university of groningen / faculty of science and engineering / mathematics and applied mathematics

Necessary and sufficient conditions for impulse controllability of switched DAEs for unknown switching signals

Bachelor's Project Applied Mathematics

July 2021

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Abstract: This thesis aims to provide necessary and sufficient conditions for impulse controllability of switched Differential algebraic equations (DAEs) when the switching signal is unknown. Several necessary and sufficient conditions for impulse controllability have been derived in the literature, under the supposition that the switching signal is fixed using geometric control theory. This thesis generalises these conditions such that they ensure impulse controllability when a switching signal has not been specified. Firstly, regular DAEs will be analysed through Wong sequences and the quasi-Weierstrass form. Secondly, a different solutional framework called the piecewise-smooth distributions will be introduced. Thirdly, some geometric notations regarding DAEs will be briefly covered. Afterwards, switched DAEs will be introduced formally and several notions for impulse controllability under unknown switching signals will be introduced. The results will firstly be derived from switched DAEs with 2-modes before generalising this to switched DAEs with $p+1$-modes.

Contents

1 Introduction

In practice, many physical applications can be modelled through systems of ODEs. These are derived from the combination of physical laws and conservation laws. For example, the motion of any object in three dimensions can be modelled through its kinematics subject to the law of conservation of energy, or an electrical circuit with an arbitrary amount of components can be modelled through the governing physical laws of the components coupled with Kirchhoff's laws*.

Moreover, Kirchhoff's laws contain no differential operator. Such laws can mathematically be interpreted as **algebraic constraints**. In the case of Kirchhoff's laws, the algebraic constraint can easily be solved and hence, one variable can be eliminated to obtain an ODE description of the system. From a mathematical viewpoint, algebraic constraints do not necessarily have to be solvable. If it is desired to study a set of differential equations coupled with an unsolvable algebraic constraint, then a system of ODE's will not accurately describe the system's behaviour over time.

This can be solved by modelling this set of differential equations coupled with an algebraic constraint as a **differential-algebraic-equation**, or **DAE** in short. This is obtained by incorporating the algebraic constraint in the system formulation. For example, consider the following simple electrical system:

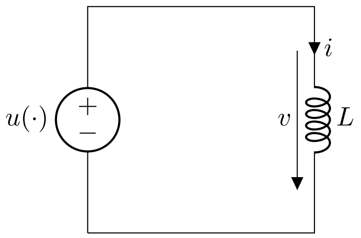
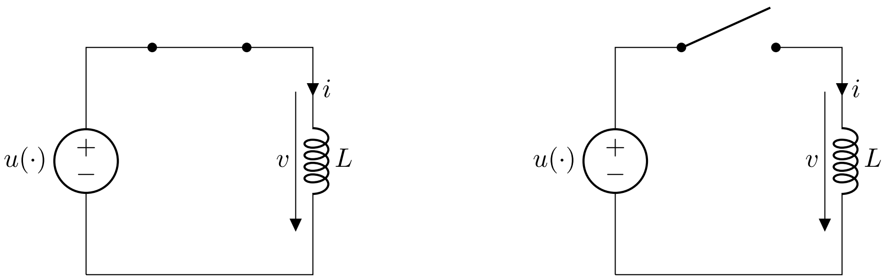


Figure 1.1: A simple electrical circuit consisting of an inductor and a voltage source.

Let $u(t)$ denote the output of the voltage source, i the current over the inductor L and v the voltage over the inductor L. The dynamics of the inductor can be modelled by $\frac{\mathrm{d}}{\mathrm{d}t}i = \frac{1}{L}v$ and Kirchhoff's second law states that $v - u = 0$. The ODE description of the system can be obtained by eliminating the algebraic constraint, giving the following ODE $\frac{\mathrm{d}}{\mathrm{d}t}i = \frac{1}{L}u$. To obtain the DAE description, let $x = \left[\, i \;\, v \,\right]^\top$, then if one writes the dynamics of the inductor combined with Kirchhoff's law in system formulation one obtains the DAE description:

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \tag{‡}$$

Observe that in the example above Kirchhoff's second law remains applicable for all time the circuit is active, since the circuit is closed for all time. However, what if one models a circuit that will not remain closed for all time? Mathematically, this corresponds to a sudden change in the algebraic constraint. In the following example it will be demonstrated that several DAE descriptions are required to model such electrical circuits:



$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

Figure 1.2: Electrical system, consisting of an inductor L connected to a voltage source $u(\cdot)$ with a switch. The circuit on the left and right will be referred to as mode 0 and mode 1 respectively.

As can be observed, both **modes** of the circuit are modelled using different DAE descriptions. Combining these descriptions yields a **switched DAE** description of the electrical circuit. Both DAEs describe different **modes** of the electric circuit, either the loop is closed or not.

*In particular, Kirchhoff's second law states that the voltage sum around a closed loop equals 0, which is a rephrasing of the law of conservation of energy

Without performing a thorough analysis, several observations can be made regarding this electrical circuit. Again, let the closed circuit be mode 0 and the open circuit be mode 1. Suppose mode 0 is active on $(-\infty, 0)$ and mode 1 active on $[0, \infty)$. If the switch from mode 0 to mode 1 occurs at $t = 0$, observe that the current drops to zero, therefore experiencing a jump discontinuity. However, $v = \frac{1}{L}\frac{\mathrm{d}}{\mathrm{d}t}i$. What phenomena will v experience at $t = 0$? From electronics it is well known that a **spark** or **impulse** can possibly jump across the switch, possibly damaging the electrical components.

Sparks are not a consequence of the loop structure changing from closed to open, but rather, are a consequence from a jump in the current. However, this is not necessary. Consider the following circuit:

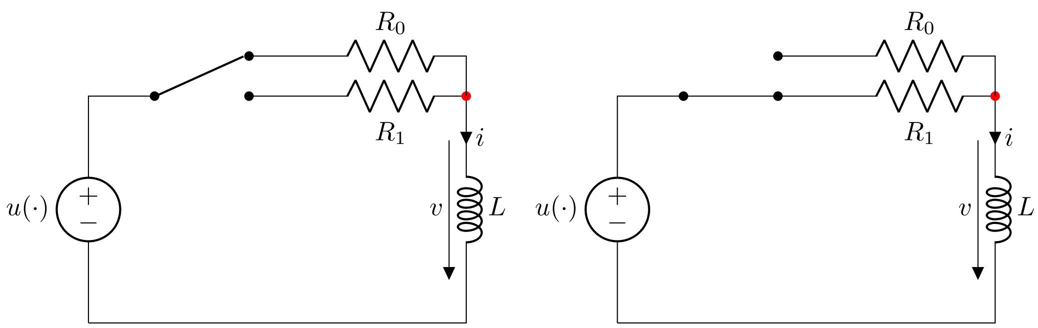


Figure 1.3: Electrical system where the switch induces an impulse in the voltage accross the inductor L. The circuit on the left and right will be referred to as mode 0 and 1, respectively.

Assume that $R_0 \neq R_1$. Let the system on the left be mode 0 and the system on the right be mode 1. Using Ohm's law, one is able to relate the current across the inductor with the input. Kirchhoff's first law states that the current that flows out of the red node is equal to the current flowing out of one of the resistors. To make this more rigorous, suppose the switching happens at $\tau \in (0, \infty)$. Kirchhoffs second law states:

$$\begin{aligned} R_0 i(t) + v(t) - u(t) = 0, \quad t \in [0, \tau) \\ R_1 i(t) + v(t) - u(t) = 0, \quad t \in [\tau, \infty) \end{aligned}$$

This constraint can be solved explicitly this in terms of one of the state variables, namely the current. Hence, the current flowing out of the red node:

$$i(t) = \begin{cases} (u(t) - v(t))/R_0 & t \in [0, \tau) \\ (u(t) - v(t)/R_1 & t \in [\tau, \infty) \end{cases} \qquad (\diamond)$$

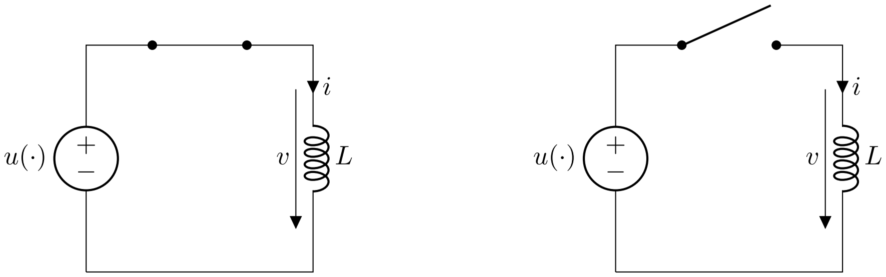
Again, the dynamics of the inductor can be modelled by $v = L\frac{\mathrm{d}}{\mathrm{d}t}i$. Let $x = [\, i \; v \,]^\top$. Hence, the system formulation is given as follows:

$$\begin{aligned} \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ R_0 & 1 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad t \in [0, \tau) \\ \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ R_1 & 1 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad t \in [\tau, \infty) \end{aligned}$$

Again, intuition seems to suggest that if the current experiences a jump discontinuity, one would expect a Dirac impulse to occur in the voltage. However, it will be shown later in this thesis that for this specific circuit no impulses can occur.

The previous examples have shown that sparks occur as a consequence of a derivative of a jump discontinuity. But how does one express them with proper mathematical rigor? As can be observed through the examples, the classical solution framework of ODE's do not well define the time derivative of a jump discontinuity. Therefore, one has to extend the solutional framework and the theory of **distributions** or **generalized functions** will be explored in order to find a suitable space of distributions one can use.

Impulses have the ability to damage electrical components, and thus, should be prevented. In the aforementioned examples, one can find inputs that prevent the impulses from happening. Consider the first circuit again:



Consider mode 0 on the interval $[0, \tau)$ and consider mode 1 on the interval $[\tau, \infty)$. Let $u(t) = 2t - \tau$ on $[0, \tau)$ and $u(t) = 0$ else. Applying Kirchhoff's second law yields that $v(t) = 2t - \tau$, and since $\frac{1}{L}v(t) = \frac{\mathrm{d}}{\mathrm{d}t}i$:

$$i(t) = \tfrac{1}{L}\int_0^t v(s)\mathrm{d}s = \tfrac{t}{L}(t - \tau)$$

Hence, one has that $i(\tau-) = 0$. Since the current will be zero at $t = \tau$, one has that the current experiences no jump discontinuity, and hence, v doesn't experience an impulse.

As observed in the aforementioned examples, impulses can occur but can also be prevented. The big problem with this approach is that both inputs explicitly require that the switching time τ is known. However, in many practical applications this is not necessarily known, and therefore impulse controllability in terms of unknown switching times has to be investigated.

This thesis will focus on deriving necessary and sufficient conditions for preventing impulses in the solution of a switched DAE using geometric control theory, as can be seen in [1], [4], [6]. The approach to the preliminaries will be similar as was done in [4]. Firstly, in section 2 several analytic notions will be explored for DAEs using a sequence of subspaces. In section 3 the distributional framework will be explored and a suitable space of distributions will be setup. Furthermore, in section 3 several geometric notions regarding DAEs will be introduced that will be of use for the analysis and switched DAEs will be introduced formally along with defining impulse controllability for fixed and unknown switching behaviours. Finally, section 4 will deal with the analysis of a switched DAE for the 2-mode case and $(p + 1)$-mode case.

2 Regular Linear descriptor systems

2.1 Introduction

DAEs, or **linear descriptor systems**, have to be properly studied before studying switched linear descriptor systems. Therefore, the homogeneous case will only be considered for now, i.e.

$$E\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n \tag{2.1}$$

Where $E, A \in \mathbb{R}^{n\times n}$ and in general, E is assumed to be singular. These systems can simply be identified by their matrix 2-tuple (E, A). The difference between a DAE and an ODE is the fact that DAEs can contain unsolvable algebraic constraints, and therefore, makes it harder to solve. For an ODE, one should take note that E is non-singular, and can then therefore be inverted. It is feasible to explore if these parts can be decoupled, as this will simplify the analysis significantly.

This desired decoupling can be achieved by transforming (E, A) into the *Quasi-Weierstrass form*, under the supposition that (E, A) is *regular*. This is defined as follows:

Definition 1. *Let $E, A \in \mathbb{R}^{n\times n}$. It is said that the matrix pair (E, A) is regular if, and only if, $(\det(sA - B))$ is not the zero polynomial.*

The Quasi-Weierstrass form can explicitly be calculated using the **Wong sequences**, and will be explored in the next subsection.

2.2 Wong Sequences

Consider the DAE as in (2.1) and assume $x(t)$ is a differentiable solution. Consider the linear subspace $\mathcal{V}_0 = \mathbb{R}^n$. If $x(t) \in \mathcal{V}_0$ for all $t \in \mathbb{R}$, then also $\dot{x}(t) \in \mathcal{V}_0$, since any linear subspace of a finite dimensional vector space is closed under differentiation with the standard Euclidean topology.

Observe that using (2.1) and that $\dot{x} \in \mathcal{V}_0$, one consequently has $x(t) \in A^{-1}E\mathcal{V}_0 =: \mathcal{V}_1$. Observe that this reasoning can be applied inductively, giving $\mathcal{V}_{i+1} = A^{-1}(E\mathcal{V}_i)$ and one therefore obtains a sequence of subspaces $\mathcal{V}_0, \mathcal{V}_1, \ldots$ in which the solution $x(t)$ lies. Additionally, the observation shows that $\mathcal{V}_0 \subseteq \mathcal{V}_1$, therefore by induction it can be shown that $\{\mathcal{V}_i\}_{i\in\mathbb{N}}$ is a nested decreasing sequence of subspaces, i.e. $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \ldots \subseteq \mathcal{V}_i \subseteq \mathcal{V}_{i+1}$.

Since $\mathcal{V}_i \subseteq \mathbb{R}^n$ for all $i \in \mathbb{N}$, one has that $\dim(\mathcal{V}_i) \leqslant n < \infty$ for all $i \in \mathbb{N}$. The dimension of $\mathcal{V}_i$ can only decrease in the first n-steps. Thus, after at most n-steps, one has that $\mathcal{V}_{n+1} = \mathcal{V}_n$, which shows that the sequence $\{\mathcal{V}_i\}_{i\in\mathbb{N}}$ will terminate in at most n steps. However, it could be that the sequence terminates in less than n-steps. Therefore, let the terminating index k^* denote the first value for which $\mathcal{V}_{k^*} = \mathcal{V}_{k^*+1}$ holds. The limiting subspace will be denoted by $\mathcal{V}^*$. Since $x(t)$ lies in every subspace of the sequence, one must have that $x(t) \in \mathcal{V}^*$ for all $t > 0$.

It seems feasible to be able to decompose $\mathbb{R}^n$ as the direct sum of $\mathcal{V}^*$ and some other subspace. In order to find this other subspace, one can define a similar sequence of subspaces, namely $\mathcal{W}_{i+1} = E^{-1}(A\mathcal{W}_i)$ with $\mathcal{W}_0 = \{0\}$, which is a nested increasing sequence of subspaces. By similar reasoning, this will terminate in at most n steps and the limiting subspace will be denoted by $\mathcal{W}^*$.

If we want to show that $\mathbb{R}^n = \mathcal{V}^* \bigoplus \mathcal{W}^*$, it can simply be checked that $n = \dim(\mathcal{V}^*) + \dim(\mathcal{W}^*)$ and $\mathcal{V}^* \cap \mathcal{W}^* = \{0\}$. For the first result, a lemma will be stated:

Lemma 1. *Assume that (E, A) is regular. Let $\sigma(E, A) = \{\lambda \in \mathbb{C} | \det(\lambda E - A) = 0\}$. Let $\lambda \in \mathbb{R} \setminus \sigma(E, A)$. Then the subspaces $\mathcal{V}_i, \mathcal{W}_i$ satisfy the following relationships:*

1. $\mathcal{V}_i = \operatorname{Im}\left((A - \lambda E)^{-1}E\right)^i$
2. $\mathcal{W}_i = \ker\left(((A - \lambda E)^{-1}E\right)^i$
3. $\dim \mathcal{V}_i + \dim \mathcal{W}_i = n$

Proof. The first 2 arguments can be proven by induction, which can be found in [1]. The last claim follows from applying the rank-nullity theorem to $\mathcal{V}_i, \mathcal{W}_i$, which proves the last claim. □

Now, it can be proven that one indeed has that $\mathbb{R}^n = \mathcal{V}^* \bigoplus \mathcal{W}^*$. This will be done in the following theorem:

Theorem 1. *Assume that (E, A) is regular, and define the Wong sequences as $\mathcal{V}_{i+1} = A^{-1}(E\mathcal{V}_i)$ and $\mathcal{W}_{i+1} = E^{-1}(A\mathcal{W}_i)$. Both of these sequences will converge in at most n iterations and convergence in the same amount of steps. Denote the limiting subspaces by $\mathcal{V}^*, \mathcal{W}^*$. Then one has that $\mathbb{R}^n = \mathcal{V}^* \bigoplus \mathcal{W}^*$.*

Proof. As a consequence of the 3th statement of Lemma 1, one must have that both sequences convergence in the same amount of steps.

Again, using the 3th statement of Lemma 1, one obtains that $\dim \mathcal{V}^* + \dim \mathcal{W}^* = n$. Hence, it only needs to be shown that $\mathcal{V}^* \cap \mathcal{W}^* = \{0\}$. Let k^* denote the terminating index for both $\mathcal{V}^*, \mathcal{W}^*$, such that $\mathcal{V}_{k^*} = \mathcal{V}^*$, $\mathcal{W}_{k^*} = \mathcal{W}^*$.

