shock wave arriving just as the queue dissipates

Duration of u'

Finally, the duration of the speed limit is also important to the mitigation of the queue. From the other two characteristics, it should be clear that if $u'_d \geq T_Q$ then the best case scenario is to occur as the stop wave and the go wave are of the lowest and highest magnitude respectively. Assume u'_r and $|u'|$ are fixed, then u'_d should be chosen such that the forwards propagating wave with magnitude $|u|$, created at the moment that the temporal reduction in speed is lifted, intersects the queue as late as possible, if at all.

Note that if $u'_d < T_b$, and u'_r is given such that both $\Omega_{\alpha\epsilon'}$ and $\Omega_{\alpha\zeta}$ catch up to the queue, then the stop wave is split into three pieces

$$\text{Stop wave} = \begin{cases} \Omega_{\beta'\epsilon'}, & \text{left most} \\ \Omega_{\beta\zeta} \text{ or } \Omega_{\beta\delta}, & \text{middle} \\ \Omega_{\alpha\beta}, & \text{right} \end{cases}$$

where the middle part of the wave is $\Omega_{\beta\zeta}$ if the change in u is high enough for upstream to remain in a free-flow state and $\Omega_{\beta\delta}$ otherwise. This can be seen in Figure 21. as $|\Omega_{\beta'\epsilon'}| < |\Omega_{\alpha\beta}| < |\Omega_{\beta\zeta}| < |\Omega_{\beta\delta}|$ it stands to reason that $|u'_d - T_b|$ should be minimised in order to minimise the formation of the queue.

All these factors can be tailored to suit the individual situation, and hence make viable techniques to reduce the disruption of a bottleneck as much as possible.

Rate of improvement

The rate of improvement to the mitigation of the congestion formed upstream of a bottleneck can be evaluated to determine whether a measure is worth putting into place.

Note that for $\rho^{c'} \leq \rho_\beta < \rho_{\beta'}$, then by (25)

$$\rho_\beta = (\rho^j - \rho^c)\eta + \rho^c \tag{49}$$

where $\eta = 1 - \dfrac{\phi_\beta}{\phi^c}$. Then,

$$\rho_\beta = b\rho'_\beta \tag{50}$$

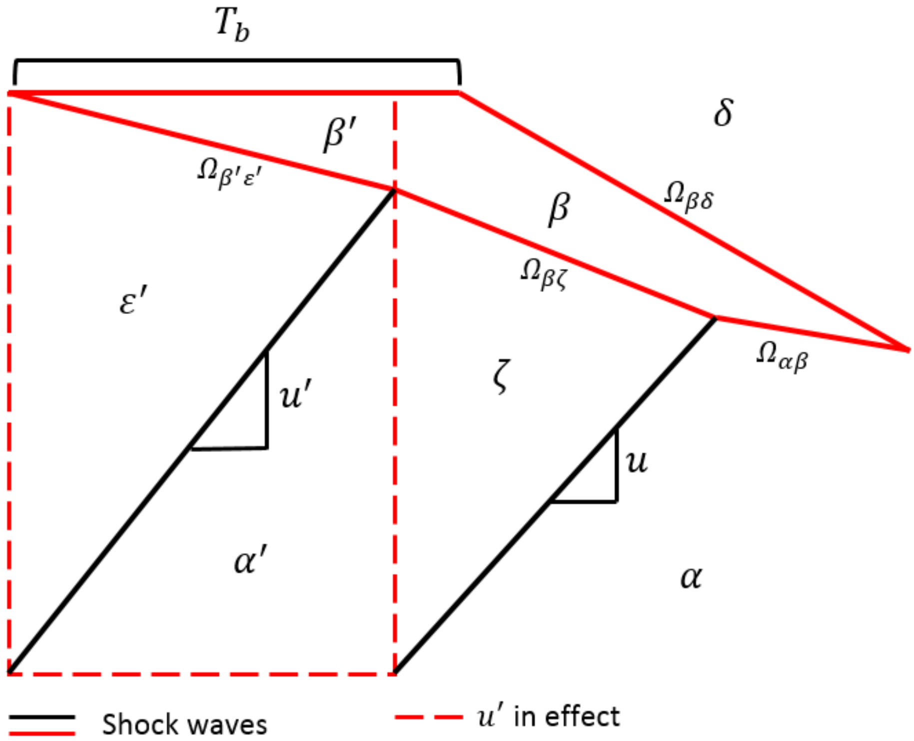

Figure 21: 3-piece stop wave

for some $b \in \mathbb{R}_{>0}$ such that

$$b = \frac{(\rho^j - a\rho^{c'})\eta + a\rho^{c'}}{(\rho^j - \rho^{c'})\eta + \rho^{c'}}, \tag{51}$$

gives the rate of change of the congested densities whilst u' is in effect. Similarly, the rate of change between the resultant shock waves is determined by (38), such that

$$|\Omega_{\alpha\beta'}| = c|\Omega_{\alpha\beta}| \tag{52}$$

for some $c \in \mathbb{R}_{>0}$, where

$$c = \frac{|\rho_\alpha - \rho_\beta|}{|\rho_\alpha - \rho_{\beta'}|} = \frac{|\rho_\alpha - b\rho_{\beta'}|}{|\rho_\alpha - \rho_{\beta'}|} \tag{53}$$

for some $\phi_\alpha > \phi_\beta$ and $\rho_\alpha < \rho^c$.

Then the rate of improvement regarding the maximum queue is given by

$$\frac{Q_{max'}}{Q_{max}} = \frac{T_b|\Omega_{\alpha\beta'}|}{T_b|\Omega_{\alpha\beta}|} = \frac{|\Omega_{\alpha\beta'}|}{|\Omega_{\alpha\beta}|} = c. \tag{54}$$

The constant $d \in \mathbb{R}_{>0}$ can also be used to determine the effect of the change in speed limit on the reach of the queue, given by

$$d = \frac{|\Omega_{\alpha\beta'}||\Omega_{\beta'\delta}|(|\Omega_{\beta\delta}| - |\Omega_{\alpha\beta}|)}{|\Omega_{\beta\delta}||\Omega_{\alpha\beta}|(|\Omega_{\beta'\delta}| - |\Omega_{\alpha\beta'}|)} \tag{55}$$

where $Q_{r'} = dQ_r$. Clearly $Q_{r'} < Q_r$ if $d < 1$ and $Q_{r'} \geq Q_r$ if $d \geq 1$.

