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# The Ricci flow on two-dimensional almost-Riemannian manifolds 

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#### Abstract

The Ricci flow has shown to be a powerful tool in the study of Riemannian geometry. In this setting, we provide explicit proofs for the isometric invariance of the Levi-Civita connection, Riemann curvature tensor and, subsequently, the Ricci curvature. The latter allows one to show that the Ricci flow is invariant under the infinite dimensional group of diffeomorphisms. This causes the Ricci flow, characterized as a heat-type non-linear partial differential equation, to be weakly parabolic. In general, existence and uniqueness theorems only apply to strongly parabolic equations. With DeTurck's trick, we show that on a closed Riemannian manifold of any dimension, existence and uniqueness of short-time solutions to the Ricci flow can still be obtained. In addition, we investigate the evolution of the Ricci flow on two-dimensional almost-Riemannian structures (2-ARS) on compact, oriented and connected smooth manifolds. These are generalized Riemannian structures on surfaces for which an orthonormal frame is obtained from a pair of vector fields that satisfy the Hörmander condition. The vector fields can become collinear at certain points, which as a collection define a singular set. If one removes the singular set from the connected manifold, one obtains two regular Riemannian structures on both parts of the surface. However, these are non-complete Riemannian surfaces with boundary, giving rise to difficulties regarding the evolution of Ricci flow on these surfaces.


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## 1 Introduction

The Ricci flow was introduced by Richard Hamilton in his 1982 paper Three-manifolds with positive Ricci curvature [1]. With the help of essential observations from Shing-Tung Yao, it was soon conjectured as being the key to proving Thurston's geometrization conjecture [2], which has the Poincaré conjecture [3] as a corollary. The former states that every closed three-dimensional manifold can be decomposed in such a way, that each remaining component admits exactly one of eight types of geometries. The Poincaré conjecture claims that every simply connected, closed three-dimensional manifold is homeomorphic to the 3 -sphere. Both conjectures belonged among the most important open questions in mathematics. Hamilton proceeded in publishing numerous papers in these directions, but it was the Russian mathematician Grigori Perelman who was eventually able to prove the geometrization conjecture in his series of papers $[4,5,6]$. Although he gained more public attention for declining multiple prizes than for the actual proof, the Ricci flow has since been a thoroughly investigated tool.

Starting with an arbitrary smooth Riemannian manifold, the Ricci flow is a geometric evolution equation in which one allows the metric to evolve along the vector field of -2 times the Ricci curvature. By changing the distances between any two points, one also alters the angles between any two points (except in two-dimensions, when the deformation is conformal) and the volume of the manifold. This happens in a way that makes the manifold more symmetric, and hence smoothens a manifold's geometry. The Ricci curvature can be regarded as a Laplacian of the metric, which causes the Ricci flow to have a strong resemblance to the usual heat equation. However, it fails to be parabolic due to the isometric invariance of the Ricci curvature. Nevertheless, without the help of any a priori curvature bounds, we can still guarantee short-time existence and uniqueness of solutions to the flow.

In [1], Hamilton proves existence and uniqueness for closed manifolds, i.e. compact without boundary, by using the Nash-Moser inverse function theorem. DeTurck simplified the proof by showing that the Ricci flow is equivalent to a quasilinear parabolic initial-value problem in [7]. Multiple works provide similar results for manifolds with distinct topological properties. On non-compact manifolds, existence was shown by Shi in [8] and uniqueness by Chen and Zhu in [9]. On manifolds with boundary and an arbitrary initial metric without any curvature constraints or prescriptions, existence and uniqueness results were established very recently by Chow in [10]. Furthermore, Topping and Giesen [11] obtained existence and uniqueness results for a non-complete initial manifold. In the generalized context of subRiemannian geometry, little research has been done on the evolution of Ricci flow. The first work in this direction is from Lovrić, Min-Oo and Ruh [12], who proved existence to Ricci flow on Riemannian foliations on compact Riemannian manifolds, which can be considered as certain sub-Riemannian manifolds. Due to recent developments on sub-Riemannian manifolds of e.g. Dong [13], who provided contributions to the closely related harmonic map heat flow, Baudoin and Garofalo [14], who generalized curvature dimension inequalities, and Agrachev and Lee [15], who studied sub-Laplacian comparison theorems, more existence results for Ricci flow on such manifolds are to be expected.

Our aim is to investigate the possibilities of evolving Ricci flow on two-dimensional almost-Riemannian manifolds. These are smooth connected two-dimensional manifolds endowed with an almost-Riemannian structure, firstly introduced by Grushin [16] in the context of hypoelliptic operators. They are the prototypes of rank-varying sub-Riemannian structures. More specifically, they are sub-Riemannian structures that can be locally defined by a set of smooth vector fields that satisfy the Hörmander condition and of which the cardinality equals the dimension of the manifold. Almost-Riemannian structures are primarily studied in dimension two, and the first results on its general properties were accomplished by Agrachev, Boscain and Sigalotti in [17]. With a generic two-dimensional almost-Riemannian structure there are only three types of points. The most common points are regular Riemannian points, on which we would like to evolve the Ricci flow. However, the set of Riemannian points is a non-complete surface with boundary. There have not yet been any results regarding Ricci flow on surfaces with these specific topological properties.

The thesis is structured as follows: Chapter 2 focusses entirely on differential geometry, since both the Ricci flow and almost-Riemannian geometry belong to this branch of mathematics. In an attempt to write a comprehensive piece, we treat everything that is of significant importance for one's understanding of the topic with great detail. In particular, we pay close attention to the isometric invariance of the Levi-Civita connection, Riemann curvature tensor, and, subsequently, the Ricci curvature. Chapter 3
is devoted to the Ricci flow. After having discussed time derivatives and deformations of geometric quantities, we obtain equations for the evolution of the Riemann curvature tensor and Ricci curvature under Ricci flow. Here, the heat-type nature of the Ricci flow becomes apparent. In the remainder of the chapter, we devote ourselves to proving short-time existence and uniqueness. We remark on the weakly parabolicity of Ricci flow, which is due to the isometric invariance of the Ricci curvature. The eventual proof is in analogy with DeTurck's proof from [7], but can be considered as a detailed version of the proof described in [18]. Chapter 4 starts with a brief introduction to sub-Riemannian structures, after which we quickly focus on (two-dimensional) almost-Riemannian structures. Having established the topology on the Riemannian points of generic almost-Riemannian manifolds, we investigate whether we can use existing results on the Ricci flow for our specific case.

We remark that we make use of the Einstein summation convention throughout the thesis. That is, if an index appears once in a lower, and once in an upper index position of a certain term, then it is understood to be summed over all possible index values. Furthermore, the reader is assumed to be acquainted with preliminary notions on topological spaces as from for example [19].

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## 2 Differential geometry

The mathematical foundation of Ricci flow and almost-Riemannian geometry is that of differential geometry. On that account, we devote this first chapter exclusively to the tools needed to define Ricci flow.

We are particularly interested in the non-Euclidean differential geometry of Riemannian manifolds. After a brief overview of the most elementary notions regarding differentiable manifolds, we will continue with providing the tools that allow us to construct Riemannian manifolds: smooth manifolds equipped with a metric. This process starts with examining tangent vectors, covectors, tensors and some of their useful properties. We will see that the Riemannian metric is in fact a tensor and enables one to define multiple (intrinsic) geometric properties of Riemannian manifolds, such as curvature. There are various kinds of curvature tensors, each with a slightly distinct geometric meaning. To define curvature tensors, however, one needs a method for taking derivatives of vector fields, which can be achieved by means of a connection. The unique connection on the tangent bundle of a Riemannian manifold, called the LeviCivita connection, is the last necessary tool to define the Riemann curvature tensor. The Ricci curvature tensor can then easily be obtained, after which the definition of Ricci flow lays at our hands.

The chapter mainly relies upon the educational textbooks on Riemannian Geometry of Lee [20, 21] and the lecture notes on analysis on manifolds of Seri [22]. To lesser extent, results from [23] and [24] are used as well.

### 2.1 Smooth manifolds

The purpose of this section is to briefly recall or familiarize the reader with smooth manifolds and related notions in differential geometry. It summarises the most relevant definitions and results for our later purpose. For more detail on smooth manifolds, we refer to [22, 25, 26]. The main idea is to equip topological spaces that locally look like Euclidean spaces with a smooth structure and define derivatives of smooth functions between such spaces. Firstly, we will look at topological spaces of which any point has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^{n}$.

Definition 2.1. A homeomorphism is bijective map $f: X \rightarrow Y$ between two topological spaces $X$ and $Y$ such that both $f$ and $f^{-1}$ are continuous.

With the topological properties of Hausdorffness and second countability, we can already define a topological manifold.

Definition 2.2. A $n$-dimensional topological manifold is a topological space $M$ that satisfies the following properties:
(i) $M$ is a Hausdorff space;
(ii) $M$ is second countable;
(iii) $M$ is locally euclidean of dimension $n$. That is, for any point $p \in M$ there exists an open subset $U \subseteq M$ with $p \in U$, an open subset $V \subseteq \mathbb{R}^{n}$ and a homeomorphism $\varphi: U \rightarrow V$.

Remark 2.3. A n-dimensional topological manifold $M$ with boundary satisfies properties (i) and (ii) of the above definition, but is instead locally homeomorphic to $\mathcal{H}^{n}:=\left\{x \in \mathbb{R}^{n} \mid x^{n} \geq 0\right\}$. One could interpret manifolds without boundary as a special case of manifolds with boundary. Therefore, when we are discussing a manifold it may either have or may not have a boundary.
Remark 2.4. A compact manifold is a manifold that is compact as a topological space. A closed manifold is a compact manifold without boundary. Lastly, a connected manifold is a manifolds that is connected as a topological space. We will frequently use compact, closed ad connected (Riemannian) manifolds in the coming chapters.

We call the pair $(U, \varphi)$ a coordinate chart, with $\varphi: U \rightarrow V \subseteq \mathbb{R}^{n}$ the coordinate map and $U$ the coordinate neighbourhood about a point $p \in U$. In order to define a smooth structure on a topological manifold $M$, we need a way to assemble the charts $(U, \varphi)$ such that they cover $M$. That is, such that $M=\bigcup_{i \in I} U_{i}$, where $I$ is some indexing set. For such a collection to make any sense, two different coordinate maps of overlapping neighbourhoods must agree on the intersection of these neighbourhoods.

Definition 2.5. Two coordinate charts $\left(U_{i}, \varphi_{i}\right)$ and $\left(U_{j}, \varphi_{j}\right)$ on a manifold $M$ are smoothly compatible if either $U_{i} \cap U_{j}=\emptyset$, or if the transition map

$$
\varphi_{i} \circ \varphi_{j}^{-1}: \varphi_{j}\left(U_{i} \cap U_{j}\right) \rightarrow \varphi_{i}\left(U_{i} \cap U_{j}\right)
$$

is $C^{\infty}$.
This additional structure enables us to define a smooth structure on a manifold.
Definition 2.6. Let $M$ be a $n$-dimensional topological manifold, then
(i) the collection

$$
\mathcal{A}=\left\{\varphi_{i}: U_{i} \rightarrow V_{i} \mid i \in I\right\}
$$

of pairwise smoothly compatible charts that cover $M$ is a smooth atlas;
(ii) we call an equivalence class of smooth atlases a smooth structure on $M$. Atlases are said to be equivalent if any two charts of these atlases are smoothly compatible;
(iii) we say that the pair $(M, \mathcal{A})$ is a $n$-dimensional smooth manifold if $\mathcal{A}$ a smooth structure on $M$.

It is clear that there exists a smooth structure $\mathcal{A}$ when we say that $M$ is a $n$-dimensional smooth manifold. Of course, we can also define smooth maps between manifolds. These maps can be seen as lifted maps between Euclidean spaces, and hence differentiability follows from differentiability as a Euclidean map.

Definition 2.7. A map $F: M \rightarrow N$ between smooth manifolds of dimension $m$ and $n$ respectively is a smooth map if for any chart $(\varphi, U)$ of $M$ and $(\phi, V)$ of $N$ the map

$$
\phi \circ F \circ \varphi^{-1}: \mathbb{R}^{m} \supseteq \varphi\left(U \cap F^{-1}(V)\right) \rightarrow \phi(F(U) \cap V) \subseteq \mathbb{R}^{n}
$$

is $C^{\infty}$ as a Euclidean function.
Although we will not pay specific attention to integrals on manifolds in this thesis, it is useful for later to define the orientation of a manifold, which is a required property to apply e.g. Stoke's theorem [22, Theorem 8.3.1].

Definition 2.8. An oriented atlas $\mathcal{A}=\left\{\left(U_{i}, \varphi_{i}\right)\right\}$ is called oriented if all the charts have the same orientation. That is, if $\operatorname{det}\left(D \varphi_{i j}\right)>0$ for all the transition functions $\varphi_{i j}=\varphi_{i} \circ \varphi_{j}^{-1}$. A manifold $M$ with an oriented atlas is called a oriented manifold ${ }^{1}$. If there exists an orientation on $M$, then we say that it is orientable.

### 2.2 Tangent vectors, covectors and tensors

This section summarizes the notions of tangent vectors, covectors, tensors and the spaces they live in. We will we see that tangent vectors and covectors are just an example of the latter. Tensors will eventually allow us to define a Riemannian manifold. In the process, we will also see how to actually define a derivative of a smooth function between smooth manifolds with which we can define submanifolds. Moreover, we will briefly examine Lie derivatives of tensor fields and some other useful properties of tensors.

[^0]
### 2.2.1 Tangent bundle and vector fields

Throughout literature, a broad variety of approaches eventually lead to equivalent notions of the actual derivative of such a smooth map. Without getting to technical, the approach here follows [22]. Recall that a germ at $p \in M$ is an equivalence class in the quotient space of smooth functions on $M$, i.e. of $C_{p}^{\infty}(M):=C^{\infty}(M) / \sim_{p}$. But as pointed out in [22, Section 2.3], the tangent space at a point $p \in M$ is isomorphic to the space of derivations $C^{\infty}(V)$, where $V \subseteq M$ is some open neighbourhood of $p$. This allows us to define tangent vectors at $p$ as derivations of $C^{\infty}(V)$ for any such neighbourhood $V \ni p$, instead of seeing them as linear maps $v: C_{p}^{\infty}(M) \rightarrow \mathbb{R}$ that act on equivalence classes. For convenience, in the next definition we assume $V$ to coincide with whole of a smooth manifold $M$.

Definition 2.9. A tangent vector at a point $p$ of a smooth manifold $M$ is the linear map

$$
v: C^{\infty}(M) \rightarrow \mathbb{R}
$$

which is also a derivation of $C^{\infty}(M)$ at $p$. That is, it satisfies the Leibniz rule

$$
\begin{equation*}
v(f g)=f(p) v(g)+g(p) v(f), \quad \forall f, g \in C^{\infty}(M) \tag{2.1}
\end{equation*}
$$

We denote the set of all tangent vectors at $p$ by $T_{p} M$, the tangent space to $M$ at $p$.
Alternatively, we can also define a tangent vector by using smooth parametrized curves in a manifold $M$.

Definition 2.10. Let $M$ be a smooth manifold, $p \in M, I=(a, b) \subset \mathbb{R}$ with $0 \in(a, b)$ and $\gamma: I \rightarrow M$ a smooth curve with $\gamma(0)=p \in M$. A tangent vector $v$ at $p \in M$ is a map

$$
v: C^{\infty}(M) \rightarrow \mathbb{R}
$$

defined by

$$
\begin{equation*}
v(f):=\left.\frac{\mathrm{d}(f \circ \gamma(t))}{\mathrm{d} t}\right|_{t=0}, \quad \forall f \in C^{\infty}(M) \tag{2.2}
\end{equation*}
$$

It is important to note that this $\gamma: I \rightarrow M$ exists and satisfies $\gamma^{\prime}(0)=v \in T_{p} M$ [22, Theorem 2.5.5], meaning that these definitions are in fact equivalent. By using local coordinates of a chart on a manifold $M$ and with these equivalent definitions at hands, we can show that $T_{p} M$ is a vector space.

Proposition 2.11. Let $M$ be a $n$-dimensional smooth manifold, then
(i) the tangent space $T_{p} M$ at $p \in M$ is a vector space of dimension $n$;
(ii) if $(U, \varphi)$ is a chart about $p$ with coordinates ${ }^{2}\left(x^{i}\right)$ for each $i \in\{1, \ldots, n\}$, the set $\left\{\left.\left.\frac{\partial}{\partial x^{i}}\right|_{p} \right\rvert\, i \in\{1, \ldots, n\}\right\}$ forms a basis for $T_{p} M$.

Proof. Let $v, w \in T_{p} M$ and $\lambda \in \mathbb{R}$. Following Definition 2.9, we know that $v+\lambda w$ is a linear mapping $v+\lambda w: C^{\infty}(M) \rightarrow \mathbb{R}$, since both $v$ and $w$ are linear mappings. Let $V \subseteq U \subseteq M$ be a neighbourhood about $p$ and let $f, g \in C^{\infty}(V)$. By using that $v, w$ satisfy Leipniz's rule we see that

$$
\begin{aligned}
(v+\lambda w)(f g) & =v(f g)+\lambda w(f g) \\
& =f(p) v(g)+g(p) v(f)+\lambda(f(p) w(f)+g(p) w(f)) \\
& =f(p)(v+\lambda w)(g)+g(p)(v+\lambda w)(f)
\end{aligned}
$$

i.e., that $v+\lambda w$ also satisfies Leipniz's rule. It follows that $T_{p} M$ is a real vector space.

[^1]To proof (ii), notice that since Definitions 2.9 and 2.10 are equivalent, for $v \in T_{p} M$ we have

$$
\begin{aligned}
v(f) & =\left.\frac{\mathrm{d}(f \circ \gamma(t))}{\mathrm{d} t}\right|_{t=0} \\
& =\left.\frac{\mathrm{d}\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma(t)\right)}{\mathrm{d} t}\right|_{t=0}
\end{aligned}
$$

since $\varphi$ is a bijection. By using the chain rule we can rewrite this to

$$
\left.\frac{\mathrm{d}\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma(t)\right)}{\mathrm{d} t}\right|_{t=0}=\frac{\partial f \circ \varphi^{-1}}{\partial x^{i}}(\varphi(p)) \frac{\mathrm{d}\left(x^{i} \circ \gamma\right)}{\mathrm{d} t}(0)
$$

Notice that $\frac{\mathrm{d}\left(x^{i} \circ \gamma\right)}{\mathrm{d} t}(0) \in \mathbb{R}$ for each $i \in\{1, \ldots, n\}$. Define $v^{i}:=\frac{\mathrm{d}\left(x^{i} \circ \gamma\right)}{\mathrm{d} t}$, then since $\varphi^{-1} \circ \varphi(p)=p$ we see that

$$
v(f)=\left.v^{i} \frac{\partial f}{\partial x^{i}}\right|_{p},
$$

and hence that

$$
v=\left.v^{i} \frac{\partial}{\partial x^{i}}\right|_{p}
$$

Indeed, the set $\left\{\left.\left.\frac{\partial}{\partial x^{i}}\right|_{p} \right\rvert\, i \in\{1, \ldots, n\}\right\}$ spans the vector space $T_{p} M$. Additionally, notice that for each $j \in\{1, \ldots n\}$ we have

$$
v\left(x^{j}\right)=\left.v^{i} \frac{\partial x^{j}}{\partial x^{i}}\right|_{p}=v^{i} \delta_{i j},
$$

showing that $v=\left.v^{i} \frac{\partial}{\partial x^{i}}\right|_{p}=0$ if and only if $\left(v^{1}, \ldots, v^{n}\right)=0$. Hence $\left\{\left.\left.\frac{\partial}{\partial x^{2}}\right|_{p} \right\rvert\, i \in\{1, \ldots, n\}\right\}$ are linearly independent and form a basis for the vector space $T_{p} M$ with $\operatorname{dim}\left(T_{p} M\right)=n=\operatorname{dim}(M)$.


Figure 1: The tangent space $T_{p} \mathbb{S}^{2}$ at a point $p \in \mathbb{S}^{2}$.

We now have the right tools and knowledge to actually define the derivative of a smooth map between manifolds, rather than stating results on differentiability. It turns out that the derivative of a smooth map is nothing more than a linear map between tangent spaces.

Definition 2.12. The differential or total derivative of a smooth map $F: M \rightarrow N$ between smooth manifolds $M$ and $N$ at $p \in M$ is the linear map

$$
\begin{equation*}
d F_{p}: T_{p} M \rightarrow T_{F(p)} N, \quad d F_{p}(v)(f):=v(f \circ F), \quad \forall f \in C^{\infty}(N) \tag{2.3}
\end{equation*}
$$

With the total derivative of Definition 2.12, we can define two particular kinds of smooth maps between manifolds. These maps will in turn provide us with a definition for submanifolds.

Definition 2.13. Let $F: M \rightarrow N$ be a smooth map between smooth manifolds of respectively dimension $m$ and $n$. Then
(i) $F$ is an immersion if $d F_{p}$ is injective for all $p \in M$ (i.e. then, $m \leq n$ );
(ii) $F$ is an embedding if $F$ is an injective immersion that is also a homeomorphism onto its range, i.e. $F(M)=N$.

Due to Definition 2.13, we can make a distinction between two kinds of submanifolds.
Definition 2.14. Let $\iota: M \hookrightarrow N$ be the inclusion map between two smooth manifolds $M$ and $N$ such that $M \subset N$. Then we say that
(i) $M$ is an immersed submanifold if the inclusion map is an immersion;
(ii) $M$ is an embedded submanifold if the inclusion map is an embedding.

In the above definition, we see that $M$ is always a manifold 'sitting inside' a larger manifold $N$ since the dimensions satisfy $m \leq n$. With the Implicit Function Theorem for Manifolds, we can also detect submanifolds when $m \geq n$ (see, for example, [22, Theorem 2.8.14]).

Let us move back back to tangent spaces again. Rather than talking about a tangent space at one point of a manifold, it is often convenient to talk about the set of tangent spaces as a whole.
Definition 2.15. The tangent bundle $T M$ of a smooth manifold $M$ is the disjoint union of tangent spaces

$$
T M:=\bigsqcup_{p \in M}\left(\{p\} \times T_{p} M\right)
$$

Elements of $T M$ are pairs $(p, v)$ where $p \in M$ is a base point and $v \in T_{p} M$ is a tangent vector.
An important fact is that the tangent bundle itself is also a smooth manifold. More precisely, if $M$ is a $n$-dimensional smooth manifold, then $T M$ is a $2 n$-dimensional smooth manifold of which the smooth structure is naturally obtained from the smooth structure of $M$ [22, Theorem 2.6.3]. But, actually this is just the prototype of the more general notion of a vector bundle.

Definition 2.16. A smooth vector bundle of rank $r$ is a triple $(E, M, \pi)$ where $E$ and $M$ are manifolds and $\pi: E \rightarrow M$ a smooth surjective map such that, for all $p \in M$, the following properties hold:
(i) the fibre over $p, E_{p}:=\pi^{-1}(p)$, has the structure of a vector space of dimension $r$;
(ii) there exists a neighbourhood $U \subseteq M$ of $p$ and a diffeomorphism $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{r}$ such that
(a) $\pi_{1} \circ \varphi=\pi$, where $\pi_{1}: U \times \mathbb{R}^{r} \rightarrow U$ is the projection on the first factor,
(b) for all $q \in U$, the map $\left.\varphi\right|_{E_{q}}: E_{q} \rightarrow\{q\} \times \mathbb{R}^{r}$ is an isomorphism of vector spaces.

The space $E$ is called the total space, $M$ the base space, $\pi$ its projection and each $\varphi$ is a local trivialisation.

Remark 2.17. Sometimes we just say the $E$ is a vector bundle over $M$.
Remark 2.18. One could also define an orientation on a vector bundle ${ }^{3}$. An orientation on $E$ means that for each fibre $E_{p}$, there exists an orientation such that each trivialization $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{r}$ is fibrewise orientation-preserving. We will use this to define orientation on almost-Riemannian manifolds in Chapter 4.

To emphasize the earlier statement that the tangent bundle is an example of a vector bundle, notice that the projection $\pi: T M \rightarrow M$ is a surjective map such that the tangent spaces are fibres: $\pi^{-1}(p)=$ $T_{p} M$. Not surprisingly, we can also define sub-bundles.
Definition 2.19. Let $(E, M, \pi)$ be a smooth vector bundle of rank $n$ and $F \subset E$ a submanifold. If $F_{p}:=F \cap E_{p}$ is $k$-dimensional subspace of the vector space $E_{p}$ for all $p \in M$ and $\left.\pi\right|_{F}: F \rightarrow M$ defines a smooth vector bundle of rank $k$, then $\left(F, M,\left.\pi\right|_{F}\right)$ is called a sub-bundle of $E$.

For our quest on defining a Riemannian manifold, we still need more tools. The first thing that we will look at are vector fields. A vector field, however, is itself an example of another notion.
Definition 2.20. A section of a vector bundle $\pi: E \rightarrow M$ is a smooth map $S: M \rightarrow E$ such that $\pi \circ S=\operatorname{id}_{M}$. We denote the set of all smooth sections on $E$ by $\Gamma(E)$

For a local chart $(U, \varphi)$ about $p \in M$ with local coordinates $\left(x^{i}\right)$, we say that the family of $n$ local sections $\left\{\frac{\partial}{\partial x^{1}}, \ldots, \frac{\partial}{\partial x^{n}}\right\}$ is a smooth local frame of the tangent bundle $T M$, since these sections yield a basis for $T_{p} M$ for each $p \in U$. If $U=M$, we call this set a global frame. We can now define vector fields as sections of the tangent bundle: a smooth map from a manifold to the tangent bundle that assigns a tangent vector to each point of a manifold.
Definition 2.21. A smooth vector field is a smooth map $X: M \rightarrow T M$ with $\pi \circ X=\operatorname{id}_{M}$. We denote the set of smooth vector fields by $\mathfrak{X}(M)$.

By using the basis and chart about a point $p \in U \subseteq M$ of a smooth manifold as in Proposition 2.11, we can express the value of a smooth vector field $X: M \rightarrow T M$ as

$$
\begin{equation*}
X_{p}=\left.X^{i}(p) \frac{\partial}{\partial x^{i}}\right|_{p} \tag{2.4}
\end{equation*}
$$

with $X^{i}: U \rightarrow \mathbb{R}$ the component functions of $X$. A tool that will be frequently used throughout this thesis is the Lie bracket of two smooth vector fields.
Definition 2.22. Let $X, Y \in \mathfrak{X}(M)$. The Lie bracket $[\cdot, \cdot]: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ of $X$ and $Y$ is the derivation given by their commutator:

$$
\begin{equation*}
[X, Y]:=X Y-Y X \tag{2.5}
\end{equation*}
$$

Proposition 2.23. Let $(U, \varphi)$ be a chart on $M$ with local coordinates $\left(x^{i}\right)$ and let $X, Y \in \mathfrak{X}(U)$. If $X=X^{i} \frac{\partial}{\partial x^{i}}$ and $Y=Y^{j} \frac{\partial}{\partial x^{j}}$ are the coordinate expressions for $X$ and $Y$, then

$$
\begin{equation*}
[X, Y]=\left(X^{i} \frac{\partial Y^{j}}{\partial x^{i}}-Y^{i} \frac{\partial X^{j}}{\partial x^{i}}\right) \frac{\partial}{\partial x^{j}} \tag{2.6}
\end{equation*}
$$

Proof. Let $f \in C^{\infty}(U)$, then we can compute the lie bracket of $X, Y \in \mathfrak{X}(U)$ by using (2.5) as

$$
\begin{aligned}
{[X, Y] f } & =X(Y(f))-Y(X(f)) \\
& =X^{i} \frac{\partial}{\partial x^{i}}\left(Y^{j} \frac{\partial f}{\partial x^{j}}\right)-Y^{j} \frac{\partial}{\partial x^{j}}\left(X^{i} \frac{\partial f}{\partial x^{i}}\right) \\
& =X^{i} \frac{\partial Y^{j}}{\partial x^{i}} \frac{\partial f}{\partial x^{j}}+X^{i} Y^{j} \frac{\partial^{2} f}{\partial x^{i} \partial x^{j}}-Y^{j} \frac{\partial X^{i}}{\partial x^{j}} \frac{\partial f}{\partial x^{i}}-Y^{j} X^{i} \frac{\partial^{2} f}{\partial x^{j} \partial x^{i}},
\end{aligned}
$$

[^2]where we used the product rule in the third step. Since the order of taking partial derivatives can be interchanged, notice that the second and fourth term cancel out. Therefore, we get
\[

$$
\begin{aligned}
{[X, Y] f } & =X^{i} \frac{\partial Y^{j}}{\partial x^{i}} \frac{\partial f}{\partial x^{j}}-Y^{j} \frac{\partial X^{i}}{\partial x^{j}} \frac{\partial f}{\partial x^{i}} \\
& =\left(X^{i} \frac{\partial Y^{j}}{\partial x^{i}}-Y^{i} \frac{\partial X^{j}}{\partial x^{i}}\right) \frac{\partial f}{\partial x^{j}}
\end{aligned}
$$
\]

where we interchanged the dummy indices $i$ and $j$ in the last step.
The Lie bracket is actually another way of writing the Lie derivative of a vector field $Y$ with respect to $X$, i.e. $\mathcal{L}_{X} Y=[X, Y]$. Later, we will also define the Lie derivative of tensor fields. For that purpose, let us briefly introduce integral curves of vector fields [22, Definition 3.31].

Definition 2.24. Let $M$ be a smooth manifold and $X \in \mathfrak{X}(M)$. A smooth curve $\gamma: \mathbb{R} \supset(a, b) \rightarrow M$ is an integral curve of $X$ if

$$
\begin{equation*}
\gamma^{\prime}(t)=X_{\gamma(t)}, \quad \forall t \in(a, b) \tag{2.7}
\end{equation*}
$$

It is often assumed that $0 \in(a, b)$ and that $y(0)=p \in M$, such that $\gamma$ is an integral curve through $p$. $\vee$
Definition 2.25. Let $M$ be a smooth manifold and $X \in \mathfrak{X}(M)$. For a given $p \in M$ we denote $I_{p}=$ $\left(t^{-}(p), t^{+}(p)\right) \subset \mathbb{R}$, with $0 \in I_{p}$, as the maximal interval on which the unique integral curve $\gamma_{p}: I_{p} \rightarrow M$ of $X$ through $p$ is defined. We call $\gamma_{p}$ the maximal integral curve of $X$ through $p$.