Suppose that $x \in \mathcal{V}_{k^*} \cap \mathcal{W}_{k^*}$. Define the linear map $\hat{E} := \big((A - \lambda E)^{-1}E\big)$ for some $\lambda \in \mathbb{R} \setminus \sigma(E, A)$. Then, using the first two statements of Lemma 1 one must have that:

$$x \in \operatorname{Im} \hat{E}^{k^*} \cap \ker \hat{E}^{k^*}$$

Hence, there exists $z \in \mathbb{R}^n$ such that $x = \hat{E}^{k^*} z$. Furthermore, since $x \in \ker \hat{E}^{k^*}$, one has that $\hat{E}^{2k^*} z = 0$, which shows that $z \in \ker \hat{E}^{2k^*}$. However, since $\mathcal{W}^* := \mathcal{W}_{k^*} = \mathcal{W}_{2k^*}$, one has that $z \in \ker \hat{E}^{k^*}$. However, this shows that $x = \hat{E}^{k^*} z = 0$. Hence, 0 is the only element in $\mathcal{V}^* \cap \mathcal{W}^*$ and the claim has been proven. □

2.3 The quasi-Weierstrass form

The key in linear algebra is to associate linear maps with linear subspaces, which will also turn out to be the key for the quasi-Weierstrass form. The quasi-Weierstrass form will decouple the DAE into an ODE and a simpler DAE, for which the latter can easily be solved through a lemma. As stated in the introduction, the Wong sequences can be used to find the desired form. The following theorem will be the core of this subsection:

Theorem 2. *Let (E, A) be regular. Then there exists invertible maps $S, T \in \mathbb{R}^{n \times n}$ such that the matrix pair (E, A) will be transformed as:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{2.2}$$

Where I is the $n_1 \times n_1$ identity matrix, N a $n_2 \times n_2$ nilpotent matrix, i.e. there exists $q \in \mathbb{N}$ such that $N^q = O$, and J is some $n_1 \times n_1$ matrix. In particular, the maps can be constructed with regards to the Wong sequences: Let $\mathcal{V}^* := \operatorname{Im}(V)$, $\mathcal{W}^* := \operatorname{Im}(W)$ be full rank matrices. Then take $T = [V, W]$ and $S = [EV, AW]^{-1}$.

Proof. See e.g. [3]. □

This result will prove its use to show that any linear descriptor system will have a solution that is unique under the supposition that (E, A) is regular. Therefore, we will now take a look at the inhomogeneous case. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a smooth map, then the inhomogeneous case is stated as follows:

$$E\dot{x} = Ax + f, \quad x(t_0) = x_0 \in \mathbb{R}^n \tag{2.3}$$

In order to easily prove this, a convenient lemma will be introduced:

Lemma 2. *Let X be a Banach space and let $T : X \to X$ be a bounded linear operator. Suppose that $\sum_{k=0}^{\infty} \|T\|^k < \infty$. Then one has that:*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \tag{2.4}$$

Proof. (See [2]) Since X is a Banach space, and furthermore $\sum_{k=0}^{\infty} \|T\|^k < \infty$, one has that $\sum_{k=0}^{\infty} T^k$ converges and $\|T^k\| \to 0$

Next, define $S_n = \sum_{k=0}^{n} T^k$, then it can be shown that $(I - T)S_n \to I$. This follows from a direct computation:

$$(I - T)S_n = \sum_{k=0}^{n} T^k - \sum_{k=1}^{n+1} T^k = I - T^{n+1} \to I$$

Additionally, one can note that $(I - T)S_n \to (I - T)S$. Hence, it has been shown that $(I - T)S = I$, and therefore the statement has been proven. □

The result in Lemma 2 can significantly be simplified if the bounded linear operator in question also happens to be nilpotent. Now, consider again equation (2.3). Using Theorem 2 and Lemma 2, it can be shown that any DAE has a solution that is uniquely determined by its initial conditions, given that (E, A) is regular and f is a smooth map.

Theorem 3. *Let (E, A) be regular and assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map. Then for all smooth f there exists a solution to (2.3), which is uniquely determined by some initial condition $x(t_0)$ for some fixed $t_0 \in \mathbb{R}$*

Proof. Firstly, consider the DAE as in (2.3) and suppose the corresponding Wong sequences have been constructed $\mathcal{V}^*, \mathcal{W}^*$. Let $\mathcal{V}^* := \operatorname{Im} V$ and let $\mathcal{W}^* := \operatorname{Im} W$.

Next, suppose that $[EV, AW]$ is invertible. Consider the change of basis given by the map T. If one applies this change of basis whilst pre-multiplying with $S := [EV, AW]^{-1}$ yields:

$$SET\dot{x} = SATx + Sf =: SATx + g$$

Decompose x such that $x = [v, w]^\top$ and decompose g such that $g = [g_1, g_2]^\top$, then if Theorem 2 is applied one obtains:

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

Which is equivalent to solving $\dot{v} = Jv + g_1$ and $N\dot{w} = w + g_2$. Clearly, v can be given using the standard so called *variation of constants formula*, which can be found e.g. [5]. Thus it leaves us to show that $N\dot{w} = w + g_2$ is uniquely determined by $w(t_0)$. Observe that:

$$N\dot{w} = w + f_2 \iff \left(N\frac{\mathrm{d}}{\mathrm{d}t} - I\right) w = f_2$$

Next, recall that $N^q = O$. Define $T = N\frac{\mathrm{d}}{\mathrm{d}t}$ with underlying Banach Space $(C^\infty(\mathbb{R}), \|\cdot\|_\infty + \|\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)\|_\infty)$. Using Lemma 2, one derives the final solution:

$$\begin{aligned} w(t) &= -\sum_{k=0}^{q-1} \left(N\frac{\mathrm{d}}{\mathrm{d}t}\right)^k f_2(t) \\ &= -\sum_{k=0}^{q-1} N^k f_2^{(k)}(t) \end{aligned} \tag{2.5}$$

Therefore, using the variation of constants formula one can conclude that the solution of $v(t)$ is uniquely determined by $v(t_0)$. Similarly, using (2.5) one can conclude that the solution of $w(t)$ is uniquely determined by $w(t_0)$. □

As can be observed, the quasi-Weierstrass form can play a powerful role for proving theorems regarding linear descriptor systems. However, it might still be unclear how the associated matrices J, N are actually computed. Therefore, it might be worthwhile investigating how these matrices can be computed.

Using Theorem 2, one can compute explicit expressions involving J, N and other known matrices. This is stated in the following corollary:

Corollary 1. *Suppose that $[EV, AW]$ is invertible, furthermore, assume Theorem 2 is applicable. Let S, T be given as in Theorem 2. Then one can explicitly compute J, N as follows:*

$$\begin{aligned} J &= (EV)^\dagger AV \\ N &= (AW)^\dagger EW \end{aligned} \tag{2.6}$$

Where † denotes the Moore-Penrose pseudo inverse (from now on referred to as pseudo inverse) which is given as follows for a full rank matrix $A \in \mathbb{R}^n$:

$$A^\dagger := (A^\top A)^{-1} A^\top \tag{2.7}$$

Proof. This can be observed by explicitly computing SET and SAT, as in Theorem 2, and equate the terms and omit the trivial equations. Therefore, if S and T are given as in Theorem 2 then one obtains:

$$\begin{aligned} [V, W]E &= [EV, AW] \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \implies EW = AWN \\ [V, W]A &= [EV, AW] \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \implies AV = EVJ \end{aligned} \tag{2.8}$$

Next, as assumed $[EV, AW]$ is invertible, therefore EV, AW have to be full rank. Thus, their pseudoinverses do exist. Therefore, if the equations in (2.8) are solved for J and N, one obtains their final result:

$$\begin{aligned} AWN = EW &\implies N = (AW)^\dagger EW \\ EVJ = AV &\implies J = (EV)^\dagger AV \end{aligned}$$

□

The next step in our analysis would concern finding a solution to the associated linear descriptor system. However, as shown, this solution is uniquely determined by some initial condition. This can cause several problems.

Consider the homogeneous system as in (2.1). Using Wong sequences, one can find $\mathcal{V}^*, \mathcal{W}^*$. As has been noted, if a solution x is in $\mathcal{V}^*$ for some fixed time $t^* > 0$, then it must be that $x(t) \in \mathcal{V}^*$ for all $t \geqslant t^*$. However, what if one considers $x(t^*) := x^0 \notin \mathcal{V}^*$? Then the solution to the corresponding initial value might not be smooth for all $t \geqslant t^*$. If a corresponding solution is not smooth for an initial value, then this initial values is called *inconsistent*. An example of this will be explored in the introduction of the mathematical preliminaries, showing that impulses can occur for inconsistent initial values.

For now, the focus will be put on *consistent* initial values, i.e. initial values for which a smooth solution can be guaranteed. In the next section this formula will be derived for the inhomogeneous case, i.e. (2.3).

2.4 Explicit solution formula for consistent initial values

In Theorem 3 a lot of the work has already been put down. The only thing that has to be done is that the solutions have to be combined into vector form. The following projectors will play a crucial role when the solutions in Theorem 3 are to be combined in vector form:

Definition 2. *Consider a regular matrix pair (E, A) and consider its quasi-Weierstrass form and the associated matrices given in Theorem 2. Then the consistency-, differential- and impulse projectors are given as follows:*

$$\begin{aligned}\Pi_{(E,A)} &= T\begin{bmatrix} I_{n_1\times n_1} & O_{n_2\times n_1} \\ O_{n_1\times n_2} & O_{n_2\times n_2}\end{bmatrix}T^{-1}\\ \Pi^{\text{diff}}_{(E,A)} &= T\begin{bmatrix} I_{n_1\times n_1} & O_{n_2\times n_1} \\ O_{n_1\times n_2} & O_{n_2\times n_2}\end{bmatrix}S \\ \Pi^{\text{imp}}_{(E,A)} &= T\begin{bmatrix} O_{n_1\times n_1} & O_{n_2\times n_1} \\ O_{n_1\times n_2} & I_{n_2\times n_2}\end{bmatrix}S\end{aligned} \tag{2.9}$$

In general, the subscripts denoting the dimensions of the identity - or null matrix will be omitted for clarity sake.

A matrix is said to be a a projector (in the usual sense) if it is idempotent. Observe that the consistency projector is always a projector in the usual sense, but the differential- and impulse projectors are not necessarily such projectors. The following computation shows that the consistency projector is idempotent:

$$\Pi^2_{(E,A)} = T\begin{bmatrix} I & O \\ O & O\end{bmatrix}(T^{-1}T)\begin{bmatrix} I & O \\ O & O\end{bmatrix}T^{-1} = T\begin{bmatrix} I & O \\ O & O\end{bmatrix}^2 T^{-1} = T\begin{bmatrix} I & O \\ O & O\end{bmatrix}T^{-1} = \Pi_{(E,A)}$$

The second observation that can be made is that these projectors are independent of the matrices T, S. This is because the projectors, and furthermore, T, S are dependent on the Wong limits $\mathcal{V}^*, \mathcal{W}^*$ and their dimensions.

Using these projectors, one can find the explicit solution formula for (2.3). This result will be stated in the following theorem:

Theorem 4. *Let (E, A) be a regular matrix pair and let $\Pi_{(E,A)}, \Pi^{diff}_{(E,A)}, \Pi^{imp}_{(E,A)}$ be as in definition 2. Define the matrices A^{diff}, E^{imp} as follows:*

$$A^{diff} := \Pi^{diff}_{(E,A)}A, \quad E^{imp} := \Pi^{imp}_{(E,A)}E$$

Then the solutions for (2.3), for $c \in \mathbb{R}^n$, are given by[†]

$$x(t) = e^{A^{diff}t}\Pi_{(E,A)}c + \int_0^t e^{A^{diff}(t-s)}\Pi^{diff}_{(E,A)}f(s)\mathrm{d}s - \sum_{i=0}^{q-1}(E^{imp})^i\Pi^{imp}_{(E,A)}f^{(i)}(t) \tag{2.10}$$

Proof. As said, Theorem 3 has already put some work down. In particular, if x solves (2.3), then $y =: \left[\begin{smallmatrix} v & w\end{smallmatrix}\right]^\top = T^{-1}x$ must solve $\dot{v} = v + [\begin{smallmatrix} I & O\end{smallmatrix}]Sf$ and $N\dot{w} = w + [\begin{smallmatrix} O & I\end{smallmatrix}]Sf$.

Observe that $A^{\text{diff}} = T[\begin{smallmatrix} J & O \\ O & O\end{smallmatrix}]T^{-1}$ and that $E^{\text{imp}} = T[\begin{smallmatrix} O & O \\ O & N\end{smallmatrix}]T^{-1}$. Using the variation of constants formula, see e.g. [5], one finds that the solution for v is given as:

$$v(t) = e^{Jt}v(0) + \int_0^t e^{J(t-s)}[\begin{smallmatrix} I & O\end{smallmatrix}]Sf(s)\mathrm{d}s$$

[†]Observe that if a different initial time t_0 is preferred rather than 0, then one can apply the translation $t \to (t - t_0)$.

Using (2.5), one can derive the solution for $w(t)$:

$$w(t) = -\sum_{k=0}^{q-1} N^k [\,O\ I\,] S f^{(k)}(t)$$

The last step is combining these solutions in vector notation, and invert the initial change of basis. The first step in the computation is as follows:

$$x(t) = T\begin{bmatrix} v \\ w \end{bmatrix} = T\begin{bmatrix} v \\ 0 \end{bmatrix} + T\begin{bmatrix} 0 \\ w \end{bmatrix} \tag{2.11}$$

Firstly, the terms in (2.11) will be computed explicitly for clarity's sake. The trick to keep in mind is to rewrite everything in a more general matrix form, so to speak. Now, only the term involving v will be computed. This yields:

$$\begin{aligned} T\begin{bmatrix} v \\ 0 \end{bmatrix} &= T\begin{bmatrix} e^{Jt}v(0) \\ 0 \end{bmatrix} + T\begin{bmatrix} \int_0^t e^{J(t-s)}[\,I\ O\,]Sf(s)\mathrm{d}s \\ 0 \end{bmatrix} \\ &= T\begin{bmatrix} e^{Jt} & O \\ O & O \end{bmatrix}\begin{bmatrix} I & O \\ O & O \end{bmatrix}\begin{bmatrix} v(0) \\ w(0) \end{bmatrix} + \int_0^t T\begin{bmatrix} e^{J(t-s)} & O \\ O & O \end{bmatrix}\begin{bmatrix} I & O \\ O & O \end{bmatrix} Sf(s)\mathrm{d}s \end{aligned}$$

Next, one should take note that:

$$\begin{bmatrix} e^{Jt} & O \\ O & O \end{bmatrix} = \begin{bmatrix} e^{Jt} & O \\ O & e^{O} \end{bmatrix}\begin{bmatrix} I & O \\ O & O \end{bmatrix} = \exp\left\{\begin{bmatrix} J & O \\ O & O \end{bmatrix} t\right\}\begin{bmatrix} I & O \\ O & O \end{bmatrix}$$

Additionally, observe that $\exp(DAD^{-1}) = D\exp(A)D^{-1}$. Hence, $D\exp(A) = \exp(DAD^{-1})D$. Using these expressions, one is able to rewrite $T[\,v\ 0\,]^\top$ again. Firstly, define $T^{-1}c := [\,v(0)\ w(0)\,]^\top$ This yields:

$$\begin{aligned} T\begin{bmatrix} v \\ 0 \end{bmatrix} &= T\begin{bmatrix} e^{Jt} & O \\ O & O \end{bmatrix}\begin{bmatrix} v(0) \\ w(0) \end{bmatrix} + \int_0^t T\begin{bmatrix} e^{J(t-s)} & O \\ O & O \end{bmatrix}\begin{bmatrix} I & O \\ O & O \end{bmatrix} Sf(s)\mathrm{d}s \\ &= e^{T\left[\begin{smallmatrix} J & O \\ O & O \end{smallmatrix}\right]T^{-1}t}T\begin{bmatrix} I & O \\ O & O \end{bmatrix} T^{-1}c + \int_0^t e^{T\left[\begin{smallmatrix} J & O \\ O & O \end{smallmatrix}\right]T^{-1}(t-s)}T\begin{bmatrix} I & O \\ O & O \end{bmatrix} Sf(s)\mathrm{d}s \\ &= e^{A^{\mathrm{diff}}t}\Pi_{(E,A)}c + \int_0^t e^{A^{\mathrm{diff}}(t-s)}\Pi^{\mathrm{diff}}_{(E,A)}f(s)\mathrm{d}s \end{aligned} \tag{2.12}$$

Therefore, one only has to compute $T[\,0\ w\,]^\top$ and can combine the beforehand derived expression for $T[\,v\ 0\,]^\top$. This is a lot more straightforward and can be computed as:

$$\begin{aligned} T\begin{bmatrix} 0 \\ w(t) \end{bmatrix} &= \begin{bmatrix} 0 \\ -T\sum_{k=0}^{q-1} N^k[\,O\ I\,]Sf^{(k)}(t) \end{bmatrix} = -\sum_{k=0}^{q-1} T\begin{bmatrix} O & O \\ O & N^k \end{bmatrix}\begin{bmatrix} O & O \\ O & I \end{bmatrix} Sf^{(k)}(t) \\ &= -\sum_{k=0}^{q-1}\left(T\begin{bmatrix} O & O \\ O & N \end{bmatrix}T^{-1}\right)^k T\begin{bmatrix} O & O \\ O & I \end{bmatrix} Sf^{(k)}(t) = -\sum_{k=0}^{q-1}\left(E^{\mathrm{imp}}\right)^k \Pi^{\mathrm{imp}}_{(E,A)}f^{(k)}(t) \end{aligned} \tag{2.13}$$

By inspection, one can observe that if one adds equation (2.12) with equation (2.13) one derives the result in Theorem 4, and hence, the statement has been proven. □

As noted at the end of section 2.3, this formula only gives a sufficiently smooth result for consistent initial conditions. For the study of switched DAEs one has to incorporate inconsistent initial conditions into the defined solution framework. This new framework will be explored in the next chapter. Once this new solution framework has been defined, several concepts regarding switched DAEs will be introduced (e.g. ITP, impulse controllability). However, only the underlying solutional space will be altered for inconsistent initial conditions, and the above solution formula will still be valid if $u(t)$ is assumed piecewise-continuous, as is mentioned [4, Remark 4.5].

