university of groningen

faculty of science and engineering

The relation between Feynman diagrams and Kontsevich graphs in non-commutative star products

Master Project Mathematics and Physics

August 2021

Student: W.A. de Kok

Mathematics supervisor: Dr. A.V. Kiselev

Physics supervisor: Prof. dr. D. Boer

Abstract

We discuss two approaches to go from classical Poisson mechanics to non-commutative associative geometries.

In regular quantum mechanics (after Heisenberg, Dirac and many others), the associative commutative algebra of coordinate functions is replaced by the non-commutative associative algebra of linear operators that act on the Hilbert space of wave functions.

In contrast, using ideas by Weyl and Wigner, Groenewold and Moyal suggested to deform the old algebra of coordinate functions on phase space, i.e. a symplectic Poisson manifold, to an associative non-commutative algebra by deforming the old product to a star product. This is known as deformation quantization.

After a breakthrough result by Kontsevich (1997): all Poisson brackets can be deform-quantized with a star product - Cattaneo and Felder (2000) rediscovered a remarkable Poisson sigma model (from quantum field theory (QFT)): its action combines a given Poisson structure with Lobachevsky hyperbolic geometry. The calculation of correlation functions in that model reproduces the perturbative expansion (in $\hbar$) of the Kontsevich star product. This calculation relates the Feynman diagram technique from QFT to the oriented Kontsevich graphs in deformation quantization.

The theory of such a calculation has already been outlined in a pedagogical, relatively simple case of affine Poisson structures in Cattaneo's IHP-lectures [CKTB]. The angle to the subject in Cattaneo's lectures is to provide sufficient theory to understand the calculation and then to work out some details of the calculation. Yet a final illustration of the relation between Feynman diagrams and Kontsevich oriented graphs was missing in Chapter 14: no graphs were drawn at order $\hbar^3$, no expansion was known at the time. We do what was left open by Cattaneo: we illustrate the relation by calculating the Kontsevich star product perturbatively with Feynman's QFT-technique up to and including order $\hbar^3$. Explicitly, we illustrate how selection rules reduce the big set of Feynman diagrams to the small set of Kontsevich graphs that eventually appear in the Kontsevich star product.

Acknowledgements

First of all, I would like to thank my supervisors Arthemy and Daniël who have consistently supported, advised and given their critical feedback to me during the whole project, despite the Covid-measures. Arthemy and Daniël both are different persons with different insights, backgrounds and ideas about the subject. The variety of input was sometimes challenging, but always made me realize that I was studying both mathematics and physics. Despite the apparent differences, they both strongly aimed to improve the scientific quality of the work and they ensured scientific integrity during the process, by giving detailed feedback.

In particular, I appreciate Daniël's continuous search for the physical relevance of problems. Also, he has given a lot of advice on planning and ideas on how to write. Furthermore, he also encouraged me to learn new topics by starting from small examples and make progress in small steps.

I would like to thank Arthemy for teaching the MasterMath course Deformation Quantization, Graph Complex and Number Theory (2020/21) and the local course in the Quantum Universe track: Geometry of Differential Equations (2020/21). I really enjoyed our step-by-step read of the lecture notes by Cattaneo [CKTB, CI]. I found it special that our meetings usually lasted for two to four hours. In addition, it helped a lot that Arthemy had a clear vision of where the project should lead to. Lastly, I want to thank him for the great variety of analogs he used during our meetings and the discussion of literature and films.

I want to thank Ricardo Buring for teaching the course Deformation Quantization, Graph Complex and Number Theory, for helpful mailing contact and feedback on my thesis.

I want to thank Cattaneo and Felder for their explanations, remarks and historical comments on their publications in our correspondence with them during the project.

I want to thank Martijn Kluitenberg for all the time we have studied together during our bachelors and masters. Specifically, during the Covid-period it was good to regularly study together, have a coffee and walk a bit.

Lastly, I want to thank my lovely wife Lydia and my family who have always been there to support me, and even tried to understand what the project is about.

Willem

Contents

1 Introduction

1.1 Classical mechanics and quantum mechanics

Physical theories can, very roughly speaking, be characterized by three different effects that are observed in nature. Namely, relativistic effects, quantum effects and gravitational effects. Every effect comes with its own constant of nature. The respective fundamental or universal constants are Planck's constant $\hbar$, the gravitional constant G and the speed of light c[1].

For example, if a physical theory applies to objects or particles moving at speeds v, that are low with respect to the speed of light c, i.e. $\frac{v}{c} \ll 1$, then the relativistic effects will be negligible. Similarly, quantum effects are important in theories with small objects. Small means smaller or comparable with the object's de Broglie wavelength $\lambda_{dB} = \frac{\hbar}{p}$, where p is the momentum of an object. For example, in thermodynamics, the de Broglie wavelength is defined as $\lambda_{dB,th} = \frac{\hbar}{2\pi m k_B T}$, where m is the mass of the object, k_B the Boltzmann constant and T the temperature. The quantum effects then become important when the average distance between particles is smaller or equal to the de Broglie wavelength $\left(\frac{V}{N}\right)^{\frac{1}{3}} \leq \lambda_{dB,th}$, where V is the volume of the system and N the number of particles.

Lastly, gravitational effects have to do with the relative (and absolute) mass of objects and the distance between objects. About this, we want to note that since Einstein we often treat gravitational effects and deformations of the geometry of spacetime as two counterparts. Physicists often like to think of these effects in a rough classification as draw in Figure 1. And some hope that all these theories can be unified into a so-called theory of everything.

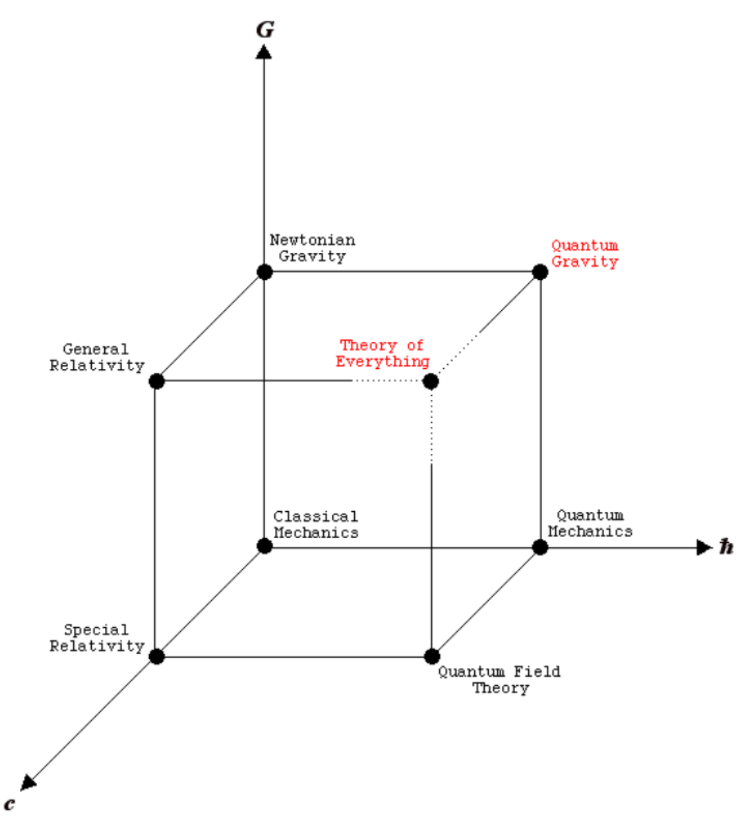


Figure 1: Rough classification of physical theories by relativistic, quantum and gravitational effects.

In this thesis, we are concerned with quantum effects and a bit with relativistic

[1]In SI units the values of the fundamental constants are $\hbar = 1.0545718 \times 10^{-34}\ m^2kgs^{-1}$, $G = 6.67408 \times 10^{-11}\ m^3kg^{-1}s^{-2}$, and $c = 299792458\ ms^{-1}$

effects. Therefore, we are mostly concerned with the lower face of the physics cube in Figure 1.

Until the end of the 19th century most of the activities of physicists, with respect to the above classification, can be located in the realm of classical mechanics and Newtonian gravity. Since the contributions of Planck, Bohr, Schrödinger, Heisenberg and many other 20th century physicists the study of quantum mechanical theories has begun.

Furthermore, Einstein, Maxwell, Lorentz, Poincaré, and Feynman are some well-known contributors to the fields of special relativity and quantum theory. The basis of most of these theories has been formed during the first decennia of the 20th century.

1.2 Quantization: Dirac and Wigner-Weyl quantization

Inherent to the rise of quantum theories, also the study of the relation between classical and quantum theories increased. A common view is that the classical description is obtained from the quantum description in the limit where the Planck constant $\hbar \to 0$ [ZJ, BFFLS, Got96]. This limit is also known as the macroscopic limit of quantum mechanical theories. We can interpret this limit as the description where we neglect the quantum effects. This is sometimes called dequantization. An example of dequantization (equivalent to taking the macroscopic limit) is: taking the limit where the wavelength of light goes to zero, in the wave theory of light, to obtain wave mechanics for optics. Nevertheless, the opposite step, going from a classical description to a quantum description, which is also called quantization, should somehow include a rule to incorporate the quantum effects. This is an ambiguous step.

To study this ambiguity of quantization, we will first give a short history of changes in formulation of classical mechanics from Newton, via Lagrange and Hamilton, to Poisson. Then, we discuss Dirac's canonical quantization which associates operators on Hilbert space with functions on phase space. However, Dirac's quantization is not complete in the sense that one has to choose an ordering. In addition, Groenewold and van Hove proved that no quantization (à la Dirac) can establish a one-to-one relation between the Poisson bracket on phase space and the anti-commuting bracket on Hilbert space (see Theorem 4.1 on page 27 below).

In contrast with Dirac's quantization, the Weyl transform, together with its inverse the Wigner transform, established a one-to-one relation between the functions on phase space and the operators on Hilbert space. The Wigner distribution associated with operators on Hilbert space allows us to describe quantum mechanics via distributions on phase space.

1.3 Deformation Quantization

The Wigner-Weyl transforms [Wig, Wey] made Groenewold realize that two of his formulated problems of quantization [Groe] could be solved simultaneously. The two problems of Groenewold are about the correspondence between physical quantities a and quantum operators $\hat{a}$, and about the possibility of understanding the statistical character of quantum mechanics by averaging over uniquely determined processes as in classical statistical mechanics (interpretation). In the renowned publication by Groenewold [Groe] he proposed a star product to remedy the problem of Dirac's quantization stated in the Groenewold-van Hove theorem.

The star product is commonly called Moyal star product, but historically it is more valid to call it the Groenewold-Moyal star product, in rest of the thesis we will stick with the latter convention.

The star product is a deformation of the usual product of functions on phase space, and allows to define the Groenewold-Moyal bracket, which is a deformation, or a quantized version, of the Poisson bracket. The quantization of the Groenewold-Moyal bracket between functions equals the anti-commuting bracket between the quantized functions for all smooth functions in the algebra of functions on phase space. The Groenewold-Moyal bracket therefore enables to describe core aspects of quantum mechanics via a classical formulation.

In the seminal papers by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer in 1978, they raised the question whether the Moyal bracket has a natural place in classical mechanics [BFFLS]. Based on developments at that time, they stated:

> “These developments encourage attempts to view quantum mechanics as a theory of functions or distributions on phase space, with deformed products and brackets. We suggest that quantization be understood as a deformation of the structure of the algebra of classical observables, rather than as a radical change in the nature of observables.”

This perspective on quantization is thus called deformation quantization. Bayen et al. claimed, as was at that time already seen sometimes, that deformation quantization would give some new insights in the ambiguous nature of the correspondence between classical and quantum theories. That being said, it would give the possibility to develop new methods for quantum (field) theories.

In the decades after the pivotal papers by Bayen et al. deformation quantization has both been studied mainly by mathematicians, but also by (mathematical) physicists. In 1997, the mathematician Kontsevich proved that every Poisson manifold (a manifold together with a Poisson structure on it) can be canonically quantized in the sense of deformation quantization [Kont03]. This was the birth of the so-called Kontsevich star product, which is calculated using directed graphs in the hyperbolic plane as representations of the differential operators on functions. The weights in the expansion of the star product are calculated by using angles between the edges in the hyperbolic plane.

1.4 A path integral approach (from quantum field theory) to the Kontsevich star product

Interestingly, in 2000, Cattaneo and Felder published a paper explaining how the Kontsevich formula can also be calculated by the use of a path integral formulation for a Poisson-sigma model [CF]. This did not come fully out of the blue, as Kontsevich had been telling around that the directed graphs, representing differential operators in the Kontsevich star product, had a lot of similarities with Feynman diagrams. He had suggested that perturbative path integral methods (in the context of quantum field theory) should be studied to relate it to the star product. Cattaneo and Felder did so, and found that a certain Poisson-sigma model would produce the desired star product.

In the paper by Cattaneo and Felder, they explain how the path integral calculation is done to obtain a star product for general Poisson manifolds. The path integral calculations include a perturbative expansion around a non-degenerate critical point. However, in most cases the critical point is degenerate due to symmetries of the system. There are several methods to compensate for the symmetries, in order to make the critical point non-degenerate. Stated differently, we need a certain gauge-fixing procedure to 'remove' the degeneracy of the critical point.

In the case of general Poisson structures, the degeneracy of the critical point can be resolved using the BV-formalism [CF]. However, when we restrict to affine Poisson manifolds, then the BRST-formalism [BRS, Tyu] will serve as a sufficient gauge-fixing procedure to resolve the degeneracy of the critical point. The path integral calculation for star products on affine Poisson manifolds is studied by Cattaneo in Chapters 10 to 14 of [CKTB].

1.5 Calculations and illustrations of star products using perturbative path integral methods and Feynman diagrams

This thesis continues where Cattaneo stopped in Chapter 14 of [CKTB]. We illustrate the relation between Feynman diagrams in quantum field theory and Kontsevich graphs in deformation quantization. We explicitly explain which Feynman diagrams are obtained from which Wick's contractions in the path integral calculation. Moreover, we illustrate how the large set of Feynman diagrams in the path integral calculation is reduced to the correct set of Kontsevich graphs which eventually appear in the star product expansion.

We do this by first calculating the Groenewold-Moyal star product via the path integral approach, where the action is quadratic. We explain the rules for drawing the Feynman diagrams based on the Wick contractions in that theory. Since the action is quadratic, there are no interaction vertices in the Feynman diagrams, or equivalently no internal vertices in the Kontsevich graphs.

As a stepping stone between the path integral calculation of the Groenewold-Moyal star product and the Kontsevich star product for affine Poisson structures, we study a non-associative star product. Whereas the Groenewold-Moyal star product is calculated via a quadratic action in the path integral, the non-associative star product is calculated via a perturbed quadratic action in the path integral. The perturbation to the quadratic action allows for interaction vertices in the Feynman diagrams.

Eventually, we discuss some crucial steps in the calculation of the star product for affine Poisson manifolds. We will explain, using the quantum field theory, how certain directed diagrams are neglected in the expansion of the Kontsevich star product for affine Poisson structure. This will result in selection rules. Specifically, we illustrate calculation of the Kontsevich star product for affine Poisson manifolds up to and including order $\hbar^3$.

We can summarize this thesis, from the perspective of physics, as a quest to finding methods to quantize phase space. Additionally, it is a quest to find physical examples for quantization methods already developed by mathematicians.

From the perspective of mathematics, this thesis can be seen as a quest to deform the commutative algebra of functions on any smooth Poisson manifolds to a deformed non-commutative algebra that is associative. Furthermore, we want to illustrate how the Kontsevich graphs, which are used to express the deformed product and deformed Poisson bracket, are related to Feynman diagrams appearing in path integral methods of quantum field theory.

2 Classical Mechanics

We study the Newtonian classical mechanics and see how over time different formulations of classical mechanics have displayed properties of, understanding of and calculation tools for mechanical systems. Eventually, we arrive at the Hamiltonian formulation and discuss its relation to Poisson brackets.

In the Newtonian setting, we consider a point mass m in Euclidean space $\mathbb{R}^n$.

Definition 2.1 (Newtonian configuration space)**.** In the Newtonian setting the system can obtain states $\mathbf{q}$ in the configuration space $\mathbb{R}^n$.

The state $\mathbf{q} \in \mathbb{R}^n$ is called the position. Under the flow of time, the position can change. Hence, we let $\mathbf{q} = \mathbf{q}(t)$ for time $t \in \mathbb{R}$. The change of position over time, is the velocity $\mathbf{v}(t) := \dot{\mathbf{q}}(t) = \frac{\mathrm{d}\mathbf{q}(t)}{\mathrm{d}t}$. The change of velocity over time is the acceleration $\mathbf{a}(t) := \dot{\mathbf{v}}(t) = \ddot{\mathbf{q}}(t) = \frac{\mathrm{d}^2\mathbf{q}}{\mathrm{d}t^2}$.

The time evolution of the position of a point mass m is described by Newton's second law of motion. Namely, the acceleration of a point mass is

$$\mathbf{a}(t) = \frac{\mathbf{F}(\mathbf{q}(t))}{m},$$

where $\mathbf{F}(t)$ is the external force applied to the particle at time t. The external force on the particle is the resultant force of pushes or pulls of other particles, and of other physical fields (e.g. electro-magnetic fields and external gravitational fields) interacting with the particle at hand. Together with the initial conditions $\mathbf{x}(t = 0) = \mathbf{x}_0$ and $\mathbf{v}(t = 0) = \mathbf{v}_0$, this is a second order differential equation in n variables.

2.1 Lagrangian formalism

From now on, we assume that the force is conservative. An equivalent condition of a force to be conservative is that it can be written as the gradient of a potential $V : \mathbb{R}^n \to \mathbb{R}$. So that, the equations for the force components are

$$F_i = -\frac{\partial V}{\partial q^i}, \tag{1}$$

for $i = 1, 2, 3, \ldots, n$.

Example 2.1 (Conservative force: gravity)**.** A well-known example of a conservative force (field) is gravity. When we look at the classical notion of gravity on Earth, in the Newtonian case, the corresponding potential of a particle of mass m is given by the height function $V = mgh$, where g the gravitational acceleration constant and h the height above the surface of the Earth. In this case, using Equation (1), we obtain the classical equation for the gravitational force $F_g = -\frac{\partial(mgh)}{\partial h} = -mg$.

Lagrange and Euler initiated the foundation for the calculus of variations in the 18th century. Hamilton later showed that the motions of particles could be described, by what we now call the Euler-Lagrange equations, which are also called the equations of motion,

$$\frac{\partial \mathcal{L}(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)}{\partial \dot{q}^i} = 0,$$

where $\mathcal{L}(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)$ is the Lagrangian of the system. The Lagrangian for mechanical systems is expressed as the sum of the kinetic energy minus the potential energy of the system $\mathcal{L} = T - V$. These equations are derived by requiring that the value of the action functional is minimal along a trajectory of motion, which results in the stationary point condition,

$$\delta S(\mathbf{q}(t)) = \int_{t_1}^{t_2} \delta\mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\mathrm{d}t = 0.$$

This formula is nowadays often called the principle of least action or Hamilton's principle.

2.2 Hamiltonian formalism

The Euler-Lagrange equations are still second order differential equations in n variables. Hamilton gives an alternative formulation, where the equations of motion become a set of first order differential equations. An assumption for Hamilton's description is that the system is closed, meaning that $\frac{\partial\mathcal{L}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)}{\partial t} = 0$. In a closed system there is no energy transfer in or out of the system. Then, from the Euler-Lagrange equations, it can be derived that [ThM]

$$\mathcal{L} - \dot{q}^i \frac{\partial\mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial\dot{q}^i} = -H = \text{constant},$$

where H is called the Hamiltonian function. This transformation from a Lagrangian description to a Hamiltonian description is called a Legendre transform.

In the Hamiltonian description, the conjugate momenta of q^i are introduced as

$$p_i = \frac{\partial\mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial\dot{q}^i},$$

so that the Hamiltonian is

$$H(\mathbf{q}, \mathbf{p}) := \sum_{i=1}^{n} \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}).$$

Definition 2.2 (Phase space $\mathcal{P}$)**.** The space which represents the state of a system in the Hamiltonian formalism, is called the phase space $\mathcal{P} = T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(\mathbf{q}, \mathbf{p}) \in \mathcal{P}$.

In this formalism, the equations of motion described by Lagrange are thus transformed into Hamilton's equations of motion, a first order system of differential equations

$$\begin{cases} \dot{q}^i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i}, \end{cases} \tag{2}$$

which is a system in $2n$ variables instead of n.

2.3 Canonical transformations and the canonical Poisson bracket

Lagrange described his system in generalized coordinates. Generalized coordinates have the defining property that a coordinate transformation $\mathbf{q} \mapsto \mathbf{Q}$ does not change the Euler-Lagrange equations. If simultaneously, in the Hamiltonian formalism, the momenta are changed by a Legrendre transform, then also the Hamilton's equations of motion are preserved.

Definition 2.3 (Canonical transformations)**.** A canonical transformation

$$(\mathbf{q}, \mathbf{p}, t) \mapsto (\mathbf{Q}(\mathbf{q}, \mathbf{p}), \mathbf{P}(\mathbf{q}, \mathbf{p}), t)$$

of coordinates in $T^*\mathbb{R}^n$ is defined by preserving the symplectic form $\mathrm{d}q \wedge \mathrm{d}p$, and is given by the generalized coordinate transform of coordinates $\mathbf{q} \mapsto \mathbf{Q}$, with its associated Legrendre transform of the momenta:

$$\mathbf{p_i} = -\frac{\partial H}{\partial q^i} = \frac{\partial L}{\partial q^i} \mapsto \mathbf{P_i} = -\frac{\partial H}{\partial Q^i} = \frac{\partial L}{\partial Q^i}, \tag{3}$$

where H and L are the Hamiltonian and Lagrangian, respectively. In such a way that

Proposition 2.1. *Canonical transformations defined in Definition 2.3 preserve the Hamilton's equation of motion given in Equation 2.*

We see that the equations of motion are preserved under a canonical transformation, because the Hamiltonian is time-independent (and hence the Hamilton's equations of motion apply):

$$\dot{\mathbf{Q}}^{\mathbf{i}} = \frac{\partial \mathbf{Q}^{\mathbf{i}}}{\partial \mathbf{q}^{\mathbf{j}}}\dot{\mathbf{q}}^{\mathbf{j}} + \frac{\partial \mathbf{Q}^{\mathbf{i}}}{\partial \mathbf{p_j}}\dot{\mathbf{p}}_{\mathbf{j}} = \frac{\partial \mathbf{Q}^{\mathbf{i}}}{\partial \mathbf{q}^{\mathbf{j}}}\frac{\partial H}{\partial \mathbf{p_j}} + \frac{\partial \mathbf{Q}^{\mathbf{i}}}{\partial \mathbf{p_j}}\frac{\partial H}{\partial \mathbf{q}^{\mathbf{j}}} = \left\{\mathbf{Q}^{\mathbf{i}}, H\right\}, \tag{4}$$

$$\dot{\mathbf{P}}_{\mathbf{i}} = \frac{\partial \mathbf{P_i}}{\partial \mathbf{q}^{\mathbf{j}}}\dot{\mathbf{q}}^{\mathbf{j}} + \frac{\partial \mathbf{P_i}}{\partial \mathbf{p_j}}\dot{\mathbf{p}}_{\mathbf{j}} = \frac{\partial \mathbf{P_i}}{\partial \mathbf{q}^{\mathbf{j}}}\frac{\partial H}{\partial \mathbf{p_j}} + \frac{\partial \mathbf{Q_i}}{\partial \mathbf{p_j}}\frac{\partial H}{\partial \mathbf{q}^{\mathbf{j}}} = \left\{\mathbf{P_i}, H\right\}, \tag{5}$$

where $\{\cdot, \cdot\}$ is the so-called canonical Poisson bracket on phase space. After we introduce the canonical Poisson bracket on phase space below, we will see that indeed the original Hamilton's equations of motion have the same form as in Equations (4) and (5). And, therefore the canonical transformation have preserved the Hamilton's equations of motion.

Theorem 2.2. *The evolution of a function $f(\mathbf{q}(t), \mathbf{p}(t)) \in C^\infty(\mathcal{P})$ is given by the canonical Poisson bracket of the function with the Hamiltonian*

$$\dot{f}(t) = \{f(\mathbf{q}(t), \mathbf{p}(t)), H(\mathbf{q}(t), \mathbf{p}(t))\} := \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i}. \tag{6}$$

Proof. The proof is given in one line,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{f}{\partial p_i}\dot{p}_i = \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i},$$

where in the last equality we use the Hamilton's equations of motion. □

Clearly, the evolution of the system will now correspond to a trajectory in phase space. Moreover, it can be observed that in this manner the Hamilton's equations in Equation (2) take the same form as the canonically transformed variables in Equation (4):

$$\begin{cases} \dot{q}^i &= \frac{\partial H}{\partial p_i} = \left\{q^i, H\right\}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} = \left\{p_i, H\right\}. \end{cases} \tag{7}$$

Moreover, we can see the evolution of the function $f(t)$, given by the canonical Poisson bracket of the function with the Hamiltonian in Equation (6), as the action of the vector field

$$\mathfrak{X}_H = \{\cdot, H\} = \frac{\partial(\cdot)}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial(\cdot)}{\partial p_i}\frac{\partial H}{\partial q^i} \tag{8}$$

on $f(t)$ In this sense, the integral curves of the Hamiltonian vector field $\mathcal{X}_H$ describes the evolution of the system. For later purposes, we note that the action of the vector field on a function can also be written using the natural pairing between one-forms and vector fields. Namely, the action of the Hamiltonian vector field of the function f on the functions g is given by the pairing of the one-form $\mathrm{d}g$ and the Hamiltonian vector field $\mathcal{X}_f$ as

$$\mathcal{X}_f(g) = \langle \mathrm{d}g, \mathcal{X}_f \rangle := \{g, f\}. \tag{9}$$

The concept of constants of motion motivated the discovery of the Jacobi identity [LPV], and therefore eventually a formalization of the canonical Poisson bracket. Constants of motion are also very important in physics (or quantum physics) in order to have a welldefined set of observables.

Definition 2.4 (Constant of Motion)**.** If the evolution of a function is zero along the Hamiltonian vector field, i.e. $\mathcal{X}_H(f) = \{f, H\} = 0$ and $\frac{\partial f}{\partial t} = 0$, then the function f is a constant of motion.

Poisson observed the following.

Theorem 2.3. *If f and g are constants of motion, then $\{f, g\}$ is also a constant of motion.*

Proof. We start by applying the definition we have above

$$\begin{aligned}
\{\{f,g\},H\} =& \left\{ \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}, H \right\} \\
=& \frac{\partial}{\partial q^j}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}\right)\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}\right) && (10)\\
=& \left\{\left\{ \frac{\partial f}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j}\frac{\partial f}{\partial p_j} \right\}, g\right\} - \left\{\left\{ \frac{\partial g}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j}\frac{\partial g}{\partial p_j j} \right\}, f\right\} && (11)\\
=& \{\{f,H\},g\} - \{\{g,H\},f\} && \text{(By Eq. (8))} \quad (12)\\
=& \{0,g\} - \{0,f\} && (f \text{ and } g \text{ are CoM})\\
=& 0,
\end{aligned}$$

Where the Leibniz rule was applied to go from line (10) to line (11). Moreover, line (12) displays the Jacobi identity if we use anti-symmetry to get $\{f, H\} = -\{H, f\}$ in the first term. □

So indeed, we see explicitly how the Jacobi identity, the Leibniz rule and the skew-symmetry of the Poisson bracket are important in the above proof. With this one can formally define what a Poisson bracket on phase space is.

Definition 2.5 (Canonical Poisson bracket on phase space)**.** The canonical Poisson bracket on phase space is a map $\{\cdot,\cdot\} : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ that satisfies the conditions for a bracket to be a Lie bracket, i.e. for any $f, g, h \in C^\infty(\mathcal{P})$:

1. bilinearity: $\{f, g\}$ is linear w.r.t. f and g,
2. skew-symmetry: $\{f, g\} = -\{g, f\}$,
3. Jacobi identity: $\sum_{\circlearrowright}\{f, \{g, h\}\} = \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$,

together with the Leibniz rules:

$$\begin{aligned}
\{f \cdot g, h\} &= \{f, h\} \cdot g + f \cdot \{g, h\}, \\
\{f, g \cdot h\} &= g \cdot \{f, h\} + \{f, g\} \cdot h.
\end{aligned}$$

Remark 2.1. Lie bracket on phase space.
First of all, note that if a bracket satisfies only the first three conditions above, then this bracket is a Lie bracket, but not a canonical Poisson bracket. On the other hand, a canonical Poisson bracket always also defines a Lie bracket on the phase space.

Remark 2.2. The Leibniz rules are sometimes also called the compatibility conditions. Or it is said that the bracket acts as a bi-derivation on phase space. These three notions are often used interchangeably.

Remark 2.3. The notion of the canonical Poisson bracket will be extended to the notion of a general Poisson bracket on any smooth manifold, and more generally to any associative algebra over any field $\mathbb{F}$ in section 3.

3 Poisson Structures

In section 2, we got familiar with the canonical Poisson bracket on phase space. In this section, we study generalizations of the Poisson bracket on phase space. Moreover, we study how Poisson brackets are described via multi-vector fields. And, we discuss some examples of different types of Poisson structures on phase space, e.g. a linear Poisson bracket on $so(3)^*$ and an affine Poisson bracket on a smooth manifold.

3.1 General Poisson bracket and Poisson algebra

First of all, we note that in Definition 2.5 the Lie bracket and a Lie algebra were already defined on phase space. Here, we give the general definition of a Lie bracket and a Lie algebra.

Definition 3.1 (Lie bracket (see [Lee03])). A Lie bracket on a vector space $\mathcal{A}$ over a field $\mathbb{F}$ is a bilinear map $\{\cdot,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ sending $(a_1, a_2) \mapsto \{a_1, a_2\}$, and satisfying the following conditions:

1. $\{a_1, a_2\} = -\{a_2, a_1\}$ (skew-symmetry),
2. $\{a_1, \{a_2, a_3\}\} + \{a_2, \{a_3, a_1\}\} + \{a_3, \{a_1, a_2\}\} = 0$ (Jacobi identity),

for $a_1, a_2, a_3 \in \mathcal{A}$.
The triple $(\mathcal{A}, \cdot, \{\cdot,\cdot\})$, where $\cdot$ is the commutative product on $\mathcal{A}$, is a Lie algebra.

Then, a Poisson bracket and Poisson algebra are defined in line with the canonical Poisson bracket. In essence, the commutative associative algebra of functions on phase space $C^\infty(\mathcal{P})$ is now replaced by an algebra $\mathcal{A}$ over a field $\mathbb{F}$, that is, a vector space together with a unital commutative associative product on it.

Definition 3.2 (Poisson bracket and Poisson algebra). Let $\mathcal{A}$ be a vector space over a field $\mathbb{F}$ together with two products on it, the commutative associative product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and a Lie bracket $\{\cdot,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. If the following properties are satisfied:

1. $(\mathcal{A}, \cdot)$ is a unital commutative associative algebra over $\mathbb{F}$,
2. $(\mathcal{A}, \{\cdot,\cdot\})$ is a Lie algebra over $\mathbb{F}$,
3. the Lie bracket satisfies the Leibniz rule: $\{a_1, a_2 \cdot a_3\} = a_2\{a_1, a_3\} + \{a_1, a_2\}a_3$ and $\{a_1 \cdot a_2, a_3\} = a_1\{a_2, a_3\} + \{a_1, a_3\}a_2$,

then the triple $(\mathcal{A}, \cdot, \{\cdot,\cdot\})$ is a Poisson algebra, and the Lie bracket is a Poisson bracket on the Poisson algebra.

Remark 3.1. The commutativity property is needed for the Leibniz rule to be satisfied.

Example 3.1. Consider the canonical Poisson bracket on phase space.

1. Let $\cdot : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ be the usual associative multiplication of function on phase space, so that $(C^\infty(\mathcal{P}), \cdot)$ is a commutative associative algebra over the field of real numbers.
2. Let $\{\cdot,\cdot\}$ be the canonical Poisson bracket on phase space, so that $(C^\infty(\mathcal{P}), \{\cdot,\cdot\})$ is a Lie algebra.

3. The canonical Poisson bracket satisfies the Leibniz rules. Indeed, let $f, g, h \in C^\infty(\mathcal{P})$, then

$$\begin{aligned}
\{f, g\cdot h\} =& \frac{\partial f}{\partial q}\frac{\partial (g\cdot h)}{\partial p} - \frac{\partial (g\cdot h)}{\partial q}\frac{\partial f}{\partial p} \\
=& \frac{\partial f}{\partial q}\cdot g\cdot \frac{\partial h}{\partial p} + \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\cdot h - g\cdot \frac{\partial h}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial g}{\partial q}\cdot h\cdot \frac{\partial f}{\partial p} \\
=& g\cdot \frac{\partial f}{\partial q}\frac{\partial h}{\partial p} + \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\cdot h - g\cdot \frac{\partial h}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\cdot h \qquad (13) \\
=& g\cdot\{f,h\} + \{f,g\}\cdot h, \\
\{f\cdot g, h\} =& \frac{\partial (f\cdot g)}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial q}\frac{\partial (f\cdot g)}{\partial p} \\
=& f\cdot \frac{\partial g}{\partial q}\frac{\partial h}{\partial p} + \frac{\partial f}{\partial q}\cdot g\cdot \frac{\partial h}{\partial p} - \frac{\partial h}{\partial q}\cdot f\cdot \frac{\partial g}{\partial p} - \frac{\partial h}{\partial q}\frac{\partial f}{\partial p}\cdot g \\
=& f\cdot \frac{\partial g}{\partial q}\frac{\partial h}{\partial p} + \frac{\partial f}{\partial q}\frac{\partial h}{\partial p}\cdot g - f\cdot \frac{\partial h}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial h}{\partial q}\frac{\partial f}{\partial p}\cdot g \qquad (14) \\
=& f\cdot\{g,h\} + \{f,h\}\cdot g,
\end{aligned}$$

where it is explicitly seen that we use the commutativity of functions (and derivatives of them) in lines (13) and (14) of the proof.

Then, the triple $(C^\infty(\mathcal{P}), \cdot, \{\cdot,\cdot\})$, is a Poisson algebra.

