# An application of the Knauf criteria to determine the existence of abrupt bifurcations to chaos in planar scattering systems 

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Student: A. Zanuttini
First supervisor: dr. M. Seri
Second assessor: dr. H. Jardon Kojakhmetov

## Abstract

This paper studies the occurrence of chaos in planar scattering systems and the means by which this type of behaviour ensues. Of particular interest are some new analytical results which will here be used to determine the existence of an abrupt bifurcation to chaos. This type of bifurcation occurs when a nonattracting chaotic set is created from nothing, upon lowering the energy parameter below a certain threshold.

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## 1 Introduction

Classical scattering describes the physical phenomena in which particles are forced to deviate from a given trajectory by the non-uniformity of the space they are moving in. Particularly, scattering is a study of asymptotic behaviour of such particles, by means of comparing how the motion of a free particle deviates upon interacting with a localised potential, meaning that either the potential decays fast towards infinity, or is zero outside of some compact region. Under the assumption that the energy is conserved, these phenomena can be modelled via the Newton equation

$$
\begin{equation*}
\ddot{\mathbf{x}}=-\nabla V(\mathbf{x}) \tag{1.1}
\end{equation*}
$$

which, rewritten as a first order system reads:

$$
\begin{align*}
\dot{\mathbf{v}} & =-\nabla V(\mathbf{x}) \\
\dot{\mathbf{x}} & =\mathbf{v} \tag{1.2}
\end{align*}
$$

where $V(\mathbf{x})$ is some arbitrary potential and $\mathbf{x}, \mathbf{v} \in \mathbb{R}^{d}$ for some $d \in \mathbb{N}$. It should be noted that $\mathbf{x}$ denotes position and $\mathbf{v}$ denotes momentum. The potential typically will have a number of hills, the collection of which forms what we call the scattering region.

Systems like (1.1) are called Hamiltonian systems. That is, there exists a function:

$$
\begin{equation*}
H: \mathbb{R}^{2 d} \rightarrow \mathbb{R}, \quad H(\mathbf{x}, \mathbf{v})=\frac{1}{2} \mathbf{v}^{2}+V(\mathbf{x}) \tag{1.3}
\end{equation*}
$$

called the Hamiltonian, which encodes the equations of motion:

$$
\begin{aligned}
& \frac{d \mathbf{v}}{d t}=-\frac{\partial H(\mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} \\
& \frac{d \mathbf{x}}{d t}=\frac{\partial H(\mathbf{x}, \mathbf{v})}{\partial \mathbf{v}}
\end{aligned}
$$

A number of famous results have been found which apply to the generic class of Hamiltonian systems, the most notable of which is that [5]:

$$
\frac{d}{d t} H(\mathbf{x}(t), \mathbf{v}(t))=\mathbf{v} \dot{\mathbf{v}}+\dot{\mathbf{x}} \nabla V(\mathbf{x})=\dot{\mathbf{x}} \dot{\mathbf{v}}-\dot{\mathbf{x}} \dot{\mathbf{v}}=0
$$

meaning that $H$ is conserved. It is often the case that the energy of a system is the Hamiltonian, hence why the assumption that energy is conserved is key in modelling scattering phenomena via (1.1). Another important related theorem is the Liouville theorem, which states that Hamiltonian systems preserve phase-space volumes [5]. The phase-space of a given dynamical system is the collection of all possible states the system may exhibit.

In this article we shall see how chaos ensues in a number of scattering problems upon the variation of a parameter. The reader may be familiar with the definition of chaos given by Devaney [5], which states that:

Definition 1.1. A map $F: X \rightarrow X$, where $(X, d)$ is some metric space, is chaotic if:

1. the periodic points of $F$ are dense in $X$, i.e. for any periodic point $x_{0}$ and for all $B_{\epsilon}\left(x_{0}\right)$ there exists a point $x_{1} \in B_{\epsilon}\left(x_{0}\right)$ such that $x_{1}$ is also a periodic point;
2. $F$ is topologically transitive, i.e. for any two open subsets $U, V \subset X$ there exists an $n>0$ such that $F^{n}(U) \cap V \neq \emptyset$;
3. $F$ has sensitive dependence on initial conditions, i.e. there exists some $\beta>0$ such that for all $x_{0}$ and $B_{\epsilon}\left(x_{0}\right)$ there exists a $y_{0} \in B_{\epsilon}\left(x_{0}\right)$ and $n>0$ such that $d\left(F^{n}\left(x_{0}\right), F^{n}\left(y_{0}\right)\right)>\beta$.

While this definition captures the essence of chaos in systems which are measure preserving and have a compact phase space, for scattering problems this does not apply: while it is true that phasespace volumes are preserved (by the Liouville theorem) the phase space is not compact (indeed it is typically $\mathbb{R}^{d}$ ). We thus ignore topological transitivity and denseness of the periodic points and focus solely on sensitive dependence on initial conditions. The resulting definition can be found in Ott [19]:

Definition 1.2. A dynamical system $\dot{\mathbf{x}}=F(\mathbf{x})$ is chaotic if, given an initial error $\Delta \mathbf{x}_{\mathbf{0}}$, the long-term error between corresponding solutions obeys:

$$
|\Delta \mathbf{x}(t)| \approx e^{\Lambda t}\left|\Delta \mathbf{x}_{0}\right|
$$

where $\Lambda$ is a diagonal matrix with nonzero entries $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{d}$. We call $\lambda_{i}$ a Lyapunov exponent.
In section 2 we shall explore the four-hill scattering problem. The potential used in this problem has $\frac{\pi}{2}$ rotational symmetry. We shall see that by varying a particular parameter, an abrupt bifurcation $[1]$ to chaos will occur. This will consist in the creation of a particular chaotic set within the phase space, resulting from the intersection of the stable and unstable manifolds of orbits which never leave the scattering region. A topological manifold is defined [20] as:

Definition 1.3. A topological space $M$ is a topological manifold of dimension $n$, or topological $n$ manifold, if it has the following properties:

## 1. $M$ is a Hausdorff space;

2. $M$ is second countable (i.e. there exists a countable collection $\mathcal{U}$ of open subsets of $M$ such that any open set in $M$ can be written as a union of elements from $\mathcal{U}$ );
3. $M$ is locally euclidian of dimension $n$, that is, for any $p \in M$ there exists an open subset $U \subset M$ with $p \in U$, and open subset $V \subset \mathbb{R}^{n}$ and a homeomorphism $\phi: U \rightarrow V$.

Later we will also talk about smooth manifolds. These are topological manifolds with an extra structure that allows us to talk about derivatives. In particular at each $x \in M$ we can attach a vector space $T_{x} M \cong \mathbb{R}^{n}$ consisting of velocity vectors at the point $x$ or, equivalently, of directional derivatives at $x$.

The stable set of a given fixed point $p$ of a map $f: D \rightarrow D^{\prime}$, is [5]:

$$
E^{s}(p)=\left\{x \in D \mid \lim _{n \rightarrow \infty} f^{n}(x)=p\right\}
$$

and the unstable set of such fixed point is [5]:

$$
E^{u}(p)=\left\{x \in D \mid \lim _{n \rightarrow-\infty} f^{n}(x)=p\right\}
$$

For a continuous dynamical system $\dot{\mathbf{x}}=f(\mathbf{x}, t)$, the above sets translate to:

$$
\begin{aligned}
& E^{s}(p)=\left\{\mathbf{x} \in D \mid \lim _{t \rightarrow \infty} \phi^{t}(\mathbf{x})=p\right\} \\
& E^{u}(p)=\left\{\mathbf{x} \in D \mid \lim _{t \rightarrow-\infty} \phi^{t}(\mathbf{x})=p\right\}
\end{aligned}
$$

where $\phi^{t}(\mathbf{x})$ is the flow of such system $\left(\phi^{t}\left(x_{0}\right)=\int_{0}^{t} \dot{\mathbf{x}}\left(x_{0}, t\right) d t\right)$. The stable (resp. unstable) sets of a given orbit can be defined as the union of the stable (resp. unstable) sets of all the points which the given orbit visits. It turns out that often these sets are manifolds, in such case they will be called stable (resp. unstable) manifolds 10 . In the case they are not, it may be possible that many useful local properties of Euclidian spaces do not apply.

As we shall see, the chaotic set is a strange saddle. This is a type of hyperbolic, nonattracting invariant chaotic set, for which almost all points are saddle points. The following definition [3] should aid in understanding what is meant by hyperbolic set:

Definition 1.4. Let $M$ be a real, smooth manifold equipped with a positive definite inner product on the tangent spaces at each point in $M$. Let $U \subset M$ be a non-empty open subset and $h: U \rightarrow f(U) \subset M$ be a $C^{1}$ diffeomorphism. A compact, $h$-invariant subset $\Lambda \subset U$ is called hyperbolic if there exist $\lambda \in(0,1), C>0$ and families of subspaces $E^{s}(x), E^{u}(x) \subset T_{x} M, x \in \Lambda$ such that, for all $x \in \Lambda$ :

- $T_{x} M=E^{s}(x) \bigoplus E^{u}(x)$ (where $\bigoplus$ denotes the direct sum),
- $\frac{d}{d x} h\left(v^{s}, t\right) \leq C \lambda^{t}\left\|v^{s}\right\|$ for all $v^{s} \in E^{s}(x)$ and $t \geq 0$,
- $\frac{d}{d x} h\left(v^{u},-t\right) \leq C \lambda^{t}\left\|v^{u}\right\|$ for all $v^{u} \in E^{u}(x)$ and $t \geq 0$,
- $\frac{d}{d x} h\left(E^{s}(x)\right)=E^{s}(f(x))$ and $\frac{d}{d x} h\left(E^{u}(x)\right)=E^{u}(h(x))$.