Existence and uniqueness of (maximal) integral curves follow from existence and uniqueness of Euclidean theorems [22, Theorem 3.3.5, Theorem 3.3.11] on ODE's. Theorem 3.3.11 of [22] guarantees the existence of a unique map $\varphi: \mathcal{D} \rightarrow M$ with $\mathcal{D} \subset \mathbb{R} \times \stackrel{M}{\circ}$ such that for all $p \in M$ one has $D \cap(\mathbb{R} \times\{p\})=I_{p} \times\{p\}$ and $\varphi(t, p)=\gamma_{p}(t)$ for all $(t, p) \in \mathcal{D}$. The map $\varphi^{X}$ is called the flow of $X$.

### 2.2.2 Cotangent bundle and tensor bundle

By using vector fields we can eventually endow a smooth manifold with an inner product to define lengths of and angles between vectors of a tangent space. As will later become clear, we therefore first need to define the dual of a tangent space.
Definition 2.26. The cotangent space $T_{p}^{*} M:=\left(T_{p} M\right)^{*}$ of a smooth manifold $M$ at $p \in M$ is the dual of the tangent space $T_{p} M$. Elements of $T_{p}^{*} M$ are called covectors or differential 1-forms at $p$, linear functionals from $T_{p} M$ to $\mathbb{R}$.

Not surprisingly, if $M$ is $n$-dimensional, then the dual space $T_{p}^{*} M$ at $p \in M$ is also $n$-dimensional. Also similar to Proposition 2.11, if $(U, \varphi)$ is a chart about $p$ with local coordinates $\left(x^{i}\right)$, then the set of covectors given by $\left\{\left.d x^{i}\right|_{p} \mid i \in\{1, \ldots, n\}\right\}$ forms a basis for $T_{p}^{*} M$. Moreover, we can define the cotangent bundle, which is then also a $2 n$-dimensional smooth manifold.
Definition 2.27. The cotangent bundle $T^{*} M$ of a smooth manifold $M$ is the disjoint union of the cotangent spaces

$$
T^{*} M:=\bigsqcup_{p \in M}\left(\{p\} \times T_{p}^{*} M\right)
$$

Elements of $T^{*} M$ are pairs $(p, \omega)$ where $p \in M$ is a base point and $\omega \in T_{p}^{*} M$ is a covector.
Remark 2.28. Note that just as with the tangent bundle and vector bundle, the cotangent bundle is a specific example of the dual bundle $E^{*}$ of a vector bundle $E$ over $M$. Its fibres are the dual spaces of the fibres of $E[18$, Section 2.3.4].

The smooth section from a manifold to the cotangent bundle that assigns a covector to each point of a manifold is what we call a covector field.

Definition 2.29. A smooth covector field is a smooth map $\omega: M \rightarrow T^{*} M$ with $\pi \circ \omega=\operatorname{id}_{M}$. We denote the set of smooth covector fields by $\mathfrak{X}^{*}(M)$.

Similarly to (2.4), we can express the value of a smooth vector field $\omega: M \rightarrow T^{*} M$ at $p \in M$ as

$$
\begin{equation*}
\omega_{p}=\left.\omega_{i}(p) d x^{i}\right|_{p} \tag{2.8}
\end{equation*}
$$

with $\omega_{i}: U \rightarrow \mathbb{R}$. Both tangent vectors as covectors are, however, examples of more general objects, called tensors. These multilinear maps will be one of our primary tools throughout this thesis.
Definition 2.30. The tensor space

$$
T_{s}^{r}\left(T_{p} M\right):=\operatorname{Mult}(\overbrace{T_{p}^{*} M, \ldots, T_{p}^{*} M}^{r \text { times }}, \underbrace{T_{p} M, \ldots, T_{p} M}_{s \text { times }})
$$

of a smooth manifold $M$ at $p \in M$ is the space of multilinear maps $\tau$. That is, the space of tensors of type $(r, s)$ :

$$
\tau: \overbrace{T_{p}^{*} M \times \cdots \times T_{p}^{*} M}^{r \text { times }} \times \underbrace{T_{p} M \times \cdots \times T_{p} M}_{s \text { times }} \rightarrow \mathbb{R}
$$

If $r=0$ and $s=k \geq 1$, then we say that $\tau \in T_{k}^{0}\left(T_{p} M\right)$ is a covariant $k$-tensor on $T_{p} M$. In contrast, if $r=k \geq 1$ and $s=0$, we call $\tau \in T_{0}^{k}\left(T_{p} M\right)$ a contravariant $k$-tensor on $T_{p} M$. A tensor of type $(r, s)$, i.e. $\tau \in T_{s}^{r}\left(T_{p} M\right)$, is sometimes defined as the pairing

$$
\begin{equation*}
\tau\left(\omega^{1}, \ldots, \omega^{r} ; v_{1}, \ldots, v_{s}\right)=:\left(\tau \mid \omega^{1}, \ldots, \omega^{r} ; v_{1}, \ldots, v_{s}\right), \quad \omega^{1}, \ldots, \omega^{r} \in T_{p}^{*} M, \quad v_{1}, \ldots, v_{s} \in T_{p} M \tag{2.9}
\end{equation*}
$$

Of course, we can also define the tensor bundle over a manifold $M$.
Definition 2.31. The $(r, s)$-tensor bundle $T_{s}^{r} M$ of a smooth manifold $M$ is the disjoint union of the tensor spaces

$$
T_{s}^{r} M:=\bigsqcup_{p \in M}\left(\{p\} \times T_{s}^{r}\left(T_{p} M\right)\right) .
$$

Remark 2.32. Note again that the tensor bundle is an example of a tensor product of vector bundles and dual bundles. More specifically, if $E_{1}^{*}, \ldots, E_{r}^{*}$ are dual bundles and $E_{1}, \ldots, E_{s}$ are vector bundles, then the tensor product $E_{1}^{*} \otimes \cdots \otimes E_{r}^{*} \otimes E_{1} \otimes \cdots \otimes E_{k}$ is a vector bundles with fibres $\left(E_{1}^{*}\right)_{p} \otimes \cdots \otimes\left(E_{r}^{*}\right)_{p} \otimes$ $\left(E_{1}\right)_{p} \otimes \cdots \otimes\left(E_{k}\right)_{p}[18$, Section 2.3.5].

Note that therefore $T M=T_{0}^{1} M$ and $T^{*} M=T_{1}^{0} M$. For completeness, we also define tensor fields. The vector and covector field that we saw before are just an example of this more general object, as one would expect.
Definition 2.33. A smooth tensor field of type $(r, s)$ is a smooth map $\tau: M \rightarrow T_{s}^{r} M$ such that $\pi \circ \tau=\mathrm{id}_{M}$. We denote the space of smooth tensor fields by $\mathcal{T}_{s}^{r}(M)$.

Then clearly $\mathfrak{X}(M)=\mathcal{T}_{0}^{1}(M)$ and $\mathfrak{X}^{*}(M)=\mathcal{T}_{1}^{0}(M)$. If $(U, \varphi)$ is a chart about $p \in M$ with local coordinates $\left(x^{i}\right)$, then we can express the value of a smooth tensor field $\tau: M \rightarrow T_{s}^{r}(M)$ as the tensor product

$$
\begin{equation*}
\tau_{p}=\left.\left.\left.\left.\tau_{i_{1} \cdots i_{s}}^{j_{1} \cdots j_{r}}(p) \frac{\partial}{\partial x^{j_{1}}}\right|_{p} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_{r}}}\right|_{p} \otimes d x^{i_{1}}\right|_{p} \otimes \cdots \otimes d x^{i_{s}}\right|_{p}, \tag{2.10}
\end{equation*}
$$

with $\tau_{i_{1} \cdots i_{s}}^{j_{1} \cdots j_{r}}: U \rightarrow \mathbb{R}$ the component defined by

$$
\begin{equation*}
\tau_{i_{1} \cdots i_{s}}^{j_{1} \cdots j_{r}}:=\tau\left(d x^{j_{1}}, \ldots, d x^{j_{r}}, \frac{\partial}{\partial x^{i_{1}}}, \ldots, \frac{\partial}{\partial x^{i_{s}}}\right) \tag{2.11}
\end{equation*}
$$

The following lemma allows us to conveniently characterize tensor fields as $C^{\infty}(M)$ multilinear maps.

Lemma 2.34. A map

$$
\tau: \overbrace{\mathfrak{X}^{*}(M) \times \cdots \times \mathfrak{X}^{*}(M)}^{r \text { times }} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s \text { times }} \rightarrow C^{\infty}(M)
$$

is induced by a $(r, s)$-tensor field $\tau \in \mathcal{T}_{s}^{r}(M)$ if and only if it is multilinear over $C^{\infty}(M)$. Similarly, a map

$$
\tau: \overbrace{\mathfrak{X}^{*}(M) \times \cdots \times \mathfrak{X}^{*}(M)}^{r \text { times }} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s \text { times }} \rightarrow \mathfrak{X}(M)
$$

is induced by a $(r+1, s)$-tensor field $\tau \in \mathcal{T}_{s}^{r+1}(M)$ if and only if it is multilinear over $C^{\infty}(M)$.
Proof. The proof of this lemma is not of significant importance for our purpose. Therefore, it is omitted and we refer to the tensor characterization lemma in [21, Lemma B.6] or [25, Lemma 12.24].

As promised, we define the Lie derivative of tensor fields [22, Remark 7.6.6], which we will use in our proof for existence and uniqueness of the Ricci flow.

Definition 2.35. Let $M$ be a smooth manifold and $\tau \in \mathcal{T}_{r}^{s}(M)$. The Lie derivative of $\tau$ along $X \in \mathfrak{X}(M)$ for $p \in M$ is

$$
\begin{equation*}
\left(\mathcal{L}_{X} \tau\right)_{p}:=\left.\frac{d}{d t}\right|_{t=0}\left(\left(\varphi_{t}^{X}\right)^{*} \tau\right)_{p} \tag{2.12}
\end{equation*}
$$

Notice that here, $\mathbb{R} \ni t \mapsto \varphi_{t}^{X}(p):=\varphi(t, p)$ denotes the maximal integral curve for $X$ starting from $p$.
Particular interesting classes of tensors are symmetric and alternating tensors. If $V$ is a real $n$ dimensional vector space and $\omega \in T_{k}^{0}(V)$, then we say that $\omega$ is a symmetric covariant $k$-tensor on $V$ if its value is unchanged by interchanging any pair of arguments. That is, if $v_{1}, \ldots, v_{k} \in V$, we have

$$
\begin{equation*}
\omega\left(v_{1}, \ldots, v_{i}, v_{j}, \ldots, v_{k}\right)=\omega\left(v_{1}, \ldots, v_{j}, v_{i}, \ldots, v_{k}\right) \tag{2.13}
\end{equation*}
$$

for all $1 \leq i<j \leq k$. Let us denote the set of all symmetric $k$-tensors on $V$ by $\Sigma^{k}(V)$. Via the projection Sym : $T_{k}^{0}(V) \rightarrow \Sigma^{k}(V)$ defined by

$$
\begin{equation*}
\operatorname{Sym} \omega\left(v_{1}, \ldots, v_{k}\right):=\frac{1}{k!} \sum_{\sigma \in S_{k}} \omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \tag{2.14}
\end{equation*}
$$

one is able to symmetrize any $\omega \in T_{k}^{0}(V)$. Here, $S_{k}$ denotes the symmetric group on $k$ elements and $\sigma \in S_{k}$ is a permutation. The symmetric product of $\omega \in \Sigma^{k}(V)$ and $\eta \in \Sigma^{l}(V)$ is the $(k+l)$-tensor given by

$$
\begin{equation*}
\omega \eta:=\operatorname{Sym}(\omega \otimes \eta) \tag{2.15}
\end{equation*}
$$

Notice that if $\omega$ and $\eta$ are covectors on $V$, they are always symmetric since they act on only one vector $v \in V$.

Lemma 2.36. If $\alpha$ and $\beta$ are covectors on a $n$-dimensional real vector space $V$, i.e. $\alpha, \beta \in T_{1}^{0}(V)$, then

$$
\begin{equation*}
\alpha \beta=\frac{1}{2}(\alpha \otimes \beta+\beta \otimes \alpha) . \tag{2.16}
\end{equation*}
$$

Proof. Let $v_{1}, v_{2} \in V$, then by (2.15) we have

$$
\begin{aligned}
\alpha \beta\left(v_{1}, v_{2}\right) & =\operatorname{Sym}(\alpha \otimes \beta)\left(v_{1}, v_{2}\right) \\
& =\frac{1}{2} \sum_{\sigma \in S_{k}} \alpha\left(v_{\sigma(1)}\right) \beta\left(v_{\sigma(2)}\right) \\
& =\frac{1}{2} \alpha\left(v_{1}\right) \beta\left(v_{2}\right)+\frac{1}{2} \beta\left(v_{1}\right) \alpha\left(v_{2}\right) \\
& =\frac{1}{2}(\alpha \otimes \beta+\beta \otimes \alpha)\left(v_{1}, v_{2}\right),
\end{aligned}
$$

which proves the proposition.
Contrastingly, we call $\omega$ an alternating covariant $k$-tensor if it changes sign whenever two arguments are interchanged:

$$
\begin{equation*}
\omega\left(v_{1}, \ldots, v_{i}, v_{j}, \ldots, v_{k}\right)=-\omega\left(v_{1}, \ldots, v_{j}, v_{i}, \ldots, v_{k}\right) \tag{2.17}
\end{equation*}
$$

for all $1 \leq i<j \leq k$. These are also called exterior forms, or $k$-covectors.

### 2.3 Riemannian manifolds

Since we have acquired the right tools, we can start off with the definition [27, Definition 3.11].
Definition 2.37. A Riemannian manifold is a pair $(M, g)$ where $M$ is a smooth manifold and $g$ is a Riemannian metric that provides each $p \in M$ with an inner product $g_{p}: T_{p} M \times T_{p} M \rightarrow \mathbb{R}$ such that for all $X, Y \in \mathfrak{X}(M)$, the map $p \mapsto g_{p}\left(X_{p}, Y_{p}\right)$ is smooth.
Remark 2.38. When it is more convenient, we say that $M$ is an Riemannian manifold.
Remark 2.39. More generally, one could also define metrics on a vector bundle, dual bundle and tensor bundle. In the latter case for two tensor fields $\alpha, \beta \in \mathcal{T}_{s}^{r}(M)$, their inner product is given by

$$
\begin{equation*}
\langle\alpha, \beta\rangle=g^{a_{1} b_{1}} \cdots g^{a_{r} b_{r}} g_{i_{1} j_{1}} \cdots g_{i_{s} j_{s}} \alpha_{a_{1} \cdots a_{r}}^{i_{1} \cdots i_{s}} \beta_{b_{1} \cdots b_{r}}^{j_{1} \cdots j_{s}} . \tag{2.18}
\end{equation*}
$$

Remark 2.40. In the more general pseudo-Riemannian manifold, we require the metric $g$ to be nondegenerate, smooth, symmetric and bilinear. Almost all of the upcoming theory is also applicable to pseudo-Riemannian manifolds, but some is not or requires extra information. For simplicity we therefore only consider Riemannian manifolds.

We see from Lemma 2.34 that the metric $g: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^{\infty}(M)$ is a $(0,2)$-tensor field, and hence lives in $\mathcal{T}_{2}^{0}(M)$. Sometimes, it is more convenient to use the usual inner product notation, i.e. $\left\langle X_{p}, Y_{p}\right\rangle:=g_{p}\left(X_{p}, Y_{p}\right)$. In our usual notation for a chart $(U, \varphi)$ with local coordinates $\left(x^{i}\right)$ about a point $p \in M$, we can write

$$
\begin{equation*}
g_{p}=\left.\left.g_{i j}(p) d x^{i}\right|_{p} \otimes d x^{j}\right|_{p} \tag{2.19}
\end{equation*}
$$

with each

$$
\begin{equation*}
g_{i j}(p)=g_{p}\left(\left.\frac{\partial}{\partial x^{i}}\right|_{p},\left.\frac{\partial}{\partial x^{j}}\right|_{p}\right) . \tag{2.20}
\end{equation*}
$$

The matrix $\left[g_{i j}\right]$ can therefore be seen as the $n \times n$ matrix with inner products as its entries. We denote the inverse of this inner product matrix by $\left[g^{i j}\right]$. We can alternatively write (2.19) as

$$
\begin{align*}
g & =g_{i j} d x^{i} \otimes d x^{j} \\
& =\frac{1}{2}\left(g_{i j} d x^{i} \otimes d x^{j}+g_{j i} d x^{i} \otimes d x^{j}\right) \\
& =\frac{1}{2}\left(g_{i j} d x^{i} \otimes d x^{j}+g_{i j} d x^{i} \otimes d x^{j}\right) \\
& =g_{i j} d x^{i} d x^{j}, \tag{2.21}
\end{align*}
$$

where the final step follows from Lemma 2.36. As showed in for example [25, Proposition 13.3], every smooth manifold can be endowed with a Riemannian metric.

Example 2.41. The most straightforward example of a Riemannian manifold is Euclidean space $\mathbb{R}^{n}$. The metric is then the usual inner product on the tangent space $T_{p} \mathbb{R}^{n}=\mathbb{R}^{n}$ [20, Section 3]. In standard coordinates we usually write

$$
\bar{g}=\delta_{i j} d x^{i} d x^{j}
$$

In the example above, we also see that $\left\{\frac{\partial}{\partial x^{2}}, \ldots, \frac{\partial}{\partial x^{n}}\right\}$ is an orthonormal frame for $T \mathbb{R}^{n}$. More generally, $\left\{E_{1}, \ldots, E_{n}\right\}$ is a smooth orthonormal frame for a tangent bundle $T M$ on a open set $U \subseteq M$ if and only if

$$
\begin{equation*}
\left\langle E_{i}, E_{j}\right\rangle=\delta_{i j} \tag{2.22}
\end{equation*}
$$

Since we can apply the Gram-Schmidt algorithm on any smooth local frame for the tangent bundle over a subset $U \subseteq M$, we can always find such a smooth orthonormal frame defined on some neighbourhood of any $p \in M$ [21, Proposition 2.8].

A useful property of a Riemannian metric $g$ on $M$ are the musical isomorphisms

$$
{ }^{b}: T M \rightarrow T^{*} M \quad \text { and } \quad \sharp: T^{*} M \rightarrow T M
$$

used to convert vectors to covectors and covectors to vectors respectively. If $\left\{\frac{\partial}{\partial x^{1}}, \ldots, \frac{\partial}{\partial x^{n}}\right\}$ is smooth local frame for $T M$ on $U$ and $\left\{d x^{1}, \ldots, d x^{n}\right\}$ its dual coframe, then by writing $g=g_{i j} d x^{i} d x^{j}$ according to equation (2.21) and $\mathfrak{X}(U) \ni X=X^{i} \frac{\partial}{\partial x^{i}}$, we define its flat by

$$
\begin{equation*}
X^{b}:=g_{i j} X^{i} d x^{j}=X_{j} d x^{j} \tag{2.23}
\end{equation*}
$$

Similarly, if $\mathfrak{X}^{*}(U) \ni \omega=\omega_{i} d x^{i}$, we define its sharp by

$$
\begin{equation*}
\omega^{\sharp}:=g^{i j} \omega_{i} \frac{\partial}{\partial x^{j}}=\omega^{j} \frac{\partial}{\partial x^{j}} . \tag{2.24}
\end{equation*}
$$

The latter raising operator will be of use when applying the trace operator on the metric.
Definition 2.42. The trace or ( $a, b$ )-contraction of a tensor ${ }^{4} \frac{\partial}{\partial x^{1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{r}} \otimes d x^{1} \otimes \cdots \otimes d x^{s} \in \mathcal{T}_{s}^{r}(M)$, with $a \leq r$ and $b \leq s$, is the linear map

$$
\operatorname{tr}_{a}^{b}: \mathcal{T}_{s}^{r}(M) \rightarrow \mathcal{T}_{s-1}^{r-1}(M)
$$

defined by ${ }^{5}$

$$
\begin{align*}
\operatorname{tr}_{a}^{b}\left(\frac{\partial}{\partial x^{1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{r}} \otimes d x^{1} \otimes \cdots \otimes d x^{s}\right):= & d x^{b}\left(\frac{\partial}{\partial x^{a}}\right) \frac{\partial}{\partial x^{1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{a-1}} \otimes \frac{\partial}{\partial x^{a+1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{r}} \\
& \otimes d x^{1} \otimes \cdots \otimes d x^{b-1} \otimes d x^{b+1} \otimes \cdots \otimes d x^{s} \tag{2.25}
\end{align*}
$$

Example 2.43. A specifically relevant example is the trace of a symmetric ( 0,2 )-tensor field $\omega_{i j} d x^{i} d x^{j}$ through the metric tensor $g$ of some Riemannian manifold $M$ :

$$
\begin{align*}
\operatorname{tr}_{g}(\omega) & :=\operatorname{tr}_{1}^{1}\left(\omega^{\sharp}\right) \\
& =\operatorname{tr}_{1}^{1}\left(g^{j k} \omega_{i j} d x^{i} \otimes \frac{\partial}{\partial x^{k}}\right)  \tag{2.24}\\
& =g^{i j} \omega_{i j} . \tag{2.25}
\end{align*}
$$

Equivalently, we may write

$$
\begin{equation*}
\operatorname{tr}_{g}(\omega)=\operatorname{tr}_{1,2}^{1,2}\left(g^{-1} \otimes \omega\right)=\operatorname{tr}_{1,2}^{1,2}\left(g^{i j} \frac{\partial}{\partial x^{i}} \otimes \frac{\partial}{\partial x^{j}} \otimes \omega_{i j} d x^{i} \otimes d x^{j}\right)=g^{i j} \omega_{i j} \tag{2.26}
\end{equation*}
$$

[^3]A characteristic of Riemannian manifolds that will be used a lot is that its properties are preserved by isometries.
Definition 2.44. Let $(M, g)$ and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds. An isometry $\varphi:(M, g) \rightarrow(\widetilde{M}, \tilde{g})$ is a diffeomorphism such that ${ }^{6} \varphi^{*} \tilde{g}=g$.

A map $\varphi:(M, g) \rightarrow(\widetilde{M}, \tilde{g})$ between Riemannian manifolds $(M, g)$ and $(\widetilde{M}, \tilde{g})$ is called a local isometry if for each point $p \in M$ there exists a neighbourhood $U \subseteq M$ such that $\left.\varphi\right|_{U}$ is an isometry onto an open subset of $\widetilde{U} \subseteq \widetilde{M}$.

When considering submanifolds of Riemannian manifolds, one naturally equips this submanifold with an induced Riemannian metric. Mathematically, this phenomena is depicted in the following lemma.
Lemma 2.45. Let $(\widetilde{M}, \tilde{g})$ be a Riemannian manifold, $M$ a smooth manifold, and $F: M \hookrightarrow \widetilde{M}$ a smooth map. The smooth $(0,2)$-tensor field $g=F^{*} \tilde{g}$ is a Riemannian metric on $M$ if and only if $F$ is an immersion.
Proof. We start by assuming that $g=F^{*} \tilde{g}$ is a Riemannian metric on $M$. Recall that $F$ is an immersion if its derivative is injective at every $p \in M$. Since $g$ is assumed to be a Riemannian metric, it is positive definite. Let $v \in \operatorname{ker} d F_{p} \subset T_{p} M$ be nonzero, then by definition of the pullback of tensor fields we have

$$
\left(F^{*} \tilde{g}\right)_{p}(v, v)=d F_{p}^{*}\left(\tilde{g}_{F(p)}(v, v)\right)=\tilde{g}_{F(p)}\left(d F_{p}(v), d F_{p}(v)\right) \geq 0
$$

This means that $d F_{p}(v)=0$ if and only if $v=0$, i.e. that $d F_{p}$ is injective for every $p \in M$. Therefore, $F$ is an immersion.

Next, we assume $F$ to be an immersion. We must check whether the induced metric $g=F^{*} \tilde{g}$ satisfies the conditions of an inner product. Therefore, let $v_{1}, v_{2} \in T_{p} M$ and observe that

$$
\begin{aligned}
g_{p}\left(v_{1}, v_{2}\right) & =\left(F^{*} \tilde{g}\right)_{p}\left(v_{1}, v_{2}\right) \\
& =d F_{p}^{*}\left(\tilde{g}_{F(p)}\left(v_{1}, v_{2}\right)\right) \\
& =\tilde{g}_{F(p)}\left(d F_{p}\left(v_{1}\right), d F_{p}\left(v_{2}\right)\right) \\
& =\tilde{g}_{F_{(p)}( }\left(d F_{p}\left(v_{2},\right), d F_{p}\left(v_{1}\right)\right) \quad \text { by linearity of } \tilde{g} \\
& =\left(F^{*} \tilde{g}\right)_{p}\left(v_{2}, v_{1}\right) \\
& =g_{p}\left(v_{2}, v_{1}\right)
\end{aligned}
$$

Indeed, $g$ is symmetric. Secondly, let $a, b \in \mathbb{R}$, and $v_{1}, v_{2}, v_{3} \in T_{p} M$. Then

$$
\begin{array}{rlrl}
g_{p}\left(a v_{1}+b v_{2}, v_{3}\right) & =\left(F^{*} \tilde{g}\right)_{p}\left(a v_{1}+b v_{2}, v_{3}\right) & \\
& =d F_{p}^{*}\left(\tilde{g}_{F(p)}\left(a v_{1}+b v_{2}, v_{3}\right)\right) & \\
& =\tilde{g}_{F(p)}\left(d F_{p}\left(a v_{1}+b v_{2}\right), d F_{p}\left(v_{3}\right)\right) & & \\
& =\tilde{g}_{F(p)}\left(a d F_{p}\left(v_{1}\right)+b d F_{p}\left(v_{2}\right), d F_{p}\left(v_{3}\right)\right) & & \text { by linearity of } d F_{p} \\
& =a \tilde{g}_{F}(p)\left(d F_{p}\left(v_{1}\right), d F_{p}\left(v_{3}\right)+b \tilde{g}_{F(p)}\left(d F_{p}\left(v_{2}\right), d F_{p}\left(v_{3}\right)\right)\right. & & \text { by linearity of } \tilde{g} \\
& =a g_{p}\left(v_{1}, v_{3}\right)+b g_{p}\left(v_{2}, v_{3}\right), &
\end{array}
$$

which shows that $g$ is a linear. Since $F$ is an immersion, i.e. its derivative $d F_{p}$ is injective for all $p \in M$, notice that for $v \in T_{p} M$ we have

$$
g_{p}(v, v)=\left(F^{*} \tilde{g}\right)_{p}(v, v)=d F_{p}^{*}\left(\tilde{g}_{F(p)}(v, v)\right)=\tilde{g}_{F(p)}\left(d F_{p}(v), d F_{p}(v)\right) \geq 0
$$

and equality follows if and only if $d F_{p}(v)=0$, i.e. if and only if $v=0$. Therefore $g=F^{*} \tilde{g}$ is an Riemannian metric.

Definition 2.46. A Riemannian submanifold $M$ is an immersed or embedded submanifold of a Riemannian manifold $\widetilde{M}$ with the metric $g=\iota^{*} \tilde{g}$ induced by the inclusion map $\iota: M \hookrightarrow \widetilde{M}$. We call $\widetilde{M}$ the ambient manifold.

[^4]Example 2.47. Consider $\mathbb{S}^{n}:=\left\{x \in \mathbb{R}^{n+1} \mid\|x\|=1\right\} \subseteq \mathbb{R}^{n+1}$. We can equip $\mathbb{S}^{n}$ with the Euclidean metric of $\mathbb{R}^{n+1}$ induced on $\mathbb{S}^{n}$ making it into a Riemannian submanifold.

For $\mathbb{S}^{2}$, we can find a local expression of the metric by using spherical coordinates as in Figure 2. Notice that $\mathbb{S}^{2}$ (except for the semi-circle from the point $(0,0,1)$ to $(0,0,-1)$ in the $x z$-plane) can be parametrized by

$$
x=\sin \theta \cos \varphi, \quad y=\sin \theta \sin \varphi, \quad z=\cos \theta, \quad\{(\theta, \varphi) \mid 0<\theta<\pi, 0<\varphi<2 \pi\}
$$

From this we can derive the basis vectors for $T_{p} \mathbb{S}^{2}$ at a point $p=(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ :

$$
\left.\frac{\partial}{\partial \theta}\right|_{p}=\left(\begin{array}{lll}
\cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta
\end{array}\right),\left.\quad \frac{\partial}{\partial \varphi}\right|_{p}=\left(\begin{array}{ll}
-\sin \theta \sin \varphi & \sin \theta \cos \varphi
\end{array}\right)
$$

This yields the matrix representation of the metric on $T_{p} \mathbb{S}^{2}$ with respect to the basis $\left\{\left.\frac{\partial}{\partial \theta}\right|_{p},\left.\frac{\partial}{\partial \varphi}\right|_{p}\right\}$ given by

$$
g_{p}=\left(\begin{array}{cc}
\left\langle\left.\frac{\partial}{\partial \theta}\right|_{p},\left.\frac{\partial}{\partial \theta}\right|_{p}\right\rangle & \left\langle\left.\frac{\partial}{\partial \theta}\right|_{p},\left.\frac{\partial}{\partial \varphi}\right|_{p}\right\rangle  \tag{2.27}\\
\left\langle\left.\frac{\partial}{\partial \varphi}\right|_{p},\left.\frac{\partial}{\partial \theta}\right|_{p}\right\rangle & \left\langle\left.\frac{\partial}{\partial \varphi}\right|_{p},\left.\frac{\partial}{\partial \varphi}\right|_{p}\right\rangle
\end{array}\right)=\left(\begin{array}{cc}
1 & 0 \\
0 & \sin ^{2} \theta
\end{array}\right) .
$$

Alternatively, we may write

$$
\begin{equation*}
g=d \theta^{2}+\sin ^{2} \theta d \varphi^{2} \tag{2.28}
\end{equation*}
$$



Figure 2: Parametrization of $\mathbb{S}^{2}$.
Given a a Riemannian manifold $(\widetilde{M}, \tilde{g})$ and a smooth submanifold $M \subset \widetilde{M}$, we call a tangent vector $v \in T_{p} \widetilde{M}$ a normal to $M$ if $\tilde{g}(v, w)=0$ for every $w \in T_{p} M$. The set of all vectors normal to $M$ at a point $p$ is the normal space at $p$ denoted by $N_{p} M=\left(T_{p} M\right)^{\perp} \subset T_{p} \widetilde{M}$. In particular, the orthogonal direct sum $T_{p} M \oplus N_{p} M=T_{p} \widetilde{M}$ at each $p \in M$. To conclude the section, we define the normal bundle of $M$ as

$$
\begin{equation*}
N M:=\bigsqcup_{p \in M}\left(\{p\} \times N_{p} M\right) . \tag{2.29}
\end{equation*}
$$

The theory of curvature that is dealt with in Section 2.6 can also be applied to Riemannian submanifolds. We briefly discuss the extend to which a submanifold curves within its ambient manifold, i.e. the second fundamental form of Riemannian submanifolds, in Section A.