Example 3.2. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and $\mathfrak{g}^*$ its dual. So that, if $e^i \in \mathfrak{g}$ are basis vectors of the Lie algebra, then $\hat{e}_i \in \mathfrak{g}^*$ acts as linear maps on the elements in $\mathfrak{g}$ such that $\hat{e}_i(e^j) = \delta_i^j$.

1. Let $\cdot : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$ be the commutative associative product on $\mathfrak{g}^*$, so that $(C^\infty(\mathfrak{g}^*), \cdot)$ is a commutative associative algebra over the field of real numbers.

2. Let $\{\cdot,\cdot\}$ be the induced (from $\mathfrak{g}$) Lie bracket on $\mathfrak{g}^*$, i.e.

$$\{\hat{e}_i, \hat{e}_j\} = a_{ij}^k \hat{e}_k, \qquad (15)$$

where a_{ij}^k the structure constants of the Lie algebra $\mathfrak{g}$. The derivation of the above equation is explained in more detail section 3.4.2. With this induced Lie bracket, $(C^\infty(\mathfrak{g}^*), \{\cdot,\cdot\})$ is a Lie algebra over the field of real numbers.

3. The Lie bracket satisfies the Leibniz rule. This can be shown in an exact similar as is done for the canonical Poisson bracket.

Thus, the triple $(C^\infty(\mathfrak{g}^*), \cdot, \{\cdot,\cdot\})$ is a Poisson algebra.

3.2 Poisson manifold and symplectic manifold

We want to get familiar with the notions Poisson manifold and symplectic manifold.

Let $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$ be the symplectic two-form on phase space $\mathcal{P} = T^*\mathbb{R}^n$, where we recall that repeated indices are summed over as in Einstein's summation convention. Let $\mathfrak{X}_f$ and $\mathfrak{X}_g$ be two Hamiltonian vector fields of f and g, respectively. Then the canonical Poisson bracket on phase space can be written in terms of the action of ω on the functions f and g:

$$\{f, g\} = \langle \omega, \mathfrak{X}_f \wedge \mathfrak{X}_g\rangle = \omega\left(\mathfrak{X}_f, \mathfrak{X}_g\right), \qquad (16)$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between two-forms and bi-vector fields (see section 3.3 for more details).

Definition 3.3 (Symplectic form)**.** A symplectic form ω is a closed two-form, i.e. $\mathrm{d}\omega = 0$, which is non-degenerate.

Symplectic two-forms can also be defined on other manifolds than on phase space.

Definition 3.4 (Symplectic manifold)**.** A symplectic manifold (M, ω) is a manifold M equipped with a symplectic form ω on it.

Example 3.3. Phase space $\mathcal{P} \simeq \mathbb{R}^{2n}$ is a symplectic manifold when we equip it with the canonical symplectic two-form $\tilde{\omega} = \mathrm{d}q^i \wedge \mathrm{d}p_i$.

There are also Poisson structures that are not defined through symplectic two-forms. In section 3.3, we see how non-symplectic Poisson structures are defined through a Poisson bi-vector field P. Let us define what a Poisson manifold is.

Definition 3.5 (Poisson manifold)**.** A Poisson manifold (M, P) is a manifold M equipped with a Poisson structure P on it.

Poisson manifolds and symplectic manifolds are related.

Proposition 3.1. *Every symplectic manifold* (M, ω) *is a Poisson manifold* (M, P)*.*

For a proof see [RouDu, LPV].This ends the description of Poisson structures via symplectic two-forms. In the next subsections, we study Poisson structures defined through bi-vector fields.

3.3 Poisson bi-vectors

In this section, we describe Poisson brackets by their related Poisson bi-vector fields. A Poisson bi-vector field acts on functions, for example f and g, in a way that a partial differential operator acts on two functions f and g. In the context of deformation quantization, star products play an important role, and can be described as a formal sum of partial differential operators. Thus, the description of Poisson bi-vectors fields helps to get intuition for the action of a star product on functions.

Instead of the canonical coordinates $(\mathbf{q}, \mathbf{p})$ on phase space $\mathbb{R}^{2n}$, consider local coordinates $\{x^i\}_{i=1}^{\infty}$ on any smooth n-dimensional manifold M. We extend the idea of the natural coupling between a one-form and a one-vector field, as in Equation (9) on page 14, to a coupling between a two-forms with bi-vector fields in Equation (20).

First of all, we will express the Poisson bracket via a Poisson bi-vector field. Let P be a bi-vector field on the manifold M defined in local coordinates as

$$P = \frac{1}{2} P^{ij}(\mathbf{x}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad i, j = 1, 2, \cdots n, \tag{17}$$

where $P(\mathbf{x})$ are smooth functions on M, called the Poisson coefficients. We can consider $\Gamma(\bigwedge^2 TM)$ to be the space of sections on the second exterior power of the tangent bundle $\bigwedge^2 TM$ via the map $P : M \to \Lambda^2 TM : \mathbf{x} \mapsto P(\mathbf{x})$.

The Poisson coefficients satisfy, because they are Poisson, the anti-symmetry property

$$P^{ij}(\mathbf{x}) = \{x^i, x^j\} = -\{x^j, x^i\} = -P^{ji}, \tag{18}$$

and the Jacobi identity for Poisson brackets translates into a Jacobi identity for the coefficients

$$\sum_{\circlearrowleft (ijk)} P^{hi}\partial_h P^{jk} = 0. \tag{19}$$

The Poisson bracket corresponding to the Poisson bi-vector field P, acting on two functions $f, g \in C^\infty(M)$, is given by the natural pairing between the bi-vector field P and a two-form $\mathrm{d}f \otimes \mathrm{d}g$ as

$$\{f, g\} = P(\mathrm{d}f, \mathrm{d}g) = \langle P, \mathrm{d}f \otimes \mathrm{d}g\rangle\,. \tag{20}$$

Thus, the bi-vector field may be called a Poisson bi-vector field.

Remark 3.2. Often we classify Poisson brackets on a manifold M as constant, linear, affine, polynomial or something else. Implicitly, we then refer to the Poisson coefficients $P^{ij}(\mathbf{x}) \in C^\infty(M)$.
This is similar to when we consider a Poisson bracket or structure on a manifold, we actually mean that we consider a manifold and a Poisson bracket on the algebra associated with that manifold.

Definition 3.6 (Poisson Manifold (M, P))**.** A Poisson manifold (M, P) is a manifold M equipped with a Poisson structure P on it.

3.4 Three examples of Poisson structures

Below we discuss three different examples of Poisson structures on different manifolds. In every example, we emphasize some of the properties of the Poisson structure. These three examples will also be studied in further detail when we study the deformation of the corresponding Poisson algebras.

3.4.1 Constant Poisson structure on $\mathbb{R}^{2n}$

Example 3.4 (Constant Poisson structure on $\mathbb{R}^{2n}$)**.** Let the manifold be $M = \mathbb{R}^2$. We consider the constant Poisson structure, where the Poisson structure coefficients are given by the Poisson matrix

$$P^{ij}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{21}$$

so that the Poisson bi-vector field is

$$P(\mathbf{x}) = \frac{1}{2}\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^1}\right). \tag{22}$$

Hence, the Poisson bracket on functions $f, g \in C^\infty(\mathbb{R}^2)$ is

$$\{f, g\} = \langle P, \mathrm{d}f \otimes \mathrm{d}g\rangle = \frac{\partial f}{\partial x^1}\frac{\partial g}{\partial x^2} - \frac{\partial f}{\partial x^2}\frac{\partial g}{\partial x^1}, \tag{23}$$

which we recognize as the canonical Poisson bracket on phase space $\mathcal{P} \simeq \mathbb{R}^2$. The Poisson structure coefficients do satisfy the Jacobi identity in Equation (19), because all terms are separately zero: $\partial_h P^{jk} = \partial(constant) = 0$.

3.4.2 Poisson bracket on the dual of a Lie algebra

The next example is a Poisson bracket on the dual of a Lie algebra. To introduce the example properly, we first introduce a bit of theory about Poisson structures on the dual of Lie algebras, which is well explained in [RouDu].

Recall that if $\mathfrak{g}$ is a real n-dimensional Lie algebra and its dual vector space is $\mathfrak{g}^*$, then $\mathfrak{g}^*$ is isomorphic as a manifold to $\mathbb{R}^n$.
A bracket can be defined on $C^\infty(\mathfrak{g}^*)$ after we have made the identification $\mathfrak{g} \simeq \mathfrak{g}^{**} \subset C^\infty(\mathfrak{g}^*)$ via $X(\in \mathfrak{g}) \mapsto F_X(\in \mathfrak{g}^{**})$, where $F_X : \mathfrak{g}^* \to \mathbb{R} : \xi \mapsto \langle \xi, X \rangle = \xi(X)$. This map has the property that $\{F_X, F_Y\} = F_{[X,Y]}$, where $\{\cdot, \cdot\}$ is the Lie bracket on $C^\infty(\mathfrak{g}^*)$ and $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{g}$. We let $\{e_1, \cdots, e_n\}$ be a basis for $\mathfrak{g}$, then the Lie bracket on the Lie algebra $\mathfrak{g}$ acts as follows

$$[e_i, e_j] = a_{ij}^k e_k, \tag{24}$$

where in any literature the tensorial constants a_{ij}^k are often called the structure constants of the Lie algebra. Let $\{F_1, \cdots, F_n\}$ be the dual basis of $\mathfrak{g}^*$, and let coordinates on $\mathfrak{g}^*$ be denoted by $\{X_1, \cdots, X_n\}$, that is equivalent to the following relations

$$\xi = X_i(\xi) F_i, \tag{25}$$

$$X_i(F_j) = \delta_{ij}, \tag{26}$$

$$X_i = F_{e_i}. \tag{27}$$

So that we have the following Lie bracket on $\mathfrak{g}^*$

$$\{X_i, X_j\} = F_{[e_i, e_j]} = a_{ij}^k F_{e_k} = a_{ij}^k X_k. \tag{28}$$

Hence, the Poisson bracket on the dual of the Lie algebra can be written in coordinates like

$$\{f, g\} = a_{ij}^k X_k \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j}. \tag{29}$$

We now apply the theory to the example of the dual of the Lie algebra of $so(3)$

Example 3.5 (Linear (Lie-)Poisson structure on $so(3)^*$)**.** Consider the Lie algebra $so(3)$ of skew-symmetric 3×3 matrices with real entries. This Lie algebra is a three-dimensional vector space with the following basis

$$e_{xy} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{zx} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_{yz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{30}$$

These matrices generate the infinitesimal rotations in the three-dimensional vector space. So, they are the basis of the Lie algebra of the Lie group $SO(3)$. The Lie bracket on the Lie algebra $so(3)$ is defined through the action on the generators $\{e_i\}_{i=1}^n$ of the algebra

$$\{e_i, e_j\} = \varepsilon_{ij}^k e_k, \tag{31}$$

where ε_{ij}^k is the Levi-Civita symbol. The indices of the Levi-Civita symbol can be raised and lowered using the Killing form mentioned below, so that the symbol ε_{ijk} is the fully anti-symmetric tensor in the indices i, j, k.

There is a way to identify $so(3)$ with its dual space $so(3)^*$ as explained in the theory in the beginning of this section. Namely, we can define a Killing form[2] $\kappa : so(3) \times so(3) \to \mathbb{R} : (X, Y) \mapsto \mathrm{tr}\ (XY)$, which is a symmetric, bi-linear and non-degenerate map[3], and the $tr(\cdot)$ is the trace of a matrix. Then, elements in $so(3)$ and $so(3)^*$ are identified by sending $X \in so(3)$ to $\kappa(X, \cdot) \in so(3)^*$.

[2] Killing forms are used for other purposes as well, but here we only use it to identify $so(3)$ and $so(3)^*$.
[3] Recall that a map $f : X \times X \to Y$ on a finite space X is non-degenerate if and only if it has a trivial kernel.

With the theory, we now know that the Lie-Poisson structure can be expressed by letting the Lie-Poisson bracket act on two elements from the dual basis of $F, G \in \mathfrak{g}^*$:

$$\{F, G\} = \varepsilon_k^{ij} x^k \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j}, \tag{32}$$

where $x_i \in C^\infty(so(3)^*)$ (like in Equation (29) in the theory above).

The corresponding Poisson bi-vector field is

$$\begin{align}
P &= P^{ij}(\mathbf{x}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{33}\\
&= \varepsilon_k^{ij} x^k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{34}\\
&= x^3 \left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^1} \right) - x^2 \left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1} \right) + \tag{35}\\
&\quad x^1 \left(\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^2} \right), \tag{36}
\end{align}$$

which corresponds to the matrix representation of the Poisson structure

$$P^{ij}(\mathbf{x}) = \begin{bmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{bmatrix}. \tag{37}$$

It is a good exercise to verify that the Jacobi identity holds, playing with the Levi-Civita symbols.

3.4.3 Affine Poisson structure $P^{ij}(\mathbf{x}) = a_k^{ij} x^k + b^{ij}$ on a smooth manifold M

Example 3.6 (Affine Poisson structure on a smooth manifold M)**.** Let M be an n-dimensional smooth manifold, and x^i local coordinate function for $i = 1,\ 2,\ \cdots,\ n$. Then, a Poisson structure is affine when

$$P^{ij}(\mathbf{x}) = a_k^{ij} x^k + b^{ij}. \tag{38}$$

By the properties of $P^{ij}(\mathbf{x})$ being Poisson, the following restrictions on a_k^{ij} and b^{ij} hold when we equate the constant terms and the terms of order x^k

$$\begin{aligned}
a_k^{ij} + a_k^{ji} &= 0 \quad \text{(anti-symmetry)},\\
b^{ij} + b^{ji} &= 0 \quad \text{(anti-symmetry)},\\
a_l^{ij} a_m^{lk} + a_l^{jk} a_m^{li} + a_l^{ki} a_m^{lj} &= 0 \quad \text{(Jacobi identity)},\\
a_l^{ij} b^{lk} + a_l^{jk} b^{li} + a_l^{ki} b^{lj} &= 0 \quad \text{(Jacobi identity)},
\end{aligned}$$

for fixed $i, j, k, m \in \{1, 2, \cdots, n\}$. The corresponding bi-vector field is

$$P(\mathbb{x}) = \left(a_k^{ij} x^k + b^{ij} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

An important property of affine Poisson structures, is that any second order derivative of it vanishes, i.e.

$$\frac{\partial^2}{\partial x^l \partial x^m} \left(a_k^{ij} x^k + b^{ij} \right) = 0,$$

for all l, m. We will see later that this property allows for an easier deformation procedure for affine Poisson structures than for general Poisson structures when using a path integral approach.

There are multiple physical systems that correspond to affine Poisson structures. In the Poisson structure, the structure constants a_k^{ij} are the Lie algebra structure constants and the constant $b^{ij}-$term corresponds to a Chevalley cocycle with respect to that Lie algebra, which is well-known [AK, KW]. The affine construction is seen as an extension of the Lie algebra structure.

Khesin [AK, KW] together with Arnold, and later with Wendt, wrote about natural sources and physical realizations of systems with an affine Poisson structure. Those systems are all classical, especially the heavy top. The role of having a non-zero b in the Poisson structure, can be compared with switching on gravity. For example, with b zero, we have a the linear part of the Poisson structure describes a gyroscope or a top in free fall, but with b non-zero gravity is 'switched on' so that we have a heavy top. In the book by Khesin and Arnold together [AK], they essentially discuss infinite dimenisional examples. That book covers fluid motion as the geodesic motion on an infinite-dimensional group of diffeomorphisms, which can also be described using affine Poisson structures. For more details, please have a look at [AK, KW].

4 Quantization: from functions on phase space to quantum operators on Hilbert space

In this section we study ‘quantization,’ which can have different meanings in different contexts. We discuss the ambiguity of the quantization step, in the sense of going from a classical to a quantum theory. And, we see how deformation quantization has been studied as an alternative to geometric quantization, i.e. the quantizations based on Dirac’s canonical quantization.

We first introduce the quantum mechanics on the Hilbert space of wave functions. Then, we look at the idea of canonical quantization presented by Dirac. Dirac’s quantization turns out not to be complete. Hence, an extra choice has to be made in order to define quantization, we have to choose an ordering prescription. This leads to Weyl quantization and other types of phase space quantizations which all are extensions of Dirac’s canonical quantization. We study the Wigner distribution function as an example of the associated distribution function.

Whereas phase space quantization assumes Dirac’s quantization as a starting point, Groenewold, in 1946, noted a bigger problem that applied to all quantizations based on Dirac’s prescription [Groe][4]. We recall the two problems he studied concerning elementary quantum mechanics. The first concerned quantization: the correspondence $a \leftrightarrow \hat{a}$ between physical quantities a and quantum operators $\hat{a}$. The second problem concerned the interpretation of quantum mechanics: can the statistical character of quantum mechanics be understood by averaging over uniquely determined processes as in classical statistical mechanics? Whereas these two problems are related, in this Chapter we focus on his findings concerning quantization. In the later rest of the thesis, we will (without explicitly mentioning it) be more concerned with the second problem by looking at Feynman’s techniques.

His correspondence principle states that there cannot exists a phase space quantization (based on Dirac’s canonical quantization) such that all functions can be mapped by an invertible map to Hermitian operators in the Hilbert space of wave functions so that the Poisson bracket is preserved. This is now known as the Groenewold-van Hove Theorem [Groe, VH]. As a remedy to this problem, he tried to find a deformed version of the canonical Poisson bracket, so that the quantized version of the deformed bracket would equal the Dirac bracket of the quantized functions. In the process of finding this bracket, he found that a deformed product between functions on phase space would give the desired deformed version of the canonical Poisson bracket. The deformed product is commonly called the Moyal star product, but historically it is better to call it the Groenewold-Moyal star product [BoeWaa]. Similarly, we will call the associated deformed canonical Poisson bracket on phase space the Groenewold-Moyal bracket.

Inspired by Groenewold and Moyal, Bayen et al. in 1978 suggested to study quantization in the sense of deformation quantization, with the idea of opening up new views on the process of quantization and on methods for developing new quantum (field) theories.

We used the second and third chapter of Robert Wezeman’s bachelor thesis [Wez] and the lecture notes of Lein [Lei] on Weyl quantization and semi-classics, to write the subsections on quantum mechanics, Dirac’s quantization, phase space

[4] He published it shortly after the war, because he did not want it to be released under the German occupation [Groenewold to von Neumann, 1945].

quantization and distribution functions.

4.1 Quantum mechanics: Hilbert space, quantum observables and time evolution

In this small section we highlight the basic differences between classical mechanics and quantum mechanics that are relevant for the rest of the thesis, namely the description of a state, the observables, and the time evolution. For those not too familiar with the Schrödinger and Heisenberg descriptions, i.e. wave function and matrix, of quantum mechanics, we refer to any standard book in quantum mechanics or also sufficient: [Lei, Wez].

In the Hamiltonian formalism for classical mechanics, we describe states through coordinates and momenta $(\mathbf{q}, \mathbf{p})$ in phase space. In contrast, in quantum mechanics, we describe states by wave functions on Hilbert space.

Definition 4.1 (Hilbert space of wave functions)**.** The Hilbert space of wave functions $\mathcal{H}$ is a complete inner product space of square integrable functions with respect to the Lebesgue measure

$$\mathcal{H} = L^2(\mathbb{R}) = \left\{ \phi : \mathbb{R} = M \ni q \to \mathbb{C} \; s.t. \int |\phi(q)|^2 \mathrm{d}q < \infty \right\}, \tag{39}$$

where the inner product between between wave functions $\phi, \psi \in \mathcal{H}$ is defined by

$$\langle \phi, \psi \rangle := \int \phi(q)^* \psi(q) \mathrm{d}q. \tag{40}$$

where $\phi(q)^*$ is the complex conjugate of $\phi(q)$.

Notation 4.1 (Dirac's bra- and ket-vectors). Dirac introduced the notation $|\phi\rangle$, which is called the ket-vector, for a vector ϕ in Hilbert space. The dual of the ket-vector $|\phi\rangle$ is a bra-vector in $\mathcal{H}^*$, denoted by $\langle\phi|$. The bra-vector is also called a co-vector.

In classical mechanics, observables are real-valued functions on phase space $f \in C^\infty(\mathcal{P})$. In contrast, in quantum mechanics, the observables corresponds to Hermitian operators on the Hilbert space of wave functions. Functions form a commutative algebra, whereas Hermitian operators form a non-commutative algebra.

Definition 4.2 (Quantum operator)**.** A quantum operator $\hat{A}$ is a linear map from the domain $\mathcal{D}$ of $\hat{A}$ to the Hilbert space of wave functions

$$\hat{A} : \mathcal{D}(\hat{A}) \subset \mathcal{H} \to \mathcal{H} \tag{41}$$

Definition 4.3 (Quantum observable)**.** A quantum observable is a physical quantity of a quantum state that can be measured. A measurement of an observable A corresponds to the action of a Hermitian quantum operator $\hat{A}$ on a quantum state $|\phi\rangle$. The possible measurement outcomes are eigenvalues of the respective Hermitian quantum operator.

The only things we can observe about a quantum state through a measurement are the quantum observables. However, measurements of a state do influence the state itself. Heisenberg expressed this for the relation between the position and momentum in his so-called uncertainty relation

$$\Delta q \Delta p \geq \frac{1}{2}\hbar. \tag{42}$$

The derivation of this uncertainty relation is usually a standard mathematical derivation in undergraduate quantum mechanics course using a time-independent description of the wave function. Nevertheless, it is not just a mathematical result. The uncertainty relation displays that one cannot simultaneously know all quantum observables of a state. Usually, we therefore search for the largest set of Hermitian quantum operators that mutually commute, so that we have the largest set of quantum observables we can know simultaneously (in the sense that the measurements with those qantum operators do not mutually influence the outcomes) about a quantum state.

In classical mechanics the evolution of a state was governed by the Hamiltonian and the associated Hamiltonian vector fields. In quantum mechanics, the time evolution is governed by the Hamiltonian operator $\hat{H}$. This is expressed through the Schrödinger equation for wave functions

$$\hat{H}\phi(t) = i\hbar\frac{\partial}{\partial t}\phi(t), \tag{43}$$

or through the evolution of operators:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t) = \frac{i}{\hbar}[\hat{H}, \hat{A}], \tag{44}$$

where $[\cdot, \cdot]$ is the anti-commuting bracket of operators. When we compare Equation (44) with the evolution of a function in the Hamiltonian formalism $\dot{f}(t) = \{f, H\}$, the immediate intuitive way to relate classical to the quantum mechanics, is to replace the Poisson bracket by the anti-commuting bracket (times $\frac{i}{\hbar}$). We will see in the next subsection that this intuitive relation is one of the core ideas of quantization.

4.2 Dirac's quantization

In 1930 Dirac famously proposed [D] a way to go from the commutative algebra of functions on phase space to the non-commutative algebra of operators on Hilbert space.

Definition 4.4 (Dirac's canonical quantization)**.** Let q^i, p_j be coordinate functions on phase space $T^*\mathbb{R}^n$ and let ϕ be a wave function in Hilbert space $\mathcal{H}$, then Dirac's prescription for quantization [D] is: replace q^i by the multiplication by q^i, i.e.

$$(\hat{q}^i\phi)(q) = q^i\phi, \tag{45}$$

and the momentum p_i is replaced by the derivative with respect to q^i

$$(\hat{p}_i\phi)(q) = -i\hbar\frac{\mathrm{d}}{\mathrm{d}q^i}\phi(q). \tag{46}$$

Furthermore, the canonical Poisson bracket on phase space is replaced by the canonical commutation bracket between quantum operators, i.e.

$$\widehat{\{q^i, p_j\}} = \frac{1}{i\hbar}\left[\hat{q}^i, \hat{p}_j\right] = \delta^i_j. \tag{47}$$

Dirac's quantization was a first attempt to quantize functions on phase space. However, it has two problems. The more general first problem has to do with the quantization of functions consisting of products of q and p. The second problem, which we will discuss in subsection 4.3, has to do with the non-existence of quantizations (based on Dirac's canonical quantizaiton) that have a well-defined quantization of the Poisson bracket between any functions on phase space.

In particular, Dirac's quantization is only well-defined for functions that are linear combinations (with coefficients) of powers of either q or p, but not with terms containing products of q and p. For example, Dirac's prescription works for a Hamiltonian function $H(q,p) = \frac{p^2}{2m} + V(q)$, giving $\hat{H}(\hat{q},\hat{p}) = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{q})$. Instead, when we quantize functions containing terms $q \cdot p$, there are in principle multiple different orderings of operators possible, like [Wez]:

$$\hat{q} \cdot \hat{p} \qquad \hat{p} \cdot \hat{q} = \hat{q} \cdot \hat{p} - i\hbar\mathbb{1} \qquad \frac{1}{2}(\hat{q} \cdot \hat{p} + \hat{p} \cdot \hat{q}) \tag{48}$$

$$\text{standard ordered} \qquad \text{anti-standard ordered} \qquad \text{symmetric ordered} \tag{49}$$

The problem of ordering arises because operators are not commuting, whereas functions are commuting (cf. Equation (47)).

So, Dirac's quantization does not have a well-defined prescription for the quantization of all phase space functions. Other quantization have extended Dirac's ideas by adding an ordering prescription, and leaving out the necessity to quantize the Poisson bracket.

Definition 4.5 (Quantization)**.** A quantization $\mathcal{Q}$ is a linear map from the space of smooth functions on phase space to a space of operators that act on the Hilbert space $\mathcal{H}$ of wave functions, i.e. $\mathcal{Q} : f \mapsto \mathcal{Q}(f)$. The quantization $\mathcal{Q}$ satisfies properties (45) and (46) of Dirac's canonical quantization, and additionally satisfies

1. $\mathcal{Q}(1) = \hat{\mathbb{1}}$,
2. $\mathcal{Q}(\overline{f}) = \mathcal{Q}(f)^*$,

where $\overline{f}$ is the complex conjugate of f.

Remark 4.1. From now on we will use the notations for the quantized version of x, namely $\mathcal{Q}(x)$ and $\hat{x}$ interchangeably in the rest of this thesis.

Remark 4.2 (Space of operators on the Hilbert space of wave functions $\mathcal{H}$). In the definition the space of operators on the Hilbert space of wave functions is not specified. It depends on what type of quantization we consider, what the properties of the operator on the Hilbert space of wave functions are. For example, the Weyl quantization will map every smooth function to a bounded operator on the Hilbert space of wave functions.

In the section 4.3, we discuss the big no-go theorem for any quantization that additionally wants to quantize the Poisson bracket to the anti-commutator. After that section, we discuss the Wigner-Weyl quantization, and similar quantizations, that, in contrast to Dirac's quantization, establish a one-to-one relation between the functions on phase space and operators on Hilbert space.

4.3 Groenewold-van Hove Theorem

Hip Groenewold in 1946 studied the relation between physical quantities and quantum operators as well as the statistical character of elementary quantum mechanics [BoeWaa, Groe]. He found the remarkable result that no quantization, as defined in Definition 4.5, can establish a one-to-one correspondence between the algebra of functions on phase space and the operators on Hilbert space. That result is known as the Groenewold-van Hove Theorem. Before we formulate this theorem, we note that there are different phrasings of the theorem and that there are stronger and weaker versions of the theorem, for details see [Got96, Got99]. We will now formulate the stronger version.

Theorem 4.1 (Groenewold-van Hove Theorem)**.** *There is no consistent quantization $\mathcal{Q}$ (in the sense of Definition 4.5 that also satisfies Equation* (47) *on page 25) of the Poisson algebra of polynomials in coordinates q^i and p_i on phase space $\mathcal{P} \simeq \mathbb{R}^{2d}$. Or, equivalently, there is no quantization such that the quantization of the Poisson bracket:*

$$\mathcal{Q}\left(\{f,g\}\right) = \frac{1}{i\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)] \tag{50}$$

is defined consistently for all polynomial functions $f, g \in C^\infty(P)$.

Proof (Counterexample). We will here discuss the original counterexample to a consistent quantization that was given by Groenewold in case of $\mathbb{R}^2 \simeq \mathcal{P} \ni (q,p)$ (The higher-dimensional cases follow directly from the counterexample in $\mathbb{R}^2$). First of all, Groenewold, noted that the following relation always holds on phase space:

$$\{q^3, p^3\} + \frac{1}{12}\{\{p^2, q^3\}, \{q^2, p^3\}\} = 0. \tag{51}$$

Secondly, note that any quantization will, independent of a choice of ordering, have the following rules of quantization:

$$\mathcal{Q}(q) = \hat{q}, \mathcal{Q}(p) = \hat{p}, \mathcal{Q}(q^2) = \hat{q}^2, \mathcal{Q}(p^2) = \hat{p}^2, \mathcal{Q}(q^3) = \hat{q}^3, \mathcal{Q}(p^3) = \hat{p}^3. \tag{52}$$

Using these relations together with Equation (50), Groenewold observed that the quantized version of the left-hand side of Equation (51) does not equal the quantizated version of the right-hand side, i.e. does not equal $\mathcal{Q}(0) = 0$. Explicitly, this means that

$$\mathcal{Q}\left(\{q^3, p^3\}\right) = \frac{1}{i\hbar}\left[\hat{q}^3, \hat{p}^3\right] \tag{53}$$

$$\mathcal{Q}\left(\frac{1}{12}\{\{p^2, q^3\}, \{q^2, p^3\}\}\right) = \frac{1}{12i\hbar}\left[\mathcal{Q}\left(\{p^2, q^3\}\right), \mathcal{Q}\left(\{q^2, p^3\}\right)\right] \tag{54}$$

$$= \frac{1}{12i\hbar}\left[\frac{1}{i\hbar}[\hat{p}^2, \hat{q}^3], \frac{1}{i\hbar}[\hat{q}^2, \hat{p}^3]\right], \tag{55}$$

where in lines (55) and (54) we used the quantization rules from Equations (50) and (52). On the other hand we verify that

$$\frac{1}{i\hbar}\left[\hat{q}^3, \hat{p}^3\right] + \frac{1}{12i\hbar}\left[\frac{1}{i\hbar}\left[\hat{p}^2, \hat{q}^3\right], \frac{1}{i\hbar}\left[\hat{q}^2, \hat{p}^3\right]\right] = -3\hbar^2. \tag{56}$$

Thus, any consistent quantization in polynomials of solely q or solely p, as in Equation (52), gives an inconsistency in the quantization of the Poisson bracket of high enough degree polynomials. Therefore, there can never exist a fully consistent quantization of any Poisson algebra of (high degree) polynomials on $\mathbb{R}^{2d}$. □

Groenewold thus observed that any quantization of the Poisson bracket of functions on phase space $f, g \in C^\infty(\mathbb{R}^{\mathcal{P}})$ is only precise up to first order in $\hbar$

$$\mathcal{Q}\left(\{f,g\}\right) = [\hat{f}, \hat{g}] + O(\hbar^2). \tag{57}$$

In contrast to quantizations that want to quantize the Poisson bracket to the anti-commutator, the Wigner-Weyl quantization establishes a one-to-one relation between functions on phase space and operators on Hilbert space.

4.4 Wigner-Weyl quantization: quantum mechanics in phase space

In this subsection, we first discuss a type of quantization that satisfies Definition 4.5, namely the Weyl quantization. The Weyl quantization is not only well-defined: through the Wigner-Weyl transforms it establishes a one-to-one relation between the functions on phase space and operators on Hilbert space. Therefore, Weyl quantization allows us to describe quantum mechanics on phase space: with a density operator of a pure state on Hilbert space, we associate a Wigner distribution on phase space.

In addition to the Weyl quantization there are multiple different types of quantizations that establish a one-to-one relation between the functions on phase space and the operators on Hilbert space. All those other quantizations also have their respective phase space distributions associated to density operators of pure states, just like the Wigner distribution. In Wezeman's thesis [Wez] the Wigner distribution and the Husimi distribution are compared as examples of phase space distributions. Moreover, Lee [Lee95] wrote a review of the theory and applications of quantum phase space distributions, to which we would refer every interested reader.

Phase space distributions are quasi-probability distributions. This means that they are similar to probability distributions, but they do not satisfy all the axioms to be a probability distribution. For example, the Wigner distribution can attain negative values, but satisfies all other axioms of being a probability distribution. In contrast, the Husimi distribution does not attain negative values, but has the wrong marginal distribution functions. This means in case for the Husimi density function ρ_H that

$$\int \mathrm{d}p\rho_H(q,p) \neq \langle q|\hat{\rho_H}|q\rangle \,, \tag{58}$$

$$\int \mathrm{d}q\rho_H(q,p) \neq \langle p|\hat{\rho_H}|p\rangle \,. \tag{59}$$

Due to the fact that no phase space distribution satisfies all axioms to be a probability distribution, phase space distributions are not unique. We could say that phase space distributions do no satisfy all axioms, because they somehow need to incoroporate Heisenberg's uncertainty relation.

Weyl quantization is one of the best known phase space quantizations based on Dirac's principles. Weyl quantization uses the Weyl transform to map Schwartz functions on phase space to bounded operators on the Hilbert space of wave functions $\mathcal{B}(\mathcal{H})$. We first define the space of Schwartz functions and the space of linear bounded operators.

Definition 4.6 (Schwartz space)**.** The space of Schwartz functions on phase space $\mathcal{P}$ consists of all functions whose semi-norm is bounded [Lei]:

$$\mathcal{S}(\mathcal{P}) = \Big\{ f \in C^\infty(\mathcal{P}) \mid \forall n, m \in \mathbb{N} \;:\; \|f\|_{n,m} < \infty \Big\} \,, \tag{60}$$

where the semi-norm is defined by

$$\|f\|_{n,m} \;:=\; \sup_{x\in\mathcal{P}} |x^n \partial_x^m f(x)| \,, \tag{61}$$

where the powers x^l are defined through a local chart $U \subset \mathbb{R}^d$ for phase space $\mathcal{P}$ by

$$x^l = \left([x_1, \ldots, x_d]^T\right)^l = [x_1^l, \ldots, x_d^l]^T, \tag{62}$$

where the superscript T indicates the transpose of the row vector.

Definition 4.7 (Space of linear bounded operators)**.** The space of all linear bounded operators between two normed spaces X and Y is denoted by $\mathcal{B}(X, Y)$. A linear operator $\hat{A}\colon X \to Y$ is bounded if there exists an $M \geq 0$ such that for all $x \in X$ the following condition is satisfied [Lei]:

$$\left\|\hat{A}x\right\|_Y \leq M\|x\|_X. \tag{63}$$

Then, the Weyl transform is defined as follows.