Afterwards, the focus will be put on switched DAEs. However, it is beneficial to introduce a lemma regarding projections that will be of use when the analysis will be shifted towards switched DAEs. Let P be a projection matrix, that is it satisfies $P = P^2$. Projections have the property that $Py = y$ if $y \in \mathrm{Im}(P)$. This property is equivalent to $(P - I)y = 0$ which is in turn equivalent to $y \in \ker(P - I)$. This property will be proven in the following lemma:

Lemma 3. *Let $P \in \mathbb{R}^{n\times n}$ be a projection matrix. Then one has that:*

$$\mathrm{Im}(P) = \ker(P - I)$$

Proof. In order to show the equality, it will be shown that $\mathrm{Im}(P) \subseteq \ker(P - I)$ and $\ker(P - I) \subseteq \mathrm{Im}(P)$, or equivalently $\mathrm{Im}(P) \subseteq \ker(P - I)$ and $\mathrm{Im}(P) \supseteq \ker(P - I)$. These two steps will be abbreviated by $(\subseteq), (\supseteq)$ respectively:

- ($\subseteq$) Let $y \in \text{Im}(P)$, thus $\exists x \in \mathbb{R}^n$ such that $y = Px$. Since P is idempotent, one has that $P^2 = P$, hence $Py = P^2x = Px = y$. Thus $Py = y$, which is equivalent to $(P - I)y = 0$, hence $y \in \ker(P - I)$. This shows that $\text{Im}(P) \subseteq \ker(P - I)$.
- ($\supseteq$) Let $y \in \ker(P - I)$, and as shown in the previous section this implies that $Py = y$. This directly shows that y is mapped to itself under P, and is trivially in $\text{Im}(P)$. Hence, it has been shown that $\text{Im}(P) \supseteq \ker(P - I)$.

Hence, one can conclude that $\text{Im}(P) = \ker(P - I)$ and the desired statement has been proven. $\square$

3 Mathematical preliminaries

3.1 Distributional framework

As was discussed in the previous section, it can not always be ensured that the solution is smooth for all time. In particular, this has to do with the choice of the initial condition. Such inconsistent initial values can ensure that the solution of the DAE contains jumps, or even impulses.

Before introducing this new solutional framework, it might be worthwile to perform an exploratory analysis on a DAE with an inconsistent initial value. However, in order to work with impulses, a more refined definition has to be given before proceeding.

Let f be a piecewise smooth function discontinuous at $t = a$. The idea will be to compute the derivative of f at $t = a$. Let $\varepsilon > 0$ be sufficiently small such that one can approximate $f(a - \frac{1}{2}\varepsilon) \approx f(a-)$ and $f(a + \frac{1}{2}\varepsilon) \approx f(a+)$. Since ε is sufficiently small, one can approximate the derivative of $t = a$ as follows:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(a) \approx \frac{f(a + \frac{1}{2}\varepsilon) - f(a - \frac{1}{2}\varepsilon)}{\varepsilon} \approx [f(a+) - f(a-)]\frac{1}{\varepsilon}$$

Formally, to find $\frac{\mathrm{d}f}{\mathrm{d}t}(a)$ one needs to let $\varepsilon \to 0$, however this is problematic as then $\frac{\mathrm{d}f}{\mathrm{d}t}(a)$ will be undefined. Therefore, the idea will be to make this idea more consistent in terms of its inverse operation: integration. If one considers the integral of $\frac{\mathrm{d}f}{\mathrm{d}t}(a)$ on the interval $(a - \frac{1}{2}\varepsilon, a + \frac{1}{2}\varepsilon)$, one expects that this is equal to the jump discontinuity. Using the midpoint approximation one can see this actually holds:

$$\int_{a-\frac{1}{2}\varepsilon}^{a+\frac{1}{2}\varepsilon} \frac{\mathrm{d}f}{\mathrm{d}t}(t)\mathrm{d}(t) \approx \int_{a-\frac{1}{2}\varepsilon}^{a+\frac{1}{2}\varepsilon} [f(a+) - f(a-)]\frac{1}{\varepsilon}\mathrm{d}t = f(a+) - f(a-)$$

This gives us an idea as to how intuitively define an impulse, or rather, *Dirac impulse* which will be done below:

$$\delta_a :\approx \lim_{\varepsilon \to 0} \begin{cases} \frac{1}{\varepsilon} & t \in (a - \frac{1}{2}\varepsilon, a + \frac{1}{2}\varepsilon) \\ 0 & \text{else} \end{cases}$$

Where $:\approx$ means loosely defined. Hence, one can express the derivative of f at $t = a$ in terms of this impulse as follows:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(a) = [f(a+) - f(a-)]\delta_a$$

Continuing on the example with inconsistent intial values, consider the following homogeneous DAE:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = x, \quad x := \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}, \ x_0 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, t \in [0, \infty) \tag{3.1}$$

It can already be observed that this initial condition is not consistent: according to Theorem 4, one has that $x(t) = 0$ for all $t > 0$ since $\Pi_{(E,A)} = O$.

It seems that $x^{(2)}$ will experience a jump in the solution, as $x(0-) = x_0$ whilst $x(0+) = 0$, thus $x^{(2)}(0-) = 1$ and $x^{(2)}(0+) = 0$. However, that is not all. Observe that $x^{(1)} = \left(x^{(2)}\right)' = x_0\delta_0 := x_0\delta$, hence $x^{(1)}$ will experience a Dirac impulse.

However, this raises serious questions as to how one must interpret the derivative of a jump. As has been learned in traditional calculus courses, a piecewise continuous function that is discontinuous at a point will also not be differentiable at that point. Therefore, the traditional solutional space of sufficiently smooth functions is insufficient, and thus the space of piecewise smooth distributions will be adopted and explored in this subsection.

Distributions, in contrast to regular functions, map test functions[‡] to a real number. This is a neat property, as this implies that any locally integrable function f induces a distribution $f_{\mathbb{D}}$, by considering the following integral:

$$f_{\mathbb{D}}(\varphi) = \int_{\mathbb{R}} \varphi(t)f(t)\mathrm{d}t \tag{3.2}$$

Back to our example given in (3.1), the key property that is utilized is the use of *indicator functions*. Observe that there exists $\varepsilon > 0$ such that the solution $x(t)$ equals $\mathbb{1}_{[-\varepsilon,0)}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ for all $t \geqslant 0$. To this end, we first define the space of piecewise-smooth functions:

[‡]A test function is a smooth function for which the complement of its kernel is compact.

Definition 3. *The space of piecewise-smooth functions, $\mathcal{C}^\infty_{pw}$, is defined as follows:*

$$\mathcal{C}^\infty_{pw} := \left\{ \alpha = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}\alpha_i \,\middle|\, \begin{array}{c} \{t_i \in \mathbb{R} | i \in \mathbb{Z}\} \text{ discrete} \\ t_i < t_{i+1} \\ \alpha_i\ \mathcal{C}^\infty\ \forall i \in \mathbb{Z} \end{array} \right\}$$

As can be observed, $x(t)$ is in $\mathcal{C}^\infty_{pw}$. However, $\mathcal{C}^\infty_{pw}$ is not a suitable candidate for our solutional space, since $\mathcal{C}^\infty_{\text{pw}}$ is not closed under differentiation. As discussed earlier, this is due to the possible presence of Dirac impulses in the solution. Therefore, the space of *piecewise-smooth distributions* will be introduced as follows:

Definition 4. *The space of piecewise-smooth distributions, $\mathbb{D}_{pw\mathcal{C}^\infty}$, is defined as:*

$$\mathbb{D}_{\mathcal{C}^\infty_{pw}} := \left\{ \alpha = f_{\mathbb{D}} + \sum_{t\in T} D_t \,\middle|\, \begin{array}{c} f \in \mathcal{C}^\infty_{pw}\ , \\ T \subseteq \mathbb{R} \text{ discrete}, \forall t \in T : \\ D_t \in \operatorname{span}\{\delta_t, \delta_t', ...\} \end{array} \right\}$$

Here, the Dirac delta (or Dirac impulse) δ_t is defined as a distribution as $\delta_t(\varphi) := \varphi(t)$. Observe that $\mathbb{D}_{pwC^\infty}$ is closed under differentiation, since linear combinations of the Dirac delta with all its derivatives are allowed to be in $\mathbb{D}_{pwC^\infty}$.

However, after defining this new solutional framework, how must one interpret the solution of a DAE with an inconsistent initial value? This can be interpreted as an initial trajectory problem, which indicates that the DAE has been inactive before the initial time. This enables us to study the impulsive behaviour of a DAE. Let the initial time be 0, then the ITP for a non-homogeneous DAE can be more symbolically defined as follows:

Definition 5. *The initial trajectory problem (ITP) for a DAE is given as follows:*

$$\begin{cases} x(t) = x_0, & t < 0 \\ E\dot{x} = Ax + f, & t \geqslant 0 \end{cases}$$

Next, several important properties of distributions will be stated:

Corollary 2. *Let D be a distribution, φ some test function and let α be some smooth function. Then the following 3 properties hold:*

1. $D'(\varphi) := -D(\varphi')$
2. $\alpha D(\varphi) = D(\alpha\varphi)$
3. $(\alpha D)' = \alpha' D + \alpha D'$

Next, there are several ways to "evaluate" some distribution D at some time τ. As explored in the motivating example, one is able to left/right evaluate D at some time. However, we are also interested in evaluating if D experiences an impulse at time τ. The following definition will make the aforementioned points concrete:

Definition 6. *Let $\tau \in \mathbb{R}$ and let $D := f_{\mathbb{D}} + \sum_{t\in T} D_t$. Then the left/right evaluation of D is given by:*

$$\begin{aligned} D(\tau-) &:= f(\tau-) = \lim_{\varepsilon\to 0} f(\tau - \varepsilon), \\ D(\tau+) &:= f(\tau+) = f(\tau) \end{aligned} \tag{3.3}$$

Moreover, the impulsive part of D at τ, denoted by $D[\tau]$, is given by:

$$D[\tau] = \begin{cases} D_\tau, & \text{if } \tau \in T \\ 0, & \text{else} \end{cases} \tag{3.4}$$

Additionally, as already suspected in the introduction of this thesis, the Dirac delta must be the distributional derivative of the indicator function. This will be proven in the following corollary:

Corollary 3. *Let D be the distribution induced by the indicator function $\mathbb{1}_{[t,\infty)}$. Let φ be a test function, that is, φ has compact support. Let the Dirac delta centered at t, i.e. $\delta_t : \varphi \to \varphi(t)$, be defined by the integral §:*

$$\delta_t(\varphi) = \int_{\mathbb{R}} \varphi(x)\delta(x-t)\mathrm{d}x = \varphi(t)$$

Then one has that $D' = \delta_t$.

Proof. Let φ be a test function. Since the support is compact, one must have that φ is equal to 0 at $\pm\infty$ Using Corollary 2, one has that:

$$D'(\varphi) := D(-\varphi') = -\int_{\mathbb{R}} \varphi'(x)\mathbb{1}_{[t,\infty)}\mathrm{d}x = \int_{\infty}^{t} \varphi'(x)\mathrm{d}x = \varphi(t) =: \delta_t(\varphi)$$

□

§This integral is not well defined in the classical sense of Riemann integration. In order to properly define this integral, a measure theoretical approach can be taken. One can define the *Dirac measure* δ_a which maps a set X to 1 if $a \in X$ and maps X to 0 if $a \notin X$. Then the integral can be reformulated as a Lebesque integral $\int_{\mathbb{R}} f(t)\mathrm{d}\delta_a(t) = f(a)$

3.2 Geometric notations for DAEs

Before introducing switched DAEs, some geometrical notions regarding linear descriptor systems have to be introduced. To this end the inhomogeneous DAE will be considered with an input, restricted to a finite interval, i.e.:

$$E\dot{x} = Ax + Bu, \quad t \in [t_0, T] \tag{3.5}$$

Observe that (3.5) can be identified with the triplet (E, A, B). Using Theorem 4, one can observe that the solution of (3.5) can be written as the linear combination of a part independent of the input and a part dependent on the input, referred to as the *autonomous* - and *strictly controllable* part, respectively. Specifically:

$$\begin{aligned} x(t) =: x_u(t, t_0; x_0) &= e^{A^{\text{diff}}(t-t_0)}\Pi_{(E,A)}x_0 + \int_{t_0}^{t} e^{A^{\text{diff}}(t-s)}B^{\text{imp}}u(s)\mathrm{d}s - \sum_{i=0}^{q-1}(E^{\text{imp}})^i B^{\text{imp}}u^{(i)}(t) \\ &\equiv x_{\text{aut}}(t, t_0; x_0) + x_u(t, t_0) \end{aligned} \tag{3.6}$$

For now, without loss of generality, the initial time is set to 0, otherwise one simply applies a translation $t \to (t - t_*)$ if t_* is desired to be the initial time. In essence, the strictly controllable part of the trajectory has the ability to steer the final solution to some other state. This raises a question: what are the states that can be reached for some smooth input? To this end, the *reachable subspace* can be defined as follows:

$$\mathcal{R} = \left\{ x_T \in \mathbb{R}^n \middle| \begin{array}{c} \exists T > 0 \ \exists \text{ smooth solution } (x, u) \text{ of (2)}, \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\} \tag{3.7}$$

Next, as argued in the first subsection, the Wong sequences can be very useful to study the solution behaviour of (2.1). However, for the non-homogeneous problem one can define the *augmented Wong sequences* as follows for $i = 0, 1, 2, \ldots$:

$$\begin{aligned} \mathcal{V}^i_{(E,A,B)} &= A^{-1}(E\mathcal{V}^{i-1}_{(E,A,B)} + \operatorname{Im} B) =: \mathcal{V}_i \\ \mathcal{W}^i_{(E,A,B)} &= E^{-1}(A\mathcal{V}^{i-1}_{(E,A,B)} + \operatorname{Im} B) =: \mathcal{W}_i \end{aligned}$$

Where $\mathcal{V}_0 := \mathbb{R}^n$ and $\mathcal{W}_0 := \{0\}$, and the terminating sequences will be denoted by $\mathcal{V}_i^*, \mathcal{W}_i^*$. Sometimes it is convenient to explicitly indicate for which matrix triplet the Wong limits are computed, but in general, one only has to indicate for which mode the augmented Wong limits are computed. For this section, the former convention will be supposed and when switched DAEs are studied the latter convention will be supposed. Furthermore, the star superscript will be removed. In general, $\mathcal{V}_{(E,A,B)}$ is called the *augmented consistency space*. For the homogeneous problem, (2.1), the Wong limit $\mathcal{V} =: \mathcal{V}_{(E,A)}$ is called the *consistency space*. More concretely, these subspaces are defined as follows:

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \middle| \begin{array}{c} \exists \text{ smooth solution } x(t) \text{ of (2.1)}, \\ \text{with } x(0) = x_0 \end{array} \right\} \tag{3.8}$$

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \middle| \begin{array}{c} \exists \text{ smooth solution } (x, u) \text{ of (3.5)}, \\ \text{with } x(0) = x_0 \end{array} \right\} \tag{3.9}$$

Since any trajectory $x_u(t, t_0; x_0)$ can be decomposed into the sum of an autonomous part $x_{\text{aut}}(t, t_0; x_0)$ and a strictly controllable part $x_u(t, t_0)$, one can conjecture that $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \mathcal{R}$. However, the statement can be enforced by changing the sum into a direct sum and by changing the reachable space into one of its subspaces. This can be proved formally and will be done in the following lemma:

Lemma 4. *Let $\mathcal{V}_{(E,A)}$ be the consistency space and let $\mathcal{V}_{(E,A,B)}$ be the augmented consistency space the DAE as defined in (3.5) . Let $\mathcal{R}$ denote the reachable subspace of the aforementioned DAE. Define $\langle A|B\rangle := \operatorname{Im}[\,B\ AB\ \ldots\ A^{n-1}B\,]$ and define $B^{\text{imp}} = \Pi^{\text{imp}}_{(E,A)}B$. Then one has that:*

$$\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}}|B^{\text{imp}}\rangle = \mathcal{V}_{(E,A)} + \mathcal{R}$$

Proof. Let $x_0 \in \mathcal{V}_{(E,A,B)}$, in other words, x_0 is a consistent initial value. Thus, there exists a smooth solution (x, u) that solves the equation $E\dot{x} = Ax + Bu$ with $x(t_0-) = x_0$ on $[t_0, T]$. Using Theorem 4 one has that $\exists c \in \mathbb{R}^n$ such that:

$$x(t_0+) := x(t_0) = \Pi_{(E,A)}c - \sum_{i=0}^{q-1}\left(E^{\text{imp}}\right)^i B^{\text{imp}}u^{(i)}(t_0) = \Pi_{(E,A)}c - \left[\,B^{\text{imp}}\ E^{\text{imp}}B^{\text{imp}}\ \ldots\ (E^{\text{imp}})^{q-1}B^{\text{imp}}\,\right]\begin{bmatrix} u(t_0) \\ \vdots \\ u^{(q-1)}(0) \end{bmatrix}$$

Observe that the aforementioned equation can almost be expressed in terms of $\langle E^{\text{imp}}|B^{\text{imp}}\rangle$, only if the nilpotency index q is swapped for the dimension of E, n. As can be observed in the proof of Theorem 4, the nilpotency index of E^{imp} is equal to the nilpotency index of E.