### 2.4 Connections

In our quest on defining the Riemannian and Ricci curvature tensors, we firstly need to study connections. Connections allow one to define the equivalent of Euclidean straight lines on Riemannian manifolds, called geodesics, as we will see in Section 2.5. As illustratively described in [21, Section 4], a connection can be seen as a set of rules for taking directional derivatives of vector fields. We are particularly interested in the Levi-Civita connection, which enables us to properly express the properties of a Riemannian metric. This unique connection on a Riemannian manifold must satisfy certain properties that will be discussed throughout the remainder of this section.

Definition 2.48. Let $\pi: E \rightarrow M$ be a smooth vector bundle over a smooth manifold $M$. A connection in $E$ is a map

$$
\nabla: \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E), \quad(X, Y) \mapsto \nabla_{X} Y
$$

satisfying the following properties:
(i) $\nabla_{X} Y$ is linear over $C^{\infty}(M)$ in $X$, i.e. for $f, g \in C^{\infty}(M)$ and $X, Z \in \mathfrak{X}(M)$,

$$
\begin{equation*}
\nabla_{f X+g Z} Y=f \nabla_{X} Y+g \nabla_{Z} Y \tag{2.30}
\end{equation*}
$$

(ii) $\nabla_{X} Y$ is linear over $\mathbb{R}$ in $Y$, i.e. for $a, b \in \mathbb{R}$ and $Y, Z \in \Gamma(E)$,

$$
\begin{equation*}
\nabla_{X}(a Y+b Z)=a \nabla_{X} Y+b \nabla_{X} Z \tag{2.31}
\end{equation*}
$$

(iii) $\nabla$ satisfies the product rule, i.e. for $f \in C^{\infty}(M)$,

$$
\begin{equation*}
\nabla_{X}(f Y)=f \nabla_{X} Y+(X f) Y \tag{2.32}
\end{equation*}
$$

We call $\nabla_{X} Y$ the covariant derivative of $Y$ in the direction of $X$.
Remark 2.49. Following [18, Remark 2.26], we could equivalently interpret a connection as the linear map $\nabla: \Gamma(E) \rightarrow \Gamma\left(T^{*} M \otimes E\right)$ which also obeys the product rule.

This definition of a connection is rather broad, and applicable to more geometric structures. In the Riemannian setting, we define a linear connection in the tangent bundle as the map $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow$ $\mathfrak{X}(M)$ that satisfies the conditions of Definition 2.48. Although at first glance one would assume through Lemma 2.34 that $\nabla$ is then a $(1,2)$-tensor field, this is not the case since it is not linear over $C^{\infty}(M)$ in the second argument. If $\left\{\frac{\partial}{\partial x^{1}}, \ldots, \frac{\partial}{\partial x^{n}}\right\}$ is some smooth local frame for $T M$ on $U \subseteq M$ of dimension $n$, we have

$$
\begin{equation*}
\nabla_{\frac{\partial}{\partial x^{i}}} \frac{\partial}{\partial x^{j}}=\Gamma_{i j}^{k} \frac{\partial}{\partial x^{k}}, \tag{2.33}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\Gamma_{i j}^{k}=d x^{k} \nabla_{\frac{\partial}{\partial x^{i}}} \frac{\partial}{\partial x^{j}}, \tag{2.34}
\end{equation*}
$$

with $\Gamma_{i j}^{k}: U \rightarrow \mathbb{R}$ the $n^{3}$ connection coefficients or Christoffel symobls of $\nabla$ for each $i, j, k \in\{1, \ldots, n\}$. For some smooth vector fields $X, Y \in \mathfrak{X}(U)$ written in terms of the frame, i.e. as $X=X^{i} \frac{\partial}{\partial x^{i}}$ and $Y=Y^{j} \frac{\partial}{\partial x^{j}}$, we then have

$$
\begin{array}{rlrl}
\nabla_{X} Y & =\nabla_{X}\left(Y^{j} \frac{\partial}{\partial x^{j}}\right) & \\
& =Y^{j} \nabla_{X} \frac{\partial}{\partial x^{j}}+\left(X Y^{j}\right) \frac{\partial}{\partial x^{j}} & & \text { by equation (2.32) } \\
& =Y^{j} \nabla_{X^{i} \frac{\partial}{\partial x^{i}}} \frac{\partial}{\partial x^{j}}+\left(X Y^{j}\right) \frac{\partial}{\partial x^{j}} & & \\
& =X^{i} Y^{j} \nabla_{\frac{\partial}{\partial x^{i}}} \frac{\partial}{\partial x^{j}}+\left(X Y^{j}\right) \frac{\partial}{\partial x^{j}} & & \text { by equation }(2.30) \\
& =X^{i} Y^{j} \Gamma_{i j}^{k} \frac{\partial}{\partial x^{k}}+\left(X Y^{j}\right) \frac{\partial}{\partial x^{j}} & & \text { by equation }(2.33) \\
& =\left(X^{i} Y^{j} \Gamma_{i j}^{k}+X Y^{k}\right) \frac{\partial}{\partial x^{k}} . & & \text { by interchanging dummy index of the second term } \tag{2.35}
\end{array}
$$

Remark 2.50. From now on, we may write $\nabla_{i}=\nabla_{\frac{\partial}{\partial x^{i}}}$.
As showed in [21, Proposition 4.2], the tangent bundle of every smooth manifolds admits a linear connection. Moreover, a connection in the tangent bundle naturally induces a connection in all tensor bundles over a smooth manifold. Therefore, we can also compute covariant derivatives of tensor fields.

Definition 2.51. A connection in the tensor bundle $T_{s}^{r} M$ is a map $\nabla: \mathfrak{X}(M) \times \mathcal{T}_{s}^{r}(M) \rightarrow \mathcal{T}_{s}^{r}(M)$, such that the following conditions are satisfied:
(i) In $T_{0}^{1} M=T M, \nabla$ agrees with the definition of a linear connection in $T M$.
(ii) In $T_{0}^{0} M=M \times \mathbb{R}$ with $X \in \mathfrak{X}(M)$ and $f \in C^{\infty}(M), \nabla$ is equivalent to regular differentiation of functions. That is,

$$
\begin{equation*}
\nabla_{X} f=X f \tag{2.36}
\end{equation*}
$$

(iii) The connection $\nabla$ obeys the product rule with respect to tensor products. That is, for two smooth tensor fields $F, G$ on $M$, we have

$$
\begin{equation*}
\nabla_{X}(F \otimes G)=\left(\nabla_{X} F\right) \otimes G+F \otimes\left(\nabla_{X} G\right) \tag{2.37}
\end{equation*}
$$

(iv) The connection $\nabla$ commutes with all contractions, i.e. if $\operatorname{tr}$ denotes the trace on any pair of indices $(h, k)$, then

$$
\begin{equation*}
\nabla_{X}\left(\operatorname{tr}_{a}^{b} Y\right)=\operatorname{tr}_{a}^{b}\left(\nabla_{X} Y\right) \tag{2.38}
\end{equation*}
$$

The following lemma shows that a unique connection in each tensor bundle exists for a given linear connection in the tangent bundle.

Lemma 2.52. Let $\nabla$ be a linear connection in $T M$ of a smooth manifold $M$. Then there exist a unique connection in each tensor bundle $T_{s}^{r} M$, also denoted by $\nabla$, that satisfies the conditions of Definition 2.51. Moreover, it satisfies the following properties.
(i) The product rule with respect to the dual pairing between $\omega \in \mathfrak{X}^{*}(M)$ and $Y \in \mathfrak{X}(M)$ is obeyed. That is,

$$
\begin{equation*}
\nabla_{X}(\omega \mid Y)=\left(\nabla_{X} \omega \mid Y\right)+\left(\omega \mid \nabla_{X} Y\right) \tag{2.39}
\end{equation*}
$$

(ii) For all $\tau \in \mathcal{T}_{s}^{r}(M), \omega^{1}, \ldots, \omega^{s} \in \mathfrak{X}^{*}(M)$ and $Y_{1}, \ldots, Y_{r} \in \mathfrak{X}(M)$, we have

$$
\begin{align*}
\left(\nabla_{X} \tau\right)\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}\right)= & X\left(\tau\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}\right)\right) \\
& -\sum_{i=1}^{r} \tau\left(\omega^{1}, \ldots, \nabla_{X} \omega^{i}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}\right)  \tag{2.40}\\
& -\sum_{j=1}^{s} \tau\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, \nabla_{X} Y_{j}, \ldots, Y_{s}\right)
\end{align*}
$$

Proof. Using the properties of Definition 2.51, the proof for (i) follows from a direct computation and by noticing that the dual pairing (in terms of the coordinate basis) $(\omega \mid Y)=\omega_{i} Y^{j}\left(d x^{i} \left\lvert\, \frac{\partial}{\partial x^{j}}\right.\right)=\omega_{i} Y^{j}$ equals $\operatorname{tr}_{1}^{1}(\omega \otimes Y)=\operatorname{tr}_{1}^{1}\left(\omega_{i} d x^{i} \otimes Y^{j} \frac{\partial}{\partial x^{j}}\right)=\omega_{i} Y^{j}$. Hence,

$$
\nabla_{X}(\omega \mid Y)=\nabla\left(\operatorname{tr}_{1}^{1}(\omega \otimes Y)\right)=\operatorname{tr}_{1}^{1}\left(\nabla_{X} \omega \otimes Y+\omega \otimes \nabla_{X} Y\right)=\left(\nabla_{X} \omega \mid Y\right)+\left(\omega \mid \nabla_{X} Y\right)
$$

For (ii), since $\tau \in \mathcal{T}_{s}^{r}(M)$, we have $\tau \otimes \omega^{1} \otimes \cdots \otimes \omega^{r} \otimes Y_{1} \otimes \cdots \otimes Y_{s} \in \mathcal{T}_{2 s}^{2 r}(M)$. Therefore, we can apply the trace operator $r+s$ times to obtain ${ }^{7}$

$$
\begin{aligned}
\nabla_{X}\left(\tau\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}\right)\right)= & \nabla_{X}\left(\operatorname{tr}_{1, \ldots, r, r+1, \ldots, s}^{1, \ldots, r, r+1, \ldots, s}\left(\tau \otimes \omega^{1} \otimes \cdots \otimes \omega^{r} \otimes Y_{1} \otimes \cdots \otimes Y_{s}\right)\right) \\
= & \operatorname{tr}_{1, \ldots, r, r+1, \ldots, s}^{1, \ldots, r, r+1, \ldots, s}\left(\left(\nabla_{X} \tau\right) \otimes \omega^{1} \otimes \cdots \otimes \omega^{r} \otimes Y_{1} \otimes \cdots \otimes Y_{s}\right. \\
& +\sum_{i=1}^{r} \tau \otimes \omega^{1} \otimes \cdots \otimes \nabla_{X} \omega^{i} \otimes \cdots \otimes \omega^{r} \otimes Y_{1} \otimes \cdots \otimes Y_{s} \\
& \left.+\sum_{j=1}^{s} \tau \otimes \omega^{1} \otimes \cdots \otimes \omega^{r} \otimes Y_{1} \otimes \cdots \otimes \nabla_{X} Y_{j} \otimes \cdots \otimes Y_{s}\right] \\
= & \left(\nabla_{X} \tau\right)\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}\right)+\sum_{i=1}^{r} \tau\left(\omega^{1}, \ldots, \nabla_{X} \omega^{i}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}\right) \\
& +\sum_{j=1}^{s} \tau\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, \nabla_{X} Y_{j}, \ldots, Y_{s}\right),
\end{aligned}
$$

where we exploited (iv) of Definition 2.51 in the second step and reversed the trace operator in the final step.

For uniqueness, assume that $\nabla$ indeed satisfies (i) and (ii) and the conditions of Definition 2.51 . Rewriting equation (2.39) shows that the covariant derivative of any smooth covector field $\omega \in \mathfrak{X}^{*}(M)$ can be computed as

$$
\left(\nabla_{X} \omega \mid Y\right)=\nabla_{X}(\omega \mid Y)-\left(\omega \mid \nabla_{X} Y\right) .
$$

We see that the connection on $\omega$ is uniquely determined by the linear connection. Moreover, (ii) shows that the covariant derivative of any tensor field is determined by the covariant derivatives of every smooth covector and vector field. Since these are determined by the linear connection on $T M$ as well, it follows that the connection in everytensor bundle is uniquely determined.

Existence follows by checking that $\nabla_{X} \tau$ is a smooth tensor field, i.e. multilinear over $C^{\infty}(M)$, and that $\nabla$ satisfies the properties of a connection as in Definition 2.48 [21, Proposition 4.15].

It is sometimes convenient to consider the covariant derivative of a tensor field in any direction as a tensor field itself by using Lemma 2.34.

Definition 2.53. The total covariant derivative of a smooth tensor field $\tau \in \mathcal{T}_{s}^{r}(M)$ is the smooth $(r, s+1)$-tensor field on $M$

$$
\nabla \tau: \overbrace{\mathfrak{X}^{*}(M) \times \cdots \times \mathfrak{X}^{*}(M)}^{r \text { times }} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s+1 \text { times }} \rightarrow C^{\infty}(M),
$$

given by

$$
\begin{equation*}
(\nabla \tau)\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}, X\right)=\left(\nabla_{X} \tau\right)\left(\omega^{1}, \ldots, \omega^{r}, Y_{1}, \ldots, Y_{s}\right) \tag{2.41}
\end{equation*}
$$

One could repetitively take total covariant derivatives of tensor fields. That is, given $\tau \in \mathcal{T}_{s}^{r}(M)$ and $X, Y \in \mathfrak{X}(M)$, we can obtain the smooth $(r, s+2)$-tensor field as

$$
\begin{equation*}
\left(\nabla_{X, Y}^{2} \tau\right)\left(\omega^{1}, \ldots, \omega^{r}, Z_{1}, \ldots, Z_{s}\right)=\nabla^{2} \tau\left(\omega^{1}, \ldots, \omega^{r}, Z_{1}, \ldots, Z_{s}, Y, X\right) . \tag{2.42}
\end{equation*}
$$

[^5]Remark 2.54. It is important to notice that $\nabla_{X} \nabla_{Y} \tau \neq \nabla_{X, Y}^{2} \tau$. In fact, we have (see [21, Proposition 4.21])

$$
\begin{equation*}
\nabla_{X, Y}^{2} \tau=\nabla_{X} \nabla_{Y} \tau-\nabla_{\nabla_{X} Y} \tau \tag{2.43}
\end{equation*}
$$

Connections in the tensor bundles can be generalized even more to any arbitrary (tensor product of) vector bundle(s) (recall Remark 2.32). For example, if $\nabla^{(1)}$ is a connection on the dual bundle $E_{1}^{*}, \nabla^{(2)}$ is a connection on the vector bundle $E_{2}$, then there is a unique connection $\nabla$ on $E_{1}^{*} \otimes E_{2}$ such that ${ }^{8}$

$$
\begin{equation*}
\left(\nabla_{X} \tau\right)(Y)=\nabla_{X}^{(2)}(\tau(Y))-\tau\left(\nabla_{X}^{(1)} Y\right), \quad X \in \mathfrak{X}(M), \quad Y \in \Gamma\left(E_{1}\right) \tag{2.44}
\end{equation*}
$$

Moreover, we can also define the pullback connection. If $f: M \rightarrow N$ is a smooth map and $\nabla$ a connection on the vector bundle $E$ over $N$, then there is a unique connection ${ }^{f} \nabla$ on $f^{*} E$ such that ${ }^{9}$

$$
\begin{equation*}
{ }^{f} \nabla_{X}(Y):=\nabla_{f_{*} X} Y, \quad X \in \mathfrak{X}(M), \quad Y \in \Gamma(E) \tag{2.45}
\end{equation*}
$$

We are one step closer to the Levi-Civita connection since we can now define whether a linear connection in the tangent bundle is compatible with the Riemannian metric. There are multiple equivalent ways to do this, but the approach here only requires the tools that we already acquired. As opposed to [24, Proposition 3.2], where compatibility with the metric is defined with parallel vector fields along a smooth curve, we provide the following definition ${ }^{10}$.

Definition 2.55. Let $(M, g)$ be a Riemannian manifold. A Riemannian connection $\nabla$ is a connection on the tangent bundle such that $\nabla g=0$. We say that $\nabla$ is compatible with $g$.
Proposition 2.56. A connection $\nabla$ is Riemannian if and only if

$$
\begin{equation*}
\nabla_{X} g(Y, Z)=g\left(\nabla_{X} Y, Z\right)+g\left(Y, \nabla_{X} Z\right), \quad \forall X, Y, Z \in \mathfrak{X}(M) \tag{2.46}
\end{equation*}
$$

Proof. By equations (2.40) and (2.41), for some smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have

$$
(\nabla g)(Y, Z, X)=\left(\nabla_{X} g\right)(Y, Z)=X(g(Y, Z))-g\left(\nabla_{X} Y, Z\right)-g\left(Y, \nabla_{X} Y\right)
$$

which equals zero if and only if (2.46) holds.
Compatibility with the metric is not enough to define a unique connection on a Riemannian manifold. As in e.g. [23, Theorem 13.9], the additional required condition can be expressed by means of the torsion tensor $T: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ of the connection defined by

$$
\begin{equation*}
T(X, Y):=\nabla_{X} Y-\nabla_{Y} X-[X, Y], \quad \forall X, Y \in \mathfrak{X}(M) \tag{2.47}
\end{equation*}
$$

Alternatively, we can require the connection to be symmetric.
Definition 2.57. A connection $\nabla$ on the tangent bundle $T M$ of a smooth manifold $M$ is symmetric if

$$
\begin{equation*}
\nabla_{X} Y-\nabla_{Y} X=[X, Y], \quad \forall X, Y \in \mathfrak{X}(M) \tag{2.48}
\end{equation*}
$$

Clearly, the connection is symmetric if and only if the torsion tensor vanishes identically. Compatibility with the metric and symmetry are enough to proof existence and uniqueness of the connection on a Riemannian manifold. Hence, we have arrived at the Fundamental Theorem of Riemannian Geometry. The proof of this theorem can be found in e.g. [21, Theorem 5.10]. The result of this theorem is translated in the following definition.

[^6]Definition 2.58. Let $(M, g)$ be a Riemannian manifold. The Levi-Civita connection of $g$ is the unique connection $\nabla$ on $T M$ that is compatible with $g$ and symmetric.

Proposition 2.59. If $\left(U,\left(x^{i}\right)\right)$ is a smooth local coordinate chart of a Riemannian manifold $M$, then the coefficients of its Levi-Civita connection $\nabla$ are given by

$$
\begin{equation*}
\Gamma_{i j}^{k}=\frac{1}{2} g^{k l}\left(\frac{\partial}{\partial x^{i}} g_{j l}+\frac{\partial}{\partial x^{j}} g_{i l}-\frac{\partial}{\partial x^{l}} g_{i j}\right) . \tag{2.49}
\end{equation*}
$$

Proof. Let $X, Y, Z \in \mathfrak{X}(M)$, then by compatibility of $\nabla$ with $g$ we have

$$
\nabla_{X} g(Y, Z)=g\left(\nabla_{X} Y, Z\right)+g\left(Y, \nabla_{X} Z\right)
$$

Since $\nabla$ is symmetric, notice that

$$
g(Y,[X, Z])=g\left(Y, \nabla_{X} Y\right)-g\left(Y, \nabla_{Z} X\right)
$$

and therefore that

$$
\nabla_{X}(g(Y, Z))=g\left(Y, \nabla_{X} Y\right)+g\left(Y, \nabla_{Z} X\right)+g(Y,[X, Z])
$$

We can derive similar expression for $\nabla_{Y} g(Z, X)$ and $\nabla_{Z} g(X, Y)$. By adding $\nabla_{X} g(Y, Z)$ and $\nabla_{Y} g(Z, X)$ and subtracting $\nabla_{Z} g(X, Y)$, we can solve the expression for $g\left(\nabla_{X} Y, Z\right)$ to obtain

$$
g\left(\nabla_{X} Y, Z\right)=\frac{1}{2}\left(\nabla_{X} g(Y, Z)+\nabla_{Y} g(Z, X)-\nabla_{Z} g(X, Y)-g(Y,[X, Z])-g(Z,[Y, X])+g(X,[Z, Y])\right)
$$

Written in terms of the coordinate vector fields, the terms with Lie brackets cancel out by Proposition 2.23. Therefore, the equation above reduces to

$$
g\left(\nabla_{i} \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{l}}\right)=\frac{1}{2}\left(\nabla_{i} g\left(\frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{l}}\right)+\nabla_{j} g\left(\frac{\partial}{\partial x^{l}}, \frac{\partial}{\partial x^{i}}\right)-\nabla_{l} g\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right)\right) .
$$

By equations (2.20) and (2.33), the above equation can be rewritten to

$$
\Gamma_{i j}^{k} g_{k l}=\frac{1}{2}\left(\frac{\partial}{\partial x^{i}} g_{j l}+\frac{\partial}{\partial x^{j}} g_{i l}-\frac{\partial}{\partial x^{l}} g_{i j}\right) .
$$

Multiplying both sides by the inverse matrix $g^{k l}$ gives the desired expression.
A nice property of the Levi-Civita connection that will be used later on, is that it is invariant under (local) isometries.

Proposition 2.60. Let $\varphi:(M, g) \rightarrow(\widetilde{M}, \tilde{g})$ be an isometry between two Riemannian manifolds equipped with Levi-Civita connections $\nabla$ for g and $\widetilde{\nabla}$ for $\tilde{g}$. Then $\varphi^{*} \widetilde{\nabla}=\nabla$, or equivalently,

$$
\begin{equation*}
\varphi_{*}\left(\nabla_{X} Y\right)=\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right), \quad \forall X, Y \in \mathfrak{X}(M) \tag{2.50}
\end{equation*}
$$

Proof. Notice that equation (2.50) is equivalent to

$$
\nabla_{X} Y=\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)\right)
$$

Now let $\varphi^{*} \widetilde{\nabla}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ be the map defined by

$$
\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Y:=\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)\right)
$$

Firstly, we check whether $\varphi^{*} \widetilde{\nabla}$ satisfies the properties of a connection from Definition 2.48. On that account, note that for $f, g \in C^{\infty}(M)$ and $X, Z \in \mathfrak{X}(M)$ the pushforward of vector fields by $\varphi$ yields

$$
\varphi_{*}(f X+g Z)=\left(f \circ \varphi^{-1}\right) \varphi_{*} X+\left(g \circ \varphi^{-1}\right) \varphi_{*} Z
$$

Therefore, if also $Y \in \mathfrak{X}(M)$, then

$$
\begin{aligned}
\left(\varphi^{*} \widetilde{\nabla}\right)_{f X+g Z} Y & =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*}(f X+g Z)}\left(\varphi_{*} Y\right)\right) \\
& =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\left(f \circ \varphi^{-1}\right) \varphi_{*} X+\left(g \circ \varphi^{-1}\right) \varphi_{*} Z}\left(\varphi_{*} Y\right)\right) \\
& =\left(\varphi^{-1}\right)_{*}\left(\left(f \circ \varphi^{-1}\right) \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)+\left(g \circ \varphi^{-1}\right) \widetilde{\nabla}_{\varphi_{*} Z}\left(\varphi_{*} Y\right)\right)
\end{aligned}
$$

where the last step follows from the fact that $\widetilde{\nabla}$ is a connection. By now applying the pushforward $\left(\varphi^{-1}\right)_{*}$ on the vectors $\left(f \circ \varphi^{-1}\right) \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right),\left(g \circ \varphi^{-1}\right) \widetilde{\nabla}_{\varphi_{*} Z}\left(\varphi_{*} Y\right) \in \mathfrak{X}(\widetilde{M})$, we see that

$$
\begin{aligned}
\left(\varphi^{-1}\right)_{*}\left(\left(f \circ \varphi^{-1}\right) \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)+\left(g \circ \varphi^{-1}\right) \widetilde{\nabla}_{\varphi_{*} Z}\left(\varphi_{*} Y\right)\right) & =f\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)\right)+g\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} Z}\left(\varphi_{*} Y\right)\right) \\
& =f\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Y+g\left(\varphi^{*} \widetilde{\nabla}\right)_{Z} Y,
\end{aligned}
$$

since clearly $f \circ \varphi^{-1} \circ \varphi=f$ and $g \circ \varphi^{-1} \circ \varphi=g$. Indeed, $\varphi^{*} \widetilde{\nabla}$ satisfies (i) of Definition 2.48.
Next, we shows that $\varphi^{*} \widetilde{\nabla}$ is linear over $\mathbb{R}$ in $Y$. Let $a, b \in \mathbb{R}$ and $X, Y, Z \in \mathfrak{X}(M)$, then the results is again a direct consequence of $\widetilde{\nabla}$ being a connection on $T \widetilde{M}$ :

$$
\begin{aligned}
\left(\varphi^{*} \widetilde{\nabla}\right)_{X}(a Y+b Z) & =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*}(a Y+b Z)\right)\right) \\
& \left.=\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(a \varphi_{*} Y+b \varphi_{*} Z\right)\right)\right) \\
& =a\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)\right)+b\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Z\right)\right) \\
& =a\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Y+b\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Z,
\end{aligned}
$$

as desired.
For (iii) of Definition 2.48, let $f \in C^{\infty}(M)$ and $X, Y \in \mathfrak{X}(M)$. By following similar reasoning as above, we have

$$
\begin{aligned}
\left(\varphi^{*} \widetilde{\nabla}\right)_{X} f Y & =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} f Y\right)\right) \\
& =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\left(f \circ \varphi^{-1}\right) \varphi_{*} Y\right)\right) \\
& =\left(\varphi^{-1}\right)_{*}\left(\left(f \circ \varphi^{-1}\right) \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)+\left(\varphi_{*} X f\right) Y\right) \\
& =f\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)\right)+\left(\varphi^{-1}\right)_{*}\left(\varphi_{*} X f\right) Y \\
& =f\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Y+(X f) Y,
\end{aligned}
$$

where the last step follows from observing that $\left(\varphi^{-1}\right)_{*}\left(\varphi_{*} X f\right)=d \varphi^{-1} \circ\left(\varphi_{*} X f\right) \circ \varphi=d \varphi^{-1} \circ d \varphi \circ X \circ$ $\varphi^{-1} \circ \varphi=X f$.

The remainder of this proof aims to show that $\varphi^{*} \widetilde{\nabla}$ is compatible with $g$ and symmetric. For the former, recall Proposition 2.56 and Definition 2.44. For $X, Y, Z \in \mathfrak{X}(M)$, we have

$$
\begin{aligned}
\left(\varphi^{*} \widetilde{\nabla}\right)_{X} g(Y, Z) & =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} g(Y, Z)\right)\right) \\
& =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\tilde{g}\left(\varphi_{*} Y, \varphi_{*} Z\right)\right)\right)
\end{aligned}
$$

since $\varphi$ is an isometry. Consequently, since $\widetilde{\nabla}$ is compatible with the metric $\tilde{g}$, we get

$$
\begin{aligned}
\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\tilde{g}\left(\varphi_{*} Y, \varphi_{*} Z\right)\right)\right) & =\left(\varphi^{-1}\right)_{*}\left(\tilde{g}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right), \varphi_{*} Z\right)+\tilde{g}\left(\varphi_{*} Y, \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Z\right)\right)\right) \\
& =\left(\varphi^{-1}\right)_{*} \tilde{g}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right), \varphi_{*} Z\right)+\left(\varphi^{-1}\right)_{*} \tilde{g}\left(\varphi_{*} Y, \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Z\right)\right) \\
& =g\left(\left(\varphi^{-1}\right)_{*} \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right), Z\right)+g\left(Y,\left(\varphi^{-1}\right)_{*} \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Z\right)\right) \\
& =g\left(\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Y, Z\right)+g\left(Y,\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Z\right),
\end{aligned}
$$

as desired. Symmetry then follows from the following computation:

$$
\begin{aligned}
\left(\varphi^{*} \widetilde{\nabla}\right)_{X} Y-\left(\varphi^{*} \widetilde{\nabla}\right)_{Y} X & =\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Y\right)\right)-\left(\varphi^{-1}\right)_{*}\left(\widetilde{\nabla}_{\varphi_{*} Y}\left(\varphi_{*} X\right)\right) \\
& =\left(\varphi^{-1}\right)_{*}\left[\varphi_{*} X, \varphi_{*} Y\right] \\
& =[X, Y] .
\end{aligned}
$$

Indeed, $\varphi^{*} \widetilde{\nabla}$ is a Levi-Civita connection of $g$, and by uniqueness, we must have that $\varphi^{*} \widetilde{\nabla}=\nabla$.
Example 2.61. In continuation of Example 2.47, the nonzero coefficients of the Levi-Civita connection $\nabla$ for $\mathbb{S}^{2}$ are given by

$$
\begin{aligned}
\Gamma_{\varphi \varphi}^{\theta} & =\frac{g^{\theta \theta}}{2}\left(\frac{\partial}{\partial \varphi} g_{\varphi \theta}+\frac{\partial}{\partial \varphi} g_{\varphi \theta}-\frac{\partial}{\partial \theta} g_{\varphi \varphi}\right) \\
& =\frac{1}{2}(0+0-2 \sin \theta \cos \theta) \\
& =-\sin \theta \cos \theta
\end{aligned}
$$

and (after similar computations)

$$
\Gamma_{\theta \varphi}^{\varphi}=\Gamma_{\varphi \theta}^{\varphi}=\frac{\cos \theta}{\sin \theta}
$$

### 2.5 Geodesics and normal coordinates

Geodesics are the equivalent of Euclidean straight lines on Riemannian manifolds. As pointed out in [21, Chapter 4], it turned out to be the most useful way to define them as curves with zero acceleration. To define them, we need another interpretation of a connection. Instead of interpreting the covariant derivative as a rule for taking directional derivatives of vector fields, we instead need a way to differentiate vector fields along curves. For our purpose, we do not necessarily need geodesics. However, we do need them to define the exponential map, which in turn will be used to define normal coordinates. Normal coordinates will come out handy for proving various properties in the next chapter. By keeping this in mind, we will keep this section relatively short and straight to the point. Therefore, we will leave out any proofs and refer to [21] when desired.