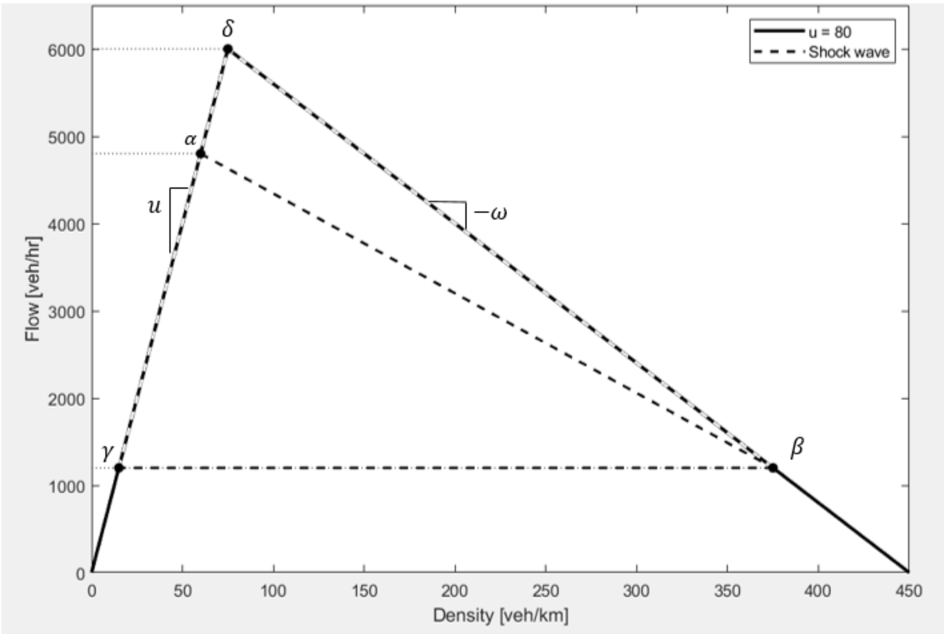

Figure 22: Fundamental diagram with all traffic states and resulting shock waves for $u = 80$.

4 Example using the Cell Transmission Model

In order to verify the theory established in the previous section, a simulation using the cell transmission model will be used to depict the density and subsequent traffic flow using a given example. As mentioned previously, the cell transmission model is a well established model used for simulating motorway traffic behaviour. For the purposes of this investigation the basic cell transmission model, as in (19), for a simple stretch of motorway is coded into Matlab and used to simulate an active bottleneck. The goal of this is to then compare the results of the model to the theoretical results for the base case and various temporal reductions in the speed limit, as calculated by the restraint on a. To illustrate this, an example is given.

Example: Bottleneck with temporal reduction in speed limit

4.1 Base case: $u = 80$

Suppose there is a stretch of motorway with a maximum flow capacity of 6000 vehicles per hour, jam density of 450 vehicles per kilometer and an free-flow speed of 80 km per hour. Traffic is flowing at an 80 percent rate of full capacity. A minor accident occurs at some position x_0 causing a reduction of traffic flow to 1200 vehicles per hour for a total of 4 minutes. After the 4 minutes have passed, the road is cleared and returns back to its original state. We can determine the evolution of traffic and the queuing formation that occurs as a result of the bottleneck assuming that the traffic follows a triangular fundamental diagram.

Firstly, we can identify the different traffic states as

- α: Free-flow conditions downstream
- β: Upstream congested conditions while bottleneck is active
- γ: Downstream free-flow conditions while bottleneck is active
- δ: Conditions once the bottleneck becomes inactive.

It is clear that $\phi_\alpha = 4800$, $u_\alpha = 80$ and hence $\rho_\alpha = 60$. Similarly, $\phi_\gamma = 1200$, $u_\gamma = 80$ and $\rho_\gamma = 15$. From Figure 22 we see that $\phi_\delta = \phi^c = 6000$, $u_\delta = 80$ and $\rho_\delta = \rho^c = \dfrac{6000}{80} = 75$. Since β is a congested state ρ_β can be computed by (49). As $\phi_\beta = 1200$, we get $\rho_\beta = 375$ and consequently $u_\beta = 3.2$.

We can use these to calculate the shock wave speed, as in (28). From this we find $\Omega_{\alpha\gamma} = \Omega_{\alpha\delta} = \Omega_{\gamma\delta} = 80 = u$, $\Omega_{\beta\delta} = -16 = -\omega$ and $\Omega_{\beta\gamma} = 0$. These can also be directly read from Figure 22. Finally,

$$\Omega_{\alpha\beta} = \frac{4800 - 1200}{60 - 375} = -\frac{80}{7} \approx -11.4km/h.$$

