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# Choosing Fair Committees and Budgets: Proportionality in Multi-Winner Elections and Participatory Budgeting 

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# Choosing Fair Committees and Budgets: <br> Proportionality in Multi-Winner Elections and Participatory Budgeting 

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#### Abstract

Participatory budgeting is a way to allow citizens to have a say in deciding how to spend public funds. It is a generalisation of multi-winner voting, where a committee is to be elected based on the preferences of multiple voters. An important topic in both multi-winner voting and participatory budgeting is fairness of committees or budgets. One way of expressing fairness and increasing the influence of minorities in a participatory budgeting project or multi-winner election, is requiring proportionality of a voting rule. But what is proportionality? Proportionality, although somewhat intuitive, is a complex concept, and can be defined in many different ways. We bring more structure in the landscape of proportionality axioms, and show the existence or non-existence of logical relations between different axioms. Furthermore, we investigate the axiomatic properties of some of the most important existing and some promising new rules for computing proportional committees or budgets and we systematically identify which axioms are satisfied by which voting rules, both in the multi-winner voting and the participatory budgeting setting.


## 1 Introduction

First introduced in Porto Alegre, Brazil in 1988, Participatory Budgeting (PB) is a growing field, both in research and application. In increasingly more cities and municipalities, citizens can have their say in the allocation of the government budgets [1]. By giving the residents of a city a vote, the community can decide over which projects will be funded by the government.
However, 'giving the people a vote' may seem easier than it is in practice. Many different voting systems have been proposed, with different forms of preference elicitation, different aggregation rules, and different output forms. Consider the input format. On one hand, we would like as much information as possible from the voters, so we could let them all define a utility function that shows the utility they would get from every single project when it is implemented. On the other hand, for the voter himself the mental load of giving such utilities is much higher than that of just denoting some 'good' projects that he likes, and this may influence the willingness of citizens to participate. Even more dilemmas arise when we look at the axioms a voting rule can satisfy. There are many desirable properties for voting rules, and since some of them are incompatible, it may not be easy to choose the appropriate voting system. We would want a rule to give a division of the available budget over the projects in such a way that it represents the voters preferences. We would like to spend the budget efficiently, so that no money is used unwisely, we would like all voters to be happy in the end, we would like to be able to compute the solution in reasonable time, we would like all voters to be encouraged to give their true preferences and not to lie about them, and we could formulate a lot of other desirable requirements. Unfortunately, there is no known rule that satisfies all of these characteristics, and some of these properties even have been shown to be incompatible (take e.g. Arrow's theorem and the Gibbard-Satterthwaite theorem, or more recently the incompatibility between efficiency, strategyproofness and a form of fairness [2]). On first sight it may seem that optimising welfare is one of the most important properties a rule should have. In a utilitarian view, we should fund the projects that give the greatest amount of happiness to as many people as possible, i.e. that maximises the total utility. However, such a utilitarian viewpoint may in some situations end up in a budget division that is intuitively not fair.
As an example, imagine we have a fictional village of 100 people of which 55 are elderly and 45 are young. Suppose the elderly people really want the sidewalk to be made broader and they want a new park to be build, and the young people do not care about sidewalks and parks, but want a new skating rink and a new festival site. Unfortunately, the public budget is such that only two of these four projects can be afforded. Suppose that every citizen gets a happiness of 1 for every project that he likes if that project is implemented, and no happiness at all from the other projects. If welfare is to be maximised, we should implement the park and the broader sidewalk, since that will give a maximal total happiness $(2 \times 55=110)$. However, intuitively it seems much fairer that one of these projects and one of the projects that the younger people like is chosen, even though that will only give a total happiness of $55+45=100$. This example shows that proportionality is a property of PB-algorithms that can be desirable over utilitarian welfare. We do not only want the majority to decide everything, minority groups should have a vote as well. This is a reason to define axioms that describe proportionality in PB , and to design rules that satisfy those axioms.
In the above example, a proportional budget would intuitively give all groups a utility that is in some way proportional to the size of the group. However, how exactly to define proportionality is not trivial. In a setting with strictly divided groups of voters, we could give all groups of voters an amount of the total budget that is proportional to the size of the group. Many situations however, are not so easily separable. People might belong to several groups, or approve different types of projects without forming clear groups. To deal with such more abstract situations, we could say that
a solution is proportional if all possible groups of agents are sufficiently happy. However, there are many possible ways of defining when a group is 'sufficiently happy'. Some of them are more strict and hence demand in some way 'more proportional' solutions, but those are also harder to satisfy. Others are easier to satisfy, but also are less demanding on the fairness of the solution. In this thesis, we will thoroughly study different notions of proportionality and try to bring structure in this broad landscape of definitions.

### 1.1 Theoretical framework

Many existing voting systems use approval voting, where each voter has to specify for each project whether he approves it or not. The set of elected projects is determined based on the number of approval votes each project gets. A well-known, easy to understand, and often used method that uses approval voting is Knapsack voting, as described in [3]. In Knapsack voting, each voter forms a subset of the projects such that the cost of this subset is at most the budget limit. The outcome is then determined by choosing one by one the projects that are approved by most voters, until the budget limit is reached.
Approval voting is only one way of preference elicitation, and although easy to use, not the most informative. Instead of only asking a set of approved projects from every voter, one could also ask for more elaborate utility functions. This may give more workload to the voters, but also gives much more information that can be used to decide on which projects to elect.
In many existing rules, indeed utility functions rather than approval sets are used to decide on the winning set of projects. For example, [4] introduce a fair version of the Knapsack rule, that maximises the Nash product $\prod_{i \in N}\left(\sum_{c \in W} u_{i}(c)\right)$, where $N$ is the set of voters, $W$ the set of elected projects and $u_{i}(c)$ the utility that voter $i$ gets when project $c$ is implemented. This method takes into account the vote of minorities (because a voter that does not yet have many elected projects he approves will add more to the Nash product than a voter who already has a lot of approved elected projects), but it is a computationally complex rule and hence it can take too much time to compute for real applications. An example of an algorithm that is relatively easy to compute (and in fact can be computed in polynomial time) is SBA [5], which computes a Condorcet-winning budget if it exists, and else finds a member of the Smith set (the minimal set of which each member beats each non-member). Both Knapsack voting and SBA are welfarist algorithms: they maximise some function of the utility vectors of the voters.
Although the field of PB is still relatively small, it is closely related to the field of multi-winner voting (MWV), of which participatory budgeting is a generalisation. In MWV, a committee of candidates has to be elected based on the preferences of the voters. In short, MWV is PB with a unit cost assumption and approval voting. The literature on MWV is much more elaborate than that on PB, which allows us to study proportionality in MWV and try to generalise it to PB. Studies of proportionality in multi-winner elections describe for example the axiom of Justified Representation (JR) [6] and the related concepts of Proportional Justified Representation (PJR) [7] or Extended Justified Representation (EJR) [6]. In short, JR requires that if a large enough group of voters agrees about a certain candidate, there is at least one candidate in the chosen committee that at least one of the group members approves. PJR and EJR are generalisations of JR and add a proportionality aspect to it: if a larger group agrees about more candidates, they should be represented by more candidates in the outcome. In [7], these concepts from multi-winner elections are generalised to concepts for PB with approval voting.
A related fairness concept is the core [8; 9, 10], and, related to that, the Lindahl equilibrium [8; 9]. A PB solution is in a Lindahl equilibrium when the costs are divided over all participants in a way
that is proportional to the utility they gain by the solution. A solution is a core solution if there is no subset of agents that can afford a different solution (with their own share of the total budget) where every agent in that subset gets more utility than in the chosen solution. Lindahl equilibria are always in the core [9]. Although the core is is a good proportionality concept, it is not known whether there always exists an outcome in the core [11].
In [10], two multi-winner election rules that both claim to enhance proportionality (Proportional Approval Voting (PAV) and Phragmén's rule) are analysed, and shown to focus on different types of proportionality. PAV induces a fair distribution of welfare, so every group of agents should get a utility proportional to the size of their group, while Phragmén's rule can be seen as inducing a fair distribution of power: every group of agents should have an influence on the proposed budget proportional to their group's size. This can for example be expressed in giving all agents the same amount of money to spend. Related to this difference in concepts of proportionality, the authors introduce two new axioms of proportionality for the field of multi-winner elections: priceability and laminar proportionality, and a new rule (Rule X) that is similar to both PAV and Phragmén's rule, and satisfies the two new axioms, as well as EJR.
The relations between different axioms can give interesting insights to their meaning. Peters and Skowron [10] show that rules that maximise welfare always fail priceability and laminar proportionality, and never satisfy the core. In [12] it is shown that the core implies EJR, PJR, and JR for multi-winner elections. These findings raise the question whether the axioms, rules, and concepts of multi-winner voting can be generalised to similar concepts for participatory budgeting, and whether the same relations hold in PB.
In [11], Rule $X$ and EJR are generalised from the context of multi-winner voting to PB , and it is shown that even in this context, Rule X satisfies EJR. The PB variant of Rule X satisfies an approximation of the core. It satisfies priceability but not exhaustiveness, an axiom that requires that once a set of projects is chosen, there is no unelected project for which there is still enough remaining money. In fact it is shown that priceability is incompatible with exhaustiveness. A stronger variant of EJR, FJR, is proposed, and a rule, Greedy Cohesive Rule (GCR), that satisfies it. PAV is also generalised for PB, and shown to fail EJR without unit cost assumption.

### 1.2 Contributions of this thesis

Although PB is a topical subject of research and the literature on it is growing, there are still a lot of gaps in the knowledge of proportionality axioms in PB. We aim to give a clear overview of a couple of promising rules and the fairness axioms that they satisfy, and to study the relations between different fairness axioms. We do not claim to give a complete overview of all axioms and rules, but aim to study some promising and relevant rules.

Proportionality of rules In Table 1, we give an overview of the axioms that four rules satisfy in MWV and PB. Phragmén's rule and PAV are well known proportional voting rules, Rule X and SBA, as mentioned above, are promising new rules. The core, PJR, EJR, FJR are all related proportionality axioms that demand that any group of voters that has certain properties gets sufficient utility from the winning budget. Priceability is another fairness axiom that ensures that the power is equally spread over the voters, laminar proportionality ensures that groups of voters get equal numbers of approved projects in the winning budget, Nash welfare is a way to spread utilities more egalitarian. We will give exact definitions of these properties in Sections 2.1 and 3.1. Purple entries in the table indicate results from the literature, green entries indicate new results. MWV indicates the multi-winner voting setting with unit costs and approval voting, PB indicates the participatory budgeting setting without
the unit cost assumption, and with ordinal utilities for SBA, approval votes for PAV and Phragmén's rule (because they are only defined for approval votes), and cardinal utilities for Rule X. Note that SBA does not have a column for MWV, because it is not well-defined in that setting: although MWV is a special case of PB, in MWV we use approval voting. However, SBA's input consists of ordinal ballots. If we would use approval ballots instead, the algorithm would have not enough information to give a sensible outcome, and depend too much on tie-breaking decisions.
Proofs and explanations of the new results will be given in Chapters 2 and 3

|  | SBA | PAV |  | Phragmén |  | Rule X |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | PB | MWV | PB | MWV | PB | MWV | PB |
| core | $X$ | $X[10]$ | $X$ | $X$ | $X$ | $X[10]$ | $X[11]$ |
| EJR | $X$ | $\checkmark[10]$ | $X[11]$ | $X[10]$ | $X$ | $\checkmark[10]$ | $\checkmark[[11]$ |
| PJR | $X$ | $\checkmark[13]$ | $X$ | $\checkmark[[10]$ | $\checkmark$ | $\checkmark[10]$ | $\checkmark$ |
| priceability | $X$ | $X[10]$ | $X$ | $\checkmark[[10]$ | $\checkmark$ | $\checkmark[10]$ | $\checkmark[[11]$ |
| laminar proportionality | $X$ | $X[10]$ | $X$ | $\checkmark[[10]$ | $X$ | $\checkmark[10]$ | $X$ |

Table 1: Different rules and the properties they satisfy, purple entries in the table indicate results from the literature, green entries indicate new results.

Logical relations between axioms Different entries in the table give rise to the conjecture that between some of the axioms there is a logical relationship. Some of these are already known, we know for example from [11] that the core implies FJR, which implies EJR, in PB, and that in MWV, EJR implies PJR, which implies JR. We study the logical relationships between PJR, EJR, the core, priceability, and laminar proportionality, both in MWV as in PB , and in this thesis will show that there are implications between them as shown in Figure 1. The arrows indicate an implicational relation, and the absence of an arrow indicates that there is no implicational relation between the axioms. The blue arrows indicate implications in PB, while the red arrows indicate implications in MWV. As mentioned in the figure, some implications only hold under certain restrictions.

### 1.3 Outline of Thesis

In Chapter 2 we will study different rules and axioms and their relations in multi-winner voting. Then, in Chapter 3, we will try to generalise those to participatory budgeting. In Chapter 4, we will look at how certain restrictions on the domain of profiles or on the definitions of axioms can make negative results positive, and in Chapter 5, we will use experimental simulations to study the influence of polarisation on proportionality.

### 1.4 Preliminaries

In the main part of this thesis, we will distinguish two settings. The first setting is that of multi-winner voting (MWV). In this setting, each voter gives a set of projects (or candidates) she approves. As is done a lot in the literature (e.g. [11], [9], [4]), we assume that a voter gets a utility of one from an approved project and a utility of zero from a non-approved project. Furthermore, every project has the same (unit) cost.


Figure 1: The relations between laminar proportionality, priceability, PJR, EJR, and the core in the multi-winner voting setting (red) and in the participatory budgeting setting (blue). Along the arrows are references to the theorems or papers in which the corresponding result is explained. Some of the implications only hold under certain conditions or restrictions: laminar instances (Definition 2.10), balanced price systems (Section 4.3.1), and unanimity affordability ( $\mathcal{P}_{u-\text { afford }}$ ) (Definition 4.3).

Secondly, we have the more general participatory budgeting (PB) setting, which is a generalisation of MWV. In this setting, projects can have different costs, and voters can have arbitrary utilities for the different projects. Nevertheless, there are some voting rules that require approval voting. In these cases we will use the term 'participatory budgeting' for the setting with approval voting but without the unit cost assumption.
In both MWV and PB, we denote the set of projects or candidates by $C=\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$ and the set of voters by $N=\left\{v_{1}, v_{2}, v_{3}, \ldots, v_{n}\right\}$. The outcome of a voting rule, i.e. the winning set of projects or candidates is called $W$. Each voter has a utility function $u_{i}: C \rightarrow\{0,1\}$ in MWV or $u_{i}: C \rightarrow[0,1]$ in PB, that assigns a utility to all projects. Utilities are assumed to be additive, so the utility of a set of projects $T$ for a set of voters $S$ is defined as $S \subseteq N: u_{S}(T)=\sum_{i \in S} \sum_{c \in T} u_{i}(c)$.
In MWV, each voter $i$ has an approval set $A_{i}:=\left\{c \in C: u_{i}(c)=1\right\}$. A profile $P$ is a vector of the approval sets of all voters: $P=\left(A_{1}, \ldots, A_{n}\right)$. We call the desired committee size $k$. Hence, with a budget limit of 1 , each candidate has a cost of $\frac{1}{k}$, and $|W| \leq k$. The set of voters that approves a set of candidates $C$ is denoted as $N(C)$, and the set of voters that approves a candidate $c$ as $N(c)$. The set of all voters that occur in a profile $P$ is called $N(P)$, but we will sometimes abuse notation and use the profile $P$ itself for the set of voters in it, so that $|P|=n$ is the number of voters in $P$, and we can have a voter $i \in P$. We will use $C(P)$ for the set of projects that occur in a profile $P$, i.e. that are approved by at least one voter in $P$, and we assume that $|C(P)| \geq k$. The sum of two profiles $P_{1}$ and $P_{2}$ is defined as the concatenation of lists $P_{1}$ and $P_{2}$. When we want to look at a profile $P$ without a certain candidate $c$,
we write $P_{-c}=\left(A_{1} \backslash\{c\}, \ldots, A_{n} \backslash\{c\}\right)$. A unanimous profile is a profile in which every voter approves the same projects: for all $i, j \in N: A_{i}=A_{j}$. In MWV, an election instance $E=(N, C, P, k)$ consists of a set of voters $N$, a set of projects $C$, a profile $P$, and a desired committee size $k$. Since $N$ and $C$ can be deducted from the profile $P$, an election instance is often denoted just as $E=(P, k)$.
In PB, there is a cost function that assigns a cost to every project: cost: $C \rightarrow \mathbb{Q}_{+}$. The cost of a set of projects $T$ is determined by $\operatorname{cost}(T)=\sum_{c \in T} \operatorname{cost}(c)$. The total budget limit is denoted by $l$. If $l$ is not mentioned, it is equal to 1 . A profile $P$ is a vector of the utility functions of all voters: $P=\left(u_{1}, \ldots, u_{n}\right)$. Hence, in PB, an election instance $E=(N, C, \operatorname{cost}, P, l)$ consists of a set of voters $N$, a set of projects $C$, a cost function that assigns a cost to every project, a profile $P$, and a budget limit $l$. If all else is clear from the context we will sometimes abbreviate this to $E=(P, l)$.

## 2 Multi-Winner Voting

In this chapter, we will study three promising MWV rules and a couple of proportionality axioms that they may or may not satisfy. Furthermore, we will investigate the relations between the different axioms. As a reminder: since we use the MWV setting, in this chapter we assume that every project or candidate has the same cost and we use approval voting.

Outline of chapter In Section 2.1 we will give definitions of the rules and axioms used in MWV. Then in Section 2.2 we will study which axioms are satisfied by the different rules, and in Section 2.3 we will study the relations between the different axioms. Finally in Section 2.4, we will mention some directions for future work.

### 2.1 Definitions

### 2.1.1 Rules

In [10], two well known voting rules are studied, Proportional Approval Votal (PAV) and Phragmén's rule, that both claim to be proportional. However, [10] shows that - while both being proportional - , the proportional representation they guarantee is of a different type.
PAV is introduced by the Danish mathematician Thorvald Thiele [14]. It chooses a subset of projects that maximises the sum of the $u_{i}$-th harmonic numbers, where $u_{i}$ is the utility that voter $i$ gets from that set. Formally, we can define it as follows:

Definition 2.1 (PAV). The outcome $W$ of PAV is the committee with $|W| \leq k$ that maximises the score

$$
\operatorname{PAV}-\operatorname{score}(W)=\sum_{i \in N}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left|W \cap A_{i}\right|}\right) .
$$

The intuition behind the proportionality of this rule is that the more projects that a voter approves are chosen, the less those projects count in the sum to be maximised. In this way, a project that is approved by a voter who approves not many other elected projects is more likely to be chosen than a project approved by a voter who already approves a lot of projects in the winning committee. In a way, PAV can be considered as a rule that distributes welfare proportionally over voters.
Competing to PAV as a proportional voting rule is Phragmén's rule [15; 16, 10], which was, as its name suggests, designed by Edvard Phragmén, who was also a Scandinavian mathematician. Phragmén's rule is a sequential rule 1 , it gives each voter virtual money at a constant rate and buys a project from that money as soon as it is affordable. More formally, its definition is as follows:

Definition 2.2 (Phragmén's rule [10]). Every voter continuously receives one unit of currency per time unit. At the first moment in time $t$ when there is a group of voters $S$ who all approve a not-yet-selected candidate $c$, and who have $\frac{n}{k}$ units of currency in total, the rule adds $c$ to the committee and asks the voters from $S$ to pay the total amount of $\frac{n}{k}$ for $c$ (i.e., the rule resets the balance of each voter from $S$ ); the other voters keep their so-far earned money. The rule stops when $k$ candidates are selected.

[^0]Instead of distributing welfare proportionally over voters, Phragmén's rule can be seen as distributing voting power fairly over the voters, by giving every voter the same amount of money.
Peters and Skowron [10] introduce a new rule, which they call Rule X, that satisfies some proportionality axioms that are satisfied by either PAV or Phragmén's rule, but not by both. Rule X is similar to Phragmén's rule, but instead of giving the voters money continuously, it gives each voter a certain budget from the beginning. It is defined as follows:

Definition 2.3 (Rule X [10]). Each voter gets an initial budget of one unit of money, which they can spend on buying projects. The price of a project is $p=\frac{n}{k}$. The rule starts with an empty committee $W=\emptyset$ and adds projects each step. Let $b_{i}(t)$ be the amount of money that voter $i$ is left with just before the start of the $t$-th iteration. In the $t$-th step, the rule selects the project that should be added to $W$ as follows. For a value $q \geq 0$, we say that a project $c \notin W$ is $q$-affordable at round $t$ if

$$
\sum_{t \in N(c)} \min \left(q, b_{i}(t)\right) \geq p
$$

If no project $c \notin W$ is $q$-affordable for any $q$, the rule stops and returns $W$. Otherwise, the rule selects a project $c \notin W$ which is $q$-affordable for a minimum $q$, and adds $c$ to the committee $W$. Note that by minimality of $q$, inequality (1) holds with equality. For each voter $i \in N(c)$, we set their budget to $b_{i}(t+1)=b_{i}(t)-\min \left(q, b_{i}(t)\right)$. (So each of these voters pays either $q$ or their entire remaining budget for $c$.) For each $i \notin N(c)$ we set $b_{i}(t+1)=b i(t)$.

### 2.1.2 Axioms

In this section, we give the definitions of the main axioms that will be used later on in the chapter, and of some specific types of profiles.
The first four definitions are closely related, and ordered here in decreasing degree of strictness. The most demanding axiom of the four is the core. If a committee is in the core, there is no group of voters that can block it, i.e. that can afford a different committee that they all prefer to the chosen committee 2.

Definition 2.4 (Core for MWV (derived from [8])). A rule $\mathcal{R}$ satisfies the core property if for every election instance $E$, the committee $W=\mathcal{R}(E)$ is in the core, i.e. there is no group of agents $S \subseteq N$ such that there is a committee $T \subseteq C$ that they can afford given their share of the budget, i.e. $|S| \geq|T| \cdot \frac{n}{k}$, with $\left|A_{i} \cap T\right|>\left|A_{i} \cap W\right|$ for all agents $i \in S$.
Example 2.1. In Figure 2 is an example election instance (taken from [11]) of a committee that does not satisfy the core. The columns represent the approval votes of the voters ( $v_{1}, \cdots, v_{6}$ ), a cross indicates that the voter approves that candidate and an empty cell indicates that a voter does not approve that candidate. In this example, $k=12$, and the elected committee $W$ is marked in blue. The set of voters $S=\left\{v_{1}, v_{2}, v_{3}\right\}$ (marked green) can block the core with committee $T=\left\{c_{1}, \cdots, c_{6}\right\}$ : their share of the budget is $3 \cdot \frac{12}{6}=6$, which is the size of $T$, and they all have four approved candidates in $T$ while they have only three approved candidates in $W$.

A slightly less demanding axiom than the core is fully justified representation, which restricts the groups of agents that can block the outcome by demanding that such a group must be cohesive in some degree, and which weakens the requirement that some voter must prefer the elected outcome over $T$ by only demanding that there must be enough representatives of this voter in the elected committee.

[^1]

Table 2: Example of a committee that does not satisfy the core and FJR but does satisfy EJR. The columns represent the approval votes of the voters $\left(v_{1}, \cdots, v_{6}\right)$, a cross indicates that the voter approves that candidate and an empty cell indicates that a voter does not approve that candidate. The committee size $k=12$.

Definition 2.5 (Fully Justified Representation (FJR) for MWV [11]). Let $S$ be a group of voters, and suppose that each member of $S$ approves at least $\beta$ candidates from some set $T \subseteq C$ with $|T| \leq \ell$, and let $|S| \geq \frac{\ell}{k} \cdot n$. Then at least one voter from $S$ must have at least $\beta$ representatives in the elected committee $W$.

Although FJR is less demanding than the core, the committee in Example 2.1 still does not satisfy it, which we can show by taking $\beta=4$ and $\ell=6$ : each member of $S$ approves four candidates from $T$, $|T| \leq 6$, and $|S|=3 \geq \frac{6}{12} \cdot 6$. However, none of the voters from $S$ has at least four representatives in $W$.
FJR is not easily satisfied, an even weaker and more often used proportionality axiom is that of extended justified representation, which is equivalent to FJR with $\beta=\ell$. Extended justified representation requires that in every group that agrees about a certain number of candidates, which they can also afford given their own share of the total budget, there is a voter who approves at least that number of candidates in the winning committee.

Definition 2.6 (Extended justified representation (EJR) for MWV [13]). An approval based voting rule $\mathcal{R}$ satisfies EJR if for every ballot profile $P$ and committee size $k$, the rule outputs a committee $W=\mathcal{R}((P, k))$ that satisfies:
For every $\ell \leq k$ and $\ell$-cohesive set of voters $S \in N$, there is a voter $i \in S$ such that $\left|W \cap A_{i}\right| \geq \ell$, where a set $S$ is $\ell$-cohesive if $|S| \geq \ell \cdot \frac{n}{k}$ and $\left|\cap_{i \in S} A_{i}\right| \geq \ell$.

The committee in Example 2.1 does satisfy EJR: there is no group of voters that agrees about four or more candidates, so for $\ell \geq 4$ there is no $\ell$-cohesive group of voters. Furthermore, every voter has three approved candidates in the winning committee $W$, so for $\ell \leq 3$ the requirement that $\left|W \cap A_{i}\right| \geq \ell$ is true for all voters.
Hence, the committee in blue is proportional in some sense (EJR) but not in other, more strict, senses (FJR or the core). In fact, $W$ is the committee that is returned by PAV in the given instance.
The requirement that there must be a voter who approves at least $\ell$ candidates in the winning committee is relaxed further in proportional justified representation, here it is only required that for at least $\ell$ candidates in the winning committee, there must be some voter in the group that approves it (which may be a different voter for each candidate, while in EJR it had to be the same voter for all $\ell$ candidates).

Definition 2.7 (Proportional justified representation (PJR) for MWV [13]). An approval based voting rule $\mathcal{R}$ satisfies PJR if for every ballot profile $P$ and committee size $k$, the rule outputs a committee $W=\mathcal{R}(P, k)$ that satisfies:
For every $\ell \leq k$ and every $\ell$-cohesive set of voters $S \subseteq N$, it holds that $\left|W \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \ell$.
The following example illustrates the difference between EJR and PJR.
Example 2.2. In the instance as shown in Table 3 (which should be read in the same way as Table 22, the group of voters $S$ witnesses that the committee $W=\left\{c_{1}, c_{2}, c_{3}\right\}$ as indicated in blue does not satisfy EJR. $S$ is 2-cohesive (it is large enough to afford two candidates and agrees about $c_{4}$ and $c_{6}$ ), but there is no agent in $S$ who approves two or more candidates from $W$. Nevertheless, $W$ does satisfy PJR. Indeed for the 2-cohesive group $S$ it holds that $\left|W \cap\left(\cup_{i \in S} A_{i}\right)\right|=3 \geq 2$, there are no other $\ell$-cohesive groups for $\ell \geq 2$, and since every voter has at least one representative in $W$, for every 1 -cohesive group the requirement is also fulfilled.

| $c_{6}$ | x | x | x |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $c_{5}$ |  |  | x | x | x |  |
| $c_{4}$ | x | x | x | x |  |  |
| $c_{3}$ |  |  |  | x | x | x |
| $c_{2}$ |  | x | x |  | x | x |
| $c_{1}$ | x |  |  |  |  | x |
|  | $v_{1}$ | $v_{2}$ | $v_{3}$ | $v_{4}$ | $v_{5}$ | $v_{6}$ |

Table 3: Example of a committee (in blue) that does not satisfy EJR but does satisfy PJR. The columns represent the approval votes of the voters $\left(v_{1}, \cdots, v_{6}\right)$, a cross indicates that the voter approves that candidate and an empty cell indicates that a voter does not approve that candidate. The committee size $k=3$.

In [10], a different kind of proportionality is proposed. By giving all voters the same amount of money, and designing a system in which they can pay for candidates they approve, each voter is given the same amount of 'voting power'. Such a price system is defined as follows:

Definition 2.8 (Price systems for MWV [10]). A price system is defined as a pair $\mathbf{p s}=\left(p,\left\langle p_{i}\right\rangle_{i \leq n}\right)$ where $p>0$ is a price and for each voter $i \in N$, there is a payment function $p_{i}: C \rightarrow[0,1]$ such that

1. a voter can only pay for candidates she approves of: if $p_{i}(c)>0$, then $c \in A_{i}$, and
2. a voter can spend at most one unit of money: $\sum_{c \in C} p_{i}(c) \leq 1$.

Now we can argue that a committee is fair or proportional if there is a price system in which all voters spend their money such that that committee is bought. The notion of priceability requires this:

Definition 2.9 (Priceability for MWV [10]). A rule $\mathcal{R}$ is priceable if for all election instances $E$, the output committee $\mathcal{R}(E)$ is priceable. A committee $W$ is priceable if there exists a price system $\mathbf{p s}=\left(p,\left\langle p_{i}\right\rangle_{i \in N}\right)$ that supports $W$, i.e.

1. for each $c \in W$, the sum of the payments for $c$ equals its price: $\sum_{i \in N} p_{i}(c)=p$;
2. no candidate outside of the committee gets any payment: for all $c \notin W, \sum_{i \in N} p_{i}(c)=0$; and
3. There exists no unelected candidate whose supporters in total have a remaining unspent budget of more than the price: for all $c \notin W, \sum_{i \in N \text { for which } c \in A_{i}}\left(1-\sum_{c^{\prime} \in W} p_{i}\left(c^{\prime}\right)\right) \leq p$.

