UNIVERSITY OF GRONINGEN

MASTER THESIS

Creating the graviton, a massless spin 2 perspective on gravity

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Abstract

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General relativity is a theory that has withstand numerous tests over the past 100 years. Although it works so well on a lot of levels there still are a few big open questions. Historically general relativity has been a geometric theory with lots of focus on curvature. This research presents an alternative, non geometric, way to derive the main feature of general relativity: the Einstein Hilbert. By constructing a theory out of a massless spin 2 particle it provides a different perspective on the theory. We start by creating a Lorentz invariant field operator and show that the gauge invariance often said to be fundamental to general relativity is a natural consequence of the deviating little group of a massless particle. Subsequently it is shown that with this gauge invariance the only linear theory you can write down is exactly Einstein Hilbert around flat space. Thereafter the full non linear theory is recovered by demand a consistent coupling to the fields own energy-momentum tensor. Finally some of these calculations are also done on the 5 dimensional extension of Einstein Hilbert: Gauss Bonnet.

Acknowledgements

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Chapter 1

Introduction

1.1 Brief history of gravity

For as long as humans have lived on this earth, they have noticed things fall to the ground. It is remarkable that the force we now consider weak has been the one that mankind has actively interacted with the most. During history all over the world scientists had tried to create a correct theory for gravity. Even the ancient greeks already had a basic understanding. Aristotle explained gravity by saying all objects move to their natural place. For basic elements earth and water this natural place was apparently the center of the universe, which was considered the center of the earth[1].

In 1687 Isaac Newton published the Philosophiae Naturalis Principia. In this work he described, among other things, that every particle in the universe attracts all other particles with a force that is proportional to the product of their masses and inversely proportional to the distance between them squared [2]:

$$F_g = G\frac{m_1 m_2}{r^2} \tag{1.1}$$

This theory is valid for most practical purposes on earth, but had some fundamental flaws. A famous example is the precession of the perihelion of mercury. This effect was observed in 1859 by Urbain Le Verrier and could not be explained by Newtonian gravity (1.1). It suffices to say a new theory had to be found, which came over 300 years after Newtons law.

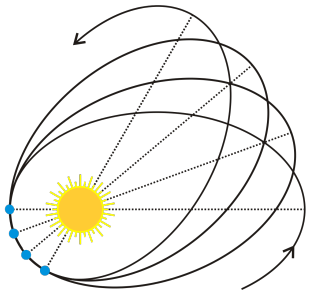

FIGURE 1.1: The orbit of mercury changed over time, inexplicable by Newtons law of gravity [3]

1.2 Relativity

After publishing his special theory of of relativity in 1905 (citation) Einstein had a big task in front of him. In contrast to the nicer behaved Maxwell

equations which were an inspiration for his theory of relativity, Newtons law of gravity described an instantaneous force at any distance. This did not match with the fundamental speed of the speed of light in the universe. Key to developing this new theory was his equivalence principle: There is no local experiment which tells you if you are at rest in a uniform gravitational field or accelerating at a constant rate. After almost a decade of hard work and conversations with his fellow physicists, he published the Einstein field equations [4]:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.2}$$

This set of general covariant equations relate the curvature of spacetime to the energy momentum tensor, which represents the amount of energy and momentum. They characterize gravity not as a force, but as the curvature of spacetime through which objects travel over geodesics according to the geodesic equation:

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}. \tag{1.3}$$

The equations had a lot of appealing properties. Obvious ones, such that in empty space where $T_{\mu\nu} = 0$ the curvature is also zero as well as more complicated ones. In weak fields where the amount of mass involved is low and speeds are much lower than the speed of light the equations still approximate Newtons universal law of gravitation. Maybe the biggest clue that the field equations were correct was that they predicted the precession of the perihelion of mercury, which was the first test of many to come.

1.3 Tests

Already in 1919 the first deflections of light by gravity were smartly observed by looking at stars close to the sun during a total eclipse [5]. Nowadays gravitational lensing is a useful tool for astronomers to be able to see objects hidden behind supermassive objects in the universe, or even to be able to study events multiple times due to the different path lengths the light has to take around an object[6]. Even in the last decade predictions from general relativity were observed. When looking for an effect analogous to the electromagnetic waves that appear with moving electric dipoles, he predicted general relativity to also be able to produce waves. This was first indirectly observed by the decay of the orbital period of a binary pulsar system [7]. In 2015, a 100 years after their prediction, the first direct observation of a gravitational wave was done by LIGO [8], showing an almost perfect fit to the predicted waveforms 1.2. Taking into account these, and many more, tests and the fact that even our daily life is affected by the effects of general relativity in the Global Positioning System it is save to say that it has been a spectacularly successful theory.

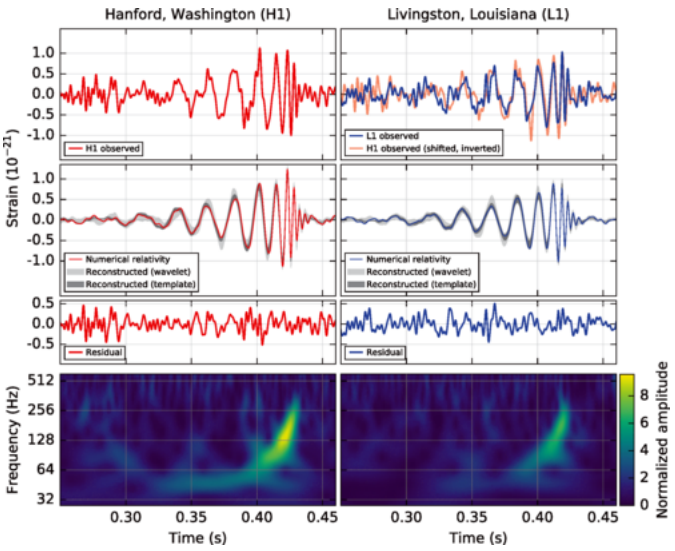

FIGURE 1.2: The detected waveforms of gravitational waves caused by the merge of a pair of black holes [8]

1.4 Problems

The theory is however not complete. In contrast to the other fundamental forces General Relativity is not compatible with the standard model because there are problems in quantizing it. Specifically perturbatively quantized general relativity is non-renormalizable which so far has not been solved. Furthermore the allows for mathematical singularities such as the center of a black hole, which you do not expect to be a physical effect [9].

1.5 An alternative route

As the theory is not complete and there are still open problems it is useful to look at it from another perspective. General Relativity started as a geometric effect, which Einstein and others build up from principles. It is however not the only way to get to this theory. This research presents another, more systematic way to derive the Einstein Hilbert action and tries to answer the question:

- Can general relativity be derived by constructing a self consistent field theory of a massless spin 2 particle?

To answer this question we are in the second chapter going to try and construct a field operator that can be used to create Lorentz invariant theories. This will bring some constraints on that action. In the third chapter we are going to find which actions we can create while complying to those constraints. The fourth action will extract from the free theory we find in the previous chapter the full non linear Einstein Hilbert action. Finally we are going to take a look at the higher dimensional extension to Einstein Hilbert: Gauss Bonnet. We are going to look at the same procedures as we did for Einstein Hilbert and see if they also hold up for that theory.

Chapter 2

On the origin of gauge invariance

2.1 Introduction

The purpose of this chapter is to show the origin of the need for gauge symmetry in a gravity theory. To show this I will first show why a similar, but different, symmetry appears in a massless spin 1 vector particle, the U(1) symmetry. The massless spin 1 vector field is the photon. The gauge symmetry in the spin 2 massless particle will follow analogously. To show this we will try to construct a field operator for the particle, made out of creation and annihilation operators. To construct a Lorentz invariant theory, the action has to contain terms with this field operator in such a way that it is Lorentz invariant. As will be clear from this chapter this is quite a straightforward task for massive particles, but not for massless particles. For this chapter we will follow parts of chapter 2 and 5 of [10].

2.2 Spin-1

A field operator is constructed out of a linear combination of annihilation and creation operators, it will have the general structure [10]:

$$\psi(x) = (2\pi)^{-\frac{3}{2}} \int d^3p \sum_{\sigma} [ba(\vec{p},\sigma)(\vec{p},\sigma)e^{ipx} + ca^{\dagger}(\vec{p},\sigma)v(\vec{p},\sigma)e^{-ipx}]. \quad (2.1)$$

In the field operator b and c are numbers, a and $a^{\dagger}$ are the annihilation and creation operators and u and v are coefficient functions. These coefficient funct The next step is to find the coefficient functions such that ψ transforms in a covariant way under lorentz transformations and can be put in a lorentz invariant theory. In other words, it would be useful if the operator transforms as a Lorentz vector:

$$U(\Lambda)\psi^{\mu}(x)U^{-1}(\Lambda) = \Lambda^{\mu}_{\nu}\psi^{\nu}(x). \quad (2.2)$$

The Λ matrices are Lorentz matrices with the property that $\Lambda^{\sigma}_{\nu}\Lambda^{\nu}_{\rho} = \delta^{\sigma}_{\rho}$, with δ^{σ}_{ρ} the Kronecker delta. This means that if you transform a term like $\psi^{\nu}\psi_{\nu}$ the Lorentz matrices cancel and the term is invariant under those Lorentz transformation.
The transformation matrices $U(\Lambda)$ commute with everything in the field operator except for the creation and annihilation operators. So to study the transformation properties of the field operator we first want to look at the

transformation properties of the creation operator. The key to this chapter will be in the difference of the transformation properties of the creation operator for a massive or a massless particle. First we will cover the massive case.

2.2.1 Massive

A massive particle state in the most general sense is represented by its momentum and its spin. In braket notation this will look like:

$$|\vec{p}, \sigma\rangle \tag{2.3}$$

The creation operator creates these single particle states by acting on the vacuum state:

$$a^\dagger(\vec{p}, \sigma)|0\rangle = |\vec{p}, \sigma\rangle. \tag{2.4}$$

As the field operator is going to be build up out of these creation and annihilation operators, it is necessary to find how these operators transform under a Lorentz transformation. To do this it is useful to define a standard momentum and relate all other states to this momentum via a standard boost. Using a massive particle it is very intuitive to use the rest frame as this standard momentum, in which $p^\mu = k^\mu = (m, 0, 0, 0)$, with m the mass of the particle. This boost allows us to write any state as:

$$|\vec{p}, \sigma\rangle = L(\vec{p})|0, \sigma\rangle. \tag{2.5}$$

With $L(\vec{p})$ the boost that takes k^μ to p^μ, i.e. $L^\mu_\nu(\vec{p})k^\nu = p^\mu$.

Every Lorentz transformation can be rewritten in such a way that it first takes the particle to the $\vec{p} = 0$ state, then performs a transformation that is part of the little group and finally boosts the particle to the desired momentum. The little group is the group of all transformations that leave the momentum of a particle unchanged:

$$U(\Lambda)|\vec{p}, \sigma\rangle = L(\Lambda p)W(\Lambda, p)L^{-1}(p)|\vec{p}, \sigma\rangle \tag{2.6}$$

$$= L(\Lambda p)W(\Lambda, p)L^{-1}(p)L(p)|0, \sigma\rangle = L(\Lambda p)W(\Lambda, p)|0, \sigma\rangle \tag{2.7}$$

Herein $W(\Lambda, p)$ is a transformation in the little group of k^μ such that $W^\mu_\nu(\Lambda, p)k^\nu = k^\mu$. For the rest frame of a massive particle only spatial rotations leave the momentum unchanged. The group of spatial rotation is SO(3), 3 dimensional rotations in the spatial dimensions. The representation of the group SO(3) are known and are characterized by the spin, J, of the particle it is acting on. The result is a (2J+1)x(2J+1) unitary matrix that mixes the $|0, \sigma >$ state into a linear combination of other states with different spins, but leaves the momentum untouched.

$$W(\Lambda, p)|0, \sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(\Lambda, p)|0, \sigma\rangle \tag{2.8}$$

We can combine (2.7) and (2.8) to see that the Lorentz transformation transforms the single particle state by the SO(3) matrices and a boost:

$$U(\Lambda)|\vec{p},\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(\Lambda,p)|\Lambda\vec{p},\sigma'\rangle. \tag{2.9}$$

We can find a relation between (2.9) and the creation operator:

$$\begin{aligned} U(\Lambda)|\vec{p},\sigma\rangle &= U(\Lambda)a^\dagger(\vec{p},\sigma)|0\rangle \\ &= U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda)U(\Lambda)|0\rangle \\ &= U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda)|0\rangle. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) will give us the transformation properties of the creation operator:

$$U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda) = \sum_{\sigma'} D_{\sigma\sigma'}a^\dagger(\vec{\Lambda p},\sigma'). \tag{2.11}$$

The annihilation operator then transforms as the hermitian conjugate:

$$U(\Lambda)a(\vec{p},\sigma)(U^{-1}(\Lambda) = \sum_{\sigma'} D^*_{\sigma\sigma'}(\Lambda,p)a(\vec{\Lambda p},\sigma'). \tag{2.12}$$

Field operator

Looking back at (2.1), everything in this operator commutes with the Lorentz matrices, except for the annihilation and creation operators. So the trick is to choose the coefficient functions u and v in such a way that they can absorb similarly to the transformation of a and $a^\dagger$.

Coefficient Functions

For a massive vector spin 1 field, the Lorentz representation is that of 4 vectors, where $D(\Lambda)^\mu_\nu = \Lambda^\mu_\nu$. Due to convention it is common to write the coefficient functions as polarization vectors with the following relation between them [10]:

$$u^\mu(\vec{p},\sigma) = (2p^0)^{-1/2}e^\mu(\vec{p},\sigma) \tag{2.13}$$

Again, we are going to describe write the polarization vectors in the rest frame and transform them to arbitrary momentum with the same Lorentz boost we saw in (2.5): $e^\mu(\vec{p},\sigma) = L^\mu_\nu(\vec{p})e^\nu(0,\sigma)$. In that rest frame these polarization vectors are given by:

$$e^\mu(0,0) = [0,0,0,1], e^\mu(0,\pm1) = \frac{1}{\sqrt{2}}[0,1,\pm i,0] \tag{2.14}$$

The spatial part of these four vectors are eigenstates of SO(3) with eigenvalues 0 and ±1. As this is exactly the little group of the particles we need them to describe we can conveniently write this little group transformation as:

$$W(\lambda,p)^\mu{}_\nu e^\nu(0,\sigma) = \sum_{\sigma'} D_{\sigma\sigma'}(\Lambda,p)e^\mu(0,\sigma'). \tag{2.15}$$

The matrices $D_{\sigma\sigma'}$ are the exact same matrices as by which the creation operator transformed in the previous chapter. We can combine this little group transformation and the earlier mentioned boost such that the general Lorentz transformation Λ^μ_ν transforms the polarisation vectors like this:

$$\lambda^\mu{}_\nu e^\nu(\vec{p},\sigma) = \sum_{\sigma'} D_{\sigma\sigma'}(\Lambda,p) e^\mu(\vec{\lambda p},\sigma'). \tag{2.16}$$

It is useful to switch around the $^\mu_\nu$ and the matrices $D_{\sigma\sigma'}$. Considering that those matrices are unitary, we can write them like:

$$\sum_{\sigma'} D^*_{\sigma\sigma'}(\Lambda,p) e^\mu(\vec{p},\sigma') = \lambda^{-1\mu}{}_\nu e^\mu(\vec{\lambda p},\sigma). \tag{2.17}$$

The building blocks of the field operator (2.1) are multiplications of the form $a(\vec{p},\sigma)e^\mu(\vec{p},\sigma)e^{ipx}$ or hermitian conjugate of that. According to the transformation rules we established, we can now transform this term.First The transformation matrices commute with everything except a. Then a transforms according to the single particle state. finally the polarization vector 'absorbs' the matrices $D_{\sigma\sigma'}$.

$$\begin{aligned} a(\vec{p},\sigma)e^\mu(\vec{p},\sigma)e^{ipx} &\to U(\Lambda)a(\vec{p},\sigma)u(\vec{p},\sigma)e^{ipx}U^{-1}(\Lambda) && (2.18)\\ &= U(\Lambda)a(\vec{p},\sigma)U^{-1}(\Lambda)e^\mu(\vec{p},\sigma)e^{ipx} && (2.19)\\ &= \sum_{\sigma'} D^*_{\sigma\sigma'}(\Lambda,p)a(\vec{\Lambda p},\sigma')e^\mu(\vec{p},\sigma)e^{ipx} && (2.20)\\ &= \lambda^{-1\mu}{}_\nu a(\vec{\Lambda p},\sigma)e^\nu(\vec{\lambda p},\sigma)e^{ipx} && (2.21) \end{aligned}$$

The hermitian conjugate of this term transforms similarly, such that for the complete field operator ψ^μ it transforms as:

$$U(\Lambda)\psi^\mu(x)U^{-1}(\Lambda) = \lambda^{-1\mu}{}_\nu \psi^\nu(\Lambda x) \tag{2.22}$$

It is now easy to construct a Lorentz invariant theory, as the only requirement is that indices of the field operator are contracted with each other. An simple example of Lorentz invariant lagrangian of a massive vector is:

$$L = \frac{1}{2}(\partial^\mu\psi^\nu)(\partial_\mu\psi_\nu) - \frac{m}{2}\psi^\mu\psi_\mu \tag{2.23}$$

As you can see, under a Lorentz transformation each term picks up a Lorentz matrix and its inverse and therefore it is Lorentz invariant.

