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Minimal Realization Theory for Fixed Switched Linear Systems

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Abstract

Minimal Realization Theory for Fixed Switched Linear Systems

by Maico Engelaar

In this paper, I will investigate whether results from minimal realization theory for linear systems can be generalized to fixed switched linear systems, under the condition that the minimal realizations are based on dynamical relations given by the corresponding system types only. For this, both linear systems and fixed switched linear systems will be introduced, observability and controllability will be considered, and finally, minimal realization theory will be formulated and generalized.

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Contents

List of Abbreviations

KCD	**K**alman **C**anonical **D**ecomposition
KCDA	**K**alman **C**anonical **D**ecomposition **A**lgorithm
KOD	**K**alman **O**bservability **D**ecomposition
KODA	**K**alman **O**bservability **D**ecomposition **A**lgorithm

List of Symbols

$\equiv$	$f \equiv g$ if and only if $f(x) = g(x)$ for all x
$\subset$	strict subset
$\subseteq$	strict or equal subset
$\triangle$	Q.E.D.
$\square$	End of algorithm, deduction, definition, example or remark
$CPD(\mathbb{R}, \mathbb{R}^n)$	The set of continuous piecewise differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$
$PC(\mathbb{R}, \mathbb{R}^n)$	The set of piecewise continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$
$\text{Im}(M)$	The image of a matrix M
$\text{Ker}(M)$	The kernel of a matrix M
$\mathbb{N}$	The set of integers
$\mathbb{N}_0$	The set of integers including 0
$\mathbb{N}_0^\infty$	The set of integers including 0 and ∞
$\mathbb{R}$	The field of real numbers
$\mathbb{R}_{>0}$	The set of real numbers greater than 0
$\mathbb{R}^n$	The set of n-dimensional real vectors
$\mathbb{R}^{n \times m}$	The set of $n \times m$-dimensional real matrices
I_n	The $n \times n$ identity matrix
0_n	The zero element of the vector space $\mathbb{R}^n$
$\mathcal{O}$	The observable matrix
$\mathcal{R}$	The reachable matrix
$\mathcal{S}$	The indexed family of switching modes
$\mathcal{T}$	The indexed family of switching times
$\#S$	The number of elements in the (index) set/indexed family S.
S^*	The (index) set S without the zero element
Σ_N	Linear system with 4-tuple N
$\Sigma_{\mathcal{M}}$	Switched linear system with 4-tuple $\mathcal{M}$
Σ_Γ	Fixed switched linear system with 4-tuple Γ

Chapter 1

Introduction

Within the mathematical field of dynamical systems, there are many different research areas. There is linear systems, non-linear systems, chaos theory, system identification, stabilization theory and many others. Among them is also the research area of *Minimal Realization Theory*, of which the objective is to express a dynamical relation by means of as optimal as possible dynamical systems. And it will be this area that will have the main focus during this paper. However, why would minimal realization theory be of interest?

Consider for example the topic of system identification. System identification is the methodology of constructing dynamical systems from input-output data. While the main objective of system identification is to construct dynamical systems that accurately represents the given input-output data, focusing solely on accuracy could lead to some issues. In most cases, more accuracy would imply more complexity. More complexity would, in turn, lead to less ease of use. Computations could become more difficult and already existing theorems and algorithms might not be applicable. Hence, it is important to find a good balance between accuracy and complexity. One possibility, that could be considered, would be to first construct a dynamical system that accurately represents the given input-output data, and afterwards apply minimal realization theory to find the most optimal system descriptions. These combined steps would then describe a possible first step within system identification, which then can be expanded upon.

Besides such a direct application, minimal realization theory can be also expanded upon. First of all understand that the idea behind minimal realization theory is to remove parts, from dynamical systems, that have no influence on the input-output behavior. Next understand that dynamical systems, besides unnecessary parts, might also contain parts that have little influence on the input-output behavior. Hence, a possible expansion would be to not only remove unnecessary parts, but also less influential parts. This subject is known as model reduction, another interesting and topical area within mathematics, which, unfortunately, will not be expanded upon, within this paper.

The above, of course, are just two of the many possible applications for minimal realization theory. Nevertheless, they clearly express the main objective of minimal realization theory, namely to find the most optimal dynamical systems to a given dynamical relation.

1.1 Which Dynamical Systems will be considered?

Considering minimal realization theory for any and all possible dynamical systems would be rather difficult and time consuming, hence, within this paper, the main focus shall lie with two particular systems, namely the *Linear Systems* and the *Switched Linear Systems*.

Linear systems are a well known subset of the dynamical systems. Many books, articles and papers have been written about them and many more applications do exists. In addition, there exists many different types of linear systems. In this paper, the main focus shall lie with the linear systems known as the *Continuous Time-Invariant Input-State-Output Linear Systems*.

Switched linear systems, on the other hand, are a well known subset of the so-called *Hybrid Dynamical Systems*. Without going into too much details, hybrid dynamical systems refer to dynamical systems in which both continuous and discrete dynamics are present. In the case of switched linear systems, the discrete dynamics will be given by the so-called *Switching Signal*, and the continuous dynamics will be given by the linear systems between which will be switched. For more information on hybrid dynamical systems, see, for example, the book by v.d. Schaft et al. [9].

Unfortunately, though the area of switched linear systems is much smaller then the area of dynamical systems, the area of switched linear systems is still immense. Hence, it will be nigh impossible to consider the entire area of switched linear systems within the time allocated to this thesis. This implies that, similar to the linear systems, a subclass needs to be taken as the main focus. In this paper this subclass will be the subclass of switched linear systems for which the switching signal is assumed to be fixed. This, of course, doesn't imply that "general" switched linear systems will be completely ignored, but by and large the main focus shall not lie with each and every possible switched linear system.

1.2 What are the Objectives?

In this paper, the main objective shall be to consider minimal realization theory for linear systems and "fixed" switched linear systems. In particular, the goal shall be to investigate whether results for the linear systems can be generalized to the "fixed" switched linear systems. Hereby, the main focus shall lie with the case that the dynamical relations are given by, respectively, linear systems and "fixed" switched linear systems. To rewrite this into an easy to understand primary research question, consider the following.

Primary Research Question: Can results from minimal realization theory for linear systems be generalized to "fixed" switched linear systems, under the assumption that the minimal realizations are based on dynamical relations given by the corresponding system types only?

In order to investigate this, within this paper, linear systems and "fixed" switched linear systems shall be formally introduced, *Observability* and *Controllability* shall be considered and finally, minimal realization theory shall be formulated and generalized. To rewrite this into easy to understand secondary research questions, consider the following.

Secondary Research Question 1: What is the mathematical definition of linear systems and "fixed" switched linear systems and what are some of their characteristics? (Chapter 2)

Secondary Research Question 2: What is observability and controllability for linear systems and "fixed" switching linear systems? (Chapter 3)

Secondary Research Question 3: What are some important results from minimal realization theory for linear systems? (Chapter 4)

Secondary Research Question 4: Which results can be obtained with regards to minimal realization theory for "fixed" switched linear systems? (Chapter 5)

In this paper, all these questions will be answered to the best of the author's ability. But first, consider the following overview regarding the topics that will be discussed within this paper.

1.3 Overview

Before being able to consider minimal realization theory for either linear systems or "fixed" switched linear systems, first the relevant system types need to be formally introduced. This will be done in Chapter 2. In particular, in Chapter 2 linear systems, "general" switched linear systems and "fixed" switched linear systems will be introduced, together with some of their characteristics. In the case of linear systems, most information will be assumed to be known to any mathematician, but possible references would be the books [1, 7, 8]. In the case of "general" switched linear systems, most information will be a "generalization" of the linear case. Finally, in the case of "fixed" switched linear systems, most information will be a simplification of the "general" switched case.

After introducing all the relevant system types, next, in Chapter 3, the concepts of observability and controllability will be introduced, for each system type separately. These two concepts will be used later in Chapter 4 and Chapter 5, when considering, respectively, minimal realization theory for linear systems and minimal realization theory for "fixed" switched linear systems. In the case of linear systems, most information will be assumed to be known to any mathematician, but possible references would, again, be the books [1, 7, 8]. In the case of "general" switched linear systems and "fixed" switched linear systems, all information, regarding observability, will be obtained from the paper by Petreczky et al. [5]. For controllability, instead, there will be only a brief statement.

After both the system types and the concepts of observability and controllability have been introduced, minimal realization theory for linear systems will be studied in Chapter 4. In this chapter a distinction will be made between the so-called function case and the so-called system case. For each case, realization theory shall be considered, minimal realization theory shall be considered and an algorithm, to find minimal realizations, shall be considered. Since the function case is well known within the literature, many references can be found. In this paper mostly the book by Antsaklis et al. [1] will be used as reference material. For the system case, most information will be a replication of the function case, but also the book by Polderman et al. [7] shall be used. Important to know is that the main objective of this chapter is to collect results from the system case, for this case will also be used in Chapter 5 to investigate whether such results can be generalized to fixed switched linear systems.

Finally, after minimal realization theory for linear systems, and in particular the linear system case, has been considered, minimal realization theory for fixed switched linear systems will be studied in Chapter 5. In this chapter only the system case shall be considered, contrary to Chapter 4. For this singular case, realization theory shall be considered, minimal realization theory shall be considered and the generalization of the linear system case shall be considered. Lastly, no references will be used within this chapter, since all information will be provided by the author himself.

Chapter 2

(Switched) Linear Systems

During the last century linear systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.1}$$

have been studied in great detail, spanning multiple articles, books and studies, see for example [1, 2, 7, 8]. This interest comes from the fact that systems of the form (2.1) have many applications, for example in control theory, mechanics, electrical engineering and many other fields. In this paper, however, the main focus does not lie *only* with systems of the form (2.1). Instead, the main focus *also* lies with systems that switch between systems of the form (2.1) as time evolves. These kinds of systems are called switched linear systems.

2.1 Overview

Before minimal realization theory can be considered, first the relevant system types need to be introduced. In this paper, the relevant system types shall be the linear systems, the "general" switched linear systems and the "fixed" switched linear systems. In particular, while the main focus does not lie with the "general" switched linear systems, "general" switched linear systems will still be introduced within this chapter, in order to create a solid foundation within the area of switched linear systems. All-in-all, in this chapter, the main objective will be to mathematically introduce linear systems (Paragraph 2.2), "general" switched linear systems (Paragraph 2.3) and "fixed" switched linear systems (Paragraph 2.4), together with some of their characteristics.

2.2 Linear Systems

In this paragraph, the idea is to formally introduce linear systems together with some of their characteristics. Hereby a distinction will be made between introducing the linear systems and considering input-output relations of linear systems. While most information in this paragraph will be assumed to be known to any mathematician, possible references would be the books [1, 7, 8].

2.2.1 Introducing Linear Systems

In this paper the general definition of linear systems shall be as follows.

Definition 2.2.1 (Linear Systems)**.** A *Continuous Linear Time-Invariant Input-State-Output System*, or *Linear System* for short, is a dynamical system given by the following equations

$$\Sigma_N : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \in \mathbb{R} \\ y(t) = Cx(t) + Du(t), & t \in \mathbb{R} \end{cases} \tag{2.2}$$

where

- $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$ are constant matrices.
- N is the 4-tuple given by $N = (A, B, C, D)$.
- $u : \mathbb{R} \to \mathbb{R}^m$ is the piecewise continuous input function, i.e. $u \in PC(\mathbb{R}, \mathbb{R}^m)$ (see Appendix A.2).
- $x : \mathbb{R} \to \mathbb{R}^n$ is the continuous piecewise differentiable state-space function, i.e. $x \in CPD(\mathbb{R}, \mathbb{R}^n)$ (see Appendix A.2).
- $y : \mathbb{R} \to \mathbb{R}^p$ is the piecewise continuous output function, i.e. $y \in PC(\mathbb{R}, \mathbb{R}^p)$. □

Remark 2.2.2. Some remarks regarding Definition 2.2.1

(i) It might happen that the dimension of the state will be equal to zero, i.e., $n = 0$. Should such a situation occur, the following should be kept in mind (see Appendix A.7).

- $x(t) = 0_0$ for all $t \in \mathbb{R}$.
- $y(t) = Du(t)$ for all $t \in \mathbb{R}$.

(ii) Similar to the dimension of the state-space, also the dimensions of the input and/or the output could be zero. These situations, however, are ignored within this paper. □

Remark 2.2.3 (Some remarks regarding simplification of notation)**.**
From now onwards, when a linear system Σ_N is given, it will be assumed that

- m, n and p represent, respectively, the dimension of the input u, the dimension of the state x and the dimension of the output y, unless stated otherwise. This statement also holds true in the case $N = N_i$, $m = m_i$, $n = n_i$ and $p = p_i$, for any $i \in \mathbb{N}_0$.
- $N = (A, B, C, D)$, unless stated otherwise. This statement also holds true in the case $N = N_i$, $A = A_i$, $B = B_i$, $C = C_i$ and $D = D_i$, for any $i \in \mathbb{N}_0$. □

Consider next the following definition regarding dimensions of linear systems.

Definition 2.2.4 (The Dimension of Linear Systems)**.** Let Σ_N be a linear system. The dimension of Σ_N will be given by the dimension of the state x, i.e.

$$\text{Dim}(\Sigma_N) := n \qquad \square$$

Consider next the following definition regarding solutions of linear systems.

Definition 2.2.5 (Solutions of Linear Systems)**.** (u, x, y) is said to be a solution to a linear system Σ_N if and only if

- $u \in PC(\mathbb{R}, \mathbb{R}^m)$.
- $x \in CPD(\mathbb{R}, \mathbb{R}^n)$ satisfies the first equation of (2.2) almost everywhere (see Appendix A.1).
- $y \in PC(\mathbb{R}, \mathbb{R}^p)$ satisfies the second equation of (2.2) for all $t \in \mathbb{R}$. $\square$

Consider next the following well-known lemma regarding solutions of linear systems.

Lemma 2.2.6. *Consider a linear system given by Σ_N and assume that the following are given.*

- *the input $u \in PC(\mathbb{R}, \mathbb{R}^m)$*
- *the initial state $x(t_b) \in \mathbb{R}^n$ at initial time $t_b \in \mathbb{R}$*

Then the unique state-space solution x is given by

$$x(t) = e^{A(t-t_b)}x(t_b) + \int_{t_b}^{t} e^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.3}$$

and the corresponding unique output solution y is given by

$$y(t) = Ce^{A(t-t_b)}x(t_b) + \int_{t_b}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2.4}$$

For an explanation on Lemma 2.2.6, see the books [7, 8].

2.2.2 Input-Output Relations

Consider the following deduction regarding the input-output relation, for linear systems, when the initial state and the initial time are both equal to zero.