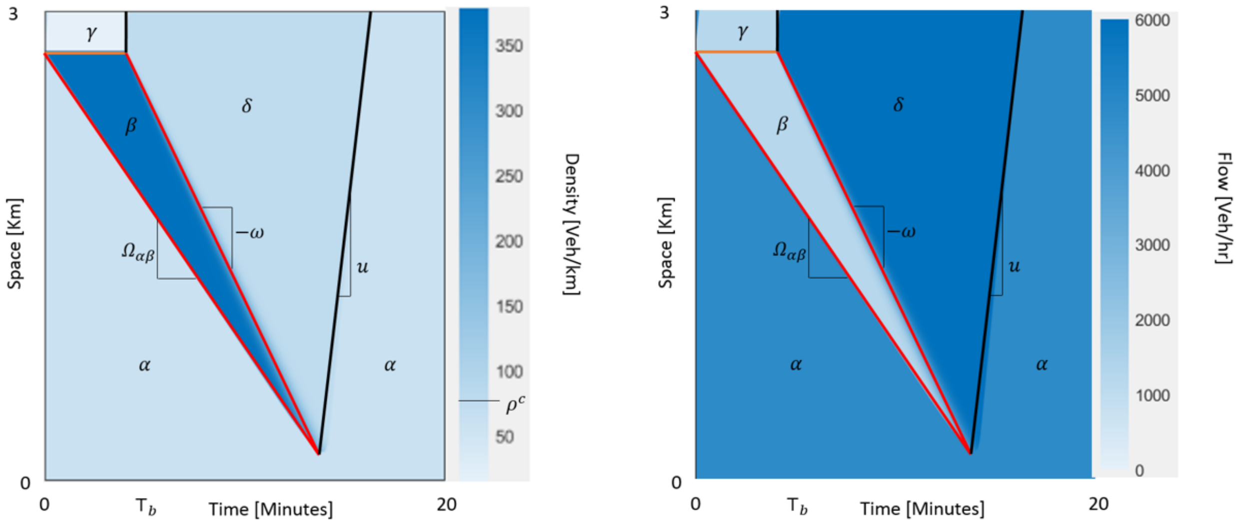

Figure 23: Density of traffic upstream of the bottleneck for $u = 80$

In order to determine the queue dynamics, we can use (30) - (34) to get

$$Q_{max} = \frac{16}{21} \approx 0.76km \qquad Q_r = \frac{8}{3} \approx 2.7km$$
$$T_d = \frac{1}{6} \text{ hour} = 10 \text{ minutes} \qquad T_Q = 14 \text{ minutes}$$

Setting up the simulation

For the purpose of the simulation, we can use the simplified cell transmission model as in (19), as it is assumed that there are no on/off ramps on the stretch of motorway considered. The main body of the code is presented in Appendix A. For the problem set-up, the following initial conditions are specified:

- $T = \frac{1}{18000}$ hours $= \frac{1}{5}$ seconds
- $\phi^c = 6000$ veh/hr
- $\rho^j = 450$ veh/km
- $D_0 = 4800$ veh/hr

where D_0 is the initial demand for the first cell for all $k = 1, ..., K$. To minimise error, T has been chosen small enough so that $\Delta x_i = 5$m for all $i = 1, ..., N$ and the step size modelling constraint in (20) is satisfied for a maximum speed of 80 km/hr. To simulate the active bottleneck, the maximum capacity of one cell is fixed at 1200 veh/hr for the first 4 minutes before returning back to ϕ^c for the remaining time steps, as the length of the bottleneck itself doesn't have an effect on the downstream queue formation. As traffic flows normally upstream of the bottleneck, the simulation focuses on the downstream congestion that is formed, although the location of the bottleneck is determined for each case depending on the expected evolution of traffic after the bottleneck becomes inactive.

From the above theory, the simulated stretch of motorway is 3km long. Figure 23 shows the evolution of traffic via a heat map of the density and subsequent traffic flow of cell i at time step k. It is clear that the formation is representative of the time space diagram in Figure 14, as expected.

The traffic flow $\phi_i(k)$ and density $\rho_i(k)$ for all $i = 1, ..., N$ at time step $k = 1, .., K$ are stored in matrices, and can subsequently be used to determine the queuing characteristics Q_{max}, Q_r, T_d and T_Q to the level of accuracy of the simulation. From examining the data the following values are found:

$$Q_{max} = 740\text{m} \qquad Q_r = 2.56 \text{ km}$$
$$T_d = 9 \text{ minutes } 59 \text{ seconds} \qquad T_Q = 13 \text{ minutes } 59 \text{ seconds}$$

From this the resultant backwards propagating shock waves upstream of the bottleneck can be read from the simulation as

$$|\Omega^s_{\alpha\beta}| = 11.1km/hr$$
$$|\Omega^s_{\beta\delta}| \approx 15.6km/hr$$

which are both smaller in magnitude than the expected values from the theory. The error estimation for the simulation is given by the above calculated shock wave speeds divided by the expected shock wave speeds

$$E_{base} = 1 - \frac{|\Omega^s_{\alpha\beta}|}{|\Omega_{\alpha\beta}|} \approx 0.028 \tag{56}$$

and hence the simulation is 97 percent accurate to the theory, given the chosen time steps and cell lengths. This is simply a result of the discritisation and the accuracy can be increased through decreasing the time steps and hence the cell lengths. However for the purposes of this report this will be viewed as negligible from this point on.

4.2 Introduction of temporal reduction in the speed limit

Using the same initial conditions, the simulation is set up so that a temporal reduction in the speed limit can be introduced. This is used to investigate the effect of introducing three different reduced speed limits, chosen based off of the theory discussed in section 3.3.1 on the magnitude of u'. For each simulation u' is introduced at the optimum distance from the bottleneck u'_r and is in effect until immediately downstream of the bottleneck. The duration of u' will also be set to T_Q, so that any effects of the reintroduction of u will not effect the queuing formation directly upstream of the bottleneck.

Complete mitigation of bottleneck: $u' = 20$

As previously shown, in order for $Q_{max} = 0$ the stop wave must be a stationary wave, namely $\Omega_{\beta'\epsilon'} = 0$. To achieve this, u' is chosen such that

$$u' = \frac{\phi_\beta}{\rho_\alpha} = 20km/hr.$$

As $Q_r = 0$, the duration of the temporal reduction in speed limit is chosen to be equal to the duration of the bottleneck T_b. From this we can determine the distance from the bottleneck to introduce u' as $u'_r = 1.34$km. Figure 25 shows the density and traffic flow of the traffic upstream of the bottleneck respectively after the introduction of u' and the re-introduction of u. Here the simulation was run over a stretch of motorway 4 km long.