Definition 4.8 (Weyl transform)**.** The Weyl transform is a map W from the space of Schwarz functions $\mathcal{S}(\mathcal{P})$ to the space of bounded operators on the Hilbert space of wave functions $\mathcal{B}(\mathcal{H})$ given by

$$\hat{A}(\hat{q}, \hat{p}) = W[A] = \left(\frac{1}{2\hbar\pi}\right)^2 \iiiint A(q,p) e^{\frac{i}{\hbar}(\xi(\hat{q}-q)+\eta(\hat{p}-p))} \mathrm{d}q \mathrm{d}p \mathrm{d}\xi \mathrm{d}\eta. \tag{64}$$

The Weyl transform is a one-to-one map and has an inverse transform: the Wigner transform.

Definition 4.9 (The Wigner Transform)**.** The Wigner transform W^{-1} is a map from the space of bounded operators on the Hilbert space of wave functions $\mathcal{B}(\mathcal{H})$ to the space of Schwarz functions on phase space $\mathcal{S}(\mathcal{P})$, and is given by

$$A(q,p) = W^{-1}[\hat{A}](q,p) = \int e^{-i\frac{py}{\hbar}} \left\langle q + \frac{y}{2}\right| \hat{A}(\hat{q}, \hat{p}) \left| q - \frac{y}{2}\right\rangle \mathrm{d}y.$$

If the operator $\hat{A}$ is a density operator of a pure state, i.e. $\hat{A} = |\psi\rangle\langle\psi|$, then the Wigner transform of that state (up to some normalization) is called the Wigner distribution of $\hat{A}$ [BoeWaa], and is given by

$$W(q,p) = \frac{1}{2\pi\hbar} \int e^{-i\frac{py}{\hbar}} \psi^*(q - \frac{y}{2})\psi(q + \frac{y}{2})\mathrm{d}y. \tag{65}$$

The Wigner distribution is bi-linear, real-valued, has the correct marginal distribution functions, but can attain negative values [Lee95, Wez].

4.4.1 Expectation value and ordering

It would be desirable if the expectation value of an operator $\hat{A}$, which we previously calculated in section 4.1, acting on a quantum state $\hat{\rho}$ (for example the density operator of a pure state) could be calculated via a corresponding density function $F(q,p,t)$ on phase space. As we have seen in the previous section, there are multiple density functions $F(q,p,t)$ corresponding to a quantum operator, e.g. the Wigner and the Husimi distribution. Hence, we can naively relate the expectation value of a state $\hat{\rho}$ when acted upon by $\hat{A}$, expressed as a trace, to an integral over a corresponding density function $F(q,p,t)$ by [Lee95]

$$\mathrm{Tr}\left\{\hat{\rho}(\hat{q}, \hat{p}, t)\hat{A}(\hat{q}, \hat{p})\right\} = \iint A(q,p)F(q,p,t)\mathrm{d}q\mathrm{d}p, \tag{66}$$

where we have now naively related $A \leftrightarrow \hat{A}$ by replacing $\hat{q}$ by q and $\hat{p}$ by p, and the trace is defined as

$$\mathrm{Tr}\Big\{\hat{\rho}\hat{A}\Big\} = \int \mathrm{d}q \, \langle q | \, \hat{\rho}\hat{A} \, | q \rangle \, . \tag{67}$$

From section 4.2, it is immediately clear that the probability distribution $F(q,p,t)$ is not unique unless one makes a certain choice ordering.

Lee points out that

$$e^{i\xi\hat{q}+i\eta\hat{p}} = e^{i\xi\hat{q}}e^{i\eta\hat{p}}e^{i\hbar\frac{\xi\eta}{2}}, \tag{68}$$

because of the non-commutativity of the operators. If we replace $\hat{A}$ in Equation (66) by the left-hand side of Equation (68), then we see how the ordering prescription could be implemented by choosing a specific $F(q,p,t)$. Lee formulates a class of quantum phase space distributions by the following Equation,

$$\mathrm{Tr}[\hat{\rho}(\hat{q},\hat{p},t)e^{i\xi\hat{q}+i\eta\hat{p}}f(\xi,\eta)] = \iint e^{i\xi q+i\eta p}F^f(q,p,t)\mathrm{d}q\mathrm{d}p, \tag{69}$$

where the function $f(\xi,\eta)$ determines which rule of association is used (see Equation (70) and Example 4.1 below). This can be transformed [Lee95] to get the formula for the density function

$$F^f(q,p) = \frac{1}{4\pi^2}\iiint f(\xi,\eta)e^{i(q'-q)\xi-ip\eta}\left\langle q'+\frac{\hbar}{2}\eta\right|\hat{\rho}\left|q'-\frac{\hbar}{2}\eta\right\rangle \mathrm{d}q'\mathrm{d}p\mathrm{d}\eta. \tag{70}$$

Let us see the above ideas of Lee in action for the example of the Wigner distribution function, from Equation (65).

Example 4.1 (Weyl quantization and the Wigner distribution function)**.** The Weyl quantization discussed previously, see Definition 4.8, yields the simplest choice of function, namely $f(\xi,\eta) = 1$ in Equation (69). The Weyl rule of association then is $e^{i\xi q+i\eta p} \leftrightarrow e^{i\xi\hat{q}+i\eta\hat{p}}$. From Equation (70), we get the corresponding distribution function

$$\begin{aligned} F^W(q,p,t) &= \frac{1}{4\pi^2}\iint e^{i\xi q-i\eta p}\mathrm{Tr}[\hat{\rho}(\hat{q},\hat{p},t)e^{i\xi\hat{q}+i\eta\hat{p}}]\mathrm{d}\xi\mathrm{d}\eta \\ &= \frac{1}{2\pi}\int e^{-i\eta p}\left\langle q+\frac{\hbar}{2}\eta\right|\hat{\rho}\left|q-\frac{\hbar}{2}\eta\right\rangle \mathrm{d}\eta \\ &= \frac{1}{2\hbar\pi}\int e^{-i\frac{yp}{\hbar}}\left\langle q+\frac{y}{2}\right|\hat{\rho}\left|q-\frac{y}{2}\right\rangle \mathrm{d}y, \end{aligned}$$

where in the last step we substituted $\eta = \frac{y}{\hbar}$, so that we recognize the Wigner distribution function.

Lee gives a very insightful overview table in his review report of quantum phase space distribution functions, where he lists six most seen distribution functions, with their rules of association [Lee95, p.154]. Further on he lists multiple properties of those distribution functions, namely whether or not they are bi-linear, real-valued, non-negative, have the correct marginal distributions and are complete.

The message we want to take away from his review is that there are many different rules of association, together with their distribution functions. The distributions all differ a little bit from each on the properties mentioned above. Also, different distribution functions turn out to be useful in calculations of different applications, such as optics, radiation fields in coherent state representations, calculation in the Morse potential and a particle in an infinite square well and even more, see [Lee95].

So, we see that every type of quantization which establishes a one-to-one relation between phase space functions and Hilbert space operators allows for an associated description of the quantum system in terms of a phase space distribution. Every quantization thus comes with its own phase space distribution, a quasi-probability distribution, which has its own advantages and disadvantages (in terms of applications). These (dis-)advantages are all caused by the fact that we go from a commutative to a non-commutative algebra.

4.5 Groenewold-Moyal bracket

Based on ideas formulated in the Groenewold-van Hove theorem and the one-to-one relation established in Wigner-Weyl quantization, Groenewold proposed an alternative to face the problem of quantizing the Poisson bracket. He proposed to deform the commutative multiplication on phase space, and thus replaces that multiplication by the Groenewold-Moyal star product, so that the deformed Poisson bracket could be defined, i.e. the Groenewold-Moyal bracket. We will see that the Groenewold-Moyal bracket is such that the Wigner-Weyl quantization can consistently relate the Groenewold-Moyal bracket to the anti-commutator.

Definition 4.10 (Groenewold-Moyal star product)**.** Let $C^\infty(\mathcal{P})[[\hbar]]$ be the algebra of formal power series over $\mathbb{R}[[\hbar]]$ with coefficients in $C^\infty(\mathcal{P})$, where $\mathbb{R}[[\hbar]]$ is the ring of formal power series in $\hbar$. Then, the Groenewold-Moyal star product $\star_{GM}$: $C^\infty(\mathcal{P})[[\hbar]] \times C^\infty(\mathcal{P})[[\hbar]] \to C^\infty(\mathcal{P})[[\hbar]]$ is defined by

$$f(\mathbf{q},\mathbf{p}) \star_{GM} g(\mathbf{q},\mathbf{p}) = f(\mathbf{q},\mathbf{p}) \exp\left(i\frac{\hbar}{2}\left(\overleftarrow{\partial}_{q^i} \overrightarrow{\partial}_{p_i} + \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{q^i} \right)\right) g(\mathbf{q},\mathbf{p}) \tag{71}$$

as the deformation of the usual commutative product of functions on phase space.

The Groenewold-Moyal star product is non-commutative, but it is associative on every symplectic manifold.

Theorem 4.2 (Associativity of the Groenewold-Moyal star product)**.** *The Groenewold-Moyal star product is associative on every symplectic manifold, thus*

$$(f(\mathbf{q},\mathbf{p}) \star_{GM} g(\mathbf{q},\mathbf{p})) \star_{GM} h(\mathbf{q},\mathbf{p}) = f(\mathbf{q},\mathbf{p}) \star_{GM} (g(\mathbf{q},\mathbf{p}) \star_{GM} h(\mathbf{q},\mathbf{p}))\,. \tag{72}$$

Proof. For convenience, we introduce some slightly different notation. Let us denote our coordinates $(\mathbf{q},\mathbf{p}) = \mathbf{x}$, such that $x^i = q^i$ and $x^{n+i} = p_i$. Then, we can rewrite the star product as

$$\begin{aligned} f(\mathbf{x}) \star_{GM} g(\mathbf{x}) &= f(\mathbf{x}) \exp\left(i\frac{\hbar}{2}\left(\overleftarrow{\partial}_{q^i} \overrightarrow{\partial}_{p_i} + \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{q^i} \right)\right) g(\mathbf{x}) & (73)\\ &= \exp\left(\hbar\omega^{ij} \frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(\mathbf{x})g(\mathbf{y})\bigg|_{\mathbf{x}=\mathbf{y}}\,, & (74) \end{aligned}$$

where ω^{ij} is the coordinate function of the symplectic form that takes into account the term $\frac{i}{2}$. Also, note that

$$\frac{\partial}{\partial x^i}\left(h(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{y}}\right) = \left(\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right) h(\mathbf{x},\mathbf{y})\right)\bigg|_{\mathbf{x}=\mathbf{y}} \tag{75}$$

Then, with the introduction of the shorthand notation

$$E\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=\exp\left(\omega^{ij}\frac{\partial}{\partial x^{i}}\frac{\partial}{\partial y^{i}}\right),\tag{76}$$

the star product looks like

$$(f\star_{GM}g)(\mathbf{x})=\left(E\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\cdot(f(\mathbf{x})g(\mathbf{y}))\right)\Big|_{\mathbf{x}=\mathbf{y}}\tag{77}$$

Then, we eventually calculate

$$\begin{aligned}((f\star_{GM}g)\star_{GM}h)(\mathbf{x})&=\left[E\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right)\cdot\left(\left(E\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\cdot(f(\mathbf{x})g(\mathbf{y}))\right)\Big|_{\mathbf{x}=\mathbf{y}}h(\mathbf{z})\right)\right]\Big|_{\mathbf{x}=\mathbf{z}}\\&=\left[E\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot E\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\cdot f(\mathbf{x})g(\mathbf{y})h(\mathbf{z})\right]\Big|_{\mathbf{x}=\mathbf{y}=\mathbf{z}}\\&=\left[E\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right)\cdot E\left(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot E\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\cdot f(\mathbf{x})g(\mathbf{y})h(\mathbf{z})\right]\Big|_{\mathbf{x}=\mathbf{y}=\mathbf{z}},\end{aligned}\tag{78}$$

where we used Equation (75) from line one to line two, and an algebraic rule for exponents from line two to line three. We could compute the $(f\star_{GM}(g\star_{GM}h))(\mathbf{x})$ in a similar fashion to arrive at exactly the same result. Hence, we have shown the associativity of the Moyal star product. □

Remark 4.3. The proof of associativity of the Groenewold-Moyal star product has also been done in an interesting geometrical way in [Zac]. The ideas presented there are also applicable to a broader scope of star products.

Definition 4.11 (Groenewold-Moyal bracket)**.** The Groenewold-Moyal bracket is defined as a map $\{\cdot,\cdot\}:C^{\infty}(\mathcal{P})[[\hbar]]\times C^{\infty}(\mathcal{P})[[\hbar]]\to C^{\infty}(\mathcal{P})[[\hbar]]$ by

$$\{f(\mathbf{q},\mathbf{p}),g(\mathbf{q},\mathbf{p})\}_{GM}:=\frac{1}{i\hbar}\left(f\star_{GM}g-g\star_{GM}f\right)(\mathbf{q},\mathbf{p})\tag{79}$$

$$=\frac{2}{\hbar}f(\mathbf{q},\mathbf{p})\sin\left(\frac{\hbar}{2}\left(\overleftarrow{\partial}_{q^i}\overrightarrow{\partial}_{p_i}-\overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{q^i}\right)\right)g(\mathbf{q},\mathbf{p}).\tag{80}$$

The Groenewold-Moyal bracket reduces to the canonical Poisson bracket on phase in the limit that $\hbar\to0$, i.e.

$$\begin{aligned}\{f,g\}_{GM}&=\frac{1}{i\hbar}\left(f\star_{GM}g-g\star_{GM}f\right)\\&=\frac{1}{i\hbar}\left((fg-gf)+\hbar\{f,g\}_{Poisson}+O(\hbar^2)\right)\\&=\{f,g\}_{Poisson}+O(\hbar).\end{aligned}\tag{81}$$

The Groenewold-Moyal bracket solves a problem stated in the Groenewold-van Hove Theorem. Namely, the quantization of the Groenewold-Moyal bracket is well-defined for all functions on phase space.

Proposition 4.3 (Equivalence of the Groenewold-Moyal bracket and the anti–commutator)**.** *The Groenewold-Moyal bracket is such that the diagram in Figure 2 is commutative.*

In Figure 2, we see how the Groenewold-Moyal bracket and the Dirac bracket form a commutative diagram together with the Wigner-Weyl transforms. If, in Figure 2, the Groenewold-Moyal bracket is replaced by the Poisson bracket, the diagram is not commutative for arbitrary functions f and g.

$$\begin{tikzcd}
f, g \in C^\infty(\mathcal{P})[[\hbar]] \arrow[r, shift left, "W"] \arrow[d, "{\{\cdot,\cdot\}_{GM}}"'] & W(f), W(g) \in \mathcal{B}(\mathcal{H})[[\hbar]] \arrow[l, shift left, "W^{-1}"] \arrow[d, "{[\cdot,\cdot]}"] \\
\{f, g\}_{GM} \in C^\infty(\mathcal{P}[[\hbar]]) \arrow[r, shift left, "W"] & W\left(\{f, g\}_{GM}\right) = [W(f), W(g)] \in \mathcal{B}(\mathcal{H})[[\hbar]] \arrow[l, shift left, "W^{-1}"]
\end{tikzcd}$$

Figure 2: Commutative diagram relating the Groenewold-Moyal bracket, the anti-commutator and the Wigner-Weyl transforms.

The diagram also clarifies the role of the Wigner-Weyl transforms. Groenewold observed that the Wigner-Weyl transforms give a one-to-one relation between functions on phase space and bounded operators on Hilbert space. Without the existence of these transforms, there would have been no way to construct such a diagram.

Note that this diagram also allows to relate the time evolution in phase space to the time evolution in Hilbert space. Replace respectively g and $W(g)$ by the Hamiltonian function H and the Hamiltonian operator $W(H)$, then the bottom two positions in the diagram will represent the time evolution in phase space and in Hilbert space respectively (where the factor of $\frac{1}{i\hbar}$ is absorbed in the quantization step).

An interesting question about the relation between the Poisson bracket and the Moyal bracket and their respective Weyl transforms was posed by Bayen et al. [BFFLS]. As discussed above, we know that the Weyl transform of the Poisson bracket is not equal to the Dirac bracket on Hilbert space, but what is it then equal to? Stated differently, what happens to the ring of formal power series expansions of operators in $\hbar$ when we apply the macroscopic limit $\hbar \to 0$? In Figure 3, what should be at the place of the question mark? Or viewed differently, what should be the Weyl transform of the classical Poisson algebra of dynamical functions on phase space? We want that by application of the macroscopic limit the diagram becomes commutative.

$$\begin{tikzcd}
\{f, g\} \in C^\infty(\mathcal{P}) \arrow[r, shift left, "W"] \arrow[d, "\text{Deformation (in } \hbar)"'] & ? \arrow[l, shift left, "W^{-1}"] \\
\{f, g\}_{GM} \in C^\infty(\mathcal{P}[[\hbar]]) \arrow[r, shift left, "W"] & W\left(\{f, g\}_{GM}\right) = [W(f), W(g)] \in \mathcal{B}(\mathcal{H})[[\hbar]] \arrow[l, shift left, "W^{-1}"] \arrow[u, dotted, "\text{Macroscopic limit } \hbar \to 0"']
\end{tikzcd}$$

Figure 3: Possible commutative diagram: what happens to the ring of bounded operators on Hilbert space in the macroscopic limit? What should the Weyl transform of the Poisson bracket be equal to?

So far, there is no guarantee that there should always be an answer to the question mark '?'. However, Bayen et al. note that certain types of quantum optics in the coherent state formalism can be described by what should be at the place of the question mark. We refer the reader interested in coherent state formalism in relation to quantum optics to [BFFLS, BM, HZ].

5 Deformation Quantization

In this Chapter, we first give a historical overview of developments in deformation quantization up to Kontsevich with many references to literature. Then, in the second subsection, we summarize the main differences between geometric quantization and deformation quantization. These two sections require some specific mathematical background, like (De Rham) cohomology theory, they might be less relevant for those without such a background. However, in these subsections, we do not go in to details important for later parts of the thesis. Therefore, they can be skipped without any problem.

In subsection 3, we discuss how one can go from formal deformations of algebras to the Kontsevich star product and therefore what the Kontsevich oriented graphs are.

5.1 Historical overview of developments in deformation quantization up to Kontsevich

As mentioned in the introduction, the seminal papers by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer [BFFLS] strongly motivated the study of deformation quantization: the study of deforming the commutative algebra of classical observables instead of the study of quantum operators on Hilbert space. Their expectation was that deformation quantization could contribute to the development of new methods for quantum (field) theories.

In the paper itself Bayen et al. used Gerstenhaber's theory of deformation to study quantum mechanics in the sense of a deformed classical mechanics. They proved that for every generic symplectic manifold (M, ω) admitting a flat connection there exists a star product deforming the corresponding algebra of functions on the symplectic manifold[5].

And they discuss examples where they calculate energy and angular momentum spectra, for example for the harmonic oscillator and the hydrogen atom. Their idea for deformation quantization and its potential is, for example, explicated in what they say about the Moyal star product and bracket:

> Many problems of quantum physics have already been translated into the Moyal idiom. Our point of view is different: we want to make the "classical" formulation autonomous in order to open up a vast field of generalizations with all kinds of interesting applications.

In 1983, De Wilde and Lecomte proved that on every generic symplectic manifold, not necessarily with a flat connection, there exists a star product. They used cohomological arguments[6]. They continue the work already done by Vey, Neroslavsky and Vlasov. Vey [Vey] proved that on symplectic manifolds with a trivial third De Rham cohomology group there exists a non-trivial deformation of the Poisson-Lie algebra. And, Neroslavsky and Vlasov [NV] had extended the proof of Vey to associative deformations. The existence of star product was proven independently by Omori, Maeda and Yoshika and also in the article by Fedosov [Fed94][7], itself a re-print of his results from mid-1980s.

Quickly after De Wilde and Lecomte, several authors contributed to classifying equivalent star product. Gerstenhaber had already been studying deformations

[5] A connection on a fibre bundle is flat if its curvature is zero.
[6] For more background about (De Rham) cohomology, see for example [DNF].
[7] One might want to read Fedosov's book for more background [Fed96].

of rings and algebras with a very mathematical approach in 1964 [Ger]. Bayen et al. also claimed that the classification of star products had to do with the second de Rham cohomology of a symplectic manifold [BFFLS]. Multiple different approaches are given by Nest and Tsygan [NT], and subsequently by Deligne [Del] and Bertelson-Cahen-Gutt [BCG].

A breakthrough contribution to the most general case has been given by Kontsevich. He first conjectured [Kont97] and later proved [Kont03] that for any finite dimensional Poisson manifold there exist star products, and also gave their classifications. Some work on parametrizations of specific star products have been obtained by Karabegov [Kar], Bertelson, Bieliavsky and Gutt [BBG], and Chloup [Chl]. Karabegov studied parametrizations of star products with separation of variables, Bertelson, Bieliavsky and Gutt studied invariant star products on manifolds with an invariant symplectic connection, whereas Chloup focused on star products on the algebra of polynomials on the dual of a semi-simple Lie algebra.

For now we stop the historical overview of deformation quantization more or less up to Kontsevich. In section 7.3, we discuss deformation quantization after Kontsevich which includes a path integral approach to deformation quantization introduced by Cattaneo and Felder [CF].

5.2 Differences between geometric quantization and deformation quantization

After a historical overview, we now want to discuss the main differences between geometric quantization and deformation quantization, similar to how it is discussed in [CKTB].

We want to remark that the approach of deformation quantization is fundamentally different from Dirac's quantization and similars. More clearly, the quantizations (apart from Groenewold-Moyal) that we have studied so far can be put in the realm of geometric quantization, which is not equal to deformation quantization. The main aim of geometric quantization is to construct a Hilbert space with an algebra of operators on it. It depends for most of its constructions on the symmetries of phase space. In its general form, it can be applied to phase spaces that are a coadjoint orbit of a Lie group. Kirillov introduced the orbit method for representations of Lie groups, which is a purely mathematical avatar of geometric quantization[8]. Within the area of phase spaces that represent coadjoint orbits of a Lie group, geometric quantization has obtained great results [Kir]. However, this approach to quantization has neither enlightened our view on quantum field theory nor has it been succesfully applied to general relativity. The core problem of geometric quantization, as we saw also in the Groenewold-van Hove theorem, in general only a small amount of classical observables can be consistently quantized by it (see section 4.3).

Deformation quantization, similar to geometric quantization, takes a symplectic manifold, or more generally, a Poisson manifold as input. However, the output is not a Hilbert space with a non-commutative associative algebra whose elements are viewed as operators on Hilbert space, but an algebra which is viewed as a formal one-parameter deformation (often the parameter is (proportional to) the well-known Planck's constant $\hbar$) of the algebra of smooth functions on the original Poisson manifold, i.e. it is a deformed Poisson algebra.

[8] We refer to Kirillov's lecture notes for those interested in the orbit method [Kir].

5.3 Deformation quantization: from formal deformations to the Kontsevich star product

In this section, we discuss the formulation of deformation quantization in a more general case than only on phase space. We keep in mind that the Groenewold-Moyal star product is a particular example of this class of deformations: namely a deformation of symplectic manifolds. We start by defining a formal deformation, and subsequently we will see how the properties of the Groenewold-Moyal deformation quantization are extended to obtain the Kontsevich star product for deformations of Poisson algebras. For this section, we used the theory presented in [CKTB] and [Esp].

5.3.1 Formal deformations and equivalence

First of all, we discuss what a formal deformation is. The building blocks of a formal deformation are an algebra $\mathcal{A}$ over the commutative ring $\Bbbk$, so that $(\mathcal{A}, \cdot)$ is an algebra over $\Bbbk$. Also, a ring $\Bbbk[[t]]$ of formal power series in t with coefficients in $\Bbbk$. Lastly, consider the algebra $A[[t]]$ of formal power series over $\Bbbk[[t]]$ with coefficients in $\mathcal{A}$. Elements $a, b \in A[[t]]$ can be written as

$$a = \sum_{n=0}^{\infty} a_n t^n, \qquad b = \sum_{n=0}^{\infty} b_n t^n, \tag{82}$$

where $a_n, b_n \in \mathcal{A}$. Therefore, their formal multiplication is given by

$$a \cdot b = \sum_{n=0}^{\infty} c_n t^n, \quad c_n = \sum_{k=0}^{\infty} a_k b_{n-k}. \tag{83}$$

Definition 5.1 (Formal deformation of a commutative associative unital algebra $(\mathcal{A}, \cdot)$ over a commutative ring $\Bbbk$)**.** A formal deformation of the multiplication $\cdot$ of $\mathcal{A}$ is a $\Bbbk[[t]]-$bilinear map

$$\star : A[[t]] \times A[[t]] \to A[[t]], \tag{84}$$

where we recall that $A[[t]]$ is the algebra of formal power series over $\Bbbk[[t]]$ with coefficients in $\mathcal{A}$. The map $\star$ reduces to the commutative associative product on $\mathcal{A}$ when we calculate $\mod t$:

$$a \star b = a \cdot b \bmod t, \tag{85}$$

for all $a, b \in A[[t]]$, and with $a \cdot b$ as defined in Equation (83). Moreover, it is extended by $\Bbbk[[t]]-$linearity.

Thus, the formal deformation of the algebra $(\mathcal{A}, \cdot)$ is the algebra $(A[[t]], \star)$.

To define a deformation of a Poisson algebra, we have to specify how the deformed multiplication, i.e. the star product, is defined. And, we have to say what kind of restrictions it should satisfy. The building blocks of the star product are a sequence of bilinear maps

$$B_n : (a, b) \mapsto B_n(a, b), \tag{86}$$

for $a, b \in \mathcal{A}$, where we define $B_0(a, b) = a \cdot b$, with $\cdot$ the commutative associative multiplication on $\mathcal{A}$. The star product is first defined on $\mathcal{A}$: the star product of two elements $a, b \in \mathcal{A}$ is

$$\begin{aligned} a \star b &= \sum_{n=0}^{\infty} B_n(a,b)t^n \\ &= a \cdot b + B_1(a,b)t + B_2(a,b)t^2 + O(t^3). \end{aligned}$$

By $\Bbbk[[t]]-$bilinearity the star product can be extended to $A[[t]]$ by

$$\left(\sum_{k=0}^{\infty} f_k t^k\right) \star \left(\sum_{l=0}^{\infty} g_l t^l\right) = \sum_{n=0}^{\infty} \left(\sum_{k+l+m=n} B_m(f_k, g_l)\right) t^n, \tag{87}$$

where $f_k, g_l \in \mathcal{A}$. An equivalent statement will be that

$$\star = \sum_{n=0}^{\infty} B_n t^n. \tag{88}$$

Consider an element g in the group G of $\Bbbk[[t]]-$module automorphisms of $A[[t]]$, which satisfies the following relation

$$g(a) = a \pmod{tA[[t]]} \tag{89}$$

for all $a \in A[[t]]$. For example, $g(a+bt+ct^2) = a+O(t)$. Observe that $g : A[[t]] \to A[[t]]$ is determined by linear maps $g_i : \mathcal{A} \to \mathcal{A}$. Namely, for all $g \in G$ and all $a \in \mathcal{A}$, we have that

$$g(a) = \sum_{n=0}^{\infty} g_n(a)t^n = a + g_1(a)t + O(t^2). \tag{90}$$

The equivalence of two star products is defined as follows.

Definition 5.2 (Equivalent star products)**.** Two star products $\star$ and $\star'$ on $A[[t]]$ are equivalent if there exists an element $g \in G$, such that

$$g(u \star v) = g(u) \star' g(v) \tag{91}$$

for all $u, v \in A[[t]]$.

5.3.2 From Groenewold-Moyal to Kontsevich

The Groenewold-Moyal star product was associative and the Groenewold-Moyal bracket was a Poisson bracket, we also want these properties for more general deformation quantizations. We will see that requiring the star product to be associative is crucial for two consequences it has (See Lemma 5.1 and Theorem 5.2).

Definition 5.3 (Associative star product on the ring of formal power series in t over A)**.** A star product $\star$ on $A[[t]]$ is associative if for three elements $a, b, c \in \mathcal{A}$, we have

$$\sum_{i+j=n} B_i\left(B_j(a,b), c\right) = \sum_{i+j=n} B_i\left(a, B_j(b,c)\right). \tag{92}$$

Associativity of $\star$ is extended to $A[[t]]$ by $k[[t]]-$bilinearity.

Remark 5.1. The mechanism for associativity of the star product on $A[[t]]$ is not as easy as it might seem on first sight. The associativity mechanism of the Kontsevich star product (via the graphical representation of the bi-linear operators) is explained in much detail in [BurKis19]. The core idea is that identities that hold for the Poisson structure translate to associativity of the star product at all orders.

The Groenewold-Moyal bracket, defined through its respective star product, had the property that the first term in the expansion equalled the Poisson bracket. This property is nicely carried over to the general case. When we define the deformed bracket for general associative star products, it turns out that the term at zeroth order in $\hbar$ is also a Poisson bracket. Moreover, we will see that the Poisson bracket will only depend on the equivalence class of the star product.

Explicitly, we define the deformed bracket, between two elements $a, b \in A[[t]]$ as

$$
\begin{aligned}
[a,b]_\star =& \frac{1}{t}\left(a \star b - b \star a\right) && (93)\\
=& \frac{1}{t}(a \cdot b - b \cdot a) + (B_1(a,b) - B_1(b,a)) + O(t). && (94)
\end{aligned}
$$

Lemma 5.1. *If the star product $\star$ on $A[[t]]$ is associative, then the deformed bracket $[\cdot,\cdot]_\star : A[[t]] \times A[[t]] \to A[[t]] : (a,b) \mapsto$ is a Lie bracket on $A[[t]]$.*

Proof. Bi-linearity and skew-symmetry are seen trivially. The bracket also satisfies the Jacobi identity, due to the fact that the star product is associative. □

Remark 5.2. The deformed bracket $[\cdot,\cdot]_\star$ is not only a Lie bracket, but even a Poisson bracket. We do not need that property below, but one can check that the bracket satisfies the Leibniz rule.

Theorem 5.2. *Let $\star$ be an associative star product on $\mathcal{A}$. Then, the lowest order term in the deformed bracket $[\cdot,\cdot]_\star$ on $A[[t]]$, defined by $\{a,b\} = B_1(a,b) - B_1(b,a)$, will be*

1. *a Poisson bracket on $\mathcal{A}$,*
2. *and will only depend on the equivalence class of $\star$.*

Proof. 1. Since $\{\cdot,\cdot\}$ is the reduction modulo t of $[\cdot,\cdot]_\star$, by Lemma 5.1 also $\{\cdot,\cdot\}$ is a Lie bracket. Moreover, since

$$
\begin{aligned}
[a, b\cdot c]_\star &= a \star (b \cdot c) - (b \cdot c) \star a\\
&= b(a \star c) + (a \star b)c - (b \star a)c - b(c \star a)\\
&= b \cdot [a,c]_\star + [a,b]_\star \cdot c,
\end{aligned}
$$

also the Leibniz rule is satisfied for $\{\cdot,\cdot\}$.

2. Let $\star$ and $\star'$ be two equivalent star products, such that $g(a \star b) = g(a) \star' g(b)$ for $u, v \in A[[t]]$, and $g \in G$. Then the following relation holds for all $a, b \in \mathcal{A}$:

$$
B_1(a,b) + g_1(ab) = B_1'(a,b) + g_1(a)b + g_1(b). \tag{95}
$$

The difference $B_1(a,b) - B_1(a,b)$ is symmetric in a and b and therefore it does not contribute to $\{\cdot,\cdot\}$. Hence, the Poisson bracket $\{\cdot,\cdot\}$ depends only on the equivalence class of $\star$.

□

We have seen the construction of formal deformations of algebras above. Also, that when the star product is associative, the deformed bracket, at first order, is equal to a Poisson structure [Vey]. This was formulated, in reverse, by Kontsevich [Kont03].

Theorem 5.3. *Let $\mathcal{A}$ be any algebra of smooth functions on a finite-dimensional differentiable manifold, then each Poisson bracket on $\mathcal{A}$ lifts to an associative formal deformation on the respective deformed algebra $A[[t]]$.*

This can be applied to the two-dimensional phase space $\mathcal{P} \simeq \mathbb{R}^2$, with the canonical Poisson bracket. The canonical Poisson bracket lifts to the Groenewold-Moyal star product, as we have seen before. In section 3, we discuss a linear Poisson bracket on the dual of the Lie algebra and affine Poisson brackets on smooth manifolds. The respective lifts will be discussed later.

Remark 5.3. Kontsevich [Kont03] remarks that the operators B_i in general are bi-differential operators with complex coefficients. This is expected as we usually associate self-adjoint operators acting on Hilbert space to real-valued classical observables on phase space in the process of quantization.

Furthermore, he notes that it is not yet fully known how deformation quantization of Poisson algebras is related to the physical interpretation. It is certain that for symplectic manifolds, the natural counterpart of the deformed algebra is quantum mechanics. For more general Poisson algebras, Kontsevich suggests that a topological open string theory would probably give a better interpretation (than quantum mechanics as usual).

5.3.3 Kontsevich star product: graphical representation and explicit formula

Kontsevich [Kont03] gives an explicit formula for the star product, as a lift of a Poisson bracket, for arbitrary Poisson structure coefficients $P^{ij}(\mathbf{x})$ on a non-empty open set of $\mathbb{R}^d$. Every term in the formula is given by the action of a poly-differential operator on Poisson structure coefficients $P^{ij}(\mathbf{x})$ and on functions $f, g \in C^\infty(\mathbb{R}^d)$. The operators can be represented by directed graphs. When the directed graphs are correctly embedded in the hyperbolic upperhalf plane, the weights of terms in the star product are given by formulas based on the angles between edges of the directed graphs. The weights of the Kontsevich star product are invariant under affine transformations of the Kontsevich graphs that are embedded in the hyperbolic upperhalf plane.

To describe the poly-differential operators, we describe the ordered labeled graphs G_n introduced by Kontsevich.

Definition 5.4 (An oriented graph Γ)**.** An (oriented) graph Γ is a pair (V_Γ, E_Γ) of two finite sets such that E_Γ is a subset of $V_\Gamma \times V_\Gamma$.

The vertices of the graph Γ belong to the set V_Γ, whereas the edges of Γ belong to E_Γ. An edge $e \in E_\Gamma$ is denoted by the vertex that it leaves v_1 and the vertex that it lands upon v_2, i.e. $e = (v_1, v_2)$. The graphs that represent the polydifferential operators in the Kontsevich formula are all finite and do not contain multiple edges.