Deduction 2.2.7. Consider a linear system Σ_N that is given by the following equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Applying a Laplace transform on both sides, results in the following equations:

$$\begin{aligned} s\hat{x}(s) - x(0) &= A\hat{x}(s) + B\hat{u}(s) \\ \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s) \end{aligned}$$

where $\hat{u}$, $\hat{x}$ and $\hat{y}$ are, respectively, the Laplace transform of u, x and y. Solving for $\hat{x}$, the following is obtained:

$$\hat{x}(s) = (sI_n - A)^{-1}x(0) + (sI_n - A)^{-1}B\hat{u}(s)$$

Substituting this into the output part and assuming that $x(0) = 0_n$, the result will be a direct relation between input and output, in the Laplace domain, given by:

$$\hat{y}(s) = C(sI_n - A)^{-1}B\hat{u}(s) + D\hat{u}(s)$$

This relation can also be written as

$$\hat{y}(s) = H(s)\hat{u}(s)$$

where $H(s) = C(sI_n - A)^{-1}B + D$. This function $H(s)$ is then called the *Transfer Function* of linear system Σ_N. □

For Deduction 2.2.7 all information was obtained from the book by Antsaklis et al. [1]. Some information, regarding the Laplace transform, is also given in Appendix A.4.

Consider next the following definition that states when two linear systems are assumed to be input-output equivalent.

Definition 2.2.8 (Input-Output Equivalent Linear Systems)**.** Two linear systems Σ_{N_1} and Σ_{N_2} are *Input-Output Equivalent* if and only if

$$S_1 = \{(u, y) \mid \exists\, x \text{ such that } (u, x, y) \text{ is a solution of } \Sigma_{N_1}\}$$

and

$$S_2 = \{(u, y) \mid \exists\, x \text{ such that } (u, x, y) \text{ is a solution of } \Sigma_{N_2}\}$$

satisfy that $S_1 = S_2$. □

As can be seen from Definition 2.2.8, it is not trivial to show that certain linear systems do have the same input-output behavior. Luckily, there does exist a lemma that makes it easier to show input-output equivalency in certain cases.

Lemma 2.2.9. *Consider two linear systems Σ_{N_1} and Σ_{N_2}. If there exists a non-singular matrix S such that $SA_1S^{-1} = A_2$, $SB_1 = B_2$, $C_1S^{-1} = C_2$, and $D_1 = D_2$, then the two linear systems Σ_{N_1} and Σ_{N_2} are input-output equivalent.*

Since the proof of Lemma 2.2.9 is rather trivial, it will be omitted in this paper. Nevertheless, for those interested, an explanation to the proof can be found in the book by Polderman et al. [7, Chapter 4.6]

A particular topic that would be of interest next, is the relation between transfer function and input-output equivalency. For this consider first the following corollary.

Corollary 2.2.10. *If two linear systems satisfy Lemma 2.2.9, they also share the same transfer function.*

Proof. Consider two linear systems Σ_{N_1} and Σ_{N_2}, and assume that they satisfy Lemma 2.2.9. Let $H_1(s)$ and $H_2(s)$ be the transfer functions of, respectively, Σ_{N_1} and Σ_{N_2}. Then the following holds true:

$$\begin{aligned}
H_2(s) &= C_2(sI_n - A_2)^{-1}B_2 + D_2 \\
&= C_1S^{-1}(sI_n - SA_1S^{-1})^{-1}SB_1 + D_1 \\
&= C_1S^{-1}(sSS^{-1} - SA_1S^{-1})^{-1}SB_1 + D_1 \\
&= C_1S^{-1}S(sI_n - A_1)^{-1}S^{-1}SB_1 + D_1 \\
&= C_1(sI_n - A_1)^{-1}B_1 + D_1 \\
&= H_1(s)
\end{aligned}$$

Hence, both linear systems have the same transfer function. △

Remark 2.2.11. The above statement also holds true for the general case. However, in order to proof the general case, some specific concepts first need to be introduced. Combining this with the fact that no further mention of the general case will be made within this paper, the general case shall be ignored. □

Consider next the situation that two linear systems share the same transfer function. It is easily shown that this does not guarantee that the linear systems are also input-output equivalent. This is because, in general, the transfer function does not contain information about the input-output relation when $x(0) \neq 0$, see also the following example.

Example 2.2.12. Consider two linear systems Σ_{N_1} and Σ_{N_2} and let

$$N_1 = \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, 0 \right)$$

$$N_2 = \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, 0 \right)$$

It is easily shown that both linear systems have the same transfer function $T(s) = \frac{1}{s}$. However, are they also input-output equivalent?

To answer this question, first assume that input u is constant 0, the initial time is 0 and the initial states, for the linear systems Σ_{N_1} and Σ_{N_2}, are, respectively, given by

$$x_1(0) = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \in \mathbb{R}^3 \text{ and } x_2(0) = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \in \mathbb{R}^3.$$

Using Lemma 2.2.6, the unique outputs of linear systems Σ_{N_1} and Σ_{N_2} are, respectively, given by

$$y_1(t) = a_1 + c_1 t \text{ and } y_2(t) = c_2.$$

Notice that if $c_1 \neq 0$, there do not exists values $a_1, a_2, b_1, b_2, c_2 \in \mathbb{R}$ such that $y_1 \equiv y_2$. Hence, it can be concluded that the two linear systems are **not** input-output equivalent, even though they share the same transfer function. □

2.3 (General) Switched Linear Systems

In this paragraph the idea is to formally introduce the "general" switched linear systems together with some of their characteristics. Similar to the linear systems, a distinction will be made between introducing the "general" switched linear systems and considering input-output relations of "general" switched linear systems. In this paragraph most of the information will be a "generalization" of the linear case.

2.3.1 Introducing (General) Switched Linear Systems

Before introducing "general" switched linear systems, first the so-called switching signal needs to be introduced.

Definition 2.3.1 (Switching Signal)**.** A *Switching Signal* $\sigma : \mathbb{R} \to \mathbb{N}_0$ is a function given by

$$\sigma(t) := \begin{cases} s_0 & \text{if } t \in (-\infty, t_1) \\ s_1 & \text{if } t \in [t_1, t_2) \\ \vdots & \\ s_l & \text{if } t \in [t_l, \infty) \end{cases} \tag{2.5}$$

where

- $l \in \mathbb{N}_0^\infty$.
- $t_1 < t_2 < \cdots < t_l$.
- any finite interval $I \subset \mathbb{R}$ can contain only finitely many t_q, $q \in \{1, 2, \ldots, l\}$ (no Zeno behavior, see Appendix A.8). □

Remark 2.3.2. Some remarks regarding Definition 2.3.1

(i) Notice that both finite and infinite switching is allowed.

(ii) The *Indexed Family* (see Appendix A.3) of all *Switching Modes* is given by $\mathcal{S} := \{s_q \in \mathbb{N}_0 \mid q \in \{0, 1, 2, \ldots, l\}\}$.

(iii) The indexed family of all *Switching Times* is given by $\mathcal{T} := \{t_q \in \mathbb{R} \mid q \in \{1, 2, \ldots, l\}\}$ if $l > 0$. In the case $l = 0$, hence no switching occurs, the indexed family of all switching times will be empty, i.e. $\mathcal{T} = \emptyset$. □

Consider the following lemma, without proof, regarding switching signals.

Lemma 2.3.3. *Any switching signal can be expressed by means of an indexed family of switching modes $\mathcal{S}$ and an indexed family of switching times $\mathcal{T}$.*

Remark 2.3.4. Some remarks regarding the notation of switching signals.

(i) In general, notation (2.5) shall be used to express switching signals. Notice, however, that switching from mode i back to mode i could lead to confusion when using notation (2.5). Hence, an alternative description would be preferable in such cases. In this paper, this alternative description will be given by the indexed family of switching modes $\mathcal{S}$ and the indexed family of switching times $\mathcal{T}$, which together completely describe a switching signal σ, see Lemma 2.3.3. This alternative description will be used if at any point confusion may arise because of notation (2.5).

(ii) To avoid tedious notations, it will be assumed that given a switching signal σ, the corresponding indexed families $\mathcal{S}$ and $\mathcal{T}$ are also given, even if they are not explicitly mentioned. □

Now that switching signals have been introduced, next consider the following definition for "general" switched linear systems.

Definition 2.3.5 (General Switched Linear Systems)**.** A *General Switched Linear System*, or *Switched Linear System* for short, is a dynamical system given by the following equations

$$\Sigma_{\mathcal{M}} : \begin{cases} \dot{x}_q(t) & = A_{\sigma(t_q^+)} x_q(t) + B_{\sigma(t_q^+)} u(t), & t \in (t_q, t_{q+1}) \\ x_q(t_q^+) & = J_{(\sigma(t_q^+), \sigma(t_q^-))} x_{q-1}(t_q^-), & q \in \{1, 2, \ldots, l\} \\ y(t) & = C_{\sigma(t)} x_{\sigma(t)}(t) + D_{\sigma(t)} u(t), & t \in \mathbb{R} \end{cases} \tag{2.6}$$

where

- $t_0 = -\infty$.
- $\mathfrak{D} := \{0, 1, 2, \ldots, f\}$, $f \in \mathbb{N}_0^\infty$, is the *Index Set* (see Appendix A.3) of linear systems between which *can* be switched.
- for $(r, q) \in \mathfrak{D} \times \mathfrak{D}$, $J_{(r,q)} \in \mathbb{R}^{n_r \times n_q}$ are constant matrices.
- $\mathcal{J} := \{J_{(r,q)} \mid (r, q) \in \mathfrak{D} \times \mathfrak{D}\}$ is the indexed family of jump matrices.
- for $q \in \mathfrak{D}$, $A_q \in \mathbb{R}^{n_q \times n_q}$, $B_q \in \mathbb{R}^{n_q \times m}$, $C_q \in \mathbb{R}^{p \times n_q}$ and $D_q \in \mathbb{R}^{p \times m}$ are constant matrices.

- $\mathcal{N} := \{n_q \in \mathbb{N} \mid q \in \mathfrak{D}\}$ is the indexed family of linear system dimensions.
- $M := \{(A_q, B_q, C_q, D_q) \mid q \in \mathfrak{D}\}$ is the indexed family of linear systems.
- $\mathcal{M}$ is the 4-tuple given by $\mathcal{M} = (M, \mathcal{J}, \mathcal{N}, \mathfrak{D})$.
- $\sigma : \mathbb{R} \to \mathfrak{D}$ is the switching signal.
- $l = \#\mathcal{T}$.
- $t_q \in \mathcal{T}$, for all $q \in \{1, 2, \ldots, l\}$ (see Appendix A.3).
- $u \in PC(\mathbb{R}, \mathbb{R}^m)$ is the piecewise continuous input function.
- for each $q \in \{0, 1, 2, \ldots, l\}$, $x_q \in CPD(\mathbb{R}, \mathbb{R}^{n_{\sigma(t_q^+)}})$ is the $(q+1)^{\text{th}}$ continuous piecewise differentiable state-space function.
- $y \in PC(\mathbb{R}, \mathbb{R}^p)$ is the piecewise continuous output function. □

Remark 2.3.6. Some remarks regarding Definition 2.3.5

(i) Notice that in the definition the switching signal σ is regarded as an input similar to u. Therefore, only the 4-tuple $\mathcal{M}$ is required to be known, to construct the corresponding switched linear system $\Sigma_{\mathcal{M}}$.

(ii) Notice that the definition allows the number of switching times and the number of linear systems, between which *can* be switched, to be either finite or infinite *independent* of each other.

(iii) Notice that the linear systems, between which *can* be switched, could be of zero dimension. If this happens, for example for linear system Σ_{N_p}, the following should be kept in mind.

- if $\sigma(t_q^+) = p$ then $x_q(t) = 0_0$ for all $t \in (t_q, t_{q+1})$.
- if $\sigma(t_q^+) = p$ then $y(t) = D_p u(t)$ for all $t \in (t_q, t_{q+1})$.
- $\forall r \in \mathfrak{D}$, $J_{(p,r)} : \mathbb{R}^{n_r} \to \mathbb{R}^0$ maps any element in $\mathbb{R}^{n_r}$ to 0_0
- $\forall r \in \mathfrak{D}$, $J_{(r,p)} : \mathbb{R}^0 \to \mathbb{R}^{n_r}$ is given by $J_{(r,p)} 0_0 = 0_{n_r}$.

(iv) Notice that between each switch the dimension of the state can change.

(v) Notice that if σ is constant, i.e. $\mathcal{T} = \emptyset$, the result would be a linear system.

(vi) Notice that if $\mathfrak{D} = \{0\}$ and $J_{(0,0)} = I_n$, i.e. one linear system and no jumps, the resulting switched linear system would essentially be a linear system. Hence why, switched linear systems can be considered "generalizations" of linear systems.

(vii) Systems of the form (2.6) are also referred to as *Switched Linear Systems with Jumps*, since the second equation in (2.6) can be interpreted as a jump map.

(viii) Notice that this definition is just one of many. Another possibility would be to let $n_q = n$ for all $q \in \mathfrak{D}$ and $J_{(r,q)} = I_n$ for all $(r,q) \in \mathfrak{D} \times \mathfrak{D}$, see, for example, the papers by Petreczky et al. [4, 6]. These types of systems are called *Switched Linear Systems without Jumps*. It is also possible to expand on the current definition by including, for example, an input at the jump map, see, for example, the paper by Petreczky et al. [5].

(ix) Notice that in the definition there is a difference between a constant switching signal and a switching signal that switches from mode i back to mode i. This is because matrix $J_{(i,i)}$, in general, is not equal to the identity matrix.

(x) Consider again remarks (v) and (vi). Notice that since $\mathfrak{D}$ and σ are two independent mathematical objects, see remark (i), both remarks, while seemingly stating the same, state completely different observations. □

Remark 2.3.7. Some general remarks regarding notation

(i) For a function $f : \mathbb{R} \to \mathbb{R}^n$, $f(t^-)$ and $f(t^+)$ denote, respectively, the left- and right-sided limit at t, assuming these limits exists.

(ii) In the case $\mathcal{T}$ is finite, $t_{l+1} = \infty$.