Example 2.1] can serve as an illustration for priceable committees. As indicated by [10], in the instance from Example 2.1, the blue committee as elected by PAV is not priceable. Voters $v_{1}, v_{2}$, and $v_{3}$ together have only three candidates in the winning committee that at least one of them approves, while voters $v_{4}, v_{5}$, and $v_{6}$ together have nine approved candidates in $W$. Hence, if both groups get the same amount of money, this committee will not be elected. Instead, the committee $W^{\prime}=\left\{c_{1}, c_{2}, c_{3}, c_{4}, c_{5}, c_{6}, c_{7}, c_{8}, c_{10}, c_{11}, c_{13}, c_{14}\right\}$ returned by Phragmén's rule is priceable since voters $v_{1}, v_{2}$, and $v_{3}$ can work together and share the cost of $c_{1}, c_{2}$, and $c_{3}$, and now both groups have six approved candidates in $W^{\prime}$.
In some election instances, a profile is structured in a special, non-random way. Examples of such instances are laminar election instances, in which voters can be divided in parties and sub-parties.

Definition 2.10 (Laminar election instances for MWV [10]). An election instance $(P, k)$ is laminar if either:

1. $P$ is unanimous and $|C(P)| \geq k$.
2. There is a candidate $c \in C(P)$ such that $c \in A_{i}$ for all $A_{i} \in P$, the profile $P_{-c}$ is not unanimous and the instance $\left(P_{-c}, k-1\right)$ is laminar (with $P_{-c}=\left(A_{1} \backslash\{c\}, \ldots, A_{n} \backslash\{c\}\right)$ ).
3. There are two laminar instances $\left(P_{1}, k_{1}\right)$ and $\left(P_{2}, k_{2}\right)$ with $C\left(P_{1}\right) \cap C\left(P_{2}\right)=\emptyset$ and $\left|P_{1}\right| \cdot k_{2}=$ $\left|P_{2}\right| \cdot k_{1}{ }^{3}$ such that $P=P_{1}+P_{2}$ and $k=k_{1}+k_{2}$.

In these laminar election instances, we could say that a committee is proportional if the candidates are equally distributed over the parties and sub-parties, proportionally to the number of voters in each party.

Definition 2.11 (Laminar proportionality for MWV [10]). A rule $\mathcal{R}$ satisfies laminar proportionality if for every laminar election instance with ballot profile $P$ and committee size $k, \mathcal{R}(P, k)=W$ where $W$ is a laminar proportional committee, i.e.

[^2]1. If $P$ is unanimous, then $W \subseteq A_{i}$ for some $i \in N$ (if everyone agrees, then part of the candidates they agree on is chosen.
2. If there is a unanimously approved candidate $c$ s.t. $\left(P_{-c}, k-1\right)$ is laminar, then $W=W^{\prime} \cup\{c\}$ where $W^{\prime}$ is a committee which is laminar proportional for $\left(P_{-c}, k-1\right)$.
3. If $P$ is the sum of laminar instances $\left(P_{1}, k_{1}\right)$ and $\left(P_{2}, k_{2}\right)$, then $W=W_{1} \cup W_{2}$ where $W_{1}$ is laminar proportional for $\left(P_{1}, k_{1}\right)$ and $W_{2}$ is laminar proportional for $\left(P_{2}, k_{2}\right)$.

Example 2.3. Table 4 shows an example of a laminar election instance $(P, k)$ with $k=4$. Namely: there is a unanimous candidate $c_{6}$, the instance ( $P^{\prime}, k^{\prime}$ ) without $c_{6}$ and with $k^{\prime}=3$ consists of two instances $\left(P_{1}, k_{1}\right)$ and $\left(P_{2}, k_{2}\right)$, where $P_{1}$ consists of the approval sets of $v_{1}$ and $v_{2}$ and $k_{1}=2$, and $P_{2}$ consists of the approval set of $v_{3}$, with $k_{2}=1$, so $\left|P_{1}\right| \cdot k_{2}=2 \cdot 1=1 \cdot 2=\left|P_{2}\right| \cdot k_{1}$. Then $\left(P_{1}, k_{1}\right)$ and $\left(P_{2}, k_{2}\right)$ are both unanimous instances with a number of candidates larger than $k_{1} \operatorname{resp} k_{2}$, so $(P, k)$ is indeed laminar. The committee $W$ as indicated in green is a laminar proportional committee: $\left\{c_{1}, c_{2}\right\}$ is a laminar proportional committee for $\left(P_{1}, k_{1}\right),\left\{c_{4}\right\}$ is a laminar proportional committee for $\left(P_{2}, k_{2}\right)$ so $\left\{c_{1}, c_{2}, c_{4}\right\}$ is a laminar proportional committee for $\left(P^{\prime}, k^{\prime}\right)$. Since $c_{6}$ is unanimously approved, then $W=\left\{c_{1}, c_{2}, c_{4}, c_{6}\right\}$ is laminar proportional for $(P, k)$. On the contrary, if we would elect candidate $c_{3}$ instead of candidate $c_{6}$, as in the committee in Table 7, the committee is not laminar proportional anymore.


Table 4: An example of a laminar proportional committee $W$ (in green) in a laminar election instance. Each column represents the approval set of a voter (written beneath it), and each box represents a candidate. For example, voter $v_{2}$ approves $c_{6}, c_{1}, c_{2}$, and $c_{3}$

A specific instance of laminar election instances are party-list profiles, in which there are no subparties. In party-list profiles, many of the proportionality axioms become equivalent, as we will show in Chapter 4 .

Definition 2.12 (party-list profiles for MWV [10]). A profile $P=\left(A_{1}, \ldots, A_{n}\right)$ is a party-list profile if for all $i, j \in N$, either $A_{i}=A_{j}$ or $A_{i} \cap A_{j}=\emptyset$

A very different form of proportionality is obtained by maximising the Nash welfare. Nash welfare tries to combine maximising welfare with some form of fairness, by giving more weight to the voters that obtain a small utility from the committee compared to voters that are already more satisfied.

Definition 2.13 (Nash welfare [4]). A rule satisfies Nash welfare if it elects a committee $W$ with $|W| \leq k$ that maximises the Nash product

$$
\operatorname{Nash}(W)=\prod_{i \in N}\left(1+\left|W \cap A_{i}\right|\right)
$$

| $c_{3}$ | x | x |  |  |
| :--- | :--- | :--- | :--- | :--- |
| $c_{2}$ |  |  | x | x |
| $c_{1}$ | x | x |  | x |
|  | $v_{1}$ | $v_{2}$ | $v_{3}$ | $v_{4}$ |

Table 5: The profile mentioned in Example 2.4. The columns represent the approval votes of the voters ( $v_{1}, \cdots, v_{4}$ ), a cross indicates that the voter approves that candidate and an empty cell indicates that a voter does not approve that candidate. The desired committee size $k=2$.

Example 2.4. In the simple election instance in Table 5, the committees $W=\left\{c_{1}, c_{2}\right\}$ and $W^{\prime}=$ $\left\{c_{1}, c_{3}\right\}$ would lead to the same cumulative welfare: both have in total five approval votes. However, voters $v_{1}, v_{2}$ and $v_{4}$ already get utility from $c_{1}$, while $c_{3}$ does not. Hence a rule satisfying the Nash welfare axiom would compensate $v_{3}$ for that and elect $W$ (since $2 \cdot 2 \cdot 2 \cdot 3=24>3 \cdot 3 \cdot 1 \cdot 2=18$ ).

Given this list of desirable axioms that voting rules can satisfy, we will in the following section investigate some promising MWV rules and study which of these axioms they indeed satisfy, and which they do not satisfy.

### 2.2 Properties of rules

In this section, we study which axioms are satisfied by three MWV rules from the literature: Proportional Approval Voting (PAV), Phragmén's rule, and Rule X (which are all three introduced in Section 2.1.1).

|  | PAV | Phragmén | Rule X |
| :--- | :--- | :--- | :--- |
| core | $X[10]$ | $X$ (Prop. 2.1$]$ | $X[10]$ |
| EJR | $\checkmark[10]$ | $X[10]$ | $\checkmark[[10]$ |
| PJR | $\checkmark[13]$ | $\checkmark[10]$ | $\checkmark[10]$ |
| priceability | $X[10]$ | $\checkmark[10]$ | $\checkmark[[10]$ |
| laminar proportionality | $X[10]$ | $\checkmark[\overline{10}]$ | $\checkmark[[10]$ |
| Nash welfare | $X$ (Prop. 2.4$]$ | $X$ (Prop. 2.2$]$ | $X$ (Prop. 2.4 |
| FJR | $X[\overline{11]}$ | $X$ (Prop. 2.5$]$ | $X[11]$ |

Table 6: Different rules and the properties they satisfy, purple entries in the table indicate results from the literature, green entries indicate new results.

In Table 6 is shown for the three mentioned rules which axioms they satisfy and which they do not satisfy. In the top of the table are the fairness axioms that are focused on in this thesis, the bottom of the table contains other relevant axioms. Purple entries in the table indicate results from the literature, green entries indicate new results. As we see, most of the results for the five axioms in the focus of this thesis are already established in the literature. Only the fact that Phragmén's rule does not satisfy the core was missing, but is very easy to see:

Proposition 2.1. Phragmén's rule fails the core.

Proof. The core implies EJR, so because Phragmen's rule does not satisfy EJR, it also does not satisfy the core.

However, as far as we know it was not shown yet whether or not the three rules maximise Nash welfare. We show that neither of them does:

## Proposition 2.2. Phragmén does not maximise Nash welfare.

Proof. Because Phragmén's rule is build to have a fair distribution of power and maximising the Nash welfare can be considered a way to get a fair distribution of welfare, it is obvious that Phragmén's rule does not necessarily maximise the Nash welfare. We can show this by the example given in [10], as shown in Figure 2 where the difference between Phragmén's rule (which distributes power fairly) and PAV (which distributes welfare fairly) is demonstrated. The Nash score for the committee returned

| $c_{4}$ | $c_{5}$ | $c_{6}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $c_{3}$ |  |  |  |  | $c_{9}$ |$c_{12} \quad c_{15}$.

(a) Phragmén's rule


Figure 2: An example of a laminar election instance and the results of Phragmén's rule and PAV [10].
by Phragmén's rule here is $5^{3} \cdot 3^{3}=3375$, and the Nash score for the committee returned by PAV is $4^{6}=4096$. Even without knowing whether the Nash score for the PAV-committee is the highest possible, it is clear that the score for Phragmén's committee is not maximal.

Proposition 2.3. PAV does not maximise Nash welfare.
Proof. The counterexample from Proposition 3.9 (in Chapter 3) showing that PAV does not always maximise the Nash product obviously does not work in a multi-winner voting setting where every project has unit cost. In fact, the two functions are very similar, and therefore it is not trivial to find and example where their maxima are different. Both PAV and Nash are a sum over the agents of a certain value that for each agent is based on the number of projects that they approve of, that are in the elected committee. For PAV, this value per agent is determined by $g(x)=\sum_{k=1}^{x} \frac{1}{k}$, the harmonic numbers, and for Nash, this value is determined by $f(x)=\ln (1+x)$. As [4] already mention, the harmonic numbers can be considered a discrete version of the logarithm. Both functions are plotted in Figure 3. Because we are interested in finding the maximum of a sum over these functions, we want to investigate their change on adding or removing a project for an agent, hence we look at their derivatives. For $f(x)$, the derivative is just $f^{\prime}(x)=\frac{1}{1+x}$. For $g(x)$, this is a bit more involved, because $g(x)$ is not a continuous function. However, if we look at how $g(x)$ changes if we add or remove one from $x$, we observe that if we increase $x$ by one, $g(x)$ increases by $\frac{1}{x+1}$, and if we decrease $x$ by one, $g(x)$ decreases by $\frac{1}{x}$. Thus, on one side, the 'derivative' is the same as for $f(x)$, on the other side it is slightly different. If we would change it into a continuous function somehow, it might very well have the same derivative as $f(x)$. However, because they are not exactly the same, probably there is a situation where the maximum of their sums over agents are different. We try to find such a


Figure 3: The functions $g(x)$ used for PAV and $f(x)$ used for Nash, where $x$ is the number of projects that an agent approves that are elected.
counterexample by trial and error.
Let us take a small set of projects, say $\left\{w_{1}, \ldots, w_{5}\right\}$, and a large group of agents $X$ that all approve those five projects in order to make sure that they are all in the winning committee. We can choose $X$ arbitrarily large in order to make sure that indeed $\left\{w_{1}, \ldots, w_{5}\right\} \subseteq W$. Now let there be two other projects, say $c_{1}$ and $c_{2}$, such that for $c_{1}$ there is a small group of voters $Y \nsubseteq X$ that approve only $c_{1}$, and that for $c_{2}$ there is a subgroup $X^{\prime} \subseteq X$ such that all $i \in X^{\prime}$ approve $c_{2}$ (and because they are in $X$, they also approve $\left\{w_{1}, \ldots, w_{5}\right\}$ ). Let $k=6$, so that we can either add $c_{1}$ to the winning committee or $c_{2}$, but not both. The idea behind this counterexample is that in order to get a different maximum, there must be a difference in the functions in whether we add one project that is the first approved elected project for some voters, or we add one project that is approved for a group of voters that is already somewhat happy. Suppose we add $c_{1}$ to the winning committee. Then the PAV score increases with $|Y|$, and the Nash score increases with $\ln (2) \cdot|Y|$. If we add $c_{2}$ to the committee instead, the PAV score increases by $\frac{\left|X^{\prime}\right|}{6}$, and the Nash score increases by $\left|X^{\prime}\right| \ln (7)-\left|X^{\prime}\right| \ln (6)$. We now look for values of $|Y|$ and $\left|X^{\prime}\right|$ such that the PAV score increases more for adding $c_{1}$ and the Nash score increases more for adding $c_{2}$ or vice versa. The system $|Y|>\frac{\left|X^{\prime}\right|}{6}$ and $\ln (2) \times|Y|<\left|X^{\prime}\right| \times \ln (7)-\left|X^{\prime}\right| \times \ln (6)$ gives us some possible solutions, for example $\left|X^{\prime}\right|=5,|Y|=1$, so if a group of five agents from $X$ approve $c_{2}$ and one agent not from $X$ approves $c_{1}$, adding $c_{1}$ to $W$ will increase the PAV score more than adding $c_{2}$, but adding $c_{1}$ to $W$ will increase the Nash score less than adding $c_{2}$. We only have to choose $X$ large enough such that $\left\{w_{1}, \ldots, w_{5}\right\}$ are elected, and then we have found a situation where the winning committee for PAV is not the committee that maximises the Nash product.

It seems that for small numbers of agents and projects, those maxima will be the same, so an interesting question for future research is what is the minimum number of voters and projects for which they can differ.

Open Question 2.1. What is the minimal number of voters and projects for which the Nash score and the PAV score have a different maximum?

Proposition 2.4. Rule $X$ does not maximise Nash welfare.
This will be shown in Section 4.2 ,

Finally, for Phragmén's rule it was not established whether or not it satisfied FJR, but in the same way we saw it did not satisfy the core, it is very easy to see that it does not satisfy FJR either.

Proposition 2.5. Phragmén's rule fails FJR.
Proof. As shown by [10], the outcome of Phragmén's rule does not necessarily satisfy EJR. We know from [11] that FJR implies EJR, so Phragmen's rule does also not satisfy FJR.

### 2.3 Relations between axioms

We study the relations between proportionality axioms in this thesis, and quite some of them are already shown in the literature, or become immediately clear from the definition of the axioms. From [11] and [13], we know that the core implies FJR, which implies EJR, which implies PJR, and from [10], we know that priceability implies PJR.
In some of the following sections we will study new relations between axioms. A summary of the findings is given here, with references to the corresponding theorems:

- Priceability does not imply laminar proportionality in MWV: Theorem 2.1.
- Laminar proportionality implies priceability of laminar election instances in MWV: Theorem 2.2.
- Laminar proportionality implies PJR, EJR, and the core in MWV: Corollary 4.2.2.
- PJR does not imply laminar proportionality in MWV: Theorem 2.3 .
- PJR does not imply priceability in MWV, not even in laminar election instances: Theorem 2.4 and Corollary 2.4.1.
- Priceability does not imply EJR or the core in MWV: Theorem 2.5 and Corollary 2.5.1.
- Neither EJR nor the core implies priceability or laminar proportionality in MWV: Theorem 2.6 and Corollary 2.6.1.


### 2.3.1 The relation between laminar proportionality and priceability in MWV

In Table 6we saw that a lot of rules satisfy both priceability and laminar proportionality or neither of them, which gives rise to the question whether there is a logical relation between the two axioms. In this section we will try to answer this question.
First of all, note that laminar proportionality cannot say anything about priceability in general because it is only defined on laminar instances, while priceability is defined on all profiles. We can however study whether laminar proportionality implies priceability in laminar election instances, and whether priceability implies laminar proportionality.

## Theorem 2.1. In $M W V$, priceability does not imply laminar proportionality

Proof. In order to show that priceability does not imply laminar proportionality, we construct a counterexample. Consider a profile $P$ as in Table 7 .
We have 3 voters, $v_{1}, v_{2}$, and $v_{3}$, and 6 candidates $c_{1}, \ldots, c_{6}$. Candidates $c_{1}, c_{2}$, and $c_{3}$ are approved by voters $v_{1}$ and $v_{2}$, candidates $c_{4}$ and $c_{5}$ are approved by voter $v_{3}$, and candidate $c_{6}$ is approved by all three voters. This profile is laminar for $k=4$, we can construct it according to the inductive


Table 7: A counterexample that shows that priceability does not imply laminar proportionality: Profile $P$ with committee $W$ (in green). It should be read in the same way as Table 4 .

Definition 2.10 by first concatenating $\left\{c_{1}, c_{2}, c_{3}\right\}(k=2)$ and $\left\{c_{4}, c_{5}\right\}(k=1)$ by Definition 2.103 and then adding $c_{6}$ by Definition 2.10 2, which results in $k=2+1+1=4$.
The elected committee $W$, as indicated in green, is not laminar proportional, because in order to be laminar proportional, $c_{6}$ would have to be included in $W$. It is, however, priceable: we can construct a price system with price $p=0.65$ : $v_{1}$ and $v_{2}$ together pay for $c_{1}, c_{2}$, and $c_{3}$ and have $2-3 \cdot 0.65=0.05$ left over, and $v_{3}$ pays for $c_{4}$ and has $1-0.65=0.35$ left over. Hence, $v_{3}$ cannot pay for $c_{5}$ anymore, and all voters together have an unspent budget of 0.4 , so they cannot pay for $c_{6}$.

Although priceable committees in laminar election instances need not be laminar proportional, we can show that the converse is true: in laminar election instances, laminar proportional committees are priceable. In order to show this, we show that we can construct a price system with price $\frac{n}{k}$ for laminar proportional committees in laminar election instances.

Theorem 2.2. Laminar proportionality implies priceability on laminar election instances.
Proof. We construct an inductive proof on the structure of laminar election instances to prove that for every committee $W$ that is laminar proportional for a laminar election instance $(P, k)$, there exists a price system $\mathbf{p s}=\left(p,\left\langle p_{i}\right\rangle_{i \in N}\right)$ where $p=\frac{|P|}{k}$.
Basis: If $P$ is unanimous with $|C(P)| \geq k$ and $W$ is laminar proportional for $(P, k)$, then $W \subseteq C(P)$, so the voters can just divide their budget over the candidates in $W$. If we set the price $p$ to $p=\frac{n}{k}$, we can let every voter pay $\frac{1}{k}$ to every candidate. Then every voter will have in total spent $k \cdot \frac{1}{k}=1$ unit of money, so have nothing left to spend on other candidates, and every candidate in $W$ will have received $n \cdot \frac{1}{k}=p$ units of money, so can exactly be afforded.
Inductive Hypothesis (IH): Suppose that $\left(P^{\prime}, k^{\prime}\right),\left(P_{1}, k_{1}\right)$, and $\left(P_{2}, k_{2}\right)$ are laminar election instances, committees $W^{\prime}, W_{1}$, and $W_{2}$ are laminar proportional for respectively $\left(P^{\prime}, k^{\prime}\right),\left(P_{1}, k_{1}\right)$, and $\left(P_{2}, k_{2}\right)$, and suppose that for $W^{\prime}$ there exists a price system with price $p^{\prime}=\frac{\left|p^{\prime}\right|}{k^{\prime}}$, for $W_{1}$ there exists a price system with price $p_{1}=\frac{\left|P_{1}\right|}{k_{1}}$, and for $W_{2}$ there exists a price system with price $p_{2}=\frac{\left|P_{2}\right|}{k_{2}}$. Furthermore, suppose that $P^{\prime}$ is not unanimous, that $C\left(P_{1}\right) \cap C\left(P_{2}\right)=\emptyset$ and that $\left|P_{1}\right| \cdot k_{2}=\left|P_{2}\right| \cdot k_{1}$.
Inductive step:

- Case 1: There is a unanimously approved candidate $c$ such that $P=P_{+c}^{\prime}$, where $P_{+c}^{\prime}=\left(A_{1} \cup\right.$ $\left.\{c\}, \ldots A_{n} \cup\{c\}\right)$. Suppose that $W$ is laminar proportional for $P, k=k^{\prime}+1$, then $W=W^{\prime} \cup\{c\}$. By the inductive hypothesis, there exists a price system $\mathbf{p s}$ for $W^{\prime}$ with price $p^{\prime}=\frac{\left|P^{\prime}\right|}{k^{\prime} \mid}$. Because $c$ is unanimously approved, in theory all voters can pay for $c$. We know that in $\mathbf{p s}^{\prime}$, there was no candidate that was not in $W^{\prime}$ for which its supporters together had enough (more than $p^{\prime}$ ) unspent budget. If we would give every voter $\frac{1}{k^{\prime}}$ more budget, which we let them spend entirely on $c, c$ will get enough money and no voter will have more unspent budged than they had before. We then only have to rescale the system so that every voter has 1 unit of currency to start with
again. Every voter now has $1+\frac{1}{k^{\prime}}$ units of money, so we divide everything by $1+\frac{1}{k^{k}}$ in order to get a per voter budget of 1 . The price $p$ now becomes $\frac{n}{k^{\prime}} 1+\frac{n}{k^{\prime}}=\frac{n}{k^{\prime}} \cdot \frac{k^{\prime}}{k^{\prime}+1}=\frac{n}{k^{\prime}+1}=\frac{n}{k}$ and all the individual payment functions are divided by $1+\frac{1}{k^{\prime}}$ as well. Because for every candidate in $W$ the sum of the individual payments was equal to the price, when we divide both the individual payments and the price by the same constant, this equality will continue to hold.
Formally we define the price system $\mathbf{p s}$ for the instance $(P, k)$ as follows: $\mathbf{p s}=\left(p,\left\langle p_{i}\right\rangle_{i \in N}\right)$ with $p=\frac{p^{\prime}}{1+\frac{1}{k^{\prime}}}=\frac{n}{k}$, and $p_{i}: C \rightarrow[0,1]$ such that $p_{i}(c)=\frac{\frac{1}{k^{\prime}}}{1+\frac{1}{k-1}}=\frac{\frac{1}{k-1}}{1+\frac{1}{(k-1)}}=\frac{1}{k}$ and $p_{i}(d)=\frac{p_{i}^{\prime}(d)}{1+\frac{1}{(k-1)}}$ for all other candidates $d \in C(P)$, where $p_{i}^{\prime}$ is the payment function of voter $i$ in the price system ps ${ }^{\prime}$.
To show that this is indeed a valid price system that supports $W$, we look at the five points of the definition of a price system that supports a committee:

1. Voters only pay for candidates they approve of, because they did so in $p s^{\prime}$, and the only candidate which they now pay for that they did not pay for before is $c$, which is unanimously approved.
2. 

$$
\begin{aligned}
\sum_{d \in C(P)} p_{i}(d) & =\sum_{d \in C(P)} \frac{p_{i}^{\prime}(d)}{1+\frac{1}{k-1}}+\frac{1}{k} \\
& =\frac{1}{1+\frac{1}{k-1}} \sum_{d \in C(P)} p_{i}^{\prime}(d)+\frac{1}{k} \\
& \leq \frac{1}{1+\frac{1}{k-1}}+\frac{1}{k} \\
& =\frac{k-1}{k}+\frac{1}{k}=1
\end{aligned}
$$

3. For each elected candidate $d \in W$, if $d \neq c$ the sum of the payments is

$$
\begin{aligned}
\sum_{i \in N} p_{i}(d) & =\sum_{i \in N} \frac{p_{i}^{\prime}(d)}{1+\frac{1}{k-1}} \\
& =\frac{1}{1+\frac{1}{k-1}} \sum_{i \in N} p_{i}^{\prime}(d) \\
& \stackrel{\mathrm{IH}}{=} \frac{1}{1+\frac{1}{k-1}} \cdot p^{\prime} \\
& =\frac{1}{1+\frac{1}{k-1}} \cdot \frac{n}{k-1} \\
& =\frac{n}{\left(1+\frac{1}{k-1}\right) \cdot(k-1)} \\
& =\frac{n}{k}=p
\end{aligned}
$$

For $c, \sum_{i \in N} p_{i}(c)=n \cdot \frac{1}{k}=\frac{n}{k=p}$
4. For any candidate outside of the committee $d \notin W$,

$$
\begin{aligned}
\sum_{i \in N} p_{i}(d) & =\sum_{i \in N} \frac{p_{i}^{\prime}(d)}{1+\frac{1}{k-1}} \\
& =\frac{1}{1+\frac{1}{k-1}} \sum_{i \in N} p_{i}^{\prime}(d) \\
& \stackrel{\mathrm{IH}}{=} \frac{1}{1+\frac{1}{k-1}} \cdot 0=0
\end{aligned}
$$

5. For any candidate outside of the committee $d \notin W$, its supporters do not have a remaining unspent budget of more than $p$ :

$$
\begin{aligned}
& \quad \sum_{i \in N \text { for which } d \in A_{i}}\left(1-\sum_{e \in W=W^{\prime} \cup\{c\}} p_{i}(e)\right) \\
& =\sum_{i \in N: d \in A_{i}}\left(1-p_{i}(c)-\sum_{e \in W^{\prime}} p_{i}(e)\right) \\
& =\sum_{i \in N: d \in A_{i}}\left(1-\frac{1}{k}-\sum_{e \in W^{\prime}} p_{i}(e)\right) \\
& =\sum_{i \in N: d \in A_{i}}\left(\frac{k-1}{k}-\sum_{e \in W^{\prime}} p_{i}(e)\right) \\
& =\sum_{i \in N: d \in A_{i}}\left(\frac{k-1}{k}-\frac{k-1}{k} \sum_{e \in W^{\prime}}\left(p_{i}(e) \frac{k}{k-1}\right)\right) \\
& =\frac{k-1}{k} \sum_{i \in N: d \in A_{i}}\left(1-\sum_{e \in W^{\prime}}\left(p_{i}(e) \frac{k}{k-1}\right)\right) \\
& =\frac{k-1}{k} \sum_{i \in N: d \in A_{i}}\left(1-\sum_{e \in W^{\prime}} p_{i}^{\prime}(e)\right) \\
& \text { IH } \leq \frac{k-1}{k} \cdot p^{\prime} \\
& =\frac{k-1}{k} \cdot \frac{n}{k-1}=\frac{n}{k}=p,
\end{aligned}
$$

so there is no unelected candidate whose supporters in total have a remaining unspent budget of more than $p$.

Hence, ps is indeed a valid price system that supports committee $W$.