Definition 5.5 (Ordered finite directed labeled Kontsevich graphs)**.** A labeled graph Γ belongs to G_n, the set of ordered finite directed labeled Kontsevich graphs, if it satisfies the following properties

1. Γ has $n + 2$ vertices and $2n$ edges,
2. the set of vertices V_Γ is $\{1, 2, \ldots, n\} \cup \{L, R\}$, where L, R are just two symbols that mean Left and Right,

3. edges of Γ are labeled by symbols $e_1^1, e_1^2, e_2^1, e_2^2, \ldots, e_n^1, e_n^2$,
4. for every $k \in \{1, \ldots, n\}$ edges labeled by e_k^1 and e_k^2 start at the vertex k,
5. for any $v \in V_\Gamma$ the ordered pair (v, v) is not an edge of Γ, i.e. we have no tadpoles.

The set G_n is finite, with $(n(n+1))^n$ elements for $n \geq 1$ and 1 element for $n = 0$.

Remark 5.4 (Physical interpretation of the absence of tadpoles in G_n). In section 7.3, we will describe the Kontsevich formula via a perturbative path integral method which is used in quantum field theory. The quantum field theoretical interpretation of the graphs will give a physical interpretation why graphs with tadpoles, see point 5 of Definition 5.5, do not appear in the Kontsevich formula.

A labeled graph $\Gamma \in G_n$ represents a bi-differential operator on the functions in the algebra $\mathcal{A}$:

$$B_{\Gamma,\alpha} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \qquad \mathcal{A} = C^\infty(U), \;\; U \text{ is an open domain in } \mathbb{R}^d. \tag{96}$$

The bi-differential operator $B_{\Gamma,\alpha}$ does not require the bi-vector field α to be Poisson.

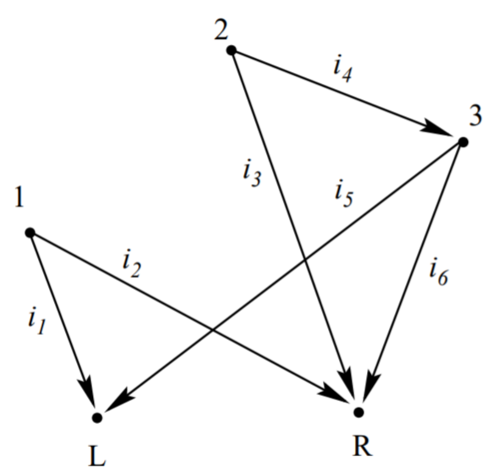


Figure 4: Example of a Kontsevich graph, from [Kont03].

In Figure 4, we give an example of such a graph Γ. In the Figure, $n = 3$ and the list of edges is

$$\left(e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2\right) = ((1, L), (1, R), (2, R), (2, 3), (3, L), (3, R)). \tag{97}$$

The edges in the Figure are decorated with indices $1 \leq i_\alpha \leq d$ instead of the labels e_*^*. The graph in the example represents the bi-differential operator

$$B_{\Gamma,\alpha} : (f, g) \mapsto \sum_{i_1,\ldots,i_6} \alpha^{i_1 i_2} \alpha^{i_3 i_4} \partial_{i_4} \left(\alpha^{i_5 i_6}\right) \partial_{i_1} \partial_{i_5}(f) \partial_{i_2} \partial_{i_3} \partial_{i_6}(g). \tag{98}$$

For any $\Gamma \in G_n$, its respective operator $B_{\Gamma,\alpha}$ is given by the general formula

$$B_{\Gamma,\alpha}(f, g) = \sum_{I : E_\Gamma \to (1,2\ldots,d)} \left[\prod_{k=1}^{n} \left(\prod_{e \in E_\Gamma, e=(\star,k)} \partial_{I(e)} \right) \alpha^{I(e_k^1) I(e_k^2)} \right] \times \tag{99}$$

$$\times \left(\prod_{e \in E_\Gamma, e=(\star,L)} \partial_{I(e)} \right) f \times \left(\prod_{e \in E_\Gamma, e=(\star,R)} \partial_{I(e)} \right) g. \tag{100}$$

Every graph $\Gamma \in G_n$ has an associated weight $W_\Gamma \in \mathbb{R}$. The weight is defined via the angles of a copy of the graph Γ in the hyperbolic plane. We will explain what angles we mean, and how the weight is constructed.

Consider two points $p, q, p \neq q$ in the Lobachevsky plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ endowed with the hyperbolic metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$. The angle $\phi^h(p,q) \in \mathbb{R}/2\pi\mathbb{Z}$ is the angle at p formed by two geodesics in $\mathbb{H}$, $l(p,\infty)$ and $l(p,q)$, that pass through p and q, and through p and the point ∞ on the absolute (See Figure 5). The direction of the measurement of the angle is counterclockwise from $l(p,\infty)$ to $l(p,q)$. The h in the notation for the angle $\phi^h(p,q)$, stands for harmonic. With

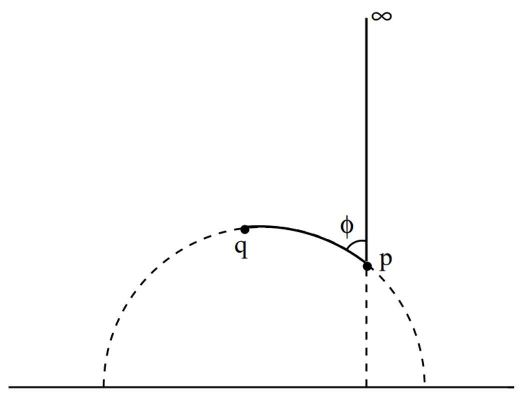


Figure 5: The angle $\phi^h(q,p)$ between two geodesics in the upper half-plane, from [Kont03].

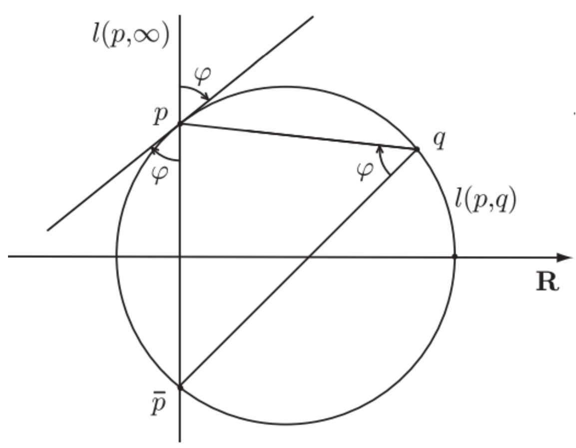


Figure 6: Geometry of angles on a circle with complex points on it, from [CKTB].

some calculus on angles on a circle and angle rules for a circle (See Figure 6), the angle is expressed as follows

$$\phi^h(p,q) = \mathrm{Arg}\left(\frac{q-p}{q-\overline{p}}\right) = \frac{1}{2i}\log\left(\frac{(q-p)(\overline{q}-p)}{(q-\overline{p})(\overline{q}-\overline{p})}\right). \tag{101}$$

The meaning of harmonic for the letter h in the angle comes from the fact that $\phi^h(p,q)$ is a harmonic function in both $p, q \in \mathbb{H}$. By continuity, the angle is also defined in the case that $p, q \in \mathbb{H} \sqcup \mathbb{R}$.

Let $\mathbb{H}_n$ be the space of n distinct points on the Lobachevsky plane, thus

$$\mathbb{H}_n = \{(p_1, \ldots, p_n) \mid p_k \in \mathbb{H},\ p_k \neq p_l \text{ for } k \neq l\}. \tag{102}$$

The space of configurations $\mathbb{H}^n \subset \mathbb{C}^n$ is a smooth $2n-$dimensional real manifold.

Summarizing, we have a graph Γ, and points $(p_1, \ldots, p_n) \in \mathbb{H}_n$, with a definition for angle. These are the ingredients to define a weight for the graph Γ. This is done as follows.

The graph Γ is mapped to the Lobachevsky plane by assigning the point $p_k \in \mathbb{H}$ to vertex $k \in \Gamma$, and the point $0 \in \mathbb{R} \subset \overline{\mathbb{H}}$ to the vertex L, and the point $1 \in \mathbb{R} \subset \overline{\mathbb{H}}$ to the vertex R. Each edge is drawn as a geodesic in the Lobachevsky plane. Every edge, i.e. an ordered pair in $\mathbb{H} \sqcup \mathbb{R}$, defines an angle $\phi^e := \phi^h(p, q)$. When points move around in the Lobachevsky plane, the angle ϕ_e^h takes values in $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 5.6 (The weight w_Γ of a graph)**.** The weight of the graph $\Gamma \in G_n$ is defined as

$$w_\Gamma := \frac{1}{n!(2\pi)^{2n}} \int_{\mathbb{H}^n/\mathrm{Aff}(\mathbb{H})} \bigwedge_{i=1}^{n} \left(\mathrm{d}\phi^h_{e_k^1} \wedge \mathrm{d}\phi^h_{e_k^2} \right), \tag{103}$$

where $\mathrm{Aff}(\mathbb{H})$ is the group of affine symmetries of $\mathbb{H}$.

Lemma 5.4. *The weight w_Γ is absolutely convergent.*

Kontsevich remarks that this Lemma is a result of the fact that w_Γ is the integral of a smooth differential form over a compact manifold with corners ([Kont03], p.188).

Theorem 5.5 (The Kontsevich star product)**.** *Let $\left(C^\infty(\mathbb{R}^d), \cdot\right)$ be a commutative algebra, with a Poisson bi-vector $P(\mathbf{x})$ on it. Then Kontsevich's star product*

$$f \star_K g := \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,P}(f, g) \tag{104}$$

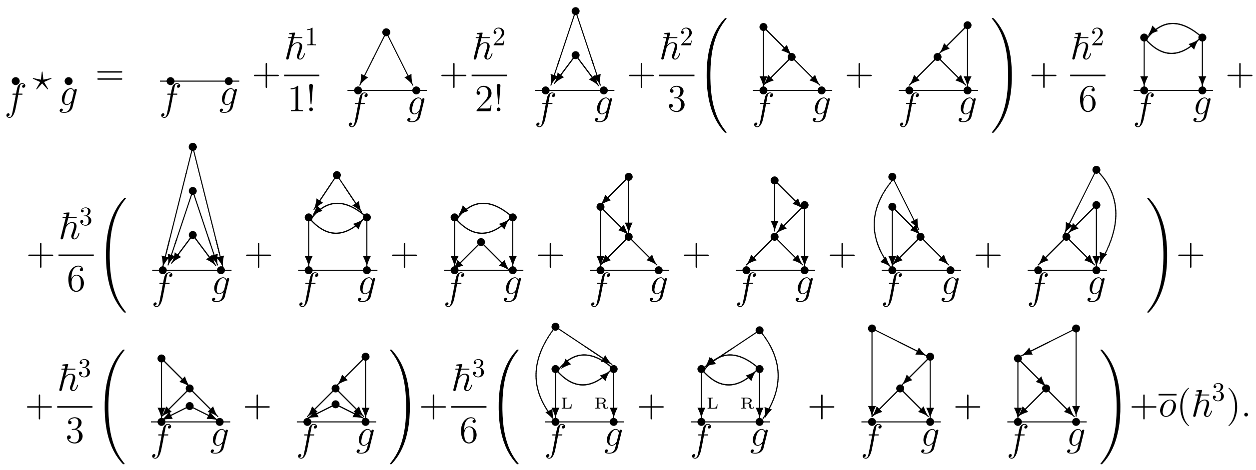
defines an associative product between functions $f, g \in C^\infty(\mathbb{R}^d)$ (even when we consider $f, g \in C^\infty(\mathbb{R}^d)[[\hbar]]$, by using $\mathbb{k}$-linearity (see Equation (87)*). A gauge equivalent star product is obtained under a change of coordinates.*

Kontsevich remarks that the proof of this theorem would only need Stokes' formula and some combinatorics of graphs. However, this theorem can also be seen as a corollary of a more general result, which he proves in Section 6, [Kont03]. The proof and different approaches were studied in the Mastermath course "Deformation quantization, graph complex and number theory" [Kis20].

The Kontsevich star product is not unique in two ways. First of all, the Kontsevich star product is only unique up to gauge equivalent star products (See Definition 5.2 and Theorem 5.2). So, there are gauge equivalent versions of the Kontsevich star product.

Secondly, there are more non-equivalent star products that also define an associative deformed product on the deformed algebra of functions on $\mathbb{R}^d$. For example, Kontsevich conjectured in 1999 [Kont99] that there exist associative star products with logarithmic weights. This conjecture was proven by Alekseev, Rossi, Torossian and Willwacher in 2014 [ARTW]. Rossi and Willwacher wrote more on associative star products with logarithmic weights later that year [RW].

The expansion of the Kontsevich star product between two functions f and g mod $\hbar^4$ is calculated in [BurKis19]:



(105)

In the paper, Buring and Kiselev present software modules to generate the Kontsevich graphs (in the set G_n in Definition 5.5), to expand the non-commutative star product by using a priori undetermined coefficients, and deriving linear relations between the weights of graphs. The main example in the paper is the calculation of the star product up to and including order $\hbar^4$. Important is that at order $\hbar^4$ there are 149 parameters that describe all the coefficients of the basic graphs in the expansion, but the actual number of graphs at order $\hbar^4$ is much greater. Using symmetries, mutual dependencies and others, the total set of graphs is reduced to the actual 149 graphs that eventually are in the expression for the star product mod $\hbar^4$. In section 7.3, we discuss a seemingly similar type of reduction. However, there we start we Feynman diagrams that are the result of the calculation of the functional/path integration and reduce to the Kontsevich graphs that actually appear in the star product.

6 Path integral methods in Quantum Field Theory

This section aims to summarize the most relevant parts of path integral and quantum field theory with the goal to perturbatively compute star products based on this theory in the next section. Most of the theory we present here, can be found in Chapter 11 of [CKTB][9] and [Sch]. As good references for the study of functional and path integral methods in quantum field theory, we refer to [Sch] and [ZJ].

Every field theory consists of at least the following two elements:

1. *a space of fields* $\mathcal{M}$: this often is a space with maps or sections,
2. *an action functional* $\mathcal{S}$: a functional on the space of fields $\mathcal{M}$ which we define by integrating a function of the fields (and their derivatives or jets[10]) on $\mathcal{M}$ over the source manifold (a manifold used to parametrize maps or sections on $\mathcal{M}$).

Often we calculate the expectation value of functions on the space of fields.

Definition 6.1 (Expectation value of a function $\mathcal{O}$ on the manifold $\mathcal{M}$)**.** The expectation value of a function $\mathcal{O}$ on the space of fields $\mathcal{M}$ is defined as follows

$$\langle\, \mathcal{O}\, \rangle := \frac{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S}\mathcal{O}}{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S}}, \tag{106}$$

The integration involved in the calculations of the expectation value is called a functional integral. This is because the integral depends on the action functional $\mathcal{S}$. In physics applications like quantum mechanics, these integrals are often called path integrals. In path integrals the functional $\mathcal{S}$ depends on the path of integration. In fact, in quantum theories, specifically in quantum field theory, the integration is performed over all possible paths between the beginning and endpoint of the integration domain. This integration over different paths is what Groenewold's second problem is concerned with (cf. section 4).

Definition 6.2 (Observable)**.** An observable of a field theory is a function on a manifold $\mathcal{M}$ whose expectation value is well-defined and real.

When calculating the expectation value of a function, this is often done perturbatively. This means that the functional $\mathcal{S}$ is expanded around a non-degenerate critical point, so that the fraction of integrals in Equation (106) becomes a formal power series in the parameter of expansion $\hbar$. The coefficients in the formal power series are given by the expectation values of Gaussian integrals. The expectation values of Gaussian integrals are well-known, which makes the calculation feasible.

6.1 Critical points and degeneracies

We recall what a critical point is, and what it means to be non-degenerate. Here we mean a critical point as it is often used in mathematics literature, in physics literature this point is commonly called a stationary point. We also classify different path integral theories based on the type of critical point we are expanding around.

[9] There are two versions of what is found in Chapters 11-14 written by Cattaneo in [CKTB]. The slightly different version includes some exercises, which are probably just left out in the publication in [CKTB]

[10] For more information about jets, we refer to [Kis12] and references therein.

Definition 6.3 (A critical point $\vec{c}$ of a function $g : \mathbb{R}^n \to \mathbb{R}$)**.** Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function in n variables. A critical point $\vec{c} \in \mathbb{R}^n$, in the domain of g, is a point where the gradient vanishes: $\vec{\nabla} g(\vec{c}) = 0$.

This definition is naturally extended to a function on a manifold. Let $\mathcal{M}$ be a manifold, and $U_\alpha \subset \mathcal{M}$ charts on $\mathcal{M}$, so that U_α are an open subsets of $\mathcal{M}$. And, we have the condition that $\mathcal{M} = \bigcup_{\alpha \in \mathfrak{I}} U_\alpha$. We have local coordinates on $U_\alpha \subset \mathcal{M}$ via the injective coordinate map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, where we assume that the dimension of U_α is n. Let $\tilde{h} : \mathcal{M} \to \mathbb{R}$ be function on $\mathcal{M}$. Note that we can, locally, see $\tilde{h}$ as a function $h : \mathbb{R}^n \to \mathbb{R}$ as we can parametrize the points on a local chart U_α via the coordinate map ϕ_α.

Definition 6.4. A critical point $x(u) \in U_x \subset \mathcal{M}$, meaning x is a point on $\mathcal{M}$ parametrized via local coordinates $u \in \mathbb{R}^n$, is a point where the gradient $\vec{\nabla} h(x(u)) = 0$.

So, in the specific case of path, or functional, integral calculation, the action $S = \int_{\mathcal{D}} \mathcal{L}$ is expanded around a critical point, which often comes down to a critical point of the function $\mathcal{L}$, where $\mathcal{D}$ is the domain of integration. The critical point is a point on $\mathcal{M}$, as $\mathcal{S} : \mathcal{M} \to \mathbb{R}$.

Definition 6.5 (Degenerate critical point)**.** A critical point $c \in \mathcal{M}$ of a function $f : \mathcal{M} \to \mathbb{R}$ is degenerate if the determinant of the Hessian matrix of f at that point is zero, i.e.

$$\det \left((\mathbf{H}f)_{ij} \right)(c) = \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)(c) = 0, \tag{107}$$

where $(\mathbf{H}f)$ is the Hessian matrix of f.

Geometrically, the determinant of the Hessian matrix evaluated at a critical point c of a function $f : \mathcal{M} \to \mathbb{R}$, is equal to the Gaussian curvature of the graph of f at $f(c) \in f(\mathcal{M})$.
Another way to say that a critical point x of a function f on a manifold $\mathcal{M}$ is degenerate, is to say that there is a continuous symmetry group of the manifold, which moves x to another point on the manifold, but leaves the function value of f unchanged. In those words, a non-degenerate critical point is an isolated critical point.

Depending on the type of degeneracy of the critical points, a different formalisms can be used to resolve the degeneracy. The degeneracy is described by the respective symmetry causing the degeneracy. In the study of path integral theory, a symmetry is a distribution of vector fields on $\mathcal{M}$ under which the action function S is invariant.

When the symmetries are given by

1. *the infinitesimal free action of a Lie algebra*, the **BRST formalism** [BRS, Tyu] can help to describe the perturbative expansion of the functional integral on the quotient space.
    - Observables are *invariant functions* with defined expectation values.
    - The Kontsevich star product for affine Poisson manifolds can be calculated with the help of the BRST-formalism.
2. *more general distributions of vector fields*, the **BV formalism** [BV] can deal with it, specifically it helps to obtain Kontsevich's formula for *general Poisson structures*.

In this thesis, we will eventually discuss the perturbative path integral calculation of the Kontsevich star product for affine Poisson manifolds. However, we will not discuss the details of the BRST-formalism, which can be found in [CKTB], Chapter 12.

6.2 Gaussian integration

In the perturbative calculation of the expectation value the coefficients in the formal power series are given by the expectation values of Gaussian integrals. Therefore, we discuss Gaussian integrals and their properties.

6.2.1 Gaussian integration on $\mathbb{R}^n$

First, we discuss Gaussian integration on $\mathbb{R}^n$, after that we extend it to infinite dimensions. Here we introduce the expectation value, partition function and Wick's theorem with respect to Gaussian distributions on $\mathbb{R}^n$. The definitions of these three concepts will be introduced here in the finite-dimensional and are extended to, or taken as a definition in, the infinite-dimensional case.

Proposition 6.1 (Gaussian integral for a non-degenerate matrix $\mathbf{A}$ on $\mathbb{R}^n$)**.** *If* $\mathbf{A}$ *is a non-degenerate - but not necessarily positive or negative definite - matrix on* $\mathbb{R}^n$ *endowed with the Lebesgue measure* $\mathrm{d}^n x$ *and the Euclidean inner product* $(\cdot,\cdot)$*, then*

$$\int_{\mathbb{R}^n} e^{\frac{i}{2}(x,\mathbf{A}x)}\,\mathrm{d}^n x = (2\pi)^{\frac{n}{2}} e^{\frac{i\pi sign\mathbf{A}}{4}} \frac{1}{\sqrt{|\det \mathbf{A}|}}, \tag{108}$$

where we denoted the signature of $\mathbf{A}$ *by sign* $\mathbf{A}$*. The signature of a diagonal matrix is the number of positive, negative and zero numbers on its main diagonal.*

Notation 6.1 (Integration with respect to Gaussian distributions)*.* As mentioned before, expectation values with respect to Gaussian distributions are well-known. Therefore, we will use the notation $\langle\cdot\rangle_0$ for integrals with respect to Gaussian distributions.

Definition 6.6 (Partition function $Z(J)$ on $\mathbb{R}^n$)**.** The partition function $Z(J) = Z[J]$, notations are used interchangeably, for a non-degenerate matrix $\mathbf{A}$ on $\mathbb{R}^n$ is the generating functional of all correlation functions, defined by

$$Z[J] = \int_{\mathbb{R}^n} e^{\frac{i}{2}(x,\mathbf{A}x)+(J,x)}\mathrm{d}^n x = (2\pi)^{\frac{n}{2}} e^{\frac{i\pi\mathrm{sign}\mathbf{A}}{4}} \frac{1}{\sqrt{|\mathrm{det}\mathbf{A}|}} e^{\frac{i}{2}\left(J,\mathbf{A}^{-1}J\right)}, \tag{109}$$

where J is an auxiliary function, known as the source field.

With the help of the partition function, the expectation value of monomials with respect to Gaussian distributions can be written as

$$\begin{aligned}
\left\langle x^{i_1}\cdots x^{i_k}\right\rangle_0 =& \frac{\int_{\mathbb{R}^n} e^{\frac{i}{2}(x,\mathbf{A}x)} x^{i_1}\cdots x^{i_k}\,\mathrm{d}^n x}{\int_{\mathbb{R}^n} e^{\frac{i}{2}(x,\mathbf{A}x)}\,\mathrm{d}^n x}\\
=& \frac{\frac{\partial}{\partial J^{i_1}}\cdots\frac{\partial}{\partial J^{i_k}} Z[J]|_{J=0}}{Z[0]}\\
=& \frac{\partial}{\partial J^{i_1}}\cdots\frac{\partial}{\partial J^{i_k}}\, e^{\frac{i}{2}\left(J,\mathbf{A}^{-1}J\right)}\Big|_{J=0},
\end{aligned}$$

for some coordinate functions $x_{i_1},\ldots,x_{i_k}$. Note that the expectation value $\left\langle x^{i_1}\cdots x^{i_k}\right\rangle_0 = 0$ if k is an odd number due to the anti-symmetry of the integral. When k is even, it becomes a sum of products of matrix elements of the inverse of $\mathbf{A}$.

Example 6.1. Two-point function and propagator.
In case $k = 2$, we calculate the expectation value of a quadratic function with respect to a Gaussian distribution, i.e.

$$\begin{aligned}\left\langle x^{i_1}x^{i_2}\right\rangle_0 =& \frac{\int_{\mathbb{R}^n} e^{\frac{i}{2}(x,\mathbf{A}x)} x^{i_1}x^{i_2} \, \mathrm{d}^n x}{\int_{\mathbb{R}^n} e^{\frac{i}{2}(x,\mathbf{A}x)} \, \mathrm{d}^n x} \\ =& i\left(\mathbf{A^{-1}}\right)^{i_1 i_2}. \end{aligned} \tag{110}$$

This is called the **contraction** or the **correlation function** of x^{i_1} and x^{i_2}. The matrix element $\left(\mathbf{A}^{-1}\right)^{i_1 i_2}$ is called a **Feynman propagator**. So, within Gaussian quantum field theories where the matrix actually is a differential operator, a Green function is a synonym for a propagator.

Example 6.2. Expectation value of an even degree monomial.
The expectation value of an even degree monomial is a sum of products of contractions:

$$\left\langle x^{i_1}\cdots x^{i_{2s}}\right\rangle_0 = i^s \sum_{\sigma\in S_{2s}} \frac{1}{2^s s!} \left(\mathbf{A^{-1}}\right)^{i_{\sigma(1)}i_{\sigma(2)}} \cdots \left(\mathbf{A^{-1}}\right)^{i_{\sigma(2s-1)}i_{\sigma(2s)}}, \tag{111}$$

with S_{2s} denoting the symmetric group of permutations of $2s$ elements. Or differently, a sum of products of Feynman propagators. So, we see that in essence, we only need to know the pairwise correlation functions to express expectation values for monomials. Furthermore, we want to note about the factor i^s that this comes from the Gaussian integration, see Equation (108).

We can simplify the above formula by using the concept of pairings.

Definition 6.7 (Pairing)**.** A pairing is a permutation $\sigma \in S_{2s}$ such that the following two properties are satisfied

1. for all $i = 1, \cdots, s$: $\sigma(2i-1) < \sigma(2i)$,
2. $\sigma(1) < \sigma(3) < \cdots < \sigma(2s-3) < \sigma(2s-1)$.

If we denote $P(s)$ by the set of pairings of $2s$ elements, then we can rewrite Equation (111) to obtain Wick's theorem.

Proposition 6.2 (Wick's theorem on $\mathbb{R}^n$)**.** *Let* $\mathbf{A}$ *be a non-degenerate matrix on* $\mathbb{R}^n$*. Then the expectation value of a monomial of degree* $2s$*, with respect to a Gaussian distribution, is given by the sum of pairings of the Feynman propagators, as*

$$\left\langle x^{i_1}\cdots x^{i_{2s}}\right\rangle_0 = i^s \sum_{\sigma\in P(s)} \left(\mathbf{A^{-1}}\right)^{i_{\sigma(1)}i_{\sigma(2)}} \cdots \left(\mathbf{A^{-1}}\right)^{i_{\sigma(2s-1)}i_{\sigma(2s)}}. \tag{112}$$

We discuss a specific example which comes in handy later in the case of the star product for symplectic manifolds.

Example 6.3. Special case.
Let $\mathbf{A} = \begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B}^T & 0 \end{pmatrix}$, where $\mathbf{B}$ is a non-degenerate $m \times m$-matrix, and $\mathbf{B}^T$ its transpose. The essential steps are shown in [CKTB], we follow them. Let $x \in \mathbb{R}^{2m}$ be regarded as $y \oplus z$, with $y, z \in \mathbb{R}^m$, and compute

$$\int_{\mathbb{R}^{2m}} e^{i(y,Bz)} \, \mathrm{d}^m y \, \mathrm{d}^m z = \int_{\mathbb{R}^{2m}} e^{\frac{i}{2}(x,Ax)} \, \mathrm{d}^{2m} x = (2\pi)^m \frac{1}{|\det(B)|}$$

and, for $K, L \in \mathbb{R}^m$,

$$Z(K,L) = \int_{\mathbb{R}^{2m}} e^{i(y,Bz)+(K,y)+(L,z)} \mathrm{d}^m y \mathrm{d}^m z = (2\pi)^m \frac{1}{|\det(B)|} e^{i(LB^{-1}K)}.$$

The expectation value vanishes if the degree in y is different from the degree in z and is a sum of products of matrix elements of B^{-1} otherwise:

$$\left\langle y^{i_1} \dots y^{i_s} z^{j_1} \dots z^{j_s} \right\rangle_0 = i^s \sum_{\sigma \in S_s} (B^{-1})^{j_{\sigma(1)} i_1} \dots (B^{-1})^{j_{\sigma(s)} i_s}.$$

6.2.2 Gaussian integration for infinite dimensions

Important is the extension of the expectation value to the case of infinite dimension. This makes sense when the symmetric operator $\mathbf{A}$ is invertible. Usually $\mathbf{A}$ is a differential operator and in this case $\mathrm{G} = \mathbf{A^{-1}}$ will denote the distributional kernel of its inverse, i.e., its Green function. We recall what a Green function is.

Definition 6.8 (Green's function $G(x,s)$ of a symmetric operator A)**.** The Green function $G(x,s)$ of a symmetric operator $\mathbf{A}$ is the solution to the equation

$$\mathbf{A}(G)(x,s) = \delta(x-s). \tag{113}$$

Green's functions are often used to calculate the response of a system $\mathbf{A}u(x) = 0$ to some external source $f(x)$, i.e. we try to solve $\mathbf{A}u(x) = f(x)$ for $u(x)$. Therefore, we calculate the Green functions, so that

$$\int \mathbf{A}(G)(x,s) f(s) \mathrm{d}s = \int f(s)\delta(x-s)\mathrm{d}s = f(x). \tag{114}$$

Due to linearity of $\mathbf{A}$, we can rewrite

$$\mathbf{A}\left(\int (G)(x,s) f(s)\mathrm{d}s\right) = f(x), \tag{115}$$

to obtain the solution for the equation $\mathbf{A}u(x) = f(x)$ as $u(x) = \int G(x,s)f(s)\mathrm{d}s$. We note that the Green function $G(x,s)$ is generally speaking no classical function, but a distribution.

This being said, we can formulate Wick's theorem for infinite-dimensional integrals.

Proposition 6.3 (Wick's theorem (infinite-dimensional case))**.** *Let $\mathbf{A}$ be a symmetric invertible differentiable operator on the space of functions on some manifold $\mathcal{M}$, then*

$$\begin{aligned}\langle \phi(x_1)\dots\phi(x_{2s})\rangle_0 =& \frac{\int e^{\frac{i}{2\hbar}\int_{\mathcal{M}} \phi \mathbf{A} \phi} \phi(x_1)\dots\phi(x_{2s}) \mathrm{D}\phi}{\int e^{\frac{i}{2\hbar}\int_{\mathcal{M}} \phi \mathbf{A} \phi} \mathrm{D}\phi} \\ &:= (i\hbar)^s \sum_{\sigma\in P(s)} G(x_{\sigma(1)}, x_{\sigma(2)}) \dots G(x_{\sigma(2s-1)}, x_{\sigma(2s)}),\end{aligned}$$

where ϕ denotes a function on $\mathcal{M}$, and $\mathrm{D}\phi$ is the "formal Lebesgue measure" on the space of functions, furthermore we require that $x_1, \dots, x_{2s}$ are all mutually distinct.

Wick's theorem can also be applied to derivatives of a field ϕ through linearity. Namely, for multi-indices $I_1, \dots, I_{2s}$ one sets

$$\begin{aligned}\langle \partial_{I_1}\phi(x_1)\dots\partial_{I_{2s}}\phi(x_{2s})\rangle_0 =& (i\hbar)^s \frac{\partial^{|I_1|}}{\partial {x_1}^{|I_1|}} \cdots \frac{\partial^{|I_{2s}|}}{\partial {x_{2s}}^{|I_{2s}|}} \times \\ & \sum_{\sigma\in P(s)} G(x_{\sigma(1)}, x_{\sigma(2)}) \dots G(x_{\sigma(2s-1)}, x_{\sigma(2s)}),\end{aligned} \tag{116}$$

where the derivatives on the right hand side are in the distributional sense.

6.3 Perturbed Gaussian integration

6.3.1 Perturbative evaluation of integrals

Let $\mathcal{M}$ be an n-dimensional manifold and S a smooth function on $\mathcal{M}$. We want to compute the integrals appearing in

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S}\mathcal{O}}{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S}},$$

(often) we do this by a method of perturbation.
First, consider the case when S has a unique critical point $x_0 \in \mathcal{M}$ which is non-degenerate. Taylor expand S around the critical point x_0 as a formal power series in the expansion parameter $\sqrt{\hbar}$

$$S(x_0 + \sqrt{\hbar}x) = S(x_0) + \frac{\hbar}{2}\,\mathrm{d}_x^2\big|_{x_0} S(x) + R_{x_0}(\sqrt{\hbar}x),$$

where we note that there is no term term at order $\sqrt{\hbar}$ because the first derivative $S'(x_0)$ is zero as x_0 is a critical point of S, and R_{x_0} is the tail of the Taylor series expansion of S starting with the cubic term in $\sqrt{\hbar}x$ for $x \in T_{x_0}\mathcal{M}$. Let $\mathbf{H}$ be the Hessian of S at x_0 with respect to the Euclidean metric, which means that $\mathrm{d}_x^2\big|_{x_0} S(x) = (x, \mathbf{H}x)$. Then the saddle-point approximation to the, often called, partition function $Z := \int_{\mathcal{M}} e^{\frac{i}{\hbar}S}\mathrm{d}^n x$ is given by the formula

$$\begin{aligned} Z &= \hbar^{\frac{n}{2}} e^{\frac{i}{\hbar}S(x_0)} \int_{\mathcal{M}} e^{\frac{i}{2}(x,\mathbf{H}x)} \sum_{r=0}^{\infty} \frac{1}{r!} R_{x_0}^r(\sqrt{\hbar}x)\mathrm{d}^n x \\ &= (2\pi\hbar)^{\frac{n}{2}} e^{\frac{i}{\hbar}S(x_0)} e^{\frac{i\pi \mathrm{sign}(\mathbf{H})}{4}} \frac{1}{\sqrt{|\det(\mathbf{H})|}} \sum_{r=0}^{\infty} \frac{1}{r!} \left\langle R_{x_0}^r(\sqrt{\hbar}x) \right\rangle_0 , \end{aligned}$$

where $\langle \cdot\,, \cdot \rangle_0$ denotes the Gaussian expectation value with respect to the non-degenerate symmetric matrix $\mathbf{H}$. This formula is the asymptotic expansion of the partition function Z as a function of $\hbar$, pre-multiplied by a term $e^{\frac{i}{\hbar}S(x_0)}$ (which is divided out when we calculate expectation values). We view the term R_{x_0} as a perturbation to the Gaussian theory defined by $\mathrm{d}^2\big|_{x_0} S$, so that the above formula for Z is referred to as the perturbative expansion of the integral.