For the next couple of definitions, we are not necessarily working on a Riemannian manifold with its Levi-Civita connection, but we move slightly back to our ordinary smooth manifold $M$ with its linear connection on the tangent bundle $\nabla$. To begin with, let us define smooth vector fields along a curve.

Definition 2.62. Let $\gamma: \mathbb{R} \supset I \rightarrow M$ a smooth curve. A smooth vector field along $\gamma$ is a smooth map $V: I \rightarrow T M$ such that $V(t) \in T_{\gamma(t)} M$ for all $t \in I$. The space of all smooth vector fields along $\gamma$ is denoted by $\mathfrak{X}(\gamma)$.

More generally, we can also define smooth tensor fields along a curve $\gamma$ in a similar way. If $\widetilde{V}: M \subseteq$ $U \rightarrow T U \subseteq T M$ is a smooth vector field such that $\gamma(I) \subseteq U$, then we can define $V: I \rightarrow M$ by setting $V(t)=\widetilde{V}_{\gamma(t)}$ for all $t \in I$. Since then $V=\widetilde{V} \circ \gamma$, it is clearly smooth. If such a smooth vector field $\widetilde{V}$ exists on a neighbourhood of $\gamma(I)$, we say that $V$ is extendible.

As shown in [21, Theorem 4.24], a connection $\nabla$ in $T M$ determines a unique operator $D_{t}: \mathfrak{X}(\gamma) \rightarrow$ $\mathfrak{X}(\gamma)$, which we call the covariant derivative along $\gamma$. This map is linear over $\mathbb{R}$, satisfies the product rule, and has the property that $D_{t} V(t)=\nabla_{\gamma^{\prime}(t)} \tilde{V}$ if $V$ is an extendible vector field along $\gamma$. With this map we can define the acceleration of $\gamma$ and consequently geodesics.
Definition 2.63. The acceleration of a smooth curve $\gamma: I \rightarrow M$ is the smooth vector field $\mathfrak{X}(\gamma) \ni D_{t} \gamma^{\prime}$ along $\gamma$.
Definition 2.64. A smooth curve $\gamma: I \rightarrow M$ is a geodesic (with respect to $\nabla$ ) if its acceleration is zero, i.e. if $D_{t} \gamma^{\prime} \equiv 0$.

Alternatively, one could characterize $\gamma$ as a geodesic if its components satisfy the geodesic equation. That is, if $\left(x^{i}\right)$ are smooth local coordinates such that $\gamma^{i}=x^{i} \circ \gamma(t)$, then

$$
\begin{equation*}
\ddot{\gamma}^{k}+\Gamma_{i j}^{k} \dot{\gamma}^{i} \dot{\gamma}^{j}=0 \tag{2.51}
\end{equation*}
$$

By using theory of ordinary differential equations, one could proof that for every $p \in M, v \in T_{p} M$ and $t_{0} \in \mathbb{R}$, there exists an open interval $I \subseteq \mathbb{R}$ and a unique geodesic $\gamma: I \rightarrow M$ such that $\gamma\left(t_{0}\right)=p$ and $\gamma^{\prime}\left(t_{0}\right)=v$ (see [21, Theorem 4.27]). We say that a geodesic $\gamma: I \rightarrow M$ is maximal, if there does not exists a geodesic $\widetilde{\gamma}: \widetilde{I} \rightarrow M$ such that $\left.\widetilde{\gamma}\right|_{I}=\gamma$. Moreover, for each $p \in M$ and $v \in T_{p} M$ there exists a unique maximal such that $\gamma(0)=p$ and $\gamma^{\prime}(0)=v$ (see [21, Corollary 4.28]). Alternatively, we call this geodesic the geodesic with initial point $p$ and initial velocity $v$ and denote it by $\gamma_{v}$.

In the remainder of this section, we again consider Riemannian manifolds ( $M, g$ ) endowed with its Levi-Civita connection (although the theory is not exclusively applicable to the Riemannian setting). Despite the fact that maximal geodesics are unique, we can relate one another with proportional initial velocities due to the rescaling lemma [21, Lemma 5.8]. Moreover, consider the following definition.
Definition 2.65. Let $\gamma_{v}$ be the unique maximal geodesic satisfying $\gamma_{v}(0)=p \in M$ and $\gamma_{v}^{\prime}(0)=v \in T_{p} M$. The exponential map at $p$ is the map $\exp _{p}: T_{p} M \rightarrow M$ defined by

$$
\begin{equation*}
\exp _{p}(v):=\gamma_{v}(1) \tag{2.52}
\end{equation*}
$$

Remark 2.66. A complete manifold or geodesically complete manifold ${ }^{11}$ is a Riemannian manifold for which every maximal geodesic is defined for all $t \in \mathbb{R}$. Equivalently, the exponential map at $p$ has all of $T_{p} M$ as its domain. From now on, we always assume that a Riemannian manifold $(M, g)$ is geodesically complete.

Clearly, since for $0 \in T_{p} M$ we have $\gamma_{0}(t)=p \in M$ for all $t \in I$, we have that $\exp _{p}(0)=\gamma_{0}(1)=p$. Geometrically, $\exp _{p}(v)$ is the point in $M$ obtained by following the geodesic that passes through $p$ for a unit time in the direction of $v$ [24, Section 3.2]. This map is in fact smooth (see [20, Proposition 5.7]) and for each $p \in M$ its differential is the identity map of $T_{p} M$ [21, Proposition 5.19]. Therefore, the differential is invertible, such that the inverse function theorem (see [22, Thoerem 2.8.7]) guarantees the existence of a star-shaped ${ }^{12}$ neighbourhood $V$ of $0 \in T_{p} M$ and a neighbourhood $U$ about $p \in M$ such that $\exp _{p}: T_{p} M \supseteq V \rightarrow U \subseteq M$ is a diffeomorphism.

We call the neighbourhood $U$ of $p$ a normal neighbourhood of $p$. Now, an orthonormal basis $\left\{\left.\left.\frac{\partial}{\partial x^{2}}\right|_{p} \right\rvert\, i \in\{1, \ldots, n\}\right\}$ for $T_{p} M$ induces a basis isomorphism $B: \mathbb{R}^{n} \rightarrow T_{p} M$ by $B\left(r^{1}, \ldots, r^{n}\right)=$ $\left.r^{i} \frac{\partial}{\partial x^{i}}\right|_{p}$. For a normal neighbourhood $U=\exp _{p}(V)$ of $p$, one could then combine this isomorphism with the exponential map to obtain a (unique, see [21, Proposition 5.23]) smooth coordinate map $\varphi=$ $\left(x^{1}, \ldots, x^{n}\right):=B^{-1} \circ\left(\left.\exp _{p}\right|_{V}\right)^{-1}: U \rightarrow \mathbb{R}^{n}$. We call these coordinates (Riemannian) normal coordinates centred at $p$.

The normal coordinates will be very useful in the next chapter due to the following proposition.

[^7]Proposition 2.67. Let $(M, g)$ be a Riemannian manifold and let $\left(U,\left(x^{i}\right)\right)$ be any normal coordinate chart centered at $p \in M$. Then
(i) in coordinates we have that $p=(0, \ldots, 0)$;
(ii) the components of the metric at $p$ are $g_{i j}(p)=\delta_{i j}$;
(iii) the Christoffel symbols vanish at $p$, i.e. $\Gamma_{i j}^{k}(p)=0$;
(iv) the first partial derivatives of the metric vanish at $p$, i.e. $\frac{\partial}{\partial x^{k}} g_{i j}(p)$.

Proof. See [21, Proposition 5.24].
Remark 2.68. Since the Christoffel symbols vanish at $p$ in a normal coordinate system, from equation (2.33) we see that then $\left.\nabla_{i} \frac{\partial}{\partial x^{j}}\right|_{p}=0$. In the light of Lemma 2.52, this implies that for $\tau \in \mathcal{T}_{s}^{r}(M)$ we have ${ }^{13}$

$$
\left(\nabla_{k} \tau_{p}\right)\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right)=\frac{\partial}{\partial x^{k}} \tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) .
$$

### 2.6 Curvature

With the Levi-Civita connection at hands, we are ready to examine curvature. Informally, we can say that curvature measures the distance between a Riemannian manifold and Euclidean space. If this distance at a point is non-zero, that means the space at this point is curved. In this sense, curvature enables us to express the extent to which a space is curved.

Definition 2.69. Suppose that $(M, g)$ is a $n$-dimensional Riemannian manifold equipped with its LeviCivita connection $\nabla$. The Riemann curvature endomorphism is the map $R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow$ $\mathfrak{X}(M)$ defined by

$$
\begin{equation*}
R(X, Y) Z:=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z . \tag{2.53}
\end{equation*}
$$

Since the Levi-Civita connection is symmetric, it follows from equations (2.48) and (2.43) that

$$
\begin{align*}
R(X, Y) Z & =\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z \\
& =\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{\left(\nabla_{X} Y-\nabla_{Y} X\right)} Z \\
& =\nabla_{X, Y}^{2} Z-\nabla_{Y, X}^{2} Z \tag{2.54}
\end{align*}
$$

By using Lemma 2.34, one could derive that $R$ is a (1,3)-tensor field [21, Proposition 7.3]. Moreover, using our smooth local frame and coframe for a coordinate neighbourhood $U \subseteq M$ at a point $p \in M$ with local coordinates $\left(x^{i}\right)$, the curvature endomorphism can be expressed as the tensor

$$
\begin{equation*}
R_{p}=\left.\left.\left.\left.R_{i j k}^{l}(p) d x^{i}\right|_{p} \otimes d x^{j}\right|_{p} \otimes d x^{k}\right|_{p} \otimes \frac{\partial}{\partial x^{l}}\right|_{p} \tag{2.55}
\end{equation*}
$$

with the coefficients $R_{i j k}^{l}$ given by

$$
\begin{equation*}
\left.R_{p}\left(\left.\frac{\partial}{\partial x^{i}}\right|_{p},\left.\frac{\partial}{\partial x^{j}}\right|_{p}\right) \frac{\partial}{\partial x^{k}}\right|_{p}=\left.R_{i j k}^{l}(p) \frac{\partial}{\partial x^{l}}\right|_{p} . \tag{2.56}
\end{equation*}
$$

[^8]Proposition 2.70. If $\left(U,\left(x^{i}\right)\right)$ is a smooth local coordinate chart of a Riemannian manifold $M$, then we can express the coefficients of the Riemann curvature endomorphism as

$$
\begin{equation*}
R_{i j k}^{l}=\frac{\partial}{\partial x^{i}} \Gamma_{j k}^{l}-\frac{\partial}{\partial x^{j}} \Gamma_{i k}^{l}+\Gamma_{j k}^{m} \Gamma_{i m}^{l}-\Gamma_{i k}^{m} \Gamma_{j m}^{l} . \tag{2.57}
\end{equation*}
$$

Proof. In terms of smooth coordinate vector fields, we use equation (2.33) to directly compute

$$
\begin{aligned}
R_{i j k}^{l} \frac{\partial}{\partial x^{l}} & =\nabla_{i} \nabla_{j} \frac{\partial}{\partial x^{k}}-\nabla_{j} \nabla_{i} \frac{\partial}{\partial x^{k}}-\nabla_{[i, j]} \frac{\partial}{\partial x^{k}} \\
& =\nabla_{i}\left(\Gamma_{j k}^{l} \frac{\partial}{\partial x^{l}}\right)-\nabla_{j}\left(\Gamma_{i k}^{l} \frac{\partial}{\partial x^{l}}\right) \\
& =\frac{\partial}{\partial x^{i}} \Gamma_{j k}^{l} \frac{\partial}{\partial x^{l}}-\frac{\partial}{\partial x^{j}} \Gamma_{i k}^{l} \frac{\partial}{\partial x^{l}}+\Gamma_{j k}^{m} \nabla_{i} \frac{\partial}{\partial x^{m}}-\Gamma_{i k}^{m} \nabla_{j} \frac{\partial}{\partial x^{m}} \\
& =\left(\frac{\partial}{\partial x^{i}} \Gamma_{j k}^{l}-\frac{\partial}{\partial x^{j}} \Gamma_{i k}^{l}+\Gamma_{j k}^{m} \Gamma_{i m}^{l}-\Gamma_{i k}^{m} \Gamma_{j m}^{l}\right) \frac{\partial}{\partial x^{l}} .
\end{aligned}
$$

Example 2.71. Recall the Christoffel symbols for $\mathbb{S}^{2}$ of Example 2.61. The nonzero coefficients of the Riemann curvature endomorphism are given by

$$
\begin{aligned}
R_{\varphi \theta \varphi}^{\theta} & =\frac{\partial}{\partial \theta}\left(\Gamma_{\varphi \varphi}^{\theta}\right)+\Gamma_{\varphi \varphi}^{\theta} \Gamma_{\varphi \theta}^{\varphi} \\
& =\sin ^{2} \theta-\cos ^{2} \theta+\cos ^{2} \theta \\
& =\sin ^{2} \theta
\end{aligned}
$$

and (by similar computations)

$$
R_{\varphi \varphi \theta}^{\theta}=-\sin ^{2} \theta, \quad R_{\theta \varphi \theta}^{\varphi}=-1, \quad R_{\theta \theta \varphi}^{\varphi}=1
$$

Similarly to Lemma 2.52 where we established how a connection acts on tensor fields, we are interested in how the curvature endomorphism acts on a tensor bundle $T_{s}^{r}(M)$.
Lemma 2.72. Let $R$ be the Riemann curvature endomorphism on the tensor bundle $T_{s}^{r}(M)$. If $\tau \in$ $\mathcal{T}_{s}^{r}(M), \omega^{1}, \ldots, \omega^{r} \in \mathfrak{X}^{*}(M)$ and $Z_{1}, \ldots, Z_{s} \in \mathfrak{X}(M)$, then

$$
\begin{align*}
(R(X, Y) \tau)\left(\omega^{1}, \ldots, \omega^{r}, Z_{1}, \ldots, Z_{s}\right)= & R(X, Y)\left(\tau\left(\omega^{1}, \ldots, \omega^{r}, Z_{1}, \ldots, Z_{s}\right)\right) \\
& -\sum_{i=1}^{r} \tau\left(\omega^{1}, \ldots, R(X, Y) \omega^{i}, \ldots, \omega^{r}, Z_{1}, \ldots, Z_{k}\right)  \tag{2.58}\\
& -\sum_{j=1}^{s} \tau\left(\omega^{1}, \ldots, \omega^{r}, Z_{1}, \ldots, R(X, Y) Z_{j}, \ldots, Z_{s}\right)
\end{align*}
$$

Proof. The proof follows from first showing that $R$ obeys the product rule with respect to tensor products and commutes with all contractions. By using these facts and following similar argumentation as in the proof of Lemma 2.52, one obtains the desired equality. For more detail, see [18, Proposition 2.43].

Definition 2.73. The Riemann curvature tensor is the ( 0,4 )-tensor field $R m$ obtained from the ( 1,3 )Riemann curvature endomorphism by lowering its last index. That is, $R^{b}=R m: \mathfrak{X}(M) \times \mathfrak{X}(M) \times$ $\mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^{\infty}(M)$, defined by

$$
\begin{equation*}
R m(X, Y, Z, W):=g(R(X, Y) Z, W) \tag{2.59}
\end{equation*}
$$

Remark 2.74. Throughout literature, the Riemann curvature tensor is sometimes just defined as in Definition 2.69. The curvature tensor, as defined above, is used when convenient.

In terms of smooth local coordinates $\left(x^{i}\right)$, the Riemann curvature tensor assigns a tensor to each point $p \in M$ as

$$
\begin{equation*}
R m_{p}=\left.\left.\left.\left.R m_{i j k l}(p) d x^{i}\right|_{p} \otimes d x^{j}\right|_{p} \otimes d x^{k}\right|_{p} \otimes d x^{l}\right|_{p} \tag{2.60}
\end{equation*}
$$

with the coefficients $R m_{i j k l}=g_{l m} R_{i j k}^{m}$. Therefore, similarly to equation (2.57), we can express these coefficients using smooth local coordinates $\left(x^{i}\right)$ as

$$
\begin{equation*}
R m_{i j k l}=g_{l m}\left(\frac{\partial}{\partial x^{i}} \Gamma_{j k}^{m}-\frac{\partial}{\partial x^{j}} \Gamma_{i k}^{m}+\Gamma_{j k}^{p} \Gamma_{i p}^{m}-\Gamma_{i k}^{p} \Gamma_{j p}^{m}\right) . \tag{2.61}
\end{equation*}
$$

What makes the Riemann curvature tensor specifically valuable, is that it enables one to define curvature intrinsically. This means that it can be measured for Riemannian manifolds that are not necessarily embedded in an ambient Euclidean space. The Riemann curvature tensor assigns a tensor to each point of a manifold and formulates the extent to which the Riemannian metric is not locally isometric to the Euclidean metric. Since Euclidean space is flat, the curvature is zero. Therefore, we say that a Riemannian manifold is flat if

$$
\begin{equation*}
\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z=\nabla_{[X, Y]} Z, \quad \forall X, Y, Z \in \mathfrak{X}(M) \tag{2.62}
\end{equation*}
$$

Just as the Levi-Civita connection, the Riemann curvature tensor is invariant under isometries [18, Theorem 4.1].

Proposition 2.75. The Riemann curvature tensor $R m$ is invariant under local isometries. That is, if $\varphi:(M, g) \rightarrow(\widetilde{M}, \tilde{g})$ is a local isometry, then $\varphi^{*} \widehat{R m}=R m$.
Proof. The proof follows from a direct computation using Definitions 2.69 and 2.73 and Proposition 2.60. Let $X, Y, Z, W \in \mathfrak{X}(M)$, then

$$
\begin{aligned}
\left(\varphi^{*} \widetilde{R m}\right)(X, Y, Z, W) & =\widetilde{R m}\left(\varphi_{*} X, \varphi_{*} Y, \varphi_{*} Z, \varphi_{*} W\right) \\
& =\tilde{g}\left(R\left(\varphi_{*} X, \varphi_{*} Y\right) \varphi_{*} Z, \varphi_{*} W\right) \\
& =\tilde{g}\left(\widetilde{\nabla}_{\varphi_{*} X} \widetilde{\nabla}_{\varphi_{*} Y}\left(\varphi_{*} Z\right)-\widetilde{\nabla}_{\varphi_{*} Y} \widetilde{\nabla}_{\varphi_{*} X}\left(\varphi_{*} Z\right)-\widetilde{\nabla}_{\left[\varphi_{*} X, \varphi_{*} Y\right]}\left(\varphi_{*} Z\right), \varphi_{*} W\right) \\
& =\tilde{g}\left(\varphi_{*}\left(\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z\right), \varphi_{*} W\right) \\
& =\left(\varphi^{*} \tilde{g}\right)(R(X, Y) Z, W) \\
& =R m(X, Y, Z, W) .
\end{aligned}
$$

Example 2.76. The coefficients of the Riemann curvature tensor on $\mathbb{S}^{2}$ are easily obtained from the curvature endomorphism coefficients:

$$
\begin{array}{ll}
R m_{\theta \varphi \theta \varphi}=g_{\varphi \varphi} R_{\theta \varphi \theta}^{\varphi}=-\sin ^{2} \theta, & \\
R m_{\theta \theta \varphi \varphi}=g_{\varphi \varphi} R_{\theta \theta \varphi}^{\varphi}=\sin ^{2} \theta \\
R m_{\varphi \theta \varphi \theta}=g_{\theta \theta} R_{\varphi \theta \varphi}^{\theta}=\sin ^{2} \theta, & \\
R m_{\varphi \varphi \theta \theta}=g_{\theta \theta} R_{\varphi \varphi \theta}^{\theta}=-\sin ^{2} \theta
\end{array}
$$

The Riemann curvature tensor satisfies multiple symmetries which often come out useful.
Lemma 2.77. Let $(M, g)$ be a Riemannian manifold. The ( 0,4 )-Riemann curvature tensor of $g$ satisfies the following symmetries for all $W, X, Y, Z \in \mathfrak{X}(M)$ :
(i) $\operatorname{Rm}(W, X, Y, Z)=-\operatorname{Rm}(X, W, Y, Z)$;
(ii) $\operatorname{Rm}(W, X, Y, Z)=-\operatorname{Rm}(W, X, Z, Y)$;
(iii) $\operatorname{Rm}(W, X, Y, Z)=-\operatorname{Rm}(X, W, Y, Z)$;
(iv) $\operatorname{Rm}(W, X, Y, Z)+\operatorname{Rm}(X, Y, W, Z)+\operatorname{Rm}(Y, W, X, Z)=0$. This is also referred to as the first Bianchi identity.

In addition, the total covariant derivative of the Riemann curvature tensor satisfies the second Bianchi identity:

$$
\begin{equation*}
\nabla R m(X, Y, Z, V, W)+\nabla R m(X, Y, V, W, Z)+\nabla R m(X, Y, W, Z, V)=0, \quad \forall X, Y, Z, V, W \in \mathfrak{X}(M) \tag{2.63}
\end{equation*}
$$

Proof. The proofs of these symmetries are not of particular interest, and we refer to e.g. [21, Proposition 7.12] for the details.

The penultimate tensor that we will look at in this section is the Ricci curvature tensor. Its geometric meaning will be thoroughly investigated in the next section. For now, we will leave it with the definition. Although it does not carry all the information of the curvature endomorphism and tensor, it is a much simpler object.

Definition 2.78. The Ricci curvature or Ricci tensor denoted by Ric is the covariant ( 0,2 )-tensor field defined as the $(1,1)$-trace of the curvature endomorphism. That is, the map Ric: $\mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow$ $C^{\infty}(M)$, defined by

$$
\begin{equation*}
\operatorname{Ric}(X, Y):=\operatorname{tr}_{1}^{1}(Z \mapsto R(Z, X) Y) \tag{2.64}
\end{equation*}
$$

For smooth local coordinates $\left(x^{i}\right)$ about $p \in M$, we have

$$
\begin{equation*}
\operatorname{Ric} c_{p}=\left.\left.\operatorname{Ric} c_{i j}(p) d x^{i}\right|_{p} \otimes d x^{j}\right|_{p} \tag{2.65}
\end{equation*}
$$

The coefficients of the Ricci tensor can be conveniently obtained from the Riemann curvature endomorphism and tensor:

$$
\begin{equation*}
R i c_{i j}=R_{k i j}^{k}=g^{k m} R m_{k i j m} \tag{2.66}
\end{equation*}
$$

Proposition 2.79. The Ricci curvature Ric is invariant under local isometries. That is, if $\varphi:(M, g) \rightarrow$ $(\widetilde{M}, \tilde{g})$ is a local isometry, then $\varphi^{*} \widetilde{\text { Ric }}=$ Ric.
Proof. The proof follows from Proposition 2.75. Observe that

$$
\begin{aligned}
\varphi^{*} \widetilde{\operatorname{Ric}}\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) & =\widetilde{\operatorname{Ric}}\left(\varphi_{*} \frac{\partial}{\partial x^{i}}, \varphi_{*} \frac{\partial}{\partial x^{j}}\right) \\
& =\widetilde{g^{-1}}\left(\varphi_{*} d x^{k}, \varphi_{*} d x^{m}\right) \widetilde{R m}\left(\varphi_{*} \frac{\partial}{\partial x^{k}}, \varphi_{*} \frac{\partial}{\partial x^{i}}, \varphi_{*} \frac{\partial}{\partial x^{j}}, \varphi_{*} \frac{\partial}{\partial x^{m}}\right) \\
& =\varphi^{*} \widetilde{g^{-1}}\left(d x^{k}, d x^{m}\right) \varphi^{*} \widetilde{R m}\left(\frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{m}}\right) \\
& =g^{-1}\left(d x^{k}, d x^{m}\right) \operatorname{Rm}\left(\frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{m}}\right) \\
& =\operatorname{Ric}\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) .
\end{aligned}
$$

Example 2.80. The Ricci coefficients for $\mathbb{S}^{2}$ are found by applying (i) of Lemma 2.77 on equation (2.66):

$$
\operatorname{Ric}_{\theta \theta}=g^{\varphi \varphi} R_{\varphi \theta \theta \varphi}=-g^{\varphi \varphi} R_{\theta \varphi \theta \varphi}=\frac{1}{\sin ^{2} \theta} \sin ^{2} \theta=1, \quad \operatorname{Ric}_{\varphi \varphi}=g^{\theta \theta} R_{\theta \varphi \varphi \theta}=-g^{\theta \theta} R_{\varphi \theta \varphi \theta}=\sin ^{2} \theta
$$

Finally, we define the scalar curvature as the trace with respect to $g$ of the Ricci curvature.
Definition 2.81. The scalar curvature $S: C^{\infty}(M) \rightarrow \mathbb{R}$ is the map defined by

$$
\begin{equation*}
S:=\operatorname{tr}_{g}(\text { Ric }) \tag{2.67}
\end{equation*}
$$

Notice that by (2.26), we have $\operatorname{tr}_{g}(R i c)=g^{i j} R i c_{i j}$.
Example 2.82. The scalar curvature for $\mathbb{S}^{2}$ is given by

$$
S=g^{i j} R i c_{i j}=1+\frac{1}{\sin ^{2} \theta} \sin ^{2} \theta=2 .
$$

## 3 Ricci flow

Informally, one could think of Ricci flow as the process of stretching or contacting the metric depending on the Ricci curvature. If the curvature is negative, the metric is stretched, and for positive curvature the metric is contracted. Moreover, the stronger the curvature, i.e. the 'more negative' or 'more positive', the faster the stretching or contracting occurs. The stretching and contracting can be thought of as a change in the distance between each pair of points of a manifold. In this way, Ricci flow has shown to be a useful tool for 'improving' or 'smoothing' the geometry of a manifold whilst preserving any present symmetries. However, in this process, Ricci flow is likely to develop singularities in finite time. Perelman developed a technique called 'surgery' to properly deal with these singularities, which helped him to prove the famous Poincaré conjecture in his series of papers $[4,5,6]$.

Despite the occurrence of singularities, Ricci flow is nevertheless a useful tool to understand the topology of a manifold. Although we will not focus on performing surgery on manifolds with singularities, we will investigate how Ricci flow can be used to 'improve' a manifold's geometry and discuss the shorttime existence and uniqueness of a solution.

Mathematically, Ricci flow allows the metric $g$ of a Riemannian manifold to evolve under the PDE

$$
\begin{equation*}
\frac{\partial}{\partial t} g(t)=-2 \operatorname{Ric}(g(t)), \quad g(0)=g_{0} \tag{3.1}
\end{equation*}
$$

From now on, we are therefore not interested in a single Riemannian manifold $(M, g)$, but rather in a family of closed manifolds $(M, g(t))$ with $t \in[0, \epsilon] \subset \mathbb{R}[18$, Section 4.1]. This chapter begins with providing the necessary intuition for what it actually means to take time derivatives of the metric. Furthermore, we discuss various important properties that we will use to eventually prove short-time existence and uniqueness. To start off, let us solve the Ricci flow on $\mathbb{S}^{2}$.
Example 3.1. In our previous examples, we examined the various curvature tensors on $\mathbb{S}^{2}$ with radius $r=1$. Now, since we consider a family of metrics dependent of time, we consider a radius $r$ dependent of time. The matrix representation on $T_{p} \mathbb{S}_{r(t)}^{2}$ is then

$$
g(t)=r^{2}(t)\left(\begin{array}{cc}
1 & 0  \tag{3.2}\\
0 & \sin ^{2} \theta
\end{array}\right)=: r^{2}(t) g_{0}
$$

One could derive that the coefficients for the Ricci tensor remain unchanged, and are hence independent of the radius such that Ric $=g_{0}$. The Ricci flow for $\mathbb{S}^{2}$ is hence given by

$$
\begin{equation*}
\frac{\partial}{\partial t} r^{2}(t) g_{0}=-2 g_{0}, \quad g(0)=r(0) g_{0}=g_{0} \tag{3.3}
\end{equation*}
$$

where we assume that the initial radius is $r(0)=1$. By taking the time derivatives on the left hand side, this becomes

$$
2 \dot{r} r g_{0}+r^{2}(t) \frac{\partial}{\partial t} g_{0}=2 \dot{r} r g_{0}+0
$$

and (3.3) becomes

$$
\begin{aligned}
2 \dot{r} r g_{0}=-2 g_{0} & \Longrightarrow \dot{r} r=-1 \\
& \Longrightarrow \int_{1}^{r} r d r=-\int_{0}^{1} 1 d t \\
& \Longrightarrow \frac{r^{2}(t)}{2}-\frac{1}{2}=-t \\
& \Longrightarrow r(t)=\sqrt{1-2 t} .
\end{aligned}
$$

Hence, we see that the solution to the Ricci flow is given by $g(t)=(1-2 t) g_{0}$. So $\mathbb{S}^{2}$ shrinks to a point as $t \rightarrow \frac{1}{2}$. In fact, one could show that any unit sphere $\left(\mathbb{S}^{n}, g_{0}\right)$ shrinks to a point since then $g(t)=(1-2(n-1) t) g_{0}[18$, Section 3.1.1.1].