- Case 2: $P=P_{1}+P_{2}$ and $k=k_{1}+k_{2}$. Take $W=W_{1} \cup W_{2}$, which is by definition laminar proportional for $(P, k)$. We have to show that $W$ is priceable for this election instance. Note that there are no overlapping candidates between $P_{1}$ and $P_{2}$, there is no voter in $P_{1}$ that approves a candidate from $C\left(P_{2}\right)$, and no voter in $P_{2}$ that approves a candidate from $C\left(P_{1}\right)$. By the inductive hypothesis, there exists a price system $\mathbf{p s}_{\mathbf{1}}=\left(p_{1},\left\{p_{1, i}\right\}_{i \in N}\right)$ for $W_{1}$ with price $p_{1}=\frac{\left|P_{1}\right|}{k_{1}}$, and for $W_{2}$ there exists a price system $\mathbf{p s}_{\mathbf{2}}=\left(p_{2},\left\{p_{2, i}\right\}_{i \in N}\right)$ with $p_{2}=\frac{\left|P_{2}\right|}{k_{2}}$. Also by the inductive hypothesis, $\left|P_{1}\right| \cdot k_{2}=\left|P_{2}\right| \cdot k_{1}$, so $p_{1}=p_{2}$. We can now define a price system $\mathbf{p s}$ that supports $W$ as follows: $\mathbf{p s}=\left(p,\left\langle p_{i}\right\rangle_{i \in N}\right)$ with $p=p_{1}=p_{2}$, and for all voters $i \in N$,

$$
p_{i}(c)=p_{1, i}^{\prime}(c)+p_{2, i}^{\prime}(c),
$$

where $p_{1, i}^{\prime}$ and $p_{2, i}^{\prime}$ are extended versions of respectively $p_{1, i}$ and $p_{2, i}$ that yield zero for the candididates that those are not defined for:

$$
\begin{aligned}
& p_{1, i}^{\prime}(c)= \begin{cases}p_{1, i}(c) & \text { if } c \in C\left(P_{1}\right) \text { and } i \in P_{1} ; \\
0 & \text { if } c \in C\left(P_{2}\right) \text { or } i \in P_{2} .\end{cases} \\
& p_{2, i}^{\prime}(c)= \begin{cases}0 & \text { if } c \in C\left(P_{1}\right) \text { or } i \in P_{1} ; \\
p_{2, i}(c) & \text { if } c \in C\left(P_{2}\right) \text { and } i \in P_{2} .\end{cases}
\end{aligned}
$$

Again, we show that this is a valid price system that supports $W$ by looking at the five points of the definition:

1. We know that $\mathbf{p s}_{\mathbf{1}}$ is a valid price system that supports $W_{1}$, so for voters $i \in P_{1}$ and candidates $c \in W_{1}$, if $p_{1, i}(c)>0$, then $c \in A_{i}$. Analogously, for voters $i \in P_{2}$ and $c \in W_{2}$ if $p_{2, i}(c)>0$, then $c \in A_{i}$. Suppose $p_{i}(c)>0$. If $c \in C\left(P_{1}\right)$, then $p_{i}(c)=p_{1, i}(c)$, so $i \in P_{1}$ because there is no voter in $P_{2}$ that approves a candidate from $C\left(P_{1}\right)$ and vice versa. Hence, for $c \in C\left(P_{1}\right)$, if $p_{i}(c)>0$, then $p_{1, i}(c)>0$ and then $c \in A_{i}$. Similarly, we can argue that for $c \in C\left(P_{2}\right)$, if $p_{i}(c)>0$, then $p_{2, i}(c)>0$ and then $c \in A_{i}$. Because $P=P_{1}+P_{2}$, $C(P)=C\left(P_{1}\right) \cup C\left(P_{2}\right)$, so for all $c \in C(P)$, if $p_{i}(c)>0$ then $c \in A_{i}$.
2. $\sum_{c \in C(P)} p_{i}(c)=\sum_{c \in C(P)} p_{1, i}^{\prime}(c)+p_{2, i}^{\prime}(c)$. We already saw that voters from $P_{1}$ do not pay for candidates from $C\left(P_{2}\right)$ and vice versa. Hence, if $i \in P_{1}$, then $\sum_{c \in C(P)} p_{i}(c)=$ $\sum_{c \in C(P)} p_{1, i}^{\prime}(c) \stackrel{\mathrm{IH}}{\leq} 1$, and if $i \in P_{2}$, then $\sum_{c \in C(P)} p_{i}(c)=\sum_{c \in C(P)} p_{2, i}^{\prime}(c) \stackrel{\mathrm{IH}}{\leq} 1$
3. For each elected candidate $c \in W$, the sum of its payments is $\sum_{i \in N} p_{i}(c)=\sum_{i \in N} p_{1, i}^{\prime}(c)+$ $p_{2, i}^{\prime}(c)$. For $c \in C\left(P_{x}\right)$ (with $x \in\{1,2\}$ ) this is $\sum_{i \in N} p_{x, i}^{\prime}(c) \stackrel{\mathrm{IH}}{=} p_{x}=p$.
4. Because $W=W_{1} \cup W_{2}$, any candidate that is not elected in the new committee, $c \notin W$, was not elected in $W_{1}$ or $W_{2}$, so did not get any payment there: for $c \in C\left(P_{x}\right), \sum_{i \in N} p_{x, i}(c)=0$. Hence, it also does not get any payment in the new system: for $c \in C\left(P_{x}\right), \sum_{i \in N} p_{i}(c)=$ $\sum_{i \in N} p_{x, i}^{\prime}(c)=\sum_{i \in N} p_{x, i}(c)=0$ (for $x \in\{1,2\}$ ).
5. All unelected candidates are only supported by voters from their own 'old' system, who did not have in total a remaining unspent budget of more than the price there, so neither will they have it now:
Without loss of generality, assume that an unelected candidate $c \notin W$ is part of $C\left(P_{1}\right)$. Then because $c \notin W$, we also have $c \notin W_{1}$, because if it was in $W_{1}$, it would also have been in $W$. Because $\mathbf{p s}_{\mathbf{1}}$ is a price system that supports $W_{1}$, we know that $\sum_{i \in N \text { for which } c \in A_{i}}(1-$ $\left.\sum_{e \in W_{1}} p_{1, i}(e)\right) \leq p_{1}$. However, for all voters $i \in N$ for which $c \in A_{i}$, we have $i \in P_{1}$, so for all $e \in W_{1}, p_{1, i}(e)=p_{i}(e)$, and for all $e \in W_{2}, p_{i}(e)=0$. This implies that

$$
\begin{aligned}
\sum_{i \in N \text { for which }}\left(1-\sum_{e \in A_{i}} p_{1, i}(e)\right) & \leq p_{1} \\
\sum_{i \in N: c \in A_{i}}\left(1-\sum_{e \in W=W_{1} \cup W_{2}} p_{i}(e)\right) & \leq p_{1}=p
\end{aligned}
$$

We can analogously show the same for $c \in C\left(P_{2}\right)$, so conclude that for all $c \in C(P)=$ $C\left(P_{1}\right) \cup C\left(P_{2}\right)$, if $c \notin W$,

$$
\sum_{i \in N: c \in A_{i}}\left(1-\sum_{e \in W} p_{i}(e)\right) \leq p
$$

By these five points, we have shown that $\mathbf{p s}$ is indeed a valid price system that supports committee $W$.

We have shown by induction over laminar election instances that, if a committee $W$ is laminar proportional in a laminar election instance $(P, k)$, it is also supported by a price system with price $\frac{|P|}{k}$. Hence we can conclude that laminar proportionality implies priceability in laminar election instances in multi-winner approval based election settings.

### 2.3.2 Priceability's and laminar proportionality's relation with PJR, EJR, and the core in multi-winner elections

In the previous section we examined the relationship between priceability and laminar proportionality. In this section, we will study the logical relations between those two axioms on one hand, and the axioms of the core, EJR, and PJR on the other hand.
As is shown in [10], a committee that is priceable also satisfies PJR. Hence, we can deduce that PJR does not imply laminar proportionality.

## Theorem 2.3. PJR does not imply laminar proportionality in $M W V$.

Proof. Take the example from Section 2.3.1 (Table 7). This profile is laminar, and committee $W$ indicated in green is priceable. Hence, it satisfies PJR. It does however not satisfy laminar proportionality, because candidate $c_{6}$ is not elected. Therefore we can conclude that PJR does not imply laminar proportionality.

PJR also does not imply priceability, as we can show by the following counterexample (visualized in Table 8):

## Theorem 2.4. PJR does not imply priceability in laminar election instances in MWV.

Proof. We give a counterexample that shows that a committee $W$ in a laminar profile can satisfy PJR without being priceable. Suppose we have 3 voters, $v_{1}, v_{2}$, and $v_{3}$, and 5 candidates $c_{1}, \ldots, c_{5}$. Candidates $c_{1}$ and $c_{2}$ are approved by voters $v_{1}$ and $v_{2}$, candidates $c_{3}$ and $c_{4}$ are approved by voter $v_{3}$, and candidate $c_{5}$ is approved by all three voters. This election instance is laminar, as can easily be checked, and the elected committee $W$, as indicated in green in Table 8 satisfies PJR: for all $\ell \leq 4$ and all $\ell$-cohesive group of voters $S$, it holds that $\left|W \cap \cup_{i \in S} A_{i}\right| \geq \ell$. It is, however, not priceable: suppose, for a contradiction, that there is a price system ps that supports $W$. The price $p$ of this system has to be such that $v_{3}$ can pay for both $c_{3}$ and $c_{4}$ (because no other voter can pay for these candidates), so $p \leq 0.5$. However, $v_{1}$ and $v_{2}$ together have a budget of 2 , which they must spend only on $c_{1}$ and $c_{2}$. Hence, $p>\frac{2}{3}$, because otherwise $v_{1}$ and $v_{2}$ together have enough unspent budget to pay for $c_{5}$ which is not in $W$. Hence $p \leq 0.5$ and $p>\frac{2}{3}$, which is a contradiction. Therefore, there is no price system that supports $W$.

Since laminar election instances are a specific type of approval elections, the same result holds for MWV elections in general.

Corollary 2.4.1. PJR does not imply priceability in MWV.
Proof. This follows directly from theorem 2.4 .


Table 8: Profile P with committee W indicated in green. It should be read in the same manner as Table 4 and Table 7 .

In [10], it is shown that in the multi-winner election setting, priceability implies PJR. We study whether it is also the case that all priceable committees must satisfy the stronger axiom of EJR. Note that we only have to look at cases where the unit cost assumption is satisfied, in PB, we have that priceable committees do not necessarily satisfy PJR, so they also cannot necessarily satisfy the stronger EJR. We see that pricable committees need not satisfy EJR. In party-list profiles however, priceable committees do satisfy EJR.
The following counterexample shows that in general, pricable committees need not satisfy EJR:
Theorem 2.5. Priceability does not imply EJR in MWV.
Proof. Take an election instance $E$ with 3 voters $N=\left\{v_{1}, v_{2}, v_{3}\right\}$ and 6 candidates $C=\left\{c_{1}, \ldots c_{6}\right\}$, and let $k=3$. Let every voter approve four projects, namely one that only that voter approves and three that all three voters approve: $A_{i}=\left\{c_{i}, c_{4}, c_{5}, c_{6}\right\}$. Suppose the winning committee of the election is $W=\left\{c_{1}, c_{2}, c_{3}\right\}$. For this committee, there is a price system with $p=1$ in which every voter $i$ pays the price of candidate $i$ and nothing else. The group $S=N$ is 3-cohesive, because $|S|=3=3 \frac{n}{k}$, and all three voters agree on the projects $\left\{c_{4}, c_{5}, c_{6}\right\}$. However, there is no voter in $S$ who approves 3 or more projects in $W$, so $W$ does not satisfy EJR.

Since the core implies EJR, this example also shows that priceable committees are not necessarily in the core.

Corollary 2.5.1. Priceability does not imply the core in MWV.
The implication in the other direction does not hold either, EJR does not imply priceability or laminar proportionality.

Theorem 2.6. Neither EJR nor the core implies priceability or laminar proportionality in laminar election instances in MWV.

Proof. The counterexample for Theorem 2.4 also shows that a committee that satisfies EJR or is in the core does not necessarily have to be priceable or laminar proportional. The committee in Table 8 does satisfy EJR and it is in the core, as can easily be checked, but it is neither priceable nor laminar proportional.

Again, since laminar election instances are a specific type of multi-winner approval elections, the same result holds for MWV in general.

Corollary 2.6.1. Neither EJR nor the core implies priceability in MWV.
Proof. This follows directly from theorem 2.6 .
In Section 4.3.2 (Corollary 4.2.2), we will show that in MWV, laminar proportionality implies all three axioms of the core, EJR, and PJR.

### 2.3.3 FJR

So far, we did not really study the FJR axiom. However, it can be easily fitted in our landscape of proportionality axioms, since we know that FJR is 'in between' EJR and the core (that the core implies FJR and FJR implies EJR). We have not studied it's relation to laminar proportionality and priceability yet. However, since laminar proportionality implies the core (Corollary 4.2.2), it also implies FJR, and since priceability does not imply EJR (Theorem 2.5), pricability does not imply FJR. Furthermore, since the core does not imply priceability or laminar proportionality (Theorem 2.6 , Corollary 2.6.1), FJR does not imply priceability or laminar proportionality either (since otherwise the core would implicitly imply them as well).

In multi-winner voting, we now have the relations as shown in Figure 4. We have shown that these are the only implicational relations between the four properties, the absence of arrows indicate there is a counterexample that shows there is no implication between the properties.
As we will show in Chapter 4 , under certain conditions there are more relations. For example, we will show in Section 4.4 that in party-list profiles (Definition 2.12) all axioms from Figure 4 are equivalent.


Figure 4: The implicational relations between laminar proportionality, priceability, PJR, EJR, and the core in multi-winner approval voting

### 2.4 Future work

In the beginning of this Chapter, we also looked at the proportionality axiom of Nash welfare, and saw that neither of the rules we studied satisfied it. PAV, however, seems to output a committee that has an nearly maximal Nash score. We found a counterexample that shows that PAV does not always maximise the Nash score (Proposition ), but for small numbers of voters and projects, the maximum Nash product and maximum PAV score seem to be produced by the same committee. It could be interesting to study what is the minimum number of voters and projects for which the maximum PAV score and maximum Nash product are different (Open Question 2.1). Another interesting topic of research is the
relation between the Nash welfare axiom and the other axioms. We have not included Nash welfare in Figure 4 as the question remains what are the relations between Nash welfare and the axioms that had our main focus, it is not entirely clear yet how it fits into the landscape of proportionality axioms. In Section 3.4, we will see that in the PB setting, a committee maximising the Nash product does not necessarily satisfy EJR or the core. We have no such result for the MWV setting, but that result leads to the conjecture that also in MWV Nash welfare does not imply EJR or the core. Investigating this, as well as studying the relation between Nash welfare and priceability, laminar proportionality, and PJR, remain topics for future research:

Open Question 2.2. Is there a logical relation in MWV between Nash welfare on the one hand, and the core, EJR, PJR, priceability, and laminar proportionality on the other hand?

## 3 Participatory Budgeting

As we already mentioned, participatory budgeting is a generalisation of multi-winner voting, with a relaxation of the unit cost assumption and utility based voting instead of approval voting. Hence, the results that are obtained in the field of MWV can help a lot in understanding the behavior of rules and axioms in PB. The rules that we studied in the previous chapter can each be generalised to a PB version (although some of them are only defined for approval voting), and for the axioms this is possible as well. We will also study a new rule that is especially designed for PB, namely the Smithconsistent Budgeting Algorithm (SBA), which is a promising algorithm because it is computable in polynomial time and always returns a Condorcet-winning budget if it exists. As we will show, however, it performs poorly on the different fairness axioms.

Outline of chapter This chapter is structured very similar to Chapter 2 . Just like in the previous chapter, we will start by giving the definitions of the rules and axioms used in PB in Section 3.1 . Section 3.2 studies which axioms are satisfied by the different rules in PB, and in Section 3.3 we will study the relations between the different axioms. Finally in Section 3.4, we will mention some directions for future work.

### 3.1 Definitions

### 3.1.1 Rules

In order for a rule to be generalisable from approval voting to utility voting, we must be able to use utility scores that can have any real value between 0 and 1 . However, for both PAV and Phragmén's rule, there is no trivial way to do this. In computing the PAV-score, we use the harmonic numbers, which are only defined on integers (e.g. we cannot have an 0.345 -th harmonic number). We could define a continuous function that approximates the harmonic numbers ${ }^{4}$, but instead we chose to keep using approval voting in the PB setting. Hence, the only generalization for PAV from MWV to PB is the relaxation of the unit cost assumption, and the definition is as follows:

Definition 3.1 (PAV for PB). The outcome $W$ of PAV is the committee with $\operatorname{cost}(W) \leq l$ that maximises the score

$$
\operatorname{PAV}-\operatorname{score}(W)=\sum_{i \in N}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left|W \cap A_{i}\right|}\right) .
$$

A similar problem arises in the generalisation of Phragmén's rule. In Phragmén's rule, whether or not a voter has to pay for a candidate is determined by whether or not the voter approves that candidate. The amount every approving agent has to pay may differ. However, having to pay or not is a binary variable, so we need a binary value to determine this. We could require that the amount a voter needs to pay is proportional to the utility they get from it, but as [11] show, this will make Phragmén's rule elect very inefficient outcomes. However, we can generalise the rule very naturally to non-unit costs. The PB definition of Phragmén's rule then is as follows:

Definition 3.2 (Phragmén's rule for PB [11]). Every voter gets continuously one unit of currency per time unit. At the first moment in time $t$ when there is a group of voters $S$ who all approve a not-yetselected candidate $c$, and who have $\operatorname{cost}(c)$ units of currency in total, the rule adds $c$ to the committee

[^3]and asks the voters from $S$ to pay the cost of $c$ (i.e., the rule resets the balance of each voter from $S$ ); the other voters keep their so-far earned money. The rule stops when it would select a project which, when implemented, would overshoot the budget limit.

In contrast to PAV and Phragmén's rule, Rule X is very well generalisable to PB , as is shown in [11]. This is done by letting the voters pay for a project proportionally to the utility they get from it. Unlike with Phragmén's rule, with Rule X this requirement does not cause inefficiency. Just as Phragmén's rule, Rule X is naturally extendable to non-unit costs. Each voter is now asked to pay for a project an amount proportional to their utility for that project if they can, namely $\rho$ units of money per unit of utility, and if they do not have this amount anymore they have to pay all the money they have left.

Definition 3.3 (Rule X for PB [11]). The rule starts by giving each voter an equal fraction of the budget. In case of a budget limit of 1 , each voter gets $\frac{1}{n}$ units of currency. We start with an empty outcome $W=\emptyset$ and sequentially add projects to $W$. To add a project $c$ to $W$, the voters have to pay for $c$. Write $p_{i}(c)$ for the amount that voter $i$ pays for $c$; we will need that $\sum_{i \in N} p_{i}(c)=\operatorname{cost}(c)$. Let $p_{i}(W)=\sum_{c \in W} p_{i}(c) \leq \frac{1}{n}$ be the total amount voter $i$ has paid so far. For $\rho \geq 0$, we say that a project $c \notin W$ is $\rho$-affordable if

$$
\sum_{i \in N} \min \left(\frac{1}{n}-p_{i}(W), u_{i}(c) \cdot \rho\right)=\operatorname{cost}(c) .
$$

If no project is $\rho$-affordable for any $\rho$, Rule X terminates and returns $W$. Otherwise it selects a project $c \notin W$ that is $\rho$-affordable for a minimum $\rho$. Individual payments are given by $p_{i}(c)=\min \left(\frac{1}{n}-\right.$ $\left.p_{i}(W), u_{i}(c) \cdot \rho\right)$.
The new rule SBA is introduced in [5]. Since SBA needs as input a ranking of projects from each voter, it is not generalisable to approval voting and hence not discussed in the chapter about MWV. (If we would transfer the ordinal ballots to approval ballots using some kind of approval threshold, the algorithm would depend largely on tie-breaking decisions.) For the total pseudo-code we refer to the original paper, here we will give a textual definition.

Definition 3.4 (SBA [5]). The rule starts by generating a majority graph $G$ of the profile, which is a graph that has all projects as vertices and an arc between two projects $c$ and $c^{\prime}$ if more than half of the voters rank $c^{\prime}$ higher than $c$. Next, the rule continues with the ranking procedure: it creates an ordered partition $V$ of the set of projects $C$. $V$ starts empty, and in every iteration the rule identifies the Schwartz se $\left[^{5}\right.$ of the majority graph $G$, adds the projects in it as a new component of $V$ and removes them from $G$. This is done until the $G$ is empty, at which point $V$ is an ordered partition of $C$. Now the rule starts in the pruning procedure to fill the set of elected projects $W$ by iterating over the components $C_{1}, \cdots C_{z}$ of $V$ and choosing the subset with maximal cost $B_{i}$ of the current component $C_{i}$ such that the cost of $B_{i} \cup W$ is still within the budget limit. If there are more such subsets with equal cost, the previous budget (for example last year's budget) is used as a tie-breaker: the one closest (minimal cost of symmetric distance) to the previous budget is chosen. This subset of $C_{i}$ is added to $W$. After having considered all components of V , the rule stops.
Example 3.1. In Figure 5, an example of the SBA procedure is displayed. The input consists of a utility profile, which is translated into a profile of ordinal ballots (of course, the input can also be an ordinal profile directly). Then the majority graph $G$ is constructed and the Ranking procedure starts: the Schwartz-set is computed and $G$ is adjusted until $G$ is empty, which leaves us with an ordered partition $V$. Then in the Pruning procedure the output set $B$ is formed by going over the items from $V$ and adding affordable projects to $B$.

[^4]

Figure 5: Example of the SBA-algorithm

In the following subsections, we will give the definitions of several (fairness) axioms in PB. These are the same axioms as we used in MWV (Section 2.1.2), but generalised to the PB setting.

### 3.1.2 PB Axioms from the literature

Just as in MWV, we have the related axioms of core, FJR, and EJR. The core naturally extends to PB, by requiring that there is no group of agents that can afford a set of projects they strictly prefer to the winning set of projects. The formal definition is then as follows:

Definition 3.5 (Core for PB [11]). For a given election instance $E=(N, C, \operatorname{cost}, P, l)$, we say that an outcome $W$ is in the core, if for every $S \subseteq N$ and $T \subseteq C$ with $|S| \geq \operatorname{cost}(T) \cdot n$ there exists $i \in S$ such that $u_{i}(W) \geq u_{i}(T)$. We say that an election rule $\mathcal{R}$ satisfies the core property if for each election instance $E$ the winning outcome $\mathcal{R}(E)$ is in the core.

It is clear that this is a generalisation of the MWV variant of the core and that applying it in the MWV setting will make it equivalent to Definition 2.4 .
In [11], the definition of FJR is given, which was originally designed for PB. The MWV version we gave in Definition 2.5 is a specific variant of it, proven by the authors to be a MWV variant of the same axiom.

Definition 3.6 (Full Justified Representation (FJR) for PB [11]). We say that a group of voters $S$ is weakly $(\beta, T)$-cohesive for $\left.\beta \in \mathbb{R}_{\geq}\right]^{6}$ and $T \subseteq C$, if $|S|>\frac{\operatorname{cost}(T)}{l} \cdot n$ and $u_{i}(T)>\beta$ for every voter $i \in S$. A rule $\mathcal{R}$ satisfies FJR if for each election instance $E$ and each weakly $(\beta, T)$-cohesive group of voters $S$ there exists a voter $i \in S$ such that $u_{i}(\mathcal{R}(E))>\beta$.

[^5]A generalisation of the MWV axiom of EJR was also given by [11], and is proven to be indeed a generalisation of that axiom.
Definition 3.7 (Extended justified representation (EJR) for PB [11]). A group of voters S is $(\alpha, T)$ cohesive for $\alpha: C \rightarrow[0 ; 1]$ and $T \subseteq C$, if $|S| \geq \frac{\operatorname{cost}(T)}{l} \cdot n$ and if it holds that $u_{i}(c) \geq \alpha(c)$ for every voter $i \in S$ and each candidate $c \in T$. A rule $\mathcal{R}$ satisfies EJR if for each election instance $E$ and each $(\alpha, T)$-cohesive group of voters $S$, there is a voter $i \in S$ such that $u_{i}(\mathcal{R}(E)) \geq \sum_{c \in T} \alpha(c) .{ }^{7}$
In [11], also the idea of price systems and priceability is generalized to PB. The generalisation to non-unit costs is very natural, we can just give each project its own price in the system instead of taking one general price. The generalisation from approval votes to arbitrary utilities is a bit more involved, but Peters et al. solve that by regarding every agent that gets some utility from a project as approving that project. The new definitions then are as follows:
Definition 3.8 (Price systems for PB [11]). A price system is a pair $\mathbf{p s}=\left(b,\left\langle p_{i}\right\rangle_{i \in N}\right)$ where $b \geq 1$ is the initial budget, and for each voter $i \in N$, there is a payment function $p_{i}: C \rightarrow \mathbb{R}$ such that

1. a voter can only pay for candidates she gets at least some utility from: if $u_{i}(c)=0$, then $p_{i}(c)=0$ for each $i \in N$ and $c \in C$, and
2. each voter can spend the same budget of $\frac{b}{n}$ units of money: $\sum_{c \in C} p_{i}(c) \leq \frac{b}{n}$ for each $i \in N$.

Definition 3.9 (Priceability for PB [11]). A rule $\mathcal{R}$ is priceable if for all election instances $E$, there exists a price system $\mathbf{p s}=\left(b,\left\langle p_{i}\right\rangle_{i \in N}\right)$ that supports $\mathcal{R}(E)$, i.e.

1. for each $c \in \mathcal{R}(E)$, the sum of the payments for $c$ equals its price: $\sum_{i \in N} p_{i}(c)=\operatorname{cost}(c)$;
2. no candidate outside of the committee gets any payment: for all $c \notin \mathcal{R}(E), \sum_{i \in N} p_{i}(c)=0$; and
3. there exists no unelected candidate whose supporters in total have a remaining unspent budget of more than its cost: for all $c \notin \mathcal{R}(E)$,

$$
\sum_{i \in N \text { for which } u_{i}(c)>0}\left(\frac{b}{n}-\sum_{c^{\prime} \in \mathcal{R}(E)} p_{i}\left(c^{\prime}\right)\right) \leq \operatorname{cost}(c) .
$$

It is easy to see that every MWV rule that satisfies the MWV version of priceability also satisfies the PB version when approved projects are regarded having a utility of 1 and non-approved projects as having a utility of 0 , and that any MWV satisfying the PB version also satisfies the MWV version. Hence, this is indeed a generalisation of the concept of priceability as introduced in [10].
The concept of Nash welfare can trivially be extended to the PB setting, by replacing the total committee size $k$ by a budget limit $l$ and the cost of a project from 1 to its actual cost, and measuring the utility an agent gets from a set of elected projects by the sum of the actual utilities of each elected project, instead of by the number of approved elected projects.

Definition 3.10 (Nash welfare [4]). A rule satisfies Nash welfare if for every election instance it returns the committee $W$ with $\operatorname{cost}(W) \leq l$ that maximises the Nash product Nash-prod $(W)=\prod_{i \in N}(1+$ $\left.u_{i}(W)\right)$.

Two rules from the previous chapter are left over without a proper generalisation to the PB setting: PJR and laminar proportionality. In the following subsections, we will give generalisations to the PB setting for those axioms.

[^6]
### 3.1.3 PJR in Participatory Budgeting

Sánchez-Fernández et al. [13] define PJR as a measure of proportionality for multi-winner approval based voting rules. We give a generalisation of PJR to the context of participatory budgeting (beyond the assumptions of unit cost and approval-based elections) based on the generalisation of EJR as in [11]. Two steps of generalisation are involved. One is that candidates do not have to have equal costs anymore as in multi-winner voting. The second generalisation is that from approval ballots to utilities: in approval voting the utility of voter $i$ for candidate $c$ can be either 1 or $0: u_{i}(c) \in\{0,1\}$, in utility based voting, the utility can be any real value in a range from 0 to $1: u_{i}(c) \in[0,1]$. To our knowledge, no generalisation of PJR to the PB context has been proposed in the literature except for the one by Aziz, Lee, and Talmon [7]. They define an axiom called Strong-BPJR-L that requires the following: For a budget limit $l$, a budget $W$ satisfies Strong-BPJR-L if for all $\ell \in[1, l]$ there does not exist a set of voters $S \subseteq N$ with $|S| \geq \ell \frac{n}{l}$, such that $\operatorname{cost}\left(\cap_{i \in S} A_{i}\right) \geq \ell$ but $\operatorname{cost}\left(\left(\cup_{i \in S} A_{i}\right) \cap W\right)<\ell$. This generalisation still uses approval votes rather than arbitrary utilities, so we could generalise it even further to be applicable to election instances with arbitrary utilities. However, note that the requirement in this definition is not that for every $\ell$-cohesive $S$ the utility of the projects they all approve that are elected is at least $\ell$, but rather the cost of this set of projects. Although this indeed a generalisation of PJR, as is shown in [7], we think the aim of PJR in MWV is to ensure a certain level of utility for every group of voters, rather than a certain cost. Only if one assumes that the cost of a project is directly proportional to a voter's utility from that project, this property is transferred in the definition of Strong-BPJR-L. We give a generalisation of PJR that does not depend on this assumption and in which utilities can be arbitrary, independent of the cost of a project. For cohesiveness, we use the notion of $(\alpha, T)$-cohesive groups as defined in Definition 3.7.

Definition 3.11 (PB-PJR). A rule $\mathcal{R}$ satisfies proportional justified representation (PB-PJR) if for each election instance $E$ and each $(\alpha, T)$-cohesive group of voters $S$,

$$
\begin{equation*}
\sum_{c \in \mathcal{R}(E)}\left(\max _{i \in S} u_{i}(c)\right) \geq \sum_{c \in T} \alpha(c) . \tag{1}
\end{equation*}
$$

The intuition behind this is that in the solution for each cohesive group $S$ (cohesive in that $S$ agrees to a certain degree about the set of projects $T$ ) there should be enough candidates to which at least one voter in $S$ assigns enough utility.

Example 3.2. As an example, consider the profile displayed in Table 9 . Suppose that the total budget limit $l=1$, and that the $W=\left\{c_{1}, c_{3}, c_{4}\right\}$ is the set of projects elected by some voting rule (note that this is, for example, the set of projects that maximises total utility). Then consider the group of voters $S=\left\{v_{1}, v_{2}\right\}$. Probably both voters in $S$ are happy that $c_{4}$ is elected, but they would both get much more utility from $c_{2}$ than from the elected $c_{1}$ or $c_{3}$. Also, if they both would get their share of the total budget $\left(\frac{1}{6}\right)$, they could together afford the set $T=\left\{c_{2}, c_{4}\right\}$. Intuitively then, $W$ is not a fair outcome considering voters $v_{1}$ and $v_{2}$. Let us look at it in the formal definition. $S$ is $(\alpha, T)$-cohesive for $\alpha\left(c_{2}\right)=7, \alpha\left(c_{4}\right)=6$ (and $\alpha\left(c_{1}\right)$ and $\alpha\left(c_{3}\right)$ arbitrary: as we already noticed, the voters in $S$ can afford $T$ with their share of the budget, and for both of them $u\left(c_{2}\right) \geq \alpha\left(c_{2}\right)$ and $u\left(c_{4}\right) \geq \alpha\left(c_{4}\right)$. However, Equation 1 is not satisfied: $\sum_{c \in W}\left(\max _{i \in S} u_{i}(c)\right)=3+2+7=12$, while $\sum_{c \in T} \alpha(c)=7+6=13$. This shows that in the given election instance, the outcome $W$ does not satisfy PB-PJR.

PB-PJR is a generalisation of PJR in MWV elections In order to show that given definition is indeed a generalisation of the PJR for multi-winner voting as defined in [13], we apply it to the multiwinner voting setting and show that PJR and PB-PJR are equivalent for approval based multi-winner voting rules.

| cost | candidate | utilities |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | ---: |
| 0.1 | $c_{4}$ | 0.7 | 0.6 | 0.6 | 0.8 | 0.4 | 0.7 |
| 0.7 | $c_{3}$ | 0.2 | 0.2 | 0.8 | 0.7 | 0.6 | 0.8 |
| 0.2 | $c_{2}$ | 0.7 | 0.9 | 0.1 | 0.2 | 0.1 | 0.3 |
| 0.2 | $c_{1}$ | 0.3 | 0.1 | 0.9 | 0.3 | 0.9 | 1 |
|  |  | $v_{1}$ | $v_{2}$ | $v_{3}$ | $v_{4}$ | $v_{5}$ | $v_{6}$ |

Table 9: Example profile used in Example 3.2. Each column contains the utilities per candidate of a voter. The cost of the candidates is indicated in the leftmost column. The budget limit $l=1$.