(iii) Notice that $\sigma(-\infty^+) = \sigma(t_1^-)$. In the case t_1 is undefined, σ will be constant and thus $\sigma(-\infty^+)$ will be equal to this constant. □

Remark 2.3.8 (Some remarks regarding simplification of notation)**.**
From now onwards, when a switched linear system $\Sigma_{\mathcal{M}}$ is given, it will be assumed that

- m and p represent, respectively, the dimension of the input u and the dimension of the output y, unless stated otherwise. This statement also holds true in the case $\mathcal{M} = \mathcal{M}_i$, $m = m_i$ and $p = p_i$, for any $i \in \mathbb{N}_0$.
- $\mathcal{M} = (M, \mathcal{J}, \mathcal{N}, \mathfrak{D})$, unless stated otherwise. This statement also holds true in the case $\mathcal{M} = \mathcal{M}_i$, $M = M_i$, $\mathcal{J} = \mathcal{J}_i$, $\mathcal{N} = \mathcal{N}_i$ and $\mathfrak{D} = \mathfrak{D}_i$, for any $i \in \mathbb{N}_0$. □

To get a better understanding of Definition 2.3.5, consider the following example.

Example 2.3.9. Consider the three linear systems Σ_{N_0}, Σ_{N_1} and Σ_{N_2}, where

$$N_0 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, 0 \right)$$
$$N_1 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, 1 \right)$$
$$N_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, 0 \right)$$

Let

$$J_{(0,0)} = I_3, \quad J_{(1,0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_{(2,0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$J_{(1,1)} = I_2, \quad J_{(0,1)} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad J_{(2,1)} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$J_{(2,2)} = I_3, \quad J_{(0,2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } J_{(1,2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Switched linear system $\Sigma_{\mathcal{M}}$ that switched between linear systems Σ_{N_0}, Σ_{N_1} and Σ_{N_2} is now given by the 4-tuple $\mathcal{M} = (M, \mathcal{J}, \mathcal{N}, \mathfrak{D})$ where $\mathfrak{D} = \{0, 1, 2\}$, $\mathcal{N} = \{\mathrm{Dim}(\Sigma_{N_i}) \mid i \in \mathfrak{D}\}$, $\mathcal{J} = \{J_{(j,i)} \mid (j, i) \in \mathfrak{D} \times \mathfrak{D}\}$ and $M = \{N_i \mid i \in \mathfrak{D}\}$. □

Consider next the following definition regarding dimensions of switched linear systems.

Definition 2.3.10 (The Dimension of Switched Linear Systems)**.** Let $\Sigma_{\mathcal{M}}$ be a switched linear system. The dimension of $\Sigma_{\mathcal{M}}$ shall be given by the indexed family of linear system dimensions i.e. $\mathrm{Dim}(\Sigma_{\mathcal{M}}) := \mathcal{N}$. □

Consider next the following definition regarding solutions of switched linear systems.

Definition 2.3.11 (Solutions of Switched Linear Systems)**.** (σ, u, x, y) is said to be a solution to a switched linear system $\Sigma_{\mathcal{M}}$ if and only if

- $\sigma : \mathbb{R} \to \mathfrak{D}$ is a switching signal.
- $u \in PC(\mathbb{R}, \mathbb{R}^m)$.
- x is of the form

$$x(t) := \begin{cases} x_0(t) & \text{if } t \in (-\infty, t_1), \\ x_1(t_1^+) & \text{if } t = t_1 \\ x_1(t) & \text{if } t \in (t_1, t_2), \\ x_2(t_2^+) & \text{if } t = t_2 \\ \vdots & \\ x_l(t) & \text{if } t \in (t_l, \infty) \end{cases} \tag{2.7}$$

  where

  - $l = \#\mathcal{T}$.
  - $t_q \in \mathcal{T}$, for all $q \in \{1, 2, \ldots, l\}$.
  - $\forall q \in \{0, 1, 2, \ldots, l\}$, (u, x_q, y_q) is a solution of $\Sigma_{N_{\sigma(t_q^+)}}$

- x satisfies the second equation of (2.6) for all $q \in \{1, 2, \dots, l\}$.
- $y \in PC(\mathbb{R}, \mathbb{R}^p)$ satisfies the third equation of (2.6) for all $t \in \mathbb{R}$. □

Remark 2.3.12. Notice that x is not well-defined as a function, since x does not have a fixed dimension. Nevertheless, to avoid tedious notations, throughout this paper notation (2.7) shall be used instead of considering each mode separately. □

Consider next the following lemma regarding solutions of switched linear systems.

Lemma 2.3.13. *Consider a switched linear system given by $\Sigma_{\mathcal{M}}$ and assume that the following are given.*

- *the switching signal $\sigma : \mathbb{R} \to \mathfrak{D}$*
- *the input $u \in PC(\mathbb{R}, \mathbb{R}^m)$*
- *the initial state $x(t_b) \in \mathbb{R}^{n_{\sigma(t_1^-)}}$ at initial time $t_b \in \mathbb{R}$, such that $t_b < t_1$*

Then the unique state-space solution x is given by

$$x(t) = \begin{cases} e^{A_{\sigma(t_1^-)}(t-t_b)} x(t_b) + \int_{t_b}^{t} e^{A_{\sigma(t_1^-)}(t-\tau)} B_{\sigma(t_1^-)} u(\tau) d\tau, & \text{if } t \in (-\infty, t_1) \\ e^{A_{\sigma(t_1^+)}(t-t_1)} J_{(\sigma(t_1^+),\sigma(t_1^-))} x(t_1^-) + \int_{t_1}^{t} e^{A_{\sigma(t_1^+)}(t-\tau)} B_{\sigma(t_1^+)} u(\tau) d\tau, & \text{if } t \in [t_1, t_2) \\ e^{A_{\sigma(t_2^+)}(t-t_2)} J_{(\sigma(t_2^+),\sigma(t_2^-))} x(t_2^-) + \int_{t_2}^{t} e^{A_{\sigma(t_2^+)}(t-\tau)} B_{\sigma(t_2^+)} u(\tau) d\tau, & \text{if } t \in [t_2, t_3) \\ \vdots \\ e^{A_{\sigma(t_l^+)}(t-t_l)} J_{(\sigma(t_l^+),\sigma(t_l^-))} x(t_l^-) + \int_{t_l}^{t} e^{A_{\sigma(t_l^+)}(t-\tau)} B_{\sigma(t_l^+)} u(\tau) d\tau, & \text{if } t \in [t_l, \infty) \end{cases} \tag{2.8}$$

where $l = \#\mathcal{T}$. The corresponding unique output solution y can be constructed by substituting equation (2.8) *into the output part of equation* (2.6).

Do notice that the proof of Lemma 2.3.13 is rather straightforward, when given Lemma 2.2.6, and is therefore omitted in this paper.

2.3.2 Input-Output Relations

For switched linear systems it is difficult to express input-output behavior, in the case the initial state and the initial time are both equal to zero, by means of a transfer function. In particular, the Laplace transform is not as straightforward as for linear systems. Hence, within this paper, transfer functions for switched linear systems will be ignored.

Instead, consider the following definition that states when two switched linear systems are assumed to be input-output equivalent.

Definition 2.3.14 (Input-Output Equivalent Switched Linear Systems)**.** Two switched linear systems $\Sigma_{\mathcal{M}_1}$ and $\Sigma_{\mathcal{M}_2}$ are *Input-Output Equivalent* if and only if

$$S_1 = \{(\sigma, u, y) \mid \exists\, x \text{ such that } (\sigma, u, x, y) \text{ is a solution of } \Sigma_{\mathcal{M}_1}\}$$

and

$$S_2 = \{(\sigma, u, y) \mid \exists\, x \text{ such that } (\sigma, u, x, y) \text{ is a solution of } \Sigma_{\mathcal{M}_2}\}$$

satisfy that $S_1 = S_2$. □

2.4 Fixed Switched Linear Systems

Now that the switched linear systems have been introduced, next would be to introduce the "fixed" switched linear systems. One possibility would be to consider, again, Definition 2.3.5 and assume that the switching signal becomes fixed. And indeed the dynamics given in Definition 2.3.5 are still applicable after fixing the switching signal. However, there does exists a more convenient description instead.

In this paragraph the goal is to consider this alternative description, together with some of its characteristics. Similar to before, a distinction will be made between introducing the "fixed" switched linear systems and considering input-output relations of "fixed" switched linear systems. In this paragraph most information will be a simplification of the general switched case.

2.4.1 Introducing Fixed Switched Linear Systems

Before introducing the alternative description for "fixed" switched linear systems, first the so-called fixed switching signal needs to be introduced.

Definition 2.4.1 (Fixed Switching Signal). A *Fixed Switching Signal* $\sigma : \mathbb{R} \to \mathbb{N}_0$ is a function given by

$$\sigma(t) := \begin{cases} 0 & \text{if } t \in (-\infty, t_1) \\ 1 & \text{if } t \in [t_1, t_2) \\ \vdots & \\ l & \text{if } t \in [t_l, \infty) \end{cases} \tag{2.9}$$

where

- $l \in \mathbb{N}_0^\infty$.
- $t_1 < t_2 < \cdots < t_l$.
- any finite interval $I \subset \mathbb{R}$ can contain only finitely many t_q, $q \in \{1, 2, \ldots, l\}$ (no Zeno behavior). □

Remark 2.4.2. Some remarks regarding Definition 2.4.1.

(i) Notice that, similar to Definition 2.3.1, both finite and infinite switching is allowed.

(ii) The indexed family of all switching times is again given by $\mathcal{T} = \{t_q \in \mathbb{R} \mid q \in \{1, 2, \ldots, l\}\}$ if $l > 0$. In the case $l = 0$, hence no switching occurs, the indexed family of all switching times will again be empty, i.e. $\mathcal{T} = \emptyset$.

(iii) Notice that in the definition the switching modes are *fixed*. The only freedom left, in this definition, is in the values of the switching times and in the number of switches. □

Consider the following lemma, without proof, regarding fixed switching signals.

Lemma 2.4.3. *Any fixed switching signal can be expressed by means of an indexed family of switching times $\mathcal{T}$.*

Remark 2.4.4. Similar to switching signals, to avoid tedious notations, it will be assumed that given a fixed switching signal σ, the corresponding indexed family $\mathcal{T}$ will also be given, even if $\mathcal{T}$ is not explicitly mentioned. □

Now that fixed switching signals have been introduced, next consider the following definition for "fixed" switched linear systems.

Definition 2.4.5 (Fixed Switched Linear System)**.** A *Switched Linear System with Fixed Switching Signal* σ, or *Fixed Switched Linear System* for short, is a dynamical system given by the following equations

$$\Sigma_\Gamma : \begin{cases} \dot{x}_q(t) &= A_q x_q(t) + B_q u(t), \quad & t \in (t_q, t_{q+1}) \\ x_q(t_q^+) &= J_q x_{q-1}(t_q^-), & q \in \{1, 2, \ldots, l\} \\ y(t) &= C_{\sigma(t)} x_{\sigma(t)}(t) + D_{\sigma(t)} u(t), & t \in \mathbb{R} \end{cases} \tag{2.10}$$

where

- $t_0 = -\infty$.
- $\mathfrak{D} := \{0, 1, 2, \ldots, l\}$, $l = \#\mathcal{T}$, is the index set of linear systems between which *will* be switched.
- for $q \in \mathfrak{D}^* := \mathfrak{D} - \{0\}$, $J_q \in \mathbb{R}^{n_q \times n_{q-1}}$ are constant matrices.
- $\mathcal{J} := \{J_q \mid q \in \mathfrak{D}^*\}$ is the indexed family of jump matrices.
- for $q \in \mathfrak{D}$, $A_q \in \mathbb{R}^{n_q \times n_q}$, $B_q \in \mathbb{R}^{n_q \times m}$, $C_q \in \mathbb{R}^{p \times n_q}$ and $D_q \in \mathbb{R}^{p \times m}$ are constant matrices.
- $\mathcal{N} := \{n_q \in \mathbb{N} \mid q \in \mathfrak{D}\}$ is the indexed family of linear system dimensions.
- $M := \{(A_q, B_q, C_q, D_q) \mid q \in \mathfrak{D}\}$ is the indexed family of linear systems.
- Γ is the 4-tuple given by $\Gamma = (\sigma, M, \mathcal{J}, \mathcal{N})$.
- $t_q \in \mathcal{T}$, for all $q \in \mathfrak{D}^*$.

- $u \in PC(\mathbb{R}, \mathbb{R}^m)$ is the piecewise continuous input function.
- for each $q \in \mathfrak{D}$, $x_q \in CDP(\mathbb{R}, \mathbb{R}^{n_q})$ is the $(q+1)^{\text{th}}$ continuous piecewise differentiable state-space function.
- $y \in PC(\mathbb{R}, \mathbb{R}^p)$ is the piecewise continuous output function. □

Remark 2.4.6. Some remarks regarding Definition 2.4.5. Do notice that also Remarks 2.3.6 (iv)-(v) hold true for fixed switched linear systems.

(i) Notice that in the definition the switching signal σ is regarded as a constant similar to the matrices. Therefore, only the 4-tuple Γ is required to be known, to construct the corresponding fixed switched linear system Σ_Γ. (See also Remark 2.3.6 (i))

(ii) Notice that the definition allows the number of switching times and the number of linear systems, between which *will* be switched, to be *both* either finite or infinite. (See also Remark 2.3.6 (ii))

(iii) Notice that the linear systems, between which *will* be switched, could be of zero dimension. If this happens, for example for linear system Σ_{N_q}, the following should be kept in mind. (See also 2.3.6 (iii))

- $x_q(t) = 0_0$ for all $t \in (t_q, t_{q+1})$.
- $y(t) = D_q u(t)$ for all $t \in (t_q, t_{q+1})$.
- if $q > 0$, $J_q : \mathbb{R}^{n_{q-1}} \to \mathbb{R}^0$ maps any element in $\mathbb{R}^{n_{q-1}}$ to 0_0.
- if $q < \#\mathcal{T}$, $J_{q+1} : \mathbb{R}^0 \to \mathbb{R}^{n_{q+1}}$ is given by $J_q 0_0 = 0_{n_{q+1}}$.

(iv) Notice that a fixed switched linear system Σ_Γ contains only linear systems between which *will* be switched, while a switched linear system $\Sigma_\mathcal{M}$ contains also linear systems between which *can* be switched. □

Remark 2.4.7 (Some remarks regarding simplification of notation)**.**
From now onwards, when a fixed switched linear system Σ_Γ is given, it will be assumed that

- m and p represent, respectively, the dimension of the input u and the dimension of the output y, unless stated otherwise. This statement also holds true in the case $\Gamma = \Gamma_i$, $m = m_i$ and $p = p_i$, for any $i \in \mathbb{N}_0$.
- $\Gamma = (\sigma, M, \mathcal{J}, \mathcal{N})$, unless stated otherwise. This statement also holds true in the case $\Gamma = \Gamma_i$, $\sigma = \sigma_i$, $M = M_i$, $\mathcal{J} = \mathcal{J}_i$ and $\mathcal{N} = \mathcal{N}_i$, for any $i \in \mathbb{N}_0$. □

Consider the following lemma, without proof, regarding switched linear systems and fixed switched linear systems.