We can see that although the queue immediately upstream of the bottleneck is entirely mitigated, the moment when u' is lifted causes a form of bottleneck itself. This is caused by the backwards propagating shock waves $\Omega_{\alpha\zeta} = -8km/hr$ and $\Omega_{\delta\zeta} = -\omega = -16$, as seen on the fundamental diagram in Figure 24. This is due to $\rho_{\alpha'} > \phi^i$, which causes $\phi_\zeta < \phi_\alpha$ and ρ_ζ to be on the congested branch of the fundamental diagram for $u = 80$ km/hr. Hence when the traffic flowing at $\phi_\alpha = 4800$ veh/hr catches up with the vehicles that are flowing at $\phi_\zeta = 3360$, a queue is formed. We can calculate the characteristics of this queue formed by the sudden change in speed as

$$Q_{max} = u'_r = 1.34\text{km} \qquad Q_r = 2.06 \text{ km}$$
$$T_d = 9 \text{ minutes } 59 \text{ seconds} \qquad T_Q = 13 \text{ minutes } 59 \text{ seconds}$$

where these characteristics were read directly from the simulation. Hence, choosing u' so low in comparison u makes a minor reduction in the length of the queue but essentially just delays the effect of the bottleneck and pushes it further upstream.

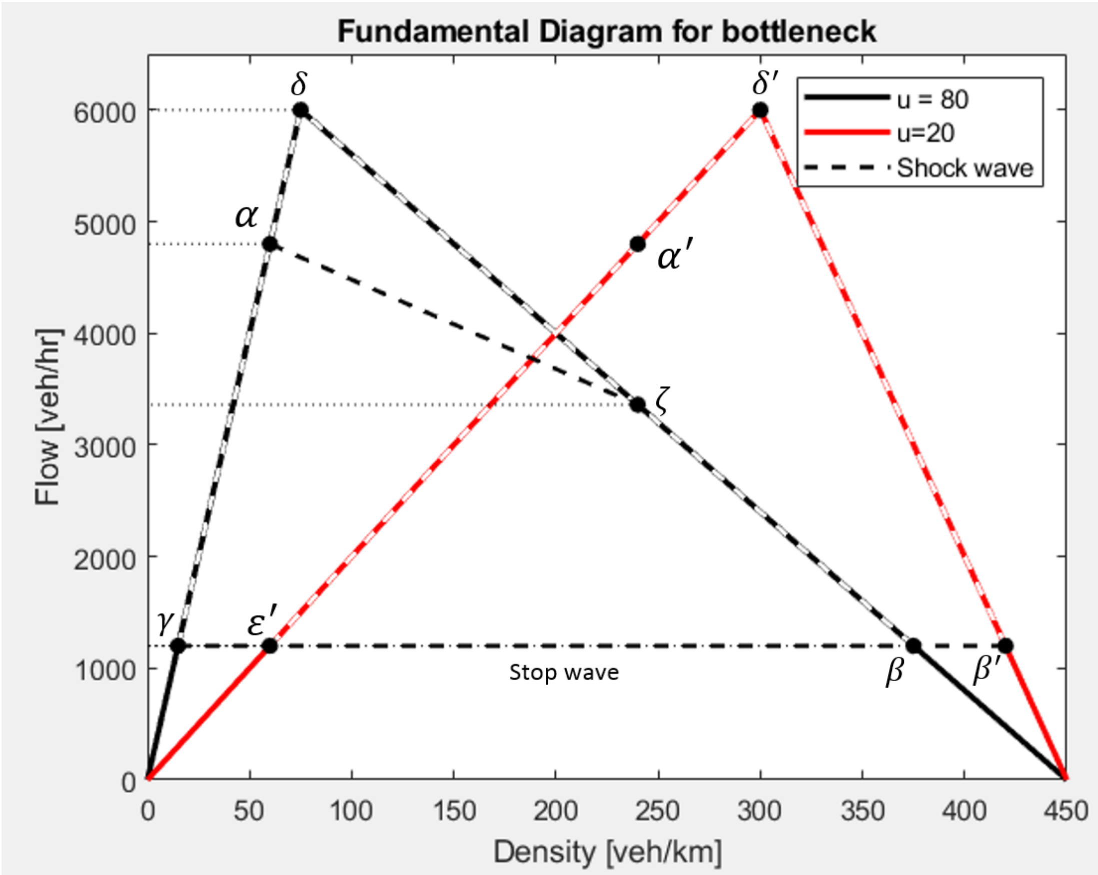

Figure 24: Fundamental diagram for $u = 80$ and $u' = 20$

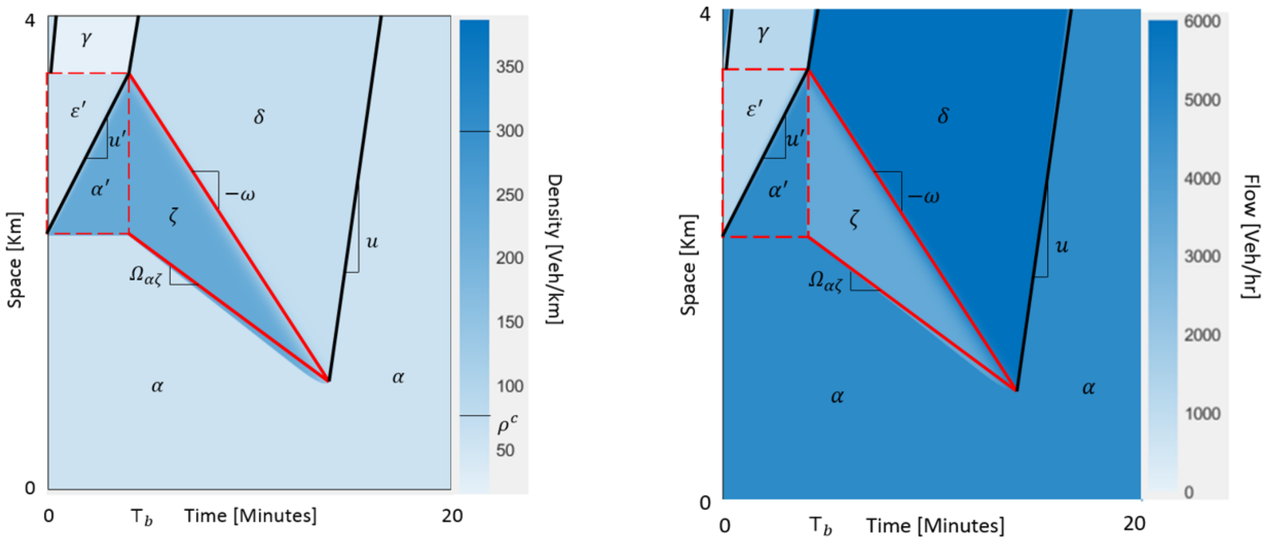

Figure 25: Density and flow of traffic upstream of the bottleneck for $u' = 20$

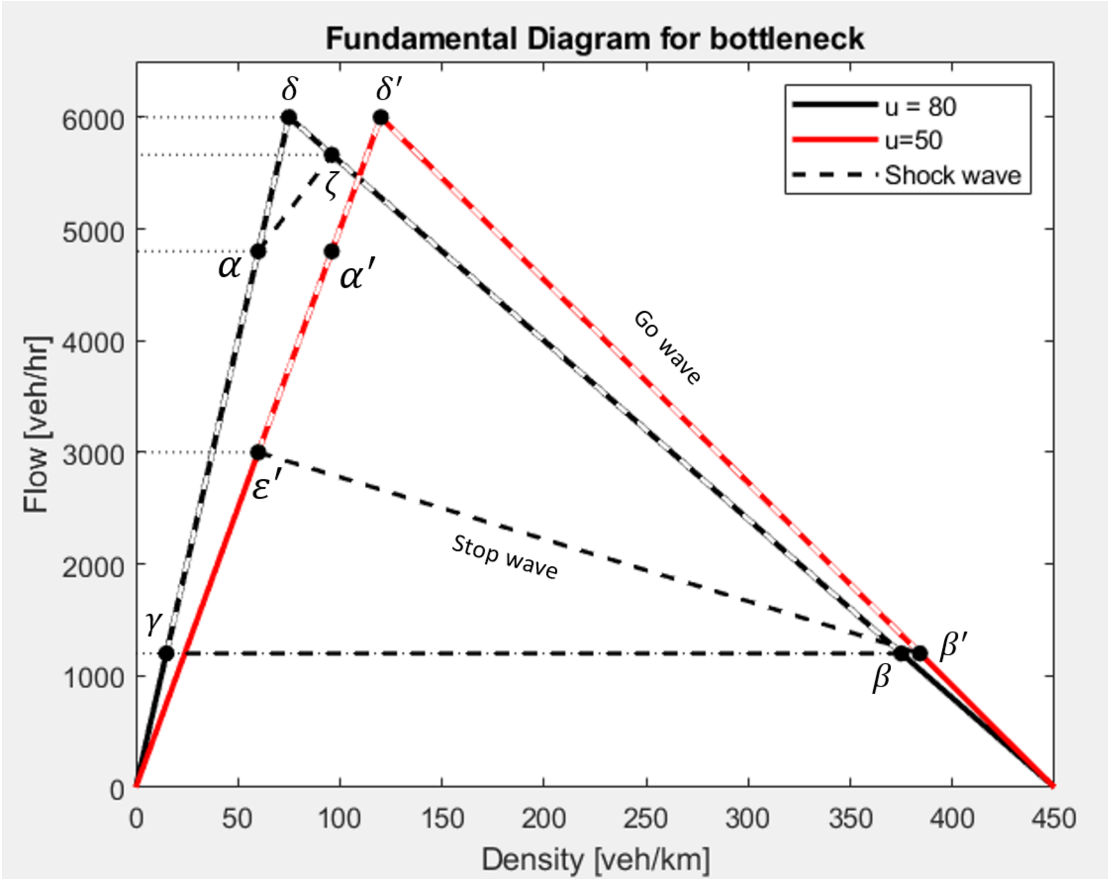

Figure 26: Fundamental diagram of stretch of road with bottleneck and imposed reduced speed limit $u' = 50$.

Minimal upstream congestion: $u' = 50$

To avoid backwards propagating shock waves from the re-introduction of u we can choose u' such that $\rho_{\alpha'} < \rho^i$. We can see from the fundamental diagram in Figure 26 that $u' = 50$km/h satisfies this condition. As the maximum flow capacity ϕ^c and the jam density ρ^j remain unchanged a new critical density can be calculated for u', namely $\rho^{c'} = \dfrac{6000}{50} = 120$. From Figure 26 we see that there are five new states introduced