Expectation values of observables may also be computed perturbatively by

$$\langle \mathcal{O} \rangle = \frac{\sum_{r=0}^{\infty} \frac{1}{r!} \left\langle \mathcal{O}(x_0 + \sqrt{\hbar}x) R_{x_0}^r(\sqrt{\hbar}x) \right\rangle_0}{\sum_{r=0}^{\infty} \frac{1}{r!} \left\langle R_{x_0}^r(\sqrt{\hbar}x) \right\rangle_0}. \tag{117}$$

As the denominator is of the form $1 + O(\hbar)$, the ratio can be computed as a formal power series in $\hbar$.

Definition 6.9 (Local function)**.** A function on a space of fields on some manifold M is called local if it is the integral on M of a function that depends at each point on finite jets, i.e. on a finite amount of derivatives, of the fields at that point.

If the action is a local function, the Hessian matrix $\mathbf{H}$ will be a differential operator. Remember that we are trying to perturbatively calculate the expectation value of an observable. In these computations we need the distributional kernel of the Hessian operator, i.e. we need the Green function, or propagator, of $\mathbf{H}$, as mentioned in section 6.2. Also, we need the Gaussian expectation values of $\mathcal{O}R^r$

and of R^r, which can be calculated with the help of Equation 116. Instead of summing over indices we will have to integrate over Cartesian products of the source manifold. The normal ordering prescription will (see e.g. Equation (7.8)) exclude all graphs with an internal line whose endpoints are equal, also called tadpoles or vacuum diagrams. The integration is restricted to the configuration space of the source manifold (meaning: the Cartesian product minus all diagonals).

Generally speaking, this is not enough to let all integrals converge. Main reason for this is the singularity appearing when the two arguments of Green's functions approach each other. Usually, this problems is resolved by renormalization. Renormalization is, in its generality, often difficult. We will not discuss it here as topological field theories, of which the Poisson sigma model is one, have the property that configuration space integrals associated with Feynman diagrams without tadpoles converge.

The asymptotic expansions also depend on the kind of critical points of the action function S we have at hand. We can classify different results based on the type of critical points. If the action has **multiple non-degenerate critical points** the asymptotic expansion is obtained by adding all contributions of saddle-point approximations around each critical point. In the case that one critical value strongly dominates the others, we may forget the contribution of the others. Another case is, when the action has **a degenerate critical point**. In a simple case, the critical points can be parametrized by a finite-dimensional manifold $\mathcal{M}_{\text{crit}}$. Then, by Fubini's theorem in the finite-dimensional case which can be naturally extended to the infinite-dimensional case, one writes,

$$\int_{\mathcal{M}} \cdots := \int_{x_0 \in \mathcal{M}_{\text{crit}}} \mu(x_0) \int_{\mathcal{M}(x_0)} \cdots,$$

where $\int_{\mathcal{M}(x_0)}$ denotes the asymptotic expansion of the integral in the complement to $T_{x_0}\mathcal{M}_{\text{crit}}$ of a neighborhood of x_0, while μ is a measure on $\mathcal{M}_{\text{crit}}$. If, in addition, we assume that the Hessian is constant on $\mathcal{M}_{\text{crit}}$, we write the expectation value of the observable as

$$\langle \mathcal{O} \rangle = \frac{\int_{x_0 \in \mathcal{M}_{\text{crit}}} \mu(x_0) \sum_{r=0}^{\infty} \frac{1}{r!} \left\langle \mathcal{O}(x_0 + \sqrt{\hbar} x) R^r_{x_0}(\sqrt{\hbar} x) \right\rangle_0}{\int_{x_0 \in \mathcal{M}_{\text{crit}}} \mu(x_0) \sum_{r=0}^{\infty} \frac{1}{r!} \left\langle R^r_{x_0}(\sqrt{\hbar} x) \right\rangle_0},$$

where $\langle\ \rangle_0 (x_0)$ denotes the Gaussian expectation value computed by expanding around x_0 orthogonally to $T_{x_0}\mathcal{M}_{\text{crit}}$. In case we choose $\mu(x_0) = \delta(x - x_0)$, we get the expectation value denoted by $\langle\ \rangle (x_0)$.

In the above considered perturbative expansion the only expansion parameter was $\hbar$. However, in more general cases, it may happen that there are other (smaller) expansion parameter, or that some coefficient appearing in S is much smaller than $\hbar$, formally that would mean some element of $\hbar^2\mathbb{R}[[\hbar]]$. If that happens, the right approach is to prescribe the Gaussian part using the quadratic, $\hbar$-independent term of $S/\hbar$ and consider all other terms as the perturbation R. Therefore, it may be the case that the perturbation R contains quadratic and linear terms as well, for examples see e.g. [ZJ, Sch].

A particular case is when we are in the setting that the action has the form

$$S(y, z) = (y, \mathbf{B}z) + f(y, z),$$

where $\mathbf{B}$ is a nondegenerate matrix and f is a function quadratic in z. If we work around a critical point $y = z = 0$, we may rescale z by $\hbar$, obtaining

$$S(y, \hbar z) = \hbar(y, \mathbf{B}z) + \hbar^2 f(y, z)$$

and consider f as the perturbation to the Gaussian theory defined by $\mathbf{B}$. An infinite-dimensional generalization is discussed in section 7.2.

6.4 Feynman diagrams

In the previous sections, we saw how expectation values of observable are calculated with respect to Gaussian distributions. In this section, we study how the expectation value of an observable can be represented by a formal sum of the famous Feynman diagrams [Fey]. We discuss how vacuum diagrams are cancelled out of the expectation values. Lastly, we discuss how the normal ordering prescription makes all diagrams with a tadpole vanish.

In the late 1940s Feynman [Fey] introduced the Feynman diagrams in a paper about quantum electrodynamics. In the years after his publication, the Feynman diagrams gained in popularity and found a wide variety of applications within quantum field theories. Nowadays, we could hardly imagine quantum field theory without Feynman diagrams. For those interested in the history of the rise of Feynman diagrams, see [Kai].

6.4.1 Interpretation of Feynman diagrams

The meaning of a Feynman diagram differs from theory to theory. However, in well-known quantum field theories, like quantum electrodynamics and quantum chromodynamics, a Feynman diagram represent an event. Namely, the diagram represents how, for a given initial state, a certain final state can be obtained. How the final state can be obtained, depends on the theory and the allowed interactions therein.

Remark 6.1 (Feynman diagrams without time and space indicated). To nuance the meaning of Feynman diagrams considered above, sometimes Feynman diagrams are drawn without time and space flow indication. Such a diagram can represent multiple different particle interactions, depending on the choice of reading the diagram. To conclude we only indicate a flow of time and space if want to consider time and space explicitly, but Feynman diagrams do not depend on such an indication.

In the next chapter, we will discuss the path integral calculation of the Kontsevich star product. This path integral calculation is done within the Poisson sigma model. The expectation value that we calculate via those path integrals are also represented by Feynman diagrams. However, the interpretation of those Feynman diagrams is different. Nevertheless, the method to obtaining Feynman diagrams is almost identical to what we can see in the theories of quantum electrodynamics and quantum chromodynamics.

To be able to see differences and similarities between Feynman diagrams in quantum electrodynamics and Feynman diagrams in the Poisson sigma model, we will now show a Feynman diagram in electrodynamics.

Example 6.4 (Feynman diagrams in electrodynamics)**.** Consider the action functional describing quantum electrodynamics

$$S = S_0 + S_{\text{int}} = \int \mathrm{d}x^4 \left(L_0 + L_1\right), \tag{118}$$

where the free Lagrangian is

$$L_0 = \overline{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}\left(F_{\mu\nu}\right)^2, \tag{119}$$

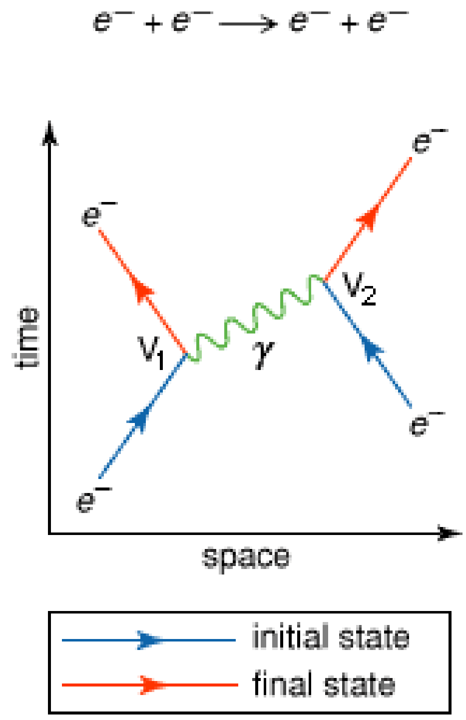


Figure 7: A Feynman diagram with two interaction vertices: electron-electron scattering

with ψ and $\overline{\psi}$ the electron and positron fields, $\not{D}$ the Dirac spinor γ^μ contracted with the gauge covariant derivative D_μ, m the mass of the electron and positron, and $F_{\mu\nu}$ the electromagnetic field tensor, and the interaction Lagrangian is

$$L_{\text{int}} = -e\overline{\psi}\gamma^\mu\psi A_\mu, \tag{120}$$

where A_μ is the covariant four-potential representing the photon in the interaction term. The interaction Lagrangian describes the possible interaction within this theory, and is a perturbation to the free Lagrangian as $|e| << 1$. Thus an interaction vertex of a Feynman diagram in electrodynamics couples an electron, a positron and a photon.

Order by order in e the expectation value of the observable

$$\mathcal{O} = \psi(x_1)\psi(x_2)\overline{\psi}(x_3)\overline{\psi}(x_4)$$

can be calculated. An interaction at second order in e that will contribute to the expectation value of this observable, is displayed in Figure 7. The order in e of a Feynman diagram can be seen from the number of interaction vertices. In Figure 7, we have a Feynman diagram with two incoming electrons, the external lines on the bottom of the diagram, we have two outgoing electrons, the external lines at the top of the diagram, we have one photon, the internal line, and we have two interaction vertices. The interaction vertices do couple an electron a positron and a photon, because a time-reversed positron is an electron.

So, this diagram represents an interaction that contributes to the expectation value of the electron-electron scattering process.

We will now mention a couple of differences betweeen the Feynman diagrams in quantum electrodynamics and the Feynman diagrams that will appear in the Poisson sigma model in next Chapter, without going in to many details of the Poisson sigma model.

First of all, in the above example, we calculated the expectation value of an observable which is the product of four fields. When we calculate, via the Poisson sigma model, the Kontsevich star product between f and g, we will have an observable that is a product of f and g. So, a product of two functions instead of four.

Secondly, the expansion parameter in the Poisson sigma model will be $\hbar$ instead of e.

Thirdly, in quantum electrodynamics, we have external lines corresponding to the fields in the observable. In the Poisson sigma model, however, we have will have so-called sinks, vertices that can only be target vertices (without have outgoing arrows themselves), that represent the functions in the observable.

Fourthly, in quantum electrodynamics, every line represents a particle, whereas directed edges (or directed lines) in the Poisson sigma model represent partial derivatives.

Fifthly, in the Feynman diagram in quantum electrodynamics, the only vertices we have are interaction vertices. In the Poisson sigma model, the Feynman diagrams have two types of vertices. Namely, internal vertices that are not a target vertex of other internal vertices, and internal vertices that are also a target vertex of other internal vertices.

This sums up the most important qualitative differences between Feynman diagrams in quantum electrodynamics and Feynman diagrams in the Poisson sigma model in next chapter.

6.4.2 Tadpole and vacuum diagrams

Before we make the step to the next chapter, we want to mention two types of diagrams that will not appear as a Feynman diagram in the expectation value of an observable. These diagrams are the tadpole diagrams and the vacuum diagrams. In this chapter, we will define what those diagrams are in case of quantum electrodynamics, and we will explain why they are not present in the expectation value of an observable. The 'rules' to decide which diagrams are or are not present in the expectation value of an observable are called selection rules and play an important role in the last chapter.

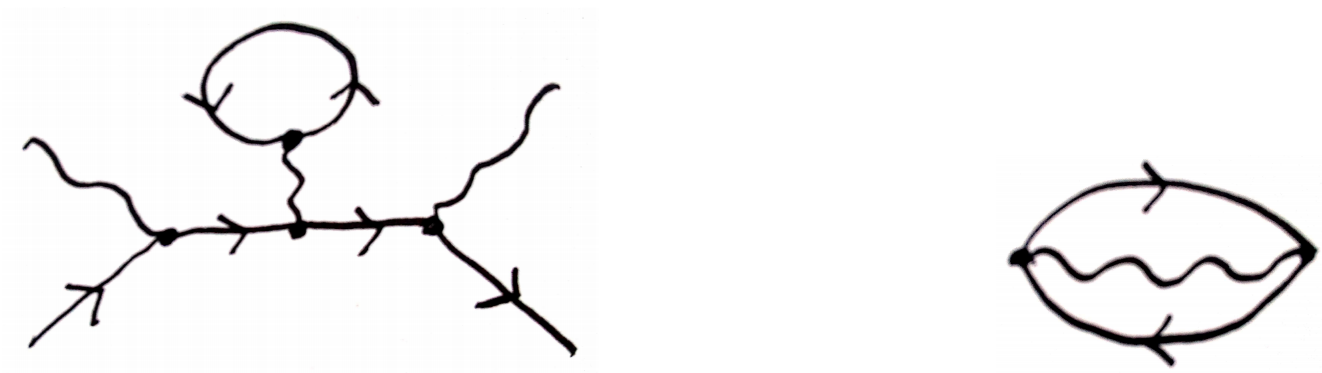
We will also encounter the tadpole and vacuum diagrams in the next chapter. In the next chapter, they will also not appear in the expectation value of the observables, but the diagrams look a bit different. In addition, in the next chapter, we will also discuss other selection rules. The selection rules discussed here apply to every path integral theory. The selection rules in next chapter are specified for those theories.

First of all, we define what a tadpole diagram is in quantum electrodynamics.

Definition 6.10 (Tadpole diagram in QED)**.** A tadpole diagram in quantum electrodynamics is a Feynman diagram which has an interaction vertex with a line connecting that vertex to itself.

An example of a tadpole diagram in quantum electrodynamics is given in Figure 8a.

When calculating the expectation value of an observable with respect to a perturbed Gaussian action, there can be an expression in the expansion of the denominator, and also in the numerator, that have Green's functions with two equal arguments. Such Green's functions correspond to the tadpoles in the Feynman diagrams. However, the value of Green's functions with equal arguments are generally not defined and thus gives an anomaly to the expression for the expectation value. This is solved by prescribing what to do with Green's functions with equal arguments.



(a) An example of a tadpole diagram in QED at order e^4 in an electron-photon scattering process.

(b) An example of a second order (in e) vacuum diagram in QED.

There are multiple approaches to deal with this problem, however in this thesis we will always use the normal ordering prescription. The normal ordering sets all the Green functions with two equal arguments to zero. Hence, all tadpole diagrams vanish.

Another type of diagram that will vanish is called a vacuum diagram or sometimes called a vacuum bubble. This is well-explained with an example in [Sch] on pages 89-92. We will explain shortly what a vacuum diagram is and how it vanishes.

Definition 6.11 (Vacuum diagram in QED)**.** A vacuum diagram in quantum electrodynamics is a Feynman diagram which has a subdiagram that contains no external lines.

An example of a vacuum diagram in quantum electrodynamics is given in Figure 8b.

Based on the formulas, we explain why the vacuum diagrams vanish. Recall the expression for the expectation value of a monomial

$$\langle \phi(x_1) \dots \phi(x_{2s}) \rangle = \frac{\int e^{\frac{i}{2\hbar}\int_{\mathcal{M}} \phi \mathbf{A} \phi} \phi(x_1) \dots \phi(x_{2s}) \mathrm{D}\phi}{\int e^{\frac{i}{2\hbar}\int_{\mathcal{M}} \phi \mathbf{A} \phi} \mathrm{D}\phi}. \tag{121}$$

In the numerator we have the observable $\mathcal{O} = \phi(x_1) \cdots \phi(x_{2s})$ under the integral, and we have the term $\phi \mathbf{A} \phi$ in the exponent. In the denominator we only have the term in the exponent. The vacuum diagrams 'disappear' in the calculation of the expectation value [Sch]. The mechanism behind the disappearance can be explained in simple terms: the numerator splits in a product of non-vacuum diagrams times vacuum diagrams. The vacuum diagrams in the numerator exactly cancel out the vacuum diagrams in the denominator.

Combining the rule of vanishing tadpole diagrams and the rule dividing out vacuum diagrams, our expectation value only contains non-tadpole non-vacuum diagrams

$$\begin{aligned}
\langle\, \mathcal{O}\, \rangle :=& \frac{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S} \mathcal{O}}{\int_{\mathcal{M}} e^{\frac{i}{\hbar}S}} \\
=& \frac{\text{(vacuum diagrams)} \times \text{(non-vacuum diagrams)}}{\text{vacuum diagrams}} \\
=& \text{non-vacuum diagrams} \\
=& \text{(non-vacuum with tadpoles)} + \text{(non-vacuum without tadpoles)} \\
=& \text{non-vacuum without tadpoles.}
\end{aligned}$$

This summarizes the subsection about for Feynman diagrams. We will see that there can be more theory specific selection rules.

7 Path integral approach to the Kontsevich star product

In the coming three subsections, we apply the techniques of quantum field theory to calculate the Groenewold-Moyal star product, a non-associative star product, and eventually the Kontsevich star product for affine Poisson manifolds. In these sections, we therefore explicitly illustrate the relation between the Feynman diagrams from quantum field theory and the Kontsevich graphs from deformation quantization. We use the contribution from Cattaneo in [CKTB], i.e. Chapters 11-14, as the main literature reference for the writing of these subsections.

With respect to Chapter 11 written by Cattaneo, we add a more deliberate explanation of what it means to be a (non-degenerate) critical point in the specific example of the Groenewold-Moyal star product discussed in section 7.1. Furthermore, we better specify the rules for drawing the Feynman diagrams, and give the naming of the vertices and edges similar to the terminology used for the description of Feynman diagrams (e.g. as used in [BurKis19]). Lastly, with respect to Chapter 11 in Cattaneo, in our section 7.2.3 we add a detailed calculation of a non-associative star[11] product $f \,\hat{\star}_{a,b;u,v}\, g(q,p)$ up to and including order $\hbar^2$.

In our section 7.3, Cattaneo's Chapters 12 and 13, where the BRST-formalism, the Poisson sigma model, Green's functions and Wick's theorem within the Poisson sigma model are discussed, are summarized. Our main contribution is that we continue writing where Cattaneo ended on page 159.

More precise, we illustrate Cattaneo's claim [CKTB, p. 159] that the expectation value of the observable $\mathcal{O}_{f,g;0,1}$ within the Poisson sigma model for affine Poisson manifolds amounts to the Kontsevich star product between functions $f, g \in C^\infty(\mathbb{R}^d)$:

$$\begin{aligned} f \star_K g(\mathbf{x}) &= \langle \mathcal{O}_{f,g:0,1} \rangle (\mathbf{x}) \\ &= \frac{\int D\xi \; D\eta \; e^{\frac{i}{\hbar}(S_0+S_1)} \mathcal{O}_{f,g;0,1}(\mathbf{x})}{\int D\xi \; D\eta \; e^{\frac{i}{\hbar}(S_0+S_1)}}, \end{aligned} \tag{122}$$

by illustrating how the calculation of the expectation value up to and including order $\hbar^3$ is done. (For further explanations of Equation (122), we refer to our section 7.3.) Moreover, we explicitly relate all the relevant (we will explain what we mean by relevant in section 7.3) Wick contractions, appearing in calculation of the expectation value, to the Feynman diagrams that represent them.

We illustrate which Feynman diagrams would in principle be present in Equation (122) due to all the Wick contractions. Then, with the use of what we call selection rules, we illustrate which diagrams are eventually not present in the Kontsevich star product for affine Poisson structures. So, we explain, using concepts of quantum field theory, theory of integration and symmetries of the Poisson structure, how to go from a large set of Feynman diagrams, that are possibly present in the expansion of Equation (122) due to all possible Wick contractions, to the set of Feynman diagrams that eventually show up in $f \star_K g(\mathbf{x})$. Thus, we relate the Feynman diagrams to the Kontsevich graphs.

Remark 7.1 (Path integration not on Minkowski space). To avoid confusion, the Gaussian path integral that gives the Groenewold-Moyal star product is defined on

[11] As we have seen in the previous chapter, we are interested in associative star products (See Lemma 5.1 and Theorem 5.2). However, the discussion of this non-associative star product helps to understand calculations with perturbed Gaussian path integrals and their representation in terms of Feynman diagrams.

a circle. The generalization to the Kontsevich star product includes intergration over a disc (with hyperbolic geometry). So, these path integra do not involve integration over Minkowski space.

7.1 The Groenewold-Moyal star product: (non-perturbed) Gaussian path integrals

We want to use the techniques from section 6 to quantize the algebra of functions on phase space $\mathcal{P} = T^*\mathbb{R}^n$ via deformation quantization. We will consider a symplectic Poisson structure on phase space, thus the quantization will come down to calculating the Moyal star product via a path integral approach. We consider the coordinates $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^n$. Consider the canonical symplectic form $\omega = \mathrm{d}p_i \mathrm{d}q^i$, which describes the classical mechanics setting. We parametrize a path $\gamma : I \to T^*\mathbb{R}^n$, with I, a one-dimensional manifold. The action functional we consider in this setting is $S(\gamma) = \int_I \gamma^*\theta$, where $\theta = p_i \mathrm{d}q^i$, the classical potential (associated with the symplectic form). If we parametrize the path $\gamma(t) = (Q(t), P(t))$ for some $t \in I$, we have the quadratic action

$$S(Q, P) = \int_{t \in I} P_i \frac{\mathrm{d}}{\mathrm{d}t} Q^i \mathrm{d}t \tag{123}$$

where the parametrization interval $I = \mathbb{R} \cup \{\infty\} \simeq S^1$ is identified with the circle via the stereographic projection of the compactified real line $I = \mathbb{R} \cup \{\infty\}$ onto the circle. In this way ∞ is identified with the north pole of the circle.

7.1.1 Choice of non-degenerate critical base point: $\infty \in S^1$

In order to perturbatively expand the path integral, we need to choose a suitable base point for expansion. The base point has to be non-degenerate.

A good choice for the critical point is $\infty \in S^1$. At that critical point the quadratic form $f(t) = P_i \frac{\mathrm{d}}{\mathrm{d}t} Q^i$ is non-degenerate. We elaborate on why the quadratic form is non-degenerate at infinity, and why the quadratic form attains a critical value at infinity.

Note that we consider the functions in the action $S(Q, P)$ to be Schwartz functions, in order to have a well-calculable action (i.e. a converging action). Therefore, the quadratic form $f(t)$ is Schwartz function which tends to zero as $t \to \infty$. This property of Schwartz functions ensures that $f'(t)$ will be zero at ∞. So, indeed ∞ is a critical point. Moreover, this critical point is isolated, i.e. non-degenerate, as any symmetry of the real line leaves ∞ untouched. So, any symmetry of S^1 will leave ∞ invariant. This makes ∞ a good choice for the base point.

7.1.2 The expectation value in terms of path integrals.

After the choice of base point, we define the manifold of paths on phase space as

$$\mathcal{M} = \{(Q, P) \in C^\infty(S^1, T^*\mathbb{R}^n)\} \tag{124}$$

and the manifold of path with a restriction on the base point as

$$\mathcal{M}(q, p) = \{(Q, P) \in C^\infty(S^1, T^*\mathbb{R}^n) : Q(\infty) = q,\ P(\infty) = p\}. \tag{125}$$

Note that the manifold $\mathcal{M}$ can be written as a product of spaces $\mathcal{M} = T^*\mathbb{R}^n \times \mathcal{M}(q, p)$ so that Fubini's theorem can be applied to split up an integration over

$\mathcal{M}$ as follows:

$$\int_{\mathcal{M}} = \int_{(q,p)\in T^*\mathbb{R}^n} \mu(q,p) \int_{\mathcal{M}(q,p)} . \tag{126}$$

Therefore, if we write the expectation value of a function $\mathcal{O}$ on $\mathcal{M}$ (which is a polynomial or a formal power series in Q and P), using Fubini's theorem, we get

$$\langle \mathcal{O} \rangle_0 (q,p) = \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S}\mathcal{O}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S}}. \tag{127}$$

Now, make a change of variables $Q = q + \tilde{Q}$, $P = p + \tilde{P}$ to perturb around the base point. In this way, $(\tilde{Q}, \tilde{P})$ vanishes at $t = \infty$. So, the action function then becomes

$$S(Q,P) = S(q+\tilde{Q}, p+\tilde{P}) = \int_{\mathbb{R}} \tilde{P}_i \frac{\mathrm{d}}{\mathrm{d}t} \tilde{Q}^i \mathrm{d}t, \tag{128}$$

and the expectation value becomes

$$\langle \mathcal{O}(Q,P) \rangle_0 (q,p) = \left\langle \mathcal{O}(q+\tilde{Q}, p+\tilde{P}) \right\rangle_0^{\sim} := \frac{\int e^{\frac{i}{\hbar}S} \mathcal{O}(q+\tilde{Q}, p+\tilde{P}) D\tilde{P} D\tilde{Q}}{\int e^{\frac{i}{\hbar}S} D\tilde{P} D\tilde{Q}}. \tag{129}$$

7.1.3 Green's function, normal ordering prescription and Wick's theorem

To be able to apply Wick's theorem, we need to know the Green function of the differential operator in the action functional, together with a normal ordering prescription.

Definition 7.1 (Green's function of the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$)**.** The Green function of the skew-symmetric operator $\frac{\mathrm{d}}{\mathrm{d}t}$ is

$$\theta(u,v) = \begin{cases} -\frac{1}{2} & \text{if } u > v, \\ \frac{1}{2} & \text{if } u < v. \end{cases} \tag{130}$$

In order to avoid anomalies, i.e. the points where the integration over time of Green's function $\theta(u,v)$ will go to infinity, we choose the **normal ordering prescription** $\theta(x,x) = 0$ for any $x \in S^1 \backslash \{\infty\}$.

This allows to get an expression for Wick's theorem on $T^*\mathbb{R}^n$.

Proposition 7.1 (Wick's theorem on $T^*\mathbb{R}^n$)**.** *The contraction of s coordinate variables $\tilde{Q}^{j_k}$ and s momentum variables $\tilde{P}_{i_k}(u_k)$ on $T^*\mathbb{R}^n$ evaluated at $u_1, \ldots, u_s, v_1, \ldots, v_s \in S^1 \backslash \{\infty\}$, with respect to the Gaussian distribution with the action functional given in Equation* (123)*, is*

$$\left\langle \tilde{P}_{i_1}(u_1) \ldots \tilde{P}_{i_s}(u_s) \tilde{Q}^{j_1}(v_1) \ldots \tilde{Q}^{j_s}(v_s) \right\rangle_0^{\sim} = (i\hbar)^s \sum_{\sigma \in \mathcal{G}} \theta(v_{\sigma(1)} - u_1) \ldots \theta(v_{\sigma(s)} - u_s) \delta_{i_1}^{j_{\sigma(1)}} \ldots \delta_{i_s}^{j_{\sigma(s)}}. \tag{131}$$

The effect of the normal ordering prescription is that any contraction of $\tilde{Q}^i(u)$ and $\tilde{P}_i(u)$ will be zero:

$$\left\langle \tilde{P}_i(u) \tilde{Q}^j(u) \right\rangle_0^{\sim} = \theta(u,u) = 0. \tag{132}$$

This means that we only have a non-zero contribution in the right-hand side of the Wick contraction in Equation (131) if a term does not have Green's function evaluated at the same point.

7.1.4 Calculation of the expectation value for the observable $\mathcal{O}_{f;u}$: function evaluation $f(q,p)$

We first calculate the expectation of the observable

$$\mathcal{O}_{f;u}(Q,P) := f(Q(u),P(u)), \qquad f \in C^\infty(T^*\mathbb{R}^n), \quad u \in S^1\backslash\{\infty\},$$

to see that it is equal to the function value of $f(q,p)$ at the base point. To do so, we Taylor expand f around the base point

$$\begin{aligned} f(Q(u),P(u)) &= f(q+\tilde{Q}(u), p+\tilde{P}(u)) \\ &= \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \sum_{|I|=r,\ |J|=s} \tilde{P}_I(u)\tilde{Q}^J(u)\partial^I\partial_J f(q,p). \end{aligned}$$

So that, the expectation value does not depend on the point u as the $\tilde{Q}s$ and $\tilde{P}s$ vanish by the normal ordering prescription, i.e.

$$\begin{aligned} \langle \mathcal{O}_{f;u}\rangle_0 (q,p) &= \left\langle \mathcal{O}_{f;u}(q+\tilde{Q}(u), p+\tilde{P}(u))\right\rangle_0^\sim \\ &= \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \sum_{|I|=r,\ |J|=s} \left\langle \tilde{P}_I(u)\tilde{Q}^J(u)\right\rangle_0^\sim \partial^I\partial_J f(q,p) \\ &= f(q,p) \end{aligned}$$

We thus see that the calculation of the expectation value of $\mathcal{O}_{f;u}$ gives the function value evaluated at the base point $f(q,p)$ as expected.

7.1.5 Calculation of the expectation value for the observable $\mathcal{O}_{f,g;u,v}$: the Groenewold-Moyal star product $(f \star_{GM} g)(q,p)$

If we choose the observable to be a product of two functions $f,g \in C^\infty(T^*\mathbb{R}^n)$ evaluated at different points $u < v$ $\left(\in S^1\backslash\{\infty\}\right)$ given by

$$\mathcal{O}_{f,g;u,v}(Q,P) := f(Q(u),P(u)) \cdot g(Q(v),P(v)),$$

the contractions in the expansion of the expectation value do not all vanish by the normal ordering prescription as was the case for $\mathcal{O}_{f;u}$. In fact, we will show that the expectation value of $\mathcal{O}_{f,g;u,v}$ will be equal to the Groenewold-Moyal star product between f and g.

Using the Taylor expansion of $\mathcal{O}_{f,g;u,v}(Q,P)$ and subsequently Wick's theorem, we calculate the expectation value of $\mathcal{O}_{f,g;u,v}(Q,P)$:

$$\begin{aligned} \langle \mathcal{O}_{f,g;u,v}\rangle_0 (q,p) &= \left\langle f(q+\tilde{Q}(u), p+\tilde{P}(u))g(q+\tilde{Q}(v), p+\tilde{P}(v))\right\rangle_0^\sim \\ &= \sum_{r_1,s_1,r_2,s_2=0}^{\infty} \frac{1}{r_1!s_1!r_2!s_2!} \sum_{|I_1|=r_1,\ |J_1|=s_1} \sum_{|I_2|=r_2,\ |J_2|=s_2} \\ &\quad \left\langle \tilde{P}_{I_1}(u)\tilde{Q}^{J_1}(u)\tilde{P}_{I_2}(v)\tilde{Q}^{J_2}(v)\right\rangle_0^\sim \partial^{I_1}\partial_{J_1} f(q,p)\partial^{I_2}\partial_{J_2} g(q,p) \\ &= \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \left(\frac{i\hbar}{2}\right)^{r+s} (-1)^s \sum_{|I|=r,\ |J|=s} \partial^I\partial_J f(q,p)\partial^I\partial_J g(q,p) \qquad \text{(by Eq. (131))} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^n \prod_{k=1}^{n} P^{i_k j_k} \left(\prod_{k=1}^{n} \partial_{i_k}\right)(f(q,p)) \times \left(\prod_{k=1}^{n} \partial_{j_k}\right)(g(q,p)) \qquad (133) \\ &= f \star_{GM} g(q,p), \end{aligned}$$

where $P^{i_k j_k}$s are the constant Poisson structure coefficients of the Groenewold-Moyal star product (consisting of entries 0, 1 and -1), so that the Groenewold-Moyal star product is recognized. The notation in terms of the Poisson structure coefficients $P^{i_k j_k}$ in Equation (133) comes in handy when we discuss the graphical representation. The factor $\frac{1}{n!}$ in Equation (133) comes from the observation that we choose $n = r+s$, and that in this change of indexing we get a factor $\frac{r!s!}{(r+s)!} = \frac{r!s!}{n!}$, so that we arrive at Equation (133).

7.1.6 The Groenewold-Moyal star product between multiple functions and associativity

As a last step, we want to show the generalization of equality between $\langle \mathcal{O}_{f,g;u,v} \rangle_0^{\sim}$ and $(f \star_{GM} g)(q,p)$. Furthermore, from the calculations below it will be immediately clear that the Groenewold-Moyal star product is associative.

Definition 7.2 (The observable $\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots,u_k}$)**.** We define the observable $\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots,u_k}$ as the multiplication of k functions $f_i \in C^\infty(\mathbb{R}^{2n})$ evaluated at k points in phase space $(Q(u_i), P(u_i)) \in T^*\mathbb{R}^{2n}$ parametrized by their respective different points on the circle $u_1 < u_2 < \cdots < u_k \in S^1 \backslash \{\infty\}$ as

$$\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots,u_k} = f_1(Q(u_1), P(u_1)) \ldots f_k(Q(u_k), P(u_k)). \tag{134}$$

Theorem 7.2. *The expectation value of $\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots,u_k}$ as defined above equals the Groenewold-Moyal star product between functions $f_1(q,p), \ldots, f_k(q,p)$, so that*

$$\langle \mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots,u_k} \rangle_0 (q,p) = f_1(q,p) \star_{GM} \ldots \star_{GM} f_k(q,p), \tag{135}$$

where we explicitly do not have a distribution of parentheses on the right-hand side of the equation, because the Groenewold-Moyal star product is associative.