Lemma 2.4.8. *Let $\Sigma_\mathcal{M}$ be a switched linear system and let $\sigma : \mathbb{R} \to \mathfrak{D}$ be a switching signal. It is possible to find Γ such that the resulting fixed switched linear system*

Σ_Γ has the same input-state-output behavior as $\Sigma_{\mathcal{M}}$ when fixing its switching signal equal to σ, i.e. there exists Γ such that

$$S_1 = \{(u,x,y) \mid (\sigma,u,x,y) \text{ is a solution of } \Sigma_{\mathcal{M}}\}$$

and

$$S_2 = \{(u,x,y) \mid (u,x,y) \text{ is a solution of } \Sigma_\Gamma\} \text{ (See Definition 2.4.11)}$$

satisfy $S_1 = S_2$.

Lemma 2.4.8 implies that Definition 2.4.5 is an alternative description for "fixed" switched linear systems. Hence the objective, as described at the beginning of paragraph 2.4, has been met.

To get a better understanding of Definition 2.4.5, consider the following continuation of Example 2.3.9.

Example 2.4.9 (Continuation Example 2.3.9)**.** Assume that $t_1 < t_2$ and let

$$\sigma(t) = \begin{cases} 0, \text{ if } t \in (-\infty, t_1) \\ 1, \text{ if } t \in [t_1, t_2) \\ 0, \text{ if } t \in [t_2, \infty) \end{cases}$$

be a switching signal. After fixing the switching signal of system $\Sigma_{\mathcal{M}}$ to be equal to σ, the result will be a "fixed" switched linear system that between $-\infty$ and t_1 follows a trajectory given by the linear system Σ_{N_0}, between t_1 and t_2 follows a trajectory given by the linear system Σ_{N_1} and between t_2 and ∞ follows a trajectory given by the linear system Σ_{N_0}.

According to Lemma 2.4.8, there exists an Γ such that the corresponding fixed switched linear system Σ_Γ has the same input-state-output behavior as $\Sigma_{\mathcal{M}}$, when fixing its switching signal equal to σ. One such possible Γ is given by $\Gamma = (\hat{\sigma}, \hat{M}, \hat{\mathcal{J}}, \hat{\mathcal{N}})$, where

- $\hat{\sigma}$ has switching times $\hat{\mathcal{T}}$
- $\hat{\mathfrak{D}} = \{0,1,2\}$
- $\hat{\mathcal{T}} = \{t_1, t_2\}^{\hat{\mathfrak{D}}^*}$ (see Appendix A.3)
- $\hat{M} = \{N_0, N_1, N_0\}^{\hat{\mathfrak{D}}}$
- $\hat{\mathcal{J}} = \{J_{(1,0)}, J_{(0,1)}\}^{\hat{\mathfrak{D}}^*}$
- $\hat{\mathcal{N}} = \{3,2,3\}^{\hat{\mathfrak{D}}}$ □

Consider next the following definition regarding dimensions of fixed switched linear systems.

Definition 2.4.10 (The Dimension of Fixed Switched Linear Systems)**.** Let Σ_Γ be a fixed switched linear system. The dimension of Σ_Γ shall be given by the indexed family of linear system dimensions i.e. $\text{Dim}(\Sigma_\Gamma) := \mathcal{N}$. □

Consider next the following definition regarding solutions of fixed switched linear systems.

Definition 2.4.11 (Solutions of Fixed Switched Linear Systems)**.** (u, x, y) is said to be a solution to a fixed switched linear system Σ_Γ if and only if

- $u \in PC(\mathbb{R}, \mathbb{R}^m)$.
- x is of the form (2.7) where
  - $l = \#\mathcal{T}$
  - $t_q \in \mathcal{T}$, for all $q \in \{1, 2, \ldots, l\}$.
  - $\forall q \in \{0, 1, 2, \ldots, l\}$, (u, x_q, y_q) is a solution of Σ_{N_q}
- x satisfies the second equation of (2.10) for all $q \in \{1, 2, \ldots, l\}$.
- $y \in PC(\mathbb{R}, \mathbb{R}^p)$ satisfies the third equation of (2.10) for all $t \in \mathbb{R}$. □

Consider next the following lemma regarding solutions of fixed switched linear systems.

Lemma 2.4.12. *Consider a fixed switched linear system given by Σ_Γ and assume that the following are given.*

- *the input $u \in PC(\mathbb{R}, \mathbb{R}^m)$*
- *the initial state $x(t_b) \in \mathbb{R}^{n_0}$ at initial time $t_b \in \mathbb{R}$, such that $t_b < t_1$*

Then the unique state-space solution x is given by

$$x(t) = \begin{cases} e^{A_0(t-t_b)}x(t_b) + \int_{t_b}^{t} e^{A_0(t-\tau)}B_0u(\tau)d\tau, & \text{if } t \in (-\infty, t_1) \\ e^{A_1(t-t_1)}J_1x(t_1^-) + \int_{t_1}^{t} e^{A_1(t-\tau)}B_1u(\tau)d\tau, & \text{if } t \in [t_1, t_2) \\ e^{A_2(t-t_2)}J_2x(t_2^-) + \int_{t_2}^{t} e^{A_2(t-\tau)}B_2u(\tau)d\tau, & \text{if } t \in [t_2, t_3) \\ \vdots & \\ e^{A_l(t-t_l)}J_lx(t_l^-) + \int_{t_l}^{t} e^{A_l(t-\tau)}B_lu(\tau)d\tau, & \text{if } t \in [t_l, \infty) \end{cases} \tag{2.11}$$

where $l = \#\mathcal{T}$. The corresponding unique output solution y can be constructed by substituting equation (2.11) *into the output part of equation* (2.10).

Do notice that the proof of Lemma 2.4.12 is rather straightforward, when given Lemma 2.2.6, and therefore is omitted in this paper.

2.4.2 Input-Output Relations

Similar to switched linear systems, for fixed switched linear systems it is difficult to express input-output behavior, in the case the initial state and the initial time are both equal to zero, by means of a transfer function. In particular, the Laplace transform is not as straightforward as for linear systems. Hence, within this paper, transfer functions for fixed switched linear systems will also be ignored.

Instead, consider the following definition that states when two fixed switched linear systems are assumed to be input-output equivalent.

Definition 2.4.13 (Input-Output Equivalent Fixed Switched Linear Systems)**.** Two fixed switched linear systems Σ_{Γ_1} and Σ_{Γ_2} are *Input-Output Equivalent* if and only if

$$S_1 = \{(u, y) \mid \exists\, x \text{ such that } (u, x, y) \text{ is a solution of } \Sigma_{\Gamma_1}\}$$

and

$$S_2 = \{(u, y) \mid \exists\, x \text{ such that } (u, x, y) \text{ is a solution of } \Sigma_{\Gamma_2}\}$$

satisfy that $S_1 = S_2$. □

A particular case of the previous definition, that will be of importance later, is the case that two fixed switched linear systems also share the same fixed switching signal. This case will have its own separate definition.

Definition 2.4.14 (Signal-Input-Output Equivalent Fixed Switched Linear Systems)**.** Two fixed switched linear systems Σ_{Γ_1} and Σ_{Γ_2} are *Signal-Input-Output Equivalent* if and only if Σ_{Γ_1} and Σ_{Γ_2} are input-output equivalent and $\sigma_1 \equiv \sigma_2$. □

Remark 2.4.15. Notice that any two fixed switched linear systems that satisfy Definition 2.4.14, will also satisfy Definition 2.4.13. The other way around, however, does not hold in general. □

Similar to linear systems, using only Definition 2.4.14 it is not trivial to show that certain fixed switched linear systems do have the same signal-input-output behavior. Luckily, also similar to linear systems, there does exist a lemma that makes it easier to show signal-input-output equivalency in certain cases.

Lemma 2.4.16. *Consider two fixed switched linear systems Σ_{Γ_1} and Σ_{Γ_2} for which $\sigma_1 \equiv \sigma_2$. If there exists a indexed family $\{S_i \in \mathbb{R}^{n_i \times n_i} \mid i \in \{0, 1, 2, \ldots, \#\mathcal{T}\}\}$ of non-singular matrices such that*

- $S_i A_i^1 S_i^{-1} = A_i^2$
- $S_i B_i^1 = B_i^2$
- $C_i^1 S_i^{-1} = C_i^2$

- $D_i^1 = D_i^2$
- $S_i J_i^1 S_{i-1}^{-1} = J_i^2$

then the two fixed switched linear systems Σ_{Γ_1} and Σ_{Γ_2} are signal-input-output equivalent. Here it should be noted that the following notations have been used:

- $(A_i^1, B_i^1, C_i^1, D_i^1) \in M_1$ *is the* i^{th} *linear system of* Σ_{Γ_1}.
- $(A_i^2, B_i^2, C_i^2, D_i^2) \in M_2$ *is the* i^{th} *linear system of* Σ_{Γ_2}.
- $J_i^1 \in \mathcal{J}_1$ *is the* i^{th} *jump map of* Σ_{Γ_1}.
- $J_i^2 \in \mathcal{J}_2$ *is the* i^{th} *jump map of* Σ_{Γ_2}.

The above lemma is a generalization of Lemma 2.2.9 and is rather trivial to proof, hence the proof is omitted in this paper. Nevertheless, for those interested, it again uses the idea explained in the book by Polderman et al. [7, Chapter 4.6].

Now that linear systems, switched linear systems and fixed switched linear systems have been introduced, in the next chapter the concepts of observability and controllability will be studied. Both these properties will be useful, later, when discussing minimal realization theory for linear systems in Chapter 4 and minimal realization theory for fixed switched linear systems in Chapter 5.

Chapter 3

Observability and Controllability

In general, observability is about how well one can "observe" the internal states of a system from the knowledge of its external signals. In many cases this boils down to the question: given the input and output data, what knowledge is available with regard to the state values? On the other hand, controllability, in general, is about how well a system allows movement between internal states. In many cases this boils down to the question: given an initial state, which states can be reached within finite time?

3.1 Overview

In this chapter the main objective is to introduce the concepts of observability (Paragraph 3.2) and controllability (Paragraph 3.3) for linear systems and fixed switched linear systems. These two concepts will be used, later, in Chapter 4 and Chapter 5, when considering, respectively, minimal realization theory for linear systems and minimal realization theory for fixed switched linear systems. Incidentally, for completeness, observability and controllability for switched linear systems will also be discussed within this chapter.

3.2 Observability

In this paragraph observability with regards to linear systems, switched linear systems and fixed switched linear systems shall be studied, starting with the linear systems. In the case of linear systems, most information is taught during any standard mathematics bachelor curriculum, but if needed, a good reference would be the book by Trentelman et al. [8, Chapter 3.3]. In the case of switched linear systems and fixed switched linear systems, most information will be obtained from the paper by Petreczky et al. [5].

Remark 3.2.1 (Remark about the paper by Petreczky et al. [5])**.** Do notice that in the paper it is assumed that all state dimensions are equal. Nevertheless, some of the observations still hold true when the state dimensions are varying. These particular observations, together with some of the more restrictive observations, shall be considered within this paragraph. □

3.2.1 Linear Systems

There are many ways to define what observability means for linear systems. Most of these definitions, however, state the exact same, using different expressions. Still, to avoid miscommunication regarding the definition of observability, it is important to clearly state what the definition shall be. In this paper the definition shall be as follows.

Definition 3.2.2 (Observability for Linear Systems)**.** Let Σ_N be a linear system. Σ_N is said to be *Observable* if and only if for all solutions (u_1, x_1, y_1) and (u_2, x_2, y_2) the following implication holds

$$(u_1, y_1) \equiv (u_2, y_2) \implies x_1 \equiv x_2. \qquad \square$$

Remark 3.2.3. Some remarks regarding Definition 3.2.2

(i) The definition has the following interpretation. A linear system is observable if and only if for the same input-output behaviour, no two different state-space solutions exists.

(ii) If a linear system is not observable, it is called *Unobservable*.

(iii) Notice that any linear system of zero dimension is observable, see also Remark 2.2.2.

(iv) It can be shown that input u plays no role in the definition of observability. Hence, also matrices B and D do not influence the observability property. [8, Chapter 3.3] □

A well-known result from linear system theory is the observability rank test. This is a test to check whether a linear system is indeed observable.

Theorem 3.2.4 (Observability Rank Test)**.** *A linear system Σ_N is observable if and only if Rank$(\mathcal{O}) = n$, where*

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Remark 3.2.5. The matrix $\mathcal{O}$ is known as the *Observable Matrix* and the kernel of $\mathcal{O}$ is known as the *Unobservable Subspace*. Notice that, if the unobservable subspace contains only the zero element, matrix $\mathcal{O}$ shall have full column rank, and vice versa, if matrix $\mathcal{O}$ has full column rank, the unobservable subspace shall contain only the zero element. This implies that the unobservable subspace is "empty" if and only if the linear system is observable. □

In the case of exact arithmetics, the observability rank test gives an easy method to check whether a given linear system is indeed observable or not.

Furthermore, because the rank test only needs finite computations, it is guaranteed that within a finite number of steps it can be checked whether a linear system is observable or not. Of course, if the dimensions of the matrices become rather large, checking for observability, using the rank test, can still be inconvenient. In those cases other methods might be preferred. However in the case of small dimensions, the observability rank test is both fast and easy to apply.

Remark 3.2.6. It should be noted that in the case of computer computations, even small dimensions can still cause incorrect results. This is because small rounding errors can easily lead to wrong conclusions. □

For more information on observability for linear systems, see, for example, the books [1, 7, 8].

3.2.2 Switched Linear Systems

In the case of switched linear systems, it is possible to define multiple different concepts of observability. In this paper, however, the following concept of observability shall be used, when considering switched linear systems.

Definition 3.2.7 (Observability for Switched Linear Systems)**.** Let $\Sigma_{\mathcal{M}}$ be a switched linear system. $\Sigma_{\mathcal{M}}$ is said to be *Observable* if and only if for all solutions $(\sigma_1, u_1, x_1, y_1)$ and $(\sigma_2, u_2, x_2, y_2)$ the following implication holds

$$(\sigma_1, u_1, y_1) \equiv (\sigma_2, u_2, y_2) \implies x_1 \equiv x_2 \qquad \square$$

Remark 3.2.8. Some remarks regarding Definition 3.2.7

(i) The definition has the following interpretation. A switched linear system is observable if and only if for the same input-output behavior and the same switching signal, no two different state-space solutions exists.

(ii) If a switched linear system is not observable, it is called *Unobservable*.