- α': $\phi_{\alpha'} = 4800$ veh/hr and $\rho_{\alpha'} = 96$ veh/km.
- β': $\phi_{\beta'} = 1200$ veh/hr and $\rho_{\beta'} = 384$ veh/km.
- δ': $\phi_{\delta'} = 6000$ veh/hr and $\rho_{\delta'} = \rho^{c'} = 120$ veh/km.
- ϵ': $\phi_{\epsilon'} = 3000$ veh/hr and $\rho_{\epsilon'} = 60$ veh/km.
- ζ: $\phi_{\zeta} = 5664$ veh/hr and $\rho_{\zeta} = 96$ veh/km.

The shock waves that cause the queue to form directly upstream of the bottleneck are given by the stop and go waves

$$\Omega_{\beta'\epsilon'} \approx -5.55km/hr \text{ and } \Omega_{\beta'\delta'} \approx -18.18km/hr.$$

From this we can easily compute

$$Q_{max'} \approx 370 \text{ m} \qquad Q_{r'} \approx 533 \text{ m}$$
$$T_{d'} \approx 1 \text{ minute } 46 \text{ seconds} \qquad T_{Q'} \approx 5 \text{ minutes } 46 \text{ seconds.}$$

Now the simulation is run to see the effect on the traffic at the moment $u = 80$ is re-introduced. From (48) $u'_r = 5.34$km, hence the simulation is run on a stretch of motorway that is 7km long. As seen in Figure 27, the queue is diminished as expected. We can also see the state of congestion ζ that is formed at the moment the temporary speed limit is lifted. As opposed to the previous simulation, this time the congestion does not form a queue but instead causes the traffic to slow down to $u_{\zeta} = \dfrac{\phi_{\zeta}}{\rho_{\zeta}} = 59$km/hr. This means that although traffic is not in a free-flow state immediately after the temporary speed limit is lifted, it is also not slowed to be lower than u'. This makes the introduction of u' a viable option for queue mitigation.