The subspaces $E^{s}(x)$ and $E^{u}(x)$ are, respectively, the stable and unstable spaces at $x$. By $h$-invariant it is instead meant that

$$
h(\Lambda) \subseteq \Lambda
$$

Another useful hyperbolic set is that created by iterating the horseshoe map. This is a particularly useful map in describing how chaos occurs in dynamical systems. The map fundamentally takes a square, stretches it vertically and folds it over itself once at each iteration. See figure (2.3) for an illustration. It is also worth noting that an there exists an invariant set of the horseshoe map, which is typically homeomorphic to a Cantor set (which we shall discuss shortly).

It should be added that typically strange saddles are homeomorphic to (multi-dimensional) Cantor sets. A Cantor set $\mathcal{C}$, built on the interval $[0,1]$, would be:

$$
\mathcal{C}=[0,1] \backslash \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^{n}-1}\left(\frac{3 k+1}{3^{n+1}}, \frac{3 k+2}{3^{n+1}}\right)
$$

which essentially consists in iteratively removing the middle third of every segment starting from the one segment $[0,1]$ up to the infinite number of segments given above (an illustration can be found for this in figure 1.1). Other variations are also possible (for instance the removal of the second and fourth fifth) and that Cantor sets may exists in multiple dimensions. For instance, the set

$$
\begin{equation*}
\mathcal{C}^{2}:=\mathcal{C} \times \mathcal{C} \tag{1.4}
\end{equation*}
$$

where $\mathcal{C}$ is defined as before and $\times$ denotes the Cartesian product, is a Cantor set in two dimensions (see figure 1.2).


Figure 1.1: A one-dimensional Cantor set obtained by iterating (1.4) up to $n=10$.

Once we have obtained the stable and unstable manifolds of the orbits which never leave the scattering region, we shall investigate a property of these, known as the capacity dimension. We shall define this in due time, for now it should suffice to know that this is a measure of the roughness of a set, in particular in the context of subsets of a phase space, this gives us an idea of how sparse such a set is. For instance, in the case of the Cantor set in figure 1.1, the capacity dimension tells us how far the set is from being a line (which would have capacity dimension one) and how far it is from being a point (capacity dimension zero). Closely related to the concept of capacity dimension is the concept of information dimension. Before we define this, we should talk about the natural measure of a set. This is defined as:

Definition 1.5. If we cover a subset of a phase space by a disjoint union of (hyper) cubes $C_{i}$ of side $\epsilon$ and define $T\left(x_{0}, t, \epsilon\right)$ to be the amount of time spent by a trajectory with initial condition $x_{0}$ in $C_{i}$ then:

$$
\begin{equation*}
\mu\left(x_{0}, C_{i}\right)=\lim _{t \rightarrow \infty} \frac{T\left(x_{0}, t, \epsilon_{i}\right)}{t} \tag{1.5}
\end{equation*}
$$

is called the natural measure of such a subset [8].
With this at hand, we may define the information dimension.
Definition 1.6. The information dimension of a given subset of a phase space 8 is defined as

$$
\begin{equation*}
d_{I}=\lim _{\epsilon \rightarrow 0} \frac{I(\epsilon)}{\log (1 / \epsilon)} \tag{1.6}
\end{equation*}
$$

where $I(\epsilon)=\sum_{i=1}^{N(\epsilon)} P_{i} \log \left(\frac{1}{P_{i}}\right)$ with $P_{i}=\mu\left(x_{0}, C_{i}\right)$ and $N(\epsilon)$ is the number of cubes of side $\epsilon$ required to cover the subset in question.

We will calculate this property for the non-attracting chaotic set. The property itself is, intuitively, a quantity describing how much the Shannon entropy grows upon refining the discretization of the set it is calculated for. The Shannon entropy is defined as 21]:

Definition 1.7. Given a random variable $X$ (i.e. a variable whose value depends on the outcome of a random phenomenon) with probability measure $p_{X}(x)$, the Shannon information is

$$
I_{X}(x)=\log \left(\frac{1}{p_{X}(x)}\right)
$$

while the Shannon entropy of $X$ is

$$
s(X)=E\left[I_{X}(X)\right]
$$

Intuitively, the Shannon information tells us how surprising the outcome $X$ is. The Shannon entropy would then be a value telling us how frequent surprising outcomes of $X$ are.

Finally, at the end of the next section, we shall review a numerical proof of the existence of the aforementioned abrupt bifurcation given by Bleher et al. 1] and use some theorems developed by Knauf [13, 14] to instead provide an analytical proof.

In section 3 we shall study a number of other scattering problems. Of particular interest will be problems which do not have rotational symmetries. We shall begin by briefly covering a scattering problem with one hill, with the aim of deriving some useful results which will be applicable to nonsymmetric potentials. We shall then see a scattering problem associated with a potential with two hills, in this case the theorems found at the end of section 2 will allow us to see that no chaos exists for such problems. We will then briefly show the existence of an abrupt bifurcation in scattering problems with three hills of the same height. Finally, we shall cover scattering systems associated with potentials which have hills of different heights, in this case we shall see that routes to chaos other than the abrupt bifurcations are possible. In this last scenario we shall see that Knauf's theorems prove useful to determine the existence of chaos under a particular set of sufficient conditions, but the onset of chaos will not require all these conditions.

## 2 The four hill scattering problem



Figure 2.1: Plot of the four-hill potential.


Figure 2.2: Level curves of the four potential hills with the trajectory of a particle (blue) which enters from the left and leaves to the right.

In this section we shall study the four equal potential hills scattering problem, firstly using numerical tools and then analytically. The system we shall study is a Hamiltonian system of the form (1.1) with:

$$
\begin{equation*}
V(x, y)=x^{2} y^{2} e^{-\left(x^{2}+y^{2}\right)} \tag{2.1}
\end{equation*}
$$

We integrated the system numerically using the Leapfrog algorithm, a second order method which has the added benefit of guaranteeing the conservation of energy. See A. 1 for the program used.

Intuitively, the problem consists of sending a particle of unit mass on a collision course with four hills of equal height whose maximum is located at $(-1,-1),(-1,1),(1,-1)$, and $(1,1)$ under the assumption that the energy is conserved (i.e. the collisions will be perfectly elastic), figure 2.16 demonstrates this behavior. This analogy is not entirely correct, since the potential defines a nontrivial force along all the trajectory, but this force is also comparatively negligible far away from the hills and for sufficiently high energies, thus the analogy is representative.

From (1.3) we see that $|\mathbf{v}|=(2(E-V(\mathbf{x})))^{\frac{1}{2}}$ and, defining the variable $\theta$ to be the angle between a given trajectory's velocity $\mathbf{v}$ and the $x$-axis, we can reduce the system's phase space to be $(x, y, \theta)$ and treat $E$ as a parameter.

It has been shown [1] that, letting $E_{m}=\max V(\mathbf{x})=e^{-2}$, for $\frac{E}{E_{m}}<1$, the system exhibits chaotic behavior. In the following subsections we shall investigate this.

### 2.1 Dynamics

We now consider trajectories which begin along a straight line perpendicular to the $x$-axis at the point $x_{0}=-4$ and move towards the potential hills parallel to the $x$-axis (thus having $\theta_{0}=\pi$ ). A script A.2 aids us in this process.

Our first observations concern the effects of varying $b:=y_{0}$ and $E$. As can be seen in figures 2.3 and 2.4 , as the $E$ decreases past $E_{m}$, the system begins to exhibit sensitive dependence upon initial conditions, a hallmark of chaotic behavior.

Defining $\phi$ to be the angle between the velocity and the $x$-axis, once a particular orbit leaves the area enclosed by a circle of radius 3 around the origin, we examine the relationship between changes in the impact parameter $b$ and changes in the scattering angle $\phi$. We find that the relationship seems to be singular on a certain set of $b$-values (see fig. 2.5). This is particularly evident for values of $E / E_{m} \leq 0.26$.

By considering the relationship between the time it takes for a trajectory to leave the scattering region and the impact parameter (see A.3), we see that the set of singular values of $\phi(b)$ seems to coincide with the set of values for which the delay time as a function of $b$ is singular (see fig. 2.6).

This would seem to indicate that the values of $b$ for which $\phi(b)$ is singular are those for which orbits remain in the scattering region for a longer period of time.

We are now interested in discussing a measure of the unpredictability of the set of singular values of $\phi(b)$, this is known as the capacity of such set.

Definition 2.1. Let $S$ be a set of points within a D-dimensional Euclidian space, consisting of real numbers. Consider D-dimensional hypercubes of side $\epsilon$ covering the set S and define $N(\epsilon)$ to be the minimum number of cubes needed to cover the set S . Then the capacity dimension is defined as [17]

$$
d_{C}=\lim _{\epsilon \rightarrow 0} \frac{\log (N(\epsilon))}{\log (1 / \epsilon)}
$$

Bleher et al. described an efficient way of calculating the capacity dimension for Hamiltonian systems where the orbits eventually leave the system [2]. This method is based on the following theorem [16:

Theorem 1. The uncertain fraction $f$ of a finite region of a $D$-dimensional phase space associated with initial condition error $\epsilon$ obeys

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \frac{\log (f(\epsilon))}{\log (\epsilon)}=\alpha \tag{2.2}
\end{equation*}
$$

if and only if the basin boundary has capacity dimension $d_{C}=D-\alpha$.
This method relates the final state of an orbit and its perturbed counterpart to the perturbation value, giving us a sense of the uncertainty of orbits within the set. The method, known as the final state method, is adapted to the present problem as follows: letting $S$ be the set of values of $b$ where $\phi(b)$ is singular, for a given $\epsilon$

1. choose randomly a large number $N$ of initial conditions within the set S ;
2. randomly pick a value $b_{i}+\epsilon$ or $b_{i}-\epsilon$ for all initial conditions $b_{i}$;
3. integrate both IVPs;
4. letting $I_{1}=(0, \pi]$ and $I_{2}=(\pi, 2 \pi]$, if $\phi\left(b_{i}\right)$ and $\phi\left(b_{i} \pm \epsilon\right)$ belong to the same interval for a fraction $\frac{k}{N}$ of the total number of pairs of chosen initial conditions and perturbed initial conditions, then we define $f(\epsilon)=\frac{k}{N}$;
5. the capacity dimension is then

$$
d_{C}=1-\lim _{\epsilon \rightarrow 0} \frac{\log (f(\epsilon))}{\log (\epsilon)}
$$

the script A.4. using this algorithm, gives a capacity dimension of $d_{C} \approx 0.67$ for $E / E_{m}=0.26$.