Theorem 3.1. PJR and PB-PJR are equivalent in approval-based multi-winner elections.
In the proof of Theorem 3.1, we make use of the following Lemma:
Lemma 3.1.1. Let $N$ be a set of agents, $C$ a set of alternatives, let each candidate $c$ have a cost $\operatorname{cost}(c)=\frac{1}{k}$, and let each voter $i$ have a utility $u_{i}(c) \in\{0,1\}$ for candidate $c$. Define the approval set $A_{i}$ of voter $i$ as $A_{i}=\left\{c \in C: u_{i}(c)=1\right\}$. Then
(a) for given $\alpha: C \rightarrow[0,1]$ and $T \subseteq C$, any group of voters $S$ that is $(\alpha, T)$-cohesive is also $\ell$-cohesive for $\ell=\left|T^{\prime}\right|$ with $T^{\prime}=\{c \in T: \alpha(c)>0\}$, and
(b) for every group $S$ that is $\ell$-cohesive there are $T \subseteq C$ with $|T|=\ell$ and $\alpha: C \rightarrow[0,1]$ with $\alpha(c)=1$ for all $c \in C$, such that $S$ is $(\alpha, T)$-cohesive.

Proof of of Lemma 3.1.1. (a): Assume that $S \subseteq N$ is $(\alpha, T)$-cohesive for some $\alpha: C \rightarrow[0,1]$ and $T \subseteq C$. By definition, in the approval based multi-winner setting this means that $|S| \geq|T| \cdot \frac{n}{k}$ and that for all $i \in S$ and all $c \in T, u_{i}(c) \geq \alpha(c)$. Now take a subset $T^{\prime}$ from $T$ of all the candidates in $T$ that have at least some utility for all voters in $S: T^{\prime}=\{c \in T: \alpha(c)>0\}$. Because $u_{i}(c) \in\{0,1\}$ (i.e. voting is approval based), $\alpha(c)>0$ and $u_{i}(c) \geq \alpha(c)$ imply that $u_{i}(c)=1$. Hence, for all $i \in S$ and all $c \in T^{\prime}, u_{i}(c)=1$. We can rewrite this as $T^{\prime} \subseteq \cap_{i \in S} A_{i}$, so $\left|T^{\prime}\right| \leq\left|\cap_{i \in S} A_{i}\right|$. Now if we call $\left|T^{\prime}\right|=\ell$, we have $\ell \leq|T|$, so $|S| \geq \ell \cdot \frac{n}{k}$ and $\left|\cap_{i \in S} A_{i}^{\alpha}\right| \geq \ell$, so $S$ is $\ell$-cohesive.
(b): Assume that $S \subseteq N$ is $\ell$-cohesive for some $\ell \leq k$. This means that $|S| \geq \ell \cdot \frac{n}{k}$ and $\left|\cap_{i \in S} A_{i}\right| \geq \ell$. We take $T \subseteq \cap_{i \in S} A_{i}$ with $|T|=\ell$ (which we can do because $\ell \leq\left|\cap_{i \in S} A_{i}\right|$. Now take $\alpha: C \rightarrow[0,1]$ such that $\alpha(c)=1$ for all $c \in C$. Then $|S| \geq|T| \cdot \frac{n}{k}$ and for all $i \in S$ and all $c \in T, u_{i}(c)=1=\alpha(c)$, so $S$ is $(\alpha, T)$-cohesive.

Proof of Theorem 3.1. We show that a rule that satisfies the new definition of PB-PJR also satisfies the old PJR for every approval based multi-winner election instance.
$\mathbf{P B}-\mathbf{P J R} \Rightarrow \mathbf{P J R}$ : Assume that a rule $\mathcal{R}$ satisfies PB-PJR. Now take an arbitrary multi-winner election instance $E$ where all projects have unit cost, and voting is approval based: the utility of a project for a voter is either 0 (when the voter does not approve the project) or 1 (when the voter does approve the project). According to $\mathrm{PB}-\mathrm{PJR}$, it is true that for all $S, \alpha: C \rightarrow[0,1]$, and $T$ with $T \subseteq C$, if $S$ is $(\alpha, T)$-cohesive $\left(|S| \geq \operatorname{cost}(T) \cdot n\right.$ and it holds that $u_{i}(c) \geq \alpha(c)$ for every voter $i \in S$ and each candidate $c \in T$ ), then

$$
\sum_{c \in \mathcal{R}(E)}\left(\max _{i \in S} u_{i}(c)\right) \geq \sum_{c \in T} \alpha(c) .
$$

Because $E$ satisfies the assumptions of unit cost and approval based voting, this boils down to the following: for all $S, \alpha: C \rightarrow[0,1]$, and $T \subseteq C$ with $|S| \geq|T| \cdot \frac{n}{k}$ and for which $u_{i}(c) \geq \alpha(c)$ for all $i \in S$ and for all $c \in T$, it is the case that

$$
\begin{equation*}
\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \sum_{c \in T} \alpha(c) . \tag{2}
\end{equation*}
$$

In order to show that $\mathcal{R}$ satisfies PJR, we have to prove that for all $S$ and $\ell \leq k$ where $S$ is $\ell$-cohesive ( $|S| \geq \ell \cdot \frac{n}{k}$ and $\left|\cap_{i \in S} A_{i}\right| \geq l$ ),

$$
\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \ell
$$

Take arbitrary $S \subseteq N$ and $\ell \leq k$ and suppose that $S$ is $\ell$-cohesive. According to Lemma 1(b), there are $T \subseteq C$ with $|T|=\ell$ and $\alpha: C \rightarrow[0,1]$ with $\alpha(c)=1$ for all $c \in C$, such that $S$ is $(\alpha, T)$-cohesive. Because $\mathcal{R}$ satisfies PB-PJR, this implies that $\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \sum_{c \in T} \alpha(c)$. However, because of our choice of $T$ and $\alpha$, we know that $\sum_{c \in T} \alpha(c)=|T|=\ell$, so $\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \ell$, which is what we had to prove.

We still have to prove that a rule that satisfies PJR also satisfies the newly defined PB-PJR in approval based multi-winner elections:
$\mathbf{P J R} \Rightarrow \mathbf{P B}-\mathbf{P J R}$ : Assume that a rule $\mathcal{R}$ satisfies PJR: in every election instance $E$, for every $\ell$-cohesive group of voters $S,\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \ell$. We want to prove that in every approval based multi-winner election instance, any $(\alpha, T)$-cohesive group $S$ satisfies $\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \sum_{c \in T} \alpha(c)$. Take arbitrary such election instance $E$, and suppose that a group $S$ is $(\alpha, T)$-cohesive. Then according to Lemma 1(a), when we take $T^{\prime}=\{c \in T: \alpha(c)>0\}, S$ is $\ell$-cohesive for $\ell=\left|T^{\prime}\right|$. Because $R$ satisfies PJR, it follows that $\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \ell=\left|T^{\prime}\right|$. By definition of $T^{\prime},\left|T^{\prime}\right| \geq \sum_{c \in T^{\prime}} \alpha(c)=\sum_{c \in T} \alpha(c)$, so $\left|\mathcal{R}(E) \cap\left(\cup_{i \in S} A_{i}\right)\right| \geq \sum_{c \in T} \alpha(c)$, which is what we had to prove.

We introduced PB-PJR, a generalisation of the axiom Proportional Justified Representation to be applied in the context of participatory budgeting. We showed that in approval based multi-winner elections with a unit cost assumption, PB-PJR is equivalent to the existing multi-winner PJR.

### 3.1.4 Laminar Proportionality in Participatory Budgeting

Laminar election instances in PB In this section, we will generalise the definition of laminar election instances and the axiom of laminar proportionality from MWV to PB. For a moment, we keep the notion of unit costs and try to generalise the definition of laminar election instances from approval voting to voting with utilities.
The basic idea of laminar proportionality is that if we know about a strict separation between different parties, we can divide the number of chosen candidates proportionally over the parties (with a bit more elaborate rules about unanimous candidates and subparties). So actually we only say things about situations in which it is very clear how to divide the committee seats fairly (situations we call 'laminar'). The point here is that if a voter that belongs to one group also votes for a candidate of another group, the interests get more complex, and that the laminar proportionality axiom only is about those situations in which the interests are clearly separated. This is slightly contradictory to the idea of utility voting. If we switch from approval voting to utility voting, voters can express their preferences more detailed, and we get more focus exactly on those cases that are ignored when we restrict the scope to laminar election instances. However, we will still explore the idea of laminar election instances in utility voting.

Laminar election instances require that if one candidate is approved by two voters, it is not possible that another candidate is approved by yet another voter and only one of the first two voters. This is like the notion of laminarity in set theory: the sets of voters that approve a candidate should be either disjoint or subsets of each other. Hence, in utility voting we could say that we do not want a voter from one party to get any utility from another party.
What we could do is define a kind of approval threshold in utility based voting, in order to transfer the utilities to approval votes, and then just use the old definition of laminar election instances. However, if we do that, we cause small utilities not to have any influence anymore, and we 'throw away' a lot of information (although the latter may not be insurmountable). Furthermore, we want to retain the idea of voters not getting any utility if candidates from other parties are selected. Therefore, it might be a solution to set this approval threshold to zero: if a voter even has the tiniest amount of utility for a candidate, we define him to approve that candidate, and only if a voter has no utility at all for a candidate, he does not approve that candidate. The problem with such an approval threshold at 0 is that we disregard the information gotten from the utility voting (all utilities above 0 are regarded equal), and that there is now a big difference between a utility 0 and a utility of 0.1 , while there is no difference between a utility of 0.1 and one of 0.2 . Nevertheless, when asking utility functions from the voters, they should expect these differences to be equal. For this reason, we do not see this as a proper way to define laminar instances for utility profiles. Note, however, that in the generalisation of priceability to PB given in [11] (Definitions 3.8 and 3.9), the exact same problem arises: the exact value of the utilities is never used, only whether or not it is positive.
What we could do in order to let utilities make a difference is to look how much different voters agree about different candidates. Suppose that in an approval voting instance we have two candidates, and three voters such that both candidates have two votes, and one voter votes for both candidates. This is not allowed in laminar election instances. However, it would be allowed if one of the voters that only votes one candidate would also vote for the other candidate. Hence, in utility voting, we want this voter to vote at least as much for this other candidate (have a utility at least as high) as the other voter does. We can capture this intuition by subtracting that part of utility from a voter's utility that is shared by the other voters. We could define unanimity in utility profiles as follows: a candidate is voted unanimously if all voters give it the same utility ( $>0$ ). Now we could say an election instance $E$ is laminar if either $E$ is unanimous, or $E$ consists of two laminar instances $E_{1}$ and $E_{2}$ (just as in Definition 2.10.3), or there is some candidate $c$ that gives all voters some utility and the instance $E^{\prime}$, where we subtract the utility of the voter with minimum utility for $c$ from all voter's utilities for $c$, is laminar. The problem remains what to do with the committee size $k$ when subtracting utilities in this last case. In approval voting, we could just take $k-1$, because we could assume that the unanimously approved candidate would be chosen anyway. However, with utility voting, it depends on the voting rule whether or not a candidate for which every voter has some utility is actually elected or not. We could choose to decrease $k$ with the amount of utility that we subtract from every voter (for that is the amount for which that candidate is 'chosen', but then we would have to round off in some way or another in the end in order to get integer numbers again 8 . Depending on the voting rule, other options could be used, but we cannot find a solution that works properly for every voting rule.
In conclusion, the main problem with generalizing the definition of laminar election instances from approval voting to utility voting is that in approval voting, we can say: "If it is clear how we should divide the seats of the committee fairly among the voters, we should divide them that way", but that in utility voting, the cases where it is really clear how we should divide the seats are very rare, and almost always the extra nuance that is provided by giving utilities rather than approval votes causes

[^7]the profile to not be that clear anymore.

Laminar election instances for approval based PB As shown above, the idea of laminar election instances is problematic for utility voting. What we can do however, is define laminar election instances and laminar proportionality for participatory budgeting with approval voting, i.e. releasing only the unit cost assumption, just as is done in the definition of Strong-BPJR-L in [7]. In order to do so, we take the budget limit $l$ instead of the committee size $k$, and use the cost of each candidate instead of the unit cost. We get the following definitions:

Definition 3.12 (Laminar election instances for PB ). An election instance $(P, l)$ (where $P$ is a profile, i.e. a list of approval sets $A_{i}$, and $l$ is the budget limit) is $P B$-laminar if either:

1. $P$ is unanimous and $\operatorname{cost}(C(P)) \geq l$.
2. There is a candidate $c \in C(P)$ such that $c \in A_{i}$ for all $A_{i} \in P$, the profile $P_{-c}$ is not unanimous and the instance $\left(P_{-c}, l-\operatorname{cost}(c)\right.$ is laminar (with $P_{-c}=\left(A_{1} \backslash\{c\}, \ldots, A_{n} \backslash\{c\}\right)$ ).
3. There are two laminar instances $\left(P_{1}, l_{1}\right)$ and $\left(P_{2}, l_{2}\right)$ with $C\left(P_{1}\right) \cap C\left(P_{2}\right)=\emptyset$ and $\left|P_{1}\right| \cdot l_{2}=\left|P_{2}\right| \cdot l_{1}$ such that $P=P_{1}+P_{2}$ and $l=l_{1}+l_{2}$.

Example 3.3. The instance $P$ in Table 4 associated with the cost function $\operatorname{cost}\left(c_{1}\right)=2, \operatorname{cost}\left(c_{2}\right)=3$, $\operatorname{cost}\left(c_{3}\right)=3, \operatorname{cost}\left(c_{4}\right)=4, \operatorname{cost}\left(c_{5}\right)=2$, and $\operatorname{cost}\left(c_{6}\right)=1$ and budget limit $l=10$ is laminar in PB. The instance $P_{1}$ with $v_{1}$ and $v_{2}$ and projects $c_{1}, c_{2}$, and $c_{3}$ with limit $l_{1}=6$ satisfies Definition 3.12, 1 , as does the instance $P_{2}$ with only voter $v_{3}$ and candidates $c_{4}$ and $c_{5}$, and limit $l_{2}=3$. Those two instances can be added by Definition $3.12 \mid 3$ since $\left|P_{1}\right| \cdot l_{2}=2 \cdot 3=1 \cdot 6=\left|P_{2}\right| \cdot l_{1}$. Then $c_{6}$ can be added by Definition 3.132 to get $P$ with limit $l=6+3+\operatorname{cost}\left(c_{6}\right)=10$.

Definition 3.13 (Laminar proportionality in PB). A rule $\mathcal{R}$ satisfies $P B$-laminar proportionality if for every laminar election instance $E$ with ballot profile $P$ and budget limit $l, \mathcal{R}(E)=W$ where $W$ is a laminar proportional committee, i.e.

1. If $P$ is unanimous, then $W \subseteq C(P)$ (if everyone agrees, then part of the candidates they agree on is chosen).
2. If there is a unanimously approved candidate $c$ s.t. $\left(P_{-c}, l-\operatorname{cost}(c)\right)$ is laminar, then $W=$ $W^{\prime} \cup\{c\}$ where $W^{\prime}$ is a committee which is laminar proportional for $\left(P_{-c}, l-\operatorname{cost}(c)\right)$.
3. If $P$ is the sum of laminar instances $\left(P_{1}, l_{1}\right)$ and $\left(P_{2}, l_{2}\right)$, then $W=W_{1} \cup W_{2}$ where $W_{1}$ is laminar proportional for $\left(P_{1}, l_{1}\right)$ and $W_{2}$ is laminar proportional for $\left(P_{2}, l_{2}\right)$.

The total cost of the committee $W$ designed in this way fits in the limit $l$. In (1), $W$ can just be chosen small enough. In (2), because $\operatorname{cost}(W)=\operatorname{cost}\left(W^{\prime}\right)+\operatorname{cost}(c)$ and $\operatorname{cost}(W)^{\prime} \leq l-\operatorname{cost}(c)$, we also have that $\operatorname{cost}(W) \leq l$. In (3), $\operatorname{cost}(W) \leq \operatorname{cost}\left(W_{1}\right)+\operatorname{cost}\left(W_{2}\right) \leq l_{1}+l_{2}=l$ It is trivial that in case of unit cost and budget limit $k$, these definitions are equivalent to the corresponding multi-winner election definitions.

Example 3.4. Elaborating on Example 3.3, the committee $W=\left\{c_{1}, c_{2}, c_{4}, c 6\right\}$ as indicated in green in Table 4 is laminar proportional in that instance with a budget limit of $l=10$. $\operatorname{In}\left(P_{1}, l_{1}\right),\left\{c_{1}, c_{2}\right\}$ is laminar proportional, as is $\left\{c_{4}\right\}$ in $\left(P_{2}, l_{2}\right)$. Hence, $\left\{c_{1}, c_{2}, c_{4}\right\}$ is laminar proportional in $\left(P_{1}+P_{2}, l_{1}+\right.$ $l_{2}$ ), and $\left\{c_{1}, c_{2}, c_{4}, c_{6}\right\}$ is laminar proportional in $(P, l)$.

### 3.2 Properties of rules

Now that we have defined the different rules and proportionality axioms in the PB setting, we can study which axioms are satisfied by which rules. Table 10 is an extension of Table 6, with the results for PB added. For the approval based rules PAV and Phragmén, the PB setting just releases the unit cost assumption, for Rule X, PB also releases the approval voting setting (we use utility based voting), and for SBA, the votes consist of rankings of the alternatives. Just like in Table 6, purple entries in the table indicate results from the literature, green entries indicate new results. References to either previous research or the corresponding propositions are given in the table. Explanations for

|  | SBA | PAV |  | Phragmén |  | Rule X |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | PB | MWV | PB | MWV | PB | MWV | PB |
| core | $x$ (Prop. 3.1) | $x[10]$ | $x$ (Prop. 3.16) | $x$ (Prop. 2.1) | $x$ (Prop. 3.16) | $x[10]$ | $x[11]$ |
| EJR | $x$ (Prop. 3.2 | $\checkmark$ [10] | $x[11]$ | $x[10]$ | $x$ (Prop. 3.16) | $\checkmark$ [10] | $\checkmark$ [11] |
| PJR | $x$ (Prop. 3.3) | $\checkmark$ [13] | $x$ (Prop. 3.8) | $\checkmark$ [10] | $\checkmark$ (Prop. 3.12) | $\checkmark$ [10 | $\checkmark$ (Prop. 3.14) |
| priceability | $x$ (Prop. 3.4 | $x[10]$ | $x$ (Prop. 3.16) | $\checkmark$ [10] | $\checkmark$ (Prop. 3.11) | $\checkmark$ [10] | $\checkmark$ [11] |
| laminar proportionality | $x$ (Prop. 3.5) | $x[10]$ | $x$ (Prop. 3.16) | $\checkmark$ [10] | $X$ (Prop. 3.13) | $\checkmark$ [10] | $X$ (Prop. 3.13) |
| Nash welfare | $x$ (Prop. 3.6 | $x$ (Prop. 2.4] | $x$ (Prop. 3.9) | $x$ (Prop. 2.2) | $X$ (Prop. 3.10) | x(Prop. 2.4 | X (Prop. 3.15) |
| FJR | $x$ (Prop. 3.7) | $x[11]$ | $x[11]$ | $x$ (Prop. 2.5) | x (Prop. 3.16) | $x[11]$ | $x$ [11] |

Table 10: Different rules and the properties they satisfy, purple entries in the table indicate results from the literature, green entries indicate new results. References to propositions or literature are included for each entry.
new results are given below. All propositions in this section are about the PB setting, with no unit cost assumption and utility voting, unless the rule itself requires approval voting.

We start by giving a set of propositions about SBA [5].
Proposition 3.1. There exist election instances where the output of SBA is not in the core.
Proof. The outcome of the SBA algorithm is not always in the core (Definition 3.5), as can be shown by the following example: Take an election instance with $C=\{w, t\}, \operatorname{cost}(w)=1=l, \operatorname{cost}(t)=\varepsilon>0$ for some small $\varepsilon$ and $n>22^{9}$. Assume that for one voter, $s, u_{s}(t)>u_{s}(w)$, so $s$ prefers $t$ to $w$, and that for all other voters $i, u_{i}(w)>u_{i}(t)$, to those voters prefer $w$ to $t$. When applying SBA to this situation, the majority graph will contain only one arc, from $w$ to $t$ because more than half of the voters prefers $w$ to $t$, hence the first Schwartz set will contain only $w$. In the pruning step of the algorithm, $w$ will be added to the budget because its cost fits in the limit, but after that $t$ cannot be added anymore, because the total amount of money is spent on $w$. Hence, the outcome of SBA is $W=\{w\}$. There exists an $S \subseteq N$ and $T \subseteq C$, namely $S=\{s\}$ and $T=\{t\}$ with $|S| \geq \frac{\operatorname{cost}(T)}{l} \cdot n: 1 \geq \varepsilon \cdot n$ if $n$ is not too large. However, there does not exist $i \in S$ with $u_{i}(W) \geq u_{i}(T)$, because $S$ only contains $s$ and $u_{s}(t)>u_{s}(w)$ so $u_{s}(T)>u_{s}(W)$. Hence the SBA outcome $\{w\}$ is not in the core.

The idea behind this counterexample is that if there is a very small project that only few people like, this is already enough to ruin the core. SBA however only looks at majorities $>\frac{1}{2} n$, so those small projects will be overlooked by the algorithm.

[^8]Proposition 3.2. SBA does not satisfy EJR.
Proof. With the generalisation of EJR to PB settings that [11] introduce (Definition 3.7) we can use the same counterexample as for the core to show that SBA does not satisfy EJR, even if we allow the extra condition for EJR from [11]: Take an election instance with $C=\{w, t\}, \operatorname{cost}(w)=1=l$, $\operatorname{cost}(t)=\varepsilon>0$ for some small $\varepsilon, n>2$. Assume that for one voter, $s, u_{s}(t)=1$ and $u_{s}(w)=0$, and that for all other voters $i, u_{i}(w)=1$ and $u_{i}(t)=0$. As shown above, the SBA outcome of this election instance is $W=\{w\}$. If we take $S=\{s\}$ and $T=\{t\}$, we have $|S|=1 \geq \varepsilon \cdot n=\operatorname{cost}(T) \cdot n$ (if $n$ is not too large). We have that for every $i \in S$ (which is only $s$ ), and every $c \in T$ (which is only $t$ ), $u_{i}(c) \geq \alpha(c)$, because $u_{s}(t)=1=\alpha(t)$. Hence, $S$ is $(\alpha, T)$-cohesive. However, there is no voter $i \in S$ such that $u_{i}(W) \geq \sum_{c \in T} \alpha(c)$ or for some $a \in C$ it holds that $u_{i}(W \cup\{a\})>\sum_{c \in T} \alpha(c)$, because the only voter in $S$ is $s$ and the only $a \in C$ that is not yet in $W$ is $t$, and $u_{s}(W)=0<1=\sum_{c \in T} \alpha(c)$ and $u_{s}(W \cup\{t\})=1=\sum_{c \in T} \alpha(c)$.

Proposition 3.3. SBA does not satisfy PB-PJR.
Proof. Using the example from the proof of proposition 3.2 where we showed that SBA fails EJR, it is easy to see that SBA also fails PJR. Because the group $S$ in the example exists of only one voter $s$, the prerequisite that there should be one voter in the group whose utility is high enough is equal to the prerequisite that the maximum utility over the voters in the group is high enough. Therefore, SBA fails PJR.

Proposition 3.4. SBA does not satisfy priceability.
Proof. According to [10], all priceable committees satisfy PJR, so if SBA fails PJR (Proposition 3.3), it fails priceability as well.

## Proposition 3.5. SBA does not satisfy laminar proportionality.

Proof. Since laminar election instances per definition use approval profiles, SBA is not a proper algorithm for such election instances. We could however see the approval ballots as very rough rankings: obviously a voter ranks all her approved ballots higher than her non-approved ballots. The outcome of the SBA depends on the majority graph of the candidates. However, in a laminar election instance, it may very well be the case that there is no majority for any project, because the parties are not large enough. Take Example 2.1. There are $n=6$ voters, $m=15$ candidates and $k=12$ seats, the approval votes are as shown in Table 2, In this example, no project has a majority of votes, projects $c_{1}, c_{2}$, and $c_{3}$ have 3 votes which is exactly half of the votes, and the others all have one vote. Hence, the majority graph of this profile will have no arcs at all, and the Schwartz-set of it will contain all candidates. This means that in the Pruning step of the SBA procedure, the selected outcome will completely depend on the previous outcome (as tie-breaker), and by no means need to be laminar proportional.

Proposition 3.6. SBA does not maximise the Nash product.
Proof. As can be shown with a counterexample, SBA does not maximise the Nash product. Take an election instance with three voters and three projects, where each project has unit cost and the budget limit is one so there is one project affordable. If the utilities are as shown in Table 11, the SBA outcome is $\{c\}$, while the outcome that would maximise the Nash welfare is $\{b\}$.

Proposition 3.7. SBA does not satisfy FJR.

| project $a$ | 1 | 1 | 50 |
| :--- | :--- | :--- | :--- |
| project $b$ | 25 | 50 | 1 |
| project $c$ | 49 | 1 | 25 |
|  | voter $v_{1}$ | voter $v_{2}$ | voter $v_{3}$ |

Table 11: Counterexample that shows that SBA does not maximise the Nash product.

Proof. As shown in proposition 3.2, the SBA outcome does not necessarily satisfy EJR. We know from [11] that FJR implies EJR, so the SBA outcome does also not necessarily satisfy FJR.

The next set of propositions is about proportional approval voting (PAV), which maximises the sum of the $x$-th harmonic numbers, where $x$ is the number of elected projects that a voter approves.

Proposition 3.8. In PB, PAV does not satisfy PB-PJR.
Proof. As [11] show by the example of Onetown, PAV does guarantee proportional representation, so in specific it does not satisfy PJR. Because PAV uses approval voting, we should use Definition 3.15 for PJR here. The group of voters that live in Leftside are $T$-cohesive for $T=\left\{L_{1}, L_{2}, L_{3}\right\}$ : they can with their share of the money afford all projects in $T$ and do all approve all projects in $T$. However, the amount of projects in the committee $W$ that PAV returns that at least one of the voters in Leftside $(S)$ approves of is $\left|W \cap_{i \in S} A_{i}\right|=2$, which is less than the number of projects in $T$.

Proposition 3.9. In PB, PAV does not maximise Nash welfare.
Proof. PAV chooses $W$ that maximises the score

$$
\operatorname{PAV}-\operatorname{score}(W)=\sum_{i \in N}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left|W \cap A_{i}\right|}\right) .
$$

In order to satisfy Nash welfare, $W$ should also maximise the Nash product

$$
\text { Nash-product }=\prod_{i \in N}\left(1+\sum_{a \in C} u_{i}(a)\right) .
$$

Note that at the maximum of this product, the sum of the logarithm of the utilities will also be maximal. Hence, instead of maximising this product, we can also maximise the sum of its logarithm:

$$
\text { Nash-product-log }=\sum_{i \in N} \log \left(1+\sum_{a \in C} u_{i}(a)\right)
$$

We will use this logarithm instead of the Nash product itself because it is more comparable to the PAV-score. As mentioned in the preliminaries, we define the utilities in approval voting to be 1 for a project if the voter if the project is approved by that voter, and 0 otherwise. Then the Nash-product-log boils down to

$$
\text { Nash-product- } \log (W)=\sum_{i \in N} \log \left(1+\left|W \cap A_{i}\right|\right) .
$$

As we will show by giving a counterexample, these two sums do not always have the same maximum. Take a setting with budget limit $l=2$, and 4 projects: $C=\{a, b, c, d\}$, with $\operatorname{cost}(a)=0$,
$\operatorname{cost}(b)=\operatorname{cost}(c)=1$, and $\operatorname{cost}(d)=2$, so the maximal feasible outcomes are $W_{1}=\{a, b, c\}$ and $W_{2}=$ $\{a, d\}$. Suppose $N=19,7$ voters approve projects $a, b$, and $c$, and the other 12 voters approve projects $a$ and $d$. The PAV-score of $W_{1}$ is PAV-score $\left(W_{1}\right)=\sum_{i \in N}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left|W \cap A_{i}\right|}\right)=7 \cdot(1+$ $\left.\frac{1}{2}+\frac{1}{3}\right)+12 \cdot 1=36 \frac{5}{6}$ and the PAV-score of $W_{2}$ is PAV-score $\left(W_{2}\right)=7 \cdot 1+12 \cdot\left(1+\frac{1}{2}\right)=25$, so PAV will select $W_{1}$. The Nash product (logarithmic version) however for $W_{1}$ is Nash-product- $\log \left(W_{1}\right)=$ $\sum_{i \in N} \log \left(1+\left|W_{1} \cap A_{i}\right|\right)=7 \cdot \log (1+3)+12 \cdot \log (1+1) \approx 7.827$, and the Nash product of $W_{2}$ is Nash-product- $\log \left(W_{2}\right)=7 \cdot \log (1+1)+12 \cdot \log (1+2) \approx 7.833$. Hence, $W_{1}$, although chosen by PAV, does not maximise the Nash product. This example shows that PAV does not satisfy Nash welfare.

The following propositions are about Phragmén's rule [17].
Proposition 3.10. In PB, Phragmén's rule does not maximise Nash welfare.
Proof. We can use the Onetown example from [11] to show that Phragmén's rule does not maximise Nash welfare. As is shown in the abovementioned paper, Phragmén's rule will implement all three $L$-projects in Onetown, but as shown in Proposition 3.17, the Nash product is maximised with two $L$ projects and the $R$-project. Hence, Phragmén's rule does not always maximise the Nash product.