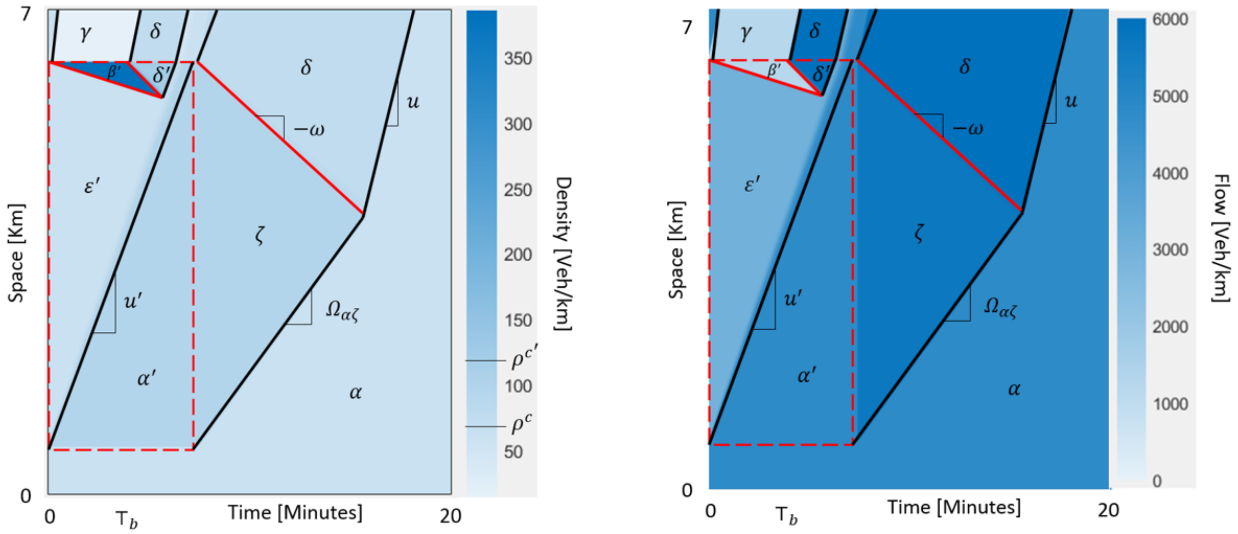

Figure 27: Density and flow of traffic on a stretch of road with bottleneck and imposed reduced speed limit $u' = 50$km/hr.

No upstream congestion: $u' = 65$

Finally, we will look at the scenario where no congestion is formed upstream of the bottleneck due to the re-introduction of $u = 80$km/hr. For this case u' can be chosen such that $\rho_{\alpha'} < \rho^c$, as in Figure 16. Hence,

$$u' > \frac{u\rho_\alpha}{\rho^c} = 64\text{km/hr}. \tag{57}$$

For ease the reduced speed limit is then chosen to be equal to 65 km/hr. As in Figure 28, only five new states of traffic occur as there is no intermediate congested state formed at the moment $u = 80$km/hr is re-introduced. These states are characterised as

- α': $\phi_{\alpha'} = 4800$ veh/hr and $\rho_{\alpha'} = 73.8$ veh/km.
- β': $\phi_{\beta'} = 1200$ veh/hr and $\rho_{\beta'} = 378.5$ veh/km.
- δ': $\phi_{\delta'} = 6000$ veh/hr and $\rho_{\delta'} = \rho^{c'} = 92.3$ veh/km.
- ϵ': $\phi_{\epsilon'} = 3900$ veh/hr and $\rho_{\epsilon'} = 60$ veh/km.

Again, the stop and go shock waves that form the queue are given by

$$\Omega_{\beta'\epsilon'} = -8.48km/hr \text{ and } \Omega_{\beta'\delta'} = -16.77km/hr$$

and we find,

$$Q_{max'} \approx 565 \text{ m} \qquad Q_{r'} \approx 1.14 \text{ km}$$
$$T_{d'} \approx 4 \text{ minutes } 53 \text{ seconds} \qquad T_{Q'} \approx 8 \text{ minutes } 53 \text{ seconds}.$$

Hence, $u'_r = 9.9$km and the simulation is run on an 11 km stretch of motorway. Figure 29 depicts the density and flow of the upstream traffic and shows that there is no congestion formed upstream of the bottleneck due to the re-introduction of $u = 80$km/hr, as expected. Whereas no congestion is formed by the change in speed limits, the effects on the queue immediately upstream of the bottleneck are diminished compared to $u' = 50$.

Overview

Having run all the simulations and analysed the effects on the traffic flow of three different magnitudes of u', one conclusion is clear namely;

introducing a temporary reduction in the speed limit is an effective method to mitigate queue formation caused by a bottleneck.