(iii) Observability of the type mentioned in this definition is also referred to as *Strong Observability*.

(iv) It can be shown that input u plays no role in the definition of observability. Hence, also matrices B_q and D_q do not influence the observability property. [5] □

Using the paper by Petreczky et al. [5], the following theorem holds true with regards to observability for switched linear systems.

Theorem 3.2.9 (Observability for Switched Linear Systems)**.** *Let $\Sigma_{\mathcal{M}}$ be a switched linear system. $\Sigma_{\mathcal{M}}$ is observable if and only if for all $q \in \mathfrak{D}$ linear system Σ_{N_q} is observable.*

For more information on observability for switched linear systems, see the paper by Petreczky et al. [5].

3.2.3 Fixed Switched Linear Systems

In the case of fixed switched linear systems, it is also possible to define multiple different concepts of observability. In this paper, however, the following concept of observability shall be used, when considering fixed switched linear systems.

Definition 3.2.10 (Observability for Fixed Switched Linear Systems)**.**
Let Σ_Γ be a fixed switched linear system. Σ_Γ is said to be *Observable* if and only if for all solutions (u_1, x_1, y_1) and (u_2, x_2, y_2), the following implication holds

$$(u_1, y_1) \equiv (u_2, y_2) \implies x_1 \equiv x_2$$ □

Remark 3.2.11. Some remarks regarding Definition 3.2.10

(i) The definition has the following interpretation. A fixed switched linear system is observable if and only if for the same input-output behavior, no two different state-space solutions exists.

(ii) If a fixed switched linear system is not observable, it is called *Unobservable*.

(iii) It can be shown that input u plays no role in the definition of observability. Hence, also matrices B_q and D_q do not influence the observability property. [5]

(iv) Notice that if a switched linear system $\Sigma_{\mathcal{M}}$ is observable, any fixed switched linear system Σ_Γ that is the result of fixing the switching signal of $\Sigma_{\mathcal{M}}$, also will be observable. However, the other way around does not hold in general, i.e., given that a fixed switched linear system Σ_Γ, that *will* switch between the linear systems M, is observable, the switched linear system $\Sigma_{\mathcal{M}}$, that *can* switch between *at least* the linear systems M, does not need to be observable, see also Example 3.2.12. □

Example 3.2.12. Consider the two linear systems Σ_{N_0} and Σ_{N_1}, where

$$N_0 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, 0 \right)$$
$$N_1 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, 0 \right).$$

Let

$$J_{(0,0)} = I_2,\ J_{(1,0)} = I_2,\ J_{(0,1)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } J_{(1,1)} = I_2.$$

Consider the following observations.

- Fixed switched linear system Σ_{Γ_1}, that *will* switch from linear system Σ_{N_0} to linear system Σ_{N_1} at t_1 using jump map $J_{(1,0)}$, can easily be proven to be observable. First notice that linear system Σ_{N_0} shall observe its first state, linear system Σ_{N_1} shall observe its second state and that the state-space does not change during the switch. Hence, the initial state can be determined uniquely, implying that Σ_{Γ_1} is observable.

- Fixed switched linear system Σ_{Γ_2}, that *will* switch from linear system Σ_{N_1} to linear system Σ_{N_0} at t_1 using jump map $J_{(0,1)}$, can easily be proven to be unobservable. First notice that during the switch all information about the first state is lost and Σ_{N_1} shall only observe its second state. Hence, the first coordinate of the initial state cannot be determined uniquely, implying that Σ_{Γ_2} is unobservable.
- Switched linear system $\Sigma_{\mathcal{M}}$, that *can* switch between *at least* the two linear systems Σ_{N_0} and Σ_{N_1}, can easily be proven to be unobservable, from the fact that both linear systems are unobservable. □

Consider the following lemma without proof. The proof comes from the fact that the solution of a fixed switched linear system is uniquely determined by the initial value, see Lemma 2.4.12

Lemma 3.2.13. *A fixed switched linear system Σ_Γ is observable if linear system Σ_{N_0} is observable.*

Using the paper by Petreczky at al. [5], the following theorem holds true with regards to observability for fixed switched linear systems.

Theorem 3.2.14 (Observability for Fixed Switched Linear Systems)**.** *Let Σ_Γ be a fixed switched linear system such that σ is **not** constant and **all** state dimensions are equal. Σ_Γ is observable if and only if there exists $k \in \{1, 2, \ldots, \#\mathcal{T}\}$ such that*

$$\mathscr{M}_1^k = \{0_{n_0}\} \tag{3.1}$$

where

$$\begin{aligned}
\mathscr{M}^i &:= \ker \mathcal{O}_{i-1} \cap \ker \mathcal{O}_i J_i, && (3.2a)\\
\mathscr{M}_k^k &:= \mathscr{M}^k, && (3.2b)\\
\mathscr{M}_i^k &:= \mathscr{M}^i \cap J_i^{-1}(e^{-A_i\tau_i}\mathscr{M}_{i+1}^k), \quad k > i \geq 1 && (3.2c)\\
\tau_i &:= t_{i+1} - t_i && (3.2d)
\end{aligned}$$

where $\mathcal{O}_i$ is the observable matrix of the linear system Σ_{N_i}, $i \in \{0, 1, 2, \ldots, k\}$.

Remark 3.2.15. Some remarks regarding Theorem 3.2.14

(i) Do notice that in the theorem it is assumed that the fixed switching signal σ is not constant. In the case the fixed switching signal σ would be constant, the resulting fixed switched linear system would be a linear system and Theorem 3.2.4 could, instead, be used to check observability.

(ii) The intuition behind $\mathscr{M}^i$ is that $\mathscr{M}^i$ would be the unobservable subspace if there would be only a single switch from mode $i-1$ to mode i.

(iii) The intuition behind the sequence (3.2) is as follows: Starting at the k^{th} switch, we go backward in time and combine the local knowledge from each of the previous switches to obtain knowledge of the initial state value [5].

(iv) A conjecture, that will not be proven within this paper, is that the above theorem also hold in the case of varying dimensions.

(v) See also Appendix A.6 regarding the concept of vector space operations. □

To get a better understanding of Theorem 3.2.14, consider the following example.

Example 3.2.16. Consider the three linear systems Σ_{N_0}, Σ_{N_1} and Σ_{N_2} where

$$N_0 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, 1 \right)$$

$$N_1 = \left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, 0 \right)$$

$$N_2 = \left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, 0 \right)$$

It can easily be shown, using the observability rank test, that linear system Σ_{N_0} is unobservable and linear systems Σ_{N_1} and Σ_{N_2} are observable. This implies that the unobservable subspaces for linear systems Σ_{N_1} and Σ_{N_2} are equal to $\{0_3\}$.

Let Σ_Γ be a fixed switched linear system such that

- $\mathfrak{D} = \{0, 1, 2\}$
- σ is a fixed switching signal given by switching times $\mathcal{T} = \{1, 2\}^{\mathfrak{D}^*}$
- $M = \{N_0, N_1, N_2\}^{\mathfrak{D}}$
- $\mathcal{J} = \{I_3, I_3\}^{\mathfrak{D}^*}$
- $\mathcal{N} = \{3, 3, 3\}^{\mathfrak{D}}$

When applying Theorem 3.2.14 with $k = 2$, the following steps are taken:

Step 1: Calculate $\mathscr{M}_2^2 = \mathscr{M}^2 = \ker \mathcal{O}_1 \cap \ker \mathcal{O}_2 J_2 = \{0_3\}$.

Step 2: Calculate $\mathscr{M}^1 = \ker \mathcal{O}_0 \cap \ker \mathcal{O}_1 J_1 = \{0_3\}$.

Step 3: Calculate $\mathscr{M}_1^2 = \mathscr{M}^1 \cap J_1^{-1}(e^{-A_1}\mathscr{M}_2^2) = \{0_3\} \cap \{0_3\} = \{0_3\}$.

From this it can be concluded, by Theorem 3.2.14, that Σ_Γ is observable. □

For more information on observability for fixed switched linear systems, see the paper by Petreczky et al. [5].

3.3 Controllability

In this paragraph controllability with regards to linear systems, switched linear systems and fixed switched linear systems shall be studied, starting with the linear systems. In the case of linear systems, most information is taught during any standard mathematics bachelor curriculum, but if needed, a good reference would be the book by Trentelman et al. [8, Chapter 3.2]. In the case of switched linear systems and fixed switched linear systems, a brief statement, with regards to controllability, will be made.

3.3.1 Linear Systems

In most textbooks, a distinction is made between what is called *Controllability*, *Null-Controllability* and *Reachability*. In the case of linear systems, it can be shown that all three concepts are equivalent. Nevertheless, for clarity, all three concepts are introduced in the following definition.

Definition 3.3.1 (Controllability for Linear Systems)**.** Let Σ_N be a linear system.

- Σ_N is said to be *Controllable* if and only if for all states $x_0, x_1 \in \mathbb{R}^n$, there exists a solution (u, x, y) such that $x(0) = x_0$ and $x(T) = x_1$, for some time $T \in \mathbb{R}_{>0}$.
- Σ_N is said to be *Null-Controllable* if and only if for all states $x_0 \in \mathbb{R}^n$, there exists a solution (u, x, y) such that $x(0) = x_0$ and $x(T) = 0_n$, for some time $T \in \mathbb{R}_{>0}$.
- Σ_N is said to be *Reachable* if and only if for all states $x_1 \in \mathbb{R}^n$, there exists a solution (u, x, y) such that $x(0) = 0_n$ and $x(T) = x_1$, for some time $T \in \mathbb{R}_{>0}$. □

Remark 3.3.2. Some remarks regarding Definition 3.3.1

(i) The definition has the following interpretations:

- A linear system is controllable if and only if any state can be reached from any state.
- A linear system is null-controllable if and only if the zero state can be reached from any state.
- A linear system is reachable if and only if any state can be reached from the zero state.

(ii) Notice that any linear system of zero dimension is controllable, null-controllable and reachable, see also Remark 2.2.2.

(iii) Notice that output y plays no role in the definitions of controllability, null-controllability and reachability. Hence, also matrices C and D do not influence the controllability, null-controllability and reachability properties.

(iv) As stated before, all three concepts are equivalent in the case of linear systems. Henceforth, in the case of linear systems, the equivalent concepts of controllability, null-controllability and reachability will be referred to as simply *Controllability*.

(v) If a linear system is not controllable, it is called *Uncontrollable*. □

A well-known result from linear system theory is the controllability rank test. This is a test to check whether a linear system is indeed controllable.

Theorem 3.3.3 (Controllability Rank Test)**.** *A linear system Σ_N is controllable if and only if Rank$(\mathcal{R}) = n$, where*

$$\mathcal{R} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

Remark 3.3.4. The matrix $\mathcal{R}$ is known as the *Reachable Matrix* and the image of $\mathcal{R}$ is known as the *Reachable Subspace*. Notice that, if the reachable subspace is equal to the entire space $\mathbb{R}^n$, matrix $\mathcal{R}$ shall have full row rank, and vice versa, if matrix $\mathcal{R}$ has full row rank, the reachable subspace shall be equal to the entire space $\mathbb{R}^n$. This implies that the reachable subspace is the entire space $\mathbb{R}^n$ if and only if the linear system is controllable. □

In the case of exact arithmetics, the controllability rank test gives an easy method to check whether a given linear system is indeed controllable or not, similar to the observability rank test. Furthermore, because the rank test only needs finite computations, it is guaranteed that within a finite number of steps it can be checked whether a linear system is controllable or not. Again, if the dimensions of the matrices become rather large, checking for controllability using the rank test can be inconvenient. In those cases other methods might be preferred. However in the case of small dimensions, the controllability rank test is both fast and easy to apply. See also Remark 3.2.6.

For more information on controllability for linear systems, see, for example, the books [1, 7, 8].

3.3.2 (Fixed) Switched Linear Systems

For switched linear system and fixed switched linear systems it is rather difficult to give a general definition for controllability. This is because already the property that each "state-piece" x_q has a different dimension, makes it difficult to understand what it means to be able to "move" from any state to any state. Fortunately, in the next chapters, controllability for switched linear systems and fixed switched linear systems will not be used, and hence controllability for both systems shall be ignored for the rest of this paper.

Now that the concepts of observability and controllability have been introduced, in the next chapter minimal realization theory for linear systems shall be considered.

Chapter 4

Minimal Realization Theory for Linear Systems

In mathematics the problem of system realization is to find an internal description to a given external description. Examples of this would be to find a linear system that has transfer function equal to a given external description $H(s)$, or to find a linear system that has input-output behavior equal to a given external description Σ_N. System realization, however, is only the beginning. For each external description there might exist infinitely many internal descriptions. Hence, the logical next step would be to consider finding the most optimal internal descriptions. This subject is known as *Minimal Realizations Theory* and will be the main subject of this and next chapter.

4.1 Overview

In this chapter the main objective shall be to consider minimal realization theory for linear systems. However, the main goal shall be to collect results for the case that the external description is given by a linear system. These results will then be used in Chapter 5, to investigated whether these results can be generalized to fixed switched linear systems.

Within this chapter a distinction will be made between having to satisfy an external description given by a function (Paragraph 4.2), and having to satisfy an external description given by a linear system (Paragraph 4.3). The main reason as to why also the function case shall be considered, is that the function case shall be the foundation on which the system case shall be constructed.

4.2 Function Case

Before considering the system case, first the function case shall be considered. Important to know is that most information, from within this paragraph, was obtained from the book by Antsaklis et al. [1].

4.2.1 Linear Realization

Given an external description $H(s)$, the following definition will explain when a linear system is indeed considered a "linear" realization of $H(s)$.

Definition 4.2.1 (Linear Realization)**.** A *Linear Realization* of $H(s)$ is any linear system Σ_N for which its transfer function is equal to $H(s)$. □

It can be shown that not all functions $H(s)$ have a linear realization. This, in turn, begs the question, when does a function $H(s)$ have a linear realization? This question is answered in the following theorem. The proof can be found in the book by Antsaklis et al. [1, Theorem 8.5].

Theorem 4.2.2 (Existence of Linear Realizations)**.** *$H(s)$ has a linear realization if and only if $H(s)$ is a matrix of rational functions and satisfies*

$$\lim_{s\to\infty} H(s) < \infty,$$

i.e. $H(s)$ has only finite entries in the limit of $s \to \infty$. This is equivalent to stating that $H(s)$ is a proper rational matrix, i.e., each entry of $H(s)$ is a rational function of which the degree of the denominator is higher or equal to the degree of the numerator.

To illustrate this theorem, consider the following example.