Observe that, since E^{imp} is nilpotent, it has eigenvalue 0 with multiplicity n, and hence, $\chi_{E^{\text{imp}}} = x^n$. Furthermore, one has that $E^q = 0$, hence its *minimal polynomial*[¶] is equal to x^q. In this case, the Cayley-Hamilton theorem states that the minimal polynomial divides the characteristic polynomial, and therefore one must have that $q \leqslant n$.

Hence, rewriting the aforementioned equation yields:

$$x(t_0) = \Pi_{(E,A)}c - \sum_{i=0}^{q-1} \left(E^{\text{imp}}\right)^i B^{\text{imp}} u^{(i)}(t_0) = \Pi_{(E,A)}c - \sum_{i=0}^{n-1} \left(E^{\text{imp}}\right)^i B^{\text{imp}} u^{(i)}(t_0)$$

$$= \Pi_{(E,A)}c - \left[\, B^{\text{imp}} \;\; E^{\text{imp}}B^{\text{imp}} \;\; \dots \;\; \left(E^{\text{imp}}\right)^{n-1}B^{\text{imp}} \,\right] \begin{bmatrix} u(t_0) \\ \vdots \\ u^{(q-1)}(t_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hence, observe that any consistent initial value $x(t_0)$ can be decomposed into an element of $\operatorname{Im}\Pi_{(E,A)}, \langle E^{\text{imp}}|B^{\text{imp}}\rangle$. From this it can be concluded that $\mathcal{V}_{(E,A,B)} = \operatorname{Im}\Pi_{(E,A)} + \langle E^{\text{imp}}|B^{\text{imp}}\rangle$. A similar approach can be taken to show that $\operatorname{Im}\Pi_{(E,A)} = \mathcal{V}_{(E,A)}$. Hence it has been shown that $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \langle E^{\text{imp}}|B^{\text{imp}}\rangle$.

If it can be shown that $\mathcal{V}_{(E,A)} \cap \langle E^{\text{imp}}|B^{\text{imp}}\rangle = \{0\}$, the direct sum has been established. Let $y \in \langle E^{\text{imp}}|B^{\text{imp}}\rangle$. Then there exists vectors $\beta_1, \dots, \beta_q \in \mathbb{R}^n$ such that:

$$y = \sum_{i=0}^{q-1} \left(E^{\text{imp}}\right)^i B^{\text{imp}}\beta_{i+1} = \Pi^{\text{imp}}_{(E,A)}B\beta_1 + \sum_{i=0}^{q-2} \Pi^{\text{imp}}_{(E,A)}E\left(E^{\text{imp}}\right)^i \beta_{i+2}$$

$$= \Pi^{\text{imp}}_{(E,A)}\left(B\beta_1 + \sum_{i=0}^{q-2} E\left(E^{\text{imp}}\right)^i \beta_{i+2}\right) \in \operatorname{Im}\Pi^{\text{imp}}_{(E,A)}$$

Hence, one can conclude that $\langle E^{\text{imp}}|B^{\text{imp}}\rangle \subseteq \operatorname{Im}\Pi^{\text{imp}}_{(E,A)} \subseteq \mathcal{W}^*_{(E,A)}$. Next, using Theorem 1, one has that $\mathcal{V}_{(E,A)} \bigoplus \mathcal{W}_{(E,A)} = \mathbb{R}^n$, and hence, one must have that $\mathcal{V}^*_{(E,A)} \cap \langle E^{\text{imp}}|B^{\text{imp}}\rangle = 0$, and thus $\mathcal{V}^*_{(E,A,B)} = \mathcal{V}^*_{(E,A)} \bigoplus \langle E^{\text{imp}}|B^{\text{imp}}\rangle$.

The reachable subspace contains all points x_T that can be reached in finite time $T > 0$ starting from the origin. Hence, any x_T can be written as:

$$x_T = \int_0^T e^{A^{\text{diff}}(t-s)}B^{\text{diff}}u(s)\mathrm{d}s - \sum_{i=0}^{q-1}(E^{\text{imp}})^i B^{\text{imp}}u^{(i)}(T)$$

As can be observed, x_T can be expressed as a linear combination of some integral and an element of $\langle E^{\text{imp}}|B^{\text{imp}}\rangle$, and therefore one must have that $\langle E^{\text{imp}}|B^{\text{imp}}\rangle \subseteq \mathcal{R}$. This shows us that $\mathcal{V}_{(E,A)} + \langle E^{\text{imp}}|B^{\text{imp}}\rangle \subseteq \mathcal{V}_{(E,A)} + \mathcal{R}$. In order to show that $\mathcal{V}_{(E,A)} + \langle E^{\text{imp}}|B^{\text{imp}}\rangle \supseteq \mathcal{V}_{(E,A)} + \mathcal{R}$ it only has to be shown that the integral term in x_T is in $\mathcal{V}_{(E,A)} = \operatorname{Im}\Pi$.

Using Lemma 3, it can equivalently be shown that:

$$\Pi\int_0^T e^{A^{\text{diff}}(s-t)}B^{\text{diff}}u(s)\mathrm{d}s = \int_0^T e^{A^{\text{diff}}(s-t)}B^{\text{diff}}u(s)\mathrm{d}s$$

Furthermore, one should take note that:

$$\Pi \cdot \Pi^{\text{diff}} = T\begin{bmatrix} I & O \\ O & O \end{bmatrix}T^{-1}T\begin{bmatrix} I & O \\ O & O \end{bmatrix}S = T\begin{bmatrix} I & O \\ O & O \end{bmatrix}S = \Pi^{\text{diff}}$$

Hence, a computation verifies that:

$$\Pi\int_0^T e^{A^{\text{diff}}(s-t)}B^{\text{diff}}u(s)\mathrm{d}s = \Pi\int_0^T \sum_{k=0}^{\infty} \tfrac{1}{k!}(A^{\text{diff}})^k B^{\text{diff}}(s-t)^k u(s)\mathrm{d}s$$

$$= \Pi\left(\int_0^T B^{\text{diff}}u(s)\mathrm{d}s + \int_0^T \sum_{k=1}^{\infty} \tfrac{1}{k!}(A^{\text{diff}})^k B^{\text{diff}}(s-t)^k u(s)\mathrm{d}s\right)$$

$$= \Pi\left(B^{\text{diff}}\int_0^T u(s)\mathrm{d}s + A^{\text{diff}}\int_0^T \sum_{k=0}^{\infty} \tfrac{1}{(k+1)!}(A^{\text{diff}})^k B^{\text{diff}}(s-t)^k u(s)\mathrm{d}s\right)$$

$$= \Pi \cdot \Pi^{\text{diff}}\left(B\int_0^T u(s)\mathrm{d}s + A\int_0^T \sum_{k=0}^{\infty} \tfrac{1}{(k+1)!}(A^{\text{diff}})^k B^{\text{diff}}(s-t)^k u(s)\mathrm{d}s\right)$$

[¶] The minimal polynomial of a matrix A is defined as the polynomial p such that p is the lowest degree polynomial for which the equality $p(A) = 0$ holds.

$$= \Pi^{\text{diff}} \left(B \int_0^T u(s)\mathrm{d}s + A \int_0^T \sum_{k=0}^{\infty} \tfrac{1}{(k+1)!} (A^{\text{diff}})^k B^{\text{diff}} (s-t)^k u(s)\mathrm{d}s \right)$$
$$= \int_0^T e^{A^{\text{diff}}(s-t)} B^{\text{diff}} u(s)$$

Hence, the integral term is in $\mathcal{V}_{(E,A)}$, which implies that $\mathcal{V}_{(E,A)} + \mathcal{R} \subseteq \mathcal{V}_{(E,A)} + \langle E^{\text{imp}} | B^{\text{imp}} \rangle$, which completes the proof. □

Impulses can also occur in the solution as was noted in the introductory example of section 2. As the solution formula already suggests, impulses will occur if the input is discontinuous at some $t > 0$. Furthermore, this statement can be reinforced. One should take note that the solution formula suggests that the nilpotency index of E determines the minimum degree of smoothness for u if impulses are to be prevented. More symbollically, if q is the nilpotency index of E, then u will cause no impulse if $u \in \mathcal{C}^q$. This can be observed through a simple example. Consider the following DAE on $[0, \infty)$:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{3.10}$$

Suppose the input is discontinuous at some fixed time $\tau > 0$ and let $x_0 \in \mathbb{R}^2$ be some fixed initial state. For this example, the solution formula will be explicitly computed in order to observe the relation between the input and the state. Firstly, let us compute the Wong limits $\mathcal{V}^*_{(E,A)}, \mathcal{W}^*_{(E,A)}$:

$$\mathcal{V}^0_{(E,A)} = \mathbb{R}^2,\ \mathcal{V}^1_{(E,A)} = \mathrm{Im}[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}],\ \mathcal{V}^2_{(E,A)} =: \mathcal{V}^*_{(E,A)} = \{0\},$$
$$\overset{\text{Thm 1}}{\Longrightarrow} \mathcal{W}^*_{(E,A)} = \mathbb{R}^2$$

Hence, pick $V = \emptyset$ and $W = I_{2\times2}$. Then, using Theorem 2, one has that $T = W$ and $S = (AW)^{-1} = W^{-1}$. Furthermore, since $\dim \mathcal{V}^*_{(E,A)} = 0$, then one has that according to Definition 2 it must be that $\Pi_{(E,A)} = \Pi^{\text{diff}}_{(E,A)} = O_{2\times2}$. Furthermore, since $ST = TS = I_{2\times2}$, one has that $\Pi^{\text{imp}}_{(E,A)} = TT^{-1}I_{2\times2} = I_{2\times2}$. Hence, one has:

$$A^{\text{diff}} = \Pi^{\text{diff}}_{(E,A)} A = O_{2\times2},\ B^{\text{diff}} = \Pi^{\text{diff}}_{(E,A)} B = O_{2\times1}$$
$$E^{\text{imp}} = \Pi^{\text{imp}}_{(E,A)} E = E, \qquad B^{\text{imp}} = \Pi^{\text{imp}}_{(E,A)} B = B$$

Furthermore, since $E^2 = O_{2\times2}$, one has that $q = 2$. The solution formula can now be simplified, as all matrices and parameters are now apparent. This yields the following:

$$x(\tau) = e^{A^{\text{diff}}\tau} \Pi_{(E,A)} x_0 + \int_0^\tau e^{A^{\text{diff}}(\tau-s)} B^{\text{imp}} u(s)\mathrm{d}s - \sum_{i=0}^{q-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t)$$
$$= O_{2\times2} x_0 + \int_0^\tau O_{2\times2} u(s)\mathrm{d}s - \sum_{i=0}^{1} E^i B u^{(i)}(\tau)$$
$$= -\begin{bmatrix} u(\tau) \\ u'(\tau) \end{bmatrix} \tag{3.11}$$

Observe that, since u is discontinuous at τ, one has that $u(t)$ is piecewise-continuous with a single jump at τ. Therefore, there exists u_1, u_2 such that u restricted to $[0, \tau)$ or $[\tau, \infty)$ is equal to u_1, u_2 respectively. Using some algebra and Corollary 2, one obtains for $u(t)$ and $u'(t)$:

$$u(t) = u_1(t)\mathbb{1}_{[0,\tau)}(t) + u_2(t)\mathbb{1}_{[\tau,\infty)}(t) = u_1(t)\mathbb{1}_{[0,\tau)}(t) + u_2(t)\mathbb{1}_{[\tau,\infty)}(t) + u_1(t)\mathbb{1}_{[\tau,\infty)}(t) - u_1(t)\mathbb{1}_{[\tau,\infty)}(t)$$
$$= u_1(t) + (u_2(t) - u_1(t))\mathbb{1}_{[\tau,\infty)}$$
$$\therefore u'(t) = u_1'(t) + (u_2'(t) - u_1'(t))\mathbb{1}_{[\tau,\infty)} + (u_2(t) - u_1(t))\delta_\tau(t)$$

Hence, as can be observed, the (distributional) derivative of the input experiences a Dirac impulse, and this impulse can only be prevented if and only if $u_1(\tau-) = u_2(\tau+)$ Furthermore, since $x^{(2)}(\tau) = u'(\tau)$, one additionally has that the state experiences a Dirac impulse aswell. Additionally, one can take note that if $u'(\tau)$ experiences a jump discontinuity. It is desired to find a space of initial states such that the solution the impulses in the solution (x, u) can be prevented. This property is known as impulse controllability.

If one interprets the DAE as an ITP, then it remains unknown how one must evaluate the impulsive part of x, as this is not given in Theorem 4. The following lemma will make this explicit:

Lemma 5. *Asssume (E, A) is regular, and let the ITP be given in Definition 5. If $u[0] = 0$, then the impulsive part of the solution with initial state $x(0-) = x_0$ can be computed as follows:*

$$x[0] = -\sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1} (I - \Pi) x_0 \delta^{(i)} - \sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1} \sum_{j=0}^{i} B^{\text{imp}} u^{(i-j)}(0+) \delta^{(j)}$$

Proof. See proof of [4, Theorem 5.1]. □

Back to the example from the introduction as in figure 1.3, using Lemma 1 and 5 and Corollary 1, one can show that the aforementioned electrical circuit won't cause any impulses. In particular, for this circuit one has that the nilpotent matrix in the quasi-Weierstrass form is equal to the zero matrix. Recall that the circuit had the following dynamics:

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ R_0 & 1 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad t \in [0, \tau)$$
$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ R_1 & 1 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad t \in [\tau, \infty)$$

For now, let $A = \begin{bmatrix} 0 & 1 \\ R_k & 1 \end{bmatrix}$, where $k = 0, 1$. Let $\lambda = 0$, then observe that $\det(-A) \neq 0$, and thus, $0 \in \mathbb{R} \setminus \sigma(E, A)$. Using Lemma 1, one finds for $\mathcal{W}^*$:

$$\mathcal{W}^* = \mathcal{W}^2 = \ker((A^{-1}E)^2) = \ker\left(\left(\frac{-1}{R_k}\begin{bmatrix} 1 & -1 \\ -R_k & 0 \end{bmatrix}\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}\right)^2\right) \cong \ker\left(\begin{bmatrix} L^2 & 0 \\ -L^2 R_k & 0 \end{bmatrix}\right) = \operatorname{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \ker(E)$$

And hence, $W = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. Using Corollary 1, one finds for N:

$$N = (AW)^\dagger EW = O, \quad \because W \in \ker(E)$$

Hence, if one applies Lemma 5 one trivially has that $x[0] = 0$ for both modes, and thus, both modes are impulse free. In the upcoming definition impulse controllability for the DAE will be given in terms of the ITP. Additionally, using the already defined subspaces one is able to derive a subspace for all states that are impulse controllable.

Definition 7. *The DAE (3.5) is called impulse controllable if for all $x_0 \in \mathbb{R}^n$ there exists a solution (x, u) of the ITP (5) with $x(0-) = x_0$ and $(x, u)[0] = 0$. All states that are impulse controllable lie in the impulse controllable space, which is defined as:*

$$C^{\text{imp}}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \,\middle|\, \begin{array}{c} \exists \text{ solution } (x,u) \in \mathbb{D}^{n+m}_{pwC^\infty} \\ \text{of (5) such that} \\ x(0-) = x_0 \text{ and} (x,u)[0] = 0 \end{array} \right\} \tag{3.12}$$

By Definition 6, one is able to rephrase this definition into a more compact statement, which says that $C^{\text{imp}}_{(E,A,B)} = \mathbb{R}^n$. Additionally, the impulse controllable subspace can be expressed in terms of the consistency spaces and other more elemental subspaces of the system. This result will be stated in the following proposition:

Proposition 1. *Consider the DAE as in (3.5) for $t_0 = 0$ and define the impulse controllable - and augmented consistency space as in (3.12),(3.9) respectively. Then one can express the impulse controllable space as follows*

$$C^{\text{imp}}_{(E,A,B)} = \mathcal{V}_{(E,A,B)} + \ker(E) = \mathcal{V}_{(E,A)} + \mathcal{R} + \ker(E) = \mathcal{V}_{(E,A)} + \langle E^{\text{imp}} | B^{\text{imp}} \rangle + \ker E \tag{3.13}$$

Before proving the proposition, an additional lemma is needed in order to reduce some work. This lemma relates the consistency projector to the matrix E:

Lemma 6. *Let the DAE be given as in (3.5). Then one has the following result:*

$$\ker(E) \subseteq \ker(\Pi_{(E,A)}) \tag{3.14}$$

Proof. The quasi-Weierstrass form will be deployed to explicitly determine the subspaces involved in (3.15). To this end, consider the full rank matrices S, T, the nilpotent matrix N and the quasi-Weierstrass form of E as in Theorem 2. Since S, T are full rank matrices, one has that $\ker(E) \cong \ker(SET)$. Hence:

$$\ker(E) \cong \ker(SET) = \ker\left(\begin{bmatrix} I_{n_1} & O \\ O & N_{n-n_1} \end{bmatrix}\right)$$

Let $\ker(N) := \operatorname{span}\{\zeta_1, \ldots, \zeta_m\} \subseteq \mathbb{R}^{n-n_1}$, where $\zeta_1, \ldots, \zeta_m \in \mathbb{R}^{n-n_1}$. Using the rank-nullity theorem, one must have that $\dim\ker(N) =: m \leqslant n - n_1$. Hence, one can simplify:

$$\ker(E) = \ker\left(\begin{bmatrix} I_{n_1} & O \\ O & N_{n-n_1} \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0_{n_1} \\ \zeta_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0_{n_1} \\ \zeta_m \end{bmatrix}\right\}$$

Here, 0_{n_1} denotes the vector in $\mathbb{R}^{n_1}$ with all entries equal to 0. Let b_i denote the standard basis of $\mathbb{R}^{n-n_1}$ for $i \in \{1, \ldots, n - n_1\}$. Similarly, for $\Pi_{(E,A)}$ as in Definition 2 one can find for its kernel:

$$\ker(\Pi_{(E,A)}) \cong \ker\left(\begin{bmatrix} I_{n_1} & O \\ O & O \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 0_{n_1} \\ b_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0_{n_1} \\ b_{n-n_1} \end{bmatrix}\right\} \supseteq \operatorname{span}\left\{\begin{bmatrix} 0_{n_1} \\ \zeta_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0_{n_1} \\ \zeta_m \end{bmatrix}\right\} = \ker(E)$$

Which finishes the proof. □

Now, using Lemma 6, one is able to prove Proposition 1.