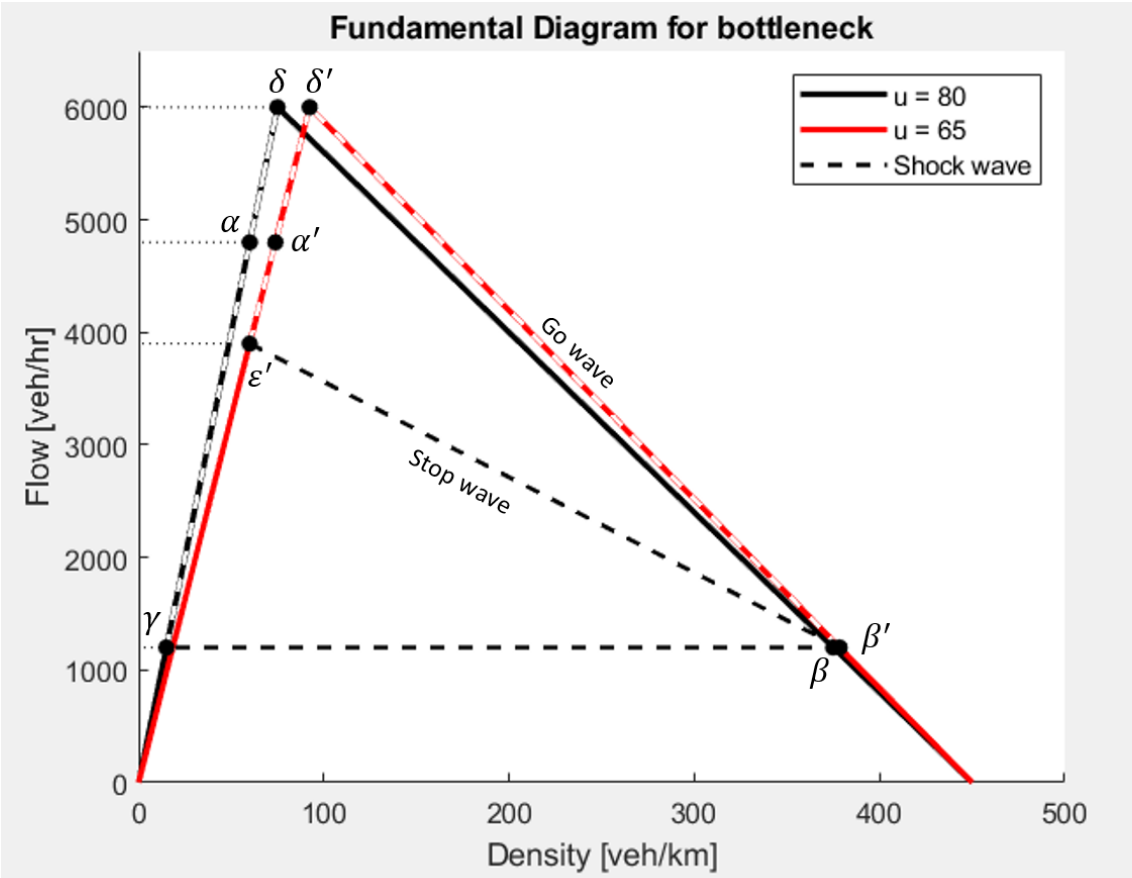

Figure 28: Fundamental diagram of a stretch of road with bottleneck and imposed reduced speed limit u'=65

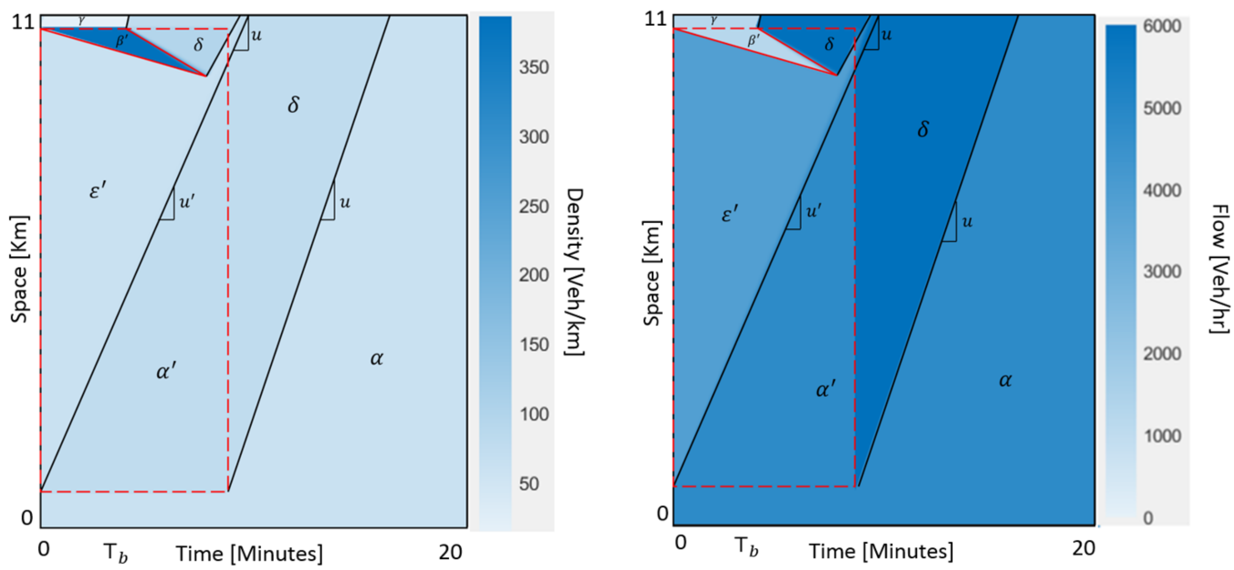

Figure 29: Density and flow of traffic on a stretch of road with bottleneck and imposed reduced speed limit $u' = 65$km/hr

Although this may be the case, there is a trade off in terms of the disruption that this temporary reduction in speed causes to the flow of the traffic at the moment that u' is lifted. Whereas the queue directly upstream of the bottleneck was entirely mitigated when $u' = 20$, a queue was formed at the instant that the speed limit returned to normal, causing such a low temporary speed limit to result in only a 24% reduction in congestion compared to the base case where no temporary speed limit is introduced. Similarly, the queue formation was reduced by 80% and 57% for $u' = 50$ and 65km/hr respectively. Even though $u' = 50$ results in the most improved result in terms of queue formation, there is slight congestion that forms at the moment u' is lifted, however the magnitude of this is not enough to cause a backwards propagating shock wave and hence only causes a temporary slowing of traffic to 59km/hr.

5 Conclusion

The goal of this research was to look into the effect that a temporary reduction in speed limit has on a traffic queue caused by an active bottleneck, for a simple stretch of road with no entrances/exits and assuming vehicles change their speed instantaneously, are not able to overtake each other and traffic flows as in a triangular fundamental diagram.