2.2.2 massless

Single particle state

Now moving on to the massless spin 1 particle, the first part of the previous section can be repeated for the massless case. The first problems arise when looking at the standard momentum for massless particles. As there is no rest frame for a massless particle as they are always traveling at c for every observer. There can be no rest momentum $p^\mu = (m,0,0,0)$ for these particles, instead of this we choose a standard momentum $p^\mu = q^\mu = (q,0,0,q)$, a particle traveling along the z-axis. From now on the standard boost $L(\vec{p})$

will take a particle from this standard momentum to the momentum $\vec{p}$

$$|\vec{p},\sigma\rangle = L(\vec{p})|\vec{q},\sigma\rangle \tag{2.24}$$

Again, we can describe a lorentz transformation as the combination of an inverse standard boost, a little group operator and a boost to the desired momentum.

$$\begin{aligned} U(\Lambda)|\vec{p},\sigma\rangle = L(\Lambda p)W(\Lambda,p)L^{-1}(p)|\vec{p},\sigma\rangle \\ = L(\Lambda p)W(\Lambda,p)L^{-1}(p)L(p)|\vec{q},\sigma\rangle = L(\Lambda p)W(\Lambda,p)|\vec{q},\sigma\rangle \end{aligned} \tag{2.25}$$

Little group

The little group of this standard momentum is different from the little group of the rest momentum we used for the massive particle. When we look at the standard momentum q^μ there is an obvious SO(2) symmetry. By rotating along the direction of motion the x and y directions get rotated into each other. This leaves the motion in the z direction, and thus our standard momentum, invariant.

A more complicated part of the little group is a combination of boosts and rotations. To see this, start with a boost in the y direction. After this boost we apply a rotation around the x axis such that the orientation of the momentum is entirely in the z direction again. Finally the particle can be boosted back to its original energy

$$\Lambda^{(1)\mu}{}_\nu q^\nu = q'^\mu = (\gamma q, 0, \gamma\beta q, q). \tag{2.26}$$

$$\Lambda^{(2)\mu}{}_\nu q'^\nu = q''^\mu = (\gamma q, 0, 0, \gamma q) \tag{2.27}$$

$$\Lambda^{(3)\mu}{}_\nu q''^\nu = q'''^{,\mu} = (q, 0, 0, q) \tag{2.28}$$

The combination of $\Lambda^{(1)\mu}{}_\nu$, $\Lambda^{(2)\mu}{}_\nu$ and $\Lambda^{(3)\mu}{}_\nu$ is an element of the little group of q^μ. In general this can be done with any boost in the x or y direction or a combination of those and form a subgroup. A general transformation of this form looks like this:

$$S^\mu{}_\nu(\alpha,\beta) = \begin{pmatrix} 1+\xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 0 & -\beta \\ \xi & \alpha & \beta & 1-\xi \end{pmatrix} \tag{2.29}$$

with α and β any real numbers and $\xi = \frac{1}{2}(\alpha^2+\beta^2)$. A quick inspection of this matrix shows that indeed $S^\mu{}_\nu q^\mu = q^\nu$ and therefore it is part of the little group of q

Structure little group

So the little group consists of the following 2 transformation matrices:

$$S^{\mu}{}_{\nu}(\alpha,\beta) = \begin{pmatrix} 1+\xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & \alpha & \beta & 1-\xi \end{pmatrix} R^{\mu}{}_{\nu}(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These transformations are both subgroups, with the $S^{\mu}{}_{\nu}$ part being an invariant subgroup, as:

$$S(\alpha,\beta)S(\alpha',\beta') = S(\alpha+\alpha',\beta+\beta') \tag{2.30}$$
$$R(\theta)R(\theta') = R(\theta+\theta') \tag{2.31}$$
$$R(\theta)S(\alpha,\beta)R^{-1}(\theta) = S(\alpha\cos\theta+\beta\sin\theta, -\alpha\sin\theta+\beta\cos\theta) \tag{2.32}$$

Close to the identity this subgroup can be described by

$$W(\alpha,\beta,\theta) = 1 + i\alpha A + i\beta B + i\theta J \tag{2.33}$$

with the generators A, B and J given by:

$$A^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} B^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \tag{2.34}$$

$$J^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.35}$$

From these generators it is possible to calculate the commutation relations:

$$[J_3, A] = iB \tag{2.36}$$

$$[J_3, b] = -iA \tag{2.37}$$

$$[A, B] = 0 \tag{2.38}$$

Because the generators A and B commute it is possible to have diagonalize a particle state for both of those eigenvalues i.e. both the following statements hold for the same state $\Psi_{a,b}$.

$$A\Psi_{q,a,b} = a\Psi_{q,a,b} \tag{2.39}$$
$$B\Psi_{q,a,b} = b\Psi_{q,a,b} \tag{2.40}$$

As we saw in equation (2.32) the rotation subgroup rotates the other two generators into each other. This means that whenever we find any state with eigenvalues α and β, you can find a continuous set of eigenvalues by

rotating the and β operators.

$$U[R(\theta)]AU^{-1}[R(\theta)] = Acos\theta - Bsin\theta \tag{2.41}$$
$$U[R(\theta)]BU^{-1}[R(\theta)] = Asin\theta - Bcos\theta \tag{2.42}$$

If such a state would exist, also states with eigenvalues $Acos\theta - Bsin\theta$ would have to exist. We have never seen this continuous degree of freedom (θ) in any particle. Therefore it is postulated that for all real particles the eigenvalues $\alpha = \beta = 0$ and as such when acting with operators A and B on a state they vanish $A\Psi_q = B\Psi_q = 0$

So the general little group element $U(W)$ can be build up like $U(S(\alpha,\beta))U(R(\theta)) = e^{i\alpha A + i\beta B}e^{iJ\theta}$ which when acting on a state $\Psi_{q,\sigma}$ produces

$$e^{i\alpha A + i\beta B}e^{iJ\theta}\Psi_{q,\sigma} = e^{i\theta\sigma}\Psi_{q,\sigma}. \tag{2.43}$$

Single particle state

Analogously to the massive case, the single particle states for massless particles transform under a general Lorentz transformation like:

$$U(\Lambda)|p,\sigma\rangle = e^{-iS(\alpha,\beta)R(\Theta)}|\Lambda p,\sigma\rangle = e^{-i\sigma\Theta}|\Lambda p,\sigma\rangle \tag{2.44}$$

By using the same logic as in (2.10) you can see the creation operator transforms as:

$$U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}e^{-i\sigma S(\alpha,\beta)R(\Theta)}a^\dagger(\vec{p}_\Lambda,\sigma). \tag{2.45}$$

Coefficient functions

Just like in the massive case, the trick now is to find coefficient functions that transform in such a way that the field operator transforms covariantly.

$$\psi(x)^\mu = (2\pi)^{-\frac{3}{2}}\int d^3p \sum_\sigma [ba(\vec{p},\sigma)u^\mu(\vec{p},\sigma)e^{ipx} + ca^\dagger(\vec{p},\sigma)v^\mu(\vec{p},\sigma)e^{-ipx}] \tag{2.46}$$

The goal is that this field transforms according to some representation of the Lorentz group, so:

$$U(\Lambda)\psi^\mu(x)U^{-1}(\Lambda) = \Lambda^\mu_\nu\psi^\nu(x). \tag{2.47}$$

As we know that the creation operator satisfies the transformation rule (2.45) the coefficient functions u and v would have to follow similar transformation rules. Now we need, for our vector case, to find the functions u_μ and v_μ that comply with these relations. Again for conventional reasons these are written as dimensionless polarization vectors e_μ with the following relation:

$$u_\mu(\vec{p},\sigma) \equiv \frac{1}{\sqrt{2p^0}}e_\mu(\vec{p},\sigma). \tag{2.48}$$

This polarization vector can then be boosted back to our standard momentum with the standard lorentz boost, e.g.

$$e^\mu(\vec{p},\sigma) = L(\vec{p})^\mu{}_\nu e^\nu(\vec{q},\sigma) \tag{2.49}$$

If the polarization vectors would behave similar to the single particle state it has to pick up a phase under the rotational part of the little group and the more complex part should act on it trivially:

$$e^\mu(\vec{k},\sigma)e^{i\sigma\theta} = R(\theta)^\mu{}_\nu e^\nu(\vec{k},\sigma) \tag{2.50}$$
$$e^\mu(\vec{k},\sigma) = S(\alpha,\beta)^\mu{}_\nu e^\nu(\vec{k},\sigma) \tag{2.51}$$

It is easy to find vectors that comply with the first condition, these are the following polarization vectors:

$$e^\mu(\vec{q},\pm 1) = \sqrt{2}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}. \tag{2.52}$$

Now to satisfy the second relation with these we find the very specific conditions $\alpha \pm i\beta = 0$, which is not a possibility as it should hold for general α, β. This is a problem, the vectors we are looking for turns out to not exist. The only solution is to still take the spin-1 vector and just accept that there is a non trivial addition in the transformation, that is to say: the polarization vectors, and thus the field operator does not transform like a proper 4-vector. Doing this the polarization vector does transform like this:

$$W(\theta,\alpha,\beta)^\mu{}_\nu e^\nu(\vec{k},\sigma) = e^{\sigma i\theta}[e^\mu(\vec{k},\sigma) + \frac{(\alpha+i\sigma\beta)}{\sqrt{2}|\vec{k}|}k^\mu]. \tag{2.53}$$

If you now do the same calculation we did in (2.21) for the massless field operator, it will transform like this:

$$U(\Lambda)\psi(x)^\mu U^{-1}(\Lambda) = \Lambda^{-1\mu}{}_\nu \psi(x)(\Lambda x) + \partial^\mu \Omega(x,\Lambda) \tag{2.54}$$

This means that the field of the massless particle does not transform like a vector at all under Lorentz transformation. The lagrangian we wrote down for them massive vector field is not valid for the massless field.

Interaction terms

We still want to write down a theory with this massless vector field and it still needs to be Lorentz invariant. It is not enough anymore to just contract the indices of the field with itself, the lagrangian also has to be invariant under the extra transformation rule in (2.54). It had a gauge invariance. An option is to let the field appear in the lagrangian in an antis-ymmetric combination:

$$F_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu. \tag{2.55}$$

Under lorentz transformations the extra term vanishes which makes $F_{\mu\nu}$ a tensor even though the field is not a vector. You can then work with this

tensor to make the Lorentz invariant term:

$$F_{\mu\nu}F^{\mu\nu}. \tag{2.56}$$

Another option is to couple the field ψ to a conserved vector J_μ such that $\partial_\mu J^\mu = 0$. Under a Lorentz transformation this coupling would transform like this:

$$A_\mu J^\mu \to (A_\mu + \partial_\mu \Omega)J^\mu. \tag{2.57}$$

$$\tag{2.58}$$

In an action this last term vanishes by integrating by parts and the term transforms like a tensor.

2.3 Spin 2

The spin 2 particle will be represented by a symmetric tensor $h_{\mu\nu}$. Building the spin 2 field operator in the same way as for spin 1, it will look like this:

$$h^{\mu\nu}(x) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{2p^0} \sum_{\sigma=-2,+2} [ba(\vec{p},\sigma)e^{\mu\nu}(\vec{p},\sigma)e^{ipx} + ca^\dagger(\vec{p},\sigma)e^{\mu\nu} * (\vec{p},\sigma)e^{-ipx}].$$

The transformation of this field will be completely analogous to that of the spin 1 field. We would now need a polarization tensor similar to the polarization vectors of the previous section, as the single particle state also behaves similar. It turns out that the particle will transform as the direct multiplication of 2 spin 1 fields. As a consequence the polarization tensor that is used to describe the field will be:

$$e^{\mu\nu}_{\pm 2} = e^\mu_{\pm 1} e^\nu_{\pm 1}. \tag{2.59}$$

Where the latter two are the polarization vectors from the previous section. The transformation of this polarization tensor is then also easy to derive, for a general lorentz group element Λ:

$$\Lambda^{\mu\nu}{}_{\rho\lambda} e^{\rho\lambda}(\vec{p}) = \Lambda^\mu{}_\rho e^\rho(\vec{p}) \Lambda^\nu{}_\lambda e^\lambda(\vec{p}) \tag{2.60}$$

$$= e^{\sigma i\theta}[e^\mu(\vec{\Lambda p}) + \frac{(\alpha + i\sigma\beta)}{\sqrt{2}\langle\vec{k}\rangle} k^\mu] W^\nu{}_\lambda \, e^\lambda \tag{2.61}$$

$$= e^{2\sigma i\theta}[e^\mu(\vec{\Lambda p}) + \frac{(\alpha + i\sigma\beta)}{\sqrt{2}\langle\vec{k}\rangle}(\Lambda p)^\mu][e^\nu(\vec{\Lambda p}) + \frac{(\alpha + i\sigma\beta)}{\sqrt{2}\langle\vec{k}\rangle}(\Lambda p)^\nu] \tag{2.62}$$

$$= e^{2\sigma i\theta}[e^\mu(\vec{\Lambda p})e^\nu(\vec{\Lambda p}) + \frac{(\alpha + i\sigma\beta)}{\sqrt{2}\langle\vec{k}\rangle}(\Lambda p)^\mu e^\nu(\vec{\Lambda p}) + \tag{2.63}$$

$$\frac{(\alpha + i\sigma\beta)}{\sqrt{2}\langle\vec{k}\rangle}(\Lambda p)^\nu e^\mu(\vec{\Lambda p}) + \frac{(\alpha + i\sigma\beta)^2}{2\langle\vec{k}\rangle^2}(\Lambda p)^\mu(\Lambda p)^\nu] \tag{2.64}$$

$$= e^{2\sigma i\theta}[e^{\mu\nu} + (\Lambda p)^\mu \xi^\nu + (\Lambda p)^\nu \xi^\mu]. \tag{2.65}$$

In the last step, the final term of the previous step is split in such a way that

$$\xi^\mu = \frac{(\alpha + i\sigma\beta)}{\sqrt{2}\langle\vec{k}\rangle} e^\mu + \frac{(\alpha + i\sigma\beta)^2}{4\langle\vec{k}\rangle^2} k^\mu. \tag{2.66}$$

When Lorentz transforming the massless spin 2 field operator a similar thing happens as in the spin 2 case, except it now picks up not 1 but 2 extra terms.

$$U(\Lambda)h^{\mu\nu}(x)U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\lambda(\Lambda^{-1})^\nu{}_\rho h^{\lambda\rho}(\Lambda x) + \partial^\nu\Omega^\mu(x,\Lambda) + \partial^\mu\Omega^\nu(x,\Lambda) \tag{2.67}$$

Interaction terms

So just like the vector, the tensor does not transform like a proper Lorentz tensor. Under a Lorentz transformation two extra terms appear. Therefore to make a lagrangian Lorentz invariant it needs to not only be contracted with other tensors, the lagrangian also needs to be invariant under the gauge transformation:

$$h^{\mu\nu} -> h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu \tag{2.68}$$

The next chapter will deal with how that lagrangian can be constructed.

Chapter 3

On the terms of a massless spin-2 action

3.1 Introduction

The previous chapter has shown that in order to be Lorentz invariant, a massless spin-2 theory also has to be invariant under linear diffeomorphisms. This chapter will focus on showing that, when expanded on a flat background, general relativity is in fact this massless spin 2 field. To show this two approaches are shown that lead to the same result. First a top down approach, wherein the usual Einstein-Hilbert action is expanded around flat space. The second half will be a bottom up approach. An analysis of a general action of such a massless spin-2 field shows that when demanded that it is invariant under the diffeormorphisms, the constraints on the coefficients show the same structure as we find in the first half.