Proof. Using the same techniques as in the calculation of the expectation value of $\mathcal{O}_{f,g;u,v}$, we calculate

$$\begin{aligned}
\langle \mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots,u_k} \rangle_0 (q,p) =& \left\langle f_1(q+\tilde{Q}(u_1), p+\tilde{P}(u_1)) \ldots f_k(q+\tilde{Q}(u_k), p+\tilde{P}(u_k)) \right\rangle_0^{\sim} = \\
=& \sum_{r_1,s_1,\ldots r_k,s_k=0}^{\infty} \frac{1}{r_1!s_1!\ldots r_k!s_k!} \sum_{|I_1|=r_1,\,|J_1|=s_1} \cdots \sum_{|I_k|=r_k,\,|J_k|=s_k} \\
& \left\langle \tilde{P}_{I_1}(u_1)\tilde{Q}^{J_1}(u_1) \ldots \tilde{P}_{I_k}(u_k)\tilde{Q}^{J_k}(u_k) \right\rangle_0^{\sim} \partial^{I_1}\partial_{J_1} f_1(q,p) \cdots \partial^{I_k}\partial_{J_k} f_k(q,p) = \\
=& \sum_{r_1,\ldots,r_k=0}^{\infty} \frac{1}{r_1!\ldots r_k!} \left(\frac{i\hbar}{2}\right)^{r_1+\cdots+r_k} (-1)^{r_2+\cdots+r_k} \\
& \sum_{|I_1|=r_1,\,\ldots|I_k|=r_k} \partial^{I_1}\partial_{I_2} f_1(q,p) \cdot \partial^{I_2}\partial_{I_3} f_2(q,p) \cdots \partial^{I_k}\partial_{I_1} f_k(q,p) = \\
=& f_1 \star_{GM} \cdots \star_{GM} f_k(q,p),
\end{aligned}$$

where we see explicitly that the calculation of the expectation value does not depend on any kind of distribution of parentheses and hence the Groenewold-Moyal star product also does not. This means that the Groenewold-Moyal star product is associative. □

7.1.7 Groenewold-Moyal star product in terms of Feynman diagrams

We have seen in section 6, that the action of the differential operator on functions in the expansion of the expectation values of an observable could also be displayed graphically with the use of some type of Feynman diagrams. We will explain what the Feynman diagrams for this set-up look like and thus what the graphical representation of differential operators in the Groenewold-Moyal star product look like.

Remark 7.2 (Feynman diagram and Kontsevich oriented graph). Since the non-tadpole Feynman diagrams in this chapter satisfy the conditions to be a Kontsevich graph, the terminology of Feynman diagrams and Kontsevich graphs are used interchangeably. Whereas we will most of the time use the term Feynman diagram, we sometimes use the term Kontsevich graph to stress the fact that the Feynman diagram in the respective path integral theory actually is a Kontsevich graph that is present in a star product.

The calculation of the expectation value of $\mathcal{O}_{f,g;u,v}$ includes the pairing of $\tilde{P}$s with $\tilde{Q}$s via Wick's theorem (see Equation (131) on page 59). A pairing of $\tilde{P}$ with $\tilde{Q}$ can be graphically described by an arrow going from $\tilde{P}$ to $\tilde{Q}$. The corresponding Feynman diagrams will be oriented graphs with $n+2$ labeled vertices (consisting of two sinks and n internal vertices) and $2n$ (optionally labeled) directed edges, where

- the **label** of a vertex is an element from the set $\{0, 1, \dots, n+1\}$.
- the **sinks** are labeled 0 and 1, corresponding to the functions $f(u)$ and $g(v)$ respectively.
- the **internal vertices** labeled with a number $k \in \{2, 3, \dots, n+1\}$, correspond to the Poisson structure coefficients of the Poisson structure $P = P^{i_k j_k}\partial_{i_k}\partial_{j_k}$, with two outgoing arrows corresponding to the partial derivatives ∂_{i_k} and ∂_{j_k} where the Poisson structure coefficients Pi_kj_k is contracted with.
- the directed edges originate from an internal vertex and land on a vertex, which is called the **target vertex**.
- the object Obj, i.e. the function f, g or the Poisson structure coefficient $P^{i_k j_k}$, at a target vertex of directed edge, labeled i_k, is derived by the respective partial derivative, i.e. $\partial_{i_k}(Obj)$.
- the **order in** $\hbar$ determines the number n of internal vertices.

In the case of the Groenewold-Moyal star product none of the internal vertices, corresponding to the Poisson structure coefficients $P^{i_k j_k}$, are target vertices. This is because the action functional S in the exponent does not contain a perturbation to the Gaussian term $P_i \frac{\mathrm{d}}{\mathrm{d}t}Q^i$. In sections 7.2 and 7.3, the Gaussian term in the action functional will be perturbed and consequently the Feynman diagrams for the star products in those sections can have internal vertices that also are target vertices.

Therefore, the Feynman diagrams in the Groenewold-Moyal star product all represent terms of the form $P^{i_1 j_1}\cdots P^{i_k j_k}\partial_{i_1}\cdots\partial_{i_k}(f)\partial_{j_1}\cdots\partial_{j_k}(g)$, so that the Groenewold-Moyal star product

$$\begin{aligned} f \star_{GM} g =& f \cdot g + \frac{i\hbar}{2}P^{i_1 j_1}\partial_{i_1}(f)\partial_{j_1}(g) + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 P^{i_1 j_1}P^{i_1 j_1}\partial_{i_1 i_2}(f)\partial_{j_1 j_2}(g) \\ &+ \frac{1}{3!}\left(\frac{i\hbar}{2}\right)^3 P^{i_1 j_1}P^{i_2 j_2}P^{i_3 j_3}\partial_{i_1 i_2 i_3}(f)\partial_{j_1 j_2 j_3}(g) + O(\hbar^4), \end{aligned}$$

graphically looks like Figure 9.

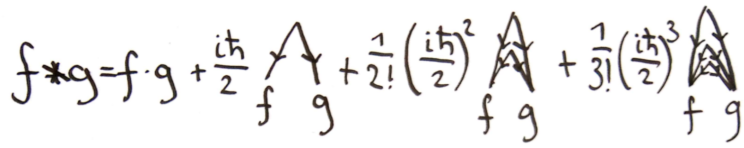


Figure 9: Graphical representation of the Groenwold-Moyal star product up to third order, where the directed edges originate from vertices corresponding to the constant Poisson tensor $P^{i_k j_k}$.

7.2 Quadratically perturbed Gaussian integral: example of calculation of $\hat{\star}_{a,b;u,v}$ up to and including order $\hbar^2$

In section 7.1, we saw that the associative Groenewold-Moyal star product between two functions f and g can be defined in terms of the expectation value of the observable $\mathcal{O}_{f,g;u,v}$. The action $S = \int_{t\in I} P_i \frac{\mathrm{d}Q^i}{\mathrm{d}t}\mathrm{d}t$ in the exponent of the expectation was a Gaussian without a perturbation. As a result the calculation of the expectation value around the unique critical point of S was independent of the evaluation points $u < v$.

We now want to study another star product between two functions, also defined in terms of the expectation value of the observable $\mathcal{O}_{f,g;u,v}$. However, in this case the action is perturbed by a Hamiltonian function H, which will make the star product in this section non-associative. This star product will also not correspond with a Poisson structure.

Definition 7.3 (Action functional perturbed with a Hamiltonian)**.** The action functional S_H, depending on a Hamiltonian $H(Q(t), P(t))$, is defined as

$$S_H = \int_{t\in I} \left(P_i \frac{\mathrm{d}Q^i}{\mathrm{d}t} + H(Q(t), P(t), t) \right) \mathrm{d}t, \tag{136}$$

where the integration domain I can be identified with S^1 as the stereographic projection of the extended real line onto a circle.

With this action functional we can calculate another star product than the Groenewold-Moyal star product. This star product will be called the non-associative star product. Its representation in terms of Feynman diagrams contains all the Feynman diagrams which were present at in the Groenewold-Moyal star product, but now with the anti-symmetric Poisson structure coefficients P^{ij} at the internal vertices replaced by a symmetric tensor G^{ij} (which is not a Poisson tensor). Hence, the . In addition to the graph present in the expression for the Groenewold-Moyal star product, there will now be more graphs allowed. Due to the perturbation by the Hamiltonian term in the action functional, there will also be graphs representing terms in the non-associative star product that have internal vertices as as target vertices.

The calculation of the non-associative star product plays a vital role in this thesis. It shows how perturbed Gaussian integration allows internal vertices to be target vertices of other internal vertices. And all the insights regarding the calculation with perturbed integration, the application of Wick's theorem, the rules for the maximal amount of incoming arrows at an internal vertex are all discussed. All these insights can all be used immediately in the final calculation of the Kontsevich star product via perturbed Gaussian calculation.

7.2.1 Evolution operator

We start the discussion of the perturbed Gaussian integration by introducing the evolution operator $U(q, p, T)$.

Definition 7.4 (Evolution operator $U(q,p,T)$)**.** Let $-\infty < a < b < \infty$ and $I = \mathbb{R} \cup \{\infty\} \simeq S^1$, let the length of the integration domain of the Hamiltonian H be T. Then, the evolution operator is defined as

$$U(q,p,T) := \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S_H}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S}} \tag{137}$$

where we recall that the non-perturbed action functional is $S = \int_{t\in I} P_i \frac{\mathrm{d}Q^i}{\mathrm{d}t}\mathrm{d}t$, and the perturbed action functional S_H is as in Definition 7.3 with the choice of Hamiltonian function

$$H(q,p,t) = h(q,p) \cdot \chi_{[a,b]}(t), \tag{138}$$

where $\chi_{[a,b]}(t)$ is the characteristic function of the interval $[a,b]$.

The evolution operator thus has a perturbed action functional: the exponent is not Gaussian anymore. However, the evolution operator does not contain an observable in it. The observable will be added later in order to define the non-associative star product.

We rewrite the evolution operator

$$\begin{aligned}
U(q,p.T) &= \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S_H}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S}} && && (139)\\
&= \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S} e^{\frac{i}{\hbar}\int H(Q(t),P(t),t)\mathrm{d}t}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S}} && \text{By definition 7.3} && (140)\\
&= \left\langle e^{\frac{i}{\hbar}\int H(Q(t),P(t),t)\mathrm{d}t} \right\rangle_0 (q,p) && \text{Using Equation (127)} && (141)\\
&= \left\langle e^{\frac{i}{\hbar}\int_a^b h(q,p)\mathrm{d}t} \right\rangle_0 (q,p) && \text{Using definition 7.4} && (142)
\end{aligned}$$

Write the integral as a limit of Riemann sums

$$\int_a^b h(Q(t),P(t),t)\mathrm{d}t = \lim_{N\to\infty} \frac{T}{N} \sum_{r=1}^{N} h\left(Q\left(a + r\frac{T}{N}\right),\ P\left(a + r\frac{T}{N}\right)\right). \tag{143}$$

So that the evolution operator can be written as a product of star products

$$U(q,p,T) = \lim_{N\to\infty} \left\langle \prod_{r=1}^{N} e^{\frac{i}{\hbar}\frac{T}{N}h\left(Q\left(a+r\frac{T}{N}\right),\, P\left(a+r\frac{T}{N}\right)\right)} \right\rangle_0 (q,p)$$

Using Equation (134) from page 61:

$$= \lim_{N\to\infty} \left\langle \mathcal{O}_{e^{\frac{i}{\hbar}\frac{T}{N}h}, \ldots, e^{\frac{i}{\hbar}\frac{T}{N}h};\, a+\frac{T}{N},\, a+\frac{2T}{N}, \ldots, a+T} \right\rangle_0 (q,p)$$

We obtain a product of $\star_{GM}$ N times by Equation (135):

$$\begin{aligned}
&= \lim_{N\to\infty} \left(e^{\frac{i}{\hbar}\frac{T}{N}h}\right) \star_{GM} \left(e^{\frac{i}{\hbar}\frac{T}{N}h}\right) \star_{GM} \cdots \star_{GM} \left(e^{\frac{i}{\hbar}\frac{T}{N}h}\right)\\
&= \lim_{N\to\infty} \left(e^{\frac{i}{\hbar}\frac{T}{N}h}\right)^{\star_{GM}N}\\
&:= \exp_{\star_{GM}}\left(\frac{i\,T}{\hbar}h\right)(q,p). && (144)
\end{aligned}$$

Remark 7.3. The final result here contains negative powers of $\hbar$. However, when using the evolution operator to calculate the non-associative star product, the time-independent Hamiltonian h, which is seen in Equation (144), will of order higher than $\hbar$. Hence, there will not be negative powers of $\hbar$ in our actual calculations with the evolution operator.

7.2.2 Non-associative star product: Perturbed Gaussian integration with an observable

After having rewritten the evolution operator in the previous section, we add an observable into the integral to define a non-associative star product. Since this new star product is non-associative, we put a hat on top of it.

Definition 7.5. (non-associative star product).
The non-associative star product $\hat{\star}_{a,b;u,v} : C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)[[\hbar]]$ for the Hamiltonian

$$H(Q(t), P(t)) = \chi_{[a,b]}(t) h(q,p). \tag{145}$$

is defined as

$$f \,\hat{\star}_{a,b;u,v}\, g(q,p) := \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S_H} \mathcal{O}_{f,g;u,v}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S_H}}, \tag{146}$$

where we recall the definition of the observable

$$\mathcal{O}_{f,g;u,v} = f(Q(u), P(u)) g(Q(v), P(v)). \tag{147}$$

Proposition 7.3. *The non-associative star product $\hat{\star}_{a,b;u,v}$ can be expressed as a ratio of expectation values*

$$f \,\hat{\star}_{a,b;u,v}\, g(q,p) = \frac{\left\langle e^{\frac{i}{\hbar} \int_a^b h \mathrm{d}t} \mathcal{O}_{f,g;u,v} \right\rangle_0 (q,p)}{\left\langle e^{\frac{i}{\hbar} \int_a^b h \mathrm{d}t} \right\rangle_0 (q,p)}. \tag{148}$$

Proof.

Rewrite definition 7.5 to obtain

$$f \,\hat{\star}_{a,b;u,v}\, g(q,p) = \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S_H} \mathcal{O}_{f,g;u,v}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S_H}}. \tag{149}$$

By definition of S_H, Equations (136) and (145), we get:

$$= \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S} e^{\frac{i}{\hbar} \int_{t \in I} \chi_{a,b}(t) h(q,p) \mathrm{d}t} \mathcal{O}_{f,g;u,v}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S} e^{\frac{i}{\hbar} \int_{t \in I} \chi_{a,b}(t) h(q,p) \mathrm{d}t}}. \tag{150}$$

We put in a factor of $1 = \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S}}$ to get a ratio of two ratios:

$$= \frac{\dfrac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S} e^{\frac{i}{\hbar} \int_a^b h(q,p) \mathrm{d}t} \mathcal{O}_{f,g;u,v}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S}}}{\dfrac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S} e^{\frac{i}{\hbar} \int_a^b h(q,p) \mathrm{d}t}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar} S}}}. \tag{151}$$

Now we recognize the definition of the evolution operator in Equation (137):

$$= \frac{\left\langle e^{\frac{i}{\hbar} \int_a^b h \mathrm{d}t} \mathcal{O}_{f,g;u,v} \right\rangle_0 (q,p)}{\left\langle e^{\frac{i}{\hbar} \int_a^b h \mathrm{d}t} \right\rangle_0 (q,p)} \tag{152}$$

□

Remark 7.4. We observe that the integration domain $[a,b]$ of the time-independent part of the Hamiltonian $h(q,p)$ does not have its boundaries at infinity. So, the function does no longer vanish at the boundaries, which is the case for Schwartz functions. Hence, the boundary values will explicitly become part of the expression for the non-associative star product. Moreover, the non-associative star product $\hat{\star}_{a,b;u,v}$ will therefore also depend on the evaluation points u and v. That is, generally speaking, the reason why $\hat{\star}_{a,b;u,v}$ is no longer associative.

Proposition 7.4. *The non-associative star product $\hat{\star}_{a,b;u,v}$ can be written in terms of Groenewold-Moyal star products, namely*

$$f\hat{\star}_{a,b;u,v}g(q,p) = \frac{\exp_{\star}\left(\frac{i}{\hbar}(u-a)h\right)\star_{GM} f \star_{GM} \exp_{\star_{GM}}\left(\frac{i}{\hbar}(v-u)h\right)\star_{GM} g \star_{GM} \exp_{\star_{GM}}\left(\frac{i}{\hbar}(b-v)h\right)}{exp_{\star_{GM}}\left(\frac{i}{\hbar}(b-a)h\right)}, \tag{153}$$

where $\exp_{\star_{GM}}$ *is defined as in Equation* (144).

Proof. The denominator in Equation (153) equals the evolution operator in terms of Groenewold-Moyal star products from section 7.2.1.

The numerator can be calculated using the evolution operators, but with a slight adaptation. First recall the definition of the observable

$$\mathcal{O}_{f,g;u,v} = f(Q(u),P(u))g(Q(v),P(v)), \tag{154}$$

to observe the splitting of the integral in three parts which comes between the functions in the observable:

$$\left\langle e^{\frac{i}{\hbar}\int_a^b h\mathrm{dt}}\mathcal{O}_{f,g;u,v}\right\rangle_0(q,p) = \left\langle e^{\frac{i}{\hbar}\int_a^u h\mathrm{dt}}f(Q(u),P(u))e^{\frac{i}{\hbar}\int_u^v h\mathrm{dt}}g(Q(v),P(v))e^{\frac{i}{\hbar}\int_v^b h\mathrm{dt}}\right\rangle_0(q,p)$$

Similar to Equation (143), the integral can be rewritten as a Riemann sum. Then, following along the lines of reasoning in of Equations (139) to (142), we get

$$\begin{aligned}
&\left\langle e^{\frac{i}{\hbar}\int_a^u h\mathrm{dt}}f(Q(u),P(u))e^{\frac{i}{\hbar}\int_u^v h\mathrm{dt}}g(Q(v),P(v))e^{\frac{i}{\hbar}\int_v^b h\mathrm{dt}}\right\rangle_0(q,p)\\
&= \lim_{N\to\infty}\Big\langle \mathcal{O}_{e^{\frac{i}{\hbar}\frac{u-a}{N}h},\dots,e^{\frac{i}{\hbar}\frac{u-a}{N}h};\,a+\frac{u-a}{N},\,a+\frac{2(u-a)}{N},\dots,u}f(Q(u),P(u))\\
&\qquad \mathcal{O}_{e^{\frac{i}{\hbar}\frac{v-u}{N}h},\dots,e^{\frac{i}{\hbar}\frac{v-u}{N}h};\,u+\frac{v-u}{N},\,u+\frac{2(v-u)}{N},\dots,v}g(Q(v),P(v))\\
&\qquad \mathcal{O}_{e^{\frac{i}{\hbar}\frac{b-v}{N}h},\dots,e^{\frac{i}{\hbar}\frac{b-v}{N}h};\,v+\frac{b-v}{N},\,v+\frac{2(b-v)}{N},\dots,b}\Big\rangle_0(q,p)\\
&= \exp_{\star_{GM}}\left(\frac{i}{\hbar}(u-a)h\right)\star_{GM} f \star_{GM} \exp_{\star_{GM}}\left(\frac{i}{\hbar}(v-u)h\right)\star_{GM} g \star_{GM} \exp_{\star_{GM}}\left(\frac{i}{\hbar}(b-v)h\right).
\end{aligned}$$

So, Equality (153) is a matter of ordering the operators ☐

So, the non-associative star product between f and g can be seen as the Groenewold-Moyal star product between f and g interacting with the Hamiltonian. In other words, the functions f and g interact with the Hamiltonian H via the Groenewold-Moyal star product $\star_{GM}$. This idea is not specific to this example, but holds in general for functional integration theories.

7.2.3 Calculation of the non-associative star product $\hat{\star}_{a,b;u,v}$ for a quadratically perturbed Gaussian integral up to and including order $\hbar^2$

In this section, we change the variables $Q(t) = q + \tilde{Q}(t)$ and $P(t) = 0 + \hbar\tilde{P}(t)$, as an expansion around the critical point $(Q(\infty),P(\infty)) = (q,p) = (q,0)$, to ensure that the Hamiltonian term $H(Q(t),P(t),t)$ in the perturbed action functional S_H really can be treated as a perturbation (of higher order in the expansion parameter $\hbar$) to the Gaussian term $S = \int_{t\in S^1} P_i(t)\frac{\mathrm{d}}{\mathrm{d}t}Q^i(t)$. So, as desired, the Hamiltonian $H(q,p,t) = \chi_{[a,b]}(t)\cdot h(Q(t),P(t))$ will have the Hamiltonian

$$h(Q(t) = q+\tilde{Q}(t), P(t) = 0+\tilde{P}(t)) = \frac{1}{2}G^{ij}(q+\tilde{Q}(t))P_i(t)P_j(t) = \frac{1}{2}G^{ij}(q+\tilde{Q}(t))\tilde{P}_i(t)\tilde{P}_j(t). \tag{155}$$

So that, indeed, the latter term of S_H, as defined in Equation (136), is of higher order than the former term of S_H:

$$S_H = \hbar \int_{-\infty}^{\infty} \tilde{P}(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Q}(t)\mathrm{d}t + \frac{\hbar^2}{2}\int_a^b G^{ij}(q+\tilde{Q}(t))\tilde{P}_i(t)\tilde{P}_j(t)\mathrm{d}t. \tag{156}$$

With this perturbed action functional S_H, we know, by proposition 7.3, how we can calculate the non-associative star product via the expectation values. This is done order by order. Before we compute the zeroth to second order, we discuss how the differential operators in the expansion of the non-associative star product are graphically represented.

We recall that this is very similar to the case for the Groenewold-Moyal star product, but now with symmetric tensors G^{ij} replacing the Poisson structure tensors P^{ij}, as said before. Furthermore, the internal vertices G^{ij} can now also be target for incoming arrows due to the perturbation to the Gaussian term in the action and the fact that $G^{ij} = G^{ij}(q+\tilde{Q}(t))$ does depend on the perturbed coordinate $\tilde{Q}(t)$.

7.2.3.1 Wick's theorem

Similar to the computation of the Groenewold-Moyal star product, we need to be able to apply Wick's theorem. Wick's theorem is a little bit different than in section 7.1.3, due to the change of variables $P(t) = \hbar\tilde{P}(t)$ and $Q(t) = q+\tilde{Q}(t)$. To state Wick's theorem we need to know Green's function of the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$ and its normal ordering prescription.

Since the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$ remains the same as for the unperturbed integration in section 7.1, its Green function does not change. Also, we keep the same normal ordering prescription as in section 7.1.3, Equation (130).

Due to the rescaling $P(t) = \hbar\tilde{P}$, the term $\tilde{P}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Q}$ will have an extra factor $\hbar$, which results in the fact that now Wick's theorem, proposition 7.1, is rescaled.

Proposition 7.5 (Wick's theorem on $T^*\mathbb{R}^n$ rescaled)**.** *The contraction of s coordinate variables $\tilde{Q}^{j_k}(v_k)$ and s momentum variables $\tilde{P}_{i_k}(u_k)$ on $T^*\mathbb{R}^n$ evaluated at coordinates $u_1,\ldots,u_s,v_1,\ldots,v_s \in S^1\backslash\{\infty\}$, with respect to the perturbed Gaussian distribution with the action functional given in Equation* (156)*, is*

$$\left\langle \tilde{P}_{i_1}(u_1)\ldots\tilde{P}_{i_s}(u_s)\tilde{Q}^{j_1}(v_1)\ldots\tilde{Q}^{j_s}(v_s)\right\rangle_0^{\sim} = i^s\sum_{\sigma\in\mathcal{G}}\theta(v_{\sigma(1)}-u_1)\ldots\theta(v_{\sigma(s)}-u_s)\delta_{i_1}^{j_{\sigma(1)}}\ldots\delta_{i_s}^{j_{\sigma(s)}}. \tag{157}$$

This is a factor of $\hbar^s$ difference with respect to Wick's theorem in section 7.1 in Equation (131) and in Proposition 7.1.

7.2.3.2 Feynman Diagrams: Graphical representation of the differential operators in the non-associative star product

Similar to the Groenewold-Moyal case, a pairing from $\tilde{P}$ to $\tilde{Q}$ can be graphically described by an arrow going from $\tilde{P}$ to $\tilde{Q}$. The calculation of the expectation value of $\mathcal{O}_{f,g;u,v}$ with respect to the perturbed Hamiltonian S_H again includes the pairing of $\tilde{P}$s with $\tilde{Q}$s via the rescaled Wick's theorem (see Equation (157) on page 67). The pairing of $\tilde{P}$ with $\tilde{Q}$ can be graphically described by an arrow going from $\tilde{P}$ to $\tilde{Q}$. The corresponding Feynman diagrams will be oriented graphs with $n+2$ labeled vertices (consisting of two sinks and n internal vertices) and $2n$ (optionally labeled) directed edges, where

- the **label** of a vertex is an element from the set $\{0,1,\ldots,n+1\}$.

- the **sinks** are labeled 0 and 1, corresponding to the functions $f(u)$ and $g(v)$ respectively.

- the **internal vertices** labeled with a number $k \in \{2, 3, \ldots, n+1\}$, correspond to the symmetric bi-derivation $G = G^{i_k j_k}\partial_{i_k}\partial_{j_k}$, with two outgoing arrows corresponding to the partial derivatives ∂_{i_k} and ∂_{j_k} where the symmetric tensor $G^{i_k j_k}$ is contracted with.

- the directed edges originate from an internal vertex and land on a vertex, which is called the **target vertex**.

- the object Obj, i.e. the function f, g or the symmetric tensor $G^{i_k j_k}$, at a target vertex of directed edge, labeled i_k, is derived by the respective partial derivative, i.e. $\partial_{i_k}(Obj)$.

- the **order in $\hbar$** determines the number n of internal vertices.

We summarize the most important differences between these Feynman diagrams, for the non-associative star product, and the Feynman diagram for the Groenewold-Moyal star product:

1. At the internal vertices **the Poisson structure coefficients $P^{i_k j_k}$ are replaced by the symmetric tensors $G^{i_k j_k}$**.

2. Since $G^{i_k j_k}$ are symmetric, **diagrams where the two target vertices v_1 and v_2 of some internal vertex k coincide, i.e. $v_1 = v_2$, are allowed**. Such a coinciding edge is called a **double edge**. In the Groenewold-Moyal case, these graphs would have been zero[12] due to the anti-symmetry of $P^{i_k j_k}$. This is the first reason why the set of possible graphs in the expansion of the non-associative star product is enlarged with respect to the Groenewold-Moyal star product.

3. **Internal vertices can now be a target vertex for other internal vertices**, because the perturbed action functional S_H, see Equation (156), contains the quadratic perturbation Hamiltonian. Within the perturbation, the symmetric tensor $G^{i_k j_k} = G^{i_k j_k}(\tilde{Q}(t))$ depends on the perturbed coordinate $\tilde{Q}(t)$, and can therefore be coupled to a perturbed momentum $\tilde{P}^{i_l}(t')$ which is contracted with another symmetric tensor $G^{i_l j_l}$. This is the second reason why the set of possible graphs in the expansion of the non-associative star product is enlarged with respect to the Groenewold-Moyal star product.

7.2.3.3 Feynman diagrams and encodings

Let Γ be a Feynman diagram with n internal vertices and 2 sinks. The Feynman diagram Γ represents the action of a differential operator, where the internal vertices in Γ represent the n symmetric bi-derivations $G^{i_k j_k}$, on the functions f and g in the observable. With every Feynman diagram Γ we associate a weight w_Γ. Once, we have an encoding for the Feynman diagram Γ, we can label the weight corresponding to that graph by its encoding.

A graph Γ is encoded by the n pairs of target vertices, which correspond to the targets of the internal vertices. We first give an example, and then remark on some possible ambiguities in the notation. Finally, we explain what choice of notation is made in the rest of the thesis.

[12] For a graphical proof of the fact that graphs with a double edge are zero graphs, see [BBK], page 274.

Example 7.1 (Encoding of a Feynman diagram with three internal vertices)**.** In Figure 10, we see two Feynman diagrams with their respective encodings given by the three pairs of target vertices of the respective arrows: the left diagram is $\Gamma_l = [i_2 j_2; i_3 j_3; i_4 j_4] = [01; 21; 21]$ and the right diagram is $\Gamma_R = [i_2 j_2; i_3 j_3; i_4 j_4] = [01; 12; 12]$.

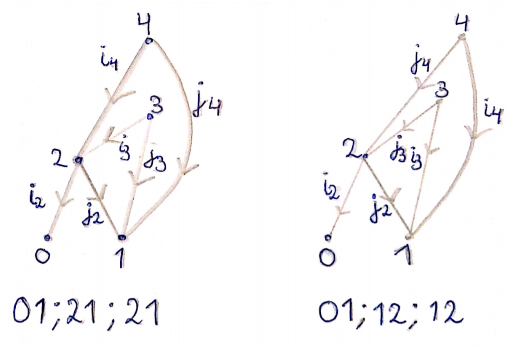


Figure 10: Two Feynman diagrams with their respective encodings.

Both diagrams represent the action of the same differential operator on the functions f and g. However, we have swapped some of the edge labelings between graphs Γ_L and Γ_R. They can be swapped without changing the action of the differential operator, because the tensors G^{ij} are symmetric under the swap of indices $i \leftrightarrow j$. This is seen more clearly when we look at the formula that corresponds to the graph.

The left graph represents the expression

$$\partial_{i_4}\partial_{i_3}(G^{i_2 j_2})G^{i_3 j_3}G^{i_4 j_4}\partial_{i_2}(f)\partial_{j_2}\partial_{j_3}\partial_{j_4}(g), \tag{158}$$

which is equal to the formula represented by the right graph: We retrieve the formula represented by the right graph when we first symmetrically swap the edge indices $i_3 \leftrightarrow j_3$ and $i_4 \leftrightarrow j_4$ in the symmetric tensor $G^{i_3 j_3}$ and $G^{i_4 j_4}$ and then change the labels $i_3 \leftrightarrow j_3$ and $i_4 \leftrightarrow j_4$ in the whole formula, to obtain Equation (159).

$$\partial_{j_4}\partial_{j_3}(G^{i_2 j_2})G^{i_3 j_3}G^{i_4 j_4}\partial_{i_2}(f)\partial_{j_2}\partial_{i_3}\partial_{i_4}(g). \tag{159}$$

The number of different encodings of Feynman diagrams representing the same differential operator, indicates that we should use a standard encoding for the graph and that the symmetry factor should be taken into account in the weight of the graph.

To decide which of the possible encodings of a graph is chosen as the representative of all the different encodings, we have to have some rules of thumb for what we call **a minimal encoding**. A first rule of thumb is that the total sum of the target vertex labelings should be minimized, see Examples below for clarification. A second rule of thumb is that if $i_l + j_l < i_k + j_k$, then the vertex l should be lower with a lower label than vertex k. If the interchange of vertex label l and vertex label k, in order to satisfy rule of thumb two, enforces a violation of rule one, it is not applied.

Example 7.2 (A good encoding)**.** The encoding of the following differential operator

$$\partial_{j_2} G^{i_1 j_1}\partial_{j_1} G^{i_2 j_2}\partial_{i_1} f \partial_{i_2} g \tag{160}$$

is correctly given in the minimal encoding by $[03; 12]$ as $0 \leq 3$, $1 \leq 2$ and $0 + 3 \leq 1 + 2$.

Example 7.3 (A bad encoding corrected)**.** The encoding of the following differential operator

$$\partial_{j_2} G^{i_1 j_1} \partial_{i_1} \partial_{j_1} G^{i_2 j_2} \partial_{i_2} f \tag{161}$$

is [33; 02], but is not in minimal notation because $3 + 3 > 0 + 2$. To solve the problem, we swap the vertex labels 2 and 3, and obtain so that the action of the differential operator on f and g reads as

$$\partial_{i_2} \partial_{j_2} G^{i_1 j_1} \partial_{j_1} G^{i_2 j_2} \partial_{i_1} f, \tag{162}$$

where the encoding [03; 22] in minimial encoding as $0 \leq 3$, $2 \leq 2$ and $0+3 \leq 2+2$.

There are graphs that have an intrinsic symmetry, so that the application of an interchange of vertex labels does not always allow for a minimal encoding. Often such an internal symmetry is immediately visible when the graph is drawn. We give an example.

Example 7.4 (Counterexample to rules of thumb for minimal encoding)**.** Let the encoding of the partial differential operator acting on f and g be as follows

$$\partial_{j_2} G^{i_1 j_1} \partial_{j_1} G^{i_2 j_2} \partial_{i_1} \partial_{i_2} f \tag{163}$$

be given by [03; 02]. This encoding will remain the same under interchange of $G^{i_1 j_1}$ and $G^{i_2 j_2}$, and thus violates the second rule of thumb. However the encoding given will serve as the right one for this graph, as there basically is no other option. Therefore, we just say that this graph is also in the minimal encoding.

Notation 7.1 (The weight w_Γ of a graph Γ)*.* The weight of a graph Γ with n internal vertices, and minimal encoding *Enc*, is denoted by w_{Enc}. The minimal encoding *Enc* assigns to every internal vertices a pairs of target vertices in accordance with the minimal encoding.

Remark 7.5*.* Thus every graph, which can possibly be given multiple different encodings, will only have one minimal encoding. Therefore, the graph will only be present once in the expansion the expectation value of $\mathcal{O}_{f,g;u,v}$. And its corresponding weight will be encoded by the minimal encoding. The number of possible labellings of internal vertices of graph Γ, that will all be encode the same differential operator, will be taken into account as a symmetry factor $\#\mathrm{Aut}(\Gamma)$ in the coefficient of the graph.

7.2.3.4 Taylor expansions

We will now calculate the actual terms in the expansion of the non-associative star product. To do so, we have to use the Taylor expansions of the terms appearing in the formula. First of all, recall that we still use coordinates expanded around the critical point[13], i.e. $P(t) = \hbar \tilde{P}(t)$ and $Q(t) = q + \tilde{Q}(t)$, so that the Taylor expansion of the observable $\mathcal{O}_{f,g;u,v}$ is

$$\mathcal{O}_{f,g;u,v} = f(Q(u),0)g(Q(v),0) = \sum_{r,s=0}^{\infty} \sum_{|I|=r,|J|=s} \frac{1}{r!s!} \tilde{Q}^I(u) \tilde{Q}^J(v) \partial_I(f(q,0)) \partial_J(g(q,0)) \tag{164}$$

with multi-indices $I = i_1 \dots i_r$ and $J = j_1 \dots j_s$. For clarity, we recall that in the above formula $(Q(u), 0) \in T^*\mathbb{R}^n$, where n is the dimension of the manifold. Also,

[13]We recall that the point ∞ is critical, since all functions are Schwartz functions.

recall that, for example, a contraction of indices i_l with i_l means a summation over the dimension of the manifold: $\sum_{i_1=1}^{n}$.