Proposition 3.11. In PB, Phragmén's rule is priceable.
Proof. As [11] mention, there is no obvious generalization of Phragmén's rule to non-approval voting. For non-unit costs, it is however directly applicable. We can easily see that Phragmén's rule for nonunit costs is still priceable (with the PB definition of priceability). We can construct a price system as follows: If the rule stopped at time $t$, let the initial budget of every voter be $t$, so the total initial budget is $b=t \cdot n$. Then every time that a candidate is elected by Phragmén's rule and added to the winning committee $W$, add the amount of money that a voter $i$ pays for it to $p_{i}(c)$. Clearly, in the price system constructed in this way, the cost for every candidate $c \in W$ is paid, every voter has spend at most $\frac{b}{n}$ units of money and pays only for voters she approves, and for each non-elected candidate $c \notin W$, its supporters do not have enough money to buy it, because if they would have, $c$ would have been added to $W$ already.

In [11] it is mentioned that it is "easy to see" that Phragmén gives proportional outcomes on instances with a district structure. However, the question remains whether this proportionality also holds for other instances. We will prove that the output of Phragmén's sequential rule will always satisfy PJR, even in the PB setting. Because Phragmén's rule uses approval voting, we should use the definition of PJR as in Section 3.3.3 here (Definition 3.15). In order to satisfy PJR for approval based participatory budgeting, the outcome $W$ of the rule should satisfy $\left|W \cap \cup_{i \in S} A_{i}\right| \geq|T|$ for every $T$-cohesive group of voters $S$.

Proposition 3.12. In PB, Phragmén's rule satisfies PJR.
Proof. Assume for a contradiction that there is a group of voters $S \subseteq N$ and a group of projects $T \subseteq C$ such that $S$ is $T$-cohesive, i.e. $|S| \geq \operatorname{cost}(T) \cdot n$ (the budget limit is 1 ) and $T \subseteq \cap_{i \in S} A_{i}$, and that for this $S$ and $T$, the outcome $W$ of Phragmén's rule does not contain enough projects that voters from $S$ like: $\left|W \cap \cup_{i \in s} A_{i}\right|<|T|$. Intuitively, voters in $S$ will get their own share of the money, and as they are $T$-cohesive, they should be able to buy enough projects from $T$. They might also pay for projects other than in $T$, but that will only increase $\left|W \cap \cup_{i \in s} A_{i}\right|$, because they can only pay for projects that they approve of. However, we are looking at the number of projects in the outcome rather than at their cost, so it might be possible that in $T$ there are many small projects, and that they gave their
money to larger projects not in $T$. But, if we follow Phragmén's rule, the amount they pay for any of those other projects should be smaller than the cost of something in $T$, because otherwise they would already have been able to pay for the project in $T$. Let us rewrite this intuition as a proof.
Let $t$ be the moment in time when the rule stops. At this moment, a project is reached (its supporters have enough virtual money to buy it) that would overshoot the budget limit if it would be implemented. We call this project $c$, so we have $\operatorname{cost}(W)+\operatorname{cost}(c)>1$, but $\operatorname{cost}(W) \leq 1$. Let $x$ be the total amount of virtual money that all voters together have earned so far, so

$$
\begin{equation*}
t \cdot n=x=\operatorname{cost}(W)+\operatorname{cost}(c)+y, \tag{3}
\end{equation*}
$$

where $y \geq 0$ is the money that voters that do not support $c$ have earned in the mean time. Because $\left|W \cap \cup_{i \in s} A_{i}\right|<|T|$, there must be some project in $T$ that is not in $W$ (because all projects in $T$ are supported by all voters in $S$ ). The voters in $S$ together have earned $\frac{x}{n} \cdot|S|$ units of money in total, and because $S$ is $T$-cohesive, $|S| \geq \operatorname{cost}(T) \cdot n$, so

$$
\begin{equation*}
\frac{x}{n} \cdot|S| \geq \operatorname{cost}(T) \cdot n \cdot \frac{x}{n}=\operatorname{cost}(T) \cdot x . \tag{4}
\end{equation*}
$$

From 3 and from the fact that $\operatorname{cost}(W)+\operatorname{cost}(c)>1$, it follows that $x>1+y$, and because $y \geq 0$, $x>1$. From 4, it now follows that the voters in $S$ together have earned enough money at time $t$ to buy all projects from $T$, but as we noted, they have not done so. Hence, either they have also paid for projects other than in $T$, or, if they did only spend their money on projects in $T, c$ must be a project from $T$, i.e. they do have the virtual money to buy $T$ but it would overshoot the total budget limit. In the first case, for every project not in $T$ that a voter or group of voters from $S$ pays for, $\left|W \cap \cup_{i \in s} A_{i}\right|$ grows by one (since they can only pay for projects they approve). In order to keep $\left|W \cap \cup_{i \in s} A_{i}\right|<|T|$, the mean amount of money they have paid at time $t$ for such a project must be greater than the mean cost of projects in $T$, otherwise the number of projects that they pay for (that they approve of and are elected) is greater than the number of projects in $T$. However, for every individual project not in $T$ that the voters from $S$ pay for, they should (as a group) pay less than the cost of any project from $T$ that is not yet elected, because otherwise they would rather (or actually: earlier) buy a cheaper project from $T$, since they all approve all projects in $T$. This is a contradiction. Hence, the voters from $S$ only spent their money on projects from $T$, and $c \in T$. Let us assume that $c$ is the last project from $T$ that is not yet elected ${ }^{10}$. Because the rule stops exactly when $c$ can be paid by its supporters, we know that at that point in time, the voters in $S$ have earned exactly $\operatorname{cost}(T)$ units of money, so $t \cdot|S|=\operatorname{cost}(T)$. Hence, the total amount of money earned at time $t$ is $x=t \cdot n=\operatorname{cost}(T) \cdot \frac{n}{|S|}$. Because $S$ is $T$-cohesive, we know that $\operatorname{cost}(T) \leq \frac{n}{|S|}$, so

$$
\begin{equation*}
x=\operatorname{cost}(T) \cdot \frac{n}{|S|} \leq \frac{|S|}{n} \frac{n}{|S|}=1 . \tag{5}
\end{equation*}
$$

However, we also had that $x>1+y>1$. This is a contradiction with (5), which proves our initial statement that $W$ satisfies PJR.

Phragmén's rule does in the PB model not satisfy laminar proportionality according to Definition 3.13. Laminar proportionality still requires any affordable unanimously approved candidates to be elected, but Phragmén's rule does not necessarily elect those, if their cost is high enough compared to the other candidates. The same holds for Rule X.

[^9]Proposition 3.13. In PB, Pragmén's rule and Rule $X$ do not satisfy laminar proportionality.
Proof. As a concrete counterexample, we can use Table 7 again: Suppose the approval votes are as shown in Table 7, candidates $c_{1}, \ldots, c_{5}$ have a cost of 0.1 and candidate $c_{6}$ has a cost of 0.7 , and the total budget is 1 . The profile is still a laminar election instance according to Definition 3.12, In this situation, Phragmén's rule will return $\left\{c_{1}, \ldots, c_{5}\right\}$ : at time $t=0.05, v_{1}$ and $v_{2}$ can buy $c_{1}$, then at $t=0.1$, they can buy $c_{2}$ and $v_{3}$ can buy $c_{4}$, at $t=0.15, v_{1}$ and $v_{2}$ can buy $c_{3}$ and at $t=0.2, v_{3}$ can buy $c_{5}$. The remaining amount of budget is 0.5 , so $c_{6}$ is not affordable anymore, and $W=\left\{c_{1}, \ldots, c_{5}\right\}$ is returned. The same committee will be returned for Rule X: each voter starts with a budget of $\frac{1}{3}$. $c_{1}, c_{2}$, and $c_{3}$ are $\rho$-affordable for $\rho=0.05, c_{4}$ and $c_{5}$ are $\rho$-affordable for $\rho=0.1$ (whereas $c_{6}$ would only be $\rho$-affordable for $\rho \geq \frac{0.7}{3}$ in the beginning), the remaining budget is not enough to buy $c_{6}$, so $W$ will be returned. A laminar proportional committee, however, would consist of $c_{6}$, two of $\left\{c_{1}, c_{2}, c_{3}\right\}$ and one of $\left\{c_{4}, c_{5}\right\}$.

Proposition 3.14. In PB, Rule $X$ satisfies PB-PJR.
Proof. In [11] Rule X was shown to satisfy EJR in the PB model. In Theorem 3.3, we will show that EJR implies PB-PJR in the PB model. Hence Rule X also satisfies PB-PJR.
Proposition 3.15. In PB, Rule $X$ does not maximise Nash welfare.
Proof. Let us consider Rule X in Onetown: Each person gets 1 unit of currency. W is empty. An L-project is $\rho$-affordable for $\rho=\frac{1}{3}$ and is chosen. Then the people from Leftside all have $\frac{2}{3}$ of a unit of currency left over, and the people from Rightside still have 1 unit of currency. In the next step, again an L-project is $\rho$-affordable for $\rho=\frac{1}{3}$ and is chosen, and in the step after that again, so just as with Phragmén's rule, all three L-projects will be chosen. Hence, Rule X does not always maximise the Nash product.

Proposition 3.16. In PB, the PAV outcome is not necessarily in the core, and it does not necessarily satisfy priceability or laminar proportionality, and the outcome of Phragmén's rule is not necessarily in the core and does not necessarily satisfy EJR or FJR.

Proof. These negative results can be derived in two steps: the corresponding result in MWV is negative and the PB versions of the axiom and rule are proper generalisations from MWV to PB. The first is shown either in the literature or in Chapter 2 of this paper, see Table 10 for the references, the latter is shown in Section 3.1. As an example, in Proposition 2.1 we saw that in MWV there are election instances where Phragmén's rule yields a committee that is not in the core. Since these instances are specific instances of PB, Definition 3.5 is a generalisation of Definition 2.4, and Definition 3.2 is a generalisation of Definition 3.2 (i.e. for MWV instances the definitions are equivalent), in these PB instances, the outcome of Phragmén's rule is not in the core.

### 3.3 Relations between axioms

From Table 10, some questions arise about relations between different axioms. We see that SBA and PAV often fail or satisfy the same axioms, and Phragmén's rule and Rule $X$ also often fail or satisfy the same axioms but different from SBA and PAV. These observations lead to the question which (implication/exclusion) relations exist between the different axioms. Some relations are already clear from the literature, but not as many as in the MWV setting. In PB, we have only found (regarding the axioms we study) that the core implies FJR, which implies EJR [11].
In some of the following sections we will study new relations between axioms. A summary of the findings is given here, with references to the corresponding theorems, corollaries, and propositions:

- In PB, priceability does not imply PB-PJR: Theorem 3.2,
- In PB, Priceability does not imply EJR or the core: Corollary 3.2.1
- In PB, EJR implies PJR: Theorem 3.3 .
- Laminar proportionality implies priceability in laminar election instances in approval based PB: Theorem 3.4
- In PB, laminar proportional committees are not necessarily in the core and do not necessarily satisfy EJR or PJR: Theorem 3.5 and Theorem 3.6 .
- In PB, Nash welfare does not imply EJR or the core: Propositions 3.17 and 3.18 .


### 3.3.1 The relation between priceability, PJR, and EJR in PB

As shown in Section 2.3.2, PJR does not imply priceability in multi-winner approval voting. Because this is a specific instance of participatory budgeting, PJR does also not imply priceability in participatory budgeting. In multi-winner approval voting, every priceable committee satisfies PJR, as is shown by [10] (Proposition 1 in their paper). This raises the question whether this relation is also present in the participatory budgeting setting. We will show that this is not the case.

## Theorem 3.2. In $P B$, priceability does not imply $P B-P J R$.

Proof. To get an intuition: the PB version of PJR is based on the utility of the voters being higher than some threshold $\alpha(c)$, while the PB version of priceability only discriminates between utilities of 0 and utilities above zero. Which value above zero a utility has does not make any difference in the property of being priceable. Therefore, it is not the case that in participatory budgeting all priceable committees also satisfy PJR. We show this by giving a counterexample:
Take a PB election instance $E$ with $N=\left\{s_{1}, s_{2}, v_{1}, v_{2}, v_{3}\right\}, C=\left\{t_{1}, t_{2}, c_{1}, c_{2}, c_{3}\right\}$, the cost of all projects is $0.2, \alpha\left(t_{1}\right)=\alpha\left(t_{2}\right)=0.4$, voters $s_{1}$ and $s_{2}$ have some utility only for the $t$-projects: $u_{s_{1}}\left(t_{1}\right)=u_{s_{1}}\left(t_{2}\right)=$ $0.6, u_{s_{1}}(c)=0$ for $c \in\left\{c_{1}, c_{2}, c_{3}\right\}, u_{s_{2}}=u_{s_{1}}$, and voters $v_{1}, v_{2}$, and $v_{3}$ only have utility for the $c$ projects: for $v \in\left\{v_{1}, v_{2}, v_{3}\right\}, u_{v}\left(t_{1}\right)=u_{v}\left(t_{2}\right)=0$ and $u_{v}(c)>0$ for $c \in\left\{c_{1}, c_{2}, c_{3}\right\}$. Furthermore, define a pricesystem ps with $b=1$ and $p_{s}(c)=0$ for $c \in\left\{c_{1}, c_{2}, c_{3}, t_{1}\right\}$, and $p_{s}\left(t_{2}\right)=0.1$ for $s \in\left\{s_{1}, s_{2}\right\}$, and with $p_{v}(t)=0$ and $p_{v}(c)=\frac{0.2}{3}$ for $v \in\left\{v_{1}, v_{2}, v_{3}\right\}, t \in\left\{t_{1}, t_{2}\right\}$, and $c \in\left\{c_{1}, c_{2}, c_{3}\right\}$.
Now, let committee $W=\left\{t_{2}, c_{1}, c_{2}, c_{3}\right\}$ be the outcome of some election rule. We have:
(1) A voter can only pay for candidates she gets at least some utility form: if $u_{i}(c)=0$, then $p_{i}(c)=0$ for each $i \in N$ and $c \in C$;
(2) Each voter can spend the same budget of $\frac{b}{n}$ units of money: $\sum_{c \in C} p_{i}(c) \leq \frac{1}{n}$ for each $i \in N$;
(3) for each $c \in W$, the sum of the payments for $c$ equals its price: $\sum_{i \in N} p_{i}(c)=\operatorname{cost}(c)$;
(4) no candidate outside of the committee gets any payment: for all $c \notin W, \sum_{i \in N} p_{i}(c)=0$;
(5) there exists no unelected candidate whose supporters in total have a remaining unspent budget of more than its cost: for all $c \notin W, \sum_{i \in N \text { for which } u_{i}(c)>0}\left(b-\sum_{c^{\prime} \in W} p_{i}\left(c^{\prime}\right)\right) \leq \operatorname{cost}(c)$.

Hence, $W$ is a priceable committee. However, if we take $S=\left\{s_{1}, s_{2}\right\}$ and $T=\left\{t_{1}, t_{2}\right\}$, we have $|S|=2=\operatorname{cost}(T) \cdot n$ and $u_{i}(c) \geq \alpha(c)$ for all $i \in S, c \in T$, so $S$ is $(\alpha, T)$-cohesive. Nevertheless, $\sum_{c \in W}\left(\max _{i \in S} u_{i}(c)\right)=0.6<0.8=\sum_{c \in T} \alpha(c)$, sW does not satisfy PB-PJR. Therefore, this counterexample shows that it is not the case that in PB (like in MWV), all priceable committees satisfy PJR.

From this result, it follows that priceability also neither implies EJR nor the core in the PB setting. For if that would have been the case, pricability would also imply PB-PJR, since the core implies EJR, and EJR implies PB-PJR (Theorem 3.3).

Corollary 3.2.1. In PB, pricability does not imply EJR or the core.
We already mentioned the following theorem a few times: Just like in multi-winner elections EJR implies PJR, in the PB setting EJR implies PB-PJR.

Theorem 3.3. In PB, EJR implies PB-PJR.
Proof. Suppose that rule $\mathcal{R}$ satisfies EJR and take an ( $\alpha, T$ )-cohesive group of voters $S$ for some $\alpha: T \rightarrow[0,1], T \subseteq C$. Because $\mathcal{R}$ satisfies EJR, there is a voter $i \in S$ such that $u_{i}(\mathcal{R}(E)) \geq \sum_{c \in T} \alpha(c)$ ${ }^{11}$ For this voter $i, \sum_{c \in \mathcal{R}(E)}\left(u_{i}(c)\right) \geq \sum_{c \in T} \alpha(c)$. From this, it is obvious that $\sum_{c \in \mathcal{R}(E)}\left(\max _{i \in S} u_{i}(c)\right) \geq$ $\sum_{c \in T} \alpha(c)$, so $\mathcal{R}$ satisfies PB-PJR.

### 3.3.2 Laminar proportionality and priceability of laminar election instances in PB

In Section 2.3.1, we saw that in multi-winner approval voting, priceability did not imply laminar proportionality, but that laminar proportionality did imply priceability on laminar election instances. Now that we have defined laminar proportionality in the participatory budgeting setting, this raises the question whether this implication still holds without the unit cost assumption.
Because laminar proportionality, and laminar election instances in general, are not defined for utility voting, we look at a PB setting with approval voting. We use the definition of priceability from [11] (Definition 3.9), and we assume that a voter has a utility greater than zero for projects he approves and a utility of zero for projects he does not approve.

## Theorem 3.4. In laminar election instances, laminar proportionality implies priceability.

Proof. We construct an inductive proof on the structure of laminar election instances, very similar to the proof for the unit cost case, to prove that for every committee $W$ that is laminar proportional for a laminar election instance $(P, l)$, where $P$ is the list of approval sets of the voters and $l$ is the budget limit, there exists a price system $\mathbf{p s}=\left(b,\left\langle p_{i}\right\rangle_{i \in N}\right)$ where $b=\operatorname{cost}(W)$.
Basis: If $P$ is unanimous with $\operatorname{cost}(C(P)) \geq l$ and $W$ is laminar proportional for $(P, l)($ with $\operatorname{cost}(W) \leq$ $l$ ), then $W \subseteq C(P)$, so the voters can just divide their budget over the candidates in $W$. If we set the initial budget to be $b=\operatorname{cost}(W)$, every voter can spend $\frac{b}{n}=\frac{\operatorname{cost}(W)}{n}$. We can now let every voter spend $\frac{\operatorname{cost}(c)}{n}$ on every candidate $c \in W$, so every candidate $c \in W$ gets exactly $\operatorname{cost}(c)$. Then every voter spends in total $\sum_{c \in W} \frac{\operatorname{cost}(c)}{n}=\frac{\operatorname{cost}(W)}{n}$, so does not have anything left to spend on other candidates.
Inductive Hypothesis: Suppose that $\left(P^{\prime}, l^{\prime}\right),\left(P_{1}, l_{1}\right)$, and $\left(P_{2}, l_{2}\right)$ are laminar election instances, committees $W^{\prime}, W_{1}$, and $W_{2}$ are laminar proportional for respectively $\left(P^{\prime}, l^{\prime}\right),\left(P_{1}, l_{1}\right)$, and $\left(P_{2}, l_{2}\right)$, and

[^10]suppose that for $W^{\prime}$ there exists a price system $\mathbf{~ p s}{ }^{\prime}$ with initial budget $b^{\prime}=\operatorname{cost}\left(W^{\prime}\right)$, for $W_{1}$ there exists a price system $\mathbf{p s} 1$ with initial budget $b_{1}=\operatorname{cost}\left(W_{1}\right)$ and for $W_{2}$ there exists a price system $\mathbf{p s}_{2}$ with initial budget $b_{2}=\operatorname{cost}\left(W_{2}\right)$. Furthermore, suppose that $P^{\prime}$ is not unanimous, that $C\left(P_{1}\right) \cap C\left(P_{2}\right)=\emptyset$ and that $\left|P_{1}\right| \cdot l_{2}=\left|P_{2}\right| \cdot l_{1}$.

## Inductive step:

- There is a unanimously approved candidate $c$ such that $P=P_{+c}^{\prime}$, where $P_{+c}^{\prime}=\left(A_{1} \cup\{c\}, \ldots A_{n} \cup\right.$ $\{c\})$ (case 2 of Definition 3.12). Suppose that $W$ is laminar proportional for $\left(P, l^{\prime}+\operatorname{cost}(c)\right)$, then $W=W^{\prime} \cup\{c\}$. By the inductive hypothesis, there exists a price system $\mathbf{p s}^{\prime}$ for $W^{\prime}$ with initial budget $b^{\prime}=\operatorname{cost}\left(W^{\prime}\right)$. Because $c$ is unanimously approved, in theory all voters can pay for $c$. We know that in $\mathbf{p s}^{\prime}$, there was no candidate that was not in $W^{\prime}$ for which its supporters together had enough (more than its cost) unspent budget. If we would give every voter $\frac{\operatorname{cost}(c)}{n}$ more budget, which we let them spend entirely on $c, c$ will get enough money and no voter will have more unspent budget than they had before. Also, the initial budget of every voter is now $\frac{b^{\prime}}{n}+\frac{\operatorname{cost}(c)}{n}=\frac{\operatorname{cost}\left(W^{\prime}\right)+\operatorname{cost}(c)}{n}=\frac{\operatorname{cost}(W)}{n}=\frac{b}{n}$ units of money, and the initial budget is $b=\operatorname{cost}(W)$ and all the individual payment functions stay the same. Because for every candidate $c$ in $W^{\prime}$ the sum of the individual payments was equal to $\operatorname{cost}(c)$, this is also the case for every candidate in $W$.
Formally we define the price system ps for the instance $(P, l)$ as follows: $\mathbf{p s}=\left(b,\left\langle p_{i}\right\rangle_{i \in N}\right)$ with $b=\operatorname{cost}(W)$ and $p_{i}: C \rightarrow[0,1]$ such that $p_{i}(c)=\frac{\operatorname{cost}(c)}{n}$ and $p_{i}(d)=p_{i}^{\prime}(d)$ for all other candidates $d \in C(P)$, where $p_{i}^{\prime}$ is the payment function of voter $i$ in the price system $\mathbf{p s}^{\prime}$.
To show that this is indeed a valid price system that supports $W$, we look at the five points of the definition of a price system that supports a committee:

1. Voters only pay for candidates they get at least some utility from because they did so in $p s^{\prime}$, and the only candidate which they now pay for that they did not pay for before is $c$, which is unanimously approved, so has some utility for all $i \in N$.
2. All voters $i \in N$ have an initial budget of $\frac{b}{n}$ :

$$
\begin{align*}
\sum_{d \in C} p_{i}(d) & =\sum_{d \in C} p_{i}^{\prime}(d)+\frac{\operatorname{cost}(c)}{n}  \tag{6}\\
& \leq \frac{b^{\prime}}{n}+\frac{\operatorname{cost}(c)}{n}  \tag{7}\\
& =\frac{b}{n} \tag{8}
\end{align*}
$$

where (7) follows from the inductive hypothesis: because $\mathbf{p s}^{\prime}$ is a price system that supports $W^{\prime}$, the sum of the payments of voter $i$ for the items in $W^{\prime}$ is smaller than or equal to $\frac{b}{n}$. Equation 8 holds because the new budget $b$ is defined as $b=\operatorname{cost}(W)=$ $\operatorname{cost}\left(W^{\prime} \cap\{c\}\right)=b^{\prime}+\operatorname{cost}(c)$.
3. For each elected candidate $d \in W$, if $d \neq c$ the sum of the payments is

$$
\begin{align*}
\sum_{i \in N} p_{i}(d) & =\sum_{i \in N} p_{i}^{\prime}(d)  \tag{9}\\
& =\operatorname{cost}(d) . \tag{10}
\end{align*}
$$

This follows from the inductive hypothesis: because $\mathbf{p s}^{\prime}$ is a price system that supports $W^{\prime}$, the sum of the payments of all voters for $d$ equals its cost. For $c, \sum_{i \in N} p_{i}(c)=n \cdot \frac{\operatorname{cost}(c)}{n}=$ $\operatorname{cost}(c)$.
4. For any non-elected candidate $d \notin W, \sum_{i \in N} p_{i}(d)=\sum_{i \in N} p_{i}^{\prime}(d)=0$.
5. For any candidate outside of the committee $d \notin W$, its supporters do not have a remaining unspent budget of more than $\operatorname{cost}(c)$ :

$$
\begin{align*}
& \sum_{i \in N \text { for which } u_{i}(d)>0}\left(b-\sum_{e \in W=W^{\prime} \cup\{c\}} p_{i}(e)\right)  \tag{11}\\
= & \sum_{i \in N: u_{i}(d)>0}\left(b-p_{i}(c)-\sum_{e \in W^{\prime}} p_{i}(e)\right)  \tag{12}\\
= & \sum_{i \in N: u_{i}(d)>0}\left(\frac{\operatorname{cost}(W)}{n}-\frac{\operatorname{cost}(c)}{n}-\sum_{e \in W^{\prime}} p_{i}^{\prime}(e)\right)  \tag{13}\\
= & \sum_{i \in N: u_{i}(d)>0}\left(\frac{\operatorname{cost}\left(W^{\prime}\right)}{n}-\sum_{e \in W^{\prime}} p_{i}^{\prime}(e)\right)  \tag{14}\\
= & \sum_{i \in N: u_{i}(d)>0}\left(b^{\prime}-\sum_{e \in W^{\prime}} p_{i}^{\prime}(e)\right)  \tag{15}\\
\leq & \operatorname{cost}(d) \tag{16}
\end{align*}
$$

so there is no unelected candidate whose supporters in total have a remaining unspent budget of more than its cost. Equation (15) follows from the inductive hypothesis because $\mathbf{p s}^{\prime}$ is a price system that supports $W^{\prime}$, so satisfies Definition 3.9.5, the other equations are just rewritings of the formula.

Hence, $\mathbf{p s}$ is indeed a valid price system that supports committee $W$.

- $P=P_{1}+P_{2}$ and $l=l_{1}+l_{2}$ (case 3 of Definition 3.12). Take $W=W_{1} \cup W_{2}$, which is by definition laminar proportional for $(P, l)$. We have to show that $W$ is priceable for this election instance. Note that there are no overlapping candidates between $P_{1}$ and $P_{2}$, there is no voter in $P_{1}$ that gets any utility from a candidate from $C\left(P_{2}\right)$, and no voter in $P_{2}$ that gets any utility from a candidate from $C\left(P_{1}\right)$. By the inductive hypothesis, there exists a price system $\mathbf{p s}_{\mathbf{1}}=\left(b_{1},\left\{p_{1, i}\right\}_{i \in N}\right)$ for $W_{1}$ with initial budget $b_{1}=\operatorname{cost}\left(W_{1}\right)$, and for $W_{2}$ there exists a price system $\mathbf{p s}_{\mathbf{2}}=\left(b_{2},\left\{p_{2, i}\right\}_{i \in N}\right)$ with $b_{2}=\operatorname{cost}\left(W_{2}\right)$. Also by the inductive hypothesis, $\left|P_{1}\right| \cdot l_{2}=\left|P_{2}\right| \cdot l_{1}$.
We can now define a price system ps that supports $W$ as follows: $\mathbf{p s}=\left(b,\left\langle p_{i}\right\rangle_{i \in N}\right)$ with $b=$ $\operatorname{cost}(W)=b_{1}+b_{2}$, and for all voters $i \in N$,

$$
p_{i}(c)=p_{1, i}^{\prime}(c)+p_{2, i}^{\prime}(c),
$$

where $p_{1, i}^{\prime}$ and $p_{2, i}^{\prime}$ are extended versions of respectively $p_{1, i}$ and $p_{2, i}$ that yield zero for the candididates that those are not defined for:

$$
\begin{aligned}
& p_{1, i}^{\prime}(c)= \begin{cases}p_{1, i}(c) & \text { if } c \in C\left(P_{1}\right) \text { and } i \in P_{1} ; \\
0 & \text { if } c \in C\left(P_{2}\right) \text { or } i \in P_{2},\end{cases} \\
& p_{2, i}^{\prime}(c)= \begin{cases}0 & \text { if } c \in C\left(P_{1}\right) \text { or } i \in P_{1} ; \\
p_{2, i}(c) & \text { if } c \in C\left(P_{2}\right) \text { and } i \in P_{2} .\end{cases}
\end{aligned}
$$

Again, we show that this is a valid price system that supports $W$ by looking at the five points of the definition:

1. We know that $\mathbf{p s}_{\mathbf{1}}$ is a valid price system that supports $W_{1}$, so for voters $i \in P_{1}$ and candidates $c \in W_{1}$, if $p_{1, i}(c)>0$, then $u_{i}(c)>0$, so $c \in A_{i}$. Analogously, for voters $i \in P_{2}$ and $c \in W_{2}$ if $p_{2, i}(c)>0$, then $u_{i}(c)>0$, so $c \in A_{i}$. Suppose $p_{i}(c)>0$. If $c \in C\left(P_{1}\right)$, then $p_{i}(c)=p_{1, i}(c)$, so $i \in P_{1}$ because there is no voter in $P_{2}$ that approves a candidate from $C\left(P_{1}\right)$ and vice versa. Hence, for $c \in C\left(P_{1}\right)$, if $p_{i}(c)>0$, then $p_{1, i}(c)>0$ and then $c \in A_{i}$. Similarly, we can argue that for $c \in C\left(P_{2}\right)$, if $p_{i}(c)>0$, then $p_{2, i}(c)>0$ and then $c \in A_{i}$. Because $P=P_{1}+P_{2}, C(P)=C\left(P_{1}\right) \cup C\left(P_{2}\right)$, so for all $c \in C(P)$, if $p_{i}(c)>0$ then $c \in A_{i}$, so $u_{i}(c)>0$.
2. $\sum_{c \in C(P)} p_{i}(c)=\sum_{c \in C(P)} p_{1, i}^{\prime}(c)+p_{2, i}^{\prime}(c)$. We already saw that voters from $P_{1}$ do not pay for candidates from $C\left(P_{2}\right)$ and vice versa. Hence, if $i \in P_{1}$, then $\sum_{c \in C(P)} p_{i}(c)=$ $\sum_{c \in C(P)} p_{1, i}^{\prime}(c) \leq \frac{b_{1}}{n}$ by the inductive hypothesis (because $\mathbf{p s}_{1}$ is a valid price system with initial budget $b_{1}$ ). Furthermore, we have $\frac{b_{1}}{n} \leq \frac{b_{1}+b_{2}}{n}=\frac{b}{n}$, so $\sum_{c \in C(P)} p_{i}(c) \leq \frac{b}{n}$. If $i \in P_{2}$, then $\sum_{c \in C(P)} p_{i}(c)=\sum_{c \in C(P)} p_{2, i}^{\prime}(c) \leq \frac{b}{n}$, in the same way.
3. For each elected candidate $c \in W$, the sum of its payments is $\sum_{i \in N} p_{i}(c)=\sum_{i \in N}\left(p_{1, i}^{\prime}(c)+\right.$ $p_{2, i}^{\prime}(c)$ ). For $c \in C\left(P_{x}\right)$ (with $x \in\{1,2\}$ ) this is $\sum_{i \in N} p_{x, i}^{\prime}(c)=\operatorname{cost}(c)$. This follows from the inductive hypothesis that $\mathbf{p s}_{1}$ and $\mathbf{p s}_{2}$ are price systems that support $W_{1}$ and $W_{2}$, so the sum of payments of all voters in these systems for an elected candidate is equal to the cost of the candidate.
4. Because $W=W_{1} \cup W_{2}$, any candidate that is not elected in the new committee, $c \notin W$, was not elected in $W_{1}$ or $W_{2}$, so did not get any payment there: for $c \in C\left(P_{x}\right), \sum_{i \in N} p_{x, i}(c)=0$. Hence, it also does not get any payment in the new system: for $c \in C\left(P_{x}\right), \sum_{i \in N} p_{i}(c)=$ $\sum_{i \in N} p_{x, i}^{\prime}(c)=\sum_{i \in N} p_{x, i}(c)=0$ (for $x \in\{1,2\}$ ).
5. All unelected candidates are only supported by voters from their own 'old' system, who did not have in total a remaining unspent budget of more than its cost there, so neither will they have it now:
Without loss of generality, assume that an unelected candidate $c \notin W$ is part of $C\left(P_{1}\right)$. Then because $c \notin W$, we also have $c \notin W_{1}$, because if it was in $W_{1}$, it would also have been in $W$. Because $\mathbf{p s}_{1}$ is a price system that supports $W_{1}$, we know that $\sum_{i \in N}$ for which $u_{i}(c)>0(1-$ $\left.\sum_{e \in W_{1}} p_{1, i}(e)\right) \leq \operatorname{cost}(c)$. However, for all voters $i \in N$ for which $c \in A_{i}$, we have $i \in P_{1}$, so for all $e \in W_{1}, p_{1, i}(e)=p_{i}(e)$, and for all $e \in W_{2}, p_{i}(e)=0$. This implies that

$$
\begin{aligned}
\sum_{i \in N \text { for which } u_{i}(c)>0}\left(1-\sum_{e \in W_{1}} p_{1, i}(e)\right) & \leq \operatorname{cost}(c) \\
\sum_{i \in N: u_{i}(c)>0}\left(1-\sum_{e \in W=W_{1} \cup W_{2}} p_{i}(e)\right) & \leq \operatorname{cost}(c) .
\end{aligned}
$$

We can analogously show the same for $c \in C\left(P_{2}\right)$, so conclude that for all $c \in C(P)=$ $C\left(P_{1}\right) \cup C\left(P_{2}\right)$, if $c \notin W$,

$$
\sum_{i \in N: u_{i}(c)>0}\left(1-\sum_{e \in W} p_{i}(e)\right) \leq \operatorname{cost}(c)
$$

By these five points, we have shown that $\mathbf{p s}$ is indeed a valid price system that supports committee $W$.