3.1.1 Top down

The first section will show that when you expand the Einstein-Hilbert action around flat space up to quadratic terms in derivatives and the fields you will find a specific combinations of contractions of derivatives and the field $h_{\mu\nu}$. To start, the Einstein-Hilbert action, up to an overall constant, is given by:

$$S = \int d^4x\sqrt{-g}R. \tag{3.1}$$

Wherein R is the Ricci scalar, given by $g^{\mu\nu}R_{\mu\nu}$ and g is the determinant of $g_{\mu\nu}$. The following definitions of curvature terms will be used:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} \tag{3.2}$$

$$R^{\lambda}{}_{\mu\rho\nu} = \partial_\rho\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}{}_{\mu\rho} + \Gamma^{\gamma}{}_{\mu\nu}\Gamma^{\lambda}{}_{\gamma\rho} - \Gamma^{\gamma}{}_{\mu\rho}\Gamma^{\lambda}{}_{\gamma\nu} \tag{3.3}$$

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}g^{\rho\gamma}(\partial_\nu g_{\mu\gamma} + \partial_\mu g_{\nu\gamma} - \partial_\gamma g_{\mu\nu}) \tag{3.4}$$

The metric tensor is expanded with a perturbation around flat space:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.5}$$

The inverse metric tensor transforms a little different, to find $\delta(g^{\mu\nu})$ we start with the identity: $g^{\mu\alpha}g_{\beta\nu} = \delta^{\alpha}_{\beta}$

$$\delta(g^{\mu\alpha}g_{\beta\mu}) = \delta(\delta^{\alpha}_{\beta}) = 0 \tag{3.6}$$
$$\delta(g^{\mu\alpha}g_{\beta\mu}) = g^{\mu\alpha}\delta(g_{\beta\nu}) + \delta(g^{\mu\alpha})g_{\beta\nu}. \tag{3.7}$$

Combining these equations gives:

$$\delta(g^{\mu\alpha})g_{\beta\mu} = -g^{\mu\alpha}\delta(g_{\beta\nu}) \tag{3.8}$$
$$\delta(g^{\mu\alpha})g_{\beta\mu}g^{\beta\nu} = -g^{\mu\alpha}\delta(g_{\beta\nu})g^{\beta\nu} \tag{3.9}$$
$$\delta(g^{\mu\alpha})\delta^{\nu}_{\mu} = -g^{\mu\alpha}\delta(g_{\beta\nu})g^{\beta\nu} \tag{3.10}$$
$$\delta(g^{\nu\alpha}) = -g^{\mu\alpha}h_{\beta\mu}g^{\beta\nu}. \tag{3.11}$$

Up to first order in h, which is all we are interested in, this last equation results into:

$$\delta(g^{\mu\nu}) = -h^{\mu\nu} \tag{3.12}$$

Where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. This also preserves the identity:

$$g^{\mu\alpha}g_{\beta\mu} = (\eta^{\mu\alpha} - h^{\mu\alpha})(\eta_{\beta\mu} + h_{\beta\mu}) = \tag{3.13}$$
$$\eta^{\mu\alpha}\eta_{\beta\mu} + \eta^{\mu\alpha}h_{\beta\mu} - h^{\mu\alpha}\eta_{\beta\mu} + O(h^2) \approx \tag{3.14}$$
$$\delta^{\alpha}_{\beta} + h^{\mu}_{\mu} - h^{\mu}_{\mu} = \delta^{\alpha}_{\beta}. \tag{3.15}$$

The action (3.1) consists of 3 parts:$\sqrt{-g}$,$g^{\mu\nu}$ and $R_{\mu\nu}$. As we want to expand this action up to 2 powers of h and the leading terms in $R_{\mu\nu}$ are of order 1 already we only need to find$\sqrt{-g}$ and $g^{\mu\nu}$ up to order 1.

The determinant

If g is the determinant of $g_{\mu\nu}$ we can write it as follows:

$$g = det(g_{\mu\nu}) = \tag{3.16}$$
$$det(\eta_{\mu\nu} + h_{\mu\nu}) = det(\eta_{\mu\nu})det(1 + \eta^{\mu\nu}h_{\mu\nu}) = \tag{3.17}$$
$$-det(1 + h) = -e^{tr(ln(1+h)} \approx -e^{h+O(h^2)} \approx -(1 + h). \tag{3.18}$$

Where in the last row the usual expansions for $log(1 + x)$ and e^x were used and the following identity [11]:

$$det(A) = e^{tr(ln(A))}. \tag{3.19}$$

The corresponding term in the action becomes:

$$\sqrt{-g} \approx (1 + h)^{\frac{1}{2}} \approx 1 + \frac{1}{2}h \tag{3.20}$$

First Order Ricci Tensor

The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + \Gamma^{\rho}{}_{\gamma\rho}\Gamma^{\gamma}{}_{\mu\nu} - \Gamma^{\rho}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\mu\rho} \tag{3.21}$$

The leading term in $\Gamma^{\rho}{}_{\mu\nu}$ is linear in h. So while in the first two terms $\Gamma^{\rho}{}_{\mu\nu}$ should be considered up to second order, in the last two terms of (3.21) we only need to evaluate $\Gamma^{\rho}{}_{\mu\nu}$ up to first order. The full Christoffel symbol is given by:

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\gamma}(\partial_\nu g_{\mu\gamma} + \partial_\mu g_{\nu\gamma} - \partial_\gamma g_{\mu\nu}). \tag{3.22}$$

After the expanding we are left with:

$$\Gamma^{\rho}{}_{\mu\nu} \approx \frac{1}{2}(\eta^{\rho\gamma} - h^{\rho\gamma})(\partial_\nu(\eta_{\mu\gamma} + h_{\mu\gamma}) + \partial_\mu(\eta_{\nu\gamma} + h_{\nu\gamma}) - \partial_\gamma(\eta_{\mu\nu} + h_{\mu\nu})). \tag{3.23}$$

As the Minkowski metric does not depend on any coordinate, $\partial_\mu \eta_{\nu\gamma} = 0$ and the resultant Christoffel symbol is:

$$\Gamma^{\rho}{}_{\mu\nu} \approx \frac{1}{2}(\eta^{\rho\gamma} - h^{\rho\gamma})(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu}). \tag{3.24}$$

Up to first order (3.24) becomes:

$$\Gamma^{(1)\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\gamma}(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu}) \tag{3.25}$$

$$= \frac{1}{2}(\partial_\nu h^{\rho}_{\mu} + \partial_\mu h^{\rho}_{\nu} - \partial^\rho h_{\mu\nu}). \tag{3.26}$$

Consequently the first two terms are:

$$\partial_\rho \Gamma^{(1)\rho}{}_{\mu\nu} = \frac{1}{2}(\partial_\rho \partial_\nu h^{\rho}_{\mu} + \partial_\rho \partial_\mu h^{\rho}_{\nu} - \partial_\rho \partial^\rho h_{\mu\nu}). \tag{3.27}$$

$$\partial_\nu \Gamma^{(1)\rho}{}_{\mu\rho} = \frac{1}{2}(\partial_\nu \partial_\rho h^{\rho}_{\mu} + \partial_\nu \partial_\mu h^{\rho}_{\rho} - \partial_\nu \partial^\rho h_{\mu\rho}). \tag{3.28}$$

Substracting these two terms results in the Ricci tensor up to first order:

$$R^{(1)}_{\mu\nu} = \partial_\rho \Gamma^{(1)\rho}{}_{\mu\nu} - \partial_\nu \Gamma^{(1)\rho}{}_{\mu\rho} \tag{3.29}$$

$$= \frac{1}{2}(\partial_\rho \partial_\nu h^{\rho}_{\mu} + \partial_\rho \partial_\mu h^{\rho}_{\nu} - \partial_\rho \partial^\rho h_{\mu\nu} - \partial_\nu \partial_\rho h^{\rho}_{\mu} - \partial_\nu \partial_\mu h^{\rho}_{\rho} + \partial_\nu \partial^\rho h_{\mu\rho}) \tag{3.30}$$

$$= \frac{1}{2}(\partial_\rho \partial_\mu h^{\rho}_{\nu} - \partial_\rho \partial^\rho h_{\mu\nu} - \partial_\nu \partial_\mu h^{\rho}_{\rho} + \partial_\nu \partial^\rho h_{\mu\rho}) \tag{3.31}$$

$$= \frac{1}{2}(\partial_\rho \partial_\mu h^{\rho}_{\nu} + \partial_\nu \partial^\rho h_{\mu\rho} - \partial_\nu \partial_\mu h - \Box h_{\mu\nu}), \tag{3.32}$$

where $\Box = \partial_\mu \partial^\mu$.

Second Order Ricci Tensor

The second order part of the Christoffel symbols looks as follows:

$$\Gamma^{(2)\rho}{}_{\mu\nu} = -\frac{1}{2} h^{\rho\gamma}(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu}). \tag{3.33}$$

The first two second order terms are the derivative of this Christoffel symbol, namely:

$$\partial_\rho \Gamma^{(2)\rho}{}_{\mu\nu} = -\frac{1}{2}\partial_\rho[h^{\rho\gamma}(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu})] \tag{3.34}$$

$$= -\frac{1}{2}[\partial_\rho h^{\rho\gamma}(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu}) \tag{3.35}$$

$$+h^{\rho\gamma}\partial_\rho(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu})]. \tag{3.36}$$

And:

$$\partial_\nu \Gamma^{(2)\rho}{}_{\mu\rho} = -\frac{1}{2}\partial_\nu[h^{\rho\gamma}(\partial_\rho h_{\mu\gamma} + \partial_\mu h_{\rho\gamma} - \partial_\gamma h_{\mu\rho})] \tag{3.37}$$

$$= -\frac{1}{2}[\partial_\nu h^{\rho\gamma}(\partial_\rho h_{\mu\gamma} + \partial_\mu h_{\rho\gamma} - \partial_\gamma h_{\mu\rho}) \tag{3.38}$$

$$+h^{\rho\gamma}\partial_\nu(\partial_\rho h_{\mu\gamma} + \partial_\mu h_{\rho\gamma} - \partial_\gamma h_{\mu\rho})] \tag{3.39}$$

$$= -\frac{1}{2}[\partial_\nu h^{\rho\gamma}\partial_\mu h_{\rho\gamma} + h^{\rho\gamma}\partial_\nu\partial_\mu h_{\rho\gamma}]. \tag{3.40}$$

Again, subtracting these two gives us the following result:

$$\partial_\rho \Gamma^{(2)\rho}{}_{\mu\nu} - \partial_\nu \Gamma^{(2)\rho}{}_{\mu\rho} = -\frac{1}{2}[\partial_\rho h^{\rho\gamma}(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu}) \tag{3.41}$$

$$+h^{\rho\gamma}(\partial_\rho\partial_\nu h_{\mu\gamma} + \partial_\rho\partial_\mu h_{\nu\gamma} - \partial_\rho\partial_\gamma h_{\mu\nu} - \partial_\nu\partial_\mu h_{\rho\gamma}) \tag{3.42}$$

$$-\partial_\nu h^{\rho\gamma}\partial_\mu h_{\rho\gamma}]. \tag{3.43}$$

The other two terms, as mentioned earlier, only require the Christoffel symbols up to first order. The first one:

$$\Gamma^{(1)\rho}{}_{\gamma\rho}\Gamma^{(1)\gamma}{}_{\mu\nu} = \frac{1}{4}(\partial_\gamma h^\rho_\rho + \partial_\rho h^\rho_\gamma - \partial^\rho h_{\gamma\rho})(\partial_\mu h^\gamma_\nu + \partial_\nu h^\gamma_\mu - \partial^\gamma h_{\mu\nu}) \tag{3.44}$$

$$= \frac{1}{4}(\partial_\gamma h(\partial_\mu h^\gamma_\nu + \partial_\nu h^\gamma_\mu - \partial^\gamma h_{\mu\nu}). \tag{3.45}$$

And the second one:

$$\Gamma^{(1)\rho}{}_{\gamma\nu}\Gamma^{(1)\gamma}{}_{\mu\rho} = \frac{1}{4}(\partial_\gamma h^\rho_\nu + \partial_\nu h^\rho_\gamma - \partial^\rho h_{\gamma\nu})(\partial_\mu h^\gamma_\rho + \partial_\rho h^\gamma_\mu - \partial^\gamma h_{\mu\rho}) \tag{3.46}$$

$$= \frac{1}{4}[\partial_\gamma h^\rho_\nu \partial_\mu h^\gamma_\rho + \partial_\gamma h^\rho_\nu \partial_\rho h^\gamma_\mu - \partial_\gamma h^\rho_\nu \partial^\gamma h_{\mu\rho} \tag{3.47}$$

$$+\partial_\nu h^\rho_\gamma \partial_\mu h^\gamma_\rho + \partial_\nu h^\rho_\gamma \partial_\rho h^\gamma_\mu - \partial_\nu h^\rho_\gamma \partial^\gamma h_{\mu\rho} \tag{3.48}$$

$$-\partial^\rho h_{\gamma\nu}\partial_\mu h^\gamma_\rho - \partial^\rho h_{\gamma\nu}\partial_\rho h^\gamma_\mu + \partial^\rho h_{\gamma\nu}\partial^\gamma h_{\mu\rho}] \tag{3.49}$$

$$= \frac{1}{4}[2\partial_\gamma h^\rho_\nu \partial_\rho h^\gamma_\mu - 2\partial_\gamma h^\rho_\nu \partial^\gamma h_{\rho\mu} + \partial_\nu h^\rho_\gamma \partial_\mu h^\gamma_\rho] \tag{3.50}$$

$$= \frac{1}{2}[\partial_\gamma h^\rho_\nu(\partial_\rho h^\gamma_\mu - \partial^\gamma h_{\rho\mu}) + \frac{1}{2}\partial_\nu h^\rho_\gamma \partial_\mu h^\gamma_\rho]. \tag{3.51}$$

Combining all these pieces gives us the total Ricci tensor:

$$R^{(2)}_{\mu\nu} = -\frac{1}{2}[\partial_\rho h^{\rho\gamma}(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu}) \tag{3.52}$$

$$+h^{\rho\gamma}(\partial_\rho\partial_\nu h_{\mu\gamma} + \partial_\rho\partial_\mu h_{\nu\gamma} - \partial_\rho\partial_\gamma h_{\mu\nu} - \partial_\nu\partial_\mu h_{\rho\gamma}) \tag{3.53}$$

$$-\frac{1}{2}\partial_\nu h^{\rho\gamma}\partial_\mu h_{\rho\gamma} - \frac{1}{2}\partial_\gamma h(\partial_\mu h^\gamma_\nu + \partial_\nu h^\gamma_\mu - \partial^\gamma h_{\mu\nu}) \tag{3.54}$$

$$+\partial_\gamma h^\rho_\nu(\partial_\rho h^\gamma_\mu - \partial^\gamma h_{\rho\mu})]. \tag{3.55}$$

The Action

Combining all these parts and keeping only the terms up to quadratic order in h, we can write down the linearized action in these four terms:

$$S^{(2)} + S^{(1)} = \int d^4x(1 + \frac{1}{2}h)(\eta^{\mu\nu} - h^{\mu\nu})(R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}) \tag{3.56}$$

$$= \int d^4x[\eta^{\mu\nu}R^{(1)}_{\mu\nu} - h^{\mu\nu}R^{(1)}_{\mu\nu} + \frac{1}{2}hR^{(1)}_{\mu\nu} + \eta^{\mu\nu}R^{(2)}_{\mu\nu}]. \tag{3.57}$$

The last line consists of all the terms quadratic order or lower. The only linear term is the first one, namely:

$$\eta^{\mu\nu}R^{(1)}_{\mu\nu} = \eta^{\mu\nu}\frac{1}{2}(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\nu\partial^\rho h_{\mu\rho} - \partial_\nu\partial_\mu h - \Box h_{\mu\nu}) \tag{3.58}$$

$$= \partial_\rho\partial_\mu h^{\rho\mu} - \Box h. \tag{3.59}$$

Next up are the 3 quadratic terms:

$$-h^{\mu\nu}R^{(1)}_{\mu\nu} = -\frac{1}{2}h^{\mu\nu}(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\nu\partial^\rho h_{\mu\rho} - \partial_\nu\partial_\mu h - \Box h_{\mu\nu}), \tag{3.60}$$

$$\frac{1}{2}h\eta^{\mu\nu}R^{(1)}_{\mu\nu} = \frac{1}{2}(h\partial_\rho\partial_\mu h^{\rho\mu} - h\Box h) \tag{3.61}$$

and

$$\eta^{\mu\nu}R^{(2)}_{\mu\nu} = \eta^{\mu\nu}\frac{1}{2}[(\frac{1}{2}\partial^\gamma h - \partial_\rho h^{\rho\gamma})(\partial_\nu h_{\mu\gamma} + \partial_\mu h_{\nu\gamma} - \partial_\gamma h_{\mu\nu}) \tag{3.62}$$

$$-h^{\rho\gamma}(\partial_\rho\partial_\nu h_{\mu\gamma} + \partial\rho\partial\mu h_{\nu\gamma} - \partial_\rho\partial_\gamma h_{\mu\nu} - \partial_\nu\partial_\mu h_{\rho\gamma}) \tag{3.63}$$

$$\frac{1}{2}\partial_\nu h^{\rho\gamma}\partial_\mu h_{\rho\gamma} - \partial^\gamma h^\rho_\nu(\partial_\rho h_{\gamma\mu} - \partial_\gamma h_{\rho\mu}] = \tag{3.64}$$

$$\frac{1}{2}[(\frac{1}{2}\partial^\gamma h - \partial_\rho h^\rho\gamma)(2\partial^\mu h_{\mu\gamma} - \partial_\gamma h) \tag{3.65}$$

$$-h^{\rho\gamma}(2\partial_\rho\partial^\mu h_{\mu\gamma} - \partial_\rho\partial_\gamma h - \Box h_{\rho\gamma}) \tag{3.66}$$

$$-\frac{1}{2}\partial^\mu h^{\rho\gamma}\partial_\mu h_{\rho\gamma} - \partial^\gamma h^{\rho\mu}(\partial_\rho h_{\gamma\mu} - \partial_\gamma h_{\rho\mu}] = \tag{3.67}$$

$$-\frac{1}{2}h\partial^\gamma\partial^\mu h_{\mu\gamma} + \frac{1}{4}h\Box h + \frac{1}{2}h^{\rho\gamma}\partial_\rho\partial^\mu h_{\mu\gamma} - \frac{1}{4}h^{\rho\gamma}\Box h_{\rho\gamma}. \tag{3.68}$$

In the last step integration by parts is used to go from the form $\partial h\partial h$ to $h\partial\partial h$, assuming the terms are integrated in the action the fields go to zero at infinity. Finally, adding all these terms together gives us the complete

inearized action:

$$S^{(2)} + S^{(1)} = \int d^4x(1 + \frac{1}{2}h)(\eta^{\mu\nu} - h^{\mu\nu})(R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}) \tag{3.69}$$

$$= \int d^4x[\eta^{\mu\nu}R^{(1)}_{\mu\nu} - h^{\mu\nu}R^{(1)}_{\mu\nu} + \frac{1}{2}hR^{(1)}_{\mu\nu} + \eta^{\mu\nu}R^{(2)}_{\mu\nu}] \tag{3.70}$$

$$= \int d^4x[\partial_\rho\partial_\mu h^{\rho\mu} - \Box h + \frac{1}{2}h\partial^\gamma\partial^\mu h_{\mu\gamma} - \frac{1}{4}h\Box h \tag{3.71}$$

$$-\frac{1}{2}h^{\rho\gamma}\partial_\rho\partial^\mu h_{\mu\gamma} + \frac{1}{4}h^{\rho\gamma}\Box h_{\rho\gamma}]. \tag{3.72}$$

The linear terms vanish because they are a total derivative so we are left with:

$$S = \int d^4x\frac{1}{2}h\partial^\gamma\partial^\mu h_{\mu\gamma} - \frac{1}{4}h\Box h - \frac{1}{2}h^{\rho\gamma}\partial_\rho\partial^\mu h_{\mu\gamma} + \frac{1}{4}h^{\rho\gamma}\Box h_{\rho\gamma}. \tag{3.73}$$

3.1.2 Bottom up

In this section another approach is taken to come to the same result. For this a general lagrangian consisting of two powers of $h^{\mu\nu}$ and two powers of ∂^μ will be demanded to be invariant under the linearized diffeomorphisms. First the different possible terms will be written down. After that their transformations and their conditions to be invariant will be studied.