Figure 2.3: Three trajectories starting from very similar initial conditions for $E / E_{m}=1$.


Figure 2.4: Same trajectories as in figure 2.3, but now with $E / E_{m}=0.26$. It can be seen that after five collisions the trajectories have completely different behaviour.


Figure 2.5: Scattering angle $\phi$ as a function of the impact parameter $b$ for $E / E_{m}=0.26$.

Adding one degree of freedom to our parameter space, we can gain further insight into the fractal structure underlying the dynamics of this scattering problem: we define $\theta$ to be the angle between the $x$-axis and the initial velocity of a particle, we then let $y(0)=0$ and treat $x(0)$ as a parameter. We can then transform $\theta$ back into a function of $v_{x}(0)$ and $v_{y}(0)$, which will allow us to use the Leapfrog method developed in A.1 without changes. In our change of variables, it is important to notice that we must use $v_{x}(0)=\sqrt{2 E /\left(1+\tan ^{2}(\theta)\right)^{2}}$ and $v_{y}(0)=\tan (\theta) v_{x}(0)$ for $\theta \in[-\pi / 2, \pi / 2]$ and $v_{x}(0)=-\sqrt{2 E /\left(1+\tan ^{2}(\pi-\theta)\right)}, v_{y}(0)=$ $-v_{x}(0) \tan (\pi-\theta)$ for $|\theta| \in[\pi / 2, \pi]$ to take the tangent across all quadrants.

Picking initial conditions $\left(\theta, x_{0}\right) \in[-\pi, \pi] \times$ $[-3,3]$, we integrate the final state of trajectories starting from these. The function A.4 will be helpful henceforth. From figures 2.8 to 2.11 , it can be seen that as $E / E_{m}$ decreases past 1 and towards zero, a set of values of $\left(\theta, x_{0}\right)$ emerges and grows which seems to resemble the stretching and folding characteristic of the horseshoe map.

The fractal dimension of the region of singular values in figures 2.8 to 2.10 can again be calculated using the final state method, this time the method is adapted as follows:

1. choose at random a large number $N$ of initial conditions $\boldsymbol{\xi}_{\mathbf{0}}=\left(\theta, x_{0}\right)$;
2. choose at random a $\delta$ such that $\left|\delta \boldsymbol{\xi}_{\mathbf{0}}\right| \leq \epsilon$;
3. obtain, again at random, either $\boldsymbol{\xi}_{\mathbf{0}}+\delta \boldsymbol{\xi}_{\mathbf{0}}$ or $\boldsymbol{\xi}_{\mathbf{0}}-\delta \boldsymbol{\xi}_{\mathbf{0}}$;
4. integrate both IVPs;
5. define $f$ as before;
6. the capacity dimension is then

$$
d_{C}=2-\lim _{\epsilon \rightarrow 0} \frac{\log (f(\epsilon))}{\log (\epsilon)}
$$

the dimension of this set of singular values has been found to be [1] $d_{C} \approx 1.66$, within numerical error this is one more than the capacity of the fractal set of singular values of $\phi(b)$. Repeated calculations of this dimension seem to show that the capacity of this set scales well as $d_{C} \sim \ln \left(1 /\left(E_{m}-E\right)\right)$ for $E / E_{m}>1$.
Remark 2.1. It is important to note that the points of contact between the blue (red, respectively) oval and the adjacent blue (red, respectively) regions in figure 2.8 coincide with orbits which begin at one of the four hilltops $\left(x_{0}, y_{0}\right)=( \pm 1, \pm 1)$ 1]. These orbits are periodic for $E \leq E_{m}$ and never leave the scattering region. Since any minor perturbation to the initial state of these aforementioned orbits will leave the scattering region, they are also unstable. Since our potential (2.1) is symmetric, all asymptotically trapped orbits spiral towards one of the unstable periodic orbits joining the four hilltops, i.e. asymptotically trapped orbits move along the stable manifold of these periodic orbits [7].


Figure 2.8: Upwards (blue) and downwards (red) scattering orbits for $E / E_{m} \approx 1$.


Figure 2.10: Same plot as figure 2.7, but for $E / E_{m}=0.26$.


Figure 2.12: Stable invariant manifold of trapped orbits intersected with the twodimensional cross section $y_{0}=0$ for $E / E_{m}=$ 0.26 .


Figure 2.9: Same plot as figure 2.7, but now with $E / E_{m}=0.6$.


Figure 2.11: Same plot as figure 2.7, but for $E / E_{m}=0.03$


Figure 2.13: Unstable invariant manifold of trapped orbits intersected with the twodimensional cross section $y_{0}=0$ for $E / E_{m}=$ 0.26 .

The similar fractal dimension obtained for the set of singular values of $\phi(b)$ and the set of uncertain values we just investigated, together with the fact that the singular values of $\phi(b)$ coincide with orbits which have greater time delay, suggests that perhaps also this latter investigated set is subject to greater time delays. Indeed, by setting a minimum time delay of $t=35$ in A.5 we obtain this set isolated from its surroundings (see figure 2.12). From our previous remark, it follows that this set is a stable invariant manifold of trapped orbits intersected with the cross section $y_{0}=0$. To obtain the unstable manifold of the invariant set of trapped orbits, it is sufficient to repeat the process through which the stable manifold was obtained, but reversing the direction of time integration. This can be done by simply choosing a negative time-step in A.5. Since the system resulting from (2.1) is conservative and symmetric under time reversal, the unstable manifold is a mirror image of the stable manifold (see figure 2.13).

### 2.2 The strange saddle

In this subsection we shall discuss the chaotic set underlying the dynamics of the scattering problem resulting from (2.1) which results from the intersection of the stable and unstable manifolds. The set in question is a strange saddle, which is defined as an invariant, non-attracting set such that almost all its points are saddle points.

Far more accurate and efficient methods than the one we will be using A.6 have been developed to approximate strange saddles, particularly worthy of mention is that proposed by Nusse and Yorke [18. These methods are typically necessary, due to the fact that the existence of chaotic behaviour would imply exponential growth in the numerical error at each iteration, which causes significant imprecision in the


Figure 2.14: Intersection of the strange saddle with the cross section $y_{0}=0$. approximation of the stable and unstable invariant manifolds discussed in the previous section. Not using one of these methods would in turn cause great errors in the calculations of properties of the chaotic set. However, given that we shall refer to results derived elsewhere as far as properties of the particular set are concerned, a simple intersection of the sets of points found in figures 2.12 and 2.13 will be sufficient.

We will now review some results concerning the dimensions of this chaotic set. We begin by defining a generalization of the information dimension.

Definition 2.2. The order $q$ Renyi dimension $D_{q}$ of a subset of a phase space is defined as

$$
\begin{equation*}
D_{q}=\lim _{\epsilon \rightarrow 0} \frac{I_{q}(\epsilon)}{\log (1 / \epsilon)} \tag{2.3}
\end{equation*}
$$

where $I_{q}(\epsilon)=\frac{1}{q-1} \sum_{i=1}^{N(\epsilon)} P_{i}^{q}$ and $P_{i}=\mu\left(x_{0}, C_{i}\right)$ as in definition 2.3.
Using a box-counting algorithm (typically used in the estimation of the capacity dimension), it has been found [1] that the Renyi dimensions of the strange saddle in figure 2.14 are $D_{q} \approx 1.2$ (with error within $(-0.1,0.1))$ for $q \in(0,2)$.

Another means of calculating the information dimension can be derived from the following theorem.
Theorem 2. Let $S$ be a strange saddle for a map $f$ and define $B$ to be a compact set containing $S$. Choose $N_{0}$ uniformly distributed points in $B$ and iterate the map $f$ a number of times. Upon iteration, all points except those on $S$ and on its stable manifold will leave $B$. Hence define

$$
\frac{1}{\tau}=\lim _{t \rightarrow \infty} \lim _{N_{0} \rightarrow \infty} \frac{1}{t} \log \left(N_{0} / N_{1}\right)
$$

to be the mean time required for the chosen points to leave B. Let $\chi_{i}$ be the ith Lyapunov exponent of $f$, then the information dimension of the stable manifold of $S$ is

$$
\begin{equation*}
d_{s}=K-\frac{1}{\tau \chi_{1}} \tag{2.4}
\end{equation*}
$$

where $K$ is the dimensionality of the map $f$ [11, 1].
Using the fact that the stable and unstable manifolds are mirror images of each other and using the fact that there is one expanding and one contracting direction of the map, and hence the second Lyapunov exponent is $\chi_{2}=-\chi_{1}$, we can derive from 2.7 that the information dimension of the unstable manifold $d_{u}$ is

$$
d_{u}=d_{s}=3-\frac{1}{\tau \chi_{1}}
$$

where $K=3$ since the phase space for our system is $(x, y, \theta)$ and is hence three-dimensional. It has been found 1 that for $E / E_{m}=0.26$ the first Lyapunov exponent for the system resulting from (2.1) is $\chi_{1} \approx 0.3208$ and $\tau \approx 9.513 \pm 0.005$, this would give $d_{s} \approx 2.67$, which is one less than the value we found for the fractal basin boundary (fig.2.12) and $D_{q} \approx 2 d_{s}-4(q \in(0,2))$, as expected.

### 2.3 Numerical proof of abrupt bifurcation

In this section we shall show that the value below which chaos occurs is indeed $E=E_{m}$ and that the way it develops is abrupt, in that fully developed chaotic scattering appears as soon as $E<E_{m}$. By fully developed chaotic scattering, we here mean that a hyperbolic nonattracting invariant set (such as the one studied in the previous section) exists.