Proof. Using Lemma 4, the second and third equality follow directly from the first equality. Again, the subspace equality will be shown by proving both inclusions, abbreviated by $(\subseteq), (\supseteq)$ respectively.

($\subseteq$) Let $x_0 \in C^{\text{imp}}_{(E,A,B)}$. Then there exists an impulse-free solution (x, u) that solves the ITP as in Definition 5. Using Lemma 5, one can compute that:

$$\begin{aligned} x[0] = 0 &= -\sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1}(I - \Pi)x_0\delta^{(i)} - \sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1} \sum_{j=0}^{i} B^{\text{imp}} u^{(i-j)}(0+)\delta^{(j)} \\ &= -\sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1}(I - \Pi)x_0\delta^{(i)} - \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (E^{\text{imp}})^{i+1} B^{\text{imp}} u^{(i-j)}(0+)\delta^{(j)} \\ &= -\sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1}(I - \Pi)x_0\delta^{(i)} - \sum_{j=0}^{n-1}\sum_{i=j}^{n-1} (E^{\text{imp}})^{i+1} B^{\text{imp}} u^{(i-j)}(0+)\delta^{(j)} \\ &= -E^{\text{imp}} \sum_{k=0}^{n-1} \left((E^{\text{imp}})^{i}(I - \Pi)x_0 + \sum_{i=k}^{n-1} (E^{\text{imp}})^{k} B^{\text{imp}} u^{(i-k)}(0+) \right) \delta^{(k)} \end{aligned}$$

Since $x[0] = 0$, one must have that for all $k \in \{0, \ldots, n-1\}$:

$$(E^{\text{imp}})^{k}(I - \Pi)x_0 + \sum_{i=k}^{n-1} (E^{\text{imp}})^{k} B^{\text{imp}} u^{(i-k)}(0+) \in \ker E$$

In particular, for $k = 0$ one obtains:

$$(I - \Pi)x_0 + \sum_{i=0}^{n-1} (E^{\text{imp}})^{k} B^{\text{imp}} u^{(i)}(0+) \in \ker E$$

Hence, there exists $\gamma \in \ker E$ such that $(I - \Pi)x_0 + \sum_{i=0}^{n-1} (E^{\text{imp}})^{k} B^{\text{imp}} u^{(k-i)}(0+) = \gamma$, then:

$$x_0 = \Pi x_0 - \sum_{i=0}^{n-1} (E^{\text{imp}})^{k} B^{\text{imp}} u^{(i)}(0+) + \gamma$$

Which shows that $C^{\text{imp}}_{(E,A,B)} \subseteq \operatorname{Im}\Pi + \langle E^{\text{imp}} | B^{\text{imp}} \rangle + \ker E = \mathcal{V}_{(E,A,B)} + \ker E$.

($\supseteq$) Suppose that $x_0 \in \mathcal{V}_{(E,A,B)}$. By definition of the augmented consistency space as in (3.9), there exists a smooth solution (x, u) of $E\dot{x} = Ax + Bu$ such that $x(0) = x_0$. Since it is a smooth solution, one by default has that $(x, u)[0] = 0$, thus $x_0 \in C^{\text{imp}}_{(E,A,B)}$. Next, suppose that $x_0 \in \ker(E)$. If one chooses the zero input $u = 0$, one has that a jump occurs at $t = 0$, where $x(0-) = x_0$ and $x(0+) = 0$. However, since the input is chosen smoothly, the input won't cause any impulses to occur in the solution. Hence, one has that $x[0] = 0$. Hence, $x_0 \in C^{\text{imp}}_{(E,A,B)}$. By linearity it follows directly that $\mathcal{V}_{(E,A,B)} + \ker(E) \subseteq C^{\text{imp}}_{(E,A,B)}$, and thus, the desired statement has been proven.

□

3.3 Switched DAEs

Equipped with our mathematical preliminaries on non-switched DAEs, we can now return to the study of switched DAEs. A switched DAE is a combination of several DAEs called *modes*, linked together through a *switching signal* σ, which is a function that maps the time t to the active mode. Let $p \in \mathbb{N}$ be the last mode of the switched DAE, then there are $p+1$ modes $\{0, \ldots, p\}$. The matrix triplet in the regular DAE (E, A, B) now changes whenever one switches mode. This triplet will be denoted by (E_i, A_i, B_i) for mode i. Thus, since the matrix triplet changes per mode, one has that the matrix triplet of a switched DAE is piecewise-constant, and therefore, one can define appropriate maps to capture this behaviour, namely $E_\sigma : t \to E_{\sigma(t)},\ A_\sigma : t \to A_{\sigma(t)},\ B_\sigma : t \to B_{\sigma(t)}$, where it has been assumed that $t \in [0, T)$. Hence, the switched DAE takes the following form:

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \quad t \in [0, T) \tag{3.15}$$

The aforementioned switched DAE with coefficient matrices $(E_\sigma, A_\sigma, B_\sigma)$ will be denoted by $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ A switched DAE can be interpreted as a repeated ITP. It is nontrivial that an arbitrary ITP has a unique solution under the supposition that (E, A) is regular. This is shown in [4]. It will be assumed throughout this thesis that the matrix pair

(E_i, A_i) is regular for all $i \in \{0, \dots, p\}$.

Observe that a switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ can be associated with the collection $\left\{\{E_k, A_k, B_k\}_{k=1}^{k=p}, \sigma\right\}$. If one interprets the switched DAE as a repeated ITP, one can rewrite this as several DAE's restricted to their own interval. However, if an arbitrary distribution is chosen, then this restriction doesn't necessarily have to exist, as is shown in [4, Remark 2.4]. Distributions chosen from $\mathbb{D}_{pw\mathcal{C}^\infty}$ do have a well defined restriction to intervals, as can be seen in [4, p. 7]. Furthermore, if the switching times do not accumulate, then the following result can be formulated:

Corollary 4. *[4, Corollary 5.2] Consider the switched DAE as in (3.15) and suppose that (E_k, A_k) is regular. Define the following class of switching signals:*

$$\mathcal{S} := \left\{ \sigma : \mathbb{R} \mapsto \{0, \dots, p\} \;\middle|\; \begin{array}{c} \sigma \text{ has locally finite many switches,} \\ \sigma \text{ constant on } (-\infty, 0) \end{array} \right\}$$

Then there exists a global solution $x \in (\mathbb{D}_{pw\mathcal{C}^\infty})^n$ which is uniquely determined by $x(0-)$.

Impulse controllability is yet to be defined in terms of the switched DAE. The following definition characterizes this:

Definition 8. *Let the switched DAE be given as in (3.15) for some fixed switching signal $\sigma \in \mathcal{S}$ on some interval $[0, T)$. The system is called impulse controllable if for all $x_0 \in \mathcal{V}_{(E_0, A_0, B_0)}$ there exists a solution pair $(x, u) \in (\mathbb{D}_{pw\mathcal{C}^\infty})^{n+m}$ for (3.15) with $x(0+) = x_0$ which is impulse free, i.e. $(x, u)[t] = 0$ for all $t \in [0, T)$.*

Observe that, by definition, if no switches occur one must have that the switched DAE is trivially impulse controllable due to the definition of the augmented consistency space. As Definition 8 states, one chooses the initial state to be in the augmented consistency space of the initial mode. Then, by definition of the augmented consistency space, there exists a smooth solution (x, u) that solves the DAE belonging to the first mode, which by definition exhibits no impulses.

It is worthwile to note the differences between the defining properties for impulse controllability for a DAE and a switched DAE. In the DAE case, see Definition 3.12, one interprets the DAE as an ITP for which the DAE was "inactive" before $t = 0$. The reason why the definition uses the condition that $(x, u)[0] = 0$, rather than $(x, u)[t] = 0$, is because the solution can already be made impulse free for $t > 0$. For the switched DAE case, see Definition 8, one interprets the switched DAE as a repeated ITP. It could be that if a solution is smooth in one mode, it is not necessarily sufficiently smooth in the next mode. For example, consider the following switched DAE for some fixed $\tau \in (0, \infty)$:

$$\Gamma : \begin{cases} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\dot{x} = x, & t \in [0, \tau) \\ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\dot{x} = x, & t \in [\tau, \infty) \end{cases}$$

Observe that $\mathcal{V}_0 = \operatorname{Im} \Pi_0 = \operatorname{Im}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\mathcal{V}_1 = \operatorname{Im} \Pi_1 = \operatorname{Im}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Hence, a solution that is smooth in mode 0 is not smooth in mode 1, unless the solution is trivial.

The modes in themselves follow the geometric notations discussed in the previous subsection, and will be labelled using a subscript instead of the usual matrix triplet, i.e. C_1^{imp} denotes the impulse controllable space of mode 1 corresponding to the matrix triplet (E_1, A_1, B_1).

To establish impulse controllability under the independence of the switching signal, one has to ensure that the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable for any chosen switching signals. Therefore, it has to be explored what kind of switching signals are appropriate to be admitted.

In many practical applications the order in which the switches occur is fixed. Thus, it seems reasonable to assume that the switching will happen in a fixed order. Hence, by possibly relabeling of the modes, one must have that the switching happens from mode 1 to mode 2 and so forth. This collection of switching signals $\mathcal{S}_n$ takes the form:

$$\mathcal{S}_p = \left\{ \sigma : \mathbb{R} \to \{0, ..., p\} \;\middle|\; \begin{array}{c} \exists \{t_i\}_{i=1}^{i=p} \text{ such that } t_0 < t_{i-1} < t_i < T =: t_{p+1} \; \forall i \in \{2, ..., p\} \\ \text{and } \sigma(t) = \sum_{i=0}^{p} i \mathbb{1}_{[t_i, t_{i+1})}(t) \end{array} \right\} \tag{3.16}$$

However, it might be feasible to also explore impulse controllability for a bigger class of switching signals. In practice one can have that due to component failure not all modes are reachable anymore. Hence, the order in which the modes will appear becomes arbitrary.

Define $\rho : \{0, \dots p\} \to \{0, \dots, p\}$ as the *mode projector*. One can think of ρ as the bijective function that permutes the original set $\{0, \dots, p\}$ in such a manner to obtain the correct order in which the switching will occur. Define $\mathcal{P}_p$ as the collection containing all possible mode projectors. In the upcoming the convention $\rho(i) := \rho_i$ will be supposed. Finally, the class of switching $\tilde{\mathcal{S}_n}$ signals take the following form:

$$\tilde{\mathcal{S}_p} = \left\{ \sigma : \mathbb{R} \to \{0, ..., p\} \;\middle|\; \begin{array}{c} \exists \{t_i\}_{i=1}^{i=p} \; \exists \rho \in \mathcal{P}_p \text{ such that } t_0 < t_{i-1} < t_i < T =: t_{p+1} \; \forall i \in \{2, ..., p\} \\ \text{and } \sigma(t) = \sum_{i=0}^{p} \rho_i \mathbb{1}_{[t_i, t_{i+1})}(t) \end{array} \right\} \tag{3.17}$$

In order to finally establish impulse controllability for open switching times, one is able to define two *system class* containing the switched DAEs $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ for all σ in $\mathcal{S}_p, \tilde{\mathcal{S}}_p$ respectively. Two system classes can be identified, namely $\Sigma_p\{(E_k, A_k, B_k)\}$ and $\tilde{\Sigma}_p\{(E_k, A_k, B_k)\}$ as defined below:

$$\Sigma_p\{(E_k, A_k, B_k)\} := \{\Sigma(E_\sigma, A_\sigma, B_\sigma) \mid \sigma \in \mathcal{S}_p\} \tag{3.18}$$

$$\tilde{\Sigma}_p\{(E_k, A_k, B_k)\} := \left\{\Sigma(E_\sigma, A_\sigma, B_\sigma) \mid \sigma \in \tilde{\mathcal{S}}_p\right\} \tag{3.19}$$

Observe that the system classes $\Sigma_p\{(E_k, A_k, B_k)\}$ and $\tilde{\Sigma}_p\{(E_k, A_k, B_k)\}$ can be identified with the collections $\left\{\{E_k, A_k, B_k\}_{k=1}^{k=p}, \mathcal{S}_p\right\}$, and $\left\{\{E_k, A_k, B_k\}_{k=1}^{k=p}, \tilde{\mathcal{S}}_p\right\}$, respectively. Impulse controllability is yet to be defined in terms of the aforementioned system classes. It seems like a natural extension to require that every switched DAE in the system class must be impulse controllable. This suggest that a distinction has to be made between *strong* - and *weak impulse controllability* of system classes, because $\Sigma_p\{(E_k, A_k, B_k)\} \subset \tilde{\Sigma}_p\{(E_k, A_k, B_k)\}$. This is done in the following definition:

Definition 9. *Let $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ be some switched DAE with $p+1$ modes for some switching signal σ. It is said that Σ is weakly impulse controllable if $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable for all $\sigma \in \mathcal{S}_p$ and Σ is strongly impulse controllable if $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable for all $\sigma \in \tilde{\mathcal{S}}_p$. In terms of the system classes, Σ is weakly impulse controllable if $\forall \bar{\Sigma} \in \Sigma_p\{(E_k, A_k, B_k)\}$ one has that $\bar{\Sigma}$ is impulse controllable. Similarly, Σ is strongly impulse controllable if $\forall \bar{\Sigma} \in \tilde{\Sigma}_p\{(E_k, A_k, B_k)\}$ one has that $\bar{\Sigma}$ is impulse controllable.*

One should also take note that if a mode projector has been fixed, one is able to relabel the system to the trivial order. Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ for some $\sigma \in \tilde{\mathcal{S}}_p$. Hence, there exists a mode projector $\rho \in \mathcal{P}_p$ such that $\sigma = \sum_{i=0}^{p} \rho(i) \mathbb{1}_{[t_i, t_{i+1})}$.

Observe that the maps $E_\sigma, A_\sigma, B_\sigma$ are equal to $E_{\rho(i)}, A_{\rho(i)}, B_{\rho(i)}$ if $t \in [t_i, t_{i+1})$. Define $P_i = E_{\rho(i)}$, $Q_i = A_{\rho(i)}$ and $R_i = B_{\rho(i)}$ and let $\gamma = \sum_{i=0}^{p} i \mathbb{1}_{[t_i, t_{i+1})}$. Observe that $\Sigma(E_\sigma, A_\sigma, B_\sigma) \cong \Sigma(P_\gamma, Q_\gamma, R_\gamma)$, whilst $\Sigma(P_\gamma, Q_\gamma, R_\gamma)$ has the trivial order.

This can even be generalized. Redefine $P_i^\rho = E_{\rho(i)}$, $Q_i^\rho = A_{\rho(i)}$ and $R_i^\rho = B_{\rho(i)}$ and let $\gamma = \sum_{i=0}^{p} i \mathbb{1}_{[t_i, t_{i+1})}$. Hence, if $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is a switched DAE for some $\sigma \in \tilde{\mathcal{S}}_p$, one has that $\Sigma(E_\sigma, A_\sigma, B_\sigma) \cong \Sigma(P_\gamma^\rho, Q_\gamma^\rho, R_\gamma^\rho)$, where ρ has been induced by σ.

This implies that in order to verify if a system is strongly impulse controllable, one has to verify that the system is weakly impulse controllable for all mode projectors $\rho \in \mathcal{P}_p$.