Example 4.2.3. Let $H(s) = \frac{1}{s}$. Since H is a proper rational function, Theorem 4.2.2 states that there exists a linear realization of $H(s)$. And indeed the linear systems Σ_{N_1} and Σ_{N_2}, where

$$N_1 = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, 0 \right)$$

$$N_2 = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, 0 \right)$$

are examples of linear realizations of $H(s)$, since, for both, their transfer function is equal to H. □

Besides illustrating the theorem, the example also shows that, indeed, for an external description there might exists multiple internal descriptions. In the following theorem it shall even be proven that for any external description $H(s)$, that is a proper rational matrix, there exists infinitely many linear realizations.

Theorem 4.2.4 (Infinitely many Linear Realizations)**.** *Consider any external description $H(s)$ that is a proper rational matrix. There exists infinitely many linear realizations of $H(s)$.*

Proof. Consider any linear realization of $H(s)$, which exists since $H(s)$ is said to be a proper rational matrix, see Theorem 4.2.2. Adding the state equation $\dot{z} = Fz + Gu$ to the linear realization of $H(s)$, will result in a new linear system that has transfer function still equal to $H(s)$, i.e. will results in a new linear realization of $H(s)$. This procedure can be repeated infinitely, hence there exists infinitely many linear realizations of $H(s)$. ⚠

Remark 4.2.5. Notice that, after the procedure, the dimension of the new linear realization of $H(s)$ is larger then the dimension of the linear realization of $H(s)$ used in the procedure. Since the procedure can be repeated infinitely, this implies that there is no upper bound for the dimension of linear realizations of $H(s)$. The lower bound, however, does exists and will have the main focus when considering minimal linear realizations of $H(s)$. □

Remark 4.2.6. From now onwards, it will be assumed that any external description $H(s)$ will be a proper rational matrix unless stated otherwise. □

4.2.2 Minimal Linear Realization

Now that linear realizations of $H(s)$ have been studied, the next topic of interest would be the minimal linear realizations of $H(s)$. For this consider the following definition.

Definition 4.2.7 (Minimal Linear Realization)**.** Let Σ_N be any linear realization of $H(s)$. Σ_N is a *Minimal Linear Realization* of $H(s)$ if and only if $\text{Dim}(\Sigma_N) \leq \text{Dim}(\Sigma_M)$ for all linear realizations Σ_M of $H(s)$. □

Consider next the following two well-known theorems within the study of minimal realization theory for linear systems. For both, the proof can be found in the book by Antsaklis et al. [1, Theorems 8.9 and 8.10].

Theorem 4.2.8 (Minimal Linear Realization Theorem: Function Case)**.** *A linear realization Σ_N of $H(s)$ is minimal if and only if Σ_N is both controllable and observable.*

Theorem 4.2.9 (Equivalent Minimal Linear Realizations)**.** *Let Σ_{N_1} and Σ_{N_2} be linear realizations of $H(s)$. If Σ_{N_1} is a minimal linear realization of $H(s)$, then Σ_{N_2} is also a minimal linear realization of $H(s)$ if and only if $D_1 = D_2$ and there exists a non-singular matrix P such that*

$$A_2 = PA_1P^{-1},\ B_2 = PB_1 \text{ and } C_2 = C_1P^{-1}.$$

Furthermore, if P exists, it is given by

$$P = \mathcal{R}_1\mathcal{R}_2^T(\mathcal{R}_2\mathcal{R}_2^T)^{-1} \text{ or } P = (\mathcal{O}_2^T\mathcal{O}_2)^{-1}\mathcal{O}_2^T\mathcal{O}_1.$$

where $\mathcal{O}_i$ and $\mathcal{R}_i$ are, respectively, the observable and reachable matrices of the linear system Σ_{N_i}, $i \in \{1,2\}$.

Consider next the following corollary, which is the result of Theorem 4.2.9 and Lemma 2.2.9.

Corollary 4.2.10. *Any two minimal linear realizations of $H(s)$ are input-output equivalent.*

To illustrate Theorem 4.2.8, consider the following continuation of Example 4.2.3.

Example 4.2.11 (Continuation Example 4.2.3)**.** Notice that linear system Σ_{N_2} is neither controllable nor observable and notice that linear system Σ_{N_1} is controllable but not observable. Hence, by Theorem 4.2.8, there exists a linear system of at most dimension 1 with transfer function equal to $H(s)$.

Consider now the linear system Σ_{N_3}, $N_3 = (0, 1, 1, 0)$. It can easily be shown that this linear system is observable, controllable and has transfer function equal to $H(s)$. Hence, by Theorem 4.2.8, linear system Σ_{N_3} is a minimal linear realization of $H(s)$. □

As might be expected, one of the issues that need to be solved, when considering minimal linear realizations of $H(s)$, is finding the "minimal dimension". While there do exists methods to derive this "minimal dimension", this is beyond the scope of this paper. Instead, the reader is referred to the book by Antsaklis et al. [1].

4.2.3 Minimal Linear Realization Algorithm: KCDA

Now that both linear realizations and minimal linear realizations of $H(s)$ have been considered, the next topic of interest would be algorithms that generate minimal linear realizations. However, instead of analyzing any and all possible algorithms, in this paragraph the focus will lie with a particular algorithm, namely the *Kalman Canonical Decomposition Algorithm* (KCDA). The main objective behind introducing this algorithm is that a similar algorithm will be considered later in the system case. Instead, for more information on general minimal linear realization algorithms using, for example, the observer form, the controller form or the singular value decomposition, see, for example, the book by Antsaklis et al. [1].

As the name KCDA already implies, this algorithm uses the *Kalman Canonical Decomposition* (KCD) to find a minimal linear realization of $H(s)$. To use this algorithm, it is required that already a linear realization of $H(s)$ is given. However, before explaining how this algorithm works, step-by-step, first consider the following explanation regarding the KCD itself.

4.2.3.1 Kalman Canonical Decomposition

Around 60 years ago R.E. Kalman wrote his article about the KCD [2]. In this article he explained that any linear system can be decomposed into observable, unobservable, controllable and uncontrollable parts. In the following theorem a quick summary of the KCD will be given.

Theorem 4.2.12 (KCD)**.** *Let Σ_{N_1} be a linear system. There exists a non-singular matrix S such that*

$$S^{-1}A_1S = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \; S^{-1}B_1 = \begin{bmatrix} B_{11} \\ B_{21} \\ 0 \\ 0 \end{bmatrix}, \; C_1S = \begin{bmatrix} 0 & C_{12} & 0 & C_{14} \end{bmatrix}$$

Furthermore

- Σ_{N_2}, $N_2 = \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \begin{bmatrix} 0 & C_{12} \end{bmatrix}, D_1 \right)$ *is controllable.*
- Σ_{N_3}, $N_3 = \left(\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_{21} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{12} & C_{14} \end{bmatrix}, D_1 \right)$ *is observable.*
- Σ_{N_4}, $N_4 = (A_{22}, B_{21}, C_{12}, D_1)$ *is controllable and observable.*
- Σ_{N_1}, Σ_{N_2}, Σ_{N_3} *and* Σ_{N_4} *all have the same transfer function.*

Proof.

- The main body and the first two points are proven in the book by Polderman et al. [7, Theorem 5.4.1].
- The third point directly follows from the previous two points and the observability/controllability rank tests.
- The fourth point can be proven by noticing, first, that for any two linear systems Σ_{N_i} and Σ_{N_j}, $i, j \in \{1, 2, 3, 4\}$, $C_i e^{A_i t} B_i \equiv C_j e^{A_j t} B_j$ (See also Appendix A.5). Next, Lemma 2.2.6 will conclude that, for initial states $x_i(0) = 0_{n_i}$, $i \in \{1, 2, 3, 4\}$, all four linear systems have equal input-output relation, i.e. have the same transfer function, see also Deduction 2.2.7. △

Consider next the following algorithm that explains how non-singular matrix S can be derived, see also [7, Proof 5.4.1].

Algorithm 4.2.13 (Non-singular matrix S)**.** The Kalman Canonical Decomposition of a linear system Σ_N is entirely dependent on the non-singular matrix S. While matrix S is not unique, there does exists a reliable algorithm of construction:

Step 1: Find matrices $\mathcal{O}$ and $\mathcal{R}$ which are, respectively, the observable and reachable matrices of the linear system Σ_N.

Step 2: Find the dimensions of $\mathrm{Im}(\mathcal{R}) \cap \mathrm{Ker}(\mathcal{O})$, $\mathrm{Im}(\mathcal{R})$ and $\mathrm{Ker}(\mathcal{O})$ and denote them, respectively, by k_1, $k_1 + k_2$, $k_1 + k_3$ and let $k_4 = n - (k_1 + k_2 + k_3)$.

Step 3: Choose vectors $a_1, \ldots, a_{k_1}, b_1, \ldots, b_{k_2}, c_1, \ldots, c_{k_3}$ and $d_1, \ldots, d_{k_4}$ such that $(a_1, \ldots, a_{k_1})$ is a basis of $\mathrm{Im}(\mathcal{R}) \cap \mathrm{Ker}(\mathcal{O})$, $(a_1, \ldots, a_{k_1}, b_1, \ldots, b_{k_2})$ is a basis of $\mathrm{Im}(\mathcal{R})$, $(a_1, \ldots, a_{k_1}, c_1, \ldots, c_{k_3})$ is a basis of $\mathrm{Ker}(\mathcal{O})$ and $(a_1, \ldots, a_{k_1}, b_1, \ldots, b_{k_2}, c_1, \ldots, c_{k_3}, d_1, \ldots, d_{k_4})$ is a basis of $\mathbb{R}^n$.

Step 4: $S = [a_1 \ \ldots \ a_{k_1} \ b_1 \ \ldots \ b_{k_2} \ c_1 \ \ldots \ c_{k_3} \ d_1 \ \ldots \ d_{k_4}]$. □

4.2.3.2 Kalman Canonical Decomposition Algorithm

Now that the KCD has been explained, next up is the KCDA. Even though the algorithm is not complicated, for clarity, the algorithm will be explained step-by-step.

Algorithm 4.2.14 (KCDA)**.** Let Σ_N be a linear realization of $H(s)$.

Step 1: Apply Algorithm 4.2.13 to Σ_N and obtain S.

Step 2: Apply Theorem 4.2.12 to Σ_N using S .

Step 3: Obtain Σ_M, where $M = N_4$.

The result will be that Σ_M is observable, controllable and has transfer function $H(s)$, see Theorem 4.2.12. Hence, by Theorem 4.2.8, Σ_M is a minimal linear realization of $H(s)$. □

Remark 4.2.15. Some remarks regarding Algorithm 4.2.14

(i) As can be seen from the algorithm, the KCDA is very simplistic once the KCD is constructed. The most difficult aspect of the entire algorithm would be finding the non-singular matrix S, which is certainly doable by using Algorithm 4.2.13.

(ii) Notice that the whole idea of this algorithm is to remove the uncontrollable and unobservable parts from the linear system. This simultaneously reduces the dimension while also preserving the transfer function. □

Before ending this paragraph, consider the following remark.

Remark 4.2.16. It should be noticed that given a linear realization of $H(s)$, there does not need to exists a minimal linear realization of $H(s)$ such that both realizations are input-output equivalent. To give an example consider, again, Example 2.2.12. In this example it was proven that the given linear systems were not input-output equivalent, even though they had the same transfer function $G(s)$, i.e. they could be considered linear realizations of $G(s)$. Assume now that for each of these linear realizations of $G(s)$ there does exists a minimal linear realization of $G(s)$, that is input-output equivalent to the corresponding linear realization of $G(s)$. The problem would be that Corollary 4.2.10 implies that the minimal linear realizations of $G(s)$ should be input-output equivalent and hence, a contradiction will emerge. This implies that the assumption is incorrect and hence, at least one of the linear realizations of $G(s)$ is not input-output equivalent to any minimal linear realization of $G(s)$. □

The above remark, in particular, implies the following lemma.

Lemma 4.2.17. *Given a linear realization Σ_N of $H(s)$, there does not need to exist a minimal linear realization of $H(s)$ that is input-output equivalent to Σ_N.*

The above lemma introduces an interesting topic to consider, namely external descriptions for which linear realizations and minimal linear realizations are always input-output equivalent. And it is exactly one such external description that shall be considered within the system case.

4.3 System Case

Now that the function case has been considered, next would be the system case. For this case most information will be a replication of the function case, in hopes of yielding similar results. Some information will also be obtained from the book by Polderman et al. [7]. Most importantly however, most results mentioned within this paragraph will again be considered in Chapter 5, in order to investigate whether these results can be generalized to fixed switched linear systems.

4.3.1 Linear Realization

Given an external description Σ_N (linear system), the following definition will explain when a linear system is indeed considered a "linear" realization of Σ_N.

Definition 4.3.1 (Linear Realization)**.** A *Linear Realization* of Σ_N is any linear system Σ_M that is input-output equivalent to Σ_N. □

Remark 4.3.2. Remember from Paragraph 2.2.2 that having the same transfer function does *not* imply input-output equivalency. Hence, for certain, Definition 4.2.1, with $H(s)$ being the transfer function of Σ_N, does not imply Definition 4.3.1. Regarding whether Definition 4.3.1 does imply Definition 4.2.1, with $H(s)$ being the transfer function of Σ_N, see Remark 2.2.11 □

It can easily be shown that for any external description Σ_N, there exists infinitely many linear realizations.

Theorem 4.3.3 (Infinitely many Linear Realizations)**.** *Consider an external description Σ_N. There exists infinitely many linear realizations of Σ_N.*

Proof. Consider any linear realization of Σ_N, which exists since linear system Σ_N itself is a linear realization. Adding the state equation $\dot{z} = Fz + Gu$ to the linear realization of Σ_N, will result in a new linear system that is input-output equivalent to the linear system Σ_N, i.e. will results in a new linear realization of Σ_N. This procedure can be repeated infinitely, hence there exists infinitely many linear realizations of Σ_N. ⚠

Remark 4.3.4. Notice that, after the procedure, the dimension of the new linear realization of Σ_N is larger then the dimension of the linear realization of Σ_N used in the procedure. Since the procedure can be repeated infinitely, this implies that there is no upper bound for the dimension of linear realizations of Σ_N. The lower bound, however, does exists and will have the main focus when considering minimal linear realizations of Σ_N. □

4.3.2 Minimal Linear Realization

Now that linear realizations of Σ_N have been studied, the next topic of interest would be the minimal linear realizations of Σ_N. For this consider the following definition.

Definition 4.3.5 (Minimal Linear Realization)**.** Let Σ_M be any linear realization of Σ_N. Σ_M is a *Minimal Linear Realization* of Σ_N if and only if $\text{Dim}(\Sigma_M) \leq \text{Dim}(\Sigma_W)$ for all linear realizations Σ_W of Σ_N. □

Consider next the following theorem regarding minimal realizations of Σ_N.