The Terms

These are the four independent contractions that are the building bricks of the action:

$$\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} \tag{3.74}$$
$$\partial_\mu h^{\mu\nu}\partial^\alpha h_{\nu\alpha} \tag{3.75}$$
$$\partial_\nu h\partial_\mu h^{\mu\nu} \tag{3.76}$$
$$\partial_\mu h\partial^\mu h. \tag{3.77}$$

You can write down different combinations, such as $h_{\mu\nu}\Box h^{\mu\nu}$, but for each one it would be possible to show it is equivalent for one of the four terms above under relabeling of indices or integrating by parts. Under the linear diffeomorphisms discussed in the previous chapter, the metric perturbation $h_{\mu\nu}$ transforms like this:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{3.78}$$

The other two variables, h and $h^{\mu\nu}$ transform like this:

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} \tag{3.79}$$
$$\to \eta^{\mu\alpha}\eta^{\nu\beta}(h_{\alpha\beta} + \partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha) = h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu \tag{3.80}$$

and

$$h = \eta^{\mu\nu}h_{\mu\nu} \to \eta^{\mu\nu}(h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu) = h + 2\partial_\mu\xi^\mu. \tag{3.81}$$

The Action

The most general action can be written out like this:

$$S = \int d^4x a\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + b\partial_\mu h^{\mu\nu}\partial^\alpha h_{\nu\alpha} + c\partial_\nu h\partial_\mu h^{\mu\nu} + d\partial_\mu h\partial^\mu h, \tag{3.82}$$

With real numbers a,b,c and d. To find the values for these numbers we are going to transform all the terms in the action and then find the correct values that leave the action invariant. The individual terms transform like

this:

$$\begin{aligned}
\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} \rightarrow &\partial_\alpha(h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu)\partial^\alpha(h^{\mu\nu}+\partial^\mu\xi^\nu+\partial^\nu\xi^\mu) && (3.83)\\
=&\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu}+\partial_\alpha h_{\mu\nu}\partial^\alpha\partial^\mu\xi^\nu+\partial_\alpha h_{\mu\nu}\partial^\alpha\partial^\nu\xi^\mu && (3.84)\\
+&\partial_\alpha\partial_\mu\xi_\nu\partial^\alpha h^{\mu\nu}+\partial_\alpha\partial_\mu\xi_\nu\partial^\alpha\partial^\mu\xi^\nu+\partial_\alpha\partial_\mu\xi_\nu\partial^\alpha\partial^\nu\xi^\mu && (3.85)\\
+&\partial_\alpha\partial_\nu\xi_\mu\partial^\alpha h^{\mu\nu}+\partial_\alpha\partial_\nu\xi_\mu\partial^\alpha\partial^\mu\xi^\nu+\partial_\alpha\partial_\nu\xi_\mu\partial^\alpha\partial^\nu\xi^\mu && (3.86)\\
=&\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu}+4\partial_\alpha h_{\mu\nu}\partial^\alpha\partial^\mu\xi^\nu && (3.87)\\
+&2\partial_\alpha\partial_\nu\xi_\mu\partial^\alpha\partial^\mu\xi^\nu+2\partial_\alpha\partial_\nu\xi_\mu\partial^\alpha\partial^\nu\xi^\mu, && (3.88)
\end{aligned}$$

$$\begin{aligned}
\partial_\mu h^{\mu\nu}\partial^\alpha h_{\nu\alpha} \rightarrow &\partial_\mu(h^{\mu\nu}+\partial^\mu\xi^\nu+\partial^\nu\xi^\mu)\partial^\alpha(h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu) && (3.89)\\
=&\partial_\mu h^{\mu\nu}\partial^\alpha h_{\mu\nu}+\partial_\mu h^{\mu\nu}\partial^\alpha\partial_\mu\xi_\nu+\partial_\mu h^{\mu\nu}\partial^\mu\partial_\nu\xi_\mu && (3.90)\\
+&\partial_\mu\partial^\alpha\xi^\nu\partial^\mu h_{\mu\nu}+\partial_\mu\partial^\mu\xi^\nu\partial^\alpha\partial_\mu\xi_\nu+\partial_\mu\partial^\alpha\xi^\nu\partial^\mu\partial_\nu\xi_\mu && (3.91)\\
+&\partial_\mu\partial^\nu\xi^\mu\partial^\alpha h_{\mu\nu}+\partial_\mu\partial^\nu\xi^\mu\partial^\alpha\partial_\mu\xi_\nu+\partial_\mu\partial^\nu\xi^\mu\partial^\alpha\partial_\nu\xi_\mu && (3.92)\\
=&\partial_\mu h^{\mu\nu}\partial^\alpha h_{\nu\alpha}+2\partial_\mu h^{\mu\nu}\partial^\alpha\partial_\nu\xi_\alpha+2\partial_\mu h^{\mu\nu}\partial^\alpha\partial_\alpha\xi_\mu && (3.93)\\
+&3\partial_\mu\partial^\mu\xi^\nu\partial^\alpha\partial_\nu\xi_\alpha+\partial_\mu\partial^\mu\xi^\nu\partial^\alpha\partial_\alpha\xi_\nu, && (3.94)
\end{aligned}$$

$$\begin{aligned}
\partial_\nu h\partial_\mu h^{\mu\nu} \rightarrow &\partial_\nu(h+2\partial_\alpha\xi^\alpha)\partial_\mu(h^{\mu\nu}+\partial^\mu\xi^\nu+\partial^\nu\xi^\mu) && (3.95)\\
=&\partial_\nu h\partial_\mu h^{\mu\nu}+\partial_\nu h\partial_\mu\partial^\mu\xi^\nu+\partial_\nu h\partial_\mu\partial^\nu\xi^\mu && (3.96)\\
+&2\partial_\nu\partial_\alpha\xi^\alpha\partial_\mu h^{\mu\nu}+2\partial_\nu\partial_\alpha\xi^\alpha\partial_\mu\partial^\mu\xi^\nu+2\partial_\nu\partial_\alpha\xi^\alpha\partial_\mu\partial^\nu\xi^\mu && (3.97)\\
=&\partial_\nu h\partial_\mu h^{\mu\nu}+2\partial_\nu h\partial_\mu\partial^\nu\xi^\mu && (3.98)\\
+&2\partial_\nu\partial_\alpha\xi^\alpha\partial_\mu h^{\mu\nu}+4\partial_\nu\partial_\alpha\xi^\alpha\partial_\mu\partial^\mu\xi^\nu, && (3.99)
\end{aligned}$$

and

$$\begin{aligned}
\partial_\mu h\partial^\mu h \rightarrow &\partial_\mu(h+2\partial_\alpha\xi^\alpha)\partial^\mu(h+2\partial_\beta\xi^\beta) && (3.100)\\
=&\partial_\mu h\partial^\mu h+2\partial_\mu h\partial^\mu\partial_\beta\xi^\beta && (3.101)\\
+&2\partial_\mu\partial_\alpha\xi^\alpha\partial^\mu h+4\partial_\mu\partial_\alpha\xi^\alpha\partial^\mu\partial_\beta\xi^\beta. && (3.102)
\end{aligned}$$

If all terms other than the original terms need to vanish, such that there are no terms left with ξ in them, the following five equations need to hold:

$$\begin{aligned}
\partial_\mu h\partial^\mu\partial^\mu\xi^\beta(4d+2c) =&0 && (3.103)\\
\partial_\nu\partial_\alpha\xi^\alpha\partial_\mu h^{\mu\nu}(2c+2b) =&0 && (3.104)\\
\partial_\alpha h_{\mu\nu}\partial^\alpha\partial\mu\xi^\nu(2b+4a) =&0 && (3.105)\\
\partial_\nu\partial_\alpha\xi^\alpha\partial_\mu\partial^\mu\xi^\nu(4c+4d+3b+2a) =&0 && (3.106)\\
\partial_\mu\partial^\mu\xi^\nu\partial^\alpha\partial_\alpha\xi_\nu(b+2a) =&0. && (3.107)
\end{aligned}$$

The first three equations give us the following relations between the numbers:

$$4d + 2c = 0 \tag{3.108}$$
$$d = -\frac{1}{2}c \tag{3.109}$$
$$2c + 2b = 0 \tag{3.110}$$
$$c = -b \tag{3.111}$$
$$2b + 4a = 0 \tag{3.112}$$
$$a = -\frac{1}{2}b. \tag{3.113}$$

This gives us the conditions to construct an invariant action:

$$a = -\frac{1}{2}b = \frac{1}{2}c = -d \tag{3.114}$$

The last two equations are redundant, but for consistency we can check them as well:

$$4c + 4d + 3b + 2a = 0 \tag{3.115}$$
$$4(2a) + 4(-a) + 3(-2a) + 2a = \tag{3.116}$$
$$8 - 4 - 6 + 2 = 0 \tag{3.117}$$
$$b + 2a = 0 \tag{3.118}$$
$$-2 + 2 = 0. \tag{3.119}$$

Every combination of coefficients that comply to these conditions will form an invariant metric under the linearized diffeomorphisms. This means that it is possible to write down the following action up to an overall constant:

$$S = \int d^4x C[\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial^\alpha h_{\nu\alpha} + \partial_\nu h\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial_\mu h\partial^\mu h]. \tag{3.120}$$

Or, if integrated by parts and absorbing $-\frac{1}{2}$ into the constant:

$$S = \int d^4x D[\frac{1}{4}h_{\mu\nu}\Box h^{\mu\nu} - \frac{1}{2}h^{\mu\nu}\partial_\mu\partial^\alpha h_{\nu\alpha} + \frac{1}{2}h\partial^\nu\partial^\mu h_{\mu\nu} - \frac{1}{4}h\Box h]. \tag{3.121}$$

With D a real number. As predicted, this is the same action we found in the previous section (3.73). We can conclude that up to quadratic order in h, the Einstein-Hilbert action and the massless spin-2 action differ only by an overall constant.

Chapter 4

From linearized to the full theory

4.1 Introduction

Now that we are convinced that a theory describing linear massless spin 2 particle is the same as general relativity up to quadratic terms in h, it is time to look at the full theory. In this chapter we will try to construct a non-linear theory out of the linear action found in chapter 2. To do this we need to find a consistent energy-momentum tensor for this action such that the field couples to itself. First we will try to add this energy-momentum tensor to the action naively but we will see that this is only possible after an endless series of corrections. To make it consistent we are using a shortcut by Deser, called the Deser trick. In the end it turns out that by coupling it to its own energy-momentum tensor we retrieve the full Einstein Hilbert action.

4.2 Energy-momentum tensor

Let us start with the action found in chapter 2:

$$S = \int d^4x \frac{1}{4} h_{\mu\nu} \Box h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} \partial_\mu \partial^\alpha h_{\nu\alpha} + \frac{1}{2} h \partial^\nu \partial^\mu h_{\mu\nu} - \frac{1}{4} h \Box h. \tag{4.1}$$

$$= \int d^4x \frac{1}{2} h^{\mu\nu} [\frac{1}{2} \Box \delta^\alpha_\mu \delta^\beta_\nu - \frac{1}{2} \partial_\mu \partial^\alpha \delta^\beta_\nu - \frac{1}{2} \partial_\nu \partial^\alpha \delta^\beta_\mu + \eta_{\mu\nu} (\partial^\alpha \partial^\beta + \frac{1}{2} \Box \eta^{\alpha\beta})] h_{\alpha\beta} \tag{4.2}$$

$$= \int d^4x h^{\mu\nu} \epsilon^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \tag{4.3}$$

Right now this is only a free theory, the equations of motion are as follows:

$$\epsilon^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = 0 \tag{4.4}$$

A 'graviton' should couple to all forms of energy and mass. As the graviton also carries momentum and energy it has to couple to itself. This makes it different from the photon which, as the force carrier of electrodynamics, couples to electric charges but does not carry this charge. In other words, the graviton has to couple to its own energy-momentum tensor where the photon does not. That is, we want to add a term to the action that, when varied with respect to $h^{\mu\nu}$ gives the energy-momentum tensor of the action. Let's say $\Lambda_{\mu\nu}$ is the energy-momentum tensor that corresponds to action

above. You could add it to the action in the following way:

$$S = \int d^4x \frac{1}{4} h_{\mu\nu} \Box h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} \partial_\mu \partial^\alpha h_{\nu\alpha} + \frac{1}{2} h \partial^\nu \partial^\mu h_{\mu\nu} - \frac{1}{4} h \Box h + h^{\mu\nu} \Xi_{\mu\nu}. \tag{4.5}$$

If $\frac{\delta(h^{\rho\gamma}\Xi_{\rho\gamma})}{\delta(h^{\mu\nu})} = \Lambda_{\mu\nu}$ the field now couples to the energy-momentum tensor of our first action (4.3) and its equations of motion look like this:

$$\epsilon^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = \Lambda_{\mu\nu}. \tag{4.6}$$

The problem now lies in that our action has changed with the addition of $\Xi_{\mu\nu}$ and $\Lambda_{\mu\nu}$ is not the energy-momentum tensor of action (4.5) anymore. Since $\Xi_{\mu\nu}$ is some tensor quadratic in $h^{\mu\nu}$, we have added cubic terms to the action that also contribute to its energy-momentum tensor. You could remedy this by adding an second term as a correction to the energy-momentum tensor so it covers the new action (4.5). Except you will run into the exact same problem, you have added a quartic term that contributes to the energy-momentum tensor. It has been claimed that continuing adding these terms results into an infinite sum that only when completed turns into a valid theory, namely the Einstein Hilbert action[12].

Deser trick

4.2.1 A new action

It can also be done in a different, simpler, way with only one addition to the action. This is called the Deser argument and we will follow a combination of his paper [12] and two other sources ([13], chapter 3.2 and [14], appendix B) in some more detail. To see this we are going to start with the following action. Two new fields are introduced. The first one is $f^{\mu\nu}$, which is symmetric, and the second one $\Gamma^\alpha_{\mu\nu}$ which is symmetric in its lower indices.

$$S = \int d^4x [f^{\mu\nu}(\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha}) + \eta^{\mu\nu}(\Gamma^\alpha_{\mu\nu}\Gamma^\rho_{\alpha\rho} - \Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\alpha\nu})] \tag{4.7}$$

No further assumptions are made for now so $\Gamma^\alpha_{\mu\nu}$ is not the usual connection, but the observant reader can recognize this action as a 'linearized' form of the first order formalism of gravity up to quadratic terms in the fields ($f\Gamma$ and $\Gamma\Gamma$ terms).