Figure 3: The sphere $\mathbb{S}^{2}$ shrinks to a point as $t \rightarrow \frac{1}{2}$. In the figure, the sphere is given at $t=0, t=\frac{5}{18}$ and $t=\frac{4}{9}$.

Remark 3.2. In the above example, note that we are now dealing with manifolds without a fixed volume. To preserve the volume, one should consider normalized Ricci flow as in e.g. [28, Thereom 1.1-1.3] or [29, Theorem 1.2]. Since the scalar curvature equals twice the Gaussian curvature $K$ and Ric $=K g$ in two dimensions ${ }^{14}$, the normalized Ricci flow is then given by

$$
\begin{equation*}
\frac{\partial}{\partial t} g(t)=(s-S(t)) g(t), \quad g(0)=g_{0} \tag{3.4}
\end{equation*}
$$

where $s$ denotes the average scalar curvature of $S$. Explicitly, $s$ is given by ${ }^{15}$

$$
s=\frac{4 \pi \chi(M)}{\operatorname{vol}(M)}
$$

Proving (short-time) existence and uniqueness of solutions to (normalized) Ricci flow on compact twodimensional Riemannian manifolds is significantly easier (see e.g. [30]). This is because in two dimensions, the deformation of the metric is conformal ${ }^{16}$, and a solution can be written $g(t)=e^{v(t)} g_{0}$, with $v$ : $M \times[0, \epsilon) \rightarrow \mathbb{R}$. If the dimension $n \geq 3$, then the Ricci flow becomes a weakly parabolic equation. In Section 3.3, we proof short-time existence and uniqueness for dimension $n \geq 3$.

### 3.1 Time derivatives and deformations of geometric quantities

To begin with, it is necessary to establish how one takes time derivatives of the metric and Levi-Civita connection. Also, we are interested in how the various curvature tensors vary under the evolving metric, i.e. not necessarily under Ricci flow. This section is mainly based upon [18, Chapter 4], if not mentioned otherwise. As stated before, we are now more interested in a closed Riemannian manifold $M$ equipped with a one-parameter family of metrics $t \mapsto g(t)$, assuming these exists for some time interval $[0, \epsilon]$. We can consider the elements $g(t)$ as sections of the smooth rank-2 positive definite symmetric bundle $\Sigma^{2}\left(T^{*} M\right)$, i.e. $g(t) \in \Gamma\left(\Sigma^{2}\left(T^{*} M\right)\right)$ [18, Section 4.1.1]. Then we define the derivate of $g(t)$ as follows:

Definition 3.3. Consider the one-parameter family of smooth metrics $g=g(t) \in \Gamma\left(\Sigma^{2}\left(T^{*} M\right)\right)$ parametrised by 'time' $t$. We define the time derivative of the metric as the map $\frac{\partial}{\partial t} g: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^{\infty}(M)$ defined by

$$
\begin{equation*}
\left(\frac{\partial}{\partial t} g\right)(X, Y):=\frac{\partial}{\partial t} g(X, Y) \tag{3.5}
\end{equation*}
$$

for any pair of time independent vector fields $X, Y \in \mathfrak{X}(M)$.

[^9]Thus, it appears that $\frac{\partial}{\partial t} g(X, Y)$ is the time derivative of the smooth function $g(X, Y) \in C^{\infty}(M)$, which implies that in local coordinates equation (2.21) becomes

$$
\begin{equation*}
\frac{\partial}{\partial t} g=\dot{g}_{i j}(t) d x^{i} d x^{j} \tag{3.6}
\end{equation*}
$$

In accordance with Proposition 2.59, in which we state that the Levi-Civita connection can locally be written in terms of the metric, $\nabla=\nabla^{(t)}$ is time dependent as well.
Definition 3.4. The time derivative of the Levi-Civita connection $\nabla$ is the map $\frac{\partial}{\partial t} \nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow$ $\mathfrak{X}(M)$ defined by

$$
\begin{equation*}
\left(\frac{\partial}{\partial t} \nabla\right)(X, Y):=\frac{\partial}{\partial t} \nabla_{X} Y \tag{3.7}
\end{equation*}
$$

for any pair of time independent vector fields $X, Y \in \mathfrak{X}(M)$.
Interestingly, although $\nabla$ is by definition not tensorial since it satisfies the product rule, its time derivative is tensorial as showed in [18, Lemma 4.3] (By checking the conditions of the tensor characterization lemma, see Lemma 2.34). Furthermore, we can differentiate the Christoffel symbols $\Gamma_{i j}^{k}=d x^{k} \nabla \frac{\partial}{\partial x^{i}} \frac{\partial}{\partial x^{j}}$ of $\nabla$ by considering them as a map $\Gamma: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}^{*}(M) \rightarrow C^{\infty}(M)$ defined by $\Gamma(X, Y, \omega)=\omega\left(\nabla_{X} Y\right)$.
Definition 3.5. The time derivative of the Christoffel symbols of $\nabla$ is the map $\frac{\partial}{\partial t} \Gamma: \mathfrak{X}(M) \times \mathfrak{X}(M) \times$ $\mathfrak{X}^{*}(M) \rightarrow C^{\infty}(M)$ defined by

$$
\begin{equation*}
\left(\frac{\partial}{\partial t} \Gamma\right)(X, Y, \omega):=\frac{\partial}{\partial t} \Gamma(X, Y, \omega) \tag{3.8}
\end{equation*}
$$

for any pair of time independent smooth vector fields $X, Y \in \mathfrak{X}(M)$ and time independent smooth covector field $\omega \in \mathfrak{X}^{*}(M)$.

Following similar argumentation as for the time derivative of $\nabla$, one could also show that $\frac{\partial}{\partial t} \Gamma$ is tensorial. Now that we have established how to take time derivatives of the metric, Levi-Civita connection and the Christoffel symbols, we can describe how the various geometric quantities evolve under the metric. To begin with, we consider the metric inverse $g^{i j}$.
Lemma 3.6. Suppose that $g(t)$ is a smooth one-parameter family of metrics on a Riemannian manifold $M$. Then ${ }^{17}$

$$
\begin{equation*}
\frac{\partial}{\partial t} g^{i j}=-g^{i k} g^{j l} \frac{\partial}{\partial t} g_{k l} \tag{3.9}
\end{equation*}
$$

Proof. Note that $g_{k l} g^{l j}=\delta_{k j}$ and therefore that

$$
\left(\frac{\partial}{\partial t} g_{k l}\right) g^{l j}+g_{k l}\left(\frac{\partial}{\partial t} g^{l j}\right)=0
$$

Multiplying by $g^{i k}$ yields

$$
0=g^{i k}\left(\frac{\partial}{\partial t} g_{k l}\right) g^{l j}+g^{i k} g_{k l}\left(\frac{\partial}{\partial t} g^{l j}\right)=g^{i k} g^{j l}\left(\frac{\partial}{\partial t} g_{k l}\right)+\delta_{i l}\left(\frac{\partial}{\partial t} g^{l j}\right)
$$

and hence that $\left(\frac{\partial}{\partial t} g^{i j}\right)=-g^{i k} g^{j l}\left(\frac{\partial}{\partial t} g_{k l}\right)$.
Subsequently, we can describe the evolution of the Christoffel symbols of the Levi-Civita connection.
Proposition 3.7. Suppose that $g(t)$ is a smooth one-parameter family of metrics on a Riemannian manifold. Then the Christoffel symbols of the Levi-Civita connection evolve by

$$
\begin{equation*}
\frac{\partial}{\partial t} \Gamma_{i j}^{k}=\frac{1}{2} g^{k l}\left(\nabla_{i} \frac{\partial}{\partial t} g_{j l}+\nabla_{j} \frac{\partial}{\partial t} g_{i l}-\nabla_{l} \frac{\partial}{\partial t} g_{i j}\right) \tag{3.10}
\end{equation*}
$$

[^10]Proof. Recall the coordinate expression of the Christoffel symbols from (2.49). If we choose normal coordinates with respect to $g\left(t_{0}\right)$, i.e. at some time $t=t_{0}$, about a point $p \in M$, we compute

$$
\begin{aligned}
\frac{\partial}{\partial t} \Gamma_{i j}^{k}(p) & =\left.\frac{1}{2} \frac{\partial}{\partial t} g^{k l}(p)\left(\frac{\partial}{\partial x^{i}} g_{j l}+\frac{\partial}{\partial x^{j}} g_{i l}-\frac{\partial}{\partial x^{l}} g_{i j}\right)\right|_{p}+\left.\frac{1}{2} g^{k l}(p) \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^{i}} g_{j l}+\frac{\partial}{\partial x^{j}} g_{i l}-\frac{\partial}{\partial x^{l}} g_{i j}\right)\right|_{p} \\
& =0+\frac{1}{2} g^{k l}(p)\left(\frac{\partial}{\partial x^{i}} \frac{\partial}{\partial t} g_{j l}(p)+\frac{\partial}{\partial x^{j}} \frac{\partial}{\partial t} g_{i l}(p)-\frac{\partial}{\partial x^{l}} \frac{\partial}{\partial t} g_{i j}(p)\right)
\end{aligned}
$$

where the first term vanishes due to Proposition 2.67 (iv). In view of Remark 2.68, we then have

$$
\frac{\partial}{\partial t} \Gamma_{i j}^{k}(p)=\frac{1}{2} g^{k l}(p)\left(\nabla_{i} \frac{\partial}{\partial t} g_{j l}(p)+\nabla_{j} \frac{\partial}{\partial t} g_{i l}(p)-\nabla_{l} \frac{\partial}{\partial t} g_{i j}(p)\right),
$$

from which the desired equality then follows. Since both sides of this equation represent tensors fields in coordinate expressions, the identity holds for any coordinate system.

In what follows, we derive expressions for the evolution of the Riemann curvature endomorphism and tensor, and the Ricci curvature.

Proposition 3.8. Suppose that $g(t)$ is a smooth one-parameter family of metric on a Riemannian manifold $M$. Then the Riemannian curvature endomorphism $R$ evolves by

$$
\begin{equation*}
\frac{\partial}{\partial t} R_{i j k}^{l}=\frac{1}{2} g^{l m}\left(\nabla_{i, m}^{2} \frac{\partial}{\partial t} g_{j k}+\nabla_{j, k}^{2} \frac{\partial}{\partial t} g_{i m}-\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j m}-\nabla_{j, m}^{2} \frac{\partial}{\partial t} g_{i k}-R_{i j k}^{p} \frac{\partial}{\partial t} g_{p m}-R_{i j m}^{p} \frac{\partial}{\partial t} g_{p k}\right) . \tag{3.11}
\end{equation*}
$$

Proof. Recall (2.57) and choose normal coordinates ( $x^{i}$ ) with respect to $g\left(t_{0}\right)$ centred about a point $p \in M$. We have

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{i j k}^{l}(p)= & \frac{\partial}{\partial x^{i}}\left(\frac{\partial}{\partial t} \Gamma_{j k}^{l}(p)\right)-\frac{\partial}{\partial x^{j}}\left(\frac{\partial}{\partial t} \Gamma_{i k}^{l}(p)\right)+\left(\frac{\partial}{\partial t} \Gamma_{j k}^{m}(p)\right) \Gamma_{i m}^{l}(p)+\Gamma_{j k}^{m}(p)\left(\frac{\partial}{\partial t} \Gamma_{i m}^{l}(p)\right) \\
& -\left(\frac{\partial}{\partial t} \Gamma_{i k}^{m}(p)\right) \Gamma_{j m}^{l}(p)-\Gamma_{i k}^{m}(p)\left(\frac{\partial}{\partial t} \Gamma_{j m}^{l}(p)\right) \\
= & \frac{\partial}{\partial x^{i}}\left(\frac{\partial}{\partial t} \Gamma_{j k}^{l}(p)\right)-\frac{\partial}{\partial x^{j}}\left(\frac{\partial}{\partial t} \Gamma_{i k}^{l}(p)\right),
\end{aligned}
$$

due to the fact that in normal coordinates $\Gamma_{a b}^{c}(p)=0$. Proposition 3.7 then implies that

$$
\begin{aligned}
\frac{\partial}{\partial x^{i}}\left(\frac{\partial}{\partial t} \Gamma_{j k}^{l}(p)\right)= & \frac{\partial}{\partial x^{i}}\left[\frac{1}{2} g^{l m}(p)\left(\nabla_{j} \frac{\partial}{\partial t} g_{k m}(p)+\nabla_{k} \frac{\partial}{\partial t} g_{j m}(p)-\nabla_{m} \frac{\partial}{\partial t} g_{j k}(p)\right)\right] \\
= & \frac{1}{2} \frac{\partial}{\partial x^{i}} g^{l m}(p)\left(\nabla_{j} \frac{\partial}{\partial t} g_{k m}(p)+\nabla_{k} \frac{\partial}{\partial t} g_{j m}(p)-\nabla_{m} \frac{\partial}{\partial t} g_{j k}(p)\right) \\
& +\frac{1}{2} g^{l m}(p)\left[\frac{\partial}{\partial x^{i}}\left(\nabla_{j} \frac{\partial}{\partial t} g_{k m}(p)+\nabla_{k} \frac{\partial}{\partial t} g_{j m}(p)-\nabla_{m} \frac{\partial}{\partial t} g_{j k}(p)\right)\right] \\
= & \frac{1}{2} g^{l m}(p)\left(\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k m}(p)+\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j m}(p)-\nabla_{i, m}^{2} \frac{\partial}{\partial t} g_{j k}(p)\right),
\end{aligned}
$$

where the last step follows from Proposition 2.67 (iii) and Remark 2.68. Hence, we have

$$
\begin{align*}
\frac{\partial}{\partial t} R_{i j k}^{l}(p)= & \frac{1}{2} g^{l m}(p)\left(\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k m}(p)+\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j m}(p)-\nabla_{i, m}^{2} \frac{\partial}{\partial t} g_{j k}(p)\right)  \tag{3.12}\\
& -\frac{1}{2} g^{l m}(p)\left(\nabla_{j, i}^{2} \frac{\partial}{\partial t} g_{k m}(p)+\nabla_{j, k}^{2} \frac{\partial}{\partial t} g_{i m}(p)-\nabla_{j, m}^{2} \frac{\partial}{\partial t} g_{i k}(p)\right) .
\end{align*}
$$

Notice that equation (2.54) and consequently (2.56) combined with Lemma 2.72 give us that

$$
\begin{align*}
\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k m}-\nabla_{j, i}^{2} \frac{\partial}{\partial t} g_{k m}= & \left(R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) \frac{\partial}{\partial t} g\right)\left(\frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{m}}\right) \\
= & R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) g_{k m}-\frac{\partial}{\partial t} g\left(R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{m}}\right) \\
& -\frac{\partial}{\partial t} g\left(\frac{\partial}{\partial x^{k}}, R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) \frac{\partial}{\partial x^{m}}\right) \\
= & 0-R_{i j k}^{p} \frac{\partial}{\partial t} g_{p m}-R_{i j m}^{p} \frac{\partial}{\partial t} g_{p k}, \tag{3.13}
\end{align*}
$$

since $R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) g_{k m}=\nabla_{i} \nabla_{j} g_{k m}-\nabla_{j} \nabla_{i} g_{k m}-\nabla_{[i, j]} g_{k m}=\frac{\partial^{2}}{\partial x^{i} \partial x^{j}} g_{k m}-\frac{\partial^{2}}{\partial x^{j} \partial x^{i}} g_{k m}-0=0$. The identity then follows by combining equations (3.12) and (3.13).
Proposition 3.9. Suppose that $g(t)$ is a smooth one-parameter family of metrics on a Riemannian manifold $M$. Then the Riemann curvature tensor $R m$ evolves by

$$
\begin{align*}
\frac{\partial}{\partial t} R m_{i j k l}= & \frac{1}{2}\left(\nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{j k}+\nabla_{j, k}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j l}-\nabla_{j, l}^{2} \frac{\partial}{\partial t} g_{i k}\right) \\
& +\frac{1}{2} g^{p q}\left(R m_{i j k p} \frac{\partial}{\partial t} g_{q l}+R m_{i j p l} \frac{\partial}{\partial t} g_{q k}\right) \tag{3.14}
\end{align*}
$$

Proof. Since $R m_{i j k l}=g_{l m} R_{i j k}^{m}$, we have

$$
\frac{\partial}{\partial t} R m_{i j k l}=\frac{\partial}{\partial t} g_{l m} R_{i j k}^{m}+g_{l m} \frac{\partial}{\partial t} R_{i j k}^{m}
$$

From Proposition 3.8 we get

$$
\begin{align*}
g_{l m} \frac{\partial}{\partial t} R_{i j k}^{m} & =g_{l m}\left[\frac{1}{2} g^{m l}\left(\nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{j k}+\nabla_{j, k}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j l}-\nabla_{j, l}^{2} \frac{\partial}{\partial t} g_{i k}-R_{i j k}^{p} \frac{\partial}{\partial t} g_{p l}-R_{i j l}^{p} \frac{\partial}{\partial t} g_{p k}\right)\right] \\
& =\frac{1}{2}\left(\nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{j k}+\nabla_{j, k}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j l}-\nabla_{j, l}^{2} \frac{\partial}{\partial t} g_{i k}\right)-\frac{1}{2}\left(R_{i j k}^{p} \frac{\partial}{\partial t} g_{p l}-R_{i j l}^{p} \frac{\partial}{\partial t} g_{p k}\right) \\
& =\frac{1}{2}\left(\nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{j k}+\nabla_{j, k}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j l}-\nabla_{j, l}^{2} \frac{\partial}{\partial t} g_{i k}\right)-\frac{1}{2} g^{p q}\left(R m_{i j k p} \frac{\partial}{\partial t} g_{q l}+R m_{i j l p} \frac{\partial}{\partial t} g_{q k}\right) . \tag{3.15}
\end{align*}
$$

Furthermore, by using Lemma 2.77 for the last term of (3.15) and since

$$
\begin{equation*}
\frac{\partial}{\partial t} g_{l q} R_{i j k}^{q}=g^{p q} R m_{i j k p} \frac{\partial}{\partial t} g_{q l} \tag{3.16}
\end{equation*}
$$

we obtain the desired equality by adding (3.15) with (3.16) .
Proposition 3.10. Suppose that $g(t)$ is a smooth one-parameter family of metrics on a Riemannian manifold $M$. Then the Ricci curvature Ric evolves by

$$
\begin{equation*}
\frac{\partial}{\partial t} R i c_{i j}=\frac{1}{2} g^{k l}\left(\nabla_{k, i}^{2} \frac{\partial}{\partial t} g_{j l}+\nabla_{k, j}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{k, l}^{2} \frac{\partial}{\partial t} g_{i j}-\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k l}\right) \tag{3.17}
\end{equation*}
$$

Proof. Since $R i c_{i j}=R_{k i j}^{k}$, the desired result can be conveniently obtained from equation (3.12). I.e., we have

$$
\frac{\partial}{\partial t} R i c_{i j}=\frac{\partial}{\partial t} R_{k i j}^{k}=\frac{1}{2} g^{k l}\left(\nabla_{k, i}^{2} \frac{\partial}{\partial t} g_{j l}+\nabla_{k, j}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{k, l}^{2} \frac{\partial}{\partial t} g_{i j}-\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j l}-\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k l}+\nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{k j}\right)
$$

Now, notice that

$$
\frac{1}{2} g^{k l}\left(\nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{k j}-\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j l}\right)=\frac{1}{2} g^{k l} \nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{k j}-\frac{1}{2} g^{l k} \nabla_{i, l}^{2} \frac{\partial}{\partial t} g_{k j}=0
$$

since both the metric and its inverse are symmetric.

### 3.2 Evolution of curvature under Ricci flow

With the expressions from the preceding section, we can substitute $\frac{\partial}{\partial t} g=-2 R i c$ to derive expressions that describe the evolution of geometric quantities under Ricci flow. It turns out that we can simplify these expression by means of the quadratic curvature tensor. Again, we will primarily follow [18, Chapter 4].

Definition 3.11. The quadratic curvature tensor is the ( 0,4 )-tensor field $B: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times$ $\mathfrak{X}(M) \rightarrow C^{\infty}(M)$ defined by

$$
\begin{equation*}
B(X, Y, Z, W):=\langle R m(X, \cdot, Y, \star), R(W, \cdot, Z, \star)\rangle \tag{3.18}
\end{equation*}
$$

In accordance with Remark 2.39, for the components we get

$$
\begin{align*}
B_{i j k l} & =\left\langle R m\left(\frac{\partial}{\partial x^{i}}, \cdot, \frac{\partial}{\partial x^{j}}, \star\right), R m\left(\frac{\partial}{\partial x^{k}}, \cdot, \frac{\partial}{\partial x^{;}}, \star\right)\right\rangle \\
& =g^{p r} g^{q s} R m_{i p j q} R m_{k r l s} \\
& =g^{p r} g^{q s} R m_{p i q j} R m_{r k s l}, \tag{3.19}
\end{align*}
$$

with the last step following from Lemma 2.77 (i) and (ii). In addition, we define the Laplacian on a connection on the tensor bundle.

Definition 3.12. For any tensor $\tau \in \mathcal{T}_{s}^{r}(M)$, we define the connection Laplacian as the trace of the second covariant derivative with respect to $g$. That is,

$$
\begin{equation*}
\Delta \tau:=\operatorname{tr}_{g}\left(\nabla^{2} \tau\right) \tag{3.20}
\end{equation*}
$$

Recall equations (2.26) and (2.42) to see that

$$
\begin{align*}
(\Delta \tau)\left(d x^{j_{1}}, \ldots, d x^{j_{r}}, \frac{\partial}{\partial x^{i_{1}}}, \ldots, \frac{\partial}{\partial x^{i_{s}}}\right) & =\operatorname{tr}_{g}\left(\nabla^{2} \tau\right)\left(d x^{j_{1}}, \ldots, d x^{j_{r}}, \frac{\partial}{\partial x^{i_{1}}}, \ldots, \frac{\partial}{\partial x^{i_{s}}}\right) \\
& =\operatorname{tr}_{1,2}^{1,2}\left(g^{-1} \otimes \nabla^{2} \tau\right)\left(d x^{j_{1}}, \ldots, d x^{j_{r}}, \frac{\partial}{\partial x^{i_{1}}}, \ldots, \frac{\partial}{\partial x^{i_{s}}}\right) \\
& =\operatorname{tr}_{1,2}^{1,2}\left(g^{p q} \frac{\partial}{\partial x^{p}} \otimes \frac{\partial}{\partial x^{q}} \otimes \nabla^{2} \tau\right)\left(d x^{j_{1}}, \ldots, d x^{j_{r}}, \frac{\partial}{\partial x^{i_{1}}}, \ldots, \frac{\partial}{\partial x^{i_{s}}}\right) \\
& =g^{p q}\left(\nabla_{p, q}^{2} \tau\right)\left(d x^{j_{1}}, \ldots, d x^{j_{r}}, \frac{\partial}{\partial x^{i_{1}}}, \ldots, \frac{\partial}{\partial x^{i_{s}}}\right) . \tag{3.21}
\end{align*}
$$

Since the Levi-Civita connection is compatible with $g$, the covariant derivatives commute with metric contractions. Therefore, we have that (see [18, Proposition 2.48])

$$
\begin{equation*}
\nabla_{k} R i c_{i j}=g^{p q} \nabla_{k} R m_{p i j q} \quad \text { and } \quad \nabla_{k, l}^{2} R i c_{i j}=g^{p q} \nabla_{k, l}^{2} R m_{p i j q} \tag{3.22}
\end{equation*}
$$

Now we can examine how the connection Laplacian acts on $R m$.

Proposition 3.13. The Laplacian of the Riemannian curvature tensor $R m$ satisfies

$$
\begin{align*}
\Delta R m_{i j k l}= & \nabla_{i, k}^{2} R i c_{j l}-\nabla_{j, k}^{2} R i c_{i l}+\nabla_{j, l}^{2} R i c_{i k}-\nabla_{i, l}^{2} R i c_{j k}  \tag{3.23}\\
& -2\left(B_{i j k l}-B_{i j l k}-B_{i l j k}+B_{i k j l}\right)+g^{p q}\left(\text { Rm}_{q j k l} R i c_{p i}+R m_{i q k l} R i c_{p j}\right)
\end{align*}
$$

Proof. The following proof is in analogy with the proofs of [18, Proposition 4.2] and [1, Lemma 7.2] and works with normal coordinates centred about an arbitrary $p \in M$. Due to respectively equation (3.21), the second Bianchi identity (2.63) and (i) of Lemma 2.77, we have that

$$
\begin{aligned}
\Delta R m_{i j k l} & =\operatorname{tr}_{g}\left(\nabla^{2} R m_{i j k l}\right) \\
& =g^{p q}\left(\nabla_{p, q}^{2} R m_{i j k l}\right) \\
& =g^{p q}\left(-\nabla_{p, i}^{2} R m_{j q k l}-\nabla_{p, j}^{2} R m_{q i k l}\right) \\
& =g^{p q}\left(\nabla_{p, i}^{2} R m_{q j k l}-\nabla_{p, j}^{2} R m_{q i k l}\right) .
\end{aligned}
$$

Next, recall equation (2.54) and notice that by applying the second Bianchi identity again we have

$$
\begin{aligned}
\nabla_{p, i}^{2} R m_{q j k l} & =\nabla_{i, p}^{2} R m_{q j k l}+\left(R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{p}}\right) R m\right)\left(\frac{\partial}{\partial x^{q}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{l}}\right) \\
& =\nabla_{i, k}^{2} R m_{j q l p}-\nabla_{i, l}^{2} R m_{j q k p}+\left(R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{p}}\right) R m\right)\left(\frac{\partial}{\partial x^{q}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{l}}\right)
\end{aligned}
$$

and consequently due to (3.22) that

$$
g^{p q} \nabla_{p, i}^{2} R m_{q j k l}=\nabla_{i, k}^{2} R i c_{j l}-\nabla_{i, l}^{2} R i c_{j k}+g^{p q}\left(R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{p}}\right) R m\right)\left(\frac{\partial}{\partial x^{q}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{l}}\right) .
$$

For the third term of the above equation, by Lemma 2.72 we have

$$
\begin{aligned}
g^{p q}\left(R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{p}}\right) R m\right)\left(\frac{\partial}{\partial x^{q}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{l}}\right)= & g^{p q}\left[R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{p}}\right)\left(R m_{q j k l}\right)\right. \\
& -R m\left(R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{p}}\right) \frac{\partial}{\partial x^{q}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{l}}\right) \\
& \left.-\cdots-R m\left(\frac{\partial}{\partial x^{q}}, \frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{k}}, R\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{p}}\right) \frac{\partial}{\partial x^{l}}\right)\right] \\
= & g^{p q}\left(0-R_{i p q}^{n} R m_{n j k l}-R_{i p q}^{n} R m_{q n k l}-R_{i p q}^{n} R m_{q j n l}-R_{i p q}^{n} R m_{q j k n}\right) \\
= & g^{p q} g^{m n}\left(R m_{p i q m} R m_{n j k l}+R m_{p i j m} R m_{q n k l}\right. \\
& \left.+R m_{p i k m} R m_{q j n l}+R m_{p i l m} R m_{q j k n}\right) .
\end{aligned}
$$

Furthermore, observe that

$$
g^{p q} g^{m n} R m_{p i q m} R m_{n j k l}=g^{m n} R_{i m p}^{p} R m_{n j k l}=g^{m n} R m_{n j k l} R c_{i m}=g^{p q} R m_{p j k l} R c_{i q}
$$

and by using (3.19) and Lemma 2.77

$$
g^{p q} g^{m n} R m_{p i j m} R m_{q n k l}=g^{p q} g^{m n}\left(R m_{p i m j} R m_{q l n k}-R m_{p i m j} R m_{q k n l}\right)=B_{i j l k}-B_{i j k l}
$$

and lastly that

$$
g^{p q} g^{m n}\left(R m_{p i k m} R m_{q j n l}+R m_{p i l m} R m_{q j k n}\right)=g^{p q} g^{m n}\left(-R m_{p i m k} R m_{q j n l}+R m_{p i m l} R m_{q j n k}\right)=-B_{i k j l}+B_{i l j k}
$$

Combining all results yields

$$
\nabla_{p, i}^{2} R m_{q j k l}=\nabla_{i, k}^{2} R m_{j q l p}-\nabla_{i, l}^{2} R m_{j q k p}-\left(B_{i j k l}-B_{i j l k}-B_{i l j k}+B_{i k j l}\right)+g^{p q} R m_{p j k l} R i c_{i q}
$$

and a similar computation for $\nabla_{p, j}^{2} R m_{q i k l}$ then yields the desired equality.

Proposition 3.13 makes it easier to derive an equation for the evolution of curvature under Ricci flow. From the next result, it becomes apparent that the curvature tensor $R m$ evolves under Ricci flow similar to as in the heat equation. In the usual heat equation, one namely has $\frac{\partial u}{\partial t}=\Delta u=\operatorname{div} \operatorname{grad} u$ for some $u: \mathbb{R}^{n} \times I \rightarrow \mathbb{R}$. This will be pivotal for our analysis on existence and uniqueness to solutions of the flow.