Furthermore, we can Taylor expand the free Hamiltonian term

$$e^{\frac{i\hbar}{2}\int_a^b G^{ij}(q+\tilde{Q}(t))\tilde{P}_i(t)\tilde{P}_j(t)\mathrm{d}t} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n\left(\int_a^b G^{ij}(q+\tilde{Q})\tilde{P}_i\tilde{P}_j\mathrm{d}t\right)^n, \tag{165}$$

where

$$\left(\int_a^b G^{ij}(q+\tilde{Q})\tilde{P}_i\tilde{P}_j\mathrm{d}t\right)^n = \int_a^b\cdots\int_a^b G^{i_1j_1}(q+\tilde{Q})\dots G^{i_nj_n}(q+\tilde{Q})\ \tilde{P}_{i_1}(t_1)\tilde{P}_{j_1}(t_1)\dots\tilde{P}_{i_n}(t_n)\tilde{P}_{j_n}(t_n)\ \mathrm{d}t_1\dots\mathrm{d}t_n, \tag{166}$$

and

$$G^{i_kj_k}(q+\tilde{Q}) = \sum_{|K|\geq 0}\frac{1}{|K|!}\tilde{Q}^K(t_K)\partial_K G^{ij}(q), \tag{167}$$

with $K = k_1k_2\dots k_{|K|}$ a multi-index, which means for example that $\tilde{Q}^{k_1k_2} = \tilde{Q}^{k_1}\tilde{Q}^{k_2}$. We clarify the notation with an example.

Example 7.5 (Clarification of notation: Taylor expansion of $G^{i_kj_k}(q+\tilde{Q})$ up to order 2)**.** The Taylor expansion of $G^{i_kj_k}(q+\tilde{Q})$ up to order 2 is written out without the implicit Einstein summation convention to avoid notational confusion:

$$G^{i_kj_k}(q+\tilde{Q}) = G^{ij}(q) + \sum_{k_1=1}^{n}\tilde{Q}^{k_1}(t_{k_1})\partial_{k_1}G(q) + \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\tilde{Q}^{k_1}(t_{k_1})\tilde{Q}^{k_2}(t_{k_2})\partial_{k_1}\partial_{k_2}G(q) + \sum_{|K|\geq 3}\frac{1}{|K|!}\tilde{Q}^K(t_k)\partial_K G^{ij}(q), \tag{168}$$

where the summation over dummy variables $k_1, k_2, k_3, \dots$ is from 1 to n, because of the dimension of the manifold $\mathbb{R}^n$.

We now have all necessary Taylor expansions at our disposal. These Taylor expansions are all multiplied when we calculate the non-associative star product

$$f\ \hat{\star}_{a,b;u,v}\ g(q,p) := \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S_H}\mathcal{O}_{f,g;u,v}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S_H}}. \tag{169}$$

Because of Wick's theorem, this calculation is simplified, because in any Wick contraction we need a similar amount of $\tilde{P}$s and $\tilde{Q}$s to have a non-zero contribution to the star product expansion. Only in the terms in the multiplication where the number of $\tilde{P}$s and $\tilde{Q}$s is equal, we have to calculate the weight of the corresponding Feynman diagram.

Note that we get an even number of $\tilde{P}$s from the Taylor expansion of the exponent in Equation (165). Therefore, at every order in $\hbar$, we solely need to find the same number of $\tilde{Q}$s from the Taylor expansions of the observable $\mathcal{O}_{f,g;u,v}$, in Equation (164), and from the Taylor expansion of $G^{ij}(q+\tilde{Q})$, see Equation (167), to contract with the $\tilde{P}$s.

7.2.3.5 Calculation of the denominator

All the ingredients for the calculation are there, so let us first calculate terms in the denominator of Equation (148),

$$\left\langle e^{\frac{i}{\hbar}\int_a^b h\mathrm{d}t}\right\rangle_0(q,p), \tag{170}$$

following the above described procedure. The first term is of order $\hbar^0$:

$$\langle 1 \rangle = 1$$

Next, we look at terms of order $\hbar^1$. Here we use that in order for the expectation to be non-zero, we need that there are as many $\tilde{Q}$s as $\tilde{P}$s to 'select' the term with second derivatives with respect to $G^{ij}(q)$. Hence, the only (possibly) non-zero term at order $\hbar^1$ is:

$$\begin{aligned}\left\langle \frac{i\hbar}{2}\int_a^b \tilde{Q}^{kl}(t_1)\partial_{kl}G^{ij}(q)\tilde{P}_i(t_1)\tilde{P}_j(t_1)\mathrm{d}t_1 \right\rangle &= \frac{i\hbar}{2}\partial_{kl}G^{ij}(q)\int_a^b \left\langle \tilde{Q}^k(t_1)\tilde{Q}^l(t_1)\tilde{P}_i(t_1)\tilde{P}_j(t_1)\right\rangle \mathrm{d}t_1 \\ &= \frac{i\hbar}{2}\partial_{kl}G^{ij}(q)\int_a^b i^2\theta(t_1-t_1)\theta(t_1-t_1)(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k)\mathrm{d}t_1 \\ &= 0,\end{aligned}$$

because of the normal ordering prescription $\theta(0) = 0$, i.e. this graph is a tadpole graph.

Definition 7.6. A directed graph where a target vertex l of an internal vertex k is equal to the internal vertex, i.e. $l = k$, is called a **tadpole graph**.

Proposition 7.6. *Consider a Feynman diagram Γ that represent the action of a differential operator on the functions f and g in the expansion of the non-associative star product. If such a directed graph contains a tadpole as a subgraph, its coefficient is zero.*

Proof. Every directed graph with a tadpole as a subgraph will include integration over $\theta(0) = 0$. □

Let us now calculate the (possibly) non-zero term at order $\hbar^2$ in the denominator:

$$\begin{aligned}&\left\langle \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2\int_a^b\int_a^b \tilde{Q}^{k_1l_1}(t_1)\partial_{k_1l_1}G^{i_1j_1}(q)\tilde{Q}^{k_2l_2}(t_2)\partial_{k_2l_2}G^{i_2j_2}(q)\tilde{P}_{i_1}(t_1)\tilde{P}_{j_1}(t_1)\tilde{P}_{i_2}(t_2)\tilde{P}_{j_2}(t_2)\mathrm{d}t_1\mathrm{d}t_2\right\rangle \\ &= \frac{-\hbar^2}{8}\partial_{k_1l_1}G^{i_1j_1}(q)\partial_{k_2l_2}G^{i_2j_2}(q)\int_a^b\int_a^b \left\langle \tilde{Q}^{k_1l_1}(t_1)\tilde{Q}^{k_2l_2}(t_2)\tilde{P}_{i_1}(t_1)\tilde{P}_{j_1}(t_1)\tilde{P}_{i_2}(t_2)\tilde{P}_{j_2}(t_2)\right\rangle \mathrm{d}t_1\mathrm{d}t_2 \\ &= \hbar^2 w_{[33;22]},\end{aligned}$$

where

$$\int_a^b\int_a^b \left\langle \tilde{Q}^{k_1l_1}(t_1)\tilde{Q}^{k_2l_2}(t_2)\tilde{P}_{i_1}(t_1)\tilde{P}_{j_1}(t_1)\tilde{P}_{i_2}(t_2)\tilde{P}_{j_2}(t_2)\right\rangle \mathrm{d}t_1\mathrm{d}t_2$$

$$= \int_a^b\int_a^b (i)^4\theta(t_1-t_2)^4\left(\delta_{i_1}^{k_2}\delta_{j_1}^{l_2}\delta_{i_2}^{k_1}\delta_{j_2}^{l_1} + \delta_{i_1}^{k_2}\delta_{j_1}^{l_2}\delta_{j_2}^{k_1}\delta_{i_2}^{l_1} + \delta_{j_1}^{k_2}\delta_{i_1}^{l_2}\delta_{i_2}^{k_1}\delta_{j_2}^{l_1} + \delta_{j_1}^{k_2}\delta_{i_1}^{l_2}\delta_{j_2}^{k_1}\delta_{i_2}^{l_1}\right)\mathrm{d}t_1\mathrm{d}t_2 \tag{171}$$

$$= \int_a^b\int_a^b 4(i)^4\theta(t_1-t_2)^4\delta_{i_1}^{k_2}\delta_{j_1}^{l_2}\delta_{i_2}^{k_1}\delta_{j_2}^{l_1}\mathrm{d}t_1\mathrm{d}t_2 \tag{172}$$

$$= 4\left(\frac{1}{2}\right)^4(b-a)^2\delta_{i_1}^{k_2}\delta_{j_1}^{l_2}\delta_{i_2}^{k_1}\delta_{j_2}^{l_1},$$

where, to go from line (171) to (172), we used that all four delta tensors will be contracted with $\partial_{k_1l_1}G^{i_1j_1}(q)\partial_{k_2l_2}G^{i_2j_2}(q)$, which give four equal terms. All these possibilities yield the same Feynman diagrams, representing the same differential operator, only with different labelings of arrows. So, we have that

$$\hbar^2 w_{[33;22]} = -\frac{\hbar^2}{8}(b-a)^2\partial_{i_2j_2}G^{i_1j_1}(q)\partial_{i_1j_1}G^{i_2j_2}(q).$$

Remark 7.6. Observe that the graph [33; 22] does not have f or g as a target vertex, i.e. no partial differential operator acting on f and g. This could also directly be seen from the fact that $\mathcal{O}_{f,g;u,v}$ is not present in the denominator. As a consequence, all terms in the denominator will only have derivatives from G^{ij}s going to other G^{ij}s. These graphs have a special name referring to the graphs as they appear for example in quantum field theories.

Definition 7.7 (Vacuum diagram)**.** Consider all directed graphs that represent a differential operator, in the expansion of the non-associative star product. If all target vertices are internal vertices, corresponding to G^{ij}s, then such a graph is called a **vacuum diagram**.

Proceeding order by order, the next contribution in the denominator will be of order $\hbar^3$ and will also be a vacuum diagram. To summarize, we find that the denominator is the following

$$\left\langle e^{\frac{i}{\hbar}\int_a^b h\mathrm{d}t} \right\rangle_0 (q,p) = 1 - \frac{\hbar^2}{32}(b-a)^2 \partial_{i_2 j_2} G^{i_1 j_1}(q) \partial_{i_1 j_1} G^{i_2 j_2}(q) + O(\hbar^3).$$

Thus we demonstrated explicitly that the lowest orders in the expansion of the denominator, given in Equation (170), are (1+ (vacuum diagrams)). This is known from the theory in section 6. We have also seen in that section that all diagrams in the denominator are vacuum diagrams, and are cancelled out by the vacuum diagrams in the numerator. This cancellation occurred because the term in the numerator splits in (non-vacuum diagrams)×(vacuum diagrams in the denominator). We refer to [Sch, Ch. 7] for more examples and clarifications.

7.2.3.6 Calculation of the numerator

The calculation of terms in the numerator will be done in a very similar way. However, we will not take a look at the terms that will correspond to vacuum diagrams as they cancel out, like just discussed. At order zero in $\hbar$, graphically speaking, we just have f placed at u and g at v, but no internal vertices corresponding to G^{ij}s. The term of $\hbar^0$ is fg. This is calculated as:

$$\langle f(q)g(q)\rangle_0 = f(q)g(q).$$

At order one in $\hbar$, we have f placed at u and g at v, with one internal vertex. There are three possible non-tadpole non-vacuum diagrams with one internal vertex. Namely, the three diagrams labeled [0, 0], [0, 1] and [1, 1], and drawn in Figure 11.

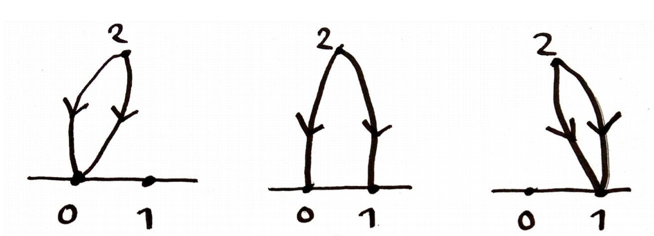


Figure 11: Three Feynman diagrams labeled, from left to right, by [0, 0], [0, 1] and [1, 1].

We explicitly calculate the weight of diagram [0, 1], and leave the calculation of the other two weights as an exercise to the reader. To calculate the weight $w_{[0,1]}$, we need the first order term in the Taylor expansion of f, i.e. $f(Q(u) = q+\tilde{Q}(u)) = \frac{1}{1!}\tilde{Q}^k(u)\partial_k(f(q))$, and the first order term in the Taylor expansion of g, i.e. $g(Q(u) = q+\tilde{Q}(u)) = \frac{1}{1!}\tilde{Q}^l(v)\partial_l(g(q))$, together with the first order term in the

Taylor expansion of the exponential $e^{\frac{i}{\hbar}S_H} = e^{i\hbar G^{ij}(q+\tilde{Q}(t))\tilde{P}_j(t)\tilde{P}_j(t)}$ where we use the zeroth order term of the Taylor expansion of the integrand, i.e. $G^{ij}(q)\tilde{P}_i(t)\tilde{P}_j(t)$, so that

$$
\begin{aligned}
&\left\langle \frac{i\hbar}{2}\int_a^b G^{ij}(q)\tilde{P}_i(t)\tilde{P}_j(t)\tilde{Q}^k(u)\tilde{Q}^l(v)\frac{1}{1!}\partial_k(f(q))\frac{1}{1!}\partial_l(g(q))\mathrm{d}t\right\rangle_0 \\
=&\frac{i\hbar}{2}G^{ij}(q)\partial_k(f(q))\partial_l(g(q))\int_a^b \left\langle \tilde{P}_i(t)\tilde{P}_j(t)\tilde{Q}^k(u)\tilde{Q}^l(v)\right\rangle_0 \mathrm{d}t \\
=&\frac{i\hbar}{2}G^{ij}(q)\partial_k(f(q))\partial_l(g(q))\int_a^b \left((i)^2 2\theta(t-u)\theta(t-v)\delta_i^l\delta_j^k\right)\mathrm{d}t \\
=&\frac{i\hbar}{2}G^{ij}(q)\partial_k(f(q))\partial_l(g(q))\left(\int_a^u + \int_u^v + \int_v^b\right)\left((i)^2 2\theta(t-u)\theta(t-v)\delta_i^l\delta_j^k\right)\mathrm{d}t \\
=&\frac{i\hbar}{2}G^{ij}(q)\partial_k(f(q))\partial_l(g(q))\left(\int_a^u \frac{1}{4} + \int_u^v\left(-\frac{1}{4}\right) + \int_v^b \frac{1}{4}\right)\mathrm{d}t\left((i)^2 2\delta_i^l\delta_j^k\right) \\
=&-\frac{i\hbar}{4}G^{ij}(q)\partial_k(f(q))\partial_l(g(q))\left((u-a)-(v-u)+(b-v)\right)\left(\delta_i^l\delta_j^k\right) \\
=&-\frac{i\hbar}{4}\left(b-a+2u-2v\right)G^{ij}(q)\partial_i(f(q))\partial_j(g(q)).
\end{aligned}
$$

Remark 7.7. The product of propagators $\theta(t-v)\ \theta(t-u)$ is plus or minus $\frac{1}{4}$ depending on where we are in the interval $[a,b]$, and thus we see explicitly that integration over propagators with different arguments is the reason for the star product being dependent on the evaluation points u and v.

In a similar fashion the other two terms at order $\hbar^1$ can be calculated, they are found in the list of coefficients after Equation (176).

Now, we show how a term of order $\hbar^2$ can be calculated, and what kind of complications we encounter at orders two and higher. We calculate a term at order 2 which has an interaction between G^{ij}s

$$
\begin{aligned}
\hbar^2 w \;=\;& \left(\frac{i\hbar}{2}\right)^2\left\langle\int_a^b\int_a^b \partial_l G^{i_1j_1}(q)\tilde{Q}^l(t_1)\tilde{P}_{i_1}(t_1)\tilde{P}_{j_1}(t_1)G^{i_2j_2}(q)\tilde{P}_{i_2}(t_2)\tilde{P}_{j_2}(t_2)\,\times\right. \\
&\left.\times\ \frac{1}{2!}\partial_{r_1r_2}(f(q))\tilde{Q}^{r_1}(u)\tilde{Q}^{r_2}(u)\partial_s(g(q))\tilde{Q}^s(v)\mathrm{d}t_1\mathrm{d}t_2\right\rangle_0 \\
=\;& -\left(\frac{\hbar^2}{8}\right)\partial_l G^{i_1j_1}(q)G^{i_2j_2}(q)\partial_{r_1r_2}(f(q))\partial_s(g(q))\,\times \\
&\times\int_a^b\int_a^b\left\langle\tilde{P}_{i_1}(t_1)\tilde{P}_{j_1}(t_1)\tilde{P}_{i_2}(t_2)\tilde{P}_{j_2}(t_2)\tilde{Q}^l(t_1)\tilde{Q}^{r_1}(u)\tilde{Q}^{r_2}(u)\tilde{Q}^s(v)\right\rangle_0\mathrm{d}t_1\mathrm{d}t_2. \qquad (173)
\end{aligned}
$$

The expectation value in the integral is found by pairing $\tilde{P}(x)$s with $\tilde{Q}(y)$s, which give non-zero contributions for $x \neq y$. Moreover we get a combinatorial factor since some $\tilde{P}$s and $\tilde{Q}$s arise from the same vertex:

$$
\begin{aligned}
&\left\langle\tilde{P}_{i_1}(t_1)\tilde{P}_{j_1}(t_1)\tilde{P}_{i_2}(t_2)\tilde{P}_{j_2}(t_2)\tilde{Q}^l(t_1)\tilde{Q}^{r_1}(u)\tilde{Q}^{r_2}(u)\tilde{Q}^s(v)\right\rangle_0 \\
=&\left(8\theta(t_2-t_1)\theta(t_2-u)\theta(t_1-u)\theta(t_1-v)\delta_{i_2}^l\delta_{j_2}^{r_1}\delta_{i_1}^{r_2}\delta_{j_1}^s + 4\theta(t_2-t_1)\theta(t_2-v)\theta(t_1-u)^2\delta_{i_2}^l\delta_{j_2}^s\delta_{i_1}^{r_1}\delta_{j_1}^{r_2}\right) \qquad (174)
\end{aligned}
$$

Remark 7.8 (Combinatorial factor). Every weight of a non-tadpole non-vacuum diagram in the numerator will have a combinatorial factor. Determining the combinatorial factor is one of the computational steps that we can systematically do when we write down the terms in the expansion of the non-associative star product or similar star products.

Next, the integration is done over t_2 and subsequently over t_1. What is new in this integration with respect to the integration in lower order calculation, is that we have an odd amount of $\theta(t_2-t_1)$s in the first term. Hence, in the first term we have to split the integration in regions where $t_1 > v$ and $t_1 \leq v$. More precisely,

$$
\begin{aligned}
&\int_a^b \theta(t_2-t_1)\theta(t_2-u)\mathrm{d}t_2 \\
&= \left(\int_a^{t_1}+\int_{t_1}^{u}+\int_u^b\right)\theta(t_2-t_1)\theta(t_2-u)\,\chi_{[a,u]}\,(t_1)\mathrm{d}t_2+\left(\int_a^{u}+\int_{u}^{t_1}+\int_{t_1}^b\right)\theta(t_2-t_1)\theta(t_2-u)\,\chi_{[u,b]}\,(t_1)\mathrm{d}t_2 \\
&= \frac{1}{4}\left((a-b+2t_1-2u)\,\chi_{[a,u]}\,(t_1)+(a-b-2t_1+2u)\,\chi_{[u,b]}\,(t_1)\right),
\end{aligned}
$$

where in the last integration, we used the results from the calculations in the lower order terms. Now, integrate the first term of the expectation value in Equation (174) over t_1,

$$
\begin{aligned}
&8\int_a^b \frac{1}{4}\left((a-b+2t_1-2u)\,\chi_{[a,u]}\,(t_1)+(a-b-2t_1+2u)\,\chi_{[u,b]}\,(t_1)\right)\theta(t_1-u)\theta(t_1-v)\delta^l_{i_2}\delta^{r_1}_{j_2}\delta^{r_2}_{i_1}\delta^s_{j_1}\mathrm{d}t_1 \\
&= 2\delta^l_{i_2}\delta^{r_1}_{j_2}\delta^{r_2}_{i_1}\delta^s_{j_1}\left(\int_a^u(a-b+2t_1-2u)\theta(t_1-u)\theta(t_1-v)\mathrm{d}t_1+\int_u^b(a-b-2t_1+2u)\theta(t_1-u)\theta(t_1-v)\mathrm{d}t_1\right) \\
&= \frac{1}{2}\delta^l_{i_2}\delta^{r_1}_{j_2}\delta^{r_2}_{i_1}\delta^s_{j_1}\left[(a-b-2u)(u-a)+(u^2-a^2)+(a-b+2u)(b-2v+u)-(b^2-2v^2+u^2)\right] \\
&= \delta^l_{i_2}\delta^{r_1}_{j_2}\delta^{r_2}_{i_1}\delta^s_{j_1}\left[-(a^2+b^2-v^2)+a(2u+b)+v(b-a-2u)\right],
\end{aligned}
$$

where in the second line the second integral was split up into a part from u to v and in a part from v to b, i.e. $\int_a^b = \int_a^u + \int_u^b$. Integration over the second term in the expectation value in Equation (174) is done via a similar approach

$$
\begin{aligned}
&\int_a^b\int_a^b 4\theta(t_2-t_1)\theta(t_2-v)\theta(t_1-u)^2\delta^l_{i_2}\delta^s_{j_2}\delta^{r_1}_{i_1}\delta^{r_2}_{j_1}\mathrm{d}t_1\mathrm{d}t_2 \\
&= 4\delta^l_{i_2}\delta^s_{j_2}\delta^{r_1}_{i_1}\delta^{r_2}_{j_1}\left(\int_a^v\frac{1}{4}(a-b+2t_1-2v)\theta(t_1-u)^2\mathrm{d}t_1+\frac{1}{4}\int_v^b(a-b-2t_1+2v)\theta(t_1-u)^2\mathrm{d}t_1\right) \\
&= \frac{1}{4}\delta^l_{i_2}\delta^s_{j_2}\delta^{r_1}_{i_1}\delta^{r_2}_{j_1}\left((a-b-2v)(v-a)+(v^2-a^2)+(a-b+2v)(b-v)-(b^2-v^2)\right) \\
&= \frac{1}{2}\delta^l_{i_2}\delta^s_{j_2}\delta^{r_1}_{i_1}\delta^{r_2}_{j_1}\left(-(a^2+b^2+v^2)+ab+v(a+b)\right)
\end{aligned}
$$

So, taking all results together, we see that the Wick contraction in Equation (173) contributes to weights of four different Feynman diagrams

$$
\begin{aligned}
\mathbf{X} &= -\frac{\hbar^2}{16}\left(-(a^2+b^2+v^2)+ab+v(a+b)\right)\partial_{i_2}G^{i_1j_1}(q)G^{i_2j_2}(q)\partial_{i_1i_2}(f(q))\partial_{j_2}(g(q)) \\
&\quad -\frac{\hbar^2}{8}\left(-(a^2+b^2-v^2)+a(2u+b)+v(b-a-2u)\right)\partial_{i_2}G^{i_1j_1}(q)G^{i_2j_2}(q)\partial_{i_1j_2}(f(q))\partial_{j_1}(g(q)) \\
&= \hbar^2 w_{[00;21]}\partial_{i_2}G^{i_1j_1}(q)G^{i_2j_2}(q)\partial_{i_1i_2}(f(q))\partial_{j_2}(g(q))+\hbar^2 w_{[01;20]}\partial_{i_2}G^{i_1j_1}(q)G^{i_2j_2}(q)\partial_{i_1j_2}(f(q))\partial_{j_1}(g(q)) \\
&= \hbar^2 w_{[00;12]}\partial_{j_2}G^{i_1j_1}(q)G^{i_2j_2}(q)\partial_{i_1i_2}(f(q))\partial_{i_2}(g(q))+\hbar^2 w_{[01;02]}\partial_{j_2}G^{i_1j_1}(q)G^{i_2j_2}(q)\partial_{i_1i_2}(f(q))\partial_{j_1}(g(q)),
\end{aligned}
\tag{175}
$$

where in the last equation we used the symmetry of $G^{i_2j_2}$ to swap the indices as explained before.

Proposition 7.7 (Non-ssociative star product $f\,\hat{\star}_{a,b;u,v}\,g(q,0)$ up to and including order $\hbar^2$)**.** *The non-associative star product $f\,\hat{\star}_{a,b;u,v}\,g(q,0)$ up to and including order $\hbar^2$ is calculated to be*

$$
\begin{aligned}
f\,\hat{\star}_{a,b;u,v}\,g(q,0) =& fg(q)+\hbar\,G^{i_1j_1}\left(w_{[00]}\partial_{i_1j_1}fg+w_{[01]}\partial_{i_1}f\partial_{j_1}g+w_{[11]}f\partial_{i_1j_1}g\right)(q)+ \\
& \hbar^2G^{i_1j_1}G^{i_2j_2}\left(w_{[00;00]}\partial_{i_1j_1i_2j_2}fg+2w_{[00;01]}\partial_{i_1j_1i_2}f\partial_{j_2}g+w_{[00;11]}\partial_{i_1j_1}f\partial_{i_2j_2}g+\right. \\
& \left.w_{[01;01]}\partial_{i_1i_2}f\partial_{j_1j_2}g+w_{[01;11]}\partial_{i_1}f\partial_{i_2j_1j_2}g+w_{[11;11]}f\partial_{i_1j_1i_2j_2}g\right)(q)+ \\
& \hbar^2\partial_{j_2}G^{i_1j_1}G^{i_2j_2}\left(w_{[00;02]}\partial_{i_1j_1i_2}fg+w_{[00;12]}\partial_{i_1j_1}f\partial_{i_2}g+w_{[01;02]}\partial_{i_1i_2}f\partial_{j_1}g+\right. \\
& \left.w_{[01;12]}\partial_{i_1}f\partial_{i_2j_1}g+w_{[11;02]}f\partial_{i_2j_1}g+w_{[11;12]}f\partial_{i_1j_1i_2}g\right)(q)+ \\
& \hbar^2\partial_{i_2j_2}G^{i_1j_1}G^{i_2j_2}\left(w_{[00;22]}\partial_{i_1j_1}fg+w_{[01;22]}\partial_{i_1}f\partial_{j_1}g+w_{[11;22]}f\partial_{i_1j_1}g\right)(q)+ \\
& \hbar^2\partial_{j_2}G^{i_1j_1}\partial_{j_1}G^{i_2j_2}\left(w_{[03;02]}\partial_{i_1i_2}fg+w_{[03;12]}\partial_{i_1}f\partial_{i_2}g+w_{[13;12]}f\partial_{i_1i_2}g\right)(q)+ \\
& \hbar^2\partial_{i_2j_2}G^{i_1j_1}\partial_{j_1}G^{i_2j_2}\left(w_{[03;22]}\partial_{i_1}fg+w_{[13;22]}f\partial_{i_1}g+\right)(q)\;+O(\hbar^3),
\end{aligned}
\tag{176}
$$

where the weights are systematically written down below:

$$
\begin{aligned}
w_{[00]} &= -\frac{i}{4}(b-a),\\
w_{[01]} &= -\frac{i}{4}(b-a+2u-2v),\\
w_{[11]} &= -\frac{i}{4}(b-a),\\
w_{[00;00]} &= -\frac{1}{64}(b-a)^2,\\
w_{[00;01]} &= -\frac{1}{8}(b-a)(b-a+2u-2v),\\
w_{[00;11]} &= -\frac{1}{32}(b-a)^2,\\
w_{[01;01]} &= -\frac{1}{32}(b-a+2u-2v)^2,\\
w_{[01;11]} &= -\frac{1}{8}(b-a)(b-a+2u-2v),\\
w_{[11;11]} &= -\frac{1}{64}(b-a)^2,\\
w_{[00;02]} &= +\frac{1}{8}(u^2+ab-u(a+b)),\\
w_{[00;12]} &= +\frac{1}{16}(v^2-v(a+b)+ab),\\
w_{[01;02]} &= -\frac{1}{16}(2v^2+4ub-2ab-v(b-a+2u)),\\
w_{[01;12]} &= -\frac{1}{8}\left(u^2-ab+u(b-a)+2v(a-u)\right),\\
w_{[11;02]} &= +\frac{1}{16}\left(u^2+ab-u(a+b)\right),\\
w_{[11;12]} &= +\frac{1}{8}(v^2+ab-v(a+b)),\\
w_{[00;22]} &= -\frac{1}{64}(b-a)^2,\\
w_{[01;22]} &= -\frac{1}{64}(b-a)(b-a+2u-2v),\\
w_{[11;22]} &= -\frac{1}{64}(b-a)^2,\\
w_{[03;02]} &= \frac{1}{16}(b-2u+a)^2,\\
w_{[03;12]} &= +\frac{1}{16}(b-2u+a)(b-2v+a),\\
w_{[13;12]} &= +\frac{1}{16}(b-2v+a)^2,\\
w_{[03;22]} &= +\frac{1}{16}(v^2+ab-2v(a+b)),\\
w_{[13;22]} &= +\frac{1}{16}(u^2+ab-2u(a+b)).
\end{aligned}
$$

We do not give an explicit proof. However, the idea behind this expansion is that we have taken into account all non-tadpole non-vacuum diagrams that can be formed by couplings of $\tilde{P}$s with $\tilde{Q}$s in all the Taylor expansions. Once all diagrams are known, we calculated the weights. As we have now explicitly calculated the weights of these diagrams at order zero, one and two, we leave the calculation of all the other weights as a (lengthy) exercise to the reader.

To conclude this section, we have calculated weights of the Feynman diagrams in the expansion of $f \,\hat{\star}_{a,b;u,v}\, g(q,0) \mod \hbar^3$ via a perturbed Gaussian integration. The perturbation allowed internal vertices in the Feynman diagrams to be a target vertex for other internal vertices. Moreover, we have seen the importance of several rules, or mechanisms, that reduce the set of Feynman diagrams that eventually appear in the star product.

All these insights prepare us to understand the calculations and illustrations of the Kontsevich star product in the next section.

7.3 Kontsevich star product for affine Poisson manifolds

This section can be seen as a summary of chapters 12 and 13 in [CKTB] and with an elaboration on (and clarification of) chapter 14 in [CKTB]. In the calculation of the Kontsevich star product for affine Poisson structures via the path integral method, we specify how the selection rules reduce the large set of Feynman diagrams to the set of Kontsevich oriented graphs that eventually appear in the Kontsevich star product.

In sections 7.1 and 7.2 we calculated the the expectation value of an observable $\mathcal{O}$ via the formula

$$\langle \mathcal{O} \rangle (q,p) = \frac{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S}\mathcal{O}}{\int_{\mathcal{M}(q,p)} e^{\frac{i}{\hbar}S}}. \tag{177}$$

In the case the action functional S was a Gaussian term, representing the Poisson bracket for the canonical symplectic structure, we obtained the Groenewold-Moyal star product. In this case the action functional S_H was a Gaussian perturbed by a free Hamiltonian, i.e. a quadratic perturbation, Equation (156)), we obtained the non-associative star product.

The non-associative star product allowed sources, corresponding to G^{ij}s, to have incoming arrows. Therefore, the Feynman diagrams in the expansion of the non-associative star product looked more like the diagrams in the Kontsevich star product for general Poisson structures (than the diagrams in the Groenewold-Moyal star product did). However, since the tensors G^{ij} were symmetric, there were still many diagrams that are not presented in the Kontsevich star product.

In this section, we discuss the model that Cattaneo and Felder [CF] have found and with which they calculated, via path integration, the Kontsevich star product for affine Poisson structures. Based on the previous two sections, we have already understood what some of the ingredients for the model should be. First of all, we need to calculate an expectation value with respect to a perturbed Gaussian integral. Secondly, the operator in the Gaussian term should give the right Green functions, which is important for the coefficients in the expansion. Thirdly, the perturbation must somehow include the affine Poisson structure P^{ij}, instead of the symmetric tensor G^{ij}, because the 'sources' in the Kontsevich diagrams represent the Poisson structures P^{ij}. Fourthly, similar to the pairing from momentum $\tilde{P}$ to coordinate $\tilde{Q}$, that represents an arrow in the Feynman diagrammatic representation, we need a coupling from some kind of momentum to a coordinate.

7.3.1 Summary of Chapter 12 and 13 in [CKTB]: BRST-formalism and intuition for the Poisson sigma model

We summarize the most important results of Chapter 12 and 13 discussed in [CKTB]. We do not aim to explain the BRST-formalism. The interested reader, might have a look at [CKTB], Chapter 12 or the orginal text by Becci, Rouet and Stora [BRS], and by Gitman and Tyutin [GT, Tyu], or for a bit more standard work on BRST, we refer to [HT]. Furthermore, we do not aim to explain the superfield formalism, which is explained in more details by Berezin [Ber]. And, we do not explain all the details of the Poisson sigma model [CF], we just want to work with the model in the next subsections using its most important features.

The model for the Kontsevich star product for affine Poisson manifolds that Cattaneo and Felder have come up with, is called the Poisson sigma model. The most important difference between the Poisson sigma and the models for the

Groenewold-Moyal star product and the non-associative product is that the Poisson sigma model includes a degenerate critical point instead of a non-degenerate critical point. This is due to the fact that the action functional in the Poisson sigma model is invariant under the free action of a non-trivial Lie group, i.e. it contains more elements than solely the identity.

In Chapter 12 of [CKTB], Cattaneo constructs the BRST-differential in order to fix a gauge, and thereby solving the degeneracy problem. Next, he introduces the trivial Poisson model (§12.3.1). He explains how the BRST-formalism for the trivial Poisson model gives us the desired Green function [CKTB, p 145-146] for the Kontsevich star product. Also, he introduces the superfields $\tilde{\xi}$ and $\tilde{\eta}$ (in Equation (12.3.3a)) which will act like the momentum and coordinate in the Wick contraction, represented by an arrow in the Feynman diagrams. Eventually, he explains how the ideas for the trivial Poisson model can be extended to a general setting applicable to the Kontsevich star product: this is done in §12.3.4 in [CKTB].