The first order formalism is a different way to describe general relativity. Instead of starting with the Ricci scalar and using the metric as the only variable and the connection as a function of the metric, both the metric and the 'connection' are independently varied. By doing that the equations of motion of the first order action set the same relation between the metric and the connection as the second order formalism assumes and therefore is equivalent to it. We can show a similar relation for these linear versions of

those theories. The equations of motions of (4.7) are:

$$\frac{\delta S}{\delta f^{\mu\nu}} =\partial_\alpha \Gamma^\alpha_{\mu\nu} - \frac{1}{2}[\partial_\nu \Gamma^\alpha_{\mu\alpha} + \partial_\mu \Gamma^\alpha_{\nu\alpha}] = 0 \tag{4.8}$$

$$\frac{\delta S}{\delta \Gamma^\xi_{\mu\nu}} =\eta^{\mu\nu}\Gamma^\rho_{\xi\rho} + \frac{1}{2}\eta^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\delta^\nu_\xi + \frac{1}{2}\eta^{\alpha\beta}\Gamma^\nu_{\alpha\beta}\delta^\mu_\xi - \eta^{\nu\beta}\Gamma^\mu_{\xi\beta} - \eta^{\mu\beta}\Gamma^\nu_{\xi\beta} \tag{4.9}$$

$$-\partial_\xi f^{\mu\nu} + \frac{1}{2}\partial_\rho f^{\rho\mu}\delta^\nu_\xi + \frac{1}{2}\partial_\rho f^{\rho\nu}\delta^\mu_\xi \tag{4.10}$$

$$=\eta^{\mu\nu}\Gamma^\rho_{\xi\rho} + \eta^{\alpha\beta}\Gamma^{(\mu}_{\alpha\beta}\delta^{\nu)}_\xi - 2\eta^{\beta(\nu}\Gamma^{\mu)}_{\xi\beta} - \partial_\xi f^{\mu\nu} + \partial_\rho f^{\rho(\mu}\delta^{\nu)}_\xi = 0 \tag{4.11}$$

In this last equation the symmetry in $\Gamma^\alpha_{\mu\nu}$ is used. Two identities can be found via the last equation, the first one by contracting it with δ^ξ_μ and the other one by contracting with $\eta_{\mu\nu}$.

$$\eta^{\xi\nu}\Gamma^\rho_{\xi\rho} + \frac{1}{2}\eta^{\alpha\beta}\Gamma^\nu_{\alpha\beta} + 2\eta^{\alpha\beta}\Gamma^\nu_{\alpha\beta} - \eta^{\nu\beta}\Gamma^\xi_{\xi\beta} \tag{4.12}$$

$$-\eta^{\xi\beta}\Gamma^\nu_{\xi\beta} - \partial_\xi f^{\nu\xi} + \frac{1}{2}\partial_\rho f^{\rho\nu} + 2\partial_\rho f^{\rho\nu} = 0 \tag{4.13}$$

$$\eta^{\alpha\beta}\Gamma^\nu_{\alpha\beta} = -\partial_\rho f^{\rho\nu} \tag{4.14}$$

and

$$4\Gamma^\rho_{\xi\rho} + \eta^{\alpha\beta}\eta_{\mu\xi}\Gamma^\mu_{\alpha\beta} - 2\Gamma^\beta_{\xi\beta} - \partial_\xi f + \partial_\rho f^\rho_\xi =0 \tag{4.15}$$

$$2\Gamma^\rho_{\xi\rho} + \eta^{\alpha\beta}\eta_{\mu\xi}\Gamma^\mu_{\alpha\beta} + \eta_{\nu\xi}f^{\rho\nu} =\partial_\xi f \tag{4.16}$$

$$\Gamma^\rho_{\xi\rho} =\frac{1}{2}\partial_\xi f, \tag{4.17}$$

where in the last step the first contraction is used. This last result can then be used to write (4.11) as:

$$-\partial^\xi f_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\partial^\xi f + \partial_\rho f^\rho_\mu \delta^\xi_\nu + \partial_\rho f^\rho_\nu \delta^\xi_\mu = 2\Gamma^\xi_{\mu\nu} - \Gamma^\alpha_{\alpha\mu}\delta^\xi_\nu - \Gamma^\alpha_{\alpha\nu}\delta^\xi_\mu. \tag{4.18}$$

When you take the ∂_ξ derivative of this equation and plug it into equation (4.8) you get:

$$-\Box f_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box f + \partial_\nu\partial_\rho f^\rho_\mu + \partial_\mu\partial_\rho f^\rho_\nu = 0. \tag{4.19}$$

By contracting with $\eta^{\mu\nu}$ you find the relation $\Box f = -2\partial^\mu\partial_\rho f^\rho_\mu$. This equation is equal to equation (4.4) after using this relation and performing the field redefinition $f_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}h - h_{\mu\nu}$.

4.2.2 Calculating the energy-momentum tensor

If $f^{\mu\nu}$ has to couple to its own energy-momentum tensor we need to find a term that when added to the action coupled to $f^{\mu\nu}$ provides this energy-momentum tensor, but does not contribute to this calculation. A way to find this tensor is covariantizing the action, that is promoting the flat metric to a general metric and introducing covariant derivatives. Then vary the action with respect to this metric.
The flat minkowski metric we used before,$\eta^{\mu\nu}$, is promoted to a general

metric $G^{\mu\nu}$. The covariant derivative that comes with this metric is Δ_μ. The two fields have to receive transformation properties with respect to this metric, the first field, $f^{\mu\nu}$ is chosen to transform as a tensor density of weight 1. As a tensor density transforms with a $\sqrt{g}$ factor it is not needed to introduce this factor in the lagrangian whenever a $f^{\mu\nu}$ appears. The other field, $\Gamma^\alpha_{\mu\nu}$, transforms as a normal tensor.
The covariant action looks like this:

$$S = \int d^4x [f^{\mu\nu}(\Delta_\alpha \Gamma^\alpha_{\mu\nu} - \Delta_\nu \Gamma^\alpha_{\mu\alpha}) + \sqrt{-G} G^{\mu\nu}(\Gamma^\alpha_{\mu\nu}\Gamma^\rho_{\alpha\rho} - \Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\alpha\nu})]. \quad (4.20)$$

The energy-momentum tensor of this action is:

$$T_{\alpha\beta} = \frac{1}{\sqrt{-G}} \frac{\delta(S)}{\delta(G^{\alpha\beta})}\bigg|_{G_{\alpha\beta}}, \quad (4.21)$$

but because our variable is trace-shifted the energy-momentum tensor of our original action is:

$$\tau_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} T^\mu_\mu \quad (4.22)$$

When expanding the covariant derivatives, the first term can be written like this:

$$\begin{aligned}
f^{\mu\nu}(\Delta_\alpha \Gamma^\alpha_{\mu\nu} - \Delta_\mu \Gamma^\alpha_{\nu\alpha}) =& f^{\mu\nu}(\partial_\alpha \Gamma^\alpha_{\mu\nu} + \Theta^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Theta^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu} - \Theta^\beta_{\nu\alpha}\Gamma^\alpha_{\beta\mu}) && (4.23)\\
&- f^{\mu\nu}(\partial_\mu \Gamma^\alpha_{\mu\nu} + \Theta^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu} - \Theta^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu} - \Theta^\beta_{\nu\mu}\Gamma^\alpha_{\alpha\beta}) && (4.24)\\
=& f^{\mu\nu}(\partial_\alpha \Gamma^\alpha_{\mu\nu} + \Theta^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - 2\Theta^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}) && (4.25)\\
&- f^{\mu\nu}(\partial_\mu \Gamma^\alpha_{\mu\nu} - \Theta^\beta_{\nu\mu}\Gamma^\alpha_{\alpha\beta}) && (4.26)\\
=& f^{\mu\nu}(\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\mu\nu} + \Theta^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} && (4.27)\\
&- 2\Theta^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu} + \Theta^\beta_{\nu\mu}\Gamma^\alpha_{\alpha\beta}). && (4.28)
\end{aligned}$$

Where $\Theta^\alpha_{\mu\nu}$ is the connection corresponding to the metric $G^{\mu\nu}$, which is defined in the usual way:

$$\Theta^\alpha_{\mu\nu} = \frac{1}{2} G^{\alpha\rho}[\partial_\mu G_{\nu\rho} + \partial_\nu G_{\mu\rho} - \partial_\rho G_{\mu\nu}] \quad (4.29)$$

Now we can start to calculate the variation of the action, we need to find all terms that depend on G and find out what happens if you vary it with

respect to to that metric:

$$\delta S = \int d^4x [f^{\mu\nu} \frac{\delta(\Delta_\sigma \Gamma^\sigma_{\mu\nu} - \Delta_\nu \Gamma^\sigma_{\mu\sigma})}{\delta(G^{\alpha\beta})} \tag{4.30}$$

$$+ \frac{\delta(\sqrt{-G}G^{\mu\nu})}{\delta(G^{\alpha\beta})}(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\sigma\nu})]\delta(G^{\alpha\beta}) \tag{4.31}$$

$$= \int d^4x [(f^{\mu\nu}\delta^\tau_\rho \Gamma^\kappa_{\mu\nu} - 2f^{\kappa\nu}\Gamma^\tau_{\rho\nu} + f^{\kappa\tau}\Gamma^\sigma_{\sigma\rho})\frac{\delta(\Theta^\rho_{\kappa\tau})}{\delta(G^{\alpha\beta})} \tag{4.32}$$

$$+(\sqrt{-G}\frac{\delta(G^{\mu\nu})}{\delta(G^{\alpha\beta})} - \frac{1}{2}\sqrt{-G}G_{\theta\tau}\frac{\delta(G^{\theta\tau})}{\delta(G^{\alpha\beta})}G^{\mu\nu}) \tag{4.33}$$

$$(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\sigma\nu})]\delta(G^{\alpha\beta}) \tag{4.34}$$

$$= \int d^4x [(f^{\mu\nu}\delta^\tau_\rho \Gamma^\kappa_{\mu\nu} - 2f^{\kappa\nu}\Gamma^\tau_{\rho\nu} + f^{\kappa\tau}\Gamma^\sigma_{\sigma\rho})\frac{\delta(\Theta^\rho_{\kappa\tau})}{\delta(G^{\alpha\beta})} \tag{4.35}$$

$$+(\sqrt{-G})(\Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\rho\alpha}\Gamma^\rho_{\sigma\beta}) \tag{4.36}$$

$$- \frac{1}{2}\sqrt{-G}G_{\alpha\beta}G^{\mu\nu}(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\sigma\nu})]\delta(G^{\alpha\beta}). \tag{4.37}$$

The first part can be worked out like this:

$$\int d^4x \delta(G^{\alpha\beta})[f^{\mu\nu}\delta^\tau_\rho \Gamma^\kappa_{\mu\nu} - 2f^{\kappa\nu}\Gamma^\tau_{\rho\nu} + f^{\kappa\tau}\Gamma^\sigma_{\sigma\rho}]\frac{\delta(\Theta^\rho_{\kappa\tau})}{\delta(G^{\alpha\beta})} = \tag{4.38}$$

$$\int d^4x \delta(G^{\alpha\beta})\frac{1}{2}[f^{\mu\nu}\delta^\tau_\rho \Gamma^\kappa_{\mu\nu} - 2f^{\kappa\nu}\Gamma^\tau_{\rho\nu} + f^{\kappa\tau}\Gamma^\sigma_{\sigma\rho}] \tag{4.39}$$

$$[\delta(G^{\rho\xi})(\partial_\kappa G_{\tau\xi} + \partial_\tau G_{\xi\kappa} - \partial_\xi G_{\tau\kappa}) \tag{4.40}$$

$$+G^{\rho\xi}(\partial_\kappa \delta(G_{\tau\xi}) + \partial_\tau \delta(G_{\xi\kappa}) - \partial_\xi \delta(G_{\tau\kappa}))]\frac{1}{\delta(G^{\alpha\beta})} \tag{4.41}$$

When setting $G^{\mu\nu}$ equal to $\eta^{\mu\nu}$ the first term will become zero, so we only continue with the second term:

$$\int d^4x \delta(G^{\alpha\beta}) \left[f^{\mu\nu}\delta^\tau_\rho \Gamma^\kappa_{\mu\nu} - 2f^{\kappa\nu}\Gamma^\tau_{\rho\nu} + f^{\kappa\tau}\Gamma^\sigma_{\sigma\rho}\right] \tag{4.42}$$

$$(G^{\rho\xi}(\partial_\kappa \delta(G_{\tau\xi}) + \partial_\tau \delta(G_{\xi\kappa}) - \partial_\xi \delta(G_{\tau\kappa}))]\frac{1}{\delta(G^{\alpha\beta})} \tag{4.43}$$

$$= \int d^4x \delta(G^{\alpha\beta})[G_{\tau(\alpha}G_{\beta)\xi}\partial_\kappa + G_{\kappa(\alpha}G_{\beta)\xi}\partial_\tau - G_{\kappa(\alpha}G_{\beta)\xi}\partial_\tau] \tag{4.44}$$

$$[f^{\mu\nu}\delta^\tau_\rho \Gamma^\kappa_{\mu\nu} - 2f^{\kappa\nu}\Gamma^\tau_{\rho\nu} + f^{\kappa\tau}\Gamma^\sigma_{\sigma\rho}]G^{\rho\xi}. \tag{4.45}$$

Now we can set $G_{\mu\nu}$ equal to $\eta_{\mu\nu}$ and the complete, but still trace shifted, energy-momentum tensor reads:

$$\frac{1}{\sqrt{-G}}\frac{\delta(S)}{\delta(G^{\alpha\beta})}\bigg|_{G_{\alpha\beta}=\eta_{\alpha\beta}} = \tau_{\alpha\beta} = (\Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\rho\alpha}\Gamma^\rho_{\sigma\beta}) \tag{4.46}$$

$$- \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\sigma\nu})] \tag{4.47}$$

$$+ \partial_\kappa(\eta_{\alpha\beta}f^{\mu\nu}\Gamma^\kappa_{\mu\nu} - 2f^{\kappa\nu}\eta_{\tau(\alpha}\Gamma^\tau_{\beta)\nu} - 2f^\nu_{(\beta}\Gamma^\tau_{\alpha)\nu} + \tag{4.48}$$

$$+ 2\eta^{\rho\kappa}f^\nu_{(\beta}\eta_{\alpha)\tau}\Gamma^\tau_{\rho\nu} + 2f^\kappa_{(\alpha}\Gamma^\sigma_{\beta)\sigma} - \eta^{\rho\kappa}f_{\alpha\beta}\Gamma^\sigma_{\sigma\rho}) \tag{4.49}$$

This makes the final energy-momentum tensor:

$$T_{\alpha\beta} = \tau_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\tau \tag{4.50}$$

$$= \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{\rho\alpha}\Gamma^{\rho}_{\sigma\beta} \tag{4.51}$$

$$+ \frac{1}{2}\partial_\kappa[\frac{1}{2}\eta^{\rho\kappa}\eta_{\alpha\beta}(\frac{1}{2}f\Gamma^{\sigma}_{\sigma\rho} - f^{\nu}_{\tau}\Gamma^{\tau}_{\rho\nu}) - 2f^{\kappa\nu}\eta_{\tau(\alpha}\Gamma^{\tau}_{\beta)\nu} \tag{4.52}$$

$$- 2f^{\nu}_{(\beta}\Gamma^{\kappa}_{\alpha)\nu} + 2\eta^{\rho\kappa}f^{\nu}_{(\beta}\eta_{\alpha)\tau}\Gamma^{\tau}_{\rho\nu} + 2f^{\kappa}_{(\alpha}\Gamma^{\sigma}_{\beta)\sigma} - \eta^{\rho\kappa}f_{\alpha\beta}\Gamma^{\sigma}_{\sigma\rho}] \tag{4.53}$$

The next step is coupling this energy-momentum tensor in the action. This will be done by adding $f^{\mu\nu}(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\rho}_{\alpha\rho} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\rho}_{\alpha\nu})$ to (4.7):

$$S = \int d^4x[f^{\mu\nu}(\partial_\alpha\Gamma^{\alpha}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}_{\mu\alpha}) + (\eta^{\mu\nu} + f^{\mu\nu})(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\rho}_{\alpha\rho} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\rho}_{\alpha\nu})]. \tag{4.54}$$

Note that this adds no factors of $\eta^{\mu\nu}$ nor any derivatives to the action, so if we were to do this procedure again we would find the exact same energy-momentum tensor. In different words, the new term does not contribute to the energy-momentum tensor like the contributions from the attempts in the beginning of this chapter. Because only the simple part from the energy momentum tensor is added to the action, it would seem that this term would only allow a small part of the energy-momentum tensor to appear in the equations of motion. However, the equations of motion of $\Gamma^{\alpha}_{\mu\nu}$ change in such a way that the desired equations are found in terms of h:

$$\epsilon^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = T_{\mu\nu}. \tag{4.55}$$

The free action is now sourced by its own energy-momentum tensor!

4.2.3 Einstein Hilbert

When adding the total derivative $\eta^{\mu\nu}[\partial_\alpha\Gamma^{\alpha}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}_{\mu\alpha}]$ and redefining the fields $\eta^{\mu\nu} + f^{\mu\nu}$ to $\sqrt{-g}g^{\mu\nu}$ the first order form of gravity, and thus the Einstein Hilbert action, is retrieved and the end goal reached:

$$S = \int d^4x\sqrt{-g}[g^{\mu\nu}(\partial_\alpha\Gamma^{\alpha}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}_{\mu\alpha}) + g^{\mu\nu}(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\rho}_{\alpha\rho} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\rho}_{\alpha\nu})] \tag{4.56}$$

$$= \int d^4x\sqrt{-g}R. \tag{4.57}$$

From the linear action, just by coupling the field to its own energy-momentum tensor, we have recovered the full non linear Einstein Hilbert action. It is quite amazing that the infinite sum mentioned in the first section turns into a single correction in the deser trick. Although one could say it is cheating a bit to start from the already known first order formalism, this is just a shortcut to get to the end result faster.