We first show that for $E>E_{m}$ the scattering is not chaotic. Assume there exist periodic orbits for energies $E>E_{m}$. We notice that due to the symmetry of the four-hill even potential (2.1), the minimum deflection from a hill must be $\pi / 2$ for there to be any periodic behaviour. Thus the first periodic orbit will have to bounce between the four hills either clockwise or counterclockwise without returning to a hill twice before having visited all the others. We thus construct a map with the aim of determining the existence of such an orbit as follows:

1. let $\xi:=-x(t=0)$ and $\theta=\arctan \left(v_{y}(t=0) / v_{x}(t=0)\right)$;
2. define $\xi^{\prime}=y(t=\tau)$ and $\theta^{\prime}=\arctan \left(v_{y}(t=\tau) /-v_{x}(t=\tau)\right)$, where $\tau$ is the time at which an orbit intersects the positive $y$-axis;
3. then $M(\xi, \theta)=\left(\xi^{\prime}, \theta^{\prime}\right)$.

We note that for a periodic orbit which moves (counter-)clockwise between hills, we should have that $M^{4}(\xi, \theta)=M^{-4}(\xi, \theta)=(\xi, \theta)$, where we denote $M^{n}(\xi, \theta)=M(M(M(\ldots M(\xi, \theta) \ldots)))$ to be the nth iterate of $M$.

Considering $M$ on the domain $D_{M}:=(0,1.1) \times(0, \pi)$, where the upper $\xi$ limitation is due to the fact that $M^{-1}$ is not defined for values of $\xi$ above that limit, iterating the whole domain once and subsequently iterating $D_{M} \cap M^{j}\left(D_{M}\right)$, for $E=E_{m}$ it has been shown [1] that $D_{M} \cap M^{3}\left(D_{M}\right)=\varnothing$. This would show that for $E>E_{m}$ there are no periodic orbits, given that if an orbit is periodic it must return to $D_{M}$ indefinitely. It is the case that under one iteration of $M, D_{M} \cap M\left(D_{M}\right)$ is not empty for $E>E_{m}$, however the order of this intersection shrinks as $E$ increases, rapidly becoming empty, thus showing that for any $E>E_{m}$ there are no periodic orbits and hence there is no chaos.

By contrast we shall now see that for $E<E_{m}$ chaotic behavior appears abruptly. When $E<E_{m}$, the (conserved) energy is now insufficient in some cases for a particle to go over the hilltop of a given potential hill. Thus, orbits which join the four hilltops at $(x, y)=( \pm 1, \pm 1)$ and are periodic and unstable, come into existence when $E<E_{m}$. The placement of these periodic orbits can be derived directly from $V(x, y)$, since

$$
\frac{\partial V( \pm 1, y)}{\partial x}=-2( \pm 1)\left(( \pm 1)^{2}-1\right) y^{2} e^{-( \pm 1)^{2}-y^{2}}=-2( \pm 1)(1-1) y^{2} e^{-1-y^{2}}=0
$$

and likewise for $\frac{\partial V(x, \pm 1)}{\partial y}$ (see figure 2.15). It is a known result [4] that a sufficient condition for the stable and unstable manifolds of any two non-parallel periodic orbits which join the hilltops to have a heteroclinic intersection, is the existence of an orbit which enters the scattering region by crossing one of these periodic orbits and leaves by crossing the other. Indeed, many of these "crossing orbits" exist already at $E=E_{m}$, for instance that in figure 2.16 which has initial conditions $\left(x_{0}, y_{0}, \theta_{0}\right)=(-4,-0.5,0)$. By symmetry of the potential, it must then hold that all the periodic orbits lying along the lines shown in figure 2.15 must have a heteroclinic intersection of their stable and unstable manifolds. This indicates the existence of a chaotic set for $E<E_{m}$.

Finally, we show that the bifurcation is abrupt, in the sense described at the beginning of this section. Labelling the four potential hills counterclockwise starting from the upper-left quadrant as $" 0 ", " 1 ", " 2 "$ and " 3 ", we can associate each orbit with a sequence of these numbers that represents the order in which the hills are visited (the validity of this symbolic dynamics will be proven in the next section). Clearly only sequences which don't repeat a number twice are possible. We call these sequences admissible. It has been found [1 that the number of distinct periodic sequences of period $l$ increases exponentially as $N_{l} \backsim 3^{l}$. These orbits are all points on the strange saddle studied in the previous section, since they form an invariant set. Thus there is fully developed chaos as soon as periodic orbits come into existence.


Figure 2.15: Lines along which the firstappearing periodic orbits lie.


Figure 2.16: A crossing orbit with $E / E_{m}=1$.

### 2.4 Analytical proof of abrupt bifurcation

In this subsection we shall use tools developed by Knauf et al. [13, 14] to provide an analytical proof of the numerical results found by Bleher et al. which we described in the previous subsection. Before we begin our discussion however, we need to introduce some key concepts. We begin by defining a number of sets. These sets are defined for an arbitrary Hamiltonian system $H: \mathbb{R}^{d} \rightarrow \mathbb{R}$ of the form $H=\frac{1}{2}|v|^{2}+V(x)$.

## Definition 2.3.

```
\(\Sigma_{E}:=\left\{(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^{d} \times \mathbb{R}^{d} \mid H(\mathbf{x}, \mathbf{v})=E\right\}=H^{-1}(E)\)
    \(P:=\left\{(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^{d} \times \mathbb{R}^{d} \mid H(\mathbf{x})>0\right\}\), (the positive energy part of the phase space)
\(b^{ \pm}:=\left\{\left(\mathbf{x}_{0}, \mathbf{v}_{\mathbf{0}}\right) \in P \mid \mathbf{x}\left( \pm \mathbb{R}^{+}, \mathbf{x}_{0}, \mathbf{v}_{0}\right)\right.\) is bounded \(\}\),
\(b_{E}^{ \pm}:=b^{ \pm} \cap \Sigma_{E}\)
    \(b:=b^{+} \cap b^{-}\)(the bound states),
\(s^{ \pm}:=P \backslash b^{ \pm}\),
    \(s:=s^{+} \cap s^{-}\)(the scattering states),
    \(t:=P \backslash(b \cup s)\) (the trapped states),
\(\mathcal{T E}:=\left\{E \in \mathbb{R}^{+} \mid t_{E} \neq \emptyset\right\}\) (the set of trapping energies)
\(\mathcal{N T}:=\mathbb{R}^{+} \backslash \mathcal{T E}\) (the set of non-trapping energies).
\(b_{E}:=b \cap \Sigma_{E}\)
\(s_{E}^{ \pm}=s^{ \pm} \cap \Sigma_{E}\)
\(s_{E}:=s \cap \Sigma_{E}\)
\(t_{E}:=t \cap \Sigma_{E}\)
```

Next, we define a region of the potential (hyper-)surface which will be central to the remainder of this proof.

Definition 2.4. For $E>0$, we call the set

$$
\begin{equation*}
\mathcal{R}_{E}:=\left\{\mathbf{x} \in \mathbb{R}^{d} \mid V(\mathbf{x}) \leq E\right\} \tag{2.5}
\end{equation*}
$$

Hill's Region. Intuitively, this is the region of the surface in figure 2.1 which is below the plane $z=E$.
Remark 2.2. It is possible that Hill's region be unbounded, while having a boundary (this follows directly from the definition of bounded set in real analysis). For instance in our case, for $E<E_{m}$ the boundary $\partial \mathcal{R}_{E}=\left\{\mathbf{x} \in \mathbb{R}^{2} \mid V(\mathbf{x})=E\right\}$ is well defined but the set extends to infinity in every other direction. We call the unbounded component of Hill's region $\mathcal{R}_{E}^{u}$. In our case it is easy to see that $\mathcal{R}_{E}=\mathcal{R}_{E}^{u}$.

Another important concept is the topological degree of a map. In Hirsch [10], this is defined pointwise as follows:

Definition 2.5. Let $(M, \omega),(N, \theta)$ be compact oriented manifolds of the same dimension, without boundaries. Assume $N$ is connected. Let $f: M \rightarrow N$ be a $C^{1}$ map and $x \in M$ be a regular point of $f$. Put $y=f(x)$. Letting $T_{x} f$ denote the tangent of $f$ at $x$, we say $f$ has positive type if the isomorphism $T_{x} f: T_{x} M \rightarrow T_{y} N$ preserves orientation (i.e. if it sends $\omega_{x}$ to $\theta_{y}$ ). In this case we write $\operatorname{deg}_{x} f=1$. In the case $f$ has negative type (i.e. $T_{x}$ does not preserve orientation), we write $\operatorname{deg}_{x} f=-1$. We call $\operatorname{deg}_{x} f$ the degree of $f$ at $x$.

Intuitively, this is the number of times that $f$ wraps $M$ around $N$. It is also stated that the degree of a map is the same for all its regular points (hence generalizing the above definition to suit our purposes).

With these definitions at hand we can finally present the first of the two theorems which will be crucial in proving the abrupt bifurcation described in the previous subsection. The theorem [14] holds for potentials $V \in C^{2}\left(\mathbb{R}^{d}, \mathbb{R}\right)$ which obey the estimate:

$$
\int_{R}^{\infty} \sup _{\|x\| \geq r}\left\|\partial^{m} F(\mathbf{x})\right\| r^{|m|} d r<\infty \quad(|m| \leq 1)
$$

for some $R \in \mathbb{R}, m$ here is a (multi-)index $m \in \mathbb{N}_{0}^{d}$ and $F:=-\nabla V$.
Theorem 3. For non-trapping energies $E \in \mathcal{N} \mathcal{T}$, the following holds true:

1. if $\partial \mathcal{R}_{E}^{u}=\emptyset$ and $d \geq 2$, then $\operatorname{deg}(E)=0$;
2. if $\partial \mathcal{R}_{E}^{u} \cong S^{d-1}$, then $\operatorname{deg}(E)=1$.