We have shown by induction over laminar election instances that, if a committee $W$ is laminar proportional in a laminar election instance $(P, l)$, it is also supported by a price system with with $b=\operatorname{cost}(W)$. Hence we can conclude that laminar proportionality implies priceability in laminar election instances in approval based participatory budgeting settings.

### 3.3.3 Laminar proportional profiles and the core in approval based PB

Because laminar proportionality seems such a promising axiom, we will study its relation with other fairness axioms, starting with the core. We use Definition 3.5 of the core for approval based elections, where we measure the utility an agent receives from a set of projects by counting the number of projects she approves in the set. Because laminar proportionality is solely defined on laminar election instances, we will restrict our domain of profiles to those, and we will follow their inductive structure in our analysis. Starting with unanimous profiles as in 3.12.1, any laminar proportional set of projects will consist of candidates that are approved by every voter, so in such instances clearly a laminar proportional committee is in the core. However, in profiles with one unanimously approved candidate, as defined in Definition 3.12, 2, we can have a set of voters that can afford a set of projects in which the unanimously approved candidate is not present, and of which all voters prefer this set of projects to the elected set of projects. This can happen even though the elected set is laminar proportional, as we show in Theorem 3.5.

Theorem 3.5. In PB, there exist laminar proportional election outcomes that are not in the core.
Proof. We will prove this theorem by giving a counterexample. Consider a situation with $N=$ $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$, a unanimously approved project $c$ with $\operatorname{cost}(c)=1$, a set of 8 projects $T=\left\{t_{1}, \ldots, t_{8}\right\}$ that cost $\frac{1}{3}$ each and are all approved by $v_{1}, v_{2}$, and $v_{3}$, and a set of 2 projects $\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ that also $\operatorname{cost} \frac{1}{3}$ and are approved by $v_{4}$. This profile is shown in Figure 6. The committee $W=\left\{c, t_{1}, \ldots, t_{6}, x_{1}, x_{2}\right\}$


Figure 6: A laminar election instance with a laminar proportional committee $W$ (indicated in green), and a set of projects $T$ (indicated with a red border)
as indicated in green in the figure is laminar proportional for limit $l=\frac{11}{3}$ (which is also its cost). However, $S=\left\{v_{1}, v_{2}, v_{3}\right\}$ is a blocking coalition. $S$ can afford $T:|S|=3>\frac{\frac{8}{3}}{\frac{11}{3}} \cdot 4=\frac{\operatorname{cost}(T)}{l} \cdot n$, and for any voter $i \in S, u_{i}(T)=8>7=u_{i}(W)$. Therefore, $W$ is not in the core.

In [11], a definition of EJR for approval based elections is given:

Definition 3.14 (EJR for approval based PB). A group of voters $S$ is $T$-cohesive for $T \subseteq C$ if $T$ is affordable with their share of the budget and they all approve all projects in $T:|S| \geq \frac{\operatorname{cost}(T)}{l} \cdot n$ and $T \subseteq \cup_{i \in S} A_{i}$. A committee $W$ satisfies EJR if for all $T$-cohesive groups $S \subseteq N$, there is a voter $i$ in $S$ who approves at least as many projects in $W$ as in $T:\left|W \cap A_{i}\right| \geq|T|$.

We construct a similar definition for PJR:
Definition 3.15 (PJR for approval based PB). A committee $W$ satisfies PJR if for all $T$-cohesive groups $S \subseteq N$, the number of projects in $W$ that is approved by at least one of the voters in $S$ is larger than the number of projects in $T:\left|W \cap \cup_{i \in S} A_{i}\right| \geq|T|$.

Using these definitions for EJR and PJR, we see that in the counterexample in Figure 12, the group $S$ is $T$-cohesive for given $T$, and there is no voter $i \in S$ such that $\left|W \cap A_{i}\right| \geq|T|$, neither is $\left|W \cap \cup_{i \in S} A_{i}\right| \geq$ $|T|$. This proves the following theorem:

Theorem 3.6. In PB, laminar proportional election outcomes do not necessarily satisfy EJR or PJR.
Note however, that it is a quite specific example. In Section 4.3.2, we will show that with certain restrictions, laminar proportional election outcomes are in the core (and hence also satisfy EJR and PJR).

### 3.3.4 Summary about the relations between laminar proportionality, priceability, PJR, EJR, and the core in PB

When comparing to the results in multi-winner approval voting, we see that some of the results still hold in the participatory budgeting setting, however, in this setting priceabile committees need not satisfy PJR and laminar proportionality does not imply either of PJR, EJR, and the core. Nevertheless as we will show in Chapter 4 , under some conditions these implications do hold and there is a restriction over price systems under which priceability implies laminar proportionality. In Figure 7, all relations between the fairness axioms we discussed so far are visualised. Note that the relations with laminar proportionality involved only apply to laminar election instances and approval based voting.

### 3.3.5 FJR

Just as in Chapter 2, the question remains what is the relation between FJR and the five axioms that had our main focus. We know that also in PB, FJR is 'in between' EJR and the core (that the core implies FJR and FJR implies EJR) from [11]. We can deduce from Corollary 3.2.1 and Theorem 3.6 that laminar proportionality and priceability do not imply FJR (since they do not imply EJR, and FJR implies EJR), and from Theorem 2.6 and Corollary 2.6.1 that the FJR does not imply LP and priceability (since the core does not imply them in MWV, so also not in PB, and the core implies FJR).

### 3.4 Future work

Just as in Chapter 2, we have regarded the axiom of Nash welfare in the beginning of this chapter and found that neither of the rules we studied satisfied it. And just as in MWV, the question remains what are the relations between Nash welfare and the other axioms. We can show that in PB Nash welfare does not imply EJR or the core:

Proposition 3.17. A committee that maximises Nash welfare does not necessarily satisfy EJR in PB.


Figure 7: The relations between laminar proportionality, priceability, PJR, EJR, and the core in the multi-winner voting setting (red) and in the participatory budgeting setting (blue).

Proof. As a counterexample, we take the city of Onetown ([11], p.4): The Nash product of the committee $\{L 1, L 2, R\}$ is $\operatorname{Nash}(\{L 1, L 2, R\})=3^{60} \cdot 2^{30} \approx 4.6 e 37$, while $\operatorname{Nash}(\{L 1, L 2, L 3\})=4^{60} \cdot 1^{30} \approx$ $1.3 e 36$, so the Nash product is maximised with the set $\{L 1, L 2, R\}$ (obviously there are no other sets with a higher Nash score). This outcome however does not satisfy EJR: if we take $S$ as the population of Leftside, $\alpha(L 1)=\alpha(L 2)=\alpha(L 3)=1, \alpha(R)=0$ and $T=\{L 1, L 2, L 3\}$, we have $|S|=$ $60 \geq \frac{60}{90} \cdot 90=\frac{\operatorname{cost}(T)}{\text { limit }} \cdot n, u_{i}(c)=1=\alpha(c)$ for every $i \in S$ and every $c \in T$. But for every voter $u_{i}(\{L 1, L 2, R\})=2<3=\sum_{c \in T} \alpha(c)$, and there is no $a \in C$ such that $u_{i}(W \cup\{a\})>\sum_{c \in T} \alpha(c)$ (if $L 3$ is added to $W$, the utility is still 3 , which is not greater than 3 ).

In general, [11] show that every rule that does not take into account the cost of the projects (but only depends on utilities and the collection of budget-feasible sets) fails proportionality.

Proposition 3.18. A committee that maximises Nash welfare is not necessarily in the core in PB.
Proof. For the same reason as for which SBA outcomes are not always in the core (Proposition 3.1), a rule that maximises the Nash product does not always yield outcomes that are in the core. With the same example as for the SBA, it can be shown that when there are small projects that are supported by only a few voters, those can 'screw up' the core.

The relation between Nash welfare and PJR, laminar proportionality, and priceability in PB is something that remains open for further research:

Open Question 3.1. Is there a logical relation in PB between Nash welfare on the one hand, and PJR, priceability, and laminar proportionality on the other hand?

Something else that has popped up during the writing of this chapter is that although PAV is so far only defined for approval voting (hence its name), we could probably think of a variant that is also applicable to utility ballots. Since the harmonic numbers are not defined for non-natural numbers, we
could either define a function that uses the logarithm, which can be seen as a continuous version of the harmonic numbers, or use some sort of approximation algorithm that counts units of utility and uses those as input for the PAV algorithm.

Open Question 3.2. Is there a natural way to generalise PAV such that it is applicable to election instances with arbitrary utilities?

## 4 Restrictions in MWV and PB

In the previous sections, we studied various rules and axioms for multi-winner elections and for participatory budgeting. We found relations between axioms and rules that satisfy certain axioms. However, we also showed that certain rules do not satisfy certain axioms, and that some axioms have no implicational relations to some other axioms, by finding a counterexample. These counterexamples do not show that there is no relation at all between certain axioms, only that in the general (MWV or PB ) setting there is no relation. In some cases, the counterexample is of a rather specific kind, and one could argue that in real life applications, such an example will rarely occur. Hence, we can study the relations between axioms and the axioms that rules satisfy under certain restrictions of the domain of profiles, or under restrictions of the definition of an axiom. The domain of laminar profiles we discussed earlier is an example of a restricted domain, and we can find more examples in the literature. One idea is that in many situations, the projects or candidates can be viewed as points in a (Euclidean) space, and that voters that like some candidate are supposed to also like candidates that lie close to it. This idea gives rise to the notion of single-peaked preferences. [18] and [19] show that in single winner voting, there always is a Condorcet winner in single-peaked profiles, and that in such profiles the rule that elects the Condorcet winner is strategyproof (while without the assumption of single-peakedness, the only reasonable strategyproof voting rule is dictatorship). Variations to normal single-peakedness are proposed in [20] and [21]. When voters do not give their preferences as utilities or rankings of the alternatives, but rather as approval sets, [18] define the notion of 'possible single-peaked', which means that an approval profile can be extended to a profile of linear orders that is single peaked. This notion is similar to the Candidate Interval (CI) domain [22; 23], to which a profile $A$ belongs if there exists a linear order of candidates such that for each voter $i \in N$, the set $A_{i}$ appears contiguously on the linear order. Note that in PB settings it is not very likely that preferences are single-peaked or in the CI domain, because different projects can be about completely different topics. There is no clear way to measure the distance between, say, a school to be built, a charity project, and an improvement of the road from one city to another. However, there could occur PB settings where such a natural ordering does exist, when the physical location of the projects matters a lot, or when projects can be placed on a political left-to-right scale.

Instead of demanding that the projects or candidates are in a linear order, we can also demand that the voters are ordered, which leads to the domain of single-crossing preferences [18]. In approval voting, we have the notion of 'possible single-crossing' [24; 25], which is similar to the Voter Interval (VI) domain [23], to which a profile $A$ belongs if there exists a linear order of voters such that for each candidate $c \in C$, the set $N(c)$ appears contiguously on the linear order (note that laminar election instances are in the VI domain). The studies of single-peaked and single-crossed preferences, respectively CI and VI domain, are mainly about computational complexity and not that much about axiomatic properties. Nevertheless, some literature exists about restrictions that improve axiomatic properties of rules. In the general setting, Rule X is not exhaustive, and does not satisfy the core. [11] propose a domain restriction to make Rule X exhaustive in the participatory budgeting setting. When every voter assigns at least some utility to every project, Rule X indeed satisfies exhaustiveness (Proposition 1 in that paper). As for the core, in MWV, Rule X satisfies core subject to priceability with equal payments: whenever Rule X fails the core, the output committee can only be blocked by proposals that are "unfair" to members of the blocking coalition. [10]. In the same way, the restriction of cohesiveness (which requires that every voter approves all candidates in the outcome), ensures that the core and EJR are equivalent [10].

Outline of chapter In this chapter we will study whether under certain restrictions, negative entries in Table 10 or absent implications in Figure 7 may become positive, either by restricting the domain of profiles or restricting the definition of an axiom or rule. In the first two sections we will look at restrictions for specific rules, in Section 4.1 for SBA and in Section 4.2 for Rule X. Then in Section 4.3 we will look at two restrictions regarding laminar proportionality: Section 4.3 .1 will deal with a restriction under which priceability implies laminar proportionality in MWV, in Section 4.3 .2 we will look at a restriction under which laminar proportionality implies the core in PB. Finally, in Section 4.4, we will study party-list profiles.

### 4.1 Restrictions for SBA

In this section, we will try to find restrictions that can make some of the negative results for SBA positive, and under which SBA does satisfy some of the proportionality axioms. However, we will see that it is difficult to find such, since SBA is not designed to be proportional.

### 4.1.1 SBA and fairness axioms

In general, because SBA is based on majority, it is not a rule that gives a fair or proportional outcome. Hence, it is clear that SBA does not satisfy any of the fairness axioms (PJR / EJR / FJR / Core / Nash), and we cannot restrict the domain in a non-trivial way such that it does satisfy any of these axioms. However, something to be mentioned about the results in Propositions 3.1 and 3.2 (where we show that the SBA outcome is not necessarily in the core and does not satisfy EJR) is that the used example is an artificial example. In real situations, the smaller projects defined in the example have a much larger chance to be chosen. In the example, the small project was not in the final budget because the larger project fitted exactly in the budget limit. However, in reality, there is more chance that the larger project(s) will not exactly fit in the limit, and therefore there will be some spare money left over for the small projects. The non-core outcomes might be seen as a result of the inseparability of projects rather than pointing out unfairness of the voting algorithms. If we take arbitrary real costs, there is smaller chance that the large projects will exactly use up the budget, so larger chance that the small projects can still be funded. Or, if we take as a domain restriction that the costs of projects cannot differ too much, projects cannot be so small that $|S|$ is large enough. We leave as an open research direction to work this out in further detail.

### 4.1.2 SBA and strategyproofness

As [18; 19] show, for single-peaked preferences there is a strategyproof rule that selects the Condorcet winner. SBA is a Condorcet-consistent algorithm [5], so it will select the Condorcet winner when there is one. We could therefore expect that SBA is strategyproof for single-peaked domains. However, we will show that this is not the case. Take a situation with three voters: $v_{1}, v_{2}$, and $v_{3}$, and three projects, $a, b$, and $c$, where the preferences are as follows: $v_{1}: a \succ b \succ c, v_{2}: b \succ a \succ c, v_{3}: c \succ b \succ a$, and suppose that the previous budget $B_{-1}=\{a\}$, i.e. last year project $a$ was implemented. This profile is single-peaked for the ordering $a \triangleright b \triangleright c$. In the majority graph corresponding to this profile, $\{b\}$ is the Schwartz set, so $b$ will be chosen first. Suppose only one of the three projects can be afforded, then for this profile $b$ will be elected. However, if voter $v_{1}$ misrepresents his preferences and instead gives the ballot $v_{1}^{\prime}: a \succ c \succ b$, the majority graph will be a cycle, so the Schwartz set will be $\{a, b, c\}$, and because $a$ was the only project in the previous budget, $a$ will be elected, which is preferred by $v_{1}$ over the outcome that would be elected with his true ballot. Note that $v_{1}$ 's fake ballot is not singlepeaked within the order specified above, so it would be interesting to study the strategyproofness of

SBA when only single-peaked preferences are allowed (but then the ordering of the projects should already be defined).

### 4.2 Restrictions for Rule $X$

In this section we will study restrictions under which Rule X might satisfy the core and Nash welfare.

### 4.2.1 A restriction under which Rule $X$ satisfies the core in PB

The restriction studied in this section is a restriction to an axiom rather than to the domain of election instances: we restrict the definition of the core. Peters and Skowron [10] show that in MWV, Rule X satisfies the core subject to priceability with equal payments. We know that in PB, Rule X does not satisfy the core but does satisfy an approximation to the core [11]. However, the question remains open whether Rule X also satisfies the core subject to priceability with equal payments in the PB setting. At the same time it is debatable whether equal payments do make sense in the PB setting. It would not really be fair if every voter had to pay the same amount for a candidate, because some voters get more utility from a candidate than others. In the MWV setting, voters either approve or disapprove of a candidate, so equal payments make more sense. In MWV, Rule X induces priceability with equal payments in the winning committee. In PB, Rule $X$ requires an equal price/utility ratio, see the proof of Theorem 2 on page 10 of [11]. Hence, for the PB setting, we want voters that get more utility to pay more, and we will define priceability with an equal ratio of price/utility, analogous to priceability with equal payment.

Definition 4.1 (Priceability with equal ratios of price/utility). An election instance $E$ with committee $T$ satisfies priceability with equal ratios of price/utility ( $\mathcal{P}_{\text {eq-price/utility }}$ ) if there exists a family of payment functions $\left\{p_{i}\right\}_{i \in N}$ with

1. $\sum_{c \in T} p_{i}(c) \leq \frac{1}{n}$ for each $i \in N$,
2. $\sum_{i \in N} p_{i}(c)=\operatorname{cost}(c)$ for each $c \in T$, and
3. for each $i, j \in N(c)$ we have $\frac{p_{i}(c)}{u_{i}(c)}=\frac{p_{j}(c)}{u_{j}(c)}$.

Note that our new definition of priceability with equal ratios of price/utility is very similar to [10]'s definition of $\mathscr{P}_{\text {price-eq. }}$. In fact, it is a generalisation of it to the participatory budgeting setting, as we will show here.

Proposition 4.1. Priceability with equal ratios of price/utility is a generalization of priceability with equal payments for the PB setting.

Proof. Note that priceability in MWV has a slightly different definition than it has in PB. In MWV, each voter has a budget of one unit of currency, while in PB, the initial budget is $b \geq 1$, and each voter has an initial budget of $\frac{b}{n}$. We show what constraints the points of the definition of $\mathcal{P}_{\text {eq-price/utility }}$ are equivalent to in the MWV setting:

1. $\sum_{c \in T} p_{i}(c) \leq \frac{1}{n}$ for each $i \in N$, where the total initial budget is 1 . In the definition of priceability and of $\mathcal{P}_{\text {price-eq }}$ in the MWV model, each voter has a total initial budget of 1 rather than $\frac{1}{n}$, to get a total initial budget of $n$. Hence, in that situation this requirement translates to $\sum_{c \in T} p_{i}(c) \leq 1$.
2. $\sum_{i \in N} p_{i}(c)=\operatorname{cost}(c)$ for each $c \in T$. In the MWV model, each candidate $c$ has the same cost, the number of candidates that can be elected is $k$, and the total budget is $n$, as mentioned above, so the cost of each candidate is $\frac{n}{k}$. Hence, the second requirement of $\mathcal{P}_{\text {eq-price/utility }}$ translates to $\sum_{i \in N} p_{i}(c)=\frac{n}{k}$.
3. For each $i, j \in N(c)$ we have $\frac{p_{i}(c)}{u_{i}(c)}=\frac{p_{j}(c)}{u_{j}(c)}$. Since the utility of a voter for projects she approves is 1 , and both $i$ and $j$ approve $c$ (since they are in $N(c)$ ), $u_{i}(c)=u_{j}(c)=1$, so this requirement in the MWV setting is $p_{i}(c)=p_{j}(c)$ for each $i, j \in N(c)$.

We see that the three requirements of $\mathcal{P}_{\text {eq-price/utility }}$ boil down to the three requirements of $\mathcal{T}_{\text {price-eq }}$ when applied in a MWV setting, so a MWV election instance $(P, k)$ and a committee $W$ satisfy $\mathcal{P}_{\text {eq-price/utility }}$ if and only if they satisfy $\mathcal{P}_{\text {price-eq. }}$. This shows that indeed $\mathcal{P}_{\text {eq-price/utility }}$ is a generalization of $\mathscr{P}_{\text {price-eq }}$ to the PB setting.

We define an allowed deviation from the core in PB as follows:
Definition 4.2. A pair $(S, T)$, with $S \subseteq N$ and $T \subseteq C$ is an allowed deviation from $W$ if
(i) $\operatorname{cost}(T) \leq \frac{|s|}{n}$
(ii) for each $i \in S$ we have that $u_{i}(T)>u_{i}(W)$, and
(iii) $T$ has property $\mathcal{P}$ when $E$ is restricted to voters in $S$.

Now we can prove that in the PB setting, Rule X satisfies the core subject to priceability with equal ratios of price/utility (very analogous to the proof of Theorem 9 on page 21 of [10]).

Proposition 4.2. Rule $X$ satisfies the core subject to priceability with equal ratios of price/utility.
Proof. Assume that there exists an instance $E$ where Rule X returns committee $W$, and that there is an allowed deviation $(S, T)$. Take a pricesystem $\mathbf{p s}$ that shows that $T$ is priceable with equal ratios of price/utility for the instance $E$ restricted to voters in $S$. Let $\rho_{c}$ be the payment per unit of utility that each voter pays for a candidate $c \in T$ in $\mathbf{p s}$, and let $W_{t}$ be the set of candidates selected by Rule X up to the $t$-th iteration. We will first prove the following invariant: For each $t$, in the $t$-th iteration Rule X selects a candidate for which each voter pays per unit of utility at $\operatorname{most}^{\min }{ }_{c \in T \backslash W_{t}} \rho_{c}$. For the sake of contradiction assume this is not the case and let $t$ be the first iteration in which Rule X selects a candidate $c_{t}$ for which some voter pays more than $\min _{c \in T \backslash W_{t}} \rho_{c}$. Let $c_{t}^{\prime}=\operatorname{argmin}_{c \in T \backslash W_{t}} \rho_{c}$. We will argue that Rule X would rather select $c_{t}^{\prime}$ than $c_{t} . T$ is an allowed deviation of the core, so according to (ii), up to the $t$-th iteration, for each voter $i, u_{i}\left(W_{t}\right)<u_{i}(T)$. Further, up to the $t$-th iteration each voter pays on average less per unit of utility for her representatives in $W_{t}$ than in $T$, so each voter pays less for $W_{t}$ than for $T$. Thus, each voter who pays for $c_{t}^{\prime}$ in $\mathbf{p s}$ can also pay for $c_{t}^{\prime}$ now. Since Rule X always selects a candidate who is affordable and who minimizes the per-voter payment-per-utility, it would rather select $c_{t}^{\prime}$ than $c_{t}$, which is a contradiction that proves our invariant.
By the invariant, each voter pays less per unit of utility for her representatives in $W$ than she would pay per utility for her representatives in $T$ according to ps. Each voter in $S$ also has a lower utility for $W$ than for $T$ (because of (ii)), so would have some money left to buy a not-yet selected candidate from $T$. This gives a contradiction and completes the proof.

### 4.2.2 Rule $X$ and Nash welfare

In the proof of proposition 3.15, we used the situation in Onetown to show that in PB, Rule X does not maximise Nash welfare: Rule X will choose $W=\left\{L_{1}, L_{2}, L_{3}\right\}$, while the Nash welfare is maximised with $W=\left\{L_{1}, L_{2}, R\right\}$. However, if we look at the situation in Twotown, where all projects have the same cost, Rule X will select the projects $W=\left\{L_{1}, L_{2}, R\right\}$ and hence maximise the Nash equilibrium. This raises the question whether in situations with unit cost, Rule X will always select the set of projects that maximises Nash welfare.
We first look at a situation like Twotown from [11], with unit cost and approval voting. We will prove that in such situations with two strictly separated parties, the outcome of Rule X does not differ more than 1 from the outcome that maximises the Nash welfare.

Proposition 4.3. In $M W V$, in a setting where there are exactly two disjoint parties, the outcome of Rule $X$ is always within a bound of 1 from optimal Nash welfare.

Proof. Suppose we have two strictly separated parties, Left and Right, where the number of voters in Left (who only vote on Left projects) is $L$ and the number of voters in Right (who only vote on Right projects) is $R$. Suppose all projects have unit cost, and we have a budget limit of $l$. If we apply Rule X in this situation, then each voter starts with a budget of $\frac{l}{n}$, each Left project is $\rho$-affordable for $\rho=\frac{1}{L}$, and each Right project is $\rho$-affordable for $\rho=\frac{1}{R}$. Because voters from Left will only approve Left projects and therefore only spend their money on Left projects, and in the same way voters from Right will only spend their money on Right projects, the total number of projects from the Left side that will be selected is $j=\frac{l}{n} \cdot L-\left(\frac{l}{n} \cdot L \bmod 1\right)$ (the number of whole projects that fits into the total budget of Left), and the total number of projects from the Right side that will be selected is $j_{r}=\frac{l}{n} \cdot R-\left(\frac{l}{n} \cdot R\right.$ $\bmod 1)$, i.e., in such a profile the outcome of Rule $X$ will be proportional to the size of the parties. Now we look at the Nash welfare of such an outcome. Note that

$$
\operatorname{Nash}(W)=\prod_{i \in N}\left(1+\left|A_{i} \cap W\right|\right)=(1+j)^{L} \cdot\left(1+j_{r}\right)^{R}
$$

For the committee that maximises this, we should find an allocation for which the derivative is 0 . If we write $j_{r}=l-j$, we have

$$
\operatorname{Nash}^{\prime}(W)=L \cdot(1+j)^{L-1} \cdot(1+l-j)^{R}-(1+j)^{L} \cdot R \cdot(1+l-j)^{R-1} .
$$

If we set this to be equal to zero, we can divide by some terms on both sides and get

$$
\begin{aligned}
L(1+l-j)-R(1+j) & =0 \\
L+l L-R & =j(L+R) \\
\frac{L+l L-R}{L+R} & =j,
\end{aligned}
$$

so if the outcome of Rule X is to maximise the Nash welfare in this situation, the number of Left projects $j$ should be equal to this $j$ that maximises the Nash welfare:

$$
j=\frac{l}{n} \cdot L-\left(\frac{l}{n} \cdot L \quad \bmod 1\right)=\frac{L+l L-R}{L+R} .
$$

Note that at $j=-1$ and at $j=l+1, \operatorname{Nash}(W)=0$, and that $-1<\frac{L+l L-R}{L+R}<l+1$, and that at $\frac{L+l L-R}{L+R}$, $\operatorname{Nash}(W)$ is positive, so this is indeed a maximum. In general, those will be different, but when the
projects exactly fit into the budget $\left(\frac{l}{n} \cdot L \bmod 1=0\right)$, the difference is only $\frac{L-R}{L+R}$. Because $L$ and $R$ are positive numbers, the absolute value of this difference is always smaller than or equal to 1 , so the outcome of Rule X does not differ more than 1 from the outcome that maximises the Nash welfare (and it does exactly maximise the Nash welfare if $L=R$ ).