Theorem 4.3.6 (Minimal Linear Realization Theorem: System Case)**.** *A linear realization Σ_M of Σ_N is minimal if and only if Σ_M is observable.*

Proof. "only if": Consider Paragraph 4.3.3. There it is explained that given an unobservable linear realization of Σ_N, a lower dimensional linear realization of Σ_N can be found, see Algorithm 4.3.10.

"if": See the book by Polderman et al. [7, Section 6.5.3] △

As might be expected one of the issues that need to be solved, when considering minimal linear realizations of Σ_N, is, again, finding the "minimal dimension". Similar to the function case, however, this subject will not be considered within this paper.

4.3.3 Minimal Linear Realization Algorithm: KODA

Now that both linear realizations and minimal linear realizations of Σ_N have been considered, the next topic of interest would be algorithms that generate minimal linear realizations. However, instead of analyzing any and all possible algorithms, in this paragraph the focus will lie with a particular algorithm, namely the *Kalman Observability Decomposition Algorithm* (KODA). This is an algorithm similar to the KCDA. The main difference, however, is that instead of separating observable, unobservable, controllable and uncontrollable parts within the linear systems, instead, KODA separates only the observable and unobservable parts.

As the name KODA already implies, this algorithm uses the *Kalman Observability Decomposition* (KOD). Hence, before explaining how this algorithm works, step-by-step, first consider the following explanation regarding the KOD itself.

4.3.3.1 Kalman Observability Decomposition

Similar to the KCD, the KOD decomposes a linear system into observable and unobservable parts. In the following theorem a quick summary of the KOD will be given.

Theorem 4.3.7 (KOD)**.** *Let Σ_{N_1} be a linear system. There exists a non-singular matrix S such that*

$$S^{-1}A_1S = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},\ S^{-1}B_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix},\ C_1S = \begin{bmatrix} 0 & C_{12} \end{bmatrix}$$

Furthermore

- Σ_{N_2}, $N_2 = (A_{22}, B_{21}, C_{12}, D_1)$ *is observable.*
- Σ_{N_1} *and* Σ_{N_2} *have the same transfer function.*
- Σ_{N_1} *and* Σ_{N_2} *are input-output equivalent*

Proof.

- The main body and the first point are proven in the book by Polderman et al. [7, Corollary 5.3.14].
- The second point can be proven by noticing, first, that for the linear systems Σ_{N_1} and Σ_{N_2}, $C_1 e^{A_1 t} B_1 \equiv C_2 e^{A_2 t} B_2$. Next, Lemma 2.2.6 will conclude that, for initial states $x_1(0) = 0_{n_1}$ and $x_2(0) = 0_{n_2}$, both linear systems have equal input-output relation, i.e. have the same transfer function, see also Deduction 2.2.7.
- The third point can be proven by first noticing, again, that for the linear systems Σ_{N_1} and Σ_{N_2}, $C_1 e^{A_1 t} B_1 \equiv C_2 e^{A_2 t} B_2$. Next, consider the following equivalency:

  $$C_1 e^{A_1(t-t_b)} x_1(t_b) \equiv C_2 e^{A_2(t-t_b)} x_2(t_b)$$

  where

  $$x_1(t_b) = S \begin{bmatrix} x_{11}(t_b) \\ x_{21}(t_b) \end{bmatrix}$$

  and $x_2(t_b) = x_{21}(t_b)$. Using the above equivalencies, the fact that S is invertable and Lemma 2.2.6, it can be concluded that the linear systems are indeed input-output equivalent. △

Remark 4.3.8. Notice that in Theorem 4.3.7 the non-controllable, but observable part is kept, while in Theorem 4.2.12, the non-controllable, but observable part is removed. This suggests that the non-controllable, but observable part is important for input-output equivalency, but does not affect the transfer function. □

Consider next the following algorithm that explains how non-singular matrix S can be derived, see also [7, Proof 5.3.14].

Algorithm 4.3.9 (Non-singular matrix S)**.** The Kalman Observability Decomposition of a linear system Σ_N is entirely dependent on the non-singular matrix S. While matrix S is not unique, there does exists a reliable algorithm of construction:

Step 1: Find matrix $\mathcal{O}$ which is the observable matrix of the linear system Σ_N.

Step 2: Find the dimension of $\mathrm{Ker}(\mathcal{O})$ and denote it by k.

Step 3: Choose vectors $s_1, \ldots, s_n$ such that $(s_1, \ldots, s_k)$ is a basis of $\mathrm{Ker}(\mathcal{O})$ and $(s_1, \ldots, s_k, s_{k+1}, \ldots, s_n)$ is a basis of $\mathbb{R}^n$.

Step 4: $S = [s_1 \ \ldots \ s_n]$. □

4.3.3.2 Kalman Observability Decomposition Algorithm

Now that the KOD has been explained, next up is the KODA. Even though the algorithm is not complicated, for clarity, the algorithm will be explained step-by-step.

Algorithm 4.3.10 (KODA)**.** Let Σ_N be a linear system.

Step 1: Apply Algorithm 4.3.9 to Σ_N and obtain S.

Step 2: Apply Theorem 4.3.7 to Σ_N using S.

Step 3: Obtain Σ_M, where $M = N_2$.

The result will be that Σ_M is observable, input-output equivalent to Σ_N and has the same transfer function as Σ_N, see Theorem 4.3.7. Furthermore, linear system Σ_M shall be strictly smaller dimensional compared to Σ_N if Σ_N is unobservable, see Algorithm 4.3.9 and Remark 3.2.5. Finally, by Theorem 4.3.6, it can be concluded that linear system Σ_M will be a minimal linear realization of Σ_N. □

Remark 4.3.11. Some remarks regarding Algorithm 4.3.10

(i) As can be seen from the algorithm, the KODA is very simplistic once the KOD is constructed. The most difficult aspect of the entire algorithm would be finding the non-singular matrix S, which is certainly doable by using Algorithm 4.3.9.

(ii) Notice that the whole idea of this algorithm is to remove the unobservable part from the linear system. This simultaneously reduces the dimension while also preserving the input-output behavior. □

Now that minimal realization theory for linear systems has been considered, in the next chapter minimal realization theory for fixed switched linear systems shall be considered.

Chapter 5

Minimal Realization Theory for Fixed Switched Linear Systems

Now that minimal realization theory for linear systems has been discussed, in this chapter minimal realization theory for fixed switched linear systems will be considered. To be more specific, in this chapter it will be investigated whether results from the system case in Chapter 4, can be generalized to fixed switched linear systems.

5.1 Overview

In this chapter the main objective will be to generalize the results from the system case in Chapter 4 to fixed switched linear systems. The main reason behind choosing *only* the system case, comes from the fact that a similar concept like transfer function does not exist for fixed switched linear systems and switched linear systems in general. Hence, instead of introducing such a concept within this paper (could be a case study on its own in all probability) only the system case will be considered. Consequently, this is also the reason why the primary research question is conditioned to only consider the system case.

Having explained that, within this chapter the following three items will be the main research topics to consider:

- Introducing realization theory for fixed switched linear systems (Paragraph 5.2)
- Introducing minimal realization theory for fixed switched linear systems (Paragraph 5.3)
- Examining whether results from the the linear system case can be generalized to fixed switched linear systems (Paragraph 5.4)

5.2 Fixed Switched Realization

Given an external description Σ_Γ (fixed switched linear system), the following definition will explain when a fixed switched linear system is indeed considered a "switched" realization of Σ_Γ.

Definition 5.2.1 (Switched Realization)**.** A *Switched Realization* of Σ_Γ is any fixed switched linear system Σ_{Γ_1} that is input-output equivalent to Σ_Γ. □

Remark 5.2.2. The above definition could also be used to define realizations for general switched linear systems. □

While the above definition certainly makes sense, it also allows for far to many possibilities. In order to be able to generalize the linear system case, a more restricted definition will be needed.

Definition 5.2.3 (Fixed Switched Realization)**.** A *Fixed Switched Realization* of Σ_Γ is any fixed switched linear system Σ_{Γ_1} that is signal-input-output equivalent to Σ_Γ. □

Remark 5.2.4. Notice that any fixed switched linear system that satisfies Definition 5.2.3 will also satisfy Definition 5.2.1, assuming that Σ_Γ remains unchanged between the two definitions, see also Remark 2.4.15. The other way around, however, does not hold in general. □

Regarding existence, notice that any fixed switched linear system is a (fixed) switched realization of itself. Combine this with the idea that any linear system has infinitely many realizations, see Theorem 4.3.3, it is easily proven that also fixed switched linear systems have infinitely many (fixed) switched realizations.

Theorem 5.2.5 (Infinitely many (Fixed) Switched Realizations)**.** *Consider an external description Σ_Γ. There exists infinitely many (fixed) switched realizations of Σ_Γ.*

Proof. Consider any (fixed) switched realization of Σ_Γ, which exists since fixed switched linear system Σ_Γ itself is a (fixed) switched realization. Consider next the (fixed) switched realization's q$^{\text{th}}$ linear system part, i.e. Σ_{N_q} and add the state equation $\dot{z} = Fz + Gu$. Simultaneously, add a zero row to J_q and a zero column to J_{q+1} if they exist. The result will be a new fixed switched linear system that is signal-input-output equivalent to the fixed switched linear system Σ_Γ, i.e. will results in a new (fixed) switched realization of Σ_Γ. This procedure can be repeated infinitely, hence there exists infinitely many (fixed) switched realizations of Σ_Γ. △

Remark 5.2.6. In a similar way to linear systems, notice that there is no upper bound for the dimension of (fixed) switched realizations. More specific, notice that there is no upper bound for any element of $\mathcal{N}$. Lower bounds, however, do exist and they will, again, be used to define minimal realizations, see also Remark 4.3.4. □

5.3 Minimal Fixed Switched Realization

Now that fixed switched realizations of Σ_Γ have been introduced, the next topic of interest would be the minimal fixed switched realizations of Σ_Γ. For this consider the following definition.

Definition 5.3.1 (Minimal Fixed Switched Realization)**.** Let Σ_{Γ_1} be any fixed switched realization of Σ_Γ. Σ_{Γ_1} is a *Minimal Fixed Switched Realization* of Σ_Γ if and only if $\text{Dim}(\Sigma_{\Gamma_1}) \leq \text{Dim}(\Sigma_{\Gamma_2})$ (element-wise, i.e. $n_i^1 \leq n_i^2$) for all fixed switched realizations Σ_{Γ_2} of Σ_Γ. □

To better understand this definition, consider the following example of a minimal fixed switched realization.

Example 5.3.2. Consider the following two fixed switched linear systems constructed from three linear systems, two jump matrices and one fixed switching signal.

Fixed Switched Linear System 1:

$$N_0^1 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, 0 \right)$$

$$N_1^1 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, 0 \right)$$

$$N_2^1 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, 1 \right)$$

$$J_1^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J_2^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\sigma(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0) \\ 1, \text{ if } t \in [0, 1) \\ 2, \text{ if } t \in [1, \infty) \end{cases}$$

Fixed Switched Linear System 2:

$$N_0^2 = (1, 1, 1, 0)$$
$$N_1^2 = (1, 2, 1, 0)$$
$$N_2^2 = (1, 0, 1, 1)$$
$$J_1^2 = 1 \quad J_2^2 = 1$$

$$\sigma(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0) \\ 1, \text{ if } t \in [0, 1) \\ 2, \text{ if } t \in [1, \infty) \end{cases}$$

Consider Lemma 2.4.12 and apply this lemma to both of the fixed switched linear systems. From the subsequent result, it can then be concluded that the fixed switched linear systems are signal-input-output equivalent. In the case this result is not readily seen, it might help to consider the initial state of fixed switched linear system 2 to be given by $x^2(t_b) = x_2^1(t_b)$. That aside, since fixed switched linear system 2 cannot be simplified any further without losing information, it can be concluded that fixed switched linear system 2 is an example of a minimal fixed switched realization of fixed switched linear system 1. □

5.4 Generalizing the Linear System Case

Now that both fixed switched realizations and minimal fixed switched realizations have been introduced, next would be to consider generalizing the linear system case. For starters consider the following candidate for a generalization of Theorem 4.3.6.

Conjecture 5.4.1. *A fixed switched realization* Σ_{Γ_1} *of* Σ_Γ *is minimal if and only if* Σ_{Γ_1} *is observable.*

Remark 5.4.2. Notice that the above conjecture does make a lot of sense when considering the "only if" direction. Namely, should a fixed switched realization not be observable, it might be possible to remove the unobservable part similar to how this is done for linear systems, see Algorithm 4.3.10. This would imply that a fixed switched realization could only be minimal if the fixed switched realization is observable. Of course, this would only hold true if removing the unobservable part would not change the input-output behavior. □

Unfortunately, though this candidate looks promising, the following example will show that Conjecture 5.4.1 turns out to be false.

Example 5.4.3. Consider the following two fixed switched linear systems constructed from two linear systems, one jump matrix and one fixed switching signal.

Fixed Switched Linear System 1:

$$
\begin{aligned}
N_0^1 &= \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, 0 \right) \\
N_1^1 &= (1, 2, 1, 0) \\
J^1 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
\sigma(t) &= \begin{cases} 0, \text{ if } t \in (-\infty, 0) \\ 1, \text{ if } t \in [0, \infty) \end{cases}
\end{aligned}
$$

Fixed Switched Linear System 2:

$$N_0^2 = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, 0 \right)$$

$$N_1^2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, 0 \right)$$

$$J^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sigma(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0) \\ 1, \text{ if } t \in [0, \infty) \end{cases}$$

Using Theorem 3.2.4, it can be shown that the linear systems $\Sigma_{N_0^i}$, $i \in \{1, 2\}$, are observable. This, in turn, will imply that both fixed switched linear systems are observable, see Lemma 3.2.13, and hence, by Conjecture 5.4.1, minimal fixed switched realizations of themselves. However, after applying Lemma 2.4.12, the subsequent result would be that both fixed switched linear systems are signal-input-output equivalent and thus there would be a contradiction. □

The above example is a clear counter example towards Conjecture 5.4.1. In particular, the above example "claims" that *just* observability is not enough to proof minimality. Hence, to resolve this, consider the following "update" of Conjecture 5.4.1.

Conjecture 5.4.4. *A fixed switched realization Σ_{Γ_1} of Σ_Γ is minimal if and only if each linear system of Σ_{Γ_1} is observable.*

Remark 5.4.5. Notice that since all linear systems need to be observable, the fixed switched realization also has to be observable, see Lemma 3.2.13. Hence, this conjecture is a stronger version of the previous conjecture. □

Unfortunately, though this candidate also looks promising, the following example will show that also Conjecture 5.4.4 turns out to be false.