The second theorem [13] we need holds for potentials $V \in C_{0}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}\right), d \geq 2$, for which all $\mathbf{x} \in \mathbb{R}^{d}$ such that $V(\mathbf{x}) \neq 0$ are contained in the union of $n$ disjoint balls

$$
B_{l}:=\left\{\mathbf{x} \in \mathbb{R}^{d} \mid\left\|\mathbf{x}-\mathbf{s}_{\mathbf{l}}\right\| \leq r_{l}\right\}, \quad(l=1, \ldots, n)
$$

and which are non-shadowing (i.e. every straight line in $\mathbb{R}^{d}$ meets at most two of these balls). Moreover, defining

$$
\begin{array}{ll}
\mathbf{v}^{ \pm}: s^{ \pm} \rightarrow \mathbb{R}^{d}, & \mathbf{v}^{ \pm}\left(\mathbf{x}_{\mathbf{0}}\right):=\lim _{t \rightarrow \pm \infty} \mathbf{v}\left(t, \mathbf{x}_{\mathbf{0}}\right) \\
\hat{\mathbf{v}}^{ \pm}: s^{ \pm} \rightarrow S^{d-1}, & \hat{\mathbf{v}}^{ \pm}(\mathbf{x}):=\frac{\mathbf{v}^{ \pm}(\mathbf{x})}{\left\|\mathbf{v}^{ \pm}(\mathbf{x})\right\|}
\end{array}
$$

and $\hat{s}_{k, l}:=\left(\mathbf{s}_{k}-\mathbf{s}_{l}\right) /\left\|\mathbf{s}_{k}-\mathbf{s}_{l}\right\|$, we restrict the initial and final directions $(\hat{\mathbf{v}})$ to the set

$$
\tilde{S}^{d-1}:=\left\{\hat{\mathbf{v}} \in S^{d-1} \left\lvert\, \measuredangle\left(\hat{\mathbf{v}}, \hat{s}_{k, l}\right)>\arcsin \left(\frac{r_{k}+r_{l}}{\left\|\mathbf{s}_{\mathbf{k}}-\mathbf{s}_{\mathbf{l}}\right\|}\right)\right., 1 \leq k \neq l \leq n\right\}
$$

[^0]Where $\measuredangle$ denotes the angle between $\hat{\mathbf{v}}$ and $\hat{s}_{k, l}$. Finally, we generalize the symbolic dynamics described in the previous section to potentials with an arbitrary number of hills. To this purpose, we use the symbol sequences

$$
\underline{k}=\left(k_{i}\right)_{i \in I} \in \mathcal{S}^{I} \quad \text { over the alphabel } \quad \mathcal{S}:=\{0, \ldots, n-1\},
$$

where

$$
I \equiv I_{l}^{r}:=\{i \in \mathbb{Z} \mid l \leq i \leq r\}
$$

Again, we call a sequence admissible if $k_{i} \neq k_{i+1}$. We also define $\mathbf{X}_{l}^{r}$ to be the admissible sequences in $\mathcal{S}^{I}$.

With the above information at hand, we can finally present the second of Knauf's theorem's which we shall be using [13].

Theorem 4. Let $n \geq 2$. $E$ be non-trapping for the individual potentials $V_{l}\left(E \in \cap_{l=1}^{n} \mathcal{N} \mathcal{T}\right)$ and $\operatorname{deg}_{l}(E) \neq 0,1 \leq l \leq n$.

Then for every interval $I_{l}^{r}, \underline{k} \in \mathbf{X}_{l}^{r}$ and $\hat{p}^{ \pm} \in \tilde{S}^{d-1}$ there is a trajectory in $\Sigma_{E}$ meeting exactly the balls $B_{k_{i}}, i \in I_{l}^{r}$ in succession.

- If $l \neq-\infty$ then this trajectory in $s_{E}^{-}$has initial direction $\hat{p}^{-}$. Otherwise it belongs to $b_{E}^{-}$.
- If $r \neq \infty$, then this trajectory in $s_{E}^{+}$has final direction $\hat{p}^{+}$. Otherwise it belongs to $b_{E}^{+}$.

In particular $E$ is a trapping energy for $V(E \in \mathcal{T E})$.
Before delving into the proof, we need to clarify one more concept: cutoff functions. These are a means to study only a certain portion of a given geometrical structure while preserving the smoothness of it. The formal definition follows from a theorem found in [20].

Theorem 5. Let $M$ be a smooth manifold and $K \subset U \subset M$ two subsets such that $K$ is closed and $U$ is open, then there exists a smooth function $\chi: M \rightarrow \mathbb{R}$, called a cutoff function, with the following properties.

$$
\begin{aligned}
& \text { 1. } 0 \leq \chi \leq 1 \text { for all } p \in M \\
& \text { 2. for all } p \in M \text { such that } \chi(p) \neq 0, p \in U
\end{aligned}
$$

$$
\text { 3. } \chi(p)=1 \text { for all } p \in K \text {. }
$$

It should be noted that the cutoff function in figure 2.17 has circular $K$ and $U$, but this need not necessarily be the case as we shall see shortly. For the sake of this proof, we will be using an extension of a particular cutoff function, which we shall now define. We now pick the norms


Figure 2.17: A cutoff function. The yellow disk at (a) is the projection of $K$ onto the plane $z=1$. The disk at (b) is $U$ and the plane (c) is $M$.

$$
\begin{align*}
\|\mathbf{x}\|_{1} & =\max \left\{\left|x_{1}-R\right|,\left|x_{2}-R\right|\right\}  \tag{2.6}\\
\|\mathbf{x}\|_{2} & =\max \left\{\left|x_{1}-R\right|,\left|x_{2}+R\right|\right\}  \tag{2.7}\\
\|\mathbf{x}\|_{3} & =\max \left\{\left|x_{1}+R\right|,\left|x_{2}+R\right|\right\}  \tag{2.8}\\
\|\mathbf{x}\|_{4} & =\max \left\{\left|x_{1}+R\right|,\left|x_{2}-R\right|\right\} \tag{2.9}
\end{align*}
$$

we then define

$$
h: \mathbb{R} \rightarrow \mathbb{R}, \quad h(t):= \begin{cases}e^{-1 / t} & t>0 \\ 0 & t \leq 0\end{cases}
$$

from which

$$
\chi_{i}(\mathbf{x}):=\frac{h\left(R-\|\mathbf{x}\|_{i}\right)}{h\left(R-\|\mathbf{x}\|_{i}\right)+h\left(\|\mathbf{x}\|_{i}-(R-\delta)\right)}
$$

is a cutoff function for any fixed, $R>0$ and $0<\delta<R$. In particular, $\chi_{i}(t)$ defines a cutoff function with $K$ and $U$ having square boundary (in the $L_{\infty}$ sense and not in the taxicab - or $L_{1}$ - sense), with height $2 R-2 \delta$ and $2 R$ respectively, centered at $( \pm R, \pm R)$. This would imply that $U$ touches the x -axis (resp. y-axis) at $(x, y)=(k, 0)$ (resp. $(x, y)=(0, k))$ for any $k \in \mathbb{R}$.

Lemma 2.1. Let $x \in M$ and $\chi_{i}$ be a cutoff function as defined above, then in the limit

$$
\lim _{R \rightarrow \pm \infty} \chi_{i}(\mathbf{x})
$$

the boundary of $U$ remains along the axes of the quadrant within which $U$ is centered.

Proof. We consider without loss of generality $\chi_{1}$ and the associated $\|.\|_{1}$ norm. Clearly the boundary of $U$ is made up of all the points which are a distance $R$ from $(R, R)$. The closest points to the $x$-axis (resp. $y$-axis) are the points $(0, k)$ and $(k, 0)$ for any $k \in \mathbb{R}$. At these points we have that:

$$
\|(0, k)\|=\|(k, 0)\|=R
$$

since any point in $U$ is at a distance less than $R$ from $(R, R)$ and thus

$$
\chi_{1}(0, R)=\chi_{1}(R, 0)=\frac{h(R-R)}{h(R-R)+h(R-(R-\delta))}=\frac{h(0)}{h(0)+h(\delta)}=0
$$

showing that

$$
\lim _{R \rightarrow \pm \infty} \chi\left(\|(0, R)\|_{1}\right)=\lim _{R \rightarrow \pm \infty} \chi\left(\|(R, 0)\|_{1}\right)=\lim _{R \rightarrow \pm \infty} 0=0
$$

and hence that $(0, k),(k, 0) \notin \partial U$. Since the shape of $U$ will not be altered by taking this limit, this completes the proof.

We hence define for all norms (2.6-9)

$$
\|\mathbf{x}\|_{i}^{q}:=\lim _{R \rightarrow \infty}\|\mathbf{x}\|_{i}
$$

and the functions

$$
\tilde{\chi}_{i}(\mathbf{x})=\lim _{R \rightarrow \infty} \chi_{i}(\mathbf{x})
$$

Remark 2.3. It should be noted that $\tilde{\chi}_{i}(x)$ is no longer a proper cutoff function (at least in the sense described in theorem 5), since the boundary in the $( \pm \infty, \pm \infty)$ direction is no longer defined. However smoothness around the $x$ and $y$-axes in the cutoff is preserved, which suffices for our purposes.

We now consider the potential (2.1) expressed as the sum of the individual potentials

$$
V(\mathbf{x})=\left(\sum_{i=1}^{4} \tilde{\chi}_{i}(x)+\psi(x)\right) V(\mathbf{x})
$$

where

$$
\psi(x)=1-\sum_{i=1}^{4} \tilde{\chi}_{i}(x)
$$

and, given that we may take $\delta$ arbitrarily small, $\psi(x)$ is near trivial and defines an arbitrarily small correction in the slope of the potential around the $x$ and $y$ axes. It has been stated that for such small corrections to the overall potential, the correction itself can be ignored in the application of theorem 4 14].