The discussion above raises the question whether Nash welfare is actually a good axiom to measure proportionality for minority groups, since it seems to focus especially on separate voters with a low utility.

Open Question 4.1. To what extend are outcomes with maximal Nash welfare conducive to minority groups?

### 4.3 Restrictions regarding laminar proportionality

### 4.3.1 A restriction under which priceability implies laminar proportionality in MWV

Note that the counterexample in Table 7 that shows that priceability does not imply laminar proportionality (in Section 2.3.1) is not efficient. By electing $c_{6}$ instead of $c_{3}$ no agent's utility would decrease, but $v_{3}$ 's utility would increase. Maybe making it efficient would make it laminar proportional, because then unanimous candidates have to be elected. Note however that electing $c_{6}$ instead of $c_{4}$ would make it efficient and still keep it priceable ( $v_{3}$ could just spend her money on $c_{6}$ instead of on $c_{4}$ ), but still it is not laminar proportional because the instance without $c_{6}$ is not laminar proportional.
However, if the payments would have been equally divided over the voters that approve a candidate, it would have been laminar proportional. In [12], a property that demands exactly this is defined: Balanced Stable Priceability (BSP). This property demands that a price system is balanced: voters that get utility from a candidate must all pay the same price for this candidate, and that the system is stable: there is no coalition of voters that wants to change their payments so that they get more utility (or pay less). As [12] show, the committees that satisfy BSP for a price $p$ are the same as the committees elected by a variant of Rule X, and because Rule X returns laminar proportional committees in laminar profiles, probably BSP implies laminar proportionality. We can show easily by induction over laminar profiles that this is indeed the case.

Theorem 4.1. Balanced Stable Priceability implies laminar proportionality in MWV.
Proof. We will give an inductive proof to show this.
Basis: for unanimous profiles with $|C(P)| \geq k$, any candidate $c$ that is in $W$ gets at least some payment in the price system that supports $W$, and hence is in $A_{i}$ for some voter $i$, and because $P$ is unanimous, $c \in A_{i}$ for all voters $i$, so $W \subseteq C(P)$.
Inductive Hypothesis: Assume laminar profiles $\left(P^{\prime}, k^{\prime}\right),\left(P_{1}, k_{1}\right)$ and $\left(P_{2}, k_{2}\right)$ are laminar and respective committees $W^{\prime}, W_{1}$, and $W_{2}$ are laminar proportional if they satisfy BSP, where $P^{\prime}$ is not unanimous.

## Inductive Step:

- Suppose $c$ is a unanimously approved candidate, such that the instance $\left(P^{\prime}, k^{\prime}\right)=(P-\{c\}, k-$ $1)$ and that $W$ satisfies BSP in the instance $(P, k)$. Then, by stability, $c \in W$. Assume for a contradiction that $c$ would not be elected, then all voters together would rather pay for $c$ and all give up one of the candidates they now pay for: then they would all get the same utility because
they all approve $c$, and would have to pay less because they can divide the price for $c$ over them all. Hence $c$ is elected in $W$. Now the committee $W$ without $c$, which we call to be the $W^{\prime}$ from the inductive hypothesis, still satisfies BSP, because every voter pays the same amount for $c$ (because the price system is balanced), and hence we can just subtract the price they all pay for $c$ from the total budget every voter gets. Then, by the inductive hypothesis, $W^{\prime}$ is laminar proportional, so $W$ itself is laminar proportional as well.
- Suppose $(P, k)$ consists of two separate laminar election instances $\left(P_{1}, k_{1}\right)$ and $\left(P_{2}, k_{2}\right)$. Define $W_{1}$ as the set of candidates in $W$ from $P_{1}$, and $W_{2}$ as the set of candidates in $W$ from $P_{2}$, so $W=W_{1} \cup W_{2}$. If $W$ satisfies BSP, then in the price system that witnesses this, voters from $P_{1}$ can only vote and pay for candidates in $P_{1}$, and voters from $P_{2}$ can only vote and pay for candidates in $P_{2}$, so we can split the price system to get a price system for both instances, which shows that both $W_{1}$ and $W_{2}$ satisfy BSP. According to the inductive hypothesis, then $W_{1}$ and $W_{2}$ are laminar proportional, so $W$ is laminar proportional.

We have thus shown by induction over laminar profiles that if a winning committee in a laminar profile satisfies BSP, it also satisfies laminar proportionality.

### 4.3.2 A restriction under which laminar proportional committees satisfy the core in PB

As shown in Section 3.3.3, laminar proportional committees in PB are not necessarily core committees. However, with certain constraints on the sets of projects that can block the core, laminar proportional election outcomes are in the core. In general, a set of projects is in the core if for every group of voters $S \subseteq N$ and set of projects $T \subseteq C$ such that $S$ can afford $T$ with their share of the budget (i.e. $\left.|S| \geq \frac{\operatorname{cost}(T)}{l} \cdot n\right)$ there is a voter $i \in S$ such that $i$ does not prefer $T$ over $W: u_{i}(W) \geq u_{i}(T)$. We will show that if for any unanimously approved candidate $c$ in $W$ either $c$ is part of $T$ or if there is some project in $T$ that costs at least as much as $c$, then there is such a voter $i \in S$ with $u_{i}(W) \geq u_{i}(T)$. We define a property of committees $\mathcal{P}_{u-\text { afford }}$ called unanimity-affordability:

Definition 4.3. $((P, l), T) \in \mathcal{P}_{u-\text { afford }}$ if for any unanimously approved candidate $c \in C(P)$ there exists $t \in T$ with $\operatorname{cost}(t) \geq \operatorname{cost}(c)$.

Since $\operatorname{cost}(c) \geq \operatorname{cost}(c)$, the definition is also satisfied if $c \in T$. Using Definition 4.2 (allowed deviations), we will show that in any laminar election instance, for any laminar proportional election outcome there exists no allowed deviation, so a laminar proportional outcome satisfies the core subject to $\mathcal{P}_{u-a f f o r d}$.

Theorem 4.2. In PB, laminar proportional committees satisfy the core subject to $\mathscr{P}_{u-a f f o r d}$.
Proof. We will prove this by induction over the structure of laminar profiles.
Basis: For unanimous profiles $P$ (Definition 3.12, 1), $W$ will consist of candidates that are approved by every voter, so clearly for every group of voters $S \subseteq N$ and $T \subseteq C$ with $|S| \geq \frac{\operatorname{cost}(T)}{l} \cdot n$, there is some voter in $S$ (namely all voters in $S$ ) who approves at least as many projects in $W$ as in $T$. This is true even in the general situation, without the restriction of $\mathscr{P}_{u-\text { afford }}$.
Inductive hypothesis: Suppose that $W^{\prime}, W_{1}$, and $W_{2}$ are arbitrary committees in respective laminar instances $\left(P^{\prime}, l^{\prime}\right),\left(P_{1}, l_{1}\right)$, and $\left(P_{2}, l_{2}\right)$ with $C\left(P_{1}\right) \cap C\left(P_{2}\right)=\emptyset$ and $\left|P_{1}\right| \cdot l_{2}=\left|P_{2}\right| \cdot l_{1}$, that $W^{\prime}, W_{1}$, and $W_{2}$ are laminar proportional and are in the core subject to $\mathscr{P}_{u-a f f o r d}$.
Inductive step:

- Suppose there is a unanimously approved candidate $c$ and $\left(P_{-c}, l-\operatorname{cost}(c)\right)=\left(P^{\prime}, l^{\prime}\right)$ is laminar, $W^{\prime}$ is a laminar proportional committee for $\left(P^{\prime}, l^{\prime}\right)$, and $W=W^{\prime} \cup\{c\}$ (Definition 3.12,2).
Assume for a contradiction that $W$ is not in the core. Then there must exist a group of voters $S$ and a set of projects $T$ such that $|S| \geq \frac{\operatorname{cost}(T) \cdot n}{l}$, with $u_{i}(T)>u_{i}(W)$ for all voters $i \in S$. Because utilities are assumed to be additive and everyone approves candidate $c$, we know that for all voters $i \in S, u_{i}(T \backslash\{c\})=u_{i}(T)-u_{i}(c)$ if $c \in T$ and $u_{i}(T \backslash\{c\})=u_{i}(T)$ otherwise, and $u_{i}\left(W^{\prime}\right)=u_{i}(W)-u_{i}(c)$, so for all $i \in S$

$$
\begin{equation*}
u_{i}(T \backslash\{c\})>u_{i}\left(W^{\prime}\right) \tag{17}
\end{equation*}
$$

We distinguish two cases, with either $c \in T$ or $c \notin T$, and show that in both cases $W$ is in the core.

1. $c \in T$ :

Because we have chosen $S$ and $T$ such that $\frac{|S|}{n} \geq \frac{\operatorname{cost}(T)}{l}$ and because $\operatorname{cost}(T) \leq l$ (since by definition $|S| \leq n$ ), we find that

$$
\begin{equation*}
\frac{\operatorname{cost}(T \backslash\{c\})}{l^{\prime}}=\frac{\operatorname{cost}(T)-\operatorname{cost}(c)}{l-\operatorname{cost}(c)} \leq \frac{\operatorname{cost}(T)}{l} \leq \frac{|S|}{n}, \tag{18}
\end{equation*}
$$

However, from the inductive hypothesis we know that $W^{\prime}$ is in the core (subject to $\mathcal{P}_{u-a f f o r d}$ ) in the election instance $\left(P^{\prime}, l^{\prime}\right)$, so for all $S^{\prime} \subseteq N^{\prime}, T^{\prime} \subseteq C^{\prime}$ with $\left|S^{\prime}\right| \geq \frac{\operatorname{cost}\left(T^{\prime}\right) \cdot n}{l^{\prime}}$, there is a voter $i^{\prime} \in S^{\prime}$ with $u_{i^{\prime}}(W) \geq u_{i^{\prime}}(T)$. In this instance, we can take $T^{\prime}=T \backslash\{c\}$ and $S^{\prime}=S$. As shown above, $S$ can afford $T^{\prime}\left(|S| \geq \frac{\operatorname{cost}\left(T^{\prime}\right) \cdot n}{l^{\prime}}\right)$, so there is a voter $i^{\prime} \in S^{\prime}$ with $u_{i}^{\prime}(W) \geq u_{i}^{\prime}(T)$. This is a contradiction with 17, which proves that $W$ is indeed in the core (subject to $\mathcal{P}_{u-\text { afford })}$ if $c \in T$.
2. $c \notin T$ :

In this case, equation 18 does not hold anymore, because not necessarily $\frac{\operatorname{cost}(T)}{l-\operatorname{cost}(c)} \leq \frac{\operatorname{cost}(T)}{l}$, in fact the first fraction is greater because $c$ has a positive cost. Now suppose that $S$ can afford $T$ and that every voter in $S$ prefers $T$ to $W$. Then, since $c$ is unanimously preferred, all voters in $S$ prefer $T$ to $W \backslash\{c\}$, and even all voters in $S$ prefer $T \backslash\{t\}$, where $t$ is an arbitrary project in $T$, to $W \backslash\{c\}$, because we are in an approval voting setting (where the utility of a project a voter approves is 1 and the utility of all projects a voter does not approve is 0 ). However, according to our inductive hypothesis, if the voters in $S$ together could afford $T \backslash\{t\}$ in the situation where the budget limit is $l-\operatorname{cost}(c)$, there would be a voter in $i \in S$ with $u_{i}(W \backslash\{c\}) \geq u_{i}(T \backslash\{t\})$, since $W \backslash\{c\}=W^{\prime}$ is in the core (subject to $\mathcal{P}_{u-a f f o r d}$ ) there. Hence, $S$ cannot afford $T \backslash\{t\}$ in the instance $\left(P^{\prime}, l-\operatorname{cost}(c)\right)$. We now have that

$$
\begin{equation*}
\frac{\operatorname{cost}(T)}{l} \leq \frac{|S|}{n}<\frac{\operatorname{cost}(T)-\operatorname{cost}(t)}{l-\operatorname{cost}(c)} \tag{19}
\end{equation*}
$$

where $t$ was an arbitrary project in $T$, so all $t \in T$ must have a lower cost than $c$. Hence, under our restriction that there exists $t \in T$ with $\operatorname{cost}(t) \geq \operatorname{cost}(c)$, there is no $S$ that can block the outcome. Hence, if $c \notin T, W$ is in the core subject to $\mathscr{P}_{u-\text { afford }}$.

- Suppose that $(P, l)$ is the sum of $\left(P_{1}, l_{1}\right)$ and $\left(P_{2}, l_{2}\right)$, i.e.that $P=P_{1}+P_{2}$ and $l=l_{1}+l_{2}$, and that $W=W_{1} \cup W_{2}$ (Definition 3.12,3). Assume for a contradiction that $W$ is not in the core. Then there must exist a group of voters $S$ and a set of projects $T$ such that $|S| \geq \frac{\operatorname{cost}(T) \cdot n}{l}$, with $u_{i}(T)>u_{i}(W)$ for all voters $i \in S$. Because $P_{1}$ and $P_{2}$ are strictly separated and voters can only
approve projects from their own election instances, each voter only gets utility from the elected candidates from his own instance, so we can divide $S$ into $S_{1}$ and $S_{2}$, and $T$ into $T_{1}$ and $T_{2}$ such that all voters from $S_{1}$ and projects from $T_{1}$ only occur in $P_{1}$ and all voters from $S_{2}$ and projects from $T_{2}$ only occur in $P_{2}$. Then we have that for all voters $i \in S_{1}, u_{i}\left(T_{1}\right)>u_{i}(W)$, and for all voters $i \in S_{2}, u_{i}\left(T_{2}\right)>u_{i}(W)$. From $|S| \geq \frac{\operatorname{cost}(T) \cdot n}{l}$ follows that

$$
\begin{equation*}
\left|S_{1}\right|+\left|S_{2}\right| \geq \frac{\left(\operatorname{cost}\left(T_{1}\right)+\operatorname{cost}\left(T_{2}\right)\right) \cdot\left(n_{1}+n_{2}\right)}{l_{1}+l_{2}} . \tag{20}
\end{equation*}
$$

Now, assume for a contradiction that both $\left|S_{1}\right|<\frac{\operatorname{cost}\left(T_{1}\right) \cdot\left(n_{1}\right)}{l_{1}}$ and $\left|S_{2}\right|<\frac{\operatorname{cost}\left(T_{2}\right) \cdot\left(n_{2}\right)}{l_{2}}$. Then from Equation 20 and the fact that $\frac{n_{1}}{l_{1}}=\frac{n_{2}}{l_{2}}$ (from the inductive hypothesis) we have that:

$$
\begin{aligned}
\frac{\operatorname{cost}\left(T_{1}\right) \cdot n_{1}}{l_{1}}+\frac{\operatorname{cost}\left(T_{2}\right) \cdot n_{2}}{l_{2}} & >\left|S_{1}\right|+\left|S_{2}\right| \\
\operatorname{cost}\left(T_{1}\right) \cdot \frac{n_{1}}{l_{1}}+\operatorname{cost}\left(T_{2}\right) \cdot \frac{n_{2}}{l_{2}} & \geq \frac{\left(\operatorname{cost}\left(T_{1}\right)+\operatorname{cost}\left(T_{2}\right)\right) \cdot\left(n_{1}+n_{2}\right)}{l_{1}+l_{2}} \\
\left(\operatorname{cost}\left(T_{1}\right)+\operatorname{cost}\left(T_{2}\right)\right) \cdot \frac{n_{2}}{l_{2}} & >\frac{\left(\operatorname{cost}\left(T_{1}\right)+\operatorname{cost}\left(T_{2}\right)\right) \cdot\left(n_{1}+n_{2}\right)}{l_{1}+l_{2}} \\
\frac{n_{2}}{l_{2}} & >\frac{n_{1}+n_{2}}{l_{1}+l_{2}} \\
\frac{n_{2} \cdot\left(l_{1}+l_{2}\right)}{l_{2} \cdot\left(l_{1}+l_{2}\right)} & >\frac{\left(n_{1}+n_{2}\right) \cdot l_{2}}{\left(l_{1}+l_{2}\right) \cdot l_{2}} \\
l_{1} \cdot n_{2}+l_{2} \cdot n_{2} & >l_{2} \cdot n_{1}+l_{2} \cdot n_{2} \\
l_{1} \cdot n_{2} & >l_{2} \cdot n_{1}=l_{2} \cdot \frac{n_{2} \cdot l_{1}}{l_{2}}=n_{2} \cdot l_{1},
\end{aligned}
$$

which is clearly a contradiction. Hence, at least one of $\left|S_{1}\right| \geq \frac{\operatorname{cost}\left(T_{1}\right) \cdot\left(n_{1}\right)}{l_{1}}$ and $\left|S_{2}\right| \geq \frac{\operatorname{cost}\left(T_{2}\right) \cdot\left(n_{2}\right)}{l_{2}}$ must be true. Without loss of generality, assume that $\left|S_{1}\right| \geq \frac{\operatorname{cost}\left(T_{1}\right) \cdot\left(n_{1}\right)}{l_{1}}$. Then, since $W_{1}$ is a core solution in the instance $\left(P_{1}, l_{1}\right)$, there exists $i \in S_{1}$ such that $u_{i}\left(W_{1}\right) \geq u_{i}\left(T_{1}\right)$. However, we already knew that for all voters $i \in S_{1}, u_{i}\left(T_{1}\right)>u_{i}(W)$. Since for voters $i$ from $S_{1} u_{i}\left(W_{1}\right)=$ $u_{i}(W)$, this is a contradiction, that shows that $W$ is in the core in the election instance $(P, l)$.

This completes the proof.
Since the core implies EJR and PJR (as showed in [11] and Theorem 3.3), any laminar profile also satisfies EJR and PJR under the same restrictions.

Corollary 4.2.1. In PB, laminar proportional committees satisfy EJR and PJR subject to $\mathscr{P}_{u-a f f o r d}$ -
Note that in multi-winner elections, where all candidates have the same cost, there always is a candidate in $T$ that has a cost that is at least the cost of $c$, so our restriction is always met. Hence in MWV, laminar committees are in the core, and therefore also satisfy EJR and PJR.
Corollary 4.2.2. In MWV, laminar proportionality implies PJR, EJR, and the core.
Proof. This follows directly from Theorem 4.2, Corollary 4.2.1, and the unit cost assumption.
Since Phragmén's rule satisfies laminar proportionality in MWV, on laminar election instances the outcome of Phragmén's rule will also satisfy the core (and hence also FJR) and EJR.

Corollary 4.2.3. In MWV, in laminar election instances, Phragmén's rule satisfies the core, FJR, and EJR.

### 4.4 Party-list profiles in MWV

Party-list profiles (Definition 2.12) are an interesting subset of approval profiles. Although they may be improbable to arise arbitrarily, their use may be natural in some situations. And because of their simplicity, axioms that in general are different turn out to be equivalent in these profiles.

Proposition 4.4. In MWV, in party-list profiles, priceability implies EJR.
Proof. Suppose we have a party-list election instance $E$ with a winning committee $W$ that is priceable, which is shown by the price system $\mathbf{p s}=\left(p,\left\langle p_{i}\right\rangle_{i \in N}\right)$. Suppose the group of voters $S \in N$ is $\ell$-cohesive: $|S| \geq \ell \cdot \frac{n}{k}$ and $\left|\cap_{i \in S} A_{i}\right| \geq \ell$. Note that if $\ell \neq 0$, this implies that all $i \in S$ vote for the same party, i.e. for all $i, j \in S, A_{i}=A_{j}$. Assume for a contradiction that for all voters $i \in S,\left|W \cap A_{i}\right|<\ell$. One cannot approve or elect negative or null amounts of projects, so $\ell$ must be greater than zero, so all voters in $S$ vote for the same party (have the same approval set). Now of all those projects that all voters in $S$ agree about $\cap_{i \in S} A_{i}$, there can only be less than $\ell$ in the winning committee $W$. Let's say there are $\ell-x$ projects from the set $\cap_{i \in S} A_{i}$ in the winning committee $W$, where $1 \leq x \leq \ell$. Then there are at least $x$ projects left over that all voters in $S$ agree on, but of which they cannot pay the price $p$ (because otherwise those projects would also be elected) so the voters in $S$ together do not have enough money left over to pay them: $x \cdot p>|S|-(\ell-x) \cdot p$. We also know that the size of $S$ is large enough to be $\ell$-cohesive: $|S| \geq \ell \cdot \frac{n}{k}$. We can now rewrite the former to obtain a restriction on the price $p$ :

$$
\begin{align*}
x \cdot p & >|S|-(\ell-x) \cdot p  \tag{21}\\
x \cdot p & >\ell \cdot \frac{n}{k}-(\ell-x) \cdot p  \tag{22}\\
0 & >\ell \cdot \frac{n}{k}-p \ell  \tag{23}\\
p \ell & >\ell \cdot \frac{n}{k}  \tag{24}\\
p & >\frac{n}{k} \tag{25}
\end{align*}
$$

If nobody from $S$ approves any of the other candidates in $W$, the other voters must pay fully for the other projects: $(k-(\ell-x)) \cdot p \leq n-|S|$, so $p \leq \frac{n-|S|}{k-\ell+x}$. Combining this with the knowledge that $p>\frac{n}{k}$ yields

$$
\begin{align*}
\frac{n}{k}<p & \leq \frac{n-|S|}{k-\ell+x}  \tag{26}\\
\frac{n(k-\ell+x)}{k} & <n-|S|  \tag{27}\\
(\ell-x) \frac{n}{k} & >|S| \geq \ell \frac{n}{k}  \tag{28}\\
\ell-x & >\ell, \tag{29}
\end{align*}
$$

which is a contradiction because $x$ is a positive number. Hence, someone from $S$ must also pay for some of the other candidates in $W$ (except from the candidates that were in $W$ and approved by all voters in $S$ ), but since voters can only pay for projects they approve, and a project approved by one voter in $S$ is approved by all voters in $S$, this is not possible.
Hence, in a party-list profile, priceable committees indeed satisfy EJR.
Proposition 4.5. In MWV, in party-list profiles, PJR and EJR are equivalent.

Proof. We already know that EJR implies PJR, so we only have to show that in party-list profiles, PJR implies EJR.
Suppose an elected committee $W$ in a party-list profile satisfies PJR. Take an arbitrary $\ell$ and $\ell$-cohesive group of voters $S$. Because $S$ is $\ell$-cohesive, we have that $\left|\cap_{i \in S} A_{i}\right| \geq \ell$, which implies if $\ell>0$ that all $i \in S$ vote for the same party, so for all $i, j \in S, A_{i}=A_{j}$. Then $\left|W \cap\left(\cup_{i \in S} A_{i}\right)\right|$ is the number of elected candidates from their party. According to PJR, this number is greater than or equal to $\ell$, so there is also some $i \in S$ for which $\left|W \cap A_{i}\right| \geq \ell$. In the situation that $\ell=0$, obviously $\left|W \cap A_{i}\right| \geq \ell$ is true for all $i \in S$. Hence, $W$ satisfies EJR.

Proposition 4.6. In $M W V$, in party-list profiles EJR implies laminar proportionality.
We will give a proof over the structure of laminar profiles. Note that party-list profiles are laminar by definition. We will give a definition of party-list profiles equivalent to Definition 2.12 that is similar to Definition 2.10, in order to make the proof more smooth. We will restrict the definition of laminar proportionality to party-list instances, which means that we will drop Definition 2.11.2. Furthermore, for convenience we will rewrite Definition 2.11] 3 such that $P$ is the sum of two or more unanimous instances instead of the sum of two laminar instances. Note that this does not change the meaning of the definition, it only takes a few of the inductive steps at once. Our new definitions of party-list profiles and laminar proportionality for party-list profiles are now as follows:

Definition 4.4 (Party-list election instances as laminar profiles). An election instance $(P, k)$ is a partylist instance if either:

1. $P$ is unanimous and $|C(P)| \geq k$.
2. There are two or more unanimous instances $\left\langle\left(P_{1}, k_{1}\right), \cdots,\left(P_{q}, k_{q}\right)\right\rangle(q \in \mathbb{N}>1$ is the number of parties) with $\left|C\left(P_{i}\right)\right| \geq k_{i}$ for all $i \leq q, C\left(P_{i}\right) \cap C\left(P_{j}\right)=\emptyset$ for all $i, j \leq q$, and $\frac{k_{1}}{\left|P_{1}\right|}=\cdots=\frac{k_{q}}{\left|P_{q}\right|}$ (we assume that there are no parties without voters, if they would exist we could ignore them) such that $P=P_{1}+\cdots+P_{q}$ and $k=k_{1}+\cdots+k_{q}$.

Definition 4.4 is equivalent to Definition 2.12, but written more like the definition of laminar election instances (Definition 2.10), with some inductive steps taken at once in part 2.

Definition 4.5 (Laminar proportionality for party-list profiles). A rule $\mathcal{R}$ satisfies laminar proportionality for party-list profiles if for every party-list election instance with ballot profile $P$ and committee size $k, \mathcal{R}((P, k))=W$ where $W$ is a laminar proportional committee, i.e.

1. If $P$ is unanimous, then $W \subseteq A_{i}$ for some $i \in N$ (if everyone agrees, then part of the candidates they agree on is chosen).
2. If $P$ is the sum of unanimous instances $\left\langle\left(P_{1}, k_{1}\right), \cdots,\left(P_{q}, k_{q}\right)\right\rangle$, then $W=W_{1} \cup \cdots \cup W_{q}$ where $W_{j}$ is laminar proportional for $\left(P_{j}, k_{j}\right)$.

Note that on party-list profiles Definition 4.5 is equivalent to Definition 2.11.
Proof of proposition 4.6. For unanimous instances $(P, k)$ that consist of one party (Definition 4.4, 1), the group of all voters $N(P)$ is $k$-cohesive (since $|N(P)|=k \cdot \frac{N(P)}{k}$ and all voters agree on all candidates), so if $W$ satisfies EJR in this instance, $\left|W \cap\left(\cup_{i \in N(P)} A_{i}\right)\right| \geq k$, hence $W \subseteq A_{i}$ for all $i \in N(P)$. Suppose that $P=P_{1}+\cdots+P_{q}$ and that $k=k_{1}+\cdots+k_{q}$ such that $\left(P_{1}, k_{1}\right), \cdots,\left(P_{q}, k_{q}\right)$ are unanimous instances as in Definition 4.4. 1 with $C\left(P_{i}\right) \cap C\left(P_{j}\right)=\emptyset$ for all $i, j \leq q$, and $\frac{k_{1}}{\left|P_{1}\right|}=\cdots=\frac{k_{q}}{\left|P_{q}\right|}$ (Definition
4.4 2). Also suppose that a committee $W$ which is elected in the instance $(P, k)$ satisfies EJR. Note that the voters within each party $P_{j}$ have the same approval sets, and the voters between parties have no overlap in approval sets. Since $|P|=\left|P_{1}\right|+\cdots+\left|P_{q}\right|, k=k_{1}+\cdots+k_{q}$, and $\frac{k_{1}}{\left|P_{1}\right|}=\cdots=\frac{k_{q}}{\mid P_{q}}$, we also have that $\frac{k_{1}}{\left|P_{1}\right|}=\cdots=\frac{k_{q}}{\left|P_{q}\right|}=\frac{k}{|P|}$. This implies that each party $P_{j}$ is $k_{j}$-cohesive: $\left|P_{j}\right|=k_{j} \cdot \frac{|P|}{k}$ and $\left|\cap_{i \in P_{j}} A_{i}\right| \geq k_{j}$. Since $W$ satisfies EJR, for each of these parties $P_{j}$ we have that there is a voter $i \in P_{j}$ such that $\left|W \cap A_{i}\right| \geq k_{j}$. Hence, $W$ contains at least $k_{j}$ candidates from party $P_{j}$. Since $|W|=k=k_{1}+\cdots+k_{q}, W$ contains exactly $k_{j}$ candidates from every party $P_{j}$. Call the $k_{j}$ candidates from party $P_{j} W_{j}$, so $W=W_{1} \cup \cdots \cup W_{q}$. For every instance ( $P_{j}, k_{j}$ ), the committee $W_{j}$ is clearly laminar proportional, since it is a unanimous instance and $W_{j} \subseteq A_{i}$ for all $i \in P_{j}$. Hence, in the instance $(P, k), W$ is laminar proportional.

From Proposition 4.6 and Theorem 2.2, the following corollary follows:
Corollary 4.2.4. In MWV, in party-list election instances, EJR implies priceability.
Theorem 4.3. In party-list profiles in MWV, EJR, PJR, laminar proportionality, priceability, and the core are equivalent.

Proof. This follows directly from Corollary 4.2.2, Propositions 4.4, 4.5, 4.6, and Corollary 4.2.4
Since PAV satisfies EJR [10] and PJR [13], Theorem 4.3 implies that in party-list profiles in MWV, the outcome of PAV also satisfies the core (and hence also FJR), priceability and laminar proportionality.

Corollary 4.3.1. In party-list profiles in MWV, PAV satisfies the core, FJR, priceability, and laminar proportionality.

The equivalence in Theorem 4.3 is only proven to hold with unit costs. Since it is a rather strong result, this raises the question whether the equivalence (or part of it) also holds in the PB situation.

Open Question 4.2. To what extend do the equivalence relations from Theorem 4.3 keep to hold in PB , that is, without the unit cost assumption?