Theorem 3.14. Suppose that $g(t)$ is a solution of the Ricci flow, then the Riemannian curvature tensor $R m$ evolves by

$$
\begin{align*}
\frac{\partial}{\partial t} R m_{i j k l}= & \Delta R m_{i j k l}+2\left(B_{i j k l}-B_{i j l k}-B_{i l j k}+B_{i k j l}\right)  \tag{3.24}\\
& -g^{p q}\left(R m_{p j k l} R i c_{q i}+R m_{i p k l} R i c_{q j}+R m_{i j k p} R i c_{q l}+R m_{i j p l} R i c_{q k}\right)
\end{align*}
$$

Proof. Using Proposition 3.9, we can substitute $\frac{\partial}{\partial t} g_{i j}=-2 R i c_{i j}$ into equation (3.14):

$$
\frac{\partial}{\partial t} R m_{i j k l}=-\nabla_{i, l}^{2} R i c_{j k}-\nabla_{j, k}^{2} R i c_{i l}+\nabla_{i, k}^{2} R i c_{j l}+\nabla_{j, l}^{2} R i c_{i k}-g^{p q}\left(R_{i j k p} R i c_{q l}+R m_{i j p l} R i c_{q k}\right) .
$$

Combining this with equation (3.23) with the indices $k$ and $l$ switched, gives us that the Laplacian satisfies

$$
\begin{aligned}
\Delta R m_{i j l k}= & -\frac{\partial}{\partial t} R m_{i j k l}-2\left(B_{i j l k}-B_{i j k l}-B_{i k j l}+B_{i l j k}\right)-g^{p q}\left(R m_{i j k p} R i c_{q l}+R m_{i j p l} R i c_{q k}\right) \\
& +g^{p q}\left(R m_{p j l k} R i c_{q i}+R m_{i p l k} R i c_{q j}\right)
\end{aligned}
$$

from which the desired result can be naturally derived by using the symmetries of Lemma 2.77.
To conclude this section, we provide an evolution equation for the Ricci curvature under Ricci flow. Also here, we see that Ricci curvature evolves as a heat-type equation.

Proposition 3.15. Suppose that $g(t)$ is a solution of the Ricci flow, then the Ricci curvature Ric evolves by

$$
\begin{equation*}
\frac{\partial}{\partial t} R i c_{i j}=\Delta R i c_{i j}+2 g^{p q} g^{r s} R m_{p i j r} R i c_{q s}-2 g^{p q} R i c_{i p} R i c_{q j} \tag{3.25}
\end{equation*}
$$

Proof. The proof is rather technical and we refer to [18, Corollarly 3.18$]$ for the details.

### 3.3 Short-time existence and uniqueness

The usual heat equation is the prototype of a strongly parabolic partial differential equation. Although the Ricci flow resembles the heat equation, we will see that after linearising equation (3.1), the PDE is only weakly parabolic. Therefore, we cannot use the usual existence theory for parabolic PDE's. Instead, we will prove existence and uniqueness to a solution of the the Ricci flow analogously to $[7]^{18}$

Our aim is first to recognize a parabolic PDE. Here, we follow the approaches of $[18,32,33]$ and [34]. Consider two smooth vector bundles $E$ and $F$ over a Riemannian manifold $(M, g)$ and an evolution equation $u:[0, T] \times M \rightarrow E$ given by

$$
\begin{equation*}
\frac{\partial u}{\partial t}=L(u), \quad u(0)=u_{0} \tag{3.26}
\end{equation*}
$$

with $L: \Gamma(E) \rightarrow \Gamma(F)$ a linear differential operator of order $k \in \mathbb{N}$. That is, it can be written in the form

$$
\begin{equation*}
L(u):=\sum_{|\alpha| \leq k} L_{\alpha} \frac{\partial u}{\partial x^{\alpha}} \tag{3.27}
\end{equation*}
$$

[^11]with $L_{\alpha} \in \operatorname{Hom}(E, F)$ a bundle homomorphism ${ }^{19}$ and $\alpha$ a multi-index. Next, we define the total symbol $\sigma_{L}: T^{*} M \rightarrow \operatorname{Hom}(E, F)$ of $L$ as the bundle homomorphism
\[

$$
\begin{equation*}
\sigma_{L}(p, \xi):=\sum_{|\alpha| \leq k} L_{\alpha} \xi^{\alpha} \tag{3.28}
\end{equation*}
$$

\]

Subsequently, the principal symbol $\hat{\sigma}_{L}$ of $L$ is the defined as the bundle homomorphism of the highest term only:

$$
\begin{equation*}
\hat{\sigma}_{L}(p, \xi):=\sum_{|\alpha|=k} L_{\alpha} \xi^{\alpha} \tag{3.29}
\end{equation*}
$$

The principal symbol tells us whether an evolution equation is parabolic.
Definition 3.16. The partial differential equation in (3.26) is strongly parabolic ${ }^{20}$ if $L$ is an elliptic operator. That is, there exists a real number $c>0$ such that for all (non-trivial) $(p, \xi) \in T^{*} M$ and $u \in \Gamma(E)$ we have

$$
\begin{equation*}
\left\langle\hat{\sigma}_{L}(p, \xi) u, u\right\rangle \geq c|\xi|^{2}|u|^{2} \tag{3.30}
\end{equation*}
$$

In the Ricci flow equation, we can regard the Ricci curvature as the operator Ric: $\Gamma\left(\Sigma_{+}^{2}\left(T^{*} M\right)\right) \rightarrow$ $\Gamma\left(\Sigma^{2}\left(T^{*} M\right)\right)$. In the light of (3.26), notice that in the Ricci flow we have $E=\Sigma_{+}^{2}\left(T^{*} M\right)$ and $F=$ $\Sigma^{2}\left(T^{*} M\right)$. In other words, the Ricci curvature sends the positive definite symmetric metric $g(t)$ to a symmetric 2-tensor. However, the operator is not a bundle homomorphism, which means that the Ricci flow is not a linear PDE. Hence, we wish to linearise the equation. Before, we will do this the Ricci curvature, we move back to general non-linear PDE's on Riemannian manifolds.
Definition 3.17. Consider a non-linear differential operator $P: \Gamma(E) \rightarrow \Gamma(F)$. The linearisation of $P$ at $u_{0}$ is the linear operator $D P: \Gamma(E) \rightarrow \Gamma(F)$ given by

$$
\begin{equation*}
\left.D P\right|_{u_{0}}(v)=\left.\frac{d}{d t} P(u(t))\right|_{t=0} \tag{3.31}
\end{equation*}
$$

where $u(0)=u_{0}$ and $u^{\prime}(0)=v$.
Example 3.18. If we consider the non-linear PDE

$$
\begin{equation*}
\frac{\partial u}{\partial t}=P\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial^{2} u}{\partial x^{2}}\right), \quad u(0)=u_{0} \tag{3.32}
\end{equation*}
$$

then we say that (3.32) is a parabolic non-linear PDE if its linearisation $\frac{\partial u}{\partial t}=\left.D P\right|_{u_{0}}(v)$ is parabolic in the light of Definition 3.16.
Lemma 3.19. The linearisation of the Ricci tensor Ric: $\Gamma\left(\Sigma_{+}^{2}\left(T^{*} M\right)\right) \rightarrow \Gamma\left(\Sigma^{2}\left(T^{*} M\right)\right)$ at $g_{0}$ is given by

$$
\begin{equation*}
\left.D R i c\right|_{g_{0}}\left(\frac{\partial}{\partial t} g\right)_{i j}=\frac{1}{2} g^{k l}\left(\nabla_{k, i}^{2} \frac{\partial}{\partial t} g_{j l}+\nabla_{k, j}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{k, l}^{2} \frac{\partial}{\partial t} g_{i j}-\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k l}\right), \quad \frac{\partial}{\partial t} g \in \Gamma\left(\Sigma_{+}^{2}\left(T^{*} M\right)\right) \tag{3.33}
\end{equation*}
$$

Proof. Let $\frac{\partial}{\partial t} g \in \Gamma\left(\Sigma_{+}^{2}\left(T^{*} M\right)\right)$ and recall Proposition 3.10 to observe that

$$
\begin{aligned}
\left.D \operatorname{Ric}\right|_{g_{0}}\left(\frac{\partial}{\partial t} g\right)_{i j} & =\left.\frac{\partial}{\partial t} \operatorname{Ric}(g(t))_{i j}\right|_{t=0} \\
& =\frac{1}{2} g^{k l}\left(\nabla_{k, i}^{2} \frac{\partial}{\partial t} g_{j l}+\nabla_{k, j}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{k, l}^{2} \frac{\partial}{\partial t} g_{i j}-\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k l}\right)
\end{aligned}
$$

[^12]Notice that the highest-order derivatives of $\frac{\partial}{\partial t} g$ are just the partial derivatives. Therefore, the principal symbol of the Ricci curvature is

$$
\begin{equation*}
\left(\hat{\sigma}_{R i c}(p, \xi) \frac{\partial}{\partial t} g\right)_{i j}=\frac{1}{2} g^{k l}\left(\xi_{k} \xi_{i} \frac{\partial}{\partial t} g_{j l}+\xi_{k} \xi_{j} \frac{\partial}{\partial t} g_{i l}-\xi_{k} \xi_{l} \frac{\partial}{\partial t} g_{i j}-\xi_{i} \xi_{j} \frac{\partial}{\partial t} g_{k l}\right), \quad(p, \xi) \in T^{*} M \tag{3.34}
\end{equation*}
$$

However, if we define $\frac{\partial}{\partial t} g_{i j}=\xi_{i} \xi_{j}$, then $\left(\hat{\sigma}_{R i c}(p, \xi) \frac{\partial}{\partial t} g\right)_{i j}=0$. In other words, the principal symbol of the Ricci curvature has non-trivial kernel, and does not satisfy the requirements of Definition 3.16. In this sense, we say that the flow is only weakly parabolic. This is caused solely by the diffeomorphism invariance of the Ricci curvature (Recall Proposition 2.79). If $\varphi:(M, g) \rightarrow(\widetilde{M}, \tilde{g})$ is a time-dependent diffeomorphism such that $\varphi^{*} \tilde{g}(t)=g(t)$ and $\tilde{g}(t)$ is a solution to the Ricci flow, then

$$
\begin{align*}
\frac{\partial}{\partial t} g(t) & =\varphi^{*}\left(\frac{\partial}{\partial t} \tilde{g}(t)\right) \\
& =\varphi^{*}(-2 \widetilde{\operatorname{Ric}}(\tilde{g}(t)))  \tag{3.35}\\
& =-2 \operatorname{Ric}(g(t))
\end{align*}
$$

We see that then $g(t)$ is also a solution to the Ricci flow, and therefore that (3.1) is invariant under the full diffeomorphism group, which is infinite dimensional. If $g(t)$ is any stationary solution to the Ricci flow, that would imply we could acquire another linear independent solution with the pullback of any diffeomorphism. However, the solution space of non-linear elliptic differential operators on compact manifolds is only finite dimensional [35, Theorem 1]. Hence, the Ricci flow is not a parabolic equation, which means we cannot directly use well known existence and uniqueness theory of such equations.

From (3.35), it also becomes apparent that Ricci flow preserves any present symmetries. Since symmetric operations can be regarded as isometric diffeomorphisms, the pullback of a solution to the Ricci flow by a symmetry still has all the initial information of the topology of the manifold in question [36, Section 6.1]. Together with the fact that metrics with positive curvature are contracted and metrics with negative curvature are stretched, the Ricci flow literally 'smoothens' a manifold's geometry.

In [7], DeTurck introduced a technique to express the weakly parabolic Ricci flow as a strongly parabolic equation, now known as DeTurck's trick. Existence and uniqueness for the Ricci flow can therefore still be guaranteed. Before moving on to this result, we state a useful property and one final definition. Due to [37, Proposition 2.61 (i)] (which is derived from [25, Proposition 12.32] using Definition 2.57 and compatibility of $g$ with $\nabla$ ), for $X=X^{i} \frac{\partial}{\partial x^{i}}$ we have

$$
\begin{equation*}
\mathcal{L}_{X} g_{i j}=\nabla_{i} X_{j}+\nabla_{j} X_{i} . \tag{3.36}
\end{equation*}
$$

To proof uniqueness of short-time Ricci flow solutions for the upcoming theorem, the following definition will come out useful.

Definition 3.20. Let $f:(M, g) \times I \rightarrow(\widetilde{M}, \tilde{g})$ be a smooth map between closed Riemannian manifolds with $I \subset \mathbb{R}$. The harmonic map Laplacian is the map

$$
\begin{equation*}
\Delta_{g, \tilde{g}} f=g^{i j}\left(\nabla_{i} f_{*}\right)\left(\frac{\partial}{\partial x^{j}}\right) \tag{3.37}
\end{equation*}
$$

We say that $f$ is harmonic if $\Delta_{g, \tilde{g}} f=0$ and call the equation

$$
\begin{equation*}
f_{*} \frac{\partial}{\partial t}:=\Delta_{g, \tilde{g}} f, \quad f(0)=f_{0} \tag{3.38}
\end{equation*}
$$

the harmonic map heat flow. In [38], Eels and Sampson proved that if $(\widetilde{M}, \tilde{g})$ has non-positive sectional curvature ${ }^{21}$, then there exists a solution to the maximal harmonic map heat flow, i.e. with $\left\{f_{t} \mid 0<t<\right.$

[^13]$T \leq \infty\}$, such that $f$ converges to a harmonic map. This implies the existence of a harmonic map in each homotopy class ${ }^{22}$. Hartman extended this result in [40] to prove uniqueness of the harmonic map in its homotopy class. The extend to which the non-positively sectional curvature of $\widetilde{M}$ is actually necessary was examined by Chang, Ding and Ye in [41]. They concluded that the maximal time of existence cannot be expected to be infinite without non-positive sectional curvature.

In our case, we only need short-time existence. Therefore, we can continue to our primary result on the Ricci flow.

Theorem 3.21. If ( $M, g_{0}$ ) is a closed Riemannian manifold, there exists an unique solution $g(t)$, defined for time $t \in[0, \epsilon]$, to the Ricci flow such that $g(0)=g_{0}$ for some $\epsilon>0$.
Proof. The proof is divided into four steps:

1. Prove that a modification of the Ricci flow, the Ricci-DeTurck flow, is parabolic and hence enjoys short-time existence and uniqueness.
2. Relate the Ricci flow to the Ricci-DeTurck flow.
3. Start with a solution to the Ricci flow and relate it to a unique solution to the Ricci-DeTurck flow by means of a reparametrisation by harmonic map heat flow.
4. Conclude that the solution to the Ricci flow must be unique.

STEP 1. Let $(M, g(t))$ be a family of Riemannian manifolds and let $\tilde{g} \in\{g(t) \mid t \in[0, \epsilon]\}$ be fixed, for some $\epsilon>0$. Define $W \in \mathfrak{X}(M)$ by

$$
\begin{equation*}
W=W^{i} \frac{\partial}{\partial x^{i}}:=g^{j k}\left(\Gamma_{j k}^{i}-\widetilde{\Gamma}_{j k}^{i}\right) \frac{\partial}{\partial x^{i}}, \tag{3.39}
\end{equation*}
$$

which is well-defined since the difference of two connections is a tensor. Also, consider the one-form obtained from $W$ with the lowering musical isomorphism:

$$
\begin{equation*}
W^{\mathrm{b}}=W_{i} d x^{i}:=g_{i j} g^{k l}\left(\Gamma_{k l}^{j}-\widetilde{\Gamma}_{k l}^{j}\right) d x^{i} \tag{3.40}
\end{equation*}
$$

Let us now define the Ricci-DeTurck flow:

$$
\begin{equation*}
\frac{\partial}{\partial t} g_{i j}=-2 R i c_{i j}+\mathcal{L}_{W} g_{i j}, \quad g(0)=g_{0} \tag{3.41}
\end{equation*}
$$

We will show that (3.41) is a quasilinear ${ }^{23}$ strongly parabolic equation. Existence and uniqueness then follows from e.g. [43, Theorem 1.1]. Recall equation (3.36) to observe that

$$
\begin{aligned}
\mathcal{L}_{W} g_{i j} & =\nabla_{i} W_{j}+\nabla_{j} W_{i} \\
& =\nabla_{i}\left(g_{j k} g^{l m}\left(\Gamma_{l m}^{k}-\widetilde{\Gamma}_{l m}^{k}\right)\right)+\nabla_{j}\left(g_{i k} g^{l m}\left(\Gamma_{l m}^{k}-\widetilde{\Gamma}_{l m}^{k}\right)\right) .
\end{aligned}
$$

By using normal coordinates about a point $p \in M$, the covariant derivatives can be interpreted as partial derivatives, such that linearising the above expression yields

$$
\begin{aligned}
\left.D \mathcal{L}_{W}\right|_{g_{0}} \frac{\partial}{\partial t} g_{i j}= & \left.\frac{\partial}{\partial t}\left(\mathcal{L}_{W} g(t)\right)_{i j}\right|_{t=0} \\
= & \frac{1}{2} g_{j k} g^{p q} \nabla_{i}\left(g^{k l}\left(\nabla_{p} \frac{\partial}{\partial t} g_{q l}+\nabla_{q} \frac{\partial}{\partial t} g_{p l}-\nabla_{l} \frac{\partial}{\partial t} g_{p q}\right)\right) \\
& +\frac{1}{2} g_{i k} g^{p q} \nabla_{j}\left(g^{k l}\left(\nabla_{p} \frac{\partial}{\partial t} g_{q l}+\nabla_{q} \frac{\partial}{\partial t} g_{p l}-\nabla_{l} \frac{\partial}{\partial t} g_{p q}\right)\right) \\
& +\left(\text { lower-order derivatives of } \frac{\partial}{\partial t} g\right) \\
= & g^{k l}\left(\nabla_{i, k}^{2} \frac{\partial}{\partial t} g_{j l}+\nabla_{j, k}^{2} \frac{\partial}{\partial t} g_{i l}-\nabla_{i, j}^{2} \frac{\partial}{\partial t} g_{k l}\right)+\left(\text { lower-order derivatives of } \frac{\partial}{\partial t} g\right),
\end{aligned}
$$

[^14]where the second step follows from Proposition 3.7. Recall Lemma 3.19 to observe that
$$
\left.D\left[-2 R i c+\mathcal{L}_{W}\right]\right|_{g_{0}}\left(\frac{\partial}{\partial t} g\right)_{i j}=g^{k l} \nabla_{k, l}^{2} \frac{\partial}{\partial t} g_{i j}+\left(\text { lower-order derivatives of } \frac{\partial}{\partial t} g\right)
$$
where we used the symmetry of the metric to rearrange the indices of the first two of equation (3.33). Now, notice that for the principal symbol we get
$$
\left(\hat{\sigma}_{-2 R i c+\mathcal{L}_{W}}(p, \xi) \frac{\partial}{\partial t} g\right)_{i j}=g^{k l} \xi_{k} \xi_{l} \frac{\partial}{\partial t} g_{i j}, \quad(p, \xi) \in T^{*}(M)
$$
which clearly has trivial kernel. We see that the Ricci-DeTurck flow in (3.41) satisfies the requirements of Definition 3.16 and hence enjoys short-time existence and uniqueness.

STEP 2. Consider a solution $g(t)$ to the Ricci-DeTurck flow and a time dependent vector field $W(t)$ defined by (3.39). Since there exists a solution to the Ricci-DeTurck flow, this one-parameter family of vector fields exists for $t \in[0, \epsilon]$. Therefore, by e.g. [22, Theorem 3.3.5] or [31, Lemma 3.15], there exists a one-parameter family of diffeomorphism $\varphi_{t}: M \rightarrow M$ constructed by solving the ODE

$$
\frac{\partial}{\partial t} \varphi_{t}(p)=-W_{\varphi_{t}(p)}(t), \quad \varphi_{0}=\operatorname{id}_{M}
$$

which we recognize as the flow of $-W$. We claim now that $\bar{g}(t)$ is a solution to the Ricci flow obtained by pulling back $g(t)$ with the diffeomorphism $\varphi_{t}$ :

$$
\bar{g}(t):=\varphi_{t}^{*} g(t), \quad t \in[0, \epsilon] .
$$

Notice that since $\varphi_{0}=\operatorname{id}_{M}$, we have $\bar{g}(0)=g(0)=g_{0}$. Moreover, we have that

$$
\begin{align*}
\frac{\partial}{\partial t} \bar{g} & =\frac{\partial}{\partial t} \varphi_{t}^{*} g(t) \\
& =\left.\frac{\partial}{\partial s}\right|_{s=0}\left(\varphi_{t+s}^{*} g(t+s)\right) \\
& =\varphi_{t}^{*}\left(\frac{\partial}{\partial t} g(t)\right)+\left.\frac{\partial}{\partial s}\right|_{s=0}\left(\varphi_{t+s}^{*} g(t)\right) \\
& =\varphi_{t}^{*}\left(-2 \operatorname{Ric}(g(t))+\mathcal{L}_{W(t)} g(t)\right)+\left.\frac{\partial}{\partial s}\right|_{s=0}\left[\left(\varphi_{t}^{-1} \circ \varphi_{t+s}\right)^{*} \varphi_{t}^{*} g(t)\right], \tag{3.42}
\end{align*}
$$

where we used that $\varphi_{t+s}^{*}=\left(\varphi_{t}^{-1} \circ \varphi_{t+s} \circ \varphi_{t}\right)^{*}$. Moreover, recall Definition 2.35 to observe that

$$
\begin{equation*}
\left.\frac{\partial}{\partial}\right|_{s=0}\left[\left(\varphi_{t}^{-1} \circ \varphi_{t+s}\right)^{*} \varphi_{t}^{*} g(t)\right]=\left.\mathcal{L}_{\frac{\partial}{\partial s}}\right|_{s=0}\left(\varphi_{t}^{-1} \circ \varphi_{s+t}\right), \tag{3.43}
\end{equation*}
$$

In addition, notice that

$$
\begin{equation*}
\left.\frac{\partial}{\partial s}\right|_{s=0}\left(\varphi_{t}^{-1} \circ \varphi_{s+t}\right)=\left(\varphi_{t}^{-1}\right)_{*}\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \varphi_{s+t}\right)=\left(\varphi_{t}^{-1}\right)_{*}\left(\frac{\partial}{\partial t} \varphi_{t}\right)=-\left(\varphi_{t}^{-1}\right)_{*} W(t)=-\varphi_{t}^{*} W(t) \tag{3.44}
\end{equation*}
$$

Combining equations (3.43) and (3.44) with (3.42) then yields

$$
\begin{align*}
\frac{\partial}{\partial t} \bar{g} & =-2 \varphi_{t}^{*} \operatorname{Ric}(g(t))+\varphi_{t}^{*} \mathcal{L}_{W(t)} g(t)-\mathcal{L}_{\varphi_{t}^{*} W(t)} \varphi_{t}^{*} g(t) \\
& =-2 \overline{\operatorname{Ric}}(\bar{g}(t)) \tag{3.45}
\end{align*}
$$

which proves the claim and concludes the existence part of the proof ${ }^{24}$.

[^15]STEP 3. Let $(M, \bar{g}(t))$ be a family of Riemannian manifolds such that $\bar{g}(t)$ is a solution to the Ricci flow. Also, let $(\widetilde{M}, \tilde{g})$ be a Riemannian manifold with fixed metric $\tilde{g}$ and corresponding Levi-Civita connection $\widetilde{\nabla}$. Define $\varphi:(M, \bar{g}(t)) \times[0, \epsilon] \rightarrow(\widetilde{M}, \tilde{g})$ by $(3.38)$ such that $\varphi_{0}:\left(M, g_{0}\right) \rightarrow(\widetilde{M}, \tilde{g})$ is a diffeomorphism ${ }^{25}$. Recall (2.44) and (2.45) to observe that if ( $x^{i}$ ) denote local coordinates about $p \in M$ and ( $y^{\alpha}$ ) denote local coordinates about $\varphi_{t}(p) \in \widetilde{M}[45$, Section 1.2.2], then

$$
\begin{align*}
\left(\varphi_{t}\right)_{*} \frac{\partial}{\partial t} & =\Delta_{\bar{g}, \tilde{g}} \varphi_{t} \\
& =\bar{g}^{i j}\left(\nabla_{i}\left(\varphi_{t}\right)_{*}\left(\frac{\partial}{\partial x^{j}}\right)\right) \\
& =\bar{g}^{i j}\left(\widetilde{\nabla}_{\left(\varphi_{t}\right)_{*} \frac{\partial}{\partial x^{i}}}\left(\left(\varphi_{t}\right)_{*} \frac{\partial}{\partial x^{j}}\right)-\left(\varphi_{t}\right)_{*}\left(\bar{\nabla}_{i} \frac{\partial}{\partial x^{j}}\right)\right) \\
& =\bar{g}^{i j}\left(\widetilde{\nabla}_{\left(\varphi_{t}\right)_{*} \frac{\partial}{\partial x^{i}}}\left(\frac{\partial \varphi_{t}^{\alpha}}{\partial x^{j}} \frac{\partial}{\partial y^{\alpha}}\right)-\left(\varphi_{t}\right)_{*} \bar{\Gamma}_{i j}^{k} \frac{\partial}{\partial x^{k}}\right) \\
& =\bar{g}^{i j}\left(\frac{\partial^{2} \varphi_{t}^{\alpha}}{\partial x^{i} \partial x^{j}} \frac{\partial}{\partial y^{\alpha}}+\frac{\partial \varphi_{t}^{\alpha}}{\partial x^{j}} \frac{\partial \varphi_{t}^{\beta}}{\partial x^{i}} \widetilde{\Gamma}_{\alpha \beta}^{\gamma} \frac{\partial}{\partial y^{\gamma}}-\bar{\Gamma}_{i j}^{k} \frac{\partial \varphi_{t}^{\alpha}}{\partial x^{k}} \frac{\partial}{\partial y^{\alpha}}\right) \tag{3.46}
\end{align*}
$$

where we used (2.33) in the third and fourth equality. Let us now define $g(t)=\left(\varphi_{t}^{-1}\right)^{*} \bar{g}(t)$ on $\widetilde{M}$. We claim that this is a unique solution to the Ricci-DeTurck flow. The uniqueness follows from the fact that the harmonic map heat flow has a unique solution. Moreover, notice that

$$
\begin{align*}
\frac{\partial}{\partial t} g & =\frac{\partial}{\partial t}\left(\left(\varphi_{t}^{-1}\right)^{*} \bar{g}(t)\right) \\
& =\left.\frac{\partial}{\partial s}\right|_{s=0}\left(\left(\varphi_{t+s}^{-1}\right)^{*} \bar{g}(t+s)\right) \\
& =\left(\varphi_{t}^{-1}\right)^{*}\left(\frac{\partial}{\partial t} \bar{g}(t)\right)+\left.\frac{\partial}{\partial s}\right|_{s=0}\left(\left(\varphi_{t+s}^{-1}\right)^{*} \bar{g}(t)\right) \\
& =-2 \operatorname{Ric}(g(t))+\left.\frac{\partial}{\partial s}\right|_{s=0}\left[\left(\varphi_{t} \circ \varphi_{t+s}^{-1}\right)^{*}\left(\varphi_{t}^{-1}\right)^{*} \bar{g}(t)\right] \\
& =-2 \operatorname{Ric}(g(t))+\mathcal{L}_{\left(\varphi_{t}\right) * \frac{\partial}{\partial t} \varphi_{t}^{-1}}\left(\varphi_{t}^{-1}\right)^{*} \bar{g}(t) \\
& =-2 \operatorname{Ric}(g(t))+\mathcal{L}_{\left(\varphi_{t}\right) * \frac{\partial}{\partial t} \varphi_{t}^{-1}} g(t) \tag{3.47}
\end{align*}
$$

where we use that the Ricci flow is invariant under diffeomorphisms in the fourth step. Also, note that the fourth and fifth step follow from similar reasoning as to Step 2. It now rests for us to show that the Lie derivative term in (3.47) is equal to the one in the Ricci-DeTurck flow, i.e. that $\left(\varphi_{t}\right)_{*} \frac{\partial}{\partial t} \varphi_{t}^{-1}=-W$ from (3.39). For that, choose local coordinates $\left(x^{i}\right)$ about $\varphi_{t}^{-1}(q) \in M$ and $\left(z^{i}\right)=\left(x^{i}\right) \circ \varphi_{t}^{-1}$ about $q \in \widetilde{M}$. In addition, let $\left(y^{\alpha}\right)$ be local coordinates fixed about $\varphi_{t} \circ \varphi_{t}^{-1}(q) \in \widetilde{M}$, then observe that

$$
\begin{aligned}
y \circ z^{-1} & =y \circ \varphi_{t} \circ x^{-1} \\
& =y \circ\left(\varphi_{t} \circ \varphi_{t}^{-1}\right) \circ z^{-1} .
\end{aligned}
$$

I.e., we see that $\bar{g}_{i j}=g_{i j}$ and therefore that $\bar{\Gamma}_{i j}^{k}=\Gamma_{i j}^{k}$ such that $\varphi_{t}^{\alpha}=\left(\varphi_{t} \circ \varphi_{t}^{-1}\right)^{\alpha}$. This means that

[^16](3.46) gives us
\[

$$
\begin{align*}
\left.\left(\varphi_{t}\right)_{*} \frac{\partial}{\partial t}\right|_{\varphi_{t}^{-1}(q)} & =\left(\Delta_{\bar{g}, \tilde{g}} \varphi_{t}\right)_{\varphi_{t}^{-1}(q)} \\
& =\bar{g}^{i j}\left(\frac{\partial^{2} \varphi_{t}^{\alpha}}{\partial x^{i} \partial x^{j}} \frac{\partial}{\partial y^{\alpha}}+\frac{\partial \varphi_{t}^{\alpha}}{\partial x^{j}} \frac{\partial \varphi_{t}^{\beta}}{\partial x^{i}} \widetilde{\Gamma}_{\alpha \beta}^{\gamma} \frac{\partial}{\partial y^{\gamma}}-\bar{\Gamma}_{i j}^{k} \frac{\partial \varphi_{t}^{\alpha}}{\partial x^{k}} \frac{\partial}{\partial y^{\alpha}}\right)_{\varphi_{t}^{-1}(q)} \\
& =g^{i j}\left(\frac{\partial^{2}\left(\varphi_{t} \circ \varphi_{t}^{-1}\right)^{\alpha}}{\partial x^{i} \partial x^{j}} \frac{\partial}{\partial y^{\alpha}}+\frac{\partial\left(\varphi_{t} \circ \varphi_{t}^{-1}\right)^{\alpha}}{\partial x^{j}} \frac{\partial\left(\varphi_{t} \circ \varphi_{t}^{-1}\right)^{\beta}}{\partial x^{i}} \widetilde{\Gamma}_{\alpha \beta}^{\gamma} \frac{\partial}{\partial y^{\gamma}}-\Gamma_{i j}^{k} \frac{\partial\left(\varphi_{t} \circ \varphi_{t}^{-1}\right)^{\alpha}}{\partial x^{k}} \frac{\partial}{\partial y^{\alpha}}\right)_{q} \\
& =\left(\Delta_{\left.g, \tilde{g}\left(\varphi_{t} \circ \varphi_{t}^{-1}\right)\right)_{q}}\right. \\
& =\left.\left(\varphi_{t} \circ \varphi_{t}^{-1}\right)_{*} \frac{\partial}{\partial t}\right|_{q} \tag{3.48}
\end{align*}
$$
\]

In other words, the harmonic map Laplacian is invariant under change of diffeomorphisms. In our specific case, since $\varphi_{t} \circ \varphi_{t}^{-1}(q)=q$, we simply have

$$
\begin{align*}
\left(\varphi_{t}\right)_{*} \frac{\partial}{\partial t} \varphi_{t}^{-1} & =g^{j k}\left(\widetilde{\Gamma}_{j k}^{i}-\Gamma_{j k}^{i}\right) \frac{\partial}{\partial x^{i}} \\
& =-W \tag{3.49}
\end{align*}
$$

which proofs the claim that $g(t)$ is a unique solution to the Ricci-DeTurck flow.
STEP 4. Now, suppose that both $\bar{g}_{1}(t)$ and $\bar{g}_{2}(t)$ are two solutions to the Ricci flow with $\bar{g}_{1}(0)=\bar{g}_{2}(0)$. Let $\widetilde{M}=M$ and $\varphi_{0}=\operatorname{id}_{M}$. Using Step 3, we could provide one with solutions to the Ricci-DeTurck flow $g_{1}(t)$ and $g_{2}(t)$ which satisfy $g_{1}(0)=\bar{g}_{1}(0)=\bar{g}_{2}(0)=g_{2}(0)$, since $\varphi_{0}=\mathrm{id}_{M}$. Consequently, since solutions to the Ricci-DeTurck flow are unique, we have that $g_{1}(t)=g_{2}(t)$ for all $t \in[0, \epsilon]$. Therefore, the time dependent vector field $W$ is also the same for the two solutions. The unique solutions to the harmonic map heat flow $\varphi_{1, t}$ and $\varphi_{2, t}$ together with (3.49) show us that

$$
\begin{equation*}
\frac{\partial}{\partial t} \varphi_{i, t}(p)=-W_{\varphi_{i, t}(p)}(t), \quad \varphi_{i, 0}=\operatorname{id}_{M} \tag{3.50}
\end{equation*}
$$

and hence that $\varphi_{1, t}(p)=\varphi_{1, t}(p)$. We see that therefore $\bar{g}_{1}=\varphi_{1, t}^{*} g_{1}=\varphi_{2, t}^{*} g_{2}=\bar{g}_{2}$, which proves uniqueness to the Ricci flow.