Chapter 5

Adding an extra dimension

5.1 Gauss Bonnet

Apart from the Hilbert-Einstein action you could write down more actions as a function of the curvature tensor, these theories are called $f(R)$ theories. In general these theories do not produce useful results because when you introduce quadratic or higher order terms of the curvature tensor, you automatically introduce higher derivatives of the metric tensor field. As higher than second order derivatives in the equations of motion cause problems like instabilities and ghosts. In 1970, Lovelock ([15]) found all possible rank-2 tensors retrieved from the variational principle that:

- Are symmetric.
- Are divergence free.
- Contain the metric and its first two derivatives.

It turns out that in 4 dimensions only the Einstein-Hilbert action is a valid candidate, all other non-problematic terms are zero. When looking at 5 dimensions a new nonzero term appears, the Gauss-Bonnet term:

$$S = \int d^5x\sqrt{-g}[R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}]. \tag{5.1}$$

This is the only combination of terms quadratic in the curvature that do not contribute to the equations of motions with derivatives higher than the second one of the field. In this chapter we are going to take a look at linearizing the Gauss Bonnet terms and trying to find if they also uniquely turn into the action that we get by forcing gauge invariance.

5.1.1 Bottom up

Quadratic

As Gauss Bonnet has two powers of the curvature tensor, the leading terms would be of the form $\partial\partial h\partial\partial h$. Similarly to (3.1.2) we can write down all contractions of this form:

$$a\Box h^{\mu\nu}\Box h_{\mu\nu} \tag{5.2}$$
$$b\Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} \tag{5.3}$$
$$c\partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} \tag{5.4}$$
$$d\partial_\mu\partial_\nu h^{\mu\nu}\Box h \tag{5.5}$$
$$e\Box h\Box h \tag{5.6}$$

These can be Incorporated into the most general action:

$$S = \int d^5x a\Box h^{\mu\nu}\Box h_{\mu\nu} + b\Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} + c\partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} \tag{5.7}$$

$$+ d\partial_\mu\partial_\nu h^{\mu\nu}\Box h + e\Box h\Box h. \tag{5.8}$$

Again transforming all these terms according to the gauge transformation from (2) gives us the following terms:

$$\Box h^{\mu\nu}\Box h_{\mu\nu} \to \Box h^{\mu\nu}\Box h_{\mu\nu} + 4\Box h^{\mu\nu}\Box\partial_\mu\xi_\nu + 2\Box\partial^\nu\xi^\mu\Box\partial_\mu\xi_\nu \tag{5.9}$$
$$+ 2\Box\partial^\nu\xi^\mu\Box\partial_\nu\xi_\mu$$

$$\Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} \to \Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} + 2\Box\partial^\mu\xi^\alpha\partial_\alpha\partial^\beta h_{\mu\beta} \tag{5.10}$$
$$+ 2\Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta h_{\mu\beta} + \Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta\partial_\beta\xi_\mu$$
$$+ 3\Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta\partial_\mu\xi_\beta$$

$$\partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} \to \partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} + 4\partial_\mu\partial_\alpha h^{\mu\alpha}\Box\partial^\beta\xi_\beta \tag{5.11}$$
$$+ 4\Box\partial_\alpha\xi^\alpha\Box\partial^\beta\xi_\beta$$

$$\partial_\mu\partial_\nu h^{\mu\nu}\Box h \to \partial_\mu\partial_\nu h^{\mu\nu}\Box h + 2\partial_\mu\partial\nu h^{\mu\nu}\Box\partial_\alpha\xi^\alpha \tag{5.12}$$
$$+ 2\Box\partial_\mu\xi^\mu\Box h + 4\Box\partial_\mu\xi^\mu\Box\partial_\nu\xi^\nu$$

$$\Box h\Box h \to \Box h\Box h + 4\Box h\partial_\nu\xi^\nu + 4\Box\partial_\mu\xi^\mu\Box\partial_\nu\xi^\nu. \tag{5.13}$$

To eliminate the terms with ξ in it the following equations need to hold:

$$\Box h^{\mu\nu}\Box\partial_\mu\xi_\nu(4a + 2b) = 0 \tag{5.14}$$

$$\Box\partial^\nu\xi^\mu\Box\partial_\mu\xi_\nu(2a + 3b + 4c + 4d + 4e) = 0 \tag{5.15}$$

$$\Box\partial^\nu\xi^\mu\Box\partial_\nu\xi_\mu(2a + b) = 0 \tag{5.16}$$

$$\Box\partial^\mu\xi^\alpha\partial_\alpha\partial_\beta h_{\mu\beta}(2b + 4c + 2d) = 0 \tag{5.17}$$

$$\Box\partial_\mu\xi^\mu\Box h(2d + 4e) = 0 \tag{5.18}$$

In other words, the parameters have the following relations:

$$a = -\frac{1}{2}b \tag{5.19}$$

$$d = -\frac{1}{2}e \tag{5.20}$$

$$c = -\frac{1}{2}(b + d) \tag{5.21}$$

This is different from the 2 derivative case, where we found a unique set of terms with only an open overall constant. For this 4 derivative case the most general gauge invariant action looks like this:

$$S = \int d^5x[-\frac{1}{2}b\Box h^{\mu\nu}\Box h_{\mu\nu} + b\Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} - \frac{1}{2}(b - 2e)\partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} \tag{5.22}$$

$$- 2e\partial_\mu\partial_\nu h^{\mu\nu}\Box h + e\Box h\Box h].$$

The difference is that in the 2 derivative case all equations of motion only contained up to 2 derivatives. For these terms all terms contribute to fourth

order derivatives in the equations of motion as seen by this:

$$\partial^\beta \partial^\alpha \frac{\partial(\partial^\sigma \partial^\xi h_{\sigma\xi} \Box h)}{\partial(\partial^\alpha \partial^\beta h_{\mu\nu})} = \partial^\mu \partial^\nu \Box h + \eta^{\mu\nu} \Box \partial^\sigma \partial^\xi h_{\sigma\xi}. \tag{5.23}$$

As they only produce fourth order derivatives, the only way to combat this is to set both parameters b and c to zero such that there are no quadratic terms with 4 derivatives

Third power

The next option is making terms third order in the field. While earlier the order of the derivatives and the field did not matter as you could switch them around by integration by parts, now we can identify 4 different forms:

$$hh\partial\partial\partial h, \tag{5.24}$$
$$h\partial h\partial\partial\partial h, \tag{5.25}$$
$$h\partial\partial h\partial\partial h, \tag{5.26}$$
$$\partial h\partial h\partial\partial h. \tag{5.27}$$

Not all of these terms are completely independent as any of them can be written as the combination of 2 others by integrating by parts:

$$h\partial\partial h\partial\partial h \to \partial h\partial h\partial\partial h + h\partial h\partial\partial\partial h, \tag{5.28}$$
$$h\partial h\partial\partial\partial h \to hh\partial\partial\partial\partial h + h\partial h\partial\partial\partial h. \tag{5.29}$$

For this part we are going to focus on all the terms of the form $h\partial\partial h\partial\partial h$. If you write down all contractions of this form you get 45 independent terms. The first step is to take the last two occurences of h and transform them under linear diffeomorphisms like this:

$$h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h \to h_{\mu\nu}\partial^\mu\partial^\nu(h_{\alpha\beta} + \partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha)\partial^\alpha\partial^\beta(h + 2\partial_\rho\xi^\rho) \tag{5.30}$$

Demanding that all terms with ξ disappear puts restrictions on the combination of terms. Within the collection of 45 terms there are 7 groups that are independently invariant under these transformation. These groups are written down in (A.2). Just like in the quadratic terms the relations can be further specified by demanding second order equations of motion. Varying these terms with respect to $h_{\sigma\kappa}$ leaves you with only second order terms, but varying with respect to $\partial^\xi\partial^\gamma h_{\theta\kappa}$ gives rise to second, third and fourth order terms:

$$\partial^\xi\partial^\gamma \frac{\delta(h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h)}{\delta(\partial^\xi\partial^\gamma h_{\theta\kappa})} = \partial^\mu\partial^\nu(h_{\mu\nu}\partial^\theta\partial^\kappa h) + \eta^{\theta\kappa}\partial^\alpha\partial^\beta(h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}) \tag{5.31}$$
$$= \partial^\mu\partial^\nu h_{\mu\nu}\partial^\theta\partial^\kappa h + \eta^{\theta\kappa}\partial^\alpha\partial^\beta h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}) \tag{5.32}$$
$$+h_{\mu\nu}\partial^\mu\partial^\nu\partial^\theta\partial^\kappa h + \eta^{\theta\kappa}h_{\mu\nu}\partial^\alpha\partial^\beta\partial^\mu\partial^\nu h_{\alpha\beta} \tag{5.33}$$
$$+2\partial^\mu h_{\mu\nu}\partial^\nu\partial^\theta\partial^\kappa h + 2\eta^{\theta\kappa}\partial^\alpha h_{\mu\nu}\partial^\beta\partial^\mu\partial^\nu h_{\alpha\beta}. \tag{5.34}$$

Demanding all terms with third or fourth order terms to disappear splits the terms in two parts, those that start with h and those that start with $h_{\mu\nu}$.

Finally, the first occurrence of h can still be transformed under the gauge transformation:

$$h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h \to 2\partial_\mu\xi_\nu\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h. \tag{5.35}$$

Naively, this is unique for each of the 45 terms, but when considering integration by parts you can write down this term as:

$$2\xi_\nu\Box\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h + 2\xi_\nu\partial^\mu\partial^\nu h_{\alpha\beta}\partial_\mu\partial^\alpha\partial^\beta h. \tag{5.36}$$

By again demanding all terms with ξ vanish the two leftover sections mix and the final unique combination of terms is:

$$\begin{aligned}
S = \int d^5x[&h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h && -2h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\rho h^\beta_\rho - 2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\partial_\alpha\partial_\beta h\\
&+ 2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\partial_\beta\partial_\rho h^\rho_\alpha && +2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\partial_\alpha\partial_\rho h^\rho_\beta + h_{\mu\nu}\partial^\mu\partial^\nu h^{\alpha\beta}\Box h^{\alpha\beta}\\
&- 2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\Box h^{\alpha\beta} && +h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\nu}\partial_\alpha\partial_\beta h - 2h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\nu}\partial_\alpha\partial_\rho h^\rho_\beta\\
&+ h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\nu}\Box h_{\alpha\beta} && -h_{\mu\nu}\partial^\mu\partial^\nu h\Box h + 2h_{\mu\nu}\partial^\mu\partial^\alpha h^\nu_\alpha\Box h\\
&+ h_{\mu\nu}\partial^\mu\partial^\nu h\partial^\alpha\partial^\beta h_{\alpha\beta} && -2h_{\mu\nu}\partial^\mu\partial^\alpha h^\nu_\alpha\partial_\beta\partial_\rho h^{\beta\rho} - 1h_{\mu\nu}\Box h^{\mu\nu}\Box h\\
&+ h_{\nu\nu}\Box h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} && +2h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\partial^\beta\partial^\rho h^\nu_\rho - 2h_{\mu\nu}\Box h^{\mu\alpha}\partial_\alpha\partial_\rho h^{\nu\rho}\\
&- 2h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\Box h^{\nu\beta} && +h_{\mu\nu}\Box h^{\mu\alpha}\Box h^\nu_\alpha + 2h_{\mu\nu}\partial^\mu\partial^\alpha h\Box h^\nu_\alpha\\
&+ h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\partial^\nu\partial^\rho h^\beta_\rho && +h_{\mu\nu}\partial^\alpha\partial^\beta h^\mu_\alpha\partial_\beta\partial_\rho h^{\rho\nu} - 2h_{\mu\nu}\partial^\mu\partial^\alpha h\partial_\alpha\partial^\beta h^\nu_\beta\\
&- 2h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\partial^\beta\partial^\nu h && +h_{\mu\nu}\partial^\mu\partial^\alpha h\partial^\nu\partial_\alpha h - h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial_\alpha\partial_\beta h^\rho_\nu\\
&+ 2h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial_\alpha\partial^\nu h_{\beta\rho} && +h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial_\alpha\partial_\rho h^\nu_\beta - 2h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial^\nu\partial_\rho h_{\alpha\beta}\\
&+ h_{\mu\nu}\partial^\mu\partial^\alpha h_{\beta\rho}\partial^\nu\partial^\rho h^\beta_\alpha && -h_{\mu\nu}\partial^\mu\partial^\alpha h_{\beta\rho}\partial^\nu\partial_\alpha h^{\rho\beta} + \frac{1}{2}h\Box h\Box h\\
&- h\Box h\partial_\mu\partial_\nu h^{\mu\nu} && +\frac{1}{2}h\partial_{\mu\nu}h^{\mu\nu}\partial\alpha\partial\beta h^{\alpha\beta} - \frac{1}{2}h\Box h_{\mu\nu}\Box h^{\mu\nu}\\
&+ 2h\Box h_{\mu\nu}\Box h^{\mu\nu} && -1h\Box h_{\mu\nu}\partial^\mu\partial^\nu h + 2h\partial_\mu\partial_\nu h\partial^\mu\partial^\rho h^\nu_\rho\\
&- 1h\partial_\mu\partial_\nu h^{\mu\rho}\partial^\nu\partial_\alpha h^\alpha_\rho && -1h\partial_\mu\partial_\nu h^{\mu\rho}\partial_\rho\partial^\alpha h^\nu_\alpha - \frac{1}{2}h\partial_\mu\partial_\nu h\partial^\mu\partial^\nu h\\
&+ \frac{1}{2}h\partial_\mu\partial_\nu h^{\alpha\beta}\partial^\mu\partial^\nu h_{\alpha\beta} && -h\partial_\mu\partial_\nu h^{\alpha\beta}\partial_\alpha\partial^\mu h^\nu_\beta + \frac{1}{2}h\partial_\mu\partial_\nu h^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu}]
\end{aligned}$$

This action is quite monstrous, but there is structure hidden in this. The Gauss Bonnet terms vanish in 4 dimensions because of a set of 5 anti symmetric indices [16], this structure should also be present in these terms, although it is not easily seen.

5.1.2 Top down

Quadratic

Just as with Einstein Hilbert we are now going to expand these higher order curvatures around flat space to see if they agree on the findings in the previous sections. When only considering terms up to 2 powers of h we only have to consider the curvature terms linear in h. These are:

$$
\begin{aligned}
R^{(1)}_{\mu\nu\rho\sigma} &= \frac{1}{2}(\partial_\rho\partial_\nu h_{\sigma\mu} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\sigma\nu} - \partial_\sigma\partial_\nu h_{\rho\mu}),\\
R^{(1)} &= \partial^\beta\partial^\mu h_{\mu\beta} - \Box h,\\
R^{(1)}_{\mu\nu} &= \frac{1}{2}(\partial^\beta\partial_\nu h_{\mu\beta} + \partial_\mu\partial^\beta h_{\beta\nu} - \partial_\mu\partial_\nu h - \Box h^{\mu\nu}).
\end{aligned}
$$

The quadratic contractions of these curvatures are up to linear order:

$$
\begin{aligned}
R^{(1)2} &= \partial^\beta\partial^\mu h_{\mu\beta}\partial^\alpha\partial^\nu h_{\alpha_\nu} + \Box h\Box h - 2\partial^\beta\partial^\mu h_{\mu\beta}\Box h,\\
R^{(1)}_{\mu\nu}R^{(1)\mu\nu} &= -\frac{1}{2}\partial_\alpha\partial^\nu h_{\mu\alpha}\partial^\beta\partial_\nu h_{\mu\beta} + \frac{1}{2}h\partial_\alpha\partial^\nu h_{\mu\alpha}\partial_\mu\partial^\beta h_{\beta\nu}\\
&\quad - \frac{1}{2}\partial_\alpha\partial^\nu h^{\mu\alpha}\partial_\mu\partial\nu h + \frac{1}{4}\Box h^{\mu\nu}\Box h_{\mu\nu} + \frac{1}{4}\Box h\Box h,\\
R^{(1)}_{\mu\nu\rho\sigma}R^{(1)\mu\nu\rho\sigma} &= \Box h_{\mu\nu}\Box h^{\mu\nu} + \partial_\rho\partial_\nu h_{\sigma\mu}\partial^\sigma\partial^\mu h^{\nu\rho} - 2\Box h_\sigma\mu\partial_\nu\partial^\mu h^{\sigma\nu}.
\end{aligned}
$$

Adding these quadratic curvature terms to each other with arbitrary parameters:

$$
\begin{aligned}
aR^{(1)2} + bR^{(1)\mu\nu}R^{(1)}_{\mu\nu} + cR^{(1)\mu\nu\rho\sigma}R^{(1)}_{\mu\nu\rho\sigma} &= (c + \frac{1}{4}b)\Box h^{\mu\nu}\Box h_{\mu\nu}\\
&\quad - (\frac{1}{2}b + 2c)\Box h_{\sigma\mu}\partial_\nu\partial^\mu h^{\sigma\nu} + (a + \frac{1}{2}b + c)\partial^\beta\partial^\mu h_{\mu\beta}\partial^\alpha\partial^\nu h_{\alpha\nu}\\
&\quad - (2a + \frac{1}{2}b)\partial^\beta\partial^\mu h_{\mu\beta}\Box h + (a + \frac{1}{4}b)\Box h\Box h.
\end{aligned}
$$

This equation is exactly the same as (5.22) if you make the following substitutions wherein the primed variables are those of (5.22):

$$b' = -c - \frac{1}{4}b \tag{5.37}$$

$$e' = a + \frac{1}{4}b \tag{5.38}$$

Because we still only want second order equations of motion, these terms should disappear. This only happens when a, b, c are exactly the Gauss Bonnet terms $1, -4, 1$. That the linearized form of the Gauss Bonnet terms break down to a structure invariant under linearized diffeomorphisms and no equations of motion higher than second order in derivatives is not a surprise, as they were constructed this way. However, it is not guaranteed that at a linear level there would be more possibilities, which we have shown do not exist. This means that at least up to quadratic order in the field and 4 derivatives we are in the same situation as with Einstein Hilbert in that the only possible terms to write down are the linearized forms of the quadratic curvature terms. More specifically they also have to be present with the exact Gauss Bonnet parameters.