Example 5.4.6. Consider the following fixed switched linear system constructed from three linear systems, two jump matrices and one fixed switching signal.

Fixed Switched Linear System:

$$N_0 = (1, 1, 1, 0)$$

$$N_1 = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, 0 \right)$$

$$N_2 = (1, 1, 1, 0)$$

$$J_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad J_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\sigma(t) = \begin{cases} 0, \text{ if } t \in (-\infty, 0) \\ 1, \text{ if } t \in [0, 1) \\ 2, \text{ if } t \in [1, \infty) \end{cases}$$

Using Theorem 3.2.4, it is readily shown that linear systems Σ_{N_1} and Σ_{N_3} are observable and linear system Σ_{N_2} is unobservable. Next, notice that none of the linear systems can be simplified any further without losing information:

1. System 1 and 3 are already 1-dimensional and reducing these systems any further would result in loss of information.

2. System 2 has two independent *optimised* roles:

   i Pass on the output of linear system 1 to linear system 3.

   ii Obtaining output from a 2-dimensional *observable* linear system.

Hence, this implies that the above fixed switched linear system is a minimal fixed switched realization of itself, contradicting Conjecture 5.4.4. □

Remark 5.4.7. Notice that nothing has been said about changing the linear systems all together, in hopes of finding three linear systems which are all observable, smaller dimensional and give the same input-output behavior when considering the corresponding fixed switched linear system. This suggestion, however, will quickly turn out to be impossible. First of all notice that changing linear system 1 doesn't matter, since linear system 2 is only interested in the output of linear system 1. Second of all, reducing linear system 2 is just not feasible. Hence, it will be impossible to remove the unobservability from linear system 2, implying that Conjecture 5.4.4 keeps being contradicted. □

Similar to the previous example, the above example is a clear counter example towards Conjecture 5.4.4. In particular, the above example "claims" that taking all linear systems to be observable, is to strong of a condition. Hence, this time, the "updated" conjecture should be stronger than Conjecture 5.4.1 but weaker then Conjecture 5.4.4. In order to construct such a conjecture, first consider the following definition.

Definition 5.4.8. Let Σ_Γ be a fixed switched linear system. Σ_Γ^i is the fixed switched linear system constructed by removing the first i linear systems within Σ_Γ. □

Conjecture 5.4.9. *A fixed switched realization Σ_{Γ_1} of Σ_Γ is minimal if and only if fixed switched linear system $\Sigma_{\Gamma_1}^i$ is observable $\forall i \in \{0, 1, 2, \ldots, \#\mathcal{T} - 1\}$.*

Remark 5.4.10. Some remarks regarding Conjecture 5.4.9

(i) The idea behind the conjecture is that for each linear system, contained within the fixed switched realization, the unobservable part, based on the observability definition for fixed switched linear systems, is removed.

(ii) Notice that Conjecture 5.4.9 is both stronger then Conjecture 5.4.1 and weaker then Conjecture 5.4.4. □

Unfortunately, though the latest conjecture is the most promising candidate of all, no proof has been found as of yet by the author. On the other hand, also no counter example has been found as of yet. Hence, a future study might resolve whether this last conjecture is correct or false. Either way, at least it is justifiable, see Remark 5.4.10.

Before ending this chapter, consider the following algorithm that can be used to find the unobservable parts within each linear system of a fixed switched linear system. The idea behind this algorithm is similar to finding the unobservable subspace for linear systems.

Algorithm 5.4.11. Let Σ_Γ be a fixed switched linear system.

Step 1: Let $i = 0$.

Step 2: Let $k = 1$.

Step 3: Apply Theorem 3.2.14 to Σ_Γ^i, obtain $\mathscr{M}_1^k$ and take $k = k + 1$.

Step 4: Repeat from step 3 until either $\mathscr{M}_1^k = \{0_{n_i}\}$ or $k = \#\mathcal{T}$.

Step 5: Let $\mathscr{S}_i = \mathscr{M}_1^k$ and repeat from step 2 until $i = \#\mathcal{T}$

The indexed family of $\mathscr{S}_i$ now contains for each linear system Σ_{N_i} the unobservable part. □

Remark 5.4.12. Some remarks regarding Algorithm 5.4.11

(i) Notice that the algorithm, technically, only works if the fixed switched linear system has finite switches.

(ii) Notice that the algorithm, technically, only works if the fixed switched linear system contains only linear systems of equal dimension. However, a conjecture, that will not be proven within this paper, states that the algorithm also works when the linear system dimensions are varying, see also Remark 3.2.15.

(iii) Now that the unobservable parts have been found, next would be to remove them from within each linear system. This, however, would require a theorem similar to Theorem 4.3.7, hence, unfortunately due to time constraint, this would be something for a future study to consider.

(iv) It still needs to be shown that removing the unobservable parts does not change the input-output behavior. However, in the case this could be proven and Conjecture 5.4.9 turns out to be true, after removing the unobservable parts the result would be a minimal fixed switched realization of Σ_Γ. □

Chapter 6

Conclusion

In this paper, the goal was to generalize results from minimal realization theory for linear systems to fixed switched linear systems, under the assumption that the minimal realizations would be based on external descriptions given by the corresponding system types only. In order to accomplish this task, first both system types were mathematically introduced, together with some of their important characteristics, among which the dimension and input-output equivalency. Afterwards, the concepts of observability and controllability were introduced for both systems types. These would then become useful when next minimal realization theory for linear systems would be considered. Furthermore, while considering minimal realization theory for linear systems, a distinction was made between the so-called function case and the so-called system case. In the end, however, only the system case was of importance, since only the system case would be considered for fixed switched linear systems. The paper was concluded with some conjectures regarding generalization for which, unfortunately, no definite answers could be given, resulting in many questions still left open, to be solved in a future study.

Appendix A

Important Concepts (Alphabetically Ordered)

In this appendix, some important concepts will be explained that are used throughout this paper. Most importantly, this appendix will contain both formal and informal explanations.

A.1 Almost Everywhere

To explain the concept of almost everywhere, first sets of zero measure need to be introduced, see also the book by Polderman et al. [7, Definition 2.3.6].

Definition A.1.1 (Set of Zero Measure)**.** A set $N \subset \mathbb{R}$ is said to have measure zero if and only if its elements can be covered by a countable union of intervals of arbitrary small total length. □

Using sets of zero measure, the following definition explains the concept of almost everywhere.

Definition A.1.2 (Almost Everywhere)**.** Let $f : \mathbb{R} \to \mathbb{R}^n$ be a function. f satisfies property A *almost everywhere* if and only if there exists a set of zero measure N such that $f(t)$ satisfies property A for all $t \in \mathbb{R} - N$. □

A.2 Functions

Consider the following definition regarding continuous piecewise differentiable functions.

Definition A.2.1 (Continuous Piecewise Differentiable Functions)**.** A function $f : \mathbb{R} \to \mathbb{R}^n$ is said to be *Continuous Piecewise Differentiable* if and only if f is both continuous and piecewise differentiable, i.e. if and only if f is continuous and

$$f(t) = \begin{cases} f_0(t) & \text{if } t \in (-\infty, t_1) \\ f_1(t) & \text{if } t \in [t_1, t_2) \\ \vdots & \\ f_k(t) & \text{if } t \in [t_k, \infty) \end{cases}$$

where

- $k \in \mathbb{N}_0^\infty$ (finite or infinite).
- $f_i : \mathbb{R} \to \mathbb{R}^n$ is differentiable for all $i \in (0, 1, 2, \ldots, k)$.
- $t_1 < t_2 < \cdots < t_k$.
- any finite interval $I \subset \mathbb{R}$ can contain only finitely many $t_q, q \in \{1, 2, \ldots, k\}$. (no Zeno behavior). □

Remark A.2.2. $CPD(\mathbb{R}, \mathbb{R}^n)$ is the set of all continuous piecewise differentiable functions $f : \mathbb{R} \to \mathbb{R}^n$. □

Consider next the following definition regarding piecewise continuous functions.

Definition A.2.3 (Piecewise Continuous Functions)**.** A function $f : \mathbb{R} \to \mathbb{R}^n$ is said to be *Piecewise Continuous* if and only if

$$f(t) = \begin{cases} f_0(t) & \text{if } t \in (-\infty, t_1) \\ f_1(t) & \text{if } t \in [t_1, t_2) \\ \vdots & \\ f_k(t) & \text{if } t \in [t_k, \infty) \end{cases}$$

where

- $k \in \mathbb{N}_0^\infty$ (finite or infinite).
- $f_i : \mathbb{R} \to \mathbb{R}^n$ is continuous for all $i \in (0, 1, 2, \ldots, k)$.
- $t_1 < t_2 < \cdots < t_k$.
- any finite interval $I \subset \mathbb{R}$ can contain only finitely many $t_q, q \in \{1, 2, \ldots, k\}$. (no Zeno behavior). □

Remark A.2.4. $PC(\mathbb{R}, \mathbb{R}^n)$ is the set of all piecewise continuous functions $f : \mathbb{R} \to \mathbb{R}^n$. □

A.3 Index Sets and Indexed Families

Consider the following formal and informal definitions regarding index sets and indexed families.

Definition A.3.1 (Index Set)**.** An *Index Set* is a set whose elements can be used to label elements of another set, i.e. is a countable (infinite) set. □

Definition A.3.2 (Indexed Family)**.** An *Indexed Family* is a set whose elements are labelled by elements of a given index set. In other words, there exists a function that associated to each element in the indexed family, an element in the index set. □

In this paper, an indexed family will have the following notation: $P = \{s_q \in S \mid q \in \mathcal{Q}\}$. Here S can be any set, but $\mathcal{Q}$ has to be an index set. Furthermore, not every element in S needs to be an element in P, but every element in $\mathcal{Q}$ needs to be associated to an element in S.

The idea is that s_q is an element of P if and only if s_q is an element in S and s_q is associated to the element q in $\mathcal{Q}$. Furthermore, $a_i \in P$ if and only if a_i is equal to the element in S that is associated to the element i in $\mathcal{Q}$, i.e. $a_i \in P$ if and only if $a_i = s_i$.

In order to make notation easier, consider the following. Let $\mathcal{Q} = \{1, 2, 3\}$ and let $S = \{a, b, c, d\}$. Assume that $P = \{s_q \in S \mid q \in \mathcal{Q}\}$ is the indexed family for which $s_1 = b$, $s_2 = a$ and $s_3 = c$. A short hand notation for P is given by $\{b, a, c\}^{\mathcal{Q}}$. Notice that this notation uses the order of elements, for both $\{b, a, c\}$ and $\mathcal{Q}$, to construct the indexed family.

A.4 Laplace Transform

Consider the following definition regarding the Laplace transform.

Definition A.4.1 (Laplace Transform)**.** The *Laplace Transform* for a function $f : \mathbb{R} \to \mathbb{R}^n$ is defined as

$$\hat{f}(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt \qquad \square$$

Remark A.4.2. Some remarks about the Laplace transform

(i) The Laplace domain of a function f is the domain of the Laplace transform of the function f i.e. is the domain of the function $\hat{f}(s)$.

(ii) According to the book by Antsaklis et al. [1, Table 3.1], the Laplace transform of $\dot{x}(t)$ is given by $s\hat{x}(s) + x(0)$. $\square$

A.5 Matrix Exponential

In this paragraph a few notes about matrix exponential will be given.

Definition A.5.1 (Matrix Exponential)**.** Given a square matrix $X \in \mathbb{R}^{n\times n}$, the matrix exponential of X is given by

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \qquad \square$$

Theorem A.5.2. *Let $S \in \mathbb{R}^{n\times n}$ be a non-singular matrix and $M \in \mathbb{R}^{n\times n}$ be a square matrix. The following holds true.*

$$e^{S^{-1}MS} = S^{-1}e^M S$$

Proof. This is a straightforward application of Definition A.5.1. △

A.6 Operations on Vector Spaces

In this paragraph, three different operations on vector spaces will be explained.

Definition A.6.1 (MV)**.** Let $M \in \mathbb{R}^{m\times n}$ be a matrix and $V \subseteq \mathbb{R}^n$ be a vector space. $MV := \{Mv \in \mathbb{R}^m \mid v \in V\}$. □

Definition A.6.2 ($M^{-1}V$)**.** Let $M \in \mathbb{R}^{m\times n}$ be a matrix and $V \subseteq \mathbb{R}^m$ be a vector space. $M^{-1}V := \{w \in \mathbb{R}^n \mid Mw \in V\}$. □

Definition A.6.3 ($V \cap W$)**.** Let $V, W \subseteq \mathbb{R}^n$ be vector spaces. $V \cap W := \{x \in \mathbb{R}^n \mid x \in V \text{ and } x \in W\}$. □

A.7 The Zero Dimensional Real Vector Space

Consider the following definitions regarding the zero dimensional real vector space.

Definition A.7.1 (The Zero Dimensional Real Vector Space)**.** The *Zero Dimensional Real Vector Space* is given by $\mathbb{R}^0 = \{0_0\}$. □

Definition A.7.2 (Valid Operation on the Zero Dimensional Real Vector Space)**.** Below some valid operations on the zero dimensional real vector space.

1. $A : \mathbb{R}^0 \to \mathbb{R}^n$ will be the map given by $A(0_0) = 0_n$.
2. $B : \mathbb{R}^n \to \mathbb{R}^0$ will be the map given by $B(x) = 0_0$ for all $x \in \mathbb{R}^n$.
3. $x : \mathbb{R} \to \mathbb{R}^0$ is a function that is given by $x(t) = 0_0$ for all $t \in \mathbb{R}$. □

Definition A.7.3 (Identity Matrix)**.** For zero dimensional real vector spaces, the identity matrix I_0 is given by $I_0(0_0) = 0_0$. □

A.8 Zeno Behavior

In this paragraph all information is obtained from the book by Liberzon [3, Chapter 1.2.2]. Hence, for more information, the reader is directed to this book.

Consider the following situation. A ball is bouncing on the floor. Each time it reaches the floor, it will bounce upwards and in the process will lose some of its energy. This implies that the time between each bounce will shorten and, eventually, the ball will come to a halt, at the accumulation point of all time instances the ball bounces on the floor. In the physical world this phenomena is easy to understand, however in a mathematical model the opposite might be true. In certain mathematical models, as the time between each bounce shortens, the number of bounces, within a finite interval, would increase and eventually would reach infinite. This phenomena is called Zeno behavior. The reason such behavior should be avoided, is because the mathematical model would need infinite computations just to reach the accumulation point and hence cannot "explain" what happens after the accumulation point.

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