## 4 Almost-Riemannian geometry \& Ricci flow

In this research, we are particularly interested in what happens when Ricci flow is practised on an almost-Riemannian structure. This type of structure is an example of a sub-Riemannian structure, just like Riemannian manifolds. The chapter provides a brief introduction of sub-Riemannian structures and almost-Riemannian structures, after which we examine the possibilities of evolving Ricci flow on two-dimensional almost-Riemannian manifolds.

### 4.1 Definitions \& properties

This first part of this section is based upon [46, 47, 27] and [48]. We always assume that $M$ is a $n$ dimensional smooth manifold, i.e. not necessarily a Riemannian manifold. At a certain point, we will put our attention only on two-dimensional smooth manifolds with an almost-Riemannian structure. Then, we will use results of [49, 17, 50] and [51].

To define a sub-Riemannian structure, we firstly have to broaden our knowledge on Lie brackets (recall Definition 2.22) and smooth vector fields. The set of all smooth vector fields over $M$ has both the structure of a real vector space, and of a $C^{\infty}(M)$-module ${ }^{26}[22$, Proposition 3.1.5]. When equipped with the Lie bracket, it turns out that $\mathfrak{X}(M)$ is also a Lie algebra over $\mathbb{R}$ [27, Proposition 3.9].
Definition 4.1. A Lie algebra over a field $\mathbb{K}$ is a pair $(V,[\cdot, \cdot])$, where $V$ is a vector space over $\mathbb{K}$ and $[\cdot, \cdot]: V \times V \rightarrow V$ is the Lie bracket that satisfies the following properties:
(i) Bilinearity over $\mathbb{K}$. I.e., for all $a, b \in \mathbb{K}$ and $x, y, z \in V,[a x+b y, y]=a[x, z]+b[y, z]$ and $[z, a x+b y]=$ $a[z, x]+b[z, y] ;$
(ii) Antisymmetry. I.e., for all $x, y \in V,[x, y]=-[y, z]$;
(iii) The Jacobi identity. I.e., for all $x, y, z \in V,[x,[y, z]]+[z,[x, y]]+[y,[z, x]]=0$.

We also need to define such algebra's for subsets of $\mathfrak{X}(M)$.
Definition 4.2. Let $\mathcal{F} \subseteq \mathfrak{X}(M)$ be a family of smooth vector fields. The Lie algebra generated by $\mathcal{F}$ is smallest Lie subalgebra of $\mathfrak{X}(M)$ containing $\mathcal{F}$. That is,

$$
\begin{equation*}
\text { Lie } \mathcal{F}:=\operatorname{span}\left\{\left[X_{1}, \ldots,\left[X_{j-1}, X_{j}\right]\right] \mid X_{i} \in \mathcal{F}, j \in \mathbb{N}\right\} \tag{4.1}
\end{equation*}
$$

If for each $p \in M$ the evaluation at $p$ of Lie $\mathcal{F}$ equals $T_{p} M$, then we say that $\mathcal{F}$ satisfies the Hörmander condition. Mathematically, that is

$$
\begin{equation*}
\operatorname{Lie}_{p} \mathcal{F}=\{X(p) \mid X \in \operatorname{Lie} \mathcal{F}\}=T_{p} M \tag{4.2}
\end{equation*}
$$

Definition 4.3. Let $M$ be connected smooth $n$-dimensional manifold. A sub-Riemannian structure on $M$ is a pair $(\mathbf{U}, f)$ for which the following properties hold:
(i) $\mathbf{U}$ is a Euclidean bundle with base space $M$. An Euclidean bundle is a vector bundle whose fibres $U_{p}$, for all $p \in M$, are equipped with a smoothly varying inner product $\langle\cdot, \cdot\rangle_{p}$ with respect to $p$;
(ii) The map $f: \mathbf{U} \rightarrow T M$ is a smooth map that is a morphism ${ }^{27}$ of vector bundles such that its restriction to any fibre of $\mathbf{U}$ is linear and the following diagram commutes:


[^17]where $\pi_{\mathbf{U}}: \mathbf{U} \rightarrow M$ and $\pi: T M \rightarrow M$ denote the projections;
(iii) The collection of smooth vector fields $\mathcal{D}=\{f \circ \sigma \mid \sigma: M \rightarrow \mathbf{U}$ is a smooth section $\}$ satisfies the Hörmander condition.

We say that the triple $(M, \mathbf{U}, f)$ is a sub-Riemannian manifold.
Example 4.4. If we take $\mathbf{U}=T M$ such that $f: T M \rightarrow T M$ and $f\left(U_{p}\right)=T_{p} M$, then $(M, \mathbf{U}, f)$ is a Riemannian manifold.

We will soon see that almost-Riemannian manifolds are also specific types of sub-Riemannian manifolds. More elaborately, they are the prototypes of rank-varying sub-Riemannian manifolds. To see what that means, consider the following definition.

Definition 4.5. Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold. Its distribution is the family of subspaces $\{\mathcal{D}(p)\}_{p \in M}$, where

$$
\begin{equation*}
\mathcal{D}(p):=\{f \circ \sigma(p) \mid f \circ \sigma \in \mathcal{D}\}=f\left(U_{p}\right) \subseteq T_{p} M \tag{4.4}
\end{equation*}
$$

The bundle rank of the sub-Riemannian structure is $k=\operatorname{rank}(\mathbf{U})$, and $r(p):=\operatorname{dim} \mathcal{D}_{p}$ is the rank of the sub-Riemannian structure at $p \in M$. So, if $r(p)$ is constant, then $(\mathbf{U}, f)$ has constant rank. If this is not the case, then $(\mathbf{U}, f)$ is called rank-varying. For every $p \in M$, the rank of a sub-Riemannian manifold $(M, \mathbf{U}, f)$ hence satisfies

$$
\begin{equation*}
r(p) \leq \min \{k, n\} . \tag{4.5}
\end{equation*}
$$

In order to define distances between two points, we need the concept of admissible curves and their lengths.

Definition 4.6. A Lipschitz curve $\gamma:[0, T] \rightarrow M$ is admissible for a sub-Riemannian structure if there exists a measurable essentially bounded function ${ }^{28} t \mapsto u(t)$, called a control, such that

$$
\begin{equation*}
\dot{\gamma}(t)=f(\gamma(t), u(t)), \quad \text { for a.e. }{ }^{29} t \in[0, T] . \tag{4.6}
\end{equation*}
$$

Moreover, we define the sub-Riemannian length of an admissible curve $\gamma$ as ${ }^{30}$

$$
\begin{equation*}
\ell(\gamma):=\int_{0}^{T}\|\dot{\gamma}(t)\| d t \tag{4.7}
\end{equation*}
$$

Having obtained a notion of length, we can define distances.
Definition 4.7. The sub-Riemannian distance between two points $p, q$ of a sub-Riemannian manifold $M$ is

$$
\begin{equation*}
d(p, q)=\inf \{\ell(\gamma) \mid \gamma:[0, T] \rightarrow M \text { is admissible, } \gamma(0)=p, \gamma(T)=q\} \tag{4.8}
\end{equation*}
$$

In fact, if $M$ is a sub-Riemannian manifold, by the Chow-Rashevskii theorem we have that $(M, d)$ is a metric space and the topology induced on $M$ is equivalent to the regular manifold topology [46, Theorem 3.31]. This ensures that there exists an admissible curve between any two points in $M$. Moreover, $(M, d)$ is complete when there exists an $\epsilon>0$ such that the closed ball $\bar{B}_{r}(p)$ is compact for every $p \in M[46$, Proposition 3.47]. One could also define geodesics in this setting. As it turns out, these are characterized as normal Pontryagin extremal trajectories. For more detail, see [46, Section 4.7].

[^18]Definition 4.8. Let $(\mathbf{U}, f)$ and $(\tilde{\mathbf{U}}, \tilde{f})$ be two sub-Riemannian structures on $M$. The structures are equivalent as distributions if the following conditions hold:
(i) There exists a Euclidean bundle $\mathbf{V}$ and two surjective bundle morphisms $p: \mathbf{V} \rightarrow \mathbf{U}$ and $\tilde{p}: \mathbf{V} \rightarrow \tilde{\mathbf{U}}$ such that the following diagram commutes;

(ii) The projections $p, \tilde{p}$ are compatible with the scalar product. That is, we have ${ }^{31}$

$$
\begin{aligned}
|u|=\min \{|v|, p(v)=u\}, & \forall u \in \mathbf{U} \\
|\tilde{u}|=\min \{|v|, \tilde{p}(v)=\tilde{u}\}, & \forall \tilde{u} \in \widetilde{\mathbf{U}}
\end{aligned}
$$

If (i) and (ii) of Definition 4.8 are satisfied, we say that the two structures $(\mathbf{U}, f)$ and $(\tilde{\mathbf{U}}, \tilde{f})$ are equivalent (as sub-Riemannian structures).
Definition 4.9. Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold. The minimal bundle rank of $M$ is defined as the infimum of bundle ranks that induce equivalent structures on $M$. The local minimal bundle rank at $p \in M$ is the minimal bundle rank of the structure restricted to a sufficiently small neighbourhood $O_{p} \ni p$.

Almost-Riemannian structures are sub-Riemannian structures such that its local minimum bundle rank at every point is equal to the dimension of the manifold. For a given sub-Riemannian $(\mathbf{U}, f)$ with a constant local minimum bundle rank $k$, we can always find a sub-Riemannian structure $(\widetilde{\mathbf{U}}, \tilde{f})$ equivalent to $(\mathbf{U}, f)$ such that rank $\widetilde{\mathbf{U}}=k[46]$. Therefore, we can define almost-Riemannian structures as follows:

Definition 4.10. Let $M$ be connected smooth $n$-dimensional manifold. A $n$-dimensional almost-Riemannian structure ( $n$-ARS) on $M$ is a pair $(\mathbf{U}, f)$ for which the following properties hold:
(i) $\mathbf{U}$ is a rank $n$ Euclidean bundle with base space $M$. A Euclidean bundle is a vector bundle whose fibres $U_{p}$, for all $p \in M$, are equipped with a smoothly varying inner product $\langle\cdot, \cdot\rangle_{p}$ with respect to $p ;$
(ii) The map $f: \mathbf{U} \rightarrow T M$ is a smooth map that is a morphism of vector bundles such that its restriction to any fibre of $\mathbf{U}$ is linear and the following diagram commutes:
where $\pi_{\mathbf{U}}: \mathbf{U} \rightarrow M$ and $\pi: T M \rightarrow M$ denote the projections;
(iii) The collection of smooth vector fields $\mathcal{D}=\{f \circ \sigma \mid \sigma: M \rightarrow \mathbf{U}$ is a smooth section $\}$ satisfies the Hörmander condition.

[^19]Moreover, we introduce the step of the distribution. As we will see, for some points $p \in M$ with a $n$-ARS, one only needs the span of $n$ vector fields to span the tangent space. For other points, we will indeed need them to obey the Hörmander condition to span $T_{p} M$.
Definition 4.11. The step of the distribution at $p \in M$ is the minimal $s \in \mathbb{N}$, with $s \geq 1$, such that $\mathcal{D}_{s}(p)=T_{p} M$, where $\mathcal{D}_{1}:=\mathcal{D}$ and $\mathcal{D}_{i+1}:=\mathcal{D}_{i}+\left[\mathcal{D}_{1}, \mathcal{D}_{i}\right]$, for $i \geq 1$.

Let us now discuss some of the properties of almost-Riemannian structures.
Definition 4.12. Let $\Omega \subset M$. An orthonormal frame on $\Omega$ for a $n$-ARS is the set of vector fields

$$
\begin{equation*}
\left\{X_{1}, X_{2}, \ldots, X_{n}\right\}=\left\{f \circ \sigma_{1}, f \circ \sigma_{2}, \ldots, f \circ \sigma_{n}\right\} \tag{4.11}
\end{equation*}
$$

where $\left\{\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n}\right\}$ is an orthonormal frame on a local trivialization $\Omega \times \mathbb{R}^{n}$ of $\mathbf{U}$ with respect to $\langle\cdot, \cdot\rangle_{p}$ for all $p \in \Omega$.

If such an orthonormal frame exists on $M$, then we say that the $n$-ARS is free.
Definition 4.13. The singular set (or singular locus) $\mathcal{Z}$ on a $n$-ARS of $M$ is the set of points $p \in M$ such that $r(p)<n$. We call these points singular points.

The singular set is what differs between an $n$-ARS and a Riemannian structure on $M$. Formally, we have that a $n$-ARS is a Riemannian structure on $M \backslash \mathcal{Z}$ [27, Theorem 3.20]. Hence, we refer to $p \in M \backslash \mathcal{Z}$ as Riemannian points.

### 4.1.1 Two-dimensional almost-Riemannian structures

From now one, we restrict ourselves to 2-ARS and hence a smooth connected two-dimensional manifold $M$. Most of the research on almost-Riemannian structures has been on this particular case, which makes it a suitable choice to see what happens when Ricci flow is evaluated on a surface endowed with a 2 -ARS. For example, while the metric and curvature explode when approaching the singular set, geodesics can pass through it. In particular, all geodesic are smooth and they coincide with the set of non-trivial normal Pontryagin extremal trajectories [46, Corollary 9.25, Proposition 9.26]. Specifically, an admissible curve $\gamma:[0, T] \rightarrow M$ is a geodesic for a 2-ARS if for every sufficiently small non-trivial interval $\left[t_{1}, t_{2}\right] \subset[0, T]$, $\left.\gamma\right|_{\left[t_{1}, t_{2}\right]}$ is a minimizer of $\ell(\cdot)$ from (4.7). If $M$ is compact, then any two points of $M$ are connected by a minimizing geodesic as a consequence of the Chow-Rashevskii theorem [51, Section 2.2].

Before moving on, recall Definition 2.8 and Remark 2.18 for the following definition
Definition 4.14. A 2-ARS $(\mathbf{U}, f)$ over $M$ is said to be oriented if $\mathbf{U}$ is oriented. It is said to be fully oriented if both $M$ and $\mathbf{U}$ are oriented.

Remark 4.15. It is in fact possible to define a non-orientable almost-Riemannian structure on orientable manifolds, and vice-versa [17, Chapter 1].

Notice that in the 2-ARS case, we have [55, Section 1.2]

$$
\begin{equation*}
\mathcal{Z}=\{p \in M \mid \operatorname{dim}(\mathcal{D}(p))=1\} \tag{4.12}
\end{equation*}
$$

As shown in [46, Section 9.4.1], a 2-ARS on a smooth connected two-dimensional manifold $M$ generically satisfies the following properties:
(i) The singular set $\mathcal{Z}$ is an embedded one-dimensional submanifold of $M$;
(ii) The points $p \in M$ at which $\operatorname{dim}\left(\mathcal{D}_{2}(p)\right)=1$ are isolated;
(iii) For every $p \in M$ we have $\mathcal{D}_{3}(p)=T_{p} M$.

We refer to the above three conditions as $\mathbf{H 0}$. Assuming that $\mathbf{H 0}$ is true, one could show that, generically, a (local) orthonormal frame is given by the vector fields $X_{1}(x, y)=\left(\begin{array}{ll}1 & 0\end{array}\right)^{\top}$ and $X_{2}(x, y)=\left(\begin{array}{ll}0 & f(x, y)\end{array}\right)^{\top}$ on $\mathbb{R}^{2}$, with $f(x, y)$ a smooth function [17, Theorem 16]. The following proposition enables us to express the local behaviour of $g$ and the scalar curvature ${ }^{32} S$ with these two vector fields [27, Theorem 3.21].

[^20]Proposition 4.16. Let $(x, y)$ denote local coordinates on $\Omega \subset M$ with an orthonormal frame for a 2 -ARS on $\Omega$ of the form

$$
\begin{equation*}
X_{1}(x, y)=\binom{1}{0}, \quad X_{2}(x, y)=\binom{0}{f(x, y)} \tag{4.13}
\end{equation*}
$$

with $f: \Omega \rightarrow \mathbb{R}$ a smooth function. The singular set is then given by $\mathcal{Z}=\{(x, y) \in \Omega \mid f(x, y)=0\}$ and on $\Omega \cap(M \backslash \mathcal{Z})$ we have the following expressions for the metric $g$ and the scalar curvature $S$ :

$$
\begin{equation*}
g=d x^{2}+\frac{1}{f^{2}} d y^{2}, \quad S=\frac{-2\left(\frac{\partial f}{\partial x}\right)^{2}+f \frac{\partial^{2} f}{\partial x^{2}}}{2 f^{2}} \tag{4.14}
\end{equation*}
$$

Proof. Observe that $X_{1}(p)$ and $X_{2}(p)$ are linearly independent if and only if $f(p)=0$. I.e., in that case we have $\operatorname{dim}(\mathcal{D}(p))=1$ such that by (4.12) we indeed have $\mathcal{Z}=\{(x, y) \in \Omega \mid f(x, y)=0\}$. Recall (2.22) to note that $\left\{X_{1}, X_{2}\right\}$ is an orthonormal frame on the Riemannian points if

$$
g_{p}\left(X_{i}(p), X_{j}(p)\right)=\delta_{i j}
$$

Clearly, we then must have that $g=d x^{2}+\frac{1}{f^{2}} d y^{2}$. For the scalar curvature, one could first compute the Christoffel symbols and, subsequently, the coefficients of the Riemann curvature endomorphism. The expression for $S$ in (4.14) can then be obtained by applying multiple consecutive traces (that is, apply equations (2.61), (2.66) and (2.67)).

Remark 4.17. This expression for the scalar curvature is especially relevant, since we will analyse the Ricci flow on almost-Riemannian surfaces in the next section and Ric $=K g=\frac{S}{2} g$ in these dimensions ${ }^{33}$.

Also under the assumption that HO is true, we can distinguish between three kinds of points for a 2-ARS on $M$ [17, Definition 19].

Definition 4.18. Consider a 2 -ARS on $M$ such that $\mathbf{H} 0$ holds. We call a point $p \in M$ a
(i) Riemannian point if $\mathcal{D}(p)=T_{p} M$;
(ii) Grushin point if $\operatorname{dim}(\mathcal{D}(p))=1$ and $\mathcal{D}_{2}(p)=T_{p} M$;
(iii) tangency point if $\operatorname{dim}(\mathcal{D}(p))=\operatorname{dim}\left(\mathcal{D}_{2}(p)\right)=1$ and $\mathcal{D}_{3}(p)=T_{p} M$.

Equivalently, if $p$ is a Grushin point, then the distribution $\mathcal{D}(p)$ is transversal to $T_{p} \mathcal{Z}$. To see this, let us adopt the assumptions of Proposition 4.16 to notice that then $\mathcal{D}(p)=\operatorname{span}\left\{\left(\begin{array}{ll}1 & 0\end{array}\right)^{\top}\right\}$ and (by working out $\left.\left[X_{1}, X_{2}\right]\right)$ that $T_{p} \mathcal{Z}=\operatorname{span}\left\{\left(-\frac{\partial f}{\partial y}(p) \quad \frac{\partial f}{\partial x}(p)\right)^{\top}\right\}$, which is transversal since $\frac{\partial f}{\partial x}(p) \neq 0$ by definition of a Grushin point. Similarly, If $p$ is a tangency point, then $\mathcal{D}(p)$ coincides with $T_{p} \mathcal{Z}$, i.e. it is tangent to $\mathcal{Z}$. In that case, we again have $\mathcal{D}(p)=\operatorname{span}\left\{\left(\begin{array}{ll}1 & 0\end{array}\right)^{\top}\right\}$, but now $T_{p} \mathcal{Z}=\operatorname{span}\left\{\left(-\frac{\partial f}{\partial y}(p) \quad 0\right)^{\top}\right\}$, since $\mathcal{D}_{2}(p) \neq T_{p} M$ for a tangency point [46, Proposition 9.36].

Example 4.19. The Grushin sphere is an example of a free almost-Riemannian structure on $\mathbb{S}^{2}$ with $(\mathbf{U}, f)$ given by $\mathbf{U}=\mathbb{S}^{2} \times \mathbb{R}^{2}$ and $f\left(\theta, \varphi, u_{1}, u_{2}\right)=\left(\theta, \varphi, u_{1}, u_{2} \tan \theta\right)$ (see also [49, Table 1]). In contrast to Example 2.47, let us now parametrize $\mathbb{S}^{2}$ by

$$
x=\sin \theta \cos \varphi, \quad y=\sin \theta \sin \varphi, \quad z=\cos \theta, \quad\left\{(\theta, \varphi) \left\lvert\,-\frac{\pi}{2}<\theta<\frac{\pi}{2}\right., 0 \leq \varphi \leq 2 \pi\right\}
$$

If we set $\mathcal{D}=\left\{X_{1}, X_{2}\right\}$ with

$$
X_{1}(\theta, \varphi)=\binom{1}{0}, \quad X_{2}(\theta, \varphi)=\binom{0}{\tan \theta}
$$

[^21]then $X_{1}$ and $X_{2}$ span $T_{p} \mathbb{S}^{2}$ except at points with $\theta=0$ (See also Figure 4). By Proposition 2.23, we have
$$
\left[X_{1}, X_{2}\right](\theta, \varphi)=\left.\frac{\partial \tan \theta}{\partial \theta} \frac{\partial}{\partial \varphi}\right|_{p}=\left.\frac{1}{\cos ^{2} \theta} \frac{\partial}{\partial \varphi}\right|_{p}=\binom{0}{\frac{1}{\cos ^{2} \theta}}
$$
such that $\mathcal{D}_{2}(p)=T_{p} M$ for all $p \in \mathbb{S}^{2}$. I.e., $\mathcal{D}$ satisfies the Hörmander condition. In particular, we have $\mathcal{Z}=\{(\theta, \varphi) \mid \theta=0\}$ and the Riemannian metric on $\mathbb{S}^{2} \backslash \mathcal{Z}$ given by
$$
g=d \theta^{2}+\frac{1}{\tan ^{2} \theta} d \varphi^{2}
$$
due to Proposition 4.16. In addition, for the scalar curvature we have
$$
S=\frac{-2\left(\frac{1}{\cos ^{2} \theta}\right)^{2}+2 \frac{\tan ^{2} \theta}{\cos ^{2} \theta}}{2 \tan ^{2} \theta}=\frac{\sin ^{2} \theta-1}{\cos ^{2} \theta \sin ^{2} \theta}=-\frac{1}{\sin ^{2} \theta} .
$$


Integral curves of $X_{1}$


Integral curves of $X_{2}$


Figure 4: The unit sphere $\mathbb{S}^{2}$ with a 2 -ARS.

### 4.2 A discussion on Ricci flow on two-dimensional almost-Riemannian manifolds

In this section, we investigate the possibilities of evolving the Ricci flow on a complete, connected smooth two-dimensional manifold $M$ endowed with a 2-ARS. One would suppose that we can best focus on $M \backslash \mathcal{Z}$, since there the structure is Riemannian. From (4.14), it becomes apparent that, generically, both the metric and curvature explode when a point $p$ approaches $\mathcal{Z}$. Hence, it seems prudent to consider the space $M_{\epsilon}:=\{p \in M \mid d(p, \mathcal{Z})>\epsilon\}$, where $d(\cdot, \cdot)$ denotes the almost-Riemannian distance ${ }^{34}$. If we additionally require $M$ to be compact and oriented, we have the following two results from [17, Lemma 24, Lemma 25]:

[^22]Lemma 4.20. Every compact orientable two-dimensional manifold admits a free 2-ARS satisfying H0 and having no tangency points.

Lemma 4.21. Let $M$ be a compact and oriented. For a 2-ARS on $M$ satisfying $\mathbf{H 0}$, the singular set $\mathcal{Z}$ is the union of finitely many curves diffeomorphic to $\mathbb{S}^{1}$. Moreover, there exists and $\epsilon_{0}>0$, such that, for every $0<\epsilon<\epsilon_{0}$, the set $M \backslash M_{\epsilon}$ is homeomorphic to $\mathcal{Z} \times[0,1]$. Under the additional assumption that $M$ contains no tangency points, $\epsilon_{0}$ can be taken in such a way that $\partial M_{\epsilon}$ is smooth for every $0<\epsilon<\epsilon_{0}$.

From now on, we therefore assume $M$ to also be compact and oriented, and choose the free 2-ARS satisfying H0 on $M$ without tangency point such that we can take $\epsilon_{0}>0$, such that $\partial M_{\epsilon}$ is smooth for every $0<\epsilon<\epsilon_{0}$. Given an orientation on $M$, let us also consider $M^{+}$as the subset of $M \backslash \mathcal{Z}$ having the same orientation as $M$, and $M^{-}$as the subset of $M \backslash \mathcal{Z}$ having the opposite orientation ${ }^{35}$. We are interested in the topology of $M_{\epsilon}^{ \pm}:=M^{ \pm} \cap M_{\epsilon}$. Then, we can examine the evolution of Ricci flow on $\left(\lim _{\epsilon \rightarrow 0} M_{\epsilon}^{ \pm}, g_{0}\right)$. First of all, notice that $M_{\epsilon}$ is compact with smooth boundary $\partial M_{\epsilon}$. To add to this, we have that $M_{\epsilon}$ is non-complete. Recall that when $M$ is compact, any two points of $M$ are connected by a minimizing geodesic due to the Chow-Rashevskii theorem. If we remove the singular set $\mathcal{Z}$, a geodesic from a point $p \in M_{\epsilon}^{+}$to $q \in M_{\epsilon}^{-}$can never be achieved. Hence, we are dealing with two non-complete compact surfaces $M_{\epsilon}^{ \pm}$with smooth boundaries $\partial M_{\epsilon}^{ \pm}$and possibly multiple connected components.

In the following two paragraphs, we discuss various results of Ricci flow on surfaces with boundary and non-complete surfaces respectively. Two of those were already mentioned in the introduction of this thesis. In order to analyse whether they may apply to our case, we ignore non-completeness in the first paragraph, and ignore the boundary in the second paragraph.

### 4.2.1 Ricci flow on surfaces with boundary

Theorem 3.21 only applies to compact Riemannian manifolds without boundary, i.e. closed manifolds. The surfaces that we are dealing with now, have significant different topological characteristics. On a surface $M$ with boundary, the aim is to impose boundary conditions in such a way, that the Ricci-DeTurck flow is a parabolic boundary value problem. The first result for Ricci flow on manifolds with boundary is from [56]. There, the author proofs short-time existence to the flow on compact manifolds with umbilic boundary and, as a specific case, totally geodesic boundary ${ }^{36}$. This result was recently improved in [10], where the authors do not necessarily start with umbilic boundary, but will become so in positive time. With umbilic boundary, we mean that the identity $I I_{\alpha \beta}=\lambda g_{\alpha \beta}$, with $\lambda$ a constant and $I I$ the second fundamental form ${ }^{37}$, holds on $\partial M$ [56, Definition 1.3]. In [57], similar results have been established for manifolds with boundary of merely constant mean curvature. For an arbitrary initial metric on a compact manifold with boundary, short-time existence and uniqueness to the Ricci flow is proved by prescribing the mean curvature and conformal class in [58]. Existence and uniqueness for normalized Ricci flow (or related curvature flows) on surfaces with boundary with various curvature constraints and assumptions are shown in $[59,60,61]$ and [62]. Finally, the un-normalized Ricci flow on surfaces with boundary is briefly dealt with in [63]. Although [56, 10, 57, 58] provide results for manifolds with boundary, it remains to be shown if they apply to two-dimensional manifolds endowed with a 2-ARS. Since [59, 60, 61, 62, 63] deal specifically with surfaces, their results are most likely to naturally apply to our case. Therefore, we will provide a brief analysis on whether this can be achieved. As mentioned before, for the moment we ignore the non-completeness of $M_{\epsilon}^{ \pm}$.