# 5 Computational Experiments: The influence of polarisation on proportionality 

### 5.1 Introduction

In welfare-maximising voting systems, proportionality is not easily guaranteed. In a polarised setting with different groups of voters with different opinions, the largest groups will get their way, because the projects that they approve will give the maximal total utility. However, when the profile is less polarised, the opinions of different agents will have much more overlap, and therefore underrepresented minority groups will be rarer.
Although some welfare-maximising algorithms have proven to fail fairness axioms, the committees they calculate might in real life situations be not that unfair, if the profile is not polarised. After all, most fairness axioms only check whether a rule always outputs a fair committee, and not how often it would do so. Hence, we will use an experimental approach to study the influence of polarisation or clustering on how often several rules satisfy several axioms (with a focus on proportionality axioms). Since polarisation does not have one single definition, we will first need to define what we mean by polarisation. ${ }^{12}$
In this experiment we will use a simple and intuitive measure. We can argue that party-list profiles (Definition 2.12) are more polarised than random approval profiles. In particular, party-list profiles can be seen as strictly clustered profiles, while more random profiles are less clustered. We will therefore start from a party-list profile and adjust it according to two parameters: $j$ is the number of voters whose ballot will be altered, and $k$ is the number of votes that is adjusted for such a voter. A vote for a project is adjusted by changing it to 'non-approved' if the voter approved that project, and changing it to 'approved' if the voter did not approve that project. Hence, in total the resulting profile will differ by $j \times k$ votes from the original profile. Note that it is possible that the resulting profile is again a party-list profile or is closer to another party-list profile than to the party-list profile from which it originated. However, since our aim is only to perform a simple experiment and it is rather time and computation consuming to check this, we assume that with a large enough sample size this will not occur too often. For these to a greater or lesser extent party-list-like profiles, we will compute their performance on different propportionality axioms.
Note that in Section 4.4, we found that in party-list profiles in MWV, the core, EJR, PJR, laminar proportionality and priceability are equivalent (Theorem 4.3). We will take one of these axioms, namely the core, and furthermore the axiom of Nash welfare, to measure the proportionality of PAV and Phragmén's rule. The other axioms and rules are left as future work.

### 5.2 Methods

The experiment consists of a number of epochs, each of which consists of the following steps:

1. Generate a random party-list profile: pick a number of parties smaller than $m$, divide the projects and voters over these parties, a budget limit $l$ such that $m \leq l \leq m \cdot n$ in the PB setting and $l<m$

[^11]in the MWV setting, and generate a random cost function (which gives every project a cost of 1 in the MWV setting and a random cost below $l$ in the PB setting).
2. Adjust the party-list profile: pick $j$ voters arbitrarily and of each of those $j$ voters, change $k$ arbitrary votes.
3. Calculate the output committee $W$ of the current rule for that profile.
4. Check whether the winning committee $W$ satisfies Nash welfare and the core.

We perform the above mentioned steps for $m=10, n=10$ for every combination of $0 \leq j \leq n$ and $0 \leq k \leq m$ multiple times for PAV and Phragmén's rule. The number of epochs ( 1 epoch runs over all possible combinations of $j$ and $k$ so consists of 100 generated profiles) differs per combination of rule and axiom, due to computational resources and time constraints, and lies between the 118 and 500 for every combination of rule and axiom. Note that for either $j=0$ or $k=0$ the value of the other parameter does not make any difference, since no votes are changed and the profile stays party-list.

### 5.3 Results

In Table 12, the average percentage of profiles in which the rules gave an outcome that satisfied respectively the core and Nash is shown.

|  | PAV |  | Phragmén |  |
| :--- | :--- | :--- | :--- | :--- |
|  | MWV | PB | MWV | PB |
| core | 100 | 97.57 | 100 | 100 |
| Nash | 91.65 | 94.92 | 0 | 0 |

Table 12: Average percentage of satisfactions of the axioms for PAV and Phragmén's rule in MWV and PB.

The core Surprisingly, both the result of PAV and Phragmén's rule are in MWV always in the core in our experiment ( 140 epochs for PAV, 222 epochs for Phragmén's rule). In PB, Phragmén's rule still in all cases ( 118 epochs) returns an outcome that is in the core, however PAV does not. In Figure 8 the percentage of runs in which the PAV outcome satisfies the core for all combinations of $j$ and $k$ is displayed in a heatmap.

Nash welfare Phragmén's rule in our experiment never returns a committee that satisfies Nash welfare, neither in MWV nor in PB (both 500 epochs).
The committee for PAV on the other hand, does satisfy Nash welfare sometimes. In Figure 9, the percentage of Nash welfare satisfactions of the committee returned by PAV in MWV is displayed, and in Figure 10 the percentage in PB.

EJR and PJR For reference, we calculated how often the committee returned by Phragmén's rule in MWV satisfied EJR and PJR, and indeed, this outcome always satisfies EJR and PJR ( 500 epochs), as suggested by the fact that it always satisfies the core (and for PJR also by the theoretical results in [10]).


Figure 8: Percentage of times that the outcome of PAV satisfies the core in PB, for different values of $j$ and $k$.

### 5.4 Discussion

Although in [10] and in Proposition 2.1 we saw that PAV and Phragmén's rule do not necessarily return a committee in the core in MWV, and in Proposition 3.16 that Phragmén's rule does not necessarily have an outcome in the core in PB, apparently with such a small number of voters and projects these rules do satisfy the core very often. This suggests that the corresponding propositions do not reach the examples generated in the above way, and leads to Open Question 5.1.

Open Question 5.1. Is there a maximal number of voters or projects under which PAV and Phragmén's rule always return a committee in the core in MWV or Phragmén's rule returns a committee in the core in PB?

More specific, the result that PAV satisfies the core in MWV in $100 \%$ of the profiles in our experiment suggests that Corollary 4.3 .1 can be generalised even further, that not only party-list profiles but also profiles that are close enough to them satisfy the core.
In Corollary 4.2.3, we saw that in laminar election instances in MWV, Phragmén's rule satisfies the core. Hence, the score of $100 \%$ core committees of Phragmén's rule in MWV could point to a high percentage of laminar instances in the profiles in our experiment, which is left as a question for further research:

Open Question 5.2. Does the process of adjusting party-list profiles as described above generate laminar election instances with a high probability?

The fact that Phragmén's rule does not give an outcome that satisfies Nash welfare (neither in MWV, nor in PB) shows that Proposition 2.2 and 3.10 are provable via a large range of examples, not only the


Figure 9: Percentage of times that the outcome of PAV satisfies Nash welfare in the MWV setting for different values of $j$ and $k$.
counterexample given in their respective proofs. It raises the question whether (with certain settings) it is impossible for the outcome of Phragmén's rule to maximise the Nash product:

Open Question 5.3. In which settings (with which numbers of voters and projects) can Phragmén's rule return a committee that satisfies Nash welfare?

We have similar results for PAV regarding the core in PB compared to PAV regarding Nash welfare in MWV. We can observe a trend that with small numbers of $j$ and $k$, so with profiles that are closer to party-list profiles, the percentage of times PAV satisfies Nash in MWV and the percentage of times PAV satisfies the core in PB is less than with more arbitrary profiles. Interestingly, this result does not keep to hold for Nash in PB, as can be seen in Figure 10, where the percentage of satisfactions does not seem to exhibit any trend with respect to $j$ and $k$. The fact that in more random profiles PAV satisfies the core ( PB ) and Nash welfare (MWV) more than in profiles closer to party-list profiles confirms our hypothesis that in less polarised profiles, the outcome is more proportional. Hence, even though certain rules may not satisfy proportionality axioms in general, in more realistic situations their proportionality performance may not be very bad. A more concrete measure of the extent to which PAV behaves in more or less polarised profiles, and a more exact relation between $j, k$, and the percentage of times PAV satisfies the proportionality axioms is left as a question for further studies. Note that in all cases except that of Phragmén satisfying Nash welfare, the axiom (core resp. Nash welfare) is satisfied most of the times (more than $84 \%$ ). This and the fact that PAV in the MWV setting and Phragmén's rule both in the MWV and PB setting always satisfy the core in our simulation are interesting and hopeful observations: although they have been proven to not always return an outcome that is in the core, they seem to do return a core solution most of the time. This leaves us with the question whether we can define a restriction on the domain of profiles, in which they do always return


Figure 10: Percentage of times that the outcome of PAV satisfies Nash welfare in the PB setting for different values of $j$ and $k$.
a committee in the core, or in what type of profiles they do not return a committee in the core (Open Question 5.1). For all abovementioned open questions, a good starting point in finding an answer could be to run similar computational simulations with different numbers of voters $n$ and projects $m$ (note that we only regarded the case with $m=n=10$ ), with more well-considered cost functions and budget limits, and with more epochs.
Furthermore, it would be interesting to run similar experiments for SBA and Rule X , and also consider priceability and laminar proportionality. Since we have proven that in party-list instances, all proportionality axioms discussed except for Nash welfare are equivalent, we expect that the further a profile deviates from a party-list profile, the larger the difference in performance on the different axioms is.

## 6 Conclusion

In this thesis, we studied different proportionality axioms for computing fair committees in the setting of MWV, and fair budgets in the setting of PB. We considered the performance of different voting rules with regard to these axioms, and investigated the relations between the different axioms. In this way, we provided novel insights into the structure of the broad landscape of proportionality axioms in multi-winner elections and participatory budgeting. We first studied rules and axioms in MWV, and then generalised those results where possible to PB. We considered different restrictions to the domain of elections or to the definitions of axioms or rules, and showed that with these restricted settings some previously negative results, in specific many results in MWV, become positive. Finally, we performed a small computational experiment to study the proportionality performance of some rules on profiles with different degrees of polarisation, viewed as a random relaxation of the party-list restriction.

Summary of results Our main results are displayed in Table 13 and Figure 11. In Table 10, we can see which rules satisfy which proportionality axioms, some of which only hold under certain restrictions as is indicated in the table. In Figure 11, which was already given in the introductory

|  | SBA |  | PAV |  |  |  | Phragmén |  |  |  | Rule X |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | PB |  | MWV |  | PB |  | MWV |  | PB |  | MWV | PB |  |  |
| core | $X$ (Prop. | 3.1 | $\checkmark$ in party-list p. (Cor. | 4.3.1) | $X$ (Prop. | 3.16 | $\checkmark$ in laminar p. (Cor. | 4.2.3 | $X$ (Prop. 3 | 3.16 | $\checkmark$ if $\mathcal{P}_{\text {price-eq }}$ [10] | $\checkmark$ if $\mathcal{P}_{\text {eq-pr }}$ | price/u |  |
| EJR | X Prop. | 3.2 | $\checkmark$ in party-list p. (Cor. | 4.3.1) | $x[11]$ |  | $\checkmark$ in laminar p. (Cor. | 4.2.3 | $x$ (Prop. | 3.16 | $\checkmark 10$ | $\checkmark \sqrt{11}$ |  |  |
| PJR | X(Prop. | 3.3) | $\sqrt{13}$ |  | $x$ (Prop. | 3.8 | $\checkmark 10$ |  | $\checkmark$ (Prop. | 3.12 | $\checkmark \boxed{10}$ | $\checkmark$ (Prop. | 3.14 |  |
| priceability | $X$ (Prop. | 3.4 | $\checkmark$ in party-list p. (Cor. | 4.3.1) | $X$ (Prop. | 3.16 | $\checkmark 10$ |  | $\checkmark$ (Prop. | 3.11 | $\checkmark \boxed{10}$ | $\checkmark \boxed{11}$ |  |  |
| lam. prop. | $X$ (Prop. | 3.5 | $\checkmark$ in party-list p. (Cor. | 4.3.1) | $X$ (Prop. | 3.16 | $\checkmark 10$ |  | $X$ (Prop. 3 | 3.13 | $\checkmark 10$ | $X$ (Prop. 3 | 3.13 |  |
| Nash | $X$ (Prop. | 3.6 | $X$ (Prop. 2.4) |  | $x$ (Prop. | 3.9 | $X$ (Prop. 2.2) |  | $X$ (Prop. 3 | 3.10 | X (Prop. 2.4) | $X$ (Prop. 3 | 3.15 |  |
| FJR | $X$ (Prop. | 3.7 | $\checkmark$ in party-list p. (Cor. | 4.3.1) | $x[11$ |  | $\checkmark$ in laminar p. (Cor. | 4.2.3 | x (Prop. | 3.16 | $x[11]$ | $x[11]$ |  |  |

Table 13: Different rules and the properties they satisfy, purple entries in the table indicate results from the literature, green entries indicate new results. References to propositions or literature are included for each entry.
chapter but is repeated here for ease of reading, all relations between the fairness axioms we studied are shown. The arrows indicate implications, some of which only hold under certain restrictions, which are written along the arrows. Each implication has a reference to either the literature or a theorem in this paper, where the result is shown. Since in party-list election instances (Section 4.4) all axioms are equivalent, all axioms imply all other axioms under that restriction. To enhance readability we excluded these implications from the figure.
How can we fit these technical results in our intuition about proportionality in MWV and PB? Can we translate our technical landscape of definitions to a more semantic one? First of all, a clear observation is that laminar proportionality implies the other axioms in MWV, and that in PB it implies priceability and under some constraints also the core, EJR, and PJR. This shows that laminar proportionality is a relatively strong proportionality axiom. However, note that it is only defined for laminar instances, which are of a specific type. In such nicely structured instances, it is intuitively clear how to divide the committee members or elected projects over the parties or sub-parties in a fair way, and this intuitive allocation is exactly the one that laminar proportionality requires. It is not surprising that if we can divide the projects or candidates neatly over pre-defined groups, the other proportionality axioms also hold.


Figure 11: The relations between laminar proportionality, priceability, PJR, EJR, and the core in the multi-winner voting setting (red) and in the participatory budgeting setting (blue), including the relations in certain restricted domains.

In more arbitrary election instances (where laminar proportionality is out of the picture), the concept of priceability has little connection to the concepts of the core, EJR, and PJR. Although in MWV, priceability still implies the least strict of the three, PJR, in PB even this connection is lost. One of the reasons for this lack of correspondence between the axioms is that in PJR, EJR, and the core, the exact utilities of the voters are taken into consideration, while in priceability only a difference is made between zero and non-zero utilities. However, we have the conjecture that when the utilities would be taken into consideration in priceability, there might be a connection to the 'JR'-axioms. In Proposition 4.2, we showed that Rule X , a rule designed to be priceable, satisfies the core subject to priceability with equal rations of price/utility. Hence, if we would redefine priceability so that a voter's payment is proportional to her gained utility, this might be a proportionality axiom that is even stronger than the core, EJR, and PJR. This would however be an axiom even more demanding than priceability, so the question remains how often it would be satisfied in real situations. This, as well as the exact definition and proofs of potential relations to other axioms we leave as an open question:

Open Question 6.1. Can we define a version of priceability that takes into account the level of utility of the voters for the projects, and if so, what is its relation to the other proportionality axioms?

Although a lot of the axioms are not satisfied by the rules we studied (see Table 10), we saw in Chapter 5 that in many cases, the output of a rule will satisfy the fairness axioms even though the rule is proven to not always return an outcome that satisfies the axiom. It remains as a question for further research how often and to what extent these rules that do not strictly satisfy an axiom will nevertheless return a committee that does satisfy the axiom, but this is a hopeful message. After all, the rules we studied turn out to be not that 'disproportional'.

Future work In Sections 2.4 and 3.4, we already mentioned some interesting topics for further studies in this field. Because the PB is still a relatively small topic in the literature of computational social choice, there is a broad range of questions that are still open to be investigated. We will mention a few of the open problems that we encountered in this project.

- To be able to have a laminar proportional committee, the election instance needs to be laminar. We can argue that in the way politics is structured, election instances are in some sense likely to be laminar. However, this is still an intuitive conjecture. How likely are laminar election instances to occur, either when we look at real election data or when we look at election data sampled from certain realistic distributions?
- Another rule that is claimed to be proportional is minimal transfers over costs (MTC), introduced in [28], as a cumulative version of the single transferable vote (STV). Since the proportionality notions used in [28] differ from the ones we studied, we did not include it in our study. Nevertheless, since the authors claim it to satisfy a strong notion of proportionality, an interesting question is whether MTC satisfies the proportionality axioms studied here.
- We can to a larger extent discuss the practical meaning of the different types of proportionality. An example of this is Open Question 4.1. is Nash welfare actually a good axiom for minority groups? Or is it only favouring separate voters with a low utility?
- We showed in section 4.4 that in MWV, in party-list instances all discussed fairness axioms are equivalent. A remaining question (Open Question 4.2) is whether this equivalence still holds in the PB setting, i.e. when projects can have different costs. Also: are there other, maybe less strict, restrictions in which the axioms are equivalent?
- We can add a lot of different nuances to the PB setting. How do the fairness axioms behave when we add diversity constraints [29; 30; 31], project groups [32], the possibility to express negative feelings [33], or several different resource types [34]?
- Although SBA is not properly defined for approval voting and therefore we only studied it in the PB setting, it would be interesting to see which axioms it satisfies if the unit cost assumption holds (so in a kind of MWV with ordinal ballots).
- The results of our computational simulation in Chapter 5 raise quite some questions for further research: Open Questions 5.1, 5.2, and 5.3, as well as the question what further simulations with different numbers of voters $n$ and projects $m$, with different cost functions and budget limits, with more epochs, with axioms like priceability and laminar proportionality, and with SBA and Rule X will reveal.
- Can we define overarching proportionality axioms like the one suggested in Open Question 6.1, and if so, are they still satisfiable by reasonable rules?

We hope to contribute to the study of these questions in future work.

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## Appendix

## A Non-proportionality axioms that PAV, Phragmén's rule and Rule $X$ satisfy

Except from just being fair, there are a lot of other desirable properties a committee or budget can have. In this section, we will shortly mention a few of those and show whether or not the rules discussed in this thesis satisfy them or not.

## A. 1 Axioms

One very intuitive requirement is that of increasing the total utility. We want rules to select committees that increase the utility, or happiness, of the voters as much as possible. Even when the utility functions of all voters are given, the total welfare can be measured in different ways. When there is some function of the utility vectors of the voters that is maximised by a rule, we call it a welfarist rule:

Definition A. 1 (Welfarist rules for MWV [10]). A rule $\mathcal{R}$ is called welfarist if for each $k$ there is a function $g_{k}$ mapping welfare vectors $w_{E}(\mathcal{R}(E))=\left(\left|A_{1} \cap \mathcal{R}(E)\right|, \ldots,\left|A_{n} \cap \mathcal{R}(E)\right|\right)$ to real values such that for each election instance $E$ with committee size $k$ we have:

$$
\mathcal{R}(E, k)=\operatorname{argmax}_{W \subseteq C:|W|=k} g_{k}\left(w_{E}(W)\right) .
$$

In many situations, we do not want there to be money left over that is not spend while there are still affordable candidates that are not elected. Hence, the axiom of exhaustiveness is defined:

Definition A. 2 (Exhaustiveness for MWV). A rule $\mathcal{R}$ is called exhaustive if for every election instance $E$ with committee size $k$, it holds that $|\mathcal{R}(E)|=k$.

Another important axiom worth mentioning is that of strategy-proofness, which requires that it is not lucrative for voters to misreport their preferences.

Definition A. 3 (Strategy-proofness in MWV). A rule $\mathcal{R}$ is strategy-proof if no agent has any incentive to misrepresent his approval set: for all election instances $E$, for all $i \in N$, the elected committee $W$ should satisfy $\left|W \cap A_{i}\right|>\left|W^{\prime} \cap A_{i}\right|$ where $W^{\prime}$ is the outcome of the rule for an election instance $E^{\prime}$ is equal to $E$ except for the approval set of agent $i$, which is true in $E$ but misrepresented in $E^{\prime}$.

The concepts of Welfarist rules, exhaustiveness, and strategyproofness can trivially be extended to the PB situation, by replacing the total committee size $k$ by a budget limit $l$ and the cost of a project from 1 to its actual cost, and measuring the utility an agent gets from a set of elected projects by the sum of the actual utilities of each elected project, instead of by the number of approved elected projects.

Definition A. 4 (Welfarist rules for PB). A rule $\mathcal{R}$ is called welfarist if for each budget limit $l$ there is a function $g_{l}$ mapping welfare vectors $w_{E}(\mathcal{R}(E))=\left(u_{1}(\mathcal{R}(E)), \ldots, u_{n}(\mathcal{R}(E))\right)$ to real values such that for each election instance $E$ with budget limit $l$ we have:

$$
\mathcal{R}(E, l)=\operatorname{argmax}_{W \subseteq C: \operatorname{cost}(W) \leq l} g_{l}\left(w_{E}(W)\right) .
$$

Definition A. 5 (Exhaustiveness for PB [7]). A rule $\mathcal{R}$ is called exhaustive if for every election instance $E$ with budget limit $l$, it holds that for each $c \notin \mathcal{R}(E), \operatorname{cost}(\mathcal{R}(E) \cup c)>l$.

Definition A. 6 (Strategy-proofness for PB ). A rule $\mathcal{R}$ is strategy-proof if no agent has any incentive to misrepresent his utilities: for all election instances $E$, for all $i \in N, u_{i}(\mathcal{R}(E))>u_{i}\left(\mathcal{R}\left(E^{\prime}\right)\right)$ where $E^{\prime}$ is equal to $E$ except for the utilities of agent $i$, which are true in $E$ but misrepresented in $E^{\prime}$.

Some (negative) relations between these axioms and the proportionality axioms in the main part of this thesis have been shown in the literature. Peters [22] shows that proportional rules (already in a weak sense of proportionality) are not strategy-proof. In [10], it is shown that in MWV, welfarist rules are not priceable, not laminar proportional and not in the core. Also, priceability is incompatible with exhaustiveness [11].

## A. 2 Properties of rules

In Table 14, we show whether or not SBA, PAV, Phragmén's rule and Rule X satisfy these three axioms, both in the MWV and the PB situation. References of explanations and proofs are given for each entry in the table, the corresponding propositions are given below.

|  | SBA | PAV |  | Phragmén |  | Rule X |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | PB | MWV | PB | MWV | PB | MWV | PB |
| welfarist | $\checkmark$ [13] | $\checkmark$ [10] | $\checkmark$ (Prop. A.6) | $X$ [10] | $X$ (Prop. A.7) | $X$ [10] | X(Prop. A.7) |
| exhaustiveness | $\checkmark$ (Prop. A.3) | $\checkmark$ (Prop. A.1) | $\checkmark$ (Prop. A.5) | $x[10]$ | $x$ (Prop. A.7) | $x$ [1] | $x$ [1] |
| strategy-proof | $x$ (Prop. A.4) | $x$ (Prop. A.2) | $x$ (Prop. A.7) | $x$ (Prop. A. 2 ) | $x$ (Prop. A.7) | $x$ (Prop. A. 2 ) | $x$ (Prop. A.7) |

Table 14: Results for PAV, Phragmén's rule, and Rule X that are not directly related to proportionality. Purple entries in the table indicate results from the literature, green entries indicate new results.

Proposition A.1. In $M W V, P A V$ is exhaustive.
Proof. PAV is exhaustive in PB (see proposition A.5), so since MWV is a generalisation of PB it is also exhaustive in MWV.

Proposition A.2. In MWV, PAV, Phragmén, and Rule $X$ are not strategyproof.
Proof. These three rules are proportional approval based rules, and as [22] shows, no proportional approval based multi-winner voting rule is strategy proof

## Proposition A.3. SBA is exhaustive.

Proof. In SBA, when all the projects are ranked in the Ranking procedure, for every group of equally ranked projects, a maximal subset such that the total cost is not exceeding the limit is added to the final outcome in the Pruning procedure. Hence, there are no projects that still fit into the budget but are not selected, for if this would be the case, the subset selected would not be maximal. Therefore, SBA is an exhaustive algorithm.

Proposition A.4. SBA fails strategyproofness.

Proof. Because SBA is based on the majority graph, in which there need not be a Condorcet winner (explain why that matters), the outcome of the SBA may depend solely on the previous budget (tiebreaking), which makes SBA non-strategy-proof. Take as a counterexample a situation with three voters, $V=\left\{v_{1}, v_{2}, v_{3}\right\}$, and three alternatives $C=\{a, b, c\}$, with equal costs such that only one of the alternatives can be selected. Assume that the previous budget was $\{a\}$ and that the preferences of the voters are as follows: $v_{1}: a \succ b \succ c, v_{2}: b \succ c \succ a$, and $v_{3}: c \succ a \succ b$. Then $a$ wins from $b, b$ wins from $c$, and $c$ wins from $a$, and the first Schwartz set consists of $a, b$, and $c$ together. The outcome of the SBA will therefore depend on the previous budget $\{a\}$. For voter $v_{2}$, it is better to submit the untruthful ballot $v_{2} *: c \succ b \succ a$ such that $c$ will win from $b$, the first Schwartz set consists of only $c$, and $c$ is selected by the SBA procedure. Hence, SBA is not strategy-proof.

Proposition A.5. In PB, PAV is exhaustive.
Proof. In Proportional Approval Voting, a committee is selected such that the PAV-score

$$
\operatorname{PAV}-\operatorname{score}(W)=\sum_{i \in N}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left|W \cap A_{i}\right|}\right)
$$

is maximised. Suppose that PAV is not exhaustive, and that in some scenario it selects a committee $W$ such that there is a non-selected candidate $c \notin W$ with $\operatorname{cost}(W \cup\{c\})<=1$. Then if $c$ is not approved by any voter, the final terms of the PAV-score are equal for $W$ and $W \cup\{c\}: \frac{1}{\left|W \cap A_{i}\right|}=\frac{1}{\left|(W \cup\{c\}) \cap A_{i}\right|}$ for every voter $i$, so PAV-score $(W)=\operatorname{PAV}$-score $(W \cup\{c\})$. If $c$ is approved by one or more voters, then $\left|W \cap A_{i}\right|<\left|(W \cup\{c\}) \cap A_{i}\right|$ for those voters (and they are equal for all other voters). Hence PAV-$\operatorname{score}(W)<\operatorname{PAV}$-score $(W \cup\{c\})$, which is a contradiction because $W$ was the outcome selected by PAV. This shows that under the assumption that every candidate is approved by at least one voter, PAV is exhaustive (and without this assumption, it can be made exhaustive by using the right type of tie-breaking).

Proposition A.6. In PB, PAV is welfarist.
Proof. In the PB setting, PAV still maximises the score PAV-score $(W)=\sum_{i \in N}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left|W \cap A_{i}\right|}\right)$. Under the assumption that in approval voting a voter's utility for a project is 1 if she approves that project and 0 otherwise, the welfare vector for election $E$ is given as $w_{E}(\mathcal{R}(E))=(\mathcal{R}(E) \cap$ $\left.A_{1}, \ldots, \mathcal{R}(E) \cap A_{n}\right)$. Then the welfare function $g\left(w_{E}(\mathcal{R}(E))\right)=\operatorname{PAV}$-score $(\mathcal{R}(E))$ clearly is a function mapping the welfare vectors to real values, that is maximised by the winning committee $\mathcal{R}(E)$.

Proposition A.7. In PB, the PAV outcome is not necessarily strategy-proof, Phragmén's rule is not welfarist, not exhaustive and not strategy-proof, and Rule X is not welfarist or strategy-proof.

Proof. These negative results can be derived in two steps: the corresponding result in MWV is negative and the PB versions of the axiom and rule are proper generalisations from MWV to PB, just like in Proposition 3.16.


[^0]:    ${ }^{1}$ Phragmén (and later researchers) constructed many versions of his rule, but the version that is commonly named 'Phragmén's rule', which is the version that we are using here, is his sequential rule.

[^1]:    ${ }^{2}$ Note that although the core is the most strict axiom of this kind, even committees that are intuitively not fair can be in the core, as is shown in [12].

[^2]:    ${ }^{3}$ In the original definition in [10], this requirement is written as $\frac{\left|P_{1}\right|}{k_{1}}=\frac{\left|P_{2}\right|}{k_{2}}$. However, in that case $k$ is not allowed to be zero. We do not often consider empty committees, nevertheless we will need the possibility for $k$ to be zero in some cases to build a laminar election instance.

[^3]:    ${ }^{4}$ In fact, the Nash score is a fairly close approximation of the PAV score, as noticed in the proof of Proposition 2.4 and in [4].

[^4]:    ${ }^{5}$ A Schwartz component $X$ of a graph $G$ with vertices $A$ is a minimal set of vertices such that for any $b \in A \backslash X$ there is no $a \in X$ such that there is an arc in $G$ from $b$ to $a$. The Schwartz set is the union of all Schwartz components.

[^5]:    ${ }^{6}$ In the original definition, there is no constraint that $\beta \geq 0$, but as for negative $\beta$ every $S$ is $(\beta, T)$-cohesive for any $T$, we added it.

[^6]:    ${ }^{7}$ In the original definition in [11], there is an alternative condition to simplify the reasoning for rules that do not necessarily spend the total budget. This extra condition relaxes EJR to hold up to one project, and allows a rule to also satisfy EJR if for some $a \in C$ it holds that $u_{i}(\mathcal{R}(E) \cup\{a\})>\sum_{c \in T} \alpha(c)$.

[^7]:    ${ }^{8} k$ has to be integer because we cannot elect half candidates. However in settings where projects can be partially budgeted, this can be an interesting option.

[^8]:    ${ }^{9}$ For $n=2$ the argument still works, the Schwartz set will consist of $b$ and $w$ together but because the algorithm chooses the maximal subset of the Schwartz set, $w$ will still be chosen first.

[^9]:    ${ }^{10}$ Just for clarity, if there are more projects from $T$ not yet elected, we get that $x \leq \operatorname{cost}(T) \cdot \frac{n}{|S|}$, so $x \leq 1$ still holds.

[^10]:    ${ }^{11}$ As mentioned the definition of [11], there is a weakening condition for situations where a rule does not utilise the whole budget, EJR up to one budget, where the definition is extended by "or for some $a \in C$ it holds that $u_{i}(\mathcal{R}(E) \cap\{a\})>$ $\sum_{c \in T} \alpha(c)$ ". We could alter our definition of PB-PJR in a similar way, and it is easy to show that this weakened version of PB-PJR is implied bu the weak version of EJR.

[^11]:    ${ }^{12}$ Polarisation can be based on the content of the policies of different political parties, or based on the distribution of voters' preferences [26]. In [26], a manner to measure polarisation capturing both notions is defined, however the paper only deals with a system with two candidates (Republican or Democrat), and makes use of real election data, to which we do not have access.

    We could also base our polarisation measure on the tendency of voters to form clusters, and use for example the Hopkins statistic [27], which is a statistical test that measures how likely the data is to contain clusters against it being uniformly randomly distributed (generated by a Poisson point process).