4 Impulse controllability for open switching times

4.1 Introduction

This thesis aims to provide necessary and sufficient conditions for impulse controllability of switched DAEs given unknown switching times. Impulse controllability will be characterized in terms of the aforementioned geometric notations of the modes in order to guarantee the independence of the switching signal. It might be worthwile investigating an example for which the impulse controllability can be ensured independent of the switching signal. To this end, consider the following switched DAE $\tilde{\Sigma}$:

$$\tilde{\Sigma}: \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), & 0 \leqslant t < t_1 \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t), & t_1 \leqslant t < t_2 \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t), & t_2 \leqslant t \end{cases} \tag{4.1}$$

Where t_i are assumed to be unknown for all $i = 1, 2, 3$. Clearly $\bar{\Sigma}$ is impulse controllable regardless of the switching times if all modes are in themselves fully impulse controllable, that is, if $C_i^{\text{imp}} = \mathbb{R}^n$ for all $i = 1, 2, 3$. In order to actually show that all modes are in itself fully controllable, several subspaces have to be computed for each mode, i.e. the kernel of E alongside (3.9),(3.8),(3.12). The following table ensures that the involved subspaces become apparent:

Table 4.1: Table of important subspaces of the switched DAE $\tilde{\Sigma}$

Mode	$\mathcal{V}_{(E_i,A_i)}$	$\mathcal{V}_{(E_i,A_i,B_i)}$	$\ker E_i$	C_i^{imp}
1	$\operatorname{Im}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\operatorname{Im}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^3$	$\operatorname{Im}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\operatorname{Im}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^3$
2	$\operatorname{Im}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\operatorname{Im}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^3$	$\operatorname{Im}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\operatorname{Im}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^3$
3	$\operatorname{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\operatorname{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\operatorname{Im}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\operatorname{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \operatorname{Im}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \operatorname{Im}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{R}^3$

In the preceding example it has been assumed that the initial time $t_0 = 0$ and the final time $T \to \infty$. Additionally, in the aforementioned example it is assumed that the order of the switches is fixed. Thus this means that in the aforementioned example impulse controllability has been ensured for all $\sigma \in \mathcal{S}_3$. Therefore, the system $\bar{\Sigma}$ is weakly impulse controllable.

Not only is $\bar{\Sigma}$ weakly impulse controllable, $\bar{\Sigma}$ is strongly impulse controllable, as the order in which the modes occur won't change that all modes are fully impulse controllable. Therefore, $\bar{\Sigma}$ is impulse controllable regardless the chosen mode projector ρ. Thus, $\bar{\Sigma}$ is impulse controllable for all $\sigma \in \tilde{\mathcal{S}}_3$. In turn this implies again that $\tilde{\Sigma}_3$ is impulse controllable, showing that $\bar{\Sigma}$ is strong impulse controllable.

Looking back at the aforementioned example it seems trivial that $\tilde{\Sigma}$ is strongly impulse controllable, as all the impulse controllable spaces of the modes equal $\mathbb{R}^3$. However, observe from the preceding table that $\mathcal{V}_{(E_k,A_k,B_k)} \subseteq C_q^{\text{imp}}$ for all $k, q \in \{1, 2, 3\}$. This implies that any smooth trajectory in mode k will cause no impulses, but can cause jumps, when switched to mode q. Before formalising this it might be fruitful to consider another example for which the impulse controllable spaces aren't necessarily equal to $\mathbb{R}^n$:

$$\bar{\Sigma}: \begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t), & 0 \leqslant t < t_1 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t), & t_1 \leqslant t < t_2 \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t), & t_2 \leqslant t \end{cases} \tag{4.2}$$

Where $\mathcal{V}_{(E_i,A_i,B_i)} =: \mathcal{V}_i$. As can be observed from Table 4.2, no mode is fully impulse controllable. However, any smooth trajectory starting in mode p will cause no impulse when switched to mode q whenever $p, q \in \{1, 2, 3\}$. This is a consequence of the fact that $\mathcal{V}_p \subseteq C_q^{\text{imp}}$ for all $p, q \in \{1, 2, 3\}$. This implies that, regardless $\sigma \in \tilde{\mathcal{S}}_3$, one must have that no impulses (perhaps jumps) occur in the state whatever the initial state is in the augmented consistency space of the first active mode. Hence, one must have that $\bar{\Sigma}$ is strongly impulse controllable, as every system in $\tilde{\Sigma}_3\{(E_k, A_k, B_k)\}$ is impulse controllable.

Table 4.2: Table of important subspaces of the switched DAE $\bar{\Sigma}$

Mode	$\mathcal{V}_i$	$\ker E_i$	C_i^{imp}
1	$\operatorname{Im}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}$	$\operatorname{Im}\begin{bmatrix}0&0\\1&0\\0&0\\0&1\\0&0\end{bmatrix}$	$\operatorname{Im}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}+\operatorname{Im}\begin{bmatrix}0&0\\1&0\\0&0\\0&1\\0&0\end{bmatrix}=\operatorname{Im}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$
2	$\operatorname{Im}\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$	$\operatorname{Im}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}$	$\operatorname{Im}\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}+\operatorname{Im}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}=\operatorname{Im}\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$
3	$\operatorname{Im}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}$	$\operatorname{Im}\begin{bmatrix}0&0\\0&0\\0&0\\1&0\\0&1\end{bmatrix}$	$\operatorname{Im}\begin{bmatrix}1&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}+\operatorname{Im}\begin{bmatrix}0&0\\0&0\\0&0\\1&0\\0&1\end{bmatrix}=\operatorname{Im}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

4.2 Switched DAE with 2 modes

For now, consider $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ on the interval $[0, T)$ as a switched DAE with 2 modes with matrix pairs $\{(E_0, A_0, B_0), (E_1, A_1, B_1)\}$ for some switching signal σ. The set of mode projectors take the form $\mathcal{P}_2 = \{\text{id}, \rho\}$, where $\rho_0 = 1, \rho_1 = 0$. For now, the *initial mode* will refer to the value of the mode projector at $i = 0$ and the *final mode* will refer to the value of the mode projector at $i = 1$. Assume that the switch happens at some arbitrary time $\tau \in [0, T)$. It seems like a natural extension to express impulse controllability of $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ in terms of the switching time. If the state reaches the impulse controllable space of the final mode just before the switch happens, one can ensure that no impulses will occur in the solution. This will be shown in the following lemma:

Lemma 7. *Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ on the interval $[0, T)$ for some $\sigma \in \mathcal{S}_2$ and let $\tau \in (0, T)$ be some fixed switching time. $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable if and only if for all $x_0 \in \mathcal{V}_0$ there exists an input $u(t)$ such that $x(\tau-) \in C_1^{imp}$.*

Proof. ($\Longrightarrow$) Assume $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable, then there exists a smooth solution $(x, u) \in \mathbb{D}_{\text{pw}C^\infty}^{n+m}$ such that the trajectory and input are impulse-free on $[0, T)$. Hence, (x, u) is an impulse-free solution of $E_1\dot{x} = A_1 x + B_1 u$ for $t \in [\tau, \infty)$. Define $g(t) := x(t + \tau)$ and $\tilde{u}(t) := u(t + \tau)$. Hence, $(g, \tilde{u})$ is an impulse-free solution of $E_1\dot{g} = A_1 g + B_1\tilde{u}$ for $t \in [0, \infty)$, thus $g(0-) := x(\tau-) \in C_1^{\text{imp}}$.

($\Longleftarrow$) Let $x_0 \in \mathcal{V}_{(E_0, A_0, B_0)}$. Assume that $x(\tau-) =: x_\tau \in C_1^{\text{imp}}$ is a given. By definition of the augmented consistency space, one has that (x, u_1) are smooth solutions on the interval $[0, \tau)$ with $x(0-) = x_0$. Hence, (x, u_1) are impulse free on $[0, \tau)$. Additionally, since $x(\tau-) \in C_1^{\text{imp}}$, one can find impulse free solutions (y, u_2) such that it solves $E_1\dot{y} = A_1 y + B_1 u_2$ with $y(0) = x_\tau$ on $t \in [0, \infty)$. Define $u(t) = u_1(t)\mathbb{1}_{[0,\tau)} + u_2(t - \tau)\mathbb{1}_{[\tau,\infty)}$ and $z(t) = x(t)\mathbb{1}_{[0,\tau)} + y(t - \tau)\mathbb{1}_{[\tau,\infty)}$. It can be shown that $\dot{z}$ contains no impulses:

$$\dot{z}(t) = \frac{\mathrm{d}}{\mathrm{d}t}[x(t) + (y(t - \tau) - x(t))\mathbb{1}_{[\tau,\infty)}] = x'(t) + (y'(t - \tau) - x'(t))\mathbb{1}_{[\tau,\infty)} + (y(t - \tau) - x(t))\delta_\tau$$

The impulse will have no effect if $y(0) - x(\tau-) = 0$, which is satisfied by construction. Hence, (z, u) is an impulse-free solution on $[0, \infty)$, thus $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable.

□

Lemma 7 will play a crucial role in determining a necessary geometric condition for weak/strong impulse controllability for switched DAEs with 2 modes. To derive such a geometric condition, observe that (3.6) suggests the solution can be written as the sum of an element of the image of the consistency projector and an element of the reachable subspace, for any mode. Lastly, as can be observed in the examples, a relationship has to be established between the aforementioned subspaces of the first mode with the impulse controllability subspace of the second mode. Before stating this as a theorem, one first has to introduce a seperate lemma:

Lemma 8. *Let $\mathcal{V}_{(E,A)} := \operatorname{Im}\Pi$ denote the consistency space of the DAE (2.1). Let $\Lambda := e^{A^{\text{diff}}}$. Then one has that $\Lambda^\ell \mathcal{V}_{(E,A)} \subseteq \mathcal{V}_{(E,A)}$ for all $\ell \in \mathbb{R}$.*

Proof. Fix $\ell \in \mathbb{R}$, and let $x_1 \in \Lambda^\ell \mathcal{V}_{(E,A)}$, hence $\exists x_0 \in \mathcal{V}_{(E,A)}$ such that $x_1 = \Lambda^\ell x_0$. Furthermore, there exists a smooth solution $\bar{x}(t)$ that solves the DAE (2.1) with $\bar{x}(0) = x_0$. Using the solution decomposition for $u = 0$, one has that $\bar{x}$ takes the following form:

$$\bar{x}(t) = \Lambda^t \Pi x_0 = \Lambda^t x_0$$

Define $\tilde{x}(t) := \bar{x}(t + \ell)$. Observe that $\tilde{x}$ still solves the DAE (2.1) whilst remaining smooth. Therefore $\tilde{x}(0) := \bar{x}(\ell) = x_1$ must be in $\mathcal{V}_{(E,A)}$. Hence, $\Lambda^\ell \mathcal{V}_{(E,A)} \subseteq \mathcal{V}_{(E,A)}$ for all $\ell \in \mathbb{R}$ □

Theorem 5. *Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ on the interval $[0, T)$ for some $\sigma \in \mathcal{S}_2$. Σ is weakly impulse controllable if and only if $\operatorname{Im}\Pi_0 \subseteq C_1^{imp} + \mathcal{R}_0$, where Π_i denotes the consistency projector of (E_i, A_i, B_i).*

Proof. ($\Longrightarrow$) Assume Σ is weakly impulse controllable. Let $(x,u) \in \mathbb{D}^{n+m}_{\text{pw}C^\infty}$ be an impulse-free solution for $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ for some fixed $\sigma \in \mathcal{S}_2$ given some initial trajectory $x(0+) = x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$. Using the solution formula decomposition, one finds that for $t \in [0,\tau)$:

$$x(t) = e^{A_0^{\text{diff}} t}\Pi_0 x_0 + x_u(t,0) =: \Lambda_0^t \Pi_0 x_0 + x_u(t,0)$$

Clearly $\Pi_0 x_0 \in \operatorname{Im}\Pi_0$ and $x_u(t,0) \in \mathcal{R}_0$ for all $t \in [0,\tau)$. Using Lemma 8, one can conclude that that $\Lambda_0^t \Pi_0 x_0 \in \operatorname{Im}\Pi_0$ for all $t \in [0,\tau)$. In particular, let $x_1 := \Lambda_0^\tau \Pi_0 x_0$, and using Lemma 7 one has that $x(\tau-) \in C_1^{\text{imp}}$, then one finds that:

$$x(\tau-) = x_1 + x_u(\tau-,0)$$
$$x_1 = x(\tau-) - x_u(\tau-,0) \in C_1^{\text{imp}} + \mathcal{R}_0$$

Which shows that if $\Sigma_0^1[E_k, A_k, B_k)$ is weak impulse controllable one has that $\operatorname{Im}\Pi_0 \subseteq C_1^{\text{imp}} + \mathcal{R}_0$.

($\Longleftarrow$) Assume that $\operatorname{Im}\Pi_0 \subseteq C_1^{\text{imp}} + \mathcal{R}_0$ and fix $x_0 \in \mathcal{V}_0$. Since $x_0 \in \mathcal{V}_0$, there must exist a smooth solution $(\tilde{x},\tilde{u})$ that solves the DAE $E_1\dot{\tilde{x}} = A_1\tilde{x} + B_1\tilde{u}$ with $\tilde{x}(0) = x_0$ for $t \in [0,\tau)$. Using the solution formula decomposition, one finds:

$$\tilde{x}(t) = \Lambda_0^t \Pi_0 x_0 + x_{\tilde{u}}(t,0)$$
$$\tilde{x}(\tau-) = \Lambda_0^\tau \Pi_0 x_0 + x_{\tilde{u}}(\tau-,0) =: x_1 + \eta$$

Where $x_1 := \Lambda_0^\tau \Pi_0 x_0$ and $\eta := x_{\tilde{u}}(\tau-,0)$. Using Lemma 8 one has that $x_1 \in \operatorname{Im}\Pi_0$. Furthermore, since $\operatorname{Im}\Pi_0 \subseteq C_1^{\text{imp}} + \mathcal{R}_0$, $\exists \alpha \in C_1^{\text{imp}}$ $\exists \beta \in \mathcal{R}_0$ such that $x_1 = \alpha + \beta$. By definition of $\mathcal{R}_0$, there exists a smooth solution $(\bar{x},\bar{u})$ such that $\bar{x}(\tau-) = \beta + \eta$ with $\bar{x}(0) = 0$. If one defines $\hat{x}(t) = \tilde{x}(t) - \bar{x}(t)$ and $\hat{u}(t) = \tilde{u}(t) - \bar{u}(t)$, then $(\hat{x},\hat{u})$ solves the DAE $E_1\dot{\hat{x}} = A_1\hat{x} + B\hat{u}$, with $\hat{x}(0) = \tilde{x}(0) - \bar{x}(0) = x_0 - 0 = x_0$ and $\hat{x}(\tau-) = \tilde{x}(\tau-) - \bar{x}(\tau-) = \alpha + \beta + \eta - (\beta + \eta) = \alpha \in C_1^{\text{imp}}$.

Hence, the solution $(\hat{x},\hat{u})$ can be extended from mode 0 to mode 1 in an impulse free manner. Since $\alpha \in C_1^{\text{imp}}$, there exists a solution $(\hat{y},\hat{k})$ such that it solves $E_1\dot{\hat{y}} = A_1\hat{y} + B_1\hat{k}$ with $\hat{y}(0) = \alpha$ for $t \in [0,\infty)$ in an impulse free manner. Define $x(t) = \hat{x}(t)\mathbb{1}_{[0,\tau)} + \hat{y}(t-\tau)\mathbb{1}_{[\tau,T)}$ and $u(t) = \hat{u}(t)\mathbb{1}_{[0,\tau)} + \hat{k}(t-\tau)\mathbb{1}_{[\tau,T)}$, then (x,u) is an impulse-free solution for $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ for all $t \in [0,T)$. Hence, Σ is weak impulse controllable. ☐

Lemma 7 was required as a stepping stone for Theorem 5, which characterizes impulse controllability for a system $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ with 2 modes. The beauty of Theorem 5 is that it is a necessary condition for impulse controllability of $\Sigma_0^1[E_k, A_k, B_k]$ for any $\sigma \in \mathcal{S}_2$ whilst being independent of the switching signal, which implies that $\Sigma[E_\sigma, A_\sigma, B_\sigma)$ is weak impulse controllable if Theorem 5 is satisfied. Furthermore, Theorem 5 can be interpreted as a condition that guarantees impulse controllability if one switches from mode 0 to mode 1. Hence, Theorem 5 can be rephrased to guarantee strong impulse controllability. One only has to guarantee impulse controllability for the other mode projector, or in other words, one has to guarantee that one can switch impulse-freely from mode 1 to mode 0. Therefore, if the theorem is rephrased one obtains:

Theorem 6. *Consider the switched DAE* $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ *on the interval* $[0,T)$ *for some* $\sigma \in \tilde{\mathcal{S}}_2$. Σ *is strong impulse controllable if and only if* $\operatorname{Im}\Pi_0 \subseteq C_1^{imp} + \mathcal{R}_0$, *and* $\operatorname{Im}\Pi_1 \subseteq C_0^{\text{imp}} + \mathcal{R}_1$.

Proof. Using Theorem 5, one already has that Σ is weak impulse controllable. Hence, in order to show that Σ is weakly impulse controllable for the other mode projector, one only has to show that impulse controllability can be ensured for any switching signal of the form:

$$\forall \tau \in (0,T): \ \sigma(t) = \mathbb{1}_{[0,\tau)}(t)$$

Thus, one has that one has to guarantee impulse controllability for switches from mode 1 to mode 0. Hence, if one rephrases Theorem 5 by swapping the subscripts, then one derives the other given expression, which ensures weak impulse controllability. Hence, Σ is strongly impulse controllable. ☐

In order to investigate impulse controllability of switched DAEs for arbitrary switching signals one has to first properly define what problem framework will be applied.

4.3 Switched DAE with $p+1$ modes

Consider a switched DAE with $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ with $p+1$ modes $\{(E_k, A_k, B_k)\}_{k=0}^p$ on the interval $[0,T)$ for some switching signal σ, where $T > 0$ is fixed.