We consider without loss of generality (due to simmetry) the potential $\tilde{\chi}_{i}(x) V(\mathbf{x})$. We note that the boundary of Hill's region will remain the same as for the individual hill in the potential (2.2) upon choosing $\delta$ small enough. With this in mind, we have the following lemma:

Lemma 2.2. The boundary of Hill's region, when nonempty, for any potential $V_{i}=\tilde{\chi}_{i}(x) V(\mathbf{x})$ is homeomorphic to $S^{1}$.

Proof. We again consider without loss of generality the potential $\tilde{\chi}_{1}(x) V(\mathbf{x})$. The map

$$
\begin{equation*}
\pi:\{\mathbf{x} \in \mathbb{R} \mid V(\mathbf{x})=E\} \rightarrow S^{1}, \quad \pi(\mathbf{x})=\frac{\mathbf{x}-\mathbf{1}}{\sqrt{\left(x_{1}-1\right)^{2}+\left(x_{2}-1\right)^{2}}} \tag{2.10}
\end{equation*}
$$

is continuous and invertible. Moreover, it shifts the center of the potential hill to the origin and projects the points of the corresponding Hill's region onto the unit circle, thus defining a homeomorphism between the two sets.

We may hence apply theorem 3. We note that since the potential only has one hill, no energy is trapping. Furthermore, we note that the boundary of Hill's region only comes into existence as $E$ is lowered past $E_{m}$. Thus, for $E>E_{m}$ we have that $\operatorname{deg}_{i}(E)=0$, while for $E<E_{m}$ we have that $\operatorname{deg}_{i}(E)=1$. The same will hold for all other potentials in the sum that gives $V(\mathbf{x})$. Thus:

1. for $E>E_{m}$, the deflection angle will be near zero (since $\operatorname{deg}_{i}(E)=0$ ) and hence no periodic orbits may exist;
2. for $E<E_{m}, E \in \mathcal{T E}$, thus showing that trapped orbits (and hence the chaotic set) appear as soon as $E<E_{m}$;
3. for any sequence of the symbols " 0 ", " $1 ", " 2 ", " 3 "$, such that no symbol repeats twice (e.g. 001) there exists an orbit visiting the hills in the order specified by such sequence, thus showing that the chaos is fully developed as soon as it appears.


Figure 2.18: The first-quadrant potential $V_{1}(\mathbf{x})=\tilde{\chi}_{1}(t) V(\mathbf{x})$


Figure 2.19: Contour plot of figure 2.18

## 3 Other scattering systems

In this section we explore a number of other scattering systems with the aim to gain further insight as to whether or not chaos occurs in them and the means by which it occurs. As we shall see, the abrupt bifurcation to chaos seen in the problem associated with (2.1) is not the only route to chaos for planar scattering systems and chaos in not present in every scattering system either.

### 3.1 One-Hill potential

In this subsection, we consider the problem associated with the quadratic potential

$$
V(x, y)=-\frac{1}{2}\left(x^{2}+y^{2}\right)+E_{m}
$$

in order to derive a particular result. The associated IVP has solution:

$$
\begin{aligned}
& x(t)=A e^{t}+B e^{-t} \\
& y(t)=C e^{t}+D e^{-t}
\end{aligned}
$$

where

$$
\begin{aligned}
A & =\frac{1}{2}\left(x_{0}+p_{x 0}\right) \\
B & =\frac{1}{2}\left(x_{0}-p_{x 0}\right) \\
C & =-\frac{1}{2}\left(1-p_{y 0}\right) \\
D & =-\frac{1}{2}\left(1+p_{y 0}\right)
\end{aligned}
$$

The scattering angle for initial conditions along the line $y_{0}=-1$ with $p_{x 0}=0$ is:

$$
\phi=\arctan \left(\frac{x_{0}}{p_{y 0}-1}\right)=\arctan \left(\frac{x_{0}}{\sqrt{2\left(E-E_{m}\right)+x_{0}^{2}+1}-1}\right)
$$

which has maximum

$$
\phi_{M}=\sqrt{1+\frac{1}{2}\left(E-E_{m}\right)^{-1}}
$$

at $x_{0}=\sqrt{2\left(E-E_{m}\right)\left(2\left(E-E_{m}\right)+1\right)}$. Clearly if $E>E_{m}, \phi_{M}<\frac{\pi}{2}$ and $\phi_{M} \rightarrow \frac{\pi}{2}$ as $E \rightarrow E_{m}$ from above. For $E<E_{m}$, the maximum deflection angle instead suddenly jumps to

$$
\phi_{M}=\pi
$$

This is a useful result as we shall see later.

### 3.2 Two-Hill potential

In this subsection we shall consider the two-hill potential:

$$
\begin{equation*}
V(x, y)=y^{2} e^{-\left(x^{2}+y^{2}\right)} \tag{3.1}
\end{equation*}
$$

which this time has maximum $E_{m}=e^{-1}$. The potential and its level sets are shown in figures 3.1-2.
One can again define two cutoff functions (the same way we did in section 2.4) for the potential at hand. These would again be:

$$
\chi_{i}(\mathbf{x}):=\frac{h\left(R-\|\mathbf{x}\|_{i}\right)}{h\left(R-\|\mathbf{x}\|_{i}\right)+h\left(\|\mathbf{x}\|_{i}-(R-\delta)\right)}
$$

where $h$ too is defined as in section 2.4. However we would now use the norms:

$$
\begin{aligned}
& \|\mathbf{x}\|_{1}=\max \left\{\left|x_{1}\right|,\left|x_{2}-R\right|\right\} \\
& \|\mathbf{x}\|_{2}=\max \left\{\left|x_{1}\right|,\left|x_{2}+R\right|\right\}
\end{aligned}
$$

and using an analogous argument to that used to prove lemma 2.1, we may define:

$$
\|\mathbf{x}\|_{i}^{h p}:=\lim _{R \rightarrow \infty}\|\mathbf{x}\|_{i}
$$

and the functions

$$
\tilde{\chi}_{i}(\mathbf{x})=\lim _{R \rightarrow \infty} \chi_{i}(\mathbf{x})
$$

and using the homeomorphisms

$$
\begin{equation*}
\pi:\{\mathbf{x} \in \mathbb{R} \mid V(\mathbf{x})=E\} \rightarrow S^{1}, \quad \pi(\mathbf{x})=\frac{\mathbf{x} \pm \mathbf{e}_{\mathbf{1}}}{\sqrt{\left(x_{1}\right)^{2}+\left(x_{2} \pm 1\right)^{2}}} \tag{3.2}
\end{equation*}
$$

where $\mathbf{e}_{\mathbf{1}}=(0,1)$, we may again show that the boundary of Hill's region for both potentials

$$
\begin{aligned}
& V_{1}(\mathbf{x})=V(\mathbf{x}) \tilde{\chi}_{1}(\mathbf{x}) \\
& V_{2}(\mathbf{x})=V(\mathbf{x}) \tilde{\chi_{2}}(\mathbf{x})
\end{aligned}
$$

is homeomorphic to the unit circle for $E<E_{m}$ and empty for $E>E_{m}$, thus showing that:

1. $E$ is a trapping energy;
2. for any admissible sequence of the symbols " 0 ", " 1 ", there exists an orbit visiting the corresponding hills in the order specified by the sequence.
It should be noted though that the latter conclusion implies that in this case, there is only one periodic orbit. This would in turn imply that the set of trapped orbits includes only this orbit, which is hyperbolic [12]. Thus there is no chaotic set underlying the dynamics of this system.


Figure 3.1: The two-hill potential


Figure 3.2: Contour plot of figure 3.1

### 3.3 Potentials with Three Equal Hills

We now consider the example potential

$$
V(x, y)=\left(x^{2}+y^{2}\right)^{2} \sin ^{2}\left(\frac{3}{2} \arctan \left(\frac{y}{x}\right)-\frac{\pi}{4}\right) e^{-x^{2}-y^{2}}
$$

which, in polar coordinates, translates to:

$$
\begin{equation*}
V(r, \theta)=r^{4} \sin ^{2}\left(\frac{3}{2} \theta-\frac{\pi}{4}\right) e^{-r^{2}} \tag{3.3}
\end{equation*}
$$

This potential has maximum $E_{m}=4 e^{-2}$. There does exist a strange saddle for this problem, which has been found to be hyperbolic [1] (see figure 3.5).


Figure 3.4: The three equal-height hills potential


Figure 3.5: Contour plot of figure 3.3

Again, applying theorems 3 and 4 from the last section, we may conclude that the bifurcation will be abrupt in this case too. We take

$$
\chi_{i}(\mathbf{x}):=\frac{h\left(R-\|\mathbf{x}\|_{i}\right)}{h\left(R-\|\mathbf{x}\|_{i}\right)+h\left(\|\mathbf{x}\|_{i}-(R-\delta)\right)}
$$

with norms
$\|\mathbf{x}\|_{1}=\left|x_{1}\right|+\left|x_{2}+R\right|$
$\|\mathbf{x}\|_{2}=\left|x_{1}-R\right|+\left|x_{2}\right|+\max \left\{\left|x_{1}-R\right|,\left|x_{2}-R\right|\right\}$
$\|\mathbf{x}\|_{3}=\left|x_{1}+R\right|+\left|x_{2}\right|+\max \left\{\left|x_{1}+R\right|,\left|x_{2}-R\right|\right\}$
we again define

$$
\|\mathbf{x}\|_{i}^{h p}:=\lim _{R \rightarrow \infty}\|\mathbf{x}\|_{i}
$$

and the functions

$$
\tilde{\chi}_{i}(\mathbf{x})=\lim _{R \rightarrow \infty} \chi_{i}(\mathbf{x})
$$



Figure 3.3: A rough approximation of the strange saddle for the problem associated with the potential (3.3) using $E / E_{m}=0.185$.

We now consider without loss of generality the hill in the upper half-plane. For this peak, the map

$$
\begin{equation*}
\pi:\{\mathbf{x} \in \mathbb{R} \mid V(\mathbf{x})=E\} \rightarrow S^{1}, \quad \pi(\mathbf{x})=\frac{\mathbf{x}-(0, \sqrt{2})}{\sqrt{\left(x_{1}\right)^{2}+\left(x_{2}-\sqrt{2}\right)^{2}}} \tag{3.4}
\end{equation*}
$$

defines a homeomorphism between the boundary of Hill's region for the upper half-plane potential and the unit circle. Because of the $\frac{2 \pi}{3}$ rotational symmetry of the potential (3.3), the same can be found for all other hills. This would show that the bifurcation is abrupt in this case too.