Third order

Due to time constraints expanding Gauss Bonnet up to third order in the field is not done in this research. However, because by construct of the lovelock terms we expect them to end up at an action that is both invariant under the linear diffeomorphisms and do not give rise to equations of

motion higher than second order in the derivative. As we found from the bottom up method that there is one of such combinations third order in h we strongly expect the Gauss Bonnet terms, when expanded, to be exactly that combination of terms. A few starting points are given in (B).

Chapter 6

Conclusion

We started the research with constructing Lorentz covariant field operators. For a massive particle this was problem less, we showed that the field operator of a massive spin 1 particles transforms exactly like a vector under Lorentz transformations. The field operator of a massless particle however, picks up an extra term or 2 extra terms for spin 1 and spin 2 respectively. For an action constructed out of these operators it means that in the massive case contracting the field with another vector is enough to make a term Lorentz invariant. For the massless field operators the action has to be invariant under these extra terms on top of that condition, showing why you need a gauge invariant theory.

In chapter 3 we looked into what the consequences of this gauge invariance are for the action. For linear terms the only combination that could be written down is a total derivative. The four quadratic terms can only be written down in a single way if you want to preserve this invariance. When starting with the Einstein Hilbert action and expanding the metric around flat space the linear and quadratic terms are the exact same as the ones we found by demanding the gauge invariance. This means that up to quadratic order the only way to construct a valid action is linearized Einstein Hilbert.

The next chapter was about trying to couple this spin 2 particle to itself. As it would couple to all mass and energy which the particle would carry energy itself it should couple to itself. They naive way to do this results in a infinite series of corrections, but following the Deser trick we could get there in one correction. The trick was to start with a different but equivalent action that was basically a linearization of the Palatini formalism of general relativity. We showed you can calculate the energy momentum tensor from this action and make a coupling in the action such that we get the correct equations of motion while not changing the calculation to this tensor. Admittedly this takes a lot of inspiration from general relativity, but it is merely a shortcut to get to the correct answer. The result of the Deser trick was a action that when adding a total derivative and doing a field redefinition is the Einstein Hilbert action.

The last chapter dealt with the higher derivative extension of general relativity which is the Gauss Bonnet action. In this chapter it is shown that an action with 4 derivatives and quadratic in contrast to the Einstein Hilbert case gives rise to a non unique set of terms with 2 free parameters. It turned out that when only allowing equations of motions with up to second order derivatives the only solution is to let all these terms be zero. For the same check as with Einstein Hilbert the Gauss Bonnet terms were linearized around flat space. First with arbitrary parameters which gave rise to the exact same set of terms as we got by working from the bottom up.

By again forcing only up to second order derivatives in the equations of motion the only option is to let all terms be zero, which is exactly when the parameters are the Gauss Bonnet parameters. For the next set of terms we looked at 3 powers of the field with 4 derivatives. From bottom up this again gave a unique set of terms that were both invariant under the gauge transformation and did not give equations of motion with higher derivatives.

This research should not be seen as a way to discredit Einstein in his geometrical derivation of his theory. In fact, it shows strength and beauty for a theory to be derivable in multiple ways. The research touches a controversial point by calling the theory a graviton, as quantization general relativity is exactly the unsolved part of the theory. It is hoped that by looking at the theory from a different perspective might spark some ideas, but for now lets be glas there is still some answers to find for years to come.

6.1 Outlook

Obviously the research has a bit of an open end. For Gauss Bonnet only part of calculation was done and it is hard to say if the other part would compute. We would expect that the first check, linearizing Gauss Bonnet, would give the same set of terms as calculating the ones we calculated from the bottom up. For the Deser trick, a wise place to start would be to start with Palatini formalism for Gauss Bonnet. The Lovelock set of higher derivative curvature actions is the exact set of actions for which the Palatini formalism is equal to the formalism where the connection is taken a function of the metric [17]. A way to start would be to see if you can find a action equivalent to the one found in 5 by considering only 3rd order terms of this Palatini action. In the appendix are some formulas that can be used as starting point.

Appendix A

Calculations for finding the gauge invariant action

This appendix holds the calculations for finding the quadratic terms for the action with 4 derivatives.

A.1 Quadratic

$$\Box h^{\mu\nu}\Box h_{\mu\nu} \to \Box(h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu)\Box(h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu) \tag{A.1}$$
$$= \Box h^{\mu\nu}\Box h_{\mu\nu} + \Box h^{\mu\nu}\Box\partial_\mu\xi_\nu + \Box h^{\mu\nu}\Box\partial_\nu\xi_\mu \tag{A.2}$$
$$+\Box\partial^\mu\xi^\nu\Box h_{\mu\nu} + \Box\partial^\mu\xi^\nu\Box\partial_\mu\xi_\nu + \Box\partial^\mu\xi^\nu\Box\partial_\nu\xi_\mu \tag{A.3}$$
$$+\Box\partial^\nu\xi^\mu\Box h_{\mu\nu} + \Box\partial^\nu\xi^\mu\Box\partial_\mu\xi_\nu + \Box\partial^\nu\xi^\mu\Box\partial_\nu\xi_\mu \tag{A.4}$$
$$= \Box h^{\mu\nu}\Box h_{\mu\nu} + 4\Box h^{\mu\nu}\Box\partial_\mu\xi_\nu + 2\Box\partial^\nu\xi^\mu\Box\partial_\mu\xi_\nu + 2\Box\partial^\nu\xi^\mu\Box\partial_\nu\xi_\mu \tag{A.5}$$

$$\Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} \to \Box(h^{\mu\alpha} + \partial^\mu\xi^\alpha + \partial^\alpha\xi^\mu)\partial_\alpha\partial^\beta(h_{\mu\beta} + \partial_\mu\xi_\beta + \partial_\beta\xi_\mu) \tag{A.6}$$
$$= \Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} + \Box h^{\mu\alpha}\partial_\alpha\partial^\beta\partial_\mu\xi_\beta + \Box h^{\mu\alpha}\partial_\alpha\partial^\beta\partial_\beta\xi_\mu \tag{A.7}$$
$$+\Box\partial^\mu\xi^\alpha\partial_\alpha\partial^\beta h_{\mu\beta} + \Box\partial^\mu\xi^\alpha\partial_\alpha\partial^\beta\partial_\mu\xi_\beta + \Box\partial^\mu\xi^\alpha\partial_\alpha\partial^\beta\partial_\beta\xi_\mu \tag{A.8}$$
$$+\Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta h_{\mu\beta} + \Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta\partial_\mu\xi_\beta + \Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta\partial_\beta\xi_\mu \tag{A.9}$$
$$= \Box h^{\mu\alpha}\partial_\alpha\partial^\beta h_{\mu\beta} + 2\Box\partial^\mu\xi^\alpha\partial_\alpha\partial^\beta h_{\mu\beta} + 2\Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta h_{\mu\beta} \tag{A.10}$$
$$+\Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta\partial_\beta\xi_\mu + 3\Box\partial^\alpha\xi^\mu\partial_\alpha\partial^\beta\partial_\mu\xi_\beta \tag{A.11}$$

$$\partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} \to \partial_\mu\partial_\alpha(h^{\mu\alpha} + \partial^\mu\xi^\alpha + \partial^\alpha\xi^\mu)\partial^\nu\partial^\beta(h_{\nu\beta} + \partial_\nu\xi_\beta + \partial_\beta\xi_\nu) \tag{A.12}$$
$$= \partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} + \partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta\partial_\nu\xi_\beta + \partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta\partial_\beta\xi_\nu \tag{A.13}$$
$$+\partial_\mu\partial_\alpha\partial^\mu\xi^\alpha\partial^\nu\partial^\beta h_{\nu\beta} + \partial_\mu\partial_\alpha\partial^\mu\xi^\alpha\partial^\nu\partial^\beta\partial_\nu\xi_\beta + \partial_\mu\partial_\alpha\partial^\mu\xi^\alpha\partial^\nu\partial^\beta\partial_\beta\xi_\nu \tag{A.14}$$
$$+\partial_\mu\partial_\alpha\partial^\alpha\xi^\mu\partial^\nu\partial^\beta h_{\nu\beta} + \partial_\mu\partial_\alpha\partial^\alpha\xi^\mu\partial^\nu\partial^\beta\partial_\nu\xi_\beta + \partial_\mu\partial_\alpha\partial^\alpha\xi^\mu\partial^\nu\partial^\beta\partial_\beta\xi_\nu \tag{A.15}$$
$$= \partial_\mu\partial_\alpha h^{\mu\alpha}\partial^\nu\partial^\beta h_{\nu\beta} + 4\partial_\mu\partial_\alpha h^{\mu\alpha}\Box\partial^\beta\xi_\beta + 4\Box\partial_\alpha\xi^\alpha\Box\partial^\beta\xi_\beta \tag{A.16}$$

$$\partial_\mu\partial_\nu h^{\mu\nu}\Box h \to \partial_\mu\partial_\nu(h^{\mu\nu}+\partial^\mu\xi^\nu+\partial^\nu\xi^\mu)\Box(h+2\partial_\alpha\xi^\alpha) \tag{A.17}$$
$$= \partial_\mu\partial_\nu h^{\mu\nu}\Box h + \partial_\mu\partial_\nu h^{\mu\nu}\Box 2\partial_\alpha\xi^\alpha \tag{A.18}$$
$$+\partial_\mu\partial_\nu\partial^\mu\xi^\nu\Box h + \partial_\mu\partial_\nu\partial^\mu\xi^\nu\Box 2\partial_\alpha\xi^\alpha \tag{A.19}$$
$$+\partial_\mu\partial_\nu\partial^\nu\xi^\mu\Box h + \partial_\mu\partial_\nu\partial^\nu\xi^\mu\Box 2\partial_\alpha\xi^\alpha \tag{A.20}$$
$$= \partial_\mu\partial_\nu h^{\mu\nu}\Box h + 2\partial_\mu\partial\nu h^{\mu\nu}\Box\partial_\alpha\xi^\alpha + 2\Box\partial_\mu\xi^\mu\Box h + 4\Box\partial_\mu\xi^\mu\Box\partial_\nu\xi^\nu \tag{A.21}$$

$$\Box h\Box h \to \Box(h+2\partial_\mu\xi^\mu)\Box(h+2\partial_\nu\xi^\nu) \tag{A.22}$$
$$= \Box h\Box h + \Box h\Box 2\partial_\nu\xi^\nu \tag{A.23}$$
$$+\Box 2\partial_\mu\xi^\mu\Box h + \Box 2\partial_\mu\xi^\mu\Box 2\partial_\nu\xi^\nu \tag{A.24}$$
$$= \Box h\Box h + 4\Box h\partial_\nu\xi^\nu + 4\Box\partial_\mu\xi^\mu\Box\partial_\nu\xi^\nu \tag{A.25}$$

A.2 Third order

Written below are all 45 independent terms of the form $h\partial\partial h\partial\partial h$ divided into groups that are invariant under the gauge transformation on the last two occurences of h.

$$A(h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\beta h - 2h_{\mu\nu}\partial^\mu\partial^\nu h_{\alpha\beta}\partial^\alpha\partial^\rho h^\beta_\rho - 2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\partial_\alpha\partial_\beta h \tag{A.26}$$
$$+2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\partial_\beta\partial_\rho h^\rho_\alpha + 2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\partial_\alpha\partial_\rho h^\rho_\beta + h_{\mu\nu}\partial^\mu\partial^\nu h^{\alpha\beta}\Box h^{\alpha\beta} \tag{A.27}$$
$$-2h_{\mu\nu}\partial^\mu\partial^\alpha h^{\nu\beta}\Box h^{\alpha\beta} + h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\nu}\partial_\alpha\partial_\beta h - 2h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\nu}\partial_\alpha\partial_\rho h^\rho_\beta \tag{A.28}$$
$$+h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\nu}\Box h_{\alpha\beta}) \tag{A.29}$$

$$B(h_{\mu\nu}\partial^\mu\partial^\nu h\Box h - 2h_{\mu\nu}\partial^\mu\partial^\alpha h^\nu_\alpha\Box h - h_{\mu\nu}\partial^\mu\partial^\nu h\partial^\alpha\partial^\beta h_{\alpha\beta} \tag{A.30}$$
$$+2h_{\mu\nu}\partial^\mu\partial^\alpha h^\nu_\alpha\partial_\beta\partial_\rho h^{\beta\rho} + 1h_{\mu\nu}\Box h^{\mu\nu}\Box h - h_{\nu\nu}\Box h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}) \tag{A.31}$$

$$C(2h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\partial^\beta\partial^\rho h^\nu_\rho - 2h_{\mu\nu}\Box h^{\mu\alpha}\partial_\alpha\partial_\rho h^{\nu\rho} - 2h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\Box h^{\nu\beta} \tag{A.32}$$
$$+h_{\mu\nu}\Box h^{\mu\alpha}\Box h^\nu_\alpha + 2h_{\mu\nu}\partial^\mu\partial^\alpha h\Box h^\nu_\alpha + h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\partial^\nu\partial^\rho h^\beta_\rho \tag{A.33}$$
$$+h_{\mu\nu}\partial^\alpha\partial^\beta h^\mu_\alpha\partial_\beta\partial_\rho h^{\rho\nu} - 2h_{\mu\nu}\partial^\mu\partial^\alpha h\partial_\alpha\partial^\beta h^\nu_\beta - 2h_{\mu\nu}\partial^\mu\partial^\alpha h_{\alpha\beta}\partial^\beta\partial^\nu h \tag{A.34}$$
$$+h_{\mu\nu}\partial^\mu\partial^\alpha h\partial^\nu\partial_\alpha h \tag{A.35}$$

$$D(h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial_\alpha\partial_\beta h^\rho_\nu - 2h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial_\alpha\partial^\nu h_{\beta\rho} - h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial_\alpha\partial_\rho h^\nu_\beta \tag{A.36}$$
$$+2h_{\mu\nu}\partial^\alpha\partial^\beta h^{\mu\rho}\partial^\nu\partial_\rho h_{\alpha\beta} - h_{\mu\nu}\partial^\mu\partial^\alpha h_{\beta\rho}\partial^\nu\partial^\rho h^\beta_\alpha + h_{\mu\nu}\partial^\mu\partial^\alpha h_{\beta\rho}\partial^\nu\partial_\alpha h^{\rho\beta}) \tag{A.37}$$

$$E(h\Box h\Box h - 2h\Box h\partial_\mu\partial_\nu h^{\mu\nu} + h\partial_{\mu\nu}h^{\mu\nu}\partial\alpha\partial\beta h^{\alpha\beta}) \tag{A.38}$$

$$F(h\Box h_{\mu\nu}\Box h^{\mu\nu} - 4h\Box h_{\mu\nu}\Box h^{\mu\nu} + 2h\Box h_{\mu\nu}\partial^{\mu}\partial^{\nu}h \tag{A.39}$$

$$-4h\partial_{\mu}\partial_{\nu}h\partial^{\mu}\partial^{\rho}h^{\nu}_{\rho} + 2h\partial_{\mu}\partial_{\nu}h^{\mu\rho}\partial^{\nu}\partial_{\alpha}h^{\alpha}_{\rho} + 2h\partial_{\mu}\partial_{\nu}h^{\mu\rho}\partial_{\rho}\partial^{\alpha}h^{\nu}_{\alpha} \tag{A.40}$$

$$+h\partial_{\mu}\partial_{\nu}h\partial^{\mu}\partial^{\nu}h) \tag{A.41}$$

$$G(h\partial_{\mu}\partial_{\nu}h^{\alpha\beta}\partial^{\mu}\partial^{\nu}h_{\alpha\beta} - 2h\partial_{\mu}\partial_{\nu}h^{\alpha\beta}\partial_{\alpha}\partial^{\mu}h^{\nu}_{\beta} + h\partial_{\mu}\partial_{\nu}h^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h^{\mu\nu}) \tag{A.42}$$