By simply generalising Theorem 5 and 6 one can already obtain sufficient conditions for weak and strong impulse controllability. Observe that Theorem 5 can be generalized as a condition for impulse controllability when one switches from mode n to mode m, where $n,m \in \{0,\dots,p\}$. The following lemma will be formulated:

Lemma 9. *Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ with $p+1$ modes on the interval $[t_0, t_f)$ for some switching signal $\sigma \in \tilde{\mathcal{S}}_p$. Fix $n, m \in \{0, \dots, p\}$ and suppose σ induces a switch from mode n to mode m. Let $\Gamma_n^m(E_\gamma, A_\gamma, B_\gamma)$ be a switched DAE with matrix pairs $\{(E_n, A_n, B_n), (E_m, A_m, B_m)\}$ for some $\gamma \in \mathcal{S}_2$. The switch from mode n to mode m of $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ can be performed impulse-freely if Γ_n^m is weak impulse controllable. Using Theorem 5 gives the following condition:*

$$\operatorname{Im} \Pi_n \subseteq C_m^{\text{imp}} + \mathcal{R}_n$$

If impulses can be prevented when one switches from mode $i-1$ to mode i $\forall i \in \{1, \dots, p\}$, one has that $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is weak impulse controllable. This yields the following result:

Theorem 7. *Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ with $p+1$ modes on the interval $[t_0, t_f)$ for some switching signal $\sigma \in \mathcal{S}_p$. If one has for all $i \in \{0, \dots, p-1\}$:*

$$\operatorname{Im} \Pi_i \subseteq C_{i+1}^{\text{imp}} + \mathcal{R}_i$$

Then Σ is weak impulse controllable for all $t \in [0, T)$.

Proof. Suppose that $\operatorname{Im} \Pi_i \subseteq C_{i+1}^{\text{imp}} + \mathcal{R}_i$ for all $i \in \{0, \dots, p-1\}$ and fix $x_0 \in \mathcal{V}_0$. Let $t_i \in (t_0, t_f)$ denote the switching time from mode $i-1$ to mode i, and furthermore, assume that t_i are distinct $\forall i \in \{1, \dots, p\}$.

Since $x_0 \in \mathcal{V}_0$, one has that the solution pair (x^0, u^0) is smooth on $[t_0, t_1)$. Using the solution decomposition one finds for x^0 at $t = t_1-$:

$$x^0(t_1-) = \Lambda_0^{t_1} \Pi_0 x_0 + x_{u^0}(t_1-, 0)$$

Using Lemma 8, one has that $y_0 := \Lambda_0 \Pi_0 x_0 \in \operatorname{Im} \Pi_0$, and hence, $\exists \alpha_0 \in C_1^{\text{imp}}$ $\exists \beta_0 \in \mathcal{R}_0$ such that $y_0 = \alpha_0 + \beta_0$. Define $\eta_0 := x_{u^0}(t_1-, 0)$. Since $\beta_0 + \eta_0 \in \mathcal{R}_0$, there exists a smooth solution $(\bar{x}^0, \bar{u}^0)$ of $E_0(\bar{x}^0)' = A_0(\bar{x}^0) + B_0 \bar{u}^0$ on $t \in [t_0, t_1)$ such that $\bar{x}^0(t_0) = 0$ and $\bar{x}^0(t) = \beta_0 + \eta_0$. Hence, the solution $(\hat{x}^0, \hat{u}^0) = (x^0 - \bar{x}^0, u^0 - \bar{u}^0)$ is a smooth solution of $E_0(\hat{x}^0)' = A_0 \hat{x}^0 + B_0 \hat{u}^0$ on $[t_0, t_1)$ with $\hat{x}^0(t_0) = x_0$ and $\hat{x}^0(t_1-) = \alpha_0 \in C_1^{\text{imp}}$.

This argument has to be generalized for all modes. Let (x^i, u^i) denote the solution for the DAE $E_i(x^i)' = A_i(x^i) + B_i(u^i)$ on the interval $t \in [t_i, t_{i+1})$ with $x^i(t_i) := \alpha_{i-1}$. One can derive the following:

$$x^i(t_{i+1}-) = \Lambda_i^{t_{i+1}-t_i} \Pi_i \alpha_{i-1} + x_{u^i}(t_{i+1}-, t_i) =: y_i + \eta_i$$

Similarly, $y_i := \Lambda_i^{t_{i+1}} \Pi_i \alpha_{i-1} \in \operatorname{Im} \Pi_i$, hence $\exists \alpha_i \in C_{i+1}^{\text{imp}}$ $\exists \beta_i \in \mathcal{R}_i$ such that $y_i = \alpha_i + \beta_i$. By definition of $\mathcal{R}_i$, there exists a smooth solution $(\bar{x}^i, \bar{u}^i)$ such that $\bar{x}^i(t_i) = 0$ and $\bar{x}^i(t_{i+1}-) = \beta_i + \eta_i$. Define $\hat{x}^i = x^i - \bar{x}^i$ and $\hat{u}^i = u^i - \bar{u}^i$. Hence, $(\hat{x}^i, \hat{u}^i)$ is an impulse-free solution of $E_i(\hat{x}^i)' = A_i(\hat{x}^i) + B_i \hat{u}^i$ for $t \in [t_i, t_{i+1})$ for all $i \in \{1, \dots, p-1\}$ with $\hat{x}^i(t_i) = \alpha_{i-1}$ and $\hat{x}^i(t_{i+1}-) = \alpha_i$. Additionally, since $\alpha_{p-1} \in C_p^{\text{imp}}$, there exists a solution $(\hat{x}^p, \hat{u}^p)$ such that it solves $E_p(\hat{x}^p)' = A_p(\hat{x}^p) + B_p \hat{u}^p$ on $t \in [t_p, t_f)$ with $y(t_p) = \alpha_{p-1}$.

Define $t_{p+1} := t_f$ and consider $x(t) = \sum_{i=0}^p x^i(t) \mathbb{1}_{[t_i, t_{i+1})}$ and $u(t) = \sum_{i=0}^p u^i(t) \mathbb{1}_{[t_i, t_{i+1})}$. Observe that (x, u) is an impulse-free solution for $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ on $[t_0, t_f)$ for any fixed switching signal $\sigma \in \mathcal{S}_p$, which shows that Σ is weak impulse controllable. □

Similarly, one can use this approach to derive a sufficient condition for strong impulse controllability. Observe that, for some fixed $n \in \{0, \dots, p\}$ one has that:

$$\operatorname{Im} \Pi_n \subseteq C_m + \mathcal{R}_n \; \forall m \in \{0, \dots, p\} \implies \operatorname{Im} \Pi_n \subseteq \bigcap_{m=0}^{p} C_m + \mathcal{R}_n$$

Using this, one can find a sufficient condition for strong impulse controllability Σ:

Theorem 8. *Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ with $p+1$ modes on the interval $[t_0, t_f)$ for some switching signal $\sigma \in \tilde{\mathcal{S}}_p$. If for all $i \in \{0, \dots, p\}$ one has:*

$$\operatorname{Im} \Pi_i \subseteq \bigcap_{k=0}^{p} C_k^{\text{imp}} + \mathcal{R}_i$$

Then one has that Σ is strongly impulse controllable.

Proof. Observe that strong impulse controllability is equivalent to weak impulse controllability for all mode projectors. Fix $\rho \in \mathcal{P}_p$. Observe that the given statement can be rewritten as:

$$\operatorname{Im} \Pi_{\rho(i)} \subseteq \bigcap_{k=0}^{p} C_k^{\text{imp}} + \mathcal{R}_{\rho(i)} \subseteq C_{\rho(i+1)}^{\text{imp}} + \mathcal{R}_{\rho(i)}$$

Since ρ has been fixed, one can relabel the modes without loss of generality, such that $\rho = \text{id}$. Then the derived statement is equivalent to the weak impulse controllability condition in Theorem 7 for some fixed $\rho \in \mathcal{P}_p$. Since the statement can be repeated for any $\rho \in \mathcal{P}_p$, one has that Σ is weak impulse controllable for all $\rho \in \mathcal{P}_p$, and thus, Σ is strong impulse controllable. □

Furthermore, this condition was derived by fixing n, but one is also able to fix m and let n be free, this yields the following sufficient condition:

Theorem 9. *Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ with $p+1$ modes on the interval $[t_0, t_f)$ for some switching signal $\sigma \in \tilde{\mathcal{S}}_p$. If for all $i \in \{0, \ldots, p\}$ one has:*

$$\operatorname{Im}\begin{bmatrix}\Pi_0 & \ldots & \Pi_p\end{bmatrix} \subseteq C_i^{\text{imp}} + \bigcap_{k=0}^{p} \mathcal{R}_k$$

Then one has that Σ is strong impulse controllable.

Proof. Fix $n, m \in \{0, \ldots, p\}$, this condition is equivalent to:

$$\operatorname{Im}\Pi_n \subseteq \operatorname{Im}\begin{bmatrix}\Pi_0 & \ldots & \Pi_p\end{bmatrix} \subseteq C_m^{\text{imp}} + \bigcap_{k=0}^{p} \mathcal{R}_k \subseteq C_m^{\text{imp}} + \mathcal{R}_n$$

As already seen, the condition $\operatorname{Im}\Pi_n \subseteq C_m^{imp} + \mathcal{R}_n$ is equivalent to Theorem 8, and hence, Σ is strongly impulse controllable. □

The problem with Theorems 7, 8 and 9 is that they are listed as sufficient conditions. Whilst it has been proven that these conditions are sufficient, no reasoning is given as to why they are not necessary. Future research has to determine if Theorems 7, 8 and 9 are or aren't necessary, either by providing a proof or a simple counterexample to the necessity.

To derive (possibly) necessary geometric conditions for which the system Σ is either weak or strong impulse controllable, this thesis will make use of a sequence of subspaces which will be similar to the backward approach as was done in [6, p. 5]. Before proceeding it seems worthwile to investigate several important results of [6] which can serve as a guide to build up the abstraction. In the upcoming it will be assumed that the mode projector is the identity map.The method starts with the impulse controllable space of the last mode, i.e. mode p, call this $\mathcal{K}_p^b$. Then $\mathcal{K}_{p-1}^b$ is the set of points that can reach the impulse controllable space of mode p when the system behaves as in mode $p-1$. Similarly, $\mathcal{K}_{p-2}^b$ are the points such that it can reach $\mathcal{K}_{p-1}^b$ when the system behaves as in mode $p-2$. If this sequence is continued, one will eventually find the last subspace in the sequence $\mathcal{K}_1^b$. This subspace contains all points that can be steered into C_p^{imp} from mode 1 and onward.

The following sequence of subspaces that will be introduced will be the same as was introduced in the backward approach in [6] for some fixed $\sigma \in \mathcal{S}_p$. Define $\tilde{\Lambda}_{i-1} := e^{A_{i-1}^{\text{diff}}(t_i - t_{i-1})}$. Consider the following sequence of subspaces going backwards, computed for $i = p, p-1, \ldots$:

$$\begin{aligned}\mathcal{K}_p^b &= C_p^{\text{imp}} \\ \mathcal{K}_{i-1}^b &= \operatorname{Im}(\Pi_{i-1}) \cap \left(\tilde{\Lambda}_{i-1}^{-1}\mathcal{K}_i^b + \mathcal{R}_{i-1}\right) + \langle E_{i-1}^{\text{imp}} | B_{i-1}^{\text{imp}}\rangle + \ker(E_{i-1})\end{aligned} \tag{4.3}$$

If $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is restricted to the interval $[t_{i-1}, t_i)$, one has that all states at $t = t_{i-1}^-$ that can reach $\mathcal{K}_i^b$ impulse freely at $t = t_i^-$ is given by the set $\mathcal{K}_{i-1}^b$. This will be shown in the following theorem:

Theorem 10. *(See [6, Lemma 19]) Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ restricted to the interval $[t_{i-1}, t_i)$ for some fixed $\sigma \in \mathcal{S}_p$. Then $\mathcal{K}_{i-1}^b$ is the largest set of points at t_{i-1}^- that can reach $\mathcal{K}_i^b$ impulse-freely.*

Proof. The proof will be split up in 2 steps, which are:

1. First, it is shown that $\forall x(t_{i-1}^-) := x_{i-1} \in \mathcal{K}_{i-1}^b$ there exists an input such that this solution reaches $\mathcal{K}_i^b$ impulse freely. Or put differently, one must have $x_u(t_i^-, t_{i-1}^-; x_{i-1}) \in \mathcal{K}_i^b$. It will be shown that there exists an input u for all x_{i-1} in either $\operatorname{Im}(\Pi_{i-1}) \cap \left(\tilde{\Lambda}_{i-1}^{-1}\mathcal{K}_i^b + \mathcal{R}_{i-1}\right)$, $\langle E_{i-1}^{\text{imp}} | B_{i-1}^{\text{imp}}\rangle$ and $\ker(E_{i-1})$ for which $x_u(t_i^-, t_{i-1}^-; x_{i-1}) \in \mathcal{K}_i^b$. Then, by using linearity, it can be concluded that all points in $\mathcal{K}_{i-1}^b$ can reach $\mathcal{K}_i^b$ impulse-freely.

2. The last step will show that $\mathcal{K}_{i-1}^b$ is indeed the largest set of points at $t = t_{i-1}^-$ for which the set $\mathcal{K}_i^b$ can be reached impulse-freely at $t = t_i^-$. This will be done by showing that if (x, u) is an impulse-free solution for $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ on $[t_{i-1}, t_i)$, one must have that $x(t_i^-) := x_i \in \mathcal{K}_i^b$. Then it will be shown that $x(t_{i-1}^-) := x_{i-1} \in \mathcal{K}_{i-1}^b$.

(Step 1) 1. Let $x_{i-1} \in \operatorname{Im}(\Pi_{i-1}) \cap \left(\tilde{\Lambda}_{i-1}^{-1}\mathcal{K}_i^b + \mathcal{R}_{i-1}\right)$. Since $x_{i-1} \in \operatorname{Im}\Pi_{i-1}$, it serves as a consistent initial value for mode $i-1$ and therefore it produces no impulses when the zero input is chosen. Hence, $\exists(\hat{x}, 0)$ such that it solves $E_{i-1}(\dot{\hat{x}}) = A_{i-1}\hat{x}$ on $[t_{i-1}, t_i)$ with $\hat{x}(t_{i-1}^-) = x_{i-1}$. Furthermore, since $x_{i-1} \in \operatorname{Im}\Pi_{i-1}$, one has that $\Pi_{i-1}x_{i-1} = x_{i-1}$. Using the solution decomposition (3.6), one has that:

$$\hat{x}(t_i^-) = e^{A_{i-1}^{\text{diff}}(t_i - t_{i-1})}\hat{x}(t_{i-1}^-) =: \tilde{\Lambda}_{i-1}\hat{x}(t_{i-1}^-)$$

Additionally, since $x_{i-1} \in \tilde{\Lambda}_{i-1}^{-1}\mathcal{K}_b^i + \mathcal{R}_{i-1}$, $\exists\alpha \in \mathcal{K}_b^i\ \exists\beta \in \mathcal{R}_{i-1}$ such that $x_{i-1} = \tilde{\Lambda}_{i-1}^{-1}\alpha + \beta$. Using the expression for $\hat{x}(t_i^-)$ one finds:

$$\hat{x}(t_i^-) = \tilde{\Lambda}_{i-1}x_{i-1} = \alpha + \tilde{\Lambda}_{i-1}\beta \in \mathcal{K}_b^i + \mathcal{R}_{i-1}$$

Observe that the reachable subspace $\mathcal{R}_{i-1}$ is A_{i-1}^{diff} invariant. Hence, it is also $\tilde{\Lambda}_{i-1}$ invariant. Hence, $\exists\xi \in \mathcal{K}_i^b\ \exists\eta \in \mathcal{R}_{i-1}$ such that $\hat{x}(t_i^-) = \xi + \eta$. Choose a smooth solution $(\tilde{x}, u)$ of $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ on $[t_{i-1}, t_i)$ such that $\tilde{x}(t_{i-1}^-) = 0$ and $\tilde{x}(t_i^-) = -\eta$. Hence, the solution $(x, u) := (\hat{x} + \tilde{x}, u)$ exhibits no impulses on $[t_{i-1}, t_i)$ such that $x(t_{i-1}^-) = \hat{x}(t_{i-1}^-) + \tilde{x}(t_{i-1}^-) = x_{i-1}$ and $x(t_i^-) = \hat{x}(t_i^-) + \tilde{x}(t_i^-) = \xi \in \mathcal{K}_i^b$.

2. Let $x_{i-1} \in \langle E_{i-1}^{\text{imp}} | B_{i-1}^{\text{imp}}\rangle$. As has been noted in the proof of Lemma 4, one has that $\langle E_{i-1}^{\text{imp}} | B_{i-1}^{\text{imp}}\rangle \subseteq \mathcal{R}_{i-1}$. All points in $\mathcal{R}_{i-1}$ are the states that can be reached smoothly on $[t_{i-1}, t_i)$ from the origin. By applying a translation, $\mathcal{R}_{i-1}$ is equivalent to all initial states that can be steered to the origin on $[t_{i-1}, t_i)$. Hence, there exists an input u such that $x_u(t_i^-, t_{i-1}^-, x_{i-1}) = 0 \in \mathcal{K}_i^b$.

3. Let $x_{i-1} \in \ker E_{i-1}$. If the zero input is applied, then the solution $x_u(t, t_{i-1}^-, x_{i-1}) = 0$ exhibits at most a jump if $t > t_{i-1}$, which cannot be expressed as the linear combination of the Dirac impulses and its derivatives. Hence, (x, u) is impulse free on $[t_{i-1}, t_i)$ and furthermore $0 \in \mathcal{K}_i^b$.