In [59] and [60] the authors require positive scalar curvature. Recall (4.14) to observe that we then must find a smooth function $f: \Omega \rightarrow \mathbb{R}$ that satisfies

$$
-2\left(\frac{\partial f}{\partial x}\right)^{2}+f \frac{\partial^{2} f}{\partial x^{2}}>0
$$

[^23]Let us assume that $f(x, y)=x^{\alpha}$, for some $\alpha \in \mathbb{R}$. Then

$$
\begin{aligned}
0<-2\left(\frac{\partial f}{\partial x}\right)^{2}+f \frac{\partial^{2} f}{\partial x^{2}} & =-2 \alpha^{2} x^{2(\alpha-1)}+\alpha(\alpha-1) x^{2 \alpha-2} \\
& =-\alpha^{2} x^{2(\alpha-1)}-\alpha x^{2(\alpha-1)} \quad \Longrightarrow \quad-1<\alpha<0
\end{aligned}
$$

But, if we choose $-1<\alpha<0$, then points lie in $\mathcal{Z}$ only when $x \rightarrow \infty$. Although this choice of $f$ does not cover all possibilities, it certainly illustrates the difficulties of finding a suitable smooth function such that we obtain positive scalar curvature on $M \backslash \mathcal{Z}$.

Hence, let us consider the results from [62]. The author obtains two results: one for an initial metric with vanishing geodesic curvature on $\partial M$ and constant Gaussian curvature in $M$, and one for an initial metric with vanishing Gaussian curvature in $M$ and constant geodesic curvature on $\partial M$. The latter situation gives rise to similar difficulties as before. For the former note that (4.14) implies that $f: \Omega \rightarrow \mathbb{R}$ is constant if it satisfies the differential equation

$$
-2\left(\frac{\partial f}{\partial x}\right)^{2}+f \frac{\partial^{2} f}{\partial x^{2}}=f^{2} C, \quad C \in \mathbb{R}
$$

However, solving this equation does not guarantee that $X_{1}$ and $X_{2}$ satisfy the Hörmander condition.
In [61], the author proves existence and uniqueness for all times to the evolution equation

$$
\begin{cases}\frac{\partial}{\partial t} g(t)=-\frac{2}{\alpha}(K(t)-\alpha \lambda) g(t), & \text { in } M \times(0, T]  \tag{4.15}\\ \frac{\partial}{\partial t} g(t)=-\frac{2}{\beta}(\kappa(t)-\beta \lambda) g(t), & \text { on } \partial M \times(0, T] \\ g(0)=g_{0}, & \text { in } M,\end{cases}
$$

with $\alpha, \beta \in \mathbb{R}$ and by letting $A$ denote the area of the surface and $r$ the line along $\partial M$,

$$
\lambda=\frac{2 \pi \chi(M)}{\alpha \int_{M} d A+\beta \int_{\partial M} d r}
$$

Using the Gauss-Bonnet formula, the author derives that $\lambda$ lies between $\chi(M)$ and $2 \chi(M)$. The solution converges exponentially to a metric with constant Gaussian curvature in $M$ and constant geodesic curvature on $\partial M$. The regular Gauss-Bonnet theorem on a compact oriented two-dimensional manifold $M$ asserts that $\int_{M} K d A+\int_{\partial M} \kappa d r=2 \pi \chi(M)$. In [17, Section 5.2], the authors derive a Gauss-Bonnet like formula for a generic oriented 2-ARS without tangency points. The Gauss-Bonnet theorem can be applied to the both of the surfaces $\left(M_{\epsilon}^{ \pm}, g_{0}\right)$. Hence, it seems promising to consider the evolution equation (4.15) on both surfaces $\left(M_{\epsilon}^{ \pm}, g_{0}\right)$ individually and argue whether the same results can be acquired for the similar normalized Ricci flow.

Lastly, let us discuss the results of [63]. As mentioned before, [59, 60, 61, 62] all apply to the normalized Ricci flow. The un-normalized Ricci flow on a surface with boundary that is dealt with in [63] is given by the following equations ${ }^{38}$ :

$$
\begin{cases}\frac{\partial}{\partial t} g(t)=-S(t) g(t), & \text { in } M \times(0, T]  \tag{4.16}\\ \kappa(\cdot, t)=\psi(\cdot, t), & \text { on } \partial M \times(0, T] \\ g(0)=g_{0}, & \text { in } M,\end{cases}
$$

with $\psi$ a real-valued smooth function on $\partial M \times[0, \infty)$. One might argue that (4.16) can be applied to the surfaces $\left(M_{\epsilon}^{ \pm}, g_{0}\right)$ to obtain short-time existence and uniqueness results. For the moment we will totally ignore the presence of the singular set $\mathcal{Z}$. Because, after a deformation of the metrics of $M_{\epsilon}^{ \pm}$without a fixed volume constraint, it is unlikely that we can still consider whole of $M$ as smooth connected manifold. As mentioned in Remark 3.2, on a surface without boundary, the deformation is conformal such that a solution is given by $g(t)=e^{v(t)} g_{0}$. This result can be extended to a surface with boundary, such that

[^24]we obtain a non-linear partial differential equation with Robin boundary conditions and initial condition $v(0)=1$. Short-time existence can indeed be obtained for a smooth $g_{0}$ with the solution $g(t)$ in the parabolic Hölder space ${ }^{39} C^{2, \gamma}$ that is smooth except at the corner ${ }^{40}$ [63, Theorem 2.2]. In conclusion, when ignoring the non-completeness of $M_{\epsilon}^{ \pm}$and singular set $\mathcal{Z}$, it is likely that we can indeed evolve (4.16) on $\left(M_{\epsilon}^{ \pm}, g_{0}\right)$ and obtain short-time existence and uniqueness results. However, the smoothness at the corner must be taken great care with especially when $\epsilon \rightarrow 0$.

### 4.2.2 Ricci flow on non-complete surfaces

On non-complete surfaces, the existence of Ricci flow has been shown in [11] and [64]. Let us start with stating part of the main result of [11, Theorem 1.3]:

Theorem 4.22. Let $\left(M, g_{0}\right)$ be a smooth Riemannian surface which need not be complete, and could have unbounded curvature. Depending on the conformal type, we define $T \in(0, \infty]$ by

$$
T:= \begin{cases}\frac{1}{8 \pi} \operatorname{vol}_{g_{0}} M, & \text { if }\left(M, g_{0}\right) \cong \mathbb{S}^{2}  \tag{4.17}\\ \frac{1}{4 \pi} \operatorname{vol}_{g_{0}} M, & \text { if }\left(M, g_{0}\right) \cong \mathbb{C} \text { or }\left(M, g_{0}\right) \cong \mathbb{R P}^{2} \\ \infty, & \text { otherwise }\end{cases}
$$

Here, it is implied that $\left(M, g_{0}\right) \cong \mathbb{S}^{2}$ have equivalent conformal type. That is, there exists a diffeomorphism $f:\left(M, g_{0}\right) \rightarrow \mathbb{S}^{2}$ such that $f$ and $f^{-1}$ are holomorphic ${ }^{41}$.

Then there exists a smooth solution to the Ricci flow $g(t)$ for $t \in[0, T]$ such that
(i) $g(0)=g_{0}$;
(ii) $g(t)$ is instantaneously complete. That is, $(M, g(t))$ is complete for all $t \in(0, T]$;
(iii) $g(t)$ is maximally stretched. That is, if $\tilde{g}(t)$ is any solution on $M$ with $\tilde{g}(0) \leq g(0)$, then $\tilde{g}(t) \leq g(t)$ for all $t \in[0, \min \{T, \widetilde{T}\}]$.
and this flow is unique in the sense that if $\tilde{g}(t)$ for $t \in[0, \widetilde{T}]$ is another solution to the Ricci flow on $M$ satisfying (i)-(iii), then $\widetilde{T} \leq T$ and $\tilde{g}(t)=g(t)$ for all $t \in[0, \widetilde{T}]$.

Informally, the authors of [11] show that if the initial manifold is non-complete, a unique solution to the (un-normalized) Ricci flow can be obtained such that the family of manifolds ( $M, g(t)$ ) is complete. It remains to be shown whether there exists almost-Riemannian manifolds $M$ such that ( $M_{\epsilon}^{ \pm}, g_{0}$ ) has equivalent conformal type to either $\mathbb{S}^{2}, \mathbb{C}$ or $\mathbb{R P}^{2}$. In the other case, i.e. $T=\infty$, when ignoring the fact that $M_{\epsilon}^{ \pm}$have smooth boundaries, a solution to the Ricci flow (3.1) can be obtained for ( $M_{\epsilon}^{ \pm}, g_{0}$ ).

In [64], the main result applies to arbitrary surfaces with upper bounded Gaussian curvature. The final time of the solution to the flow depends only on the supremum of the Gaussian curvature. Similar to [11], the initial surface may be non-complete, and $(M, g(t))$ is complete for $t \in(0, T]$. We already concluded that the Scalar curvature (and hence the Gaussian curvature) is likely to be negative. Hence, an upper bound does not seem to be a very unrealistic requirement, implying that we can also use this result for our case. A rigorous result on Gaussian curvature upper bounds for a generic 2-ARS is, however, not yet available.

### 4.2.3 A conjecture on the Ricci flow on two-dimensional almost-Riemannian manifolds

As it turns out, to our knowledge, there are not yet enough results on Ricci flow to guarantee existence and uniqueness on non-complete surfaces with boundary. It seems to be the most promising to consider (4.16) on $\left(M_{\epsilon}^{ \pm}, g_{0}\right)$, for a sufficiently small $\epsilon>0$, and combine [63, Theorem 2.2] with [11, Theorem 1.3]

[^25]or [64, Theorem 1.1]. However, due to the deformations, it will be unlikely that $M$ as a whole can still be regarded as a smooth surfaces endowed with a 2 -ARS. This will already be more realistic when one considers the normalized Ricci flow on both surfaces, but this requires further analysis on whether the results of e.g. [61] can be combined with [11]. Nevertheless, let us conclude this thesis with a conjecture on the un-normalized Ricci flow on $\left(M_{\epsilon}^{ \pm}, g_{0}\right)$.

Conjecture. Let $\left(M, g_{0}\right)$ be a complete, connected, compact and orientable two-dimensional surface endowed with a generic 2 -ARS without tangency points such that ( $M_{\epsilon}^{ \pm}, g_{0}$ ) are compact non-complete surfaces with boundary. Consider the evolution equations

$$
\begin{cases}\frac{\partial}{\partial t} g(t)=-S(t) g(t), & \text { in } M_{\epsilon}^{ \pm} \times(0, T]  \tag{4.18}\\ \kappa(\cdot, t)=\psi(\cdot, t), & \text { on } \partial M_{\epsilon}^{ \pm} \times(0, T] \\ g(0)=g_{0}, & \text { in } M_{\epsilon}^{ \pm}\end{cases}
$$

There exists a unique short-time solution $g(t)$ to (4.18) such that $(M, g(t))$ are complete for all $t \in(0, T]$.

## 5 Conclusion

In this thesis, we have studied differential geometry and, in particular, Riemannian geometry to eventually investigate the possible evolution of Ricci flow on two-dimensional almost-Riemannian manifolds.

The thesis seems to be an useful handbook for understanding the basics of Ricci flow. The first chapter covers all the tools that are required to define Ricci flow and describe its evolution on Riemannian manifolds of any dimension. This enabled us to provide all proofs with adequate argumentation. Although we do not propose any new techniques regarding the short-time existence and uniqueness, one could argue that the proof depicted in this thesis excels through its clarity.

In the final chapter of the thesis, we introduce almost-Riemannian structures as the prototype of rank-varying sub-Riemannian structures. In two dimensions, they have been shown to generically satisfy certain properties. We proceeded in studying a generic free 2-ARS without tangency points on a compact and oriented manifold and considered all the points tat lie more than an epsilon distance away from the singular set. Under these circumstances, we established the topology on the two surfaces disjoint from the singular set. These surfaces both consist out of Riemannian points only, but are non-complete and have smooth boundary. This complicates the evolution of Ricci flow on these surfaces, since no results are available on surfaces with these kind of topological properties.

The thesis is concluded with an analysis of previous works in which either manifolds with boundary, or non-complete surfaces were subject to the (normalized) Ricci flow. We investigate whether the results may apply naturally to our case. The most promising is to consider the un-normalized Ricci flow on both surfaces individually, and combine the results of [63, Theorem 2.2] with [11, Theorem 1.3] or [64, Theorem 1.1]. This, however, requires a very precise analysis on the actual methods that were used in all of the proofs to see if they can be combined at all. Moreover, due to the deformations of the flow, it is not likely that the manifold as a whole can still be regarded as almost-Riemannian. In addition, to use [64, Theorem 1.1], one would require curvature upper bounds on almost-Riemannian manifolds. Obtaining curvature upper bounds for a generic 2-ARS seems to be a realistic and promising point of interest for further research in this direction, since the curvature is likely to be negative under these circumstances.

## A A note on Riemannian submanifolds

The following section is based on [21, Chapter 8]. To understand curvature of a Riemannian submanifold $(M, g)$ of a Riemannian manifold $(\widetilde{M}, \tilde{g})$ one first needs to understand how the Levi-Civita connection of $M$ is related to that of $\widetilde{M}$. To that end, let us first define the tangential and respectively normal projections given by

$$
\begin{equation*}
\pi^{\perp}:\left.T \widetilde{M}\right|_{M} \rightarrow T M, \quad \pi^{\perp}:\left.T \widetilde{M}\right|_{M} \rightarrow N M \tag{A.1}
\end{equation*}
$$

If $X \in \Gamma\left(\left.T \widetilde{M}\right|_{M}\right)$, then we write $X^{\perp}=\pi^{\perp} X$ and $X^{\perp}=\pi^{\perp} X$. If $X, Y \in \mathfrak{X}(M)$, we can extend them to vector fields on open neighbourhoods of $\widetilde{M}$ by applying the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{M}$ :

$$
\begin{equation*}
\widetilde{\nabla}_{X} Y=\left(\widetilde{\nabla}_{X} Y\right)^{\perp}+\left(\widetilde{\nabla}_{X} Y\right)^{\perp} \tag{A.2}
\end{equation*}
$$

Now, recall the normal bundle of (2.29) for the following definition.
Definition A.1. The second fundamental form of $M$ is the map $I I: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \Gamma(N M)$ is defined by

$$
\begin{equation*}
I I(X, Y):=\left(\widetilde{\nabla}_{X} Y\right)^{\perp} \tag{A.3}
\end{equation*}
$$

The second fundamental form satisfies certain nice properties, as shown in [21, Proposition 8.1]. More of its geometric meaning can be understood by studying the curvature of curves. Recall the definition of the acceleration of a smooth curve from Definition 2.63.

Definition A.2. Let $M$ be a Riemannian manifold and $\gamma: I \rightarrow M$ a smooth unit-speed curve in $M$. The geodesic curvature of $\gamma$ is the length of the acceleration vector field. That is, the map $\kappa: I \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\kappa(t):=\left|D_{t} \gamma^{\prime}(t)\right| \tag{A.4}
\end{equation*}
$$

It measures how far the curve is from being a geodesic. The extend to which a Riemannian submanifold curves within its ambient manifold can be expressed by means of the intrinsic curvature and extrinsic curvature. If $\gamma: I \rightarrow M$ is regular curve with $M \subset \widetilde{M}$, then the intrinsic curvature $\kappa$ is a curve in $M$, and its extrinsic curvature $\tilde{\kappa}$ is a curve in $\widetilde{M}$. The relationship between the two can be computed with the second fundamental form (See [21, Proposition 8.10]). We remark that a submanifold $M$ of $\widetilde{M}$ is called totally geodesic if every geodesic corresponding to $\tilde{g}$ that is tangent to $M$ at some time $t_{0}$, stays in $M$ for all $t \in\left(t_{0}-\epsilon, t_{0}+\epsilon\right)$.

In the case that $M \subset \widetilde{M}$ is a hypersurface, i.e. a submanifold of codimension 1 , we can use a related but slightly different object to the second fundamental form. That is, the scalar second fundamental form $h$, where $h_{N}: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^{\infty}(M)$ is symmetric covariant $(0,2)$-tensor field $h_{N} \in \Gamma\left(\Sigma^{2} T^{*} M\right)$, defined by

$$
\begin{equation*}
h_{N}(X, Y):=\langle N, I I(X, Y)\rangle, \quad N \in \Gamma(N M) \tag{A.5}
\end{equation*}
$$

The map $s_{N}: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ associated with the scalar second fundamental form is a (1,1)-tensor field and referred to as the shape operator of $M$, which is related to $h_{N}$ through

$$
\begin{equation*}
\left\langle s_{N}(X), Y\right\rangle=h_{N}(X, Y), \quad X, Y \in \mathfrak{X}(M) \tag{A.6}
\end{equation*}
$$

Subsequently, we define the mean curvature as

$$
\begin{equation*}
H:=\frac{1}{n} \operatorname{tr}(s)=\frac{1}{n} \operatorname{tr}_{g}(h), \tag{A.7}
\end{equation*}
$$

the Gaussian curvature as

$$
\begin{equation*}
K:=\operatorname{det}(s)=\frac{\operatorname{det}(h)}{\operatorname{det}(g)} \tag{A.8}
\end{equation*}
$$

and remark that the coefficients of the Riemann curvature tensor satisfy the Gauss equation [21, Equation 8.21]

$$
\begin{equation*}
R m_{i j k l}=h_{i l} h_{j k}-h_{i k} h_{j l} . \tag{A.9}
\end{equation*}
$$

From Gauss's Theorema Egregium it becomes apparent that when $M$ is an embedded two-dimensional submanifold of $\mathbb{R}^{3}$, the Gaussian curvature is an intrinsic local isometric invariant of $(M, g)$. To be precise, under those circumstances the Gaussian curvature equals half the scalar curvature, or, equivalently: $S=2 K$ [21, Theorem 8.27, Corollary 8.28]. Moreover, for the Ricci curvature we then have the following.

Lemma A.3. Let $(M, g)$ be a two-dimensional Riemannian manifold, then

$$
\begin{equation*}
R i c_{i j}=\frac{S}{2} g_{i j} . \tag{A.10}
\end{equation*}
$$

Proof. By using the Bianchi identities (Lemma 2.77), one could derive that the Riemann curvature tensor in two-dimensions only has one independent component:

$$
R m_{1212}=R m_{2121}=-R m_{1221}=-R m_{2112}
$$

From (A.9), it becomes apparent that then

$$
R m_{1212}=h_{12} h_{21}-h_{11} h_{22}=-\operatorname{det}(h)
$$

Consequently, notice that (A.8) then implies that

$$
R m_{1212}=-K \operatorname{det}(g),
$$

and hence that

$$
R m_{i j k l}=K\left(g_{i k} g_{j l}-g_{i l} g_{j k}\right) .
$$

Recall that $R i c_{i j}=g^{k l} R m_{i k j l}$ and that $K=\frac{S}{2}$ to observe that therefore

$$
R i c_{i j}=g^{k l} R m_{i k j l}=K g^{k l}\left(g_{i j} g_{k l}-g_{i l} g_{k j}\right)=\frac{S}{2} g_{i j},
$$

as desired.

## B Covariant derivative of tensor fields in normal coordinates

We claim that

$$
\begin{equation*}
\left(\nabla_{k} \tau_{p}\right)\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right)=\frac{\partial}{\partial x^{k}} \tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) \tag{B.1}
\end{equation*}
$$

To begin with, lemma 2.52 (ii) implies that for $\tau \in \mathcal{T}_{s}^{r}(M)$ we have

$$
\begin{aligned}
\left(\nabla_{k} \tau_{p}\right)\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right)= & \frac{\partial}{\partial x^{k}} \tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) \\
& -\sum_{j_{l}=j_{1}}^{j_{r}} \tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.\nabla_{k} d x^{j_{l}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) \\
& -\sum_{i_{m}=i_{1}}^{i_{s}} \tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\nabla_{k} \frac{\partial}{\partial x^{i_{m}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) .
\end{aligned}
$$

Now, notice that

$$
\begin{aligned}
& \tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.\nabla_{k} d x^{j_{l}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) \\
& =\left.\left.\left.\left.\tau_{i_{1} \cdots i_{s}}^{j_{1} \cdots j_{r}}(p) \frac{\partial}{\partial x^{j_{1}}}\right|_{p} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_{r}}}\right|_{p} \otimes d x^{i_{1}}\right|_{p} \otimes \cdots \otimes d x^{i_{s}}\right|_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.\nabla_{k} d x^{j_{l}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) \\
& =\left.\left.\left.\left.\left.\tau_{i_{1} \cdots i_{s}}^{j_{1} \cdots j_{r}}(p) d x^{j_{1}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{j_{1}}}\right|_{p}\right) \cdots \nabla_{k} d x^{j_{l}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{j_{l}}}\right|_{p}\right) \cdots d x^{j_{r}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{j_{r}}}\right|_{p}\right) d x^{i_{1}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}\right) \cdots d x^{i_{s}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) .
\end{aligned}
$$

By lemma 2.40 (i), observe that

$$
\left.\nabla_{k} d x^{j_{l}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{j_{l}}}\right|_{p}\right)=\nabla_{k}\left(\left.d x^{j l}\right|_{p}\left|\frac{\partial}{\partial x^{j_{l}}}\right|_{p}\right)-\left.d x^{j_{l}}\right|_{p}\left(\left.\nabla_{k} \frac{\partial}{\partial x^{j_{l}}}\right|_{p}\right)=\nabla_{k}\left(\delta_{j_{l}}^{i_{l}}\right)-0=0,
$$

and consequently that

$$
\tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.\nabla_{k} d x^{j_{l}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right)=0
$$

for each $j_{l} \in\left\{j_{1}, \ldots, j_{r}\right\}$. In addition,

$$
\begin{aligned}
& \tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\nabla_{k} \frac{\partial}{\partial x^{i_{m}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right) \\
& =\left.\left.\left.\left.\tau_{i_{1} \cdots i_{s}}^{j_{1} \cdots j_{r}}(p) d x^{j_{1}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{j_{1}}}\right|_{p}\right) \cdots d x^{j_{r}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{j_{r}}}\right|_{p}\right) d x^{i_{1}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}\right) \cdots \underbrace{\left.d x^{i_{m}}\right|_{p}\left(\left.\nabla_{k} \frac{\partial}{\partial x^{i_{m}}}\right|_{p}\right)}_{=0} \cdots d x^{i_{s}}\right|_{p}\left(\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right),
\end{aligned}
$$

such that also

$$
\tau_{p}\left(\left.d x^{j_{1}}\right|_{p}, \ldots,\left.d x^{j_{r}}\right|_{p},\left.\frac{\partial}{\partial x^{i_{1}}}\right|_{p}, \ldots,\left.\nabla_{k} \frac{\partial}{\partial x^{i_{m}}}\right|_{p}, \ldots,\left.\frac{\partial}{\partial x^{i_{s}}}\right|_{p}\right)=0
$$

for each $i_{m} \in\left\{i_{1}, \ldots, j_{s}\right\}$. This concludes the proof for (B.1).

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[^0]:    ${ }^{1}$ Here, $D \varphi_{i j}$ denotes the Jacobian matrix of $\varphi_{i j}$.

[^1]:    ${ }^{2}$ Here $x^{i}=r^{i} \circ \varphi$ where $r^{i}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ denote the standard coordinates on $\mathbb{R}^{n}$.

[^2]:    ${ }^{3}$ Remember that we already defined an oriented manifold in Definition 2.8. An orientation can also be defined on a vector space [22, Definition 8.1.3]. Moreover, one could also define orientation preserving maps [22, Page 128].

[^3]:    ${ }^{4}$ We use a slightly more convenient notation for this definition.
    ${ }^{5}$ Here we use $\mathbb{R} \ni d x^{k}\left(\frac{\partial}{\partial x^{h}}\right)=:\left(d x^{k} \left\lvert\, \frac{\partial}{\partial x^{h}}\right.\right)$ as the dual pairing.

[^4]:    ${ }^{6}$ Here $\varphi^{*} \tilde{g}$ denotes the pullback of $\tilde{g}$ by $\varphi$. For more detail, see [22, Section 6.2].

[^5]:    ${ }^{7}$ Here we use that $\operatorname{tr}_{1, \ldots, r, r+1, \ldots, s}^{1, \ldots, r+1, \ldots, s}$ acts $r$ times on the $\omega^{i}$ 's and $\frac{\partial}{\partial x^{i}}$ components of $\tau$, and $s$ times on the $Y_{j}$ 's and the $d x^{j}$ components of $\tau$.

[^6]:    ${ }^{8}$ This result can be derived from [18, Proposition 2.32-2.33] by using the properties of Lemma 2.52
    ${ }^{9}$ Here $f_{*} X$ denotes the pushforward of $X$ by $f$. For more detail, see [22, Section 6.2]. For more detail on the pullback bundle structure, see [18, Section 2.8]
    ${ }^{10}$ In the next chapter we will define the derivative of a vector field along a smooth curve. Parallel vector fields along smooth curves are defined by using this map as elements of its kernel.

[^7]:    ${ }^{11}$ Due to the Hopf-Rinow theorem, saying that $M$ is geodesically complete is equivalent to saying $M$ is complete as a metric space [21, Theorem 6.19].
    ${ }^{12}$ An open subset $V$ containing the origin is called star-shaped if $V$ also contains the line segments from 0 to $q$ for any $q \in V$.

[^8]:    ${ }^{13}$ See Section B for a justification of this result.

[^9]:    ${ }^{14}$ See also Lemma A. 3 in Section A.
    ${ }^{15}$ Here, $\chi(M)$ denotes the Euler characteristic of $M$, which can be obtained by integrating the Scalar or Gaussian curvature. For more information we refer to the Gauss-Bonnet theorem for surfaces [21, Chapter 9].
    ${ }^{16}$ Two metrics $g_{1}$ and $g_{2}$ are said to be conformal if there exists a positive $f \in C^{\infty}(M)$ such that $g_{1}=f g_{2}$ [21, Page 59].

[^10]:    ${ }^{17}$ Note that due to Definition 3.3, we have $\frac{\partial}{\partial t} g_{k l}=\left(\frac{\partial}{\partial t} g\right)\left(\frac{\partial}{\partial x^{k}}, \frac{\partial}{\partial x^{l}}\right)$.

[^11]:    ${ }^{18}$ With that we mean conceptually. Notation wise, we more or less follow [18] and [31].

[^12]:    ${ }^{19}$ Linear maps between fibres, see e.g. [21, Page 261].
    ${ }^{20}$ We will often just say parabolic to mean strongly parabolic.

[^13]:    ${ }^{21}$ See e.g. [21, Page 250]

[^14]:    ${ }^{22}$ Homotopies are studied in the branch of algebraic topology. A homotopy class is an equivalence class of continuous functions that can be continuously deformed into one another. For more information, we refer to [39].
    ${ }^{23}$ That is, it is linear with respect to all the highest order derivatives of the unknown function (see e.g. [42, Section 4.4]).

[^15]:    ${ }^{24}$ Notice that the final step follows from the fact that the Lie derivate of $g$ is invariant under diffeomorphisms, see for example [31, Section 2.2].

[^16]:    ${ }^{25}$ In fact, for all $t$ in the domain $[0, \epsilon]$ we have that $\varphi$ is a diffeomorphism due to a kind of elliptic regularity. The proof of this involves technicalities which do not lie within the scope of this research. In this context, it is however dealt with in [18, Section 3.2] and on general non-linear partial differential equations in e.g., [44, Section 8.3].

[^17]:    ${ }^{26}$ We can also say that $\mathfrak{X}(M)$ is a module over the ring $C^{\infty}(M)$. An algebra can be constructed from a module by equipping it with a bilinear operation. These objects are algebraic structures studied in abstract algebra. See e.g. [52].
    ${ }^{27}$ Morphisms are studied in category theory, the abstract algebra of functions. Morphisms are also called arrows, and are structure preserving maps between two objects in a category [53, Section 1.1].

[^18]:    ${ }^{28}$ That is, almost equal to a bounded function. This is a concept from measure theory, see for example [54, Section 2.11]
    ${ }^{29}$ That is, almost everywhere. Again, a term from measure theory. It means that only in a subspace of measure zero, a certain property holds everywhere.
    ${ }^{30}$ See [46, Definition 3.8] for a precise definition of the sub-Riemannian norm.

[^19]:    ${ }^{31}$ Here, we have that $|u|=\sqrt{\langle u, u\rangle_{p}}$ for $u \in U_{p}$.

[^20]:    ${ }^{32}$ In two dimensions, the scalar curvature equals twice the Gaussian curvature [21, Corollary 8.28].

[^21]:    ${ }^{33}$ For a proof, we refer to Lemma A.3.

[^22]:    ${ }^{34}$ This is the same as the sub-Riemannian distance from Definition 4.7.

[^23]:    ${ }^{35}$ To be precise, the orientations for $M^{ \pm}$are defined by the volume form $d A_{s}$ associated with $M \backslash \mathcal{Z}$.
    ${ }^{36}$ See Section A.
    ${ }^{37}$ Also, see Definition A. 1 in Section A.

[^24]:    ${ }^{38}$ Recall that on a surface the Ricci curvature equals Ric $=K g$, and $K=\frac{1}{2} S$. As opposed to the previous chapter, we now use $T$ to denote the final time to avoid confusion with the epsilon of $M_{\epsilon}$.

[^25]:    ${ }^{39}$ For a definition, see e.g. [44, Chapter 5].
    ${ }^{40}$ This is in fact described in [59] from the same author.
    ${ }^{41}$ See, for example, [65, Page 32] for more information on conformal types. Also, a holomorphic function is a complex differentiable function.