Demanding no equations of motion with higher than second order derivatives groups these 7 groups into 2 big groups with the following relation between the constants:

$$D = -A \tag{A.43}$$

$$C = A \tag{A.44}$$

$$B = -A \tag{A.45}$$

$$G = -F \tag{A.46}$$

$$E = -F \tag{A.47}$$

If now the gauge transformation invariance is done on the first occurence of h, the final relation is:

$$F = -\frac{1}{2}A \tag{A.48}$$

Appendix B

Formula's usable for future research

To help someone picking up at the open end of this research, here are some formula's that might be useful:

B.0.1 Third Power

This is the action that we want to expand around, if only interested in terms that are third order in the field h, we need to consider 6 terms:

$$S = \int d^5x\sqrt{-g}[R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}].$$

$$\begin{aligned} S^{(3)} = &\int d^5x[R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}]^{(3)} \\ &+\frac{1}{2}h[R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}]^{(2)}. \end{aligned}$$

$$\begin{aligned} R^{(2)} =&\frac{1}{2}[-3h^{\mu\nu}\partial_\rho\partial_\mu h^\rho_\nu + 2h^{\rho\gamma}\partial_\rho\partial_\gamma h + 2h^{\mu\nu}\Box h_{\mu\nu} - h^{\mu\nu}\partial_\nu\partial^\rho h_{\mu\rho} \\ &+2\partial^\mu h\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial^\gamma h\partial_\gamma h - 2\partial_\rho h^{\rho\gamma}\partial^\mu h_{\mu\gamma} \\ &+\frac{1}{2}\partial^\mu h^{\rho\gamma}\partial_\mu h_{\rho\gamma} - \partial^\gamma h^{\rho\mu}\partial_\rho h_{\gamma\mu}] \end{aligned}$$

$$\begin{aligned} R^{(2)}_{\alpha\beta} =& -\frac{1}{2}[\partial_\rho h^{\rho\gamma}\partial_\alpha h_{\beta\gamma} + \partial_\rho h^{\rho\gamma}\partial_\beta h_{\alpha\gamma} - \partial_\rho h^{\rho\gamma}\partial_\gamma h_{\alpha\beta} \\ &+h^{\rho\gamma}\partial_\rho\partial_\alpha h_{\beta\gamma} + h^{\rho\gamma}\partial_\rho\partial_\beta h_{\alpha\gamma} - h^{\rho\gamma}\partial_\rho\partial_\gamma h_{\alpha\beta} - h^{\rho\gamma}\partial_\rho\partial_\beta h_{\rho\gamma} \\ &-\frac{1}{2}\partial h^{\rho\gamma}\partial_\beta h^{\rho\gamma} - \frac{1}{2}\partial^\gamma h\partial_\alpha h_{\gamma\beta} - \frac{1}{2}\partial^\gamma h\partial_\beta h_{\gamma\alpha} \\ &+\frac{1}{2}\partial_\gamma h\partial^\gamma h_{\alpha\beta} + \partial^\gamma h_{\rho\beta}\partial^\rho h_{\gamma\alpha} - \partial^\gamma h_{\rho\beta}\partial_\gamma h_{\rho\alpha}] \end{aligned}$$

$$\begin{aligned} \Gamma^\gamma_{\nu\sigma}\Gamma^\mu_{\gamma\rho} = &\frac{1}{4}\partial_\nu h^\gamma_\sigma\partial_\gamma h^\mu_\rho + \frac{1}{4}\partial_\nu h^\gamma_\sigma\partial_\rho h^\mu_\gamma - \frac{1}{4}\partial_\nu h^\gamma_\sigma\partial^\mu h_{\gamma\rho} \\ &+\frac{1}{4}\partial_\sigma h^\gamma_\nu\partial_\gamma h^\mu_\rho + \frac{1}{4}\partial_\sigma h^\gamma_\nu\partial_\rho h^\mu_\gamma - \frac{1}{4}\partial_\sigma h^\gamma_\nu\partial^\mu h_{\gamma\rho} \\ &-\frac{1}{4}\partial^\gamma h_{\nu\sigma}\partial_\gamma h^\mu_\rho - \frac{1}{4}\partial^\gamma h_{\nu\sigma}\partial_\rho h^\mu_\gamma + \frac{1}{4}\partial^\gamma h_{\nu\sigma}\partial^\mu h_{\gamma\rho} \end{aligned}$$

$$
\begin{aligned}
(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})^{(3)} =& R^{\mu\nu\rho\sigma(1)}R^{(2)}_{\mu\nu\rho\sigma} + R^{\mu\nu\rho\sigma(2)}R^{(1)}_{\mu\nu\rho\sigma} \\
=& R^{\mu\nu\rho\sigma(1)}(\eta_{\alpha\mu}R^{\alpha(2)}_{\nu\rho\sigma} + h_{\alpha\mu}R^{\alpha(1)}_{\nu\rho\sigma}) \\
&+(\eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\xi\sigma}R^{\mu(2)}_{\alpha\beta\xi} - h^{\alpha\nu}\eta^{\beta\rho}\eta^{\xi\sigma}R^{\mu(1)}_{\alpha\beta\xi} \\
&-\eta^{\alpha\nu}h^{\beta\rho}\eta^{\xi\sigma}R^{\mu(1)}_{\alpha\beta\xi} - \eta^{\alpha\nu}\eta^{\beta\rho}h^{\xi\sigma}R^{\mu(1)}_{\alpha\beta\xi})R^{(1)}_{\mu\nu\rho\sigma} \\
=& (2\eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\xi\sigma}R^{\mu(2)}_{\alpha\beta\xi} - h^{\alpha\nu}\eta^{\beta\rho}\eta^{\xi\sigma}R^{\mu(1)}_{\alpha\beta\xi} \\
&-\eta^{\alpha\nu}h^{\beta\rho}\eta^{\xi\sigma}R^{\mu(1)}_{\alpha\beta\xi} - \eta^{\alpha\nu}\eta^{\beta\rho}h^{\xi\sigma}R^{\mu(1)}_{\alpha\beta\xi} + \eta^{\nu\xi}\eta^{\rho\beta}\eta^{\sigma\gamma}h^{\mu}_{\alpha}R^{\alpha(1)}_{\xi\beta\gamma})R^{(1)}_{\mu\nu\rho\sigma}
\end{aligned}
$$

$$
\begin{aligned}
(RR)^{(3)} = 2R^{(1)}R^{(2)} =& [-3h^{\mu\nu}\partial_\rho\partial_\mu h^\rho_\nu + 2h^{\rho\gamma}\partial_\rho\partial_\gamma h + 2h^{\mu\nu}\Box h_{\mu\nu} - h^{\mu\nu}\partial_\nu\partial^\rho h_{\mu\rho} \\
&+2\partial^\mu h\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial^\gamma h\partial_\gamma h - 2\partial_\rho h^{\rho\gamma}\partial^\mu h_{\mu\gamma} \\
&+\frac{1}{2}\partial^\mu h^{\rho\gamma}\partial_\mu h_{\rho\gamma} - \partial^\gamma h^{\rho\mu}\partial_\rho h_{\gamma\mu}][\partial_\rho\partial_\mu h^{\rho\mu} - \Box h]
\end{aligned}
$$

$$
R^{(1)}_{\mu\nu} = \frac{1}{2}[\partial^\rho\partial_\mu h_\rho\nu + \partial^\rho\partial_\nu h_{\rho\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}]
$$

$$
\begin{aligned}
(R^{\mu\nu}R_{\mu\nu})^{(3)} =& \eta^{\alpha\mu}\eta^{\beta\nu}R^{(2)}_{\mu\nu}R^{(1)}_{\alpha\beta} + R^{(1)}_{\mu\nu}(\eta^{\alpha\mu}\eta^{\beta\nu}R^{(2)}_{\alpha\beta} - h^{\alpha\mu}\eta^{\beta\nu}R^{(1)}_{\alpha\beta} - \eta^{\alpha\mu}h^{\beta\nu}R^{(1)}_{\alpha\beta}) \\
=& R^{(1)}_{\mu\nu}[2\eta^{\alpha\mu}\eta^{\beta\nu}R^{(2)}_{\alpha\beta} - h^{\alpha\mu}\eta^{\beta\nu}R^{(1)}_{\alpha\beta} - \eta^{\alpha\mu}h^{\beta\nu}R^{(1)}_{\alpha\beta}] \\
=& R^{(1)}_{\mu\nu}[2\eta^{\alpha(\mu}\eta^{\nu)\beta}R^{(2)}_{\alpha\beta} - 2h^{\alpha(\mu}\eta^{\nu)\beta}R^{(1)}_{\alpha\beta}]
\end{aligned}
$$

As $R^{(2)}_{\mu\nu}$ is symmetric

$$
-2h^{\alpha\mu}\eta^{\beta\nu}R^{(1)}_{\alpha\beta} = -h^{\alpha\mu}\partial_\rho\partial_\alpha h^{\rho\nu} - h^{\alpha\mu}\partial^\rho\partial^\nu h_{\rho\alpha} + h^{\alpha\mu}\partial_\alpha\partial^\nu h + h^{\alpha\mu}\Box h^\nu_\alpha
$$

$$
\begin{aligned}
2\eta^{\mu\alpha}\eta^{\nu\beta}R_{\alpha\beta} =& -\frac{1}{2}[2\partial_\rho h^{\rho\gamma}\partial^{(\mu}h^{\nu)}_\gamma - \partial_\rho h^{\rho\gamma}\partial_\gamma h^{\mu\nu} + 2h^{\rho\gamma}\partial_\rho\partial^{(\mu}h^{\nu)}_\gamma \\
&-h^{\rho\gamma}\partial_\gamma\rho h^{\mu\nu} + h^{\rho\gamma}\partial^\mu\partial^\nu h_{\rho\gamma}h_{\rho\gamma} + \frac{1}{2}^{\mu}h^{\rho\gamma}\partial^\nu h_{\rho\gamma} \\
&+\partial_\gamma h\partial^{(\mu}h^{\nu)\gamma} - \frac{1}{2}\partial_\gamma h\partial^\gamma h^{\mu\nu} - \partial_\gamma h^{\rho\nu}\partial_\rho h^{\gamma\mu} \\
&+\partial_\gamma h^{\rho\nu}\partial^\gamma h^\mu_\rho]
\end{aligned}
$$

B.0.2 Bootstrapping Gauss Bonnet

An idea to extend the Deser trick to Gauss Bonnet is to write down the Palatini form of the Gauss Bonnet terms and keep the terms that contain 3 fields as with Einstein Hilbert the terms with 2 fields were kept.

$$
\begin{aligned}
R^2 =& \quad g^{\mu\nu}R_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta} && \text{(B.1)} \\
R_{\mu\nu}R^{\mu\nu} =& \quad R_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} && \text{(B.2)} \\
R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} =& \quad g_{\lambda\mu}R^\lambda_{\nu\alpha\beta}g^{\xi\nu}g^{\theta\alpha}g^{\delta\beta}R^\mu_{\xi\theta\delta} && \text{(B.3)}
\end{aligned}
$$

'linearizing' these up to 3 fields:

$$R^2 \rightarrow \quad 2f^{\mu\nu}\eta^{\alpha\beta}(\partial_\rho\Gamma^\rho_{\mu\nu}\partial_\xi\Gamma^\xi_{\alpha\beta} - \partial_\rho\Gamma^\rho_{\mu\nu}\partial_\beta\Gamma^\xi_{\alpha\xi} - \partial_\nu\Gamma^\rho_{\mu\rho}\partial_\xi\Gamma^\xi_{\alpha\beta} + \partial_\nu\Gamma^\rho_{\mu\rho}\partial_\beta\Gamma^\xi_{\alpha\xi}) \tag{B.4}$$

$$+ \quad 2\eta^{\mu\nu}\eta^{\alpha\beta}(\partial_\rho\Gamma^\rho_{\mu\nu}\Gamma^\xi_{\theta\xi}\Gamma^\theta_{\alpha\beta} - \partial_\rho\Gamma^\rho_{\mu\nu}\Gamma^\xi_{\theta\beta}\Gamma^\theta_{\alpha\xi} - \partial_\nu\Gamma^\rho_{\mu\rho}\Gamma^\xi_{\theta\xi}\Gamma^\theta_{\alpha\beta} + \partial_\nu\Gamma^\rho_{\mu\rho}\Gamma^\xi_{\theta\beta}\Gamma^\theta_{\alpha\xi}) \tag{B.5}$$

$$R_{\mu\nu}R^{\mu\nu} \rightarrow \quad (f^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\mu}f^{\beta\nu})(\partial_\rho\Gamma^\rho_{\mu\nu}\partial_\xi\Gamma^\xi_{\alpha\beta} - \partial_\rho\Gamma^\rho_{\mu\nu}\partial_\beta\Gamma^\xi_{\alpha\xi} - \partial_\nu\Gamma^\rho_{\mu\rho}\partial_\xi\Gamma^\xi_{\alpha\beta} + \partial_\nu\Gamma^\rho_{\mu\rho}\partial_\beta\Gamma^\xi_{\alpha\xi}) \tag{B.6}$$

$$+ \quad \eta^{\alpha\mu}\eta^{\beta\nu}(\partial_\rho\Gamma^\rho_{\mu\nu}\Gamma^\xi_{\theta\xi}\Gamma^\theta_{\alpha\beta} - \partial_\rho\Gamma^\rho_{\mu\nu}\Gamma^\xi_{\theta\beta}\Gamma^\theta_{\alpha\xi} - \partial_\nu\Gamma^\rho_{\mu\rho}\Gamma^\xi_{\theta\xi}\Gamma^\theta_{\alpha\beta} + \partial_\nu\Gamma^\rho_{\mu\rho}\Gamma^\xi_{\theta\beta}\Gamma^\theta_{\alpha\xi} \tag{B.7}$$

$$+ \quad \partial_\rho\Gamma^\rho_{\alpha\beta}\Gamma^\xi_{\theta\xi}\Gamma^\theta_{\mu\nu} - \partial_\rho\Gamma^\rho_{\alpha\beta}\Gamma^\xi_{\theta\nu}\Gamma^\theta_{\mu\xi} - \partial_\beta\Gamma^\rho_{\alpha\rho}\Gamma^\xi_{\theta\xi}\Gamma^\theta_{\mu\nu} + \partial_\beta\Gamma^\rho_{\alpha\rho}\Gamma^\xi_{\theta\nu}\Gamma^\theta_{\mu\xi}) \tag{B.8}$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \rightarrow \quad (-f_{\lambda\mu}\eta^{\xi\nu}\eta^{\theta\alpha}\eta^{\delta\beta} + \eta_{\lambda\mu}f^{\xi\nu}\eta^{\theta\alpha}\eta^{\delta\beta} + \eta_{\lambda\mu}\eta^{\xi\nu}f^{\theta\alpha}\eta^{\delta\beta} + \eta_{\lambda\mu}\eta^{\xi\nu}\eta^{\theta\alpha}f^{\delta\beta}) \tag{B.9}$$

$$(\quad \partial_\alpha\Gamma^\lambda_{\nu\beta}\partial_\theta\Gamma^\mu_{\xi\delta} - \partial_\alpha\Gamma^\lambda_{\nu\beta}\partial_\delta\Gamma^\mu_{\xi\theta} - \partial_\beta\Gamma^\lambda_{\nu\alpha}\partial_\theta\Gamma^\mu_{\xi\delta} + \partial_\beta\Gamma^\lambda_{\nu\alpha}\partial_\delta\Gamma^\mu_{\xi\theta}) \tag{B.10}$$

$$+ \quad \eta_{\lambda\mu}\eta^{\xi\nu}\eta^{\theta\alpha}\eta^{\delta\beta}(\partial_\alpha\Gamma^\lambda_{\nu\beta}\Gamma^\mu_{\gamma\theta}\Gamma^\gamma_{\xi\delta} - \partial_\alpha\Gamma^\lambda_{\nu\beta}\Gamma^\mu_{\gamma\delta}\Gamma^\gamma_{\xi\theta} \tag{B.11}$$

$$- \quad \partial_\beta\Gamma^\lambda_{\nu\alpha}\Gamma^\mu_{\gamma\theta}\Gamma^\gamma_{\xi\delta} + \partial_\beta\Gamma^\lambda_{\nu\alpha}\Gamma^\mu_{\gamma\delta}\Gamma^\gamma_{\xi\theta} \tag{B.12}$$

$$+ \quad \partial_\theta\Gamma^\mu_{\xi\delta}\Gamma^\lambda_{\epsilon\alpha}\Gamma^\epsilon_{\nu\beta} - \partial_\theta\Gamma^\mu_{\xi\delta}\Gamma^\lambda_{\epsilon\beta}\Gamma^\epsilon_{\nu\alpha} \tag{B.13}$$

$$- \quad \partial_\delta\Gamma^\mu_{\xi\theta}\Gamma^\lambda_{\epsilon\alpha}\Gamma^\epsilon_{\nu\beta} + \partial_\delta\Gamma^\mu_{\xi\theta}\Gamma^\lambda_{\epsilon\beta}\Gamma^\epsilon_{\nu\alpha} \tag{B.14}$$

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