Finally, by linearity one must have that $\forall x_{i-1} \in \mathcal{K}_{i-1}^b$, there exists an input u such that the state can be steered into $\mathcal{K}_i^b$.

(Step 2) Let (x, u) be any impulse free solution of $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ on $[t_{i-1}, t_i)$ such that $x(t_i^-) \in \mathcal{K}_i^b$. All it rests us to show that $x(t_{i-1}^-) \in \mathcal{K}_{i-1}^b$. Since it must be that the solution is impulse-free, it must certainly be that $x_{i-1} \in C_{i-1}^{\text{imp}} := \operatorname{Im}\Pi_{i-1} + \left\langle E_k^{\text{imp}} | B_k^{\text{imp}}\right\rangle + \ker E_{i-1}$. Hence, $\exists\xi \in \operatorname{Im}\Pi_{i-1}\ \exists\eta \in \left\langle E_k^{\text{imp}} | B_k^{\text{imp}}\right\rangle \exists\zeta \in \ker E_{i-1}$ such that $x_{i-1} = \xi + \eta + \zeta$. As was noted in the proof of Lemma 4, one has that $\langle E_{i-1}^{\text{imp}} | B_{i-1}^{\text{imp}}\rangle \subseteq \mathcal{W}^*_{(E_{i-1}, A_{i-1})}$). Using Theorem 1 and Lemma 6, one has that $x_{\text{aut}}(t, t_{i-1}^-, \eta) = x_u(t, t_{i-1}^-, \zeta) = 0$ for all $t \in (t_{i-1}, t_i)$. From this, it can be directly concluded that $x_u(t, t_{i-1}^-, \zeta) \in \mathcal{R}_{i-1}$. Hence, one derives for $x(t_i^-)$:

$$\begin{aligned} x_i = x(t_i^-) := x_u(t_i^-, t_{i-1}^-, x_{i-1}) &= x_u(t_i^-, t_{i-1}^-, \xi + \eta + \zeta) = x_u(t_i^-, t_{i-1}^-, \xi) + x_u(t_i^-, t_{i-1}^-, \eta) + x_u(t_i^-, t_{i-1}^-, \zeta) \\ &= x_{\text{aut}}(t_i^-, t_{i-1}^-, \xi) + x_u(t_i^-, t_{i-1}^-, \eta) \\ &= \tilde{\Lambda}_{i-1}\Pi_{i-1}\xi + x_u(t_i^-, t_{i-1}^-, \eta) \\ &= \tilde{\Lambda}_{i-1}\xi + x_u(t_i^-, t_{i-1}^-, \eta) \end{aligned}$$

Hence, if $\gamma = \tilde{\Lambda}_{i-1}^{-1}x_u(t_i^-, t_{i-1}^-, \eta) \in \mathcal{R}_{i-1}$, one can rewrite the aforementioned expression in terms of ξ:

$$\xi = \tilde{\Lambda}_{i-1}^{-1}x_i - \gamma$$

This directly shows that $\xi \in \tilde{\Lambda}_{i-1}^{-1}\mathcal{K}_i^b + \mathcal{R}_{i-1}$, and hence, $\xi \in \operatorname{Im}(\Pi)_{i-1} \cap \left(\tilde{\Lambda}_{i-1}^{-1}\mathcal{K}_i^b + \mathcal{R}_{i-1}\right)$. Hence, it has been shown that $x_{i-1} \in \mathcal{K}_{i-1}^b$.

□

As was stated in the beginning of this subsection, the space $\mathcal{K}_0^b$ contains the states in the initial mode for which the impulse controllable space of the last mode can be reached in an impulse free manner. As all trajectories in the initial mode start in its respective augmented consistency space, it seems reasonable to expect that weak impulse controllability can be ensured if this augmented consistency space is contained in $\mathcal{K}_0^b$.

Theorem 11. *Consider the switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ for some fixed $\sigma \in \mathcal{S}_p$ defined on the interval $[0, T]$. $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is called impulse controllable if and only if $\mathcal{V}_0 \subseteq \mathcal{K}_0^b$.*

Proof. See [6, Theorem 21]. □

These results have to be generalized, as right now they are dependent on some fixed switching signal $\sigma \in \mathcal{S}_p$. To this end, the notation will be slightly changed, from $\mathcal{K}_i^b$ to ${}^\rho\mathcal{K}_i^\sigma$. Furthermore, the convention will be made that ${}^{\text{id}}\mathcal{K}_i^\sigma = \mathcal{K}_i^\sigma$. The sequence will be slightly altered to the following:

$${}^\rho\mathcal{K}_i^\sigma = C_{\rho(p)}^{\text{imp}}$$

$$^{\rho}\mathcal{K}_{i-1}^{\sigma} = \operatorname{Im}(\Pi_{\rho(i-1)}) \cap \left(\tilde{\Lambda}_{\rho(i-1)}^{-1} {}^{\rho}\mathcal{K}_{i}^{\sigma} + \mathcal{R}_{\rho(i-1)}\right) + \langle E_{\rho(i-1)}^{\text{imp}} | B_{\rho(i-1)}^{\text{imp}} \rangle + \ker(E_{\rho(i-1)})$$

Weak impulse controllability can be ensured by considering the aforementioned sequence of subspaces intersected over all switching signals σ belonging to $\mathcal{S}_p$. Then, the following result can be conjectured

Theorem 12. *Let $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ be some switched DAE defined on $[0,T)$ for some switching signal $\sigma \in \mathcal{S}_p$. Σ is weak impulse controllable if and only if:*

$$\mathcal{V}_0 \subseteq \bigcap_{\sigma \in \mathcal{S}_p} \mathcal{K}_0^{\sigma}$$

Proof. ($\Longrightarrow$) Assume Σ is weak impulse controllable. Fix $\Sigma(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_p\{(E_k, A_k, B_k)\}$. By definition, $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable. Hence, Theorem 11 is applicable, and thus one must have that $\mathcal{V}_0 \subseteq \mathcal{K}_0^{\sigma}$. Observe that the result holds for any switched DAE $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ as long as $\sigma \in \mathcal{S}_p$. Hence, $\mathcal{V}_0 \subseteq \mathcal{K}_0^{\sigma}$ for all $\sigma \in \mathcal{S}_p$. Therefore, one has that $\mathcal{V}_0 \subseteq \bigcap_{\sigma \in \mathcal{S}_p} \mathcal{K}_0^{\sigma}$.

($\Longleftarrow$) Fix $\Sigma(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_p\{(E_k, A_k, B_k)\}$. It has to be shown that $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable for all $\sigma \in \mathcal{S}_p$. Trivially, one has that:

$$\mathcal{V}_0 \subseteq \bigcap_{\gamma \in \mathcal{S}_p} \mathcal{K}_0^{\gamma} \subseteq \mathcal{K}_0^{\sigma}$$

Hence, Theorem 11 is applicable and hence $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable for all $\sigma \in \mathcal{S}_p$. Hence, Σ is weak impulse controllable.

□

The approach to strong impulse controllability is more careful, as in the preceding theorem it has been assumed that the initial mode is always mode 0. Therefore, mode projectors have to be grouped with respect to their initial mode. This gives rise to an equivalence relation $\sim$. Let $\rho, \mu \in \mathcal{P}_p$. Two mode projectors are said to be equivalent if they share the same initial mode, or symbolically:

$$\mu \sim \rho \iff \mu(0) = \rho(0)$$

This gives rise to p equivalence classes $[0], \ldots, [p]$ all having $(p-1)!$ elements. All mode projectors ρ in $[j]$ have the property that $\rho(0) = j$. Next, it is feasible to introduce a class of switching signals with a fixed mode projector. Suppose $\tilde{\mathcal{S}}_p$ is restricted to only one mode projector, which will be denoted by $\mathcal{S}_p^{\rho}$, or more symbolically:

$$\mathcal{S}_p^{\rho} = \left\{ \sigma : \mathbb{R} \to \{0, ..., p\} \,\middle|\, \begin{array}{c} \exists \{t_i\}_{i=1}^{i=p} \text{ such that } t_0 < t_{i-1} < t_i < T =: t_{p+1} \ \forall i \in \{1, ..., p\} \\ \text{and } \sigma(t) = \sum_{i=0}^{p} \rho(i) \mathbb{1}_{[t_i, t_{i+1})}(t) \end{array} \right\} \tag{4.4}$$

Then the following conjecture can be formulated:

Theorem 13. *Let $\Sigma(E_\sigma, A_\sigma, B_\sigma$ be some switched DAE defined on $[0,T)$ for some switching signal $\sigma \in \tilde{\mathcal{S}}_p$. Σ is strong impulse controllable if and only if for all $i \in \{0, \ldots, p\}$:*

$$\mathcal{V}_i \subseteq \bigcap_{\rho \in [i]} \bigcap_{\sigma \in \mathcal{S}_p^{\rho}} {}^{\rho}\mathcal{K}_0^{\sigma}$$

Proof. ($\Longrightarrow$) Assume Σ is strongly impulse controllable. Fix $\mu \in \mathcal{P}_p$ and let $\sigma \in \mathcal{S}_p^{\mu} \subseteq \tilde{\mathcal{S}}_p$ be arbitrary. Define $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ to be the switched DAE with matrix pairs (E_k, A_k, B_k) and the switching signal σ on the interval $[0,T)$. The result in Theorem 12 can be altered to give:

$$\mathcal{V}_{\mu(0)} \subseteq \bigcap_{\sigma \in \mathcal{S}_p^{\mu}} {}^{\mu}\mathcal{K}_0^{\sigma} \tag{4.5}$$

Suppose that $q := \mu(0) \in \{0, \ldots, p\}$, and let $\sim$ be the equivalence relation as defined before. Let $[q]$ be the equivalence class of all mode projectors having the initial mode equal to q. Observe that (4.5) can be rewritten as:

$$\mathcal{V}_q \subseteq \bigcap_{\sigma \in \mathcal{S}_p^{\mu}} {}^{\mu}\mathcal{K}_0^{\sigma} \ \forall \mu \in [q] \implies \mathcal{V}_q \subseteq \bigcap_{\mu \in [q]} \bigcap_{\sigma \in \mathcal{S}_p^{\mu}} {}^{\mu}\mathcal{K}_0^{\sigma} \tag{4.6}$$

Since $\mu \in \mathcal{P}_p$ is fixed, it could be that μ is in any of the equivalence classes $[0], \ldots, [p]$ since $\mathcal{P}_p = \bigcup_{i=0}^{p} [i]$. Hence, (4.6) must hold for all equivalence classes. Hence, for all $q \in \{0, \ldots, p\}$ it must be that:

$$\mathcal{V}_i \subseteq \bigcap_{\mu \in [i]} \bigcap_{\sigma \in \mathcal{S}_p^{\mu}} {}^{\mu}\mathcal{K}_0^{\sigma} \quad \forall i \in \{0, \ldots, p\} \tag{4.7}$$

($\Longleftarrow$) Fix $\Sigma(E_\sigma, A_\sigma, B_\sigma) \in \tilde{\Sigma}_p\{(E_k, A_k, B_k)\}$. Since $\sigma \in \tilde{\mathcal{S}}_p$, $\exists \rho \in \mathcal{P}_p$ such that $\sigma(t) = \sum_{i=0}^{p} \rho(i) \mathbb{1}_{[t_i, t_{i+1})}(t)$ with $t_0 = 0, t_{p+1} = T$. If it can be shown that Σ is weak impulse controllable for all mode projectors $\rho \in \mathcal{P}_p$, one can conclude that Σ is strongly impulse controllable. Using the given condition, one has that:

$$\mathcal{V}_{\rho(0)} \subseteq \bigcap_{\rho \in [\rho(0)]} \bigcap_{\sigma \in \mathcal{S}_p^\rho} {}^\rho\mathcal{K}_0^\sigma \subseteq \bigcap_{\sigma \in \mathcal{S}_p^\rho} {}^\rho\mathcal{K}_0^\sigma$$

Observe that ($\diamond$) implies Theorem 11 for all mode projectors $\rho \in \mathcal{P}_p$, and hence, Σ is weak impulse controllable for all mode projectors $\rho \in \mathcal{P}_p$. Thus Σ is strong impulse controllable, which finishes the proof. □

From the examples in the introduction of this section, several sufficient conditions can be derived for weak and strong impulse controllability of $\Sigma(E_\sigma, A_\sigma, B_\sigma)$, these conditions will be listed under the following two conjectures:

Corollary 5. *Let $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ be some switched DAE defined on $[0, T)$ for some switching signal $\sigma \in \mathcal{S}_p$. Σ is weak impulse controllable if:*

$$\mathcal{V}_i \subseteq C_{i+1}^{\text{imp}} \quad \forall i \in \{0, \dots, p\}$$

Proof. Since $\mathcal{R}_i \subseteq \mathcal{V}_i$ and $\operatorname{Im} \Pi_i \subseteq \mathcal{V}_i$, one has that $\operatorname{Im} \Pi_i \subseteq \mathcal{V}_i \subseteq C_{i+1}^{\text{imp}} + \mathcal{R}_i$. Using Theorem 7, one has that Σ is weak impulse controllable. □

Corollary 6. *Let $\Sigma(E_\sigma, A_\sigma, B_\sigma)$ be some switched DAE defined on $[0, T)$ for some switching signal $\sigma \in \tilde{\mathcal{S}}_p$. Σ is strong impulse controllable if:*

1. $\mathcal{V}_\ell^* \subseteq C_q^{\text{imp}} \quad \forall \ell, q \in \{0, \dots, p\}$
2. $\mathcal{V}_0 + \ldots + \mathcal{V}_p \subseteq \bigcap_{q=0}^{p} C_q^{\text{imp}}$.
3. $\bigcap_{k=0}^{p} C_k^{\text{imp}} = \mathbb{R}^n$

Proof. Observe that the first 2 statements are equivalent. As noted before, $\mathcal{R}_i \subseteq \mathcal{V}_i$. One must have that $\mathcal{V}_i \subseteq \bigcup_{k=0}^{q} \mathcal{V}_k \subseteq \bigcap_{q=0}^{p} C_q^{\text{imp}} \subseteq \bigcap_{q=0}^{p} C_q^{\text{imp}} + \mathcal{R}_i$. Hence Theorem 8 is applicable, which shows that the first two statements imply that Σ is strong impulse controllable. The third statement indicates that all modes in themselves are fully controllable. If the third statement holds, then the second statement is trivially satisfied, and hence, the third statement also implies that Σ is strong impulse controllable. □

5 Conclusion

In this thesis several necessary conditions and sufficient conditions have been derived for impulse controllability of switched DAEs for arbitrary switching. In practical applications, the order in which the switching occurs is fixed, but if a switch is faulty, this fixed order cannot be ensured, hence, giving rise to different orders in which the switching occurs. By possibly requiring that the switching signal can switch in any order, one has to make a distinction between weak - and strong impulse controllability. By first focussing the research on switched DAEs with 2 modes, one is able to derive necessary geometric conditions for weak - and strong impulse controllability of switched DAEs with 2 modes, as listed under Theorems 5 and 6. These conditions could be generalized to derive sufficient conditions for weak - and strong impulse controllability of switched DAEs with $p+1$ modes, as listed under Theorems 7 and 8. Afterwards, the backwards method from [6] has been generalized to derive several necessary conditions for weak - and strong impulse controllability of switched DAEs with $p+1$ modes, as listed in Theorem 12 and 13. From the examples in the results one can derive several other sufficient conditions for weak - and strong impulse controllability, which are listed under Corollary's 5 and 6.

The problems with the necessary conditions listed under Theorems 12, 13 is the uncomputability of the results. This is due to the fact that the space $\mathcal{S}_p, \tilde{\mathcal{S}}_p$ contains infinitely many switching signals, for which the sequence ${}^{\rho}\mathcal{K}_i^{\sigma}$ has to be computed. The biggest problem in the sequence ${}^{\rho}\mathcal{K}_i^{\sigma}$ is the presence of the evolution operator $\tilde{\Lambda}_{i-1}^{-1}$. For future research, to obtain a sequence that is computationally more efficient one must either show that $\operatorname{Im}(\Pi_{\rho(i-1)}) \cap \left(\tilde{\Lambda}_{\rho(i-1)}^{-1}\,{}^{\rho}\mathcal{K}_i^{\sigma} + \mathcal{R}_{\rho(i-1)}\right)$ is $\tilde{\Lambda}_{i-1}$ invariant, or alter the sequence slightly such that the aforementioned statement becomes $\tilde{\Lambda}_{i-1}$ invariant for the new choice of ${}^{\rho}\mathcal{K}_i^{\sigma}$.

Additionally, a different approach can be taken to derive conditions for impulse controllability. By representing the modes as nodes on a graph and connecting the nodes with edges only if a switch between these nodes can be performed impulse-freely, one can associate a graph with a switched DAE. For example, if the graph appears to be strongly connected, then it seems reasonable to expect that the switched DAE is also strong impulse controllable. For future research, in order to obtain computationally more efficient conditions one can take a look at a graph theory approach to switched DAEs.

Lastly, since several conditions in this thesis establish either weak or strong impulse controllablility of switched DAEs, it is yet to be investigated what kind of input or controller that ensures that the state is impulse free. When such a controller is to be developed, one should take into account that in practice switches do not necessarily switch instantaneous, and therefore, extra modes have to be developed that describes the behaviour of the solution in between modes. Therefore a further direction of research is the development of a dynamic controller which prevents the occurance of impulses in the solution.

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