Through the use of shock wave analysis, the impact of a temporal reduction in the speed limit on both general traffic flow and the queuing formation caused by a downstream bottleneck have been investigated. Three characteristics of the reduced speed limit were found to be of importance namely; the magnitude of u', the reach of u' and finally the duration that u' is in effect. It was found that the shortest queue was given when the forward propagating shock wave caused by the introduction of u' did not intersect the congested traffic at any point. From this an optimum distance upstream of the bottleneck was determined for the reduced speed limit to take effect, which is given by

$$u'_r \geq \sqrt{(u'T_q)^2 - (T_q)^2} + Q_r.$$

Similarly, in order to avoid the forward propagating shock wave from the re-introduction of u, the optimum duration of u' must be at least as long as the total queuing time.

Finally, the magnitude of u' was found to be optimised when $\rho_{\alpha'} < \rho^i$, where some mild congestion is formed at the moment u' is lifted, however not severe enough to cause a queue to form. In this case u' is chosen such that

$$\frac{u\rho_\alpha}{\rho^i} < u' < u.$$

A further constraint can be placed on u' to ensure that the traffic remains in a free-flow state at the moment u' is lifted, namely

$$\frac{u\rho_\alpha}{\rho^c} < u' < u,$$

although this is found to be less effective at reducing the queuing directly upstream of the bottleneck.

In conclusion, the use of a temporary reduction in the speed limit is an effective tool for the mitigation of queuing caused by a bottleneck, provided the characteristics of u' are chosen carefully. Although it may not be fully effective, it is a simple technique that requires minimum effort to be applied.

It should be noted that the cell transmission model is a first-order model and hence the assumption of the instantaneous change in speed can lead to unrealistic acceleration and deceleration of vehicles, which may cause discrepancies between a real life situation and the simulation. In order to amend this a second-order traffic model could be used to simulate the given scenario to determine the effect that the deceleration and acceleration has on the evolution of the traffic. Alongside this further research on a stretch of road that allows for overtaking as well as entrances/exits is recommended with the given scenario as well as investigating the effects of a sudden increase in demand, which often occur at peak travel times.

References

[1] S. Siri A. Ferrara S. Sicone. *Freeway Traffic and Modelling Control.* Springer, (2018). ISBN: 978-3-319-75961-6.

[2] C.F. Daganzo. "The Cell Transmission Model: A Dynamic Representation of Highway Traffic Consistent with the Hydrodynamic Theory". In: *Transportation Research Part B: Methodological* 28 (1993), pp. 269–287.

[3] W. Zhang H. Rakha. *Consistency of Shock-wave and Queuing Theory Procedures for Analysis of Roadway Bottlenecks.* (2005).

[4] I.S.P. Hoogendoorn. *Traffic Flow Theory and Simulation.* TU Delft, (2021).

[5] F. Kessels. *Traffic Flow Modelling: Introduction to Traffic Flow Theory Through a Genealogy of Models.* Springer, (2019). ISBN: 978-3-030-08765-4.

[6] V.L. Knoop. *Introduction to Traffic FLow Theory.* TU Delft, (2018). ISBN: 978-94-6366-062-4.

[7] J. Lebacque. "The Godunov scheme and what it means for first order traffic flow models". In: *Proceedings of the 13th international symposium on transportation and traffic theory* (1996), pp. 647–677.

[8] A. Kesting M. Treiber. *Traffic Flow Dynamics.* Springer, (2013). ISBN: 978-3-642-32459-8.

[9] G.B. Whitham M.J. Lighthill. "On kinematic waves II: a theory of traffic flow on long crowded roads". In: *Proc R Soc A* 299 (1955), pp. 317–345.

[10] G.B. Whitham M.J. Lighthill. "On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads". In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 229 (1955), pp. 317–345.

[11] G.F. Newell. "A Simplified Theory of Kinematic Waves in Highway traffic, Part 1: General Theory". In: *Transportation Research Part B: Methodological* 27B (1993), pp. 281–287.

[12] E.W. Montroll R.E. Chandler R. Herman. "Traffic Dynamics: Studies in Car Following". In: *Operations Research* 6 (1958), pp. 165–184.

[13] P.I. Richards. "Shock waves on the Highway". In: *Operations Research* 4 (1956), pp. 42–51.

[14] S.K.Godunov. "A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics". In: *Mathematicheskii Sbornik* 47 (1959), pp. 271–306.

Appendix A – Main body of CTM in Matlab

```
%%CHECK RUNNING CONDITIONS%%
if v_free*T > CL
    fprintf 'space discretisation condition not satisfied'
elseif v_free*T<=CL
    fprintf 'space discretisation condition satisfied'
end
%====Initial Conditions====%
D_0 =0.8*q_max;
ro_0(1,:)=D_0/v_free;             %Initial condition, density in each cell i at k=1.
u(:,:) = v_free;                  %Velocity in each cell i at time k     [km/hr]
Q(:,:) = q_max;                   %Max capacity of cell i at time k    [Veh/time step]

%%SET TEMPORAL BLOCKAGE
Q(2:T_jam,N-1)= 0.2*q_max;       %Reduction of capacity to 20% (and hence supply) for
4 minutes


ro(1,:)=ro_0(1,:);               %Set initial inflow of traffic
for k=1:K
    for i=1:N
        D(k,i) = min(u(k,i).*ro(k,i),Q(k,i));          %Demand of cell i at time step k
        S(k,i) = min(abs(w).*(ro_jam-ro(k,i)),Q(k,i));%Supply of cell i at time step k
    end
    phi(k,1) = min(S(k,i),D_0);                          %Set traffic flow for initial
cell at time step k
    for i=2:N
        phi(k,i) = min(D(k,i-1),S(k,i));                 %Traffic flow for cell i (2:N)
at time step k
    end
    phi(k,i+1) = D(k,i);                                 %Traffic flow for cell i+1 set
to demand of previous cell
    for i=1:N
    ro(k+1,i)= ro(k,i) +(T/CL)*(phi(k,i)-phi(k,i+1));   %Density for cell i at the
next time step k+1
    end
    ro(k+1,1) = min(ro(k+1,1),ro_jam);      %Fail safe to stop initial cell going over
capacity
end
```