It can be inferred, by applying the Knauf criteria, that all potentials with more than two hills always present the abrupt bifurcation, on condition that it is impossible to draw a straight line segment through all the hilltops (i.e. the hilltops must not be colinear) and that the height of all hills is the same. Thus in the next subsection, we explore the case of potentials with hills of unequal height.
Remark 3.1. In case the potential is colinear we have a situation which is analogous to that described in section 3.2 : in this case the only periodic orbits will be between two hills, thus if we let $N$ be the number of hilltops, the strange saddle will be composed of a finite number of points, precluding the possibility of chaos.


Figure 3.6: Three colinear hills.

### 3.4 Potentials with Hills of Unequal Height

In this subsection we consider the general class of scattering systems with three hills of unequal height. A schematic of the disposition of these hills can be seen in figure 3.7. In this figure, we have labelled the hills " 0 ", " 1 " and " 2 ". The angle $\phi_{m}$ represents the minimum deflection necessary for an orbit which begins by moving from hill 1 to hill 0 to be deflected back to hill 2 . We assume that all the individual hills' contours are homeomorphic to the unit circle, which guarantees that we may apply theorems 3 and 4. Labeling the maximum height of each hill: $E_{m 0}$ (maximum height of hill 0 ), $E_{m 1}$ (maximum height of hill 1) and $E_{m 2}$ (maximum height of hill 2), we may assume without loss of generality that $E_{m 0}<E_{m 1} \leq E_{m 2}$. We now divide the problem into the following cases:


Figure 3.7: disposition of the three hills in the plane.

- Case 1: $\phi_{m}>\frac{\pi}{2}$;
- Case 2: $\phi_{m}<\frac{\pi}{2}$.

This is a meaningful distinction: as we have seen in section 3.1, the maximum possible deflection angle reaches the separation value $\frac{\pi}{2}$ when $\mathcal{R}_{E}\left(V_{l}\right)$ is a point. For case 1 , this would imply that chaos is impossible for $E>E_{m 0}$ and (by applying theorems 3 and 4) that chaos is fully developed as soon as $E<E_{m 0}$, thus defining an abrupt bifurcation.

Case 2 is a little more complicated than case 1 . As we have seen in section 3.1, the maximum deflection angle of a single hill $\phi_{M}$ is a smooth, monotonically increasing function of the energy $E$ for $E>E_{m 0}$. It would thus be possible to have orbits which go from hill 1 to hill 2 (or viceversa) while being deflected by hill 0 . To this purpose, we define two subcases:

- Case 2.a: $E_{m 0}$ is sufficiently small that $\phi_{m}>\phi_{M 0}\left(E_{m 1}\right)$;
- Case 2.b: $\phi_{m}<\phi_{M 0}\left(E_{m 1}\right)$.

In case 2.a, there will be an energy $E_{m}^{c} \in\left(E_{m 0}, E_{m 1}\right)$ such that $\phi_{M 0}\left(E_{m}^{c}\right)=\phi_{m}$. In this case, there will be two types of orbits joining hills 1 and 2 : one will be the orbit passing through hill 0 , the other will be the orbit connecting hills 1 and 2 directly. In this case using the symbols " 0 ", " 1 " and " 2 " to denote each hill in the potential, we can describe the two types of orbits by a sequence of the symbols " $a_{i}$ " and " $b_{i}$ " where $A=\{12,21\}=\left\{a_{1}, a_{2}\right\}$ and $B=\{102,201\}=\left\{b_{1}, b_{2}\right\}$. All sequences of the symbols $a_{i}$ and $b_{i}$ are possible, with exception to those which have two distinct elements of the same set following each other (e.g. $a_{1} a_{2} \ldots$ ) and fully developed chaos is present 1. A key difference with every other case we have treated insofar where fully developed chaos exists, is the way in which it is developed: Hill's region will not develop a third boundary before fully developed chaos ensues. Thus for this case there is no abrupt bifurcation. Instead we expect to see a sequence of saddle-center bifurcations (the sudden appearance of periodic orbits out of nothing), resonance bifurcations (which happen when the eigenvalues of the linearized system are complex numbers such that $z^{p}=1$ for some $p \in \mathbb{N}$ near a center) and period-doubling bifurcations depending on the number of hills present in the potential and their relative height.

In case 2 b the periodic orbits discussed in the last paragraph appear as soon as $E<E_{m 2}$. Thus, in this case, the bifurcation is abrupt and happens at $E=E_{m 2}$.

### 3.4.1 Example: case 1

We consider the potential

$$
V(x, y)=0.15 e^{-x^{2}-(y-2)^{2}}+0.2 e^{\left(-(x+\sqrt{3})^{2}-(y+1)^{2}\right)}+0.2 e^{\left(-(x-\sqrt{3})^{2}-(y+1)^{2}\right)}
$$

the distance between the peaks of the two hills of maximum height (the second and third terms in the summation) can be found to be $d_{M}=2 \sqrt{3}$. Thus the radius of the circle whose diameter is the distance between these two potentials is $r=\sqrt{3}<2=d_{m}$ which is the distance from the circle's center of the hill with minimum height. The maximum heights of each peak are (in ascending order) $E_{m 1}=0.15, E_{m 2}=0.2$ and $E_{m 3}=0.2$. Figure 3.8 shows the deflection function for the problem associated with this potential for $E=1.14 \cdot E_{m 1} \approx 0.17<E_{m 2}=E_{m 3}$. The function would appear to be smooth, showing no sign of chaotic behaviour. Figure 3.9 is instead the deflection function calculated for $E=0.86 \cdot E_{m 1}$, which shows a number of singularities, implying the presence of chaos. It is easy to see, using theorems 3 and 4 that the chaos in fully developed in this case.


Figure 3.9: Deflection angle as a function of the impact parameter for $E=1.14 \cdot E_{m 1}$.


Figure 3.10: Deflection angle as a function of the impact parameter for $E=0.86 \cdot E_{m 1}$.

### 3.4.2 Example: case 2

We now consider a four-hill potential again, this time given by:

$$
\begin{equation*}
V(x, y)=\frac{1}{10} e^{-x^{2}-(y-2)^{2}}+\frac{1}{10} e^{-x^{2}-(y+2)^{2}}+\frac{1}{5} e^{-(x-3)^{2}-y^{2}}+\frac{1}{5} e^{-(x+3)^{2}-y^{2}} . \tag{3.5}
\end{equation*}
$$

This potential has four hills, two of which have maximum $E_{m 1}=0.1+0.1 \cdot e^{-8}+0.4 \cdot e^{-13}$ and two of which have maximum $E_{m 2}=0.2+0.2 \cdot e^{-36}+0.2 \cdot e^{-13}$. Figures 3.11-3.13 show successive blowups of the scattering angle as a function of the impact parameter for $E / E_{m 1}=1.02$. The function appears to be singular on a fractal set, hence suggesting fully developed chaotic scattering for only two exposed peaks. Numerical experiments A.7) show that $E_{m}^{c} \approx E_{m 1} \cdot 1.052$ within third decimal accuracy.


Figure 3.11: The potential (3.5).


Figure 3.13: Zoom-in of figure 3.12.


Figure 3.12: Deflection angle as a function of the impact parameter for $E=1.02 \cdot E_{m 1}$.


Figure 3.14: Zoom-in of figure 3.13.

## 4 Conclusion

In this paper we have studied a collection of planar scattering systems, with particular interest directed towards the qualitative change in behaviour resulting from the variation of a particular parameter, namely the energy.

In section 2 we have seen how chaotic behaviour can occur in a planar scattering system with $\frac{\pi}{2}$ rotational symmetry and for which all hills were of equal height. In these conditions, we have noticed that chaos occurs via the abrupt bifurcation route to chaos. This consists in the sudden creation of a hyperbolic chaotic set out of nothing upon lowering the energy parameter below the value of the maximum height of a given number of hilltops (at least three). We have studied some properties of this set and its stable and unstable invariant manifolds. We then used new analytic criteria to obtain a formal proof of the existence of this bifurcation for the system studied.

In section 3 we have briefly investigated a number of other planar scattering systems. Of particular interest, we have seen that for potentials with two hills, there can be no chaotic behaviour. We also found alternatives to the abrupt bifurcation route to chaos occurring in planar scattering systems with three potential hills, where not all hills were of the same height and the lowest one - which must be of a sufficiently low height - was within a circle containing all three hills and whose diameter was equal to the distance between the two tallest hills. In this case we have seen that the route to chaos will not be via an abrupt bifurcation, but rather via some combination of period-doubling bifurcations, resonance bifurcations and saddle-center bifurcations (i.e. the remaining types of bifurcations present in generic Hamiltonian systems). Future research in the filed of classical potential scattering could be aimed at deriving criteria useful for determining the existence of a particular collection of these other types of bifurcations, the order in which they may occur and the type of set responsible for the chaotic dynamics exhibited by systems similar to that associated with the potential (3.5). Restricting the
discussion to the existence of abrupt bifurcations, it would instead be of interest to derive analytical criteria to determine if the deflection angle of a hill is sufficient for an orbit crossing it to be deflected to a neighboring higher hill, in systems associated with potentials with hills of unequal height.

## A MATLAB scripts and programs

## A. 1 Numerical Integration of (2.1) by Leapfrog method

```
function [r, phi, t, y, x]=scattering_2(x0,y0, dx0, dy0,h,p)
x (1) =x0;
y(1)=y0;
dx (1)=dx0;
dy (1)=dy0;
gradV1(1)=y(1)^2*(-2*exp(-x(1)^2-y(1)^2)*x(1)^ 3+2*exp(-x(1)^2-y(1)^ 2) *x
    (1));
gradV2(1)=x (1)^ 2*(-2*exp(-x(1)^2-y(1)^2)*y(1)^ 3+2*exp(-x(1)^2-y(1)^2)*y
    (1));
t (1)=0;
r (1) =0;
i=2;
while r (i - 1)<=5.1
    x (i ) =x (i -1)+dx (i - 1)*h-gradV1 (i - 1)*h ^ 2*0.5;
    y(i )=y(i-1)+dy(i -1)*h-gradV2(i - 1)*h ^ 2*0.5;
    gradV1(i)=y(i )^ 2*(-2*exp(-x(i ) ^2-y(i ) ^ 2) *x(i )
        *x(i));
```



```
        *y(i ));
    dx (i ) = dx (i - 1) -0.5*h*(gradV1 (i)+gradV1 (i - 1));
    dy (i)=dy (i - 1) -0.5*h*(gradV2(i)+gradV2(i -1));
    r(i)=sqrt(x(i )^2+y(i )^2);
    t(i)=t (i - 1) +h;
    phi(i)=atan2(y(i),x(i));
    i=i +1;
end
x=x;
y=y;
t=t;
if p==1
    figure (1)
    hold on
    plot(x,y)
    xlabel('x')
    ylabel('y')
    X = linspace(-2.2, 2.2, 100);
    Y}=\mathrm{ linspace( - 2.2, 2.2, 100);
    [x, y] = meshgrid(X, Y);
    fxy = (x.^2) .*(y.^2) .* exp(-x.^2-y.^2);
    contour(X,Y,fxy);
    xlim}([\begin{array}{ll}{-2.5}&{2.5])}
    ylim([-2.5,2.5])
    hold off
end
end
```


## A. 2 Scattering angle vs impact parameter

```
% Plot phi as a function of the impact parameter
clear all
delta=10^(-3);
b0=-3;
```

```
bend=3;
tot_iter=abs(bend-b0)/delta;
Em=1/exp (2);
E=0.26*Em;
%E=1.626*Em;
for i=1:tot_iter
    x0=-4;
    y0=b0+delta*i ;
    vx0=sqrt ((E-(x0^2) * ( y0 ) ^2) *exp(-x0^2-(y0 ) ^2)) *2);
    vy0=0;
    [r,phi]=scattering_2(x0,y0,vx0,vy0,0.2,0);
    phi_p(i)=phi(end);
    b(i)=y0;
end
figure(2)
%plot(b,phi_p)
scatter(b,phi_p,1)
xlabel('b')
ylabel('phi')
%txt = {'E/Em=0.26'};
%text(-2.25,2,txt)
```


## A. 3 Delay time

\% Plot time-delay as a function of the impact parameter
clear all
delta $=0.001$;
$\mathrm{b} 0=-3.001$;
$\mathrm{Em}=1 / \exp (2)$;
$\mathrm{E}=0.9 * \mathrm{Em}$;
$\% \mathrm{E}=1.626 * \mathrm{Em}$;
\%phi=atan2 (y, x) ;
for $i=1: 6000$
$\mathrm{x} 0=-4$;
$y 0=b 0+$ delta $*$;
$\operatorname{vx} 0=\operatorname{sqrt}\left(\left(\mathrm{E}-(\mathrm{x} 0 \wedge 2) *\left((\mathrm{y} 0)^{\wedge} 2\right) * \exp \left(-\mathrm{x} 0 \wedge 2-(\mathrm{y} 0)^{\wedge} 2\right)\right) * 2\right) ;$
$\operatorname{vy} 0=0$;
$[\mathrm{r}, \mathrm{phi}, \mathrm{t}]=\mathrm{sc}$ attering$-2(\mathrm{x} 0, \mathrm{y} 0, \mathrm{vx} 0, \mathrm{vy} 0,0.2,0) ;$
delay $(\mathrm{i})=\mathrm{t}$ (end);
b(i) $=\mathrm{y} 0$;
end
figure (3)
\%plot(b,delay)
scatter (b, delay, 1 )
xlabel ('b')
ylabel('time delay')
$x \lim ([-3,3])$
$y \lim ([0,125])$

## A. 4 Fractal dimension of the singular values of $\phi(b)$

```
% fractal dimension using the uncertainty method
clear all
Em=exp(-2);
E=0.26*Em;
x0=-3;
sample_size=1000;
a=-0.535;
b}=-0.24
m=b-a;
for m=1:100
    for j=2:8
        J=j -1
        dummy1=0;
        eps(j-1,m)=10^(-j);
        b (1:sample_size)=m*rand (sample_size, 1)+a;
        b}(\mathrm{ sample_size +1:2*sample_size)=-m*rand (sample_size, 1)-a;
        b(randi([1, sample_size], sample_size));
```

```
    for k=1:length(b)
            b_perturbed (k)=b (k) +(-1)^(randi ([1, 2], 1, 1))*eps (j - 1);
        end
        for i=1:sample_size
            [r,phi]=scattering_2(x0,b(i),sqrt((E- (x0^2) *((b(i) )^ 2) *exp(-
                x0^2-(b(i ) )^2)) *2),0,0.2,0);
            [r_eps, phi_eps]=scattering_2(x0,b_perturbed(i) , sqrt((E-(x0^2)
                *((b_perturbed (i) )^2)*exp(-x0^2-(b_perturbed (i) )^ 2)) *2)
                ,0,0.2,0);
            if sign(phi(end))=}=\operatorname{sign}(\textrm{phi_eps(end))
                dummy1=dummy1;
            else
                dummy1=dummy1+1;
            end
            end
            f_eps(j - 1,m)=dummy1/sample_size ;
    end
end
f_eps=mean(f_eps,2);
eps=mean(eps,2);
coeff=polyfit(log(eps),log(f_eps),1);
d_c=1-coeff (1);
loglog(eps,f_eps)
xlabel('\epsilon')
ylabel('f(\epsilon)')
```


## A. 5 Final states as a function of $\theta$ and $x_{0}$

```
function [manifold]=SvUM(n_points,min_time_delay,h, scaling, x_int,
    theta_int,plt)
x_int_size=x_int(2)-x_int(1);
theta_int_size=theta_int(2)-theta_int(1);
delta1=x_int_size/n_points;
delta2=theta_int_size/n_points;
y0=0;
Em=1/exp(2);
E=scaling*Em;
j=1;
k=1;
for i=1:n_points
    x0=x_int (1)+i*delta1;
    for l=1:n_points
        theta0=theta_int(1)+l*delta2;
        if abs(theta0)<=pi && abs(theta0) >pi/2
            vx0=-sqrt (2*(E- (x0^2) *(y0^2)*exp(-x0^2-y0^2)) / (1+tan (pi-
                theta0)^2));
            vy0=-tan(pi-theta0 ) *vx0;
        else
            vx0=sqrt (2*(E- (x0^2) * (y0^2) *exp (-x0^2-y0^2)) / (1+tan (thetat 0)
                ^2));
            vy0=tan(theta0)*vx0;
        end
        [r,phi, t]=scattering_2(x0,y0,vx0,vy0,h,0);
        phi_p=phi(end);
        if abs(t(end))>min_time_delay
            if phi_p>0 && phi_p<=pi
                x0_up (j)=x0;
                theta0_up (j)=theta0;
                j=j+1;
            else
                x0_down(k)=x0;
                theta0_down(k)=theta0;
```

```
                k=k+1;
                end
            end
    end
end
if plt==1
        figure(4)
        scatter(x0_up, theta0_up, 1,'filled')
        hold on
        scatter(x0_down, theta0_down,1,' filled ')
        xlabel('x0')
        ylabel('theta0')
        alpha(0.7)
end
x0=[x0_up,x0_down ];
theta0 = [theta0_up, theta0_down ];
manifold =[theta0', x0'];
```


## A. 6 Obtaining the strange saddle

```
% obtain the chaotic set
% obtain stable & unstable invariant manifolds
[stable_invariant_manifold] = SvUM(512,35,0.2,0.26,[-3,3],[0, pi],0);
[unstable_invariant_manifold] = SvUM(512,35,-0.2,0.26,[-3,3],[0, pi],0);
% intersect the former two
chaotic_set=intersect(stable_invariant_manifold,
    unstable_invariant_manifold, 'rows');
% extract the x coordinates
x0_chaotic_set=chaotic_set (:, 2);
% extract the y coordinates
theta0_chaotic_set=chaotic_set (:, 1);
% plot
figure(5)
scatter(x0_chaotic_set, theta0_chaotic_set, 1,'filled')
xlabel('x0')
ylabel('theta0')
```


## A. 7 Minimum energy required to get a sufficient scattering angle for (3.5)

Note: when calling to the function scattering_2.m, this script is in truth calling to a variation of this function where we have changed the sixth, seventh, seventeenth and eighteenth lines of code to suit the problem (3.5).

```
% Finding minimum energy such that deflection angle meets requirements
clear all
delta=0.1;
b0=0;
Em=0.1+0.1*exp (-8)+0.4*\operatorname{exp}(-13);
scaling(1)=2;
E=scaling*Em;
theta 0 = -0.187167041810999*pi;
PHI=-10;
k=2;
while PHI<-theta0
    scaling(k)=scaling(k-1)-0.001;
    E=Em*scaling(k);
    for i=1:10
        x0=-2.95+delta*i;
        y0=0;
        V=0.1* exp (-x0^2-(y0-2)^2) +0.1* exp (-8)+0.4* exp (-13);
        vx0=sqrt (2*(E-V)/(1+\operatorname{tan}(\operatorname{theta}0)^2));
        vy0=tan(theta0)*vx0;
        [r, phi, t, ytest, xtest]=scattering_2(x0,y0, vx0, vy0,0.2,0);
        PHI_temp(i)=phi(end);
        end
        PHI=max(PHI_temp);
        k=k+1
end
```


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[^0]:    ${ }^{1}$ Note that $T_{x} M$ and $T_{y} N$ are as in definition 1.3.