In Chapter 13 of [CKTB], Cattaneo tries to motivate why the specific choice of the action functional for Poisson sigma model is the right one. The most important takeaway from this Chapter is that they try to incorporate the affine Poisson structure in the model. As we explained in a previous paragraph, the affine Poisson structure should come in the model as a perturbation to the Gaussian term. Cattaneo tries to explain what the most general possible perturbations are. Moreover, he tries to clarify why the BRST-formalism only is available when the Poisson structure is affine, and that the BRST-formalism is not enough if the Poisson structure is not strictly affine. The BV-formalism, which can be seen as an extension of the BRST-formalism, will be enough for general Poisson manifolds, this is discussed in [CF].

7.3.2 Poisson sigma model

After the summary of Chapters 12 and 13, we describe the Poisson sigma model for affine Poisson manifolds.

Let $(M, P) \simeq (\mathbb{R}^n, P)$ be an affine Poisson manifold, where P is the affine Poisson structure given by the coordinate functions $P^{ij}(\mathbf{x}) = a_k^{ij} x^k + b^{ij}$ with $\mathbf{x} \in \mathbb{R}^n$.

Remark 7.9. The $(2,1)-$tensor a_k^{ij} and $(2,0)-$tensor b^{ij} are constant in the sense that they do not depend on a point $\mathbf{x} \in \mathbb{R}^{2n}$. However, they behave as tensors under coordinate transformations from local open neighborhoods $U_\alpha(\ni \mathbf{x})$ to other local neighborhoods $U_\beta(\ni \mathbf{x})$ that satisfy $U_\alpha \cap U_\beta \neq \emptyset$.

7.3.2.1 Naming of Poisson sigma model

The terms Poisson and sigma in the naming of the Poisson sigma model refer to two aspects of the model. Namely, Poisson refers to the Poisson manifold (M, P) in the model. And sigma refers to some 'underlying' space Σ in the model. The model includes maps from Σ to the manifold M and maps acting on the respective tangent bundles. The space Σ is underlying in the sense that the physical application and interpretation happens on the manifold M, and Σ can be seen as a 'tool' in the full model.

In the specific example studied here, the underlying space Σ is the hyperbolic upper-half plane $\mathcal{H}$, which is equivalent to the Poincare disk (with hyperbolic geometry). In this sense the underlying space can be seen as a two-dimensional extension of the circle S^1, which served as the underlying space for the path integral calculation of the Groenewold-Moyal star product and the non-associative star product.

Moreover, the linear part $a_k^{ij}x^k$ of the Poisson structure P gives a Lie algebra structure to the dual of $\mathbb{R}^n$ whereas the constant part b^{ij} is a 2-cocycle in the respective Poisson cohomology with trivial coefficients. Therefore, for affine Poisson manifold, we can denote this Lie algebra by $\mathfrak{g}$ and let the fields of the model take values in $\mathfrak{g}$ and $\mathfrak{g}^*$.

7.3.2.2 Fields and boundary conditions within the model

The Poisson sigma model for affine Poisson manifolds has two real (physical) fields: the bosonic fields X and η [CF]. The field X is a map from the hyperbolic upper-half plane $\mathbb{H}$ (or equally from the disc $D = \{u \in \mathbb{R}^2 |\ |u| \leq 1\}$ with hyperbolic geometry) to the manifold $M(\simeq \mathfrak{g}^*)$. The field η is a differential one-form on $\mathbb{H}$ and takes values in the pull-back by X of the cotangent bundle T^*M, in terms of the Lie algebra we can say that X takes values in $\mathfrak{g}$. Seeing it as a section, η is a section of $X^*(T^*M) \otimes T^*\mathbb{H}$. In coordinates, the field X is given by the functions $X^i(u)$ and the field η is given by differential one-forms $\eta_i(u) = \eta_{i,\mu}(u)\mathrm{d}u^\mu$. A boundary condition is imposed on the field η [CKTB], and that is that $\eta(u) = 0$ for all $u \in \partial\mathbb{H}$, $\eta_i(u)$ vanishes tangent to the boundary $\partial\mathbb{H}$, which is an important requirement for the partial integration of the action later on in Equation (181).

7.3.3 Action functional and gauge-fixing

With these fields the action functional for the model is given by

$$S[X,\eta] = \int_{\mathbb{H}} \left(\eta_i(u) \wedge \mathrm{d}X^i(u) + \frac{1}{2}P^{ij}(X(u))\eta_i \wedge \eta_j \right), \tag{178}$$

where

$$P^{ij}(X(u) = \mathbf{x}) = a_k^{ij}x^k + b^{ij} \tag{179}$$

is the affine Poisson structure on $\mathfrak{g}^*$. This action functional has a degenerate critical point at $X(\infty) = x$, because of the boundary conditions and gauge symmetries that leave the action invariant [CF].

The result of applying the BRST-formalism is that, with the help of the BRST-operator δ_{BRST}, we have a well-defined gauge-fixing procedure [CKTB, p 143]. With this BRST-operator, we have a gauge fixing function $F(X,\eta) = \mathrm{d}^*\eta$, together with a gauge-fixing fermion $\Psi_F = \int_{\mathbb{H}} \langle \overline{c}, \mathrm{d}^*\eta \rangle$ so that the gauge-fixed action is given by

$$S_F[X,\eta] = \int_{\mathbb{H}} \left(\eta_i \wedge \mathrm{d}X^i + \frac{1}{2}P^{ij}(X)\eta_i \wedge \eta_j + \lambda^i \mathrm{d}^*\eta_i - \overline{c}^k \mathrm{d}^* \left(\mathrm{d}c_k + \partial_k P^{ij}(X)\eta_i c_j \right) \right), \tag{180}$$

where c_k and $\overline{c}^k$ are ghost and anti-ghost variables respectively, which are odd generators of respectively the graded commutative algebra $\Pi\mathfrak{g}$ and its dual $\Pi\mathfrak{g}^*$, and λ_i is the Lagrange multiplier satisfying $\delta_{\mathrm{BRST}}\overline{c}_i = \lambda_i$ and $\delta\lambda_i = 0$, and $\mathrm{d}^* = \star\mathrm{d}\star$, with $\star$, the Hodge-star operator[14]. The gauge-fixing procedure is well-defined in the sense that the action is invariant under the gauge symmetries, and the expectation value of an observable with respect to this action will not depend

[14]The Hodge star operator $\star$ we are concerned with here acts as a linear operator on the exterior algebra of $\mathbb{R}^n$, mapping $k-$vectors to $(n-k)-$vectors for $0 \leq k \leq n$. It depends on the conformal structure and the orientation. In our case, in coordinates $\star 1 = \mathrm{d}x^1 \wedge \mathrm{d}x^2$, $\star\mathrm{d}x^1 = -\mathrm{d}x^2$ and $\star\mathrm{d}x^2 = \mathrm{d}x^1$.

on the choice of gauge. All of the above defined mathematical objects are all by-products of the BRST-formalism, either in the general construct of the BRST-differential or specifically for this Poisson sigma model.

With this action, we can fix the value of the field X at infinity: $X(\infty) = x$, which is a non-degenerate critical point of the gauge-fixed action. We make a change of coordinates $X = x + \xi$, where the perturbation ξ to the critical point should vanish at the critical point, i.e. $\xi(\infty) = 0$.

Observe that the action functional is of the form $S(y, z) = (y, \mathbf{B}z) + f(y, z)$, where $\mathbf{B}$ is a non-degenerate matrix and $f(y, z)$ is a function quadratic in z, where y is seen as the collection of ξ, $\bar{c}$, and λ, and z as the collection of η and c. Thus the action functional has a standard term, similar to the Gaussian term for the Groenewold-Moyal and the non-associative star product, and a term that is seen as a perturbation to the standard term. We rewrite the gauge-fixed action as a sum of the non-perturbed term S_0 and the perturbation S_1 as $S_F = S_0 + S_1$, where

$$\begin{aligned} S_0[X = x + \xi, \eta] &= \int_{\mathbb{H}} \left(\eta_i \wedge \mathrm{d}\xi^i + \lambda^i \mathrm{d}^* \eta_i - \bar{c}^k \mathrm{d}^* \mathrm{d} c_k\right), \\ S_1[X = x + \xi, \eta] &= \int_{\mathbb{H}} \left(\frac{1}{2} P^{ij}(x + \xi)\eta_i \wedge \eta_j - \bar{c}^k \mathrm{d}^* \left(\partial_k P^{ij}(x + \xi)\eta_i c_j\right)\right). \end{aligned}$$

We rewrite the perturbation S_1 by integration by parts of the second term, so that

$$S_1[X = x + \xi, \eta] = \int_{\mathbb{H}} \int \left(\frac{1}{2} P^{ij}(x + \xi)\eta_i \wedge \eta_j - \mathrm{d}^* \bar{c}^k \cdot \partial_k P^{ij}(x + \xi)\eta_i c_j\right). \quad (181)$$

The final step before the action functional is in the most workable, most simplified form [CF, CKTB], is to introduce "superfields". Superfields are strange mathematical objects, because they are a sum of different degree forms and of fields as well as anti-fields. The superfields are

$$\begin{aligned} \tilde{\xi}^i &= \xi^i - \mathrm{d}^* \bar{c}^i \\ \tilde{\eta}_i &= c_i + \eta_i, \end{aligned} \quad (182)$$

and they will have the same role as $\tilde{Q}$ and $\tilde{P}$ in sections 7.1 and 7.2. The perturbation action functional in terms of the superfields reads

$$S_1[\tilde{\xi}, \tilde{\eta}] = \int_{\mathbb{H}} \int \mathrm{d}^2\theta \frac{1}{2} P^{ij}(x + \tilde{\xi})\tilde{\eta}_i \wedge \tilde{\eta}_j, \quad (183)$$

with the understanding that the integration over $\mathrm{d}^2\theta$ means integration over the supervariables, which is called Berezin's integral, thus selecting the appropriate two-forms from the superfields. For more details, see [Ber] and [CF, p 603]. Cattaneo remarks that since the whole perturbation now only depends on the superfields, and not on the initial (anti-)fields and (anti-)ghosts, the expectation values only depend on the superpropagator (definition below).

7.3.4 Green's functions and Wick's theorem

So, the next step to computing the expectation value of a suitable observable is to know what the superpropagator, i.e the contraction of $\tilde{\xi}$ and $\tilde{\eta}$, is. The superpropagator is given at the end of the calculations by Cattaneo in Chapter 12 of [CKTB], Equation (12.3.5), which we use as a definition here[15].

[15]The superpropagator for general Poisson manifolds is given in [CF, p 602] in case the BV-formalism applies (instead of the BRST-formalism).

Definition 7.8 (Superpropagator and its Green function)**.** The superpropagator between superfields $\tilde{\xi}^i(z)$ and $\tilde{\eta}_j(w)$ for $z, w \in \mathbb{H}$ is given by

$$\left\langle \tilde{\xi}^i(z)\tilde{\eta}_j(w) \right\rangle_0 = i\hbar\delta^i_j\vartheta(z,w) \qquad \text{for } (z \neq w) \in \mathbb{H}, \tag{184}$$

where the Green function for the upper-half plane is given by

$$\vartheta(z,w) = \frac{\mathrm{d}\phi_h}{2\pi}, \qquad \text{for } (z \neq w) \in \mathbb{H} \tag{185}$$

where d is the differential on the configuration space of two distinct points on $\mathbb{H}$ and $\phi_h(z,w)$ the angle between the vertical geodesic through w and geodesic arc, in the hyperbolic geometry of the upper-half plane, through the points z and w, which can be given by the formula

$$\phi_h(z,w) = \frac{1}{2i}\ln\frac{(z-w)(z-\overline{w})}{(\overline{z}-\overline{w})(\overline{z}-w)}, \tag{186}$$

which follows from the geometry of angles in Figure 6 on page 42.

The angle $\phi_h(z,w)$ is, of course, not well-defined if the points are equal $z = w$. However, we would like to be able to treat formulas where the arguments of the superpropagators coincide, i.e. $z = w$. This regularization is done, as we have seen before, with the help of a normal ordering prescription.

Proposition 7.8 (Normal ordering prescription)**.** *The normal ordering prescriptions for Green's functions in the superpropagator* $\left\langle \tilde{\xi}^i(z)\tilde{\eta}_j(w) \right\rangle_0$ *are given by*

$$\vartheta(y,y) = 0 \tag{187}$$

for $y \in \mathbb{H}$.

With the superpropagator given, we can write an expression for Wick's theorem as

$$\left\langle \tilde{\xi}^{i_1}(z_1)\cdots\tilde{\xi}^{i_s}(z_s)\cdot\tilde{\eta}_{j_1}(w_1)\cdots\tilde{\eta}_{j_s}(w_s) \right\rangle_0 = (i\hbar)^s\vartheta(z_{\sigma(1)},w_1)\cdots\vartheta(z_{\sigma(s)},w_s)\delta^{i_{\sigma(1)}}_{j_1}\cdots\delta^{i_{\sigma(s)}}_{j_s} \tag{188}$$

for $z_1,\dots,z_s,w_1,\dots,w_s \in \mathbb{H}$. Therefore, apart from the introduction of the superfields, the steps to come to a calculation and a graphical representation of the star product up till now has been very much the same in sections 7.1 and 7.2.

7.3.5 The observable and the Taylor expansions

The next thing to do is to define the observable. The observable which will give the Kontsevich star product for affine Poisson manifolds is the following

$$\mathcal{O}_{f_1,\dots,f_k;u_1,\dots u_k} = f_1(x+\tilde{\xi}(u_1))\cdots f_k(x+\tilde{\xi}(u_k)), \tag{189}$$

where $f_1,\dots,f_k \in C^\infty(M) = C^\infty(\mathbb{R}^n)$ and $u_1,\dots,u_k \in \partial\mathbb{H} \simeq \mathbb{R}$, and for convenience we choose $u_1 < \cdots < u_k$.

Remark 7.10 (Observable on the boundary $\partial\mathbb{H}$). We see that the observable is only defined on the boundary of the hyperbolic plane. It might be tempting to conclude that we only need the boundary, i.e. the circle, to define the Kontsevich star product. However, one should not forget that when the expectation value of the observable is calculated, this is done with respect to a action functional which has a dependence on points in the disc that are not on the boundary.

Remark 7.11. This remark shortly explains why the observable $\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots u_k}$ is a well-defined function using arguments from the BRST-formalism as explained in [CKTB, p 142-143].

The above function has a well-defined expectation value, i.e. is an observable, because $\delta_{\mathrm{BRST}}X(u) = 0$ for $u \in \partial\mathbb{H}$, where δ_{BRST} is the BRST-differential constructed in the BRST-formalism. The condition $\delta_{\mathrm{BRST}}X(u) = 0$ is required (as shown in Theorem 12.2.5. in [CKTB, p 144]) to make the ghost C vanish on the boundary of the upper-half plane $\partial\mathbb{H}$. Therefore, the function $\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots u_k}$ is a $\delta_{\mathrm{BRST}}-$closed function. Hence, the expectation value of the function is gauge-fixing independent, which makes it well-defined.

The following thing that is relevant are the Taylor expansions of the observable $\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots u_k}$ and of the Poisson tensor in the perturbed action functional $P^{ij}(x+\tilde{\xi})$. The observable $\mathcal{O}_{f_1,\ldots,f_k;u_1,\ldots u_k}$ is a product of the Taylor expansions of the following functions

$$f(X = x + \tilde{\xi}(u)) = \sum_{|K|\geq 0} \frac{1}{|K|!}\tilde{\xi}^K(u)\partial_K(f(x)), \tag{190}$$

where $K = k_1k_2\ldots k_{|K|}$ is a multi-index, where the notation is as explained in Example 7.5 on page 71. And, similarly, the Taylor expansion of the Poisson tensor is given by

$$\begin{aligned} P^{ij}(X = x + \tilde{\xi}(u)) = &\sum_{|K|\geq 0} \frac{1}{|K|!}\tilde{\xi}^K(u)\partial_K(P^{ij}(x)) \\ =&b^{ij} + \tilde{\xi}^{k_1}(u)\frac{\partial}{\partial x^{k_1}}\left(a_k^{ij}x^k\right) \\ =&b^{ij} + a_k^{ij}\tilde{\xi}^k(u). \end{aligned} \tag{191}$$

Since the affine Poisson tensor is of order 1 in $\tilde{\xi}^k$, every interaction vertex - in the Feynman diagrams - will be the target vertex of at most one other interaction vertex. The order of the generalized coordinates $\tilde{\xi}$ of a vertex determines how many incoming arrows can come on that vertex, because every generalized momentum $\tilde{\eta}$ has to couple with a generalized coordinate $\tilde{\xi}$ if it wants to represent an arrow in the Feynman diagrams. For general Poisson manifolds, the order in $\tilde{\xi}$ of a vertex is generally speaking not bounded. Hence, in that case there is no restriction on the number of incoming arrows on an internal vertex.

7.3.6 The Kontsevich star product as a functional integral

All the ingredients are there to calculate the Kontsevich star product for affine Poisson manifolds via the path integral approach.

Theorem 7.9 (Kontsevich star product for an affine Poisson manifold)**.** *The Kontsevich star product for an affine Poisson manifold* $(\mathbb{R}^n, P)$ *is calculated with a perturbed path integral approach, within the Poisson sigma model (see section 7.3.2), and is given for two functions* $f, g \in C^\infty(\mathbb{R}^{2n})$ *by*

$$f \star_K g(\mathbf{x}) = \langle \mathcal{O}_{f,g:0,1}\rangle(\mathbf{x}) = \frac{\int D\xi\ D\eta\ e^{\frac{i}{\hbar}(S_0+S_1)}\mathcal{O}_{f,g;0,1}(\mathbf{x})}{\int D\xi\ D\eta\ e^{\frac{i}{\hbar}(S_0+S_1)}}. \tag{192}$$

where the standard action functional S_0 *in the total gauge-fixed action (cf. Equation* (178) *and* (179)*) is given by*

$$S_0[X = x + \xi, \eta] = \int_{\mathbb{H}} \left(\eta_i \wedge \mathrm{d}\xi^i + \lambda^i \mathrm{d}^*\eta_i - \bar{c}^k\mathrm{d}^*\mathrm{d}c_k\right) \tag{193}$$

and the perturbation term in action functional, S_1, is given in terms of the superfields, defined in Equation (182)*, by*

$$S_1[\tilde{\xi}, \tilde{\eta}] = \int_{\mathbb{H}} \int \mathrm{d}^2\theta \frac{1}{2} P^{ij}(x+\tilde{\xi})\tilde{\eta}_i \wedge \tilde{\eta}_j, \tag{194}$$

and where the affine Poisson structure is $P^{ij}(\mathbf{x}) = a_k^{ij}x^k + b^{ij}$. The perturbation term S_1 is given in terms of the superfields to simplify - the Poisson structure vertices with two outgoing arrows are now clearly visible in S_1 - the calculations [CF], it is not necessary to also rewrite S_0 in terms of the superfields [CKTB, p 158].

We do not give the proof of this theorem here. The proof of this theorem for general Poisson structure, when the BV-formalism is used instead of the BRST-formalism, is the main result of [CF]. The full proof for the affine case is not given in [CKTB], but it argued that based on the examples discussed there it is understood that the $\langle \mathcal{O}_{f,g;u,v}\rangle$ gives $f \star_K g(\mathbf{x})$.

To summarize the proof, we can say that the Kontsevich weights in $f \star_K g(\mathbf{x})$ are reproduced in the perturbative integral by integration of the Green functions with coordinates in the hyperbolic plane. And, the formula for the weight w_Γ of a diagram Γ with n internal edges and labeled directed edges, where the two outgoing arrows from a vertex j are labeled by $(j, v_1(j))$, $(j, v_2(j))$, is given explicitly in [CF, p 603] as an integration of a form containing differentials of the angles (appearing in the Green functions) (cf. Equation (186)), namely

$$(-1)^n w(\Gamma) = \frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \frac{1}{2^n}\left(\frac{i\hbar}{2\pi}\right)^{2n} \int \wedge_{j=1}^n \mathrm{d}\phi_h(u_j, u_{v_1(j)}) \wedge \mathrm{d}\phi_h(u_j, u_{v_2(j)}). \tag{195}$$

This is a factor of $\frac{1}{\prod k_j!}$, with k_j the number of incoming arrows on vertex j, difference with the usual Kontsevich weights, however this will be canceled out by the combinatorial factor arising from the Wick's contractions.

The Poisson structure coefficients, building blocks of the Kontsevich graphs, are incorporated in the Poisson sigma model by placing them in the action functional. The arrows in the Kontsevich star product correspond to the contraction of generalized coordinates $\tilde{\xi}$ and generalized momenta $\tilde{\eta}$. Moreover, in principle, the path integral calculation of the Kontsevich star product results in an expansion containing many more Feynman diagrams than there are Kontsevich diagrams in the Kontsevich star product. However, using selection rules based on the theory of quantum field theory, integration theory for the weights and symmetries of the Poisson structure coefficients, many diagrams are divided out (vacuum diagrams), set to zero (tadpole diagrams) or will vanish.

In the next section, we illustrate which Feynman diagrams, in principle, appear in the expansion of $\langle \mathcal{O}_{f,g;u,v}\rangle$. Then, we illustrate which selection rules eventually determine which Feynman diagrams do not appear in the expansion of $f \star_K g(\mathbf{x})$. This is done for the diagrams up to and including order $\hbar^3$.

7.3.7 Illustration of the selection rules applied to Feynman diagrams in $\mathcal{O}_{f,g;u,v}$ up to (and including) order $\hbar^3$

In this section, we illustrate which Feynman diagrams are, in principle, present in the calculation of $\langle \mathcal{O}_{f,g;u,v}\rangle$, whose formula is given in Equation (192). Then, we discuss which diagrams will eventually not be present in the expansion of $\langle \mathcal{O}_{f,g;u,v}\rangle$ for general Poisson manifolds, due to selection rules (given below). We finish by illustrating which diagrams will be not be present when we switch, consecutively, from general to affine and then to symplectic Poisson manifolds.

The rules for drawing the Feynman diagrams are very similar to what we have seen before in the previous sections, but now with some extra selection rules.

The calculation of the expectation value in Equation (192) includes the pairing of $\tilde{\xi}$s with $\tilde{\eta}$s via Wick's theorem in Equation (188). A pairing from $\tilde{\eta}$ to $\tilde{\xi}$ can be graphically described by an arrow going from $\tilde{\eta}$ to $\tilde{\xi}$. The corresponding Feynman diagrams will be oriented graphs with $n+2$ labeled vertices (consisting of two sinks and n internal vertices) and $2n$ (optionally labeled) directed edges, where

- the **label** of a vertex is an element from the set $\{0, 1, \ldots, n+1\}$.
- the **sinks** are labeled 0 and 1, corresponding to the functions $f(u)$ and $g(v)$ respectively.
- the **internal vertices** labeled with a number $k \in \{2, 3, \ldots, n+1\}$, correspond to the Poisson structures $P = P^{i_k j_k} \partial_{i_k} \partial_{j_k}$, with two outgoing arrows corresponding to the partial derivatives ∂_{i_k} and ∂_{j_k} where the Poisson structure coefficient $P^{i_k j_k}$ is contracted with.
- the directed edges originate from an internal vertex and land on a vertex, which is called the **target vertex**.
- the object Obj, i.e. the function f, g or the Poisson structure coefficients $P^{i_k j_k}$, at a target vertex of directed edge, labeled i_l, $l \neq k$, is derived by the respective partial derivative, i.e. $\partial_{i_l}(Obj)$.
- (Specific for affine Poisson structures:) the internal vertices can have at most one incoming arrow, because the Poisson structure is of first degree in $\tilde{\xi}$, see selection rule below. For general Poisson manifolds, generally speaking, there is no restriction on the number of incoming arrows on an internal vertex.
- the **order in** $\hbar$ determines the number n of internal vertices.

In Figures 13 and 14, we have given some examples of diagrams appearing in $\langle \mathcal{O}_{f,g;u,v} \rangle$ at orders $\hbar^1$, $\hbar^2$ and $\hbar^3$ together with their.

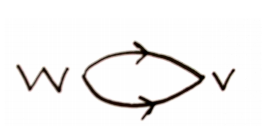
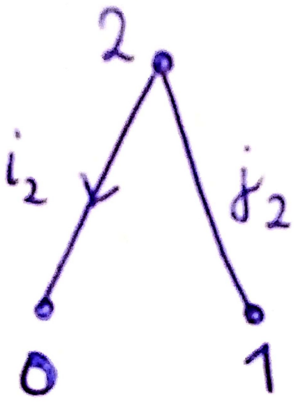


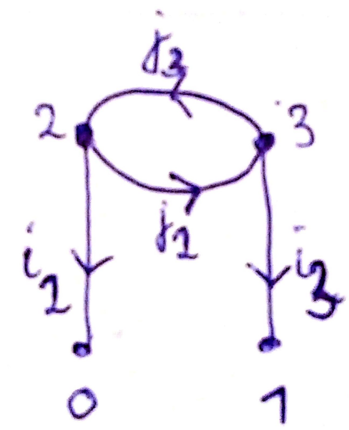
Figure 12: Double edge from vertex w to vertex v

We have given the recipe for drawing the Feynman diagrams. Behind the recipe are some selection rules, that determine which Feynman diagrams will not appear as representations of differential operators in the expansion of the expectation value of the observable $\mathcal{O}_{f,g;0,1}$. We will indicate when a rule only applies to the case of affine Poisson manifolds or symplectic Poisson manifolds, such rule does not apply to general Poisson manifolds.

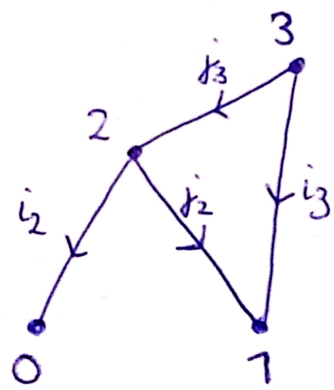
Selection Rule 7.1 (Double edge diagrams)**.** If in a Feynman diagram there exists a vertex v which is a target vertex for both outgoing arrows of some other vertex w, i.e. the diagram contains a double edge from w to v (see Figure 12),



(a) [01], $P^{i_2 j_2}\partial_{i_2}(f)\partial_{j_2}(g)$.



(b) [03; 12], $\partial_{j_3}P^{i_2 j_2}\partial_{j_2}P^{i_3 j_3}\partial_{i_2}(f)\partial_{i_3}(g)$.



(c) [01; 12], $\partial_{j_3}P^{i_2 j_2}P^{i_3 j_3}\partial_{i_2}(f)\partial_{j_2}\partial_{i_3}(g)$.

Figure 13: Three allowed Kontsevich graphs in the expansion of the affine star product between f and g together with their respective encodings, and the formulas they represent.

then this diagram will be zero, because the Poisson structure coefficient is anti-symmetric in the interchange $i \leftrightarrow j$. Thus a diagram with such a vertex will represent a term

$$P^{ij}\partial_i\partial_j(Obj), \tag{196}$$

where the object Obj is either a function f or g or a Poisson structure coefficients, placed at the target vertex labeled by v. The interchange of partial derivatives $\partial_i \leftrightarrow \partial_j$ is symmetric, i.e. leaves the equation intact. However, because of the anti-symmetry of P^{ij}, we have that the relabeling that interchanges $i \leftrightarrow j$ will give the

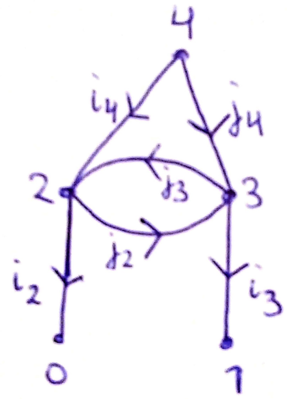
$$\begin{aligned} P^{ij}\partial_i\partial_j(h) =& P^{ji}\partial_j\partial_i(h) \\ =& P^{ji}\partial_i\partial_j(h) \\ =& -P^{ij}\partial_i\partial_j(h) \end{aligned}$$

Selection Rule 7.2 (Tadpole diagrams)**.** Tadpole diagrams all vanish, because of the normal ordering prescription (cf. Equation (7.8)).

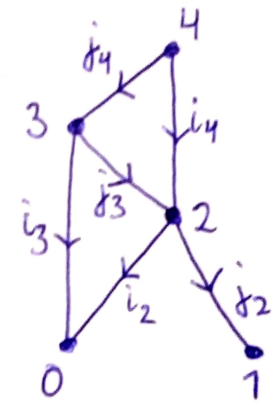
Selection Rule 7.3 (Vacuum diagrams)**.** Vacuum diagrams do not occur in the expansion of the expectation value of $\mathcal{O}_{f,g;0,1}$, because they are divided out of the expectation value as explained in Section 6, and in [Sch].

Selection Rule 7.4 (Lame diagrams: zero integrand in w_Γ)**.** A lame diagram is a diagram where not all sinks are a target vertex. The weight of a lame diagram is zero, and therefore they do not occur in the expansion of the expectation value of $\mathcal{O}_{f,g;0,1}$. This is explained in detail in the course [Kis20].

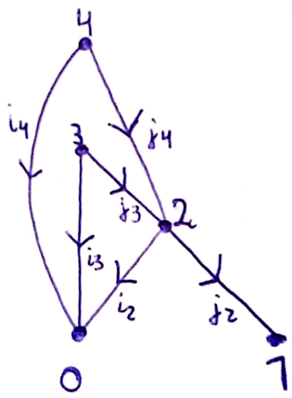
We here give a short sketch of why those weights, see Equation (195), are zero. On the configuration space of n points modulo the action of the affine group (translations along the real line and rescaling of the configuration of n points



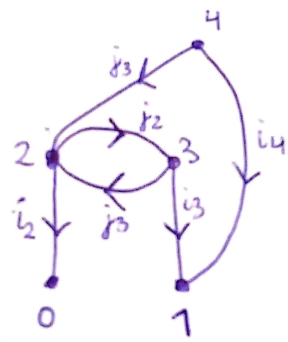
(a) [03; 12; 23], $\partial_{i_4}\partial_{j_3}P^{i_2j_2}\partial_{j_4}\partial_{j_2}P^{i_3j_3}P^{i_4j_4}\partial_{i_2}(f)\partial_{i_3}(g)$.



(b) [01; 02; 23], $\partial_{i_4}\partial_{j_3}P^{i_2j_2}\partial_{j_4}P^{i_3j_3}P^{i_4j_4}\partial_{i_3}\partial_{i_2}(f)\partial_{j_2}(g)$.



(c) [01; 02; 02], $\partial_{j_4}\partial_{j_3}P^{i_2j_2}P^{i_3j_3}P^{i_4j_4}\partial_{i_4}\partial_{i_3}\partial_{i_2}(f)\partial_{j_2}(g)$.



(d) [03; 12; ; 12], $\partial_{j_4}\partial_{j_3}P^{i_2j_2}\partial_{j_2}P^{i_3j_3}P^{i_4j_4}\partial_{i_2}(f)\partial_{i_4}\partial_{i_3}(g)$.

Figure 14: Four Kontsevich graphs, at order $\hbar^3$, that are in the expansion of the Kontsevich star product between f and g for general Poisson structures, see [BurKis19], but which are not in the expansion of the star product for affine Poisson structures. The graphs in Figure (14b), (14c), (14c) all have corresponding mirror graphs, see [BurKis19], which also are not allowed in the affine case. Below the graphs are the respective encodings, and the formulas they represent.

by dilation from the origin $(0, 0) \in \mathbb{H}$), choose coordinates in a clever way to parametrize the hyperbolic angles, see Equation (186), namely: we put the zeroth sink at $(0, 0) \in \mathbb{H}$. We fix $y := y_0$ to be the second Cartesian coordinate of "some internal vertex." The real coordinate of the forgotten sink, i.e. the sink which is not a target of any internal vertex, is indeed a coordinate on the configuration space. But none of the geodesics runs into that forgotten sink, hence none of the angles depend on that coordinate of the zeroth sink. So, in the expression for the weight of the lame diagram, i.e. Equation (195), the degree of the form we integrate over equals the dimension of the configuration space, thus of degree $2n - 2$, and the angles are expressed in terms of $(2n - 2) - 1$ (differentials of) coordinates. Therefore, at least one differential is repeated twice, with the result that the form we integrate over is zero.

Selection Rule 7.5 (Zero diagrams)**.** A Feynman diagram for which the associated poly-differential operator vanishes, by being equal to minus itself, is called a zero diagram. The set of these diagrams contain the set of double edge diagrams, but it is strictly larger for diagrams with three or more internal vertices.

In [BurKis19, p 7-8], it is explained how a zero diagram is obtained. Namely, one starts with a minimally encoded diagram. Then, run over the group S_n of permutations of the internal vertices in the graph at hand, which represent relabelings of the internal vertices. If one minimally encoding relabeled diagram (in the set of all relabeled diagrams) equals the original minimal encoding, but now with an opposite sign, then the graph is zero.

Selection Rule 7.6 (Zero weight diagrams: non-zero integrand in w_Γ, but $w_\Gamma = 0$)**.** Zero weight diagrams are diagrams, where the integrand, in the expression for the weight of the diagram, is not zero but where the integral itself is zero. For more details, see [BurKis19].

These integrals can, for example, be compared with the integration over the non-zero integrand x over the real line: $\int_{\mathbb{R}} x \mathrm{d}x = 0$. The integrand is non-zero, but the integral is zero.

Selection Rule 7.7 (Specific for affine: Diagrams with internal vertices that are a target for two or more arrows)**.** Diagrams with at least one vertex that is a target for two or more arrows do not occur in the expansion of the expectation value of $\mathcal{O}_{f,g;0,1}$. This is caused by the fact that the Taylor expansion of the affine Poisson structure coefficient is of order 1 in $\tilde{\xi}^k$. Hence, in any contraction in the expansion of the expectation value of $\mathcal{O}_{f,g;0,1}$ there can contract at most one generalized momentum $\tilde{\eta}^l$ from some internal vertex labeled by l with the generalized coordinate $\tilde{\xi}^k$ corresponding to the internal vertex labeled by k.

Selection Rule 7.8 (Specific for symplectic: Diagrams with internal vertices that are a target for one or more arrows)**.** Diagrams with at least one vertex that is a target for one or more arrows do not occur in the expansion of the expectation value of $\mathcal{O}_{f,g;0,1}$. This is caused by the fact that the Taylor expansion of the affine Poisson structure coefficient is of order zero in $\tilde{\xi}^k$. Hence, in any contraction in the expansion of the expectation value of $\mathcal{O}_{f,g;0,1}$ there cannot contract any generalized momentum $\tilde{\eta}^l$ from some internal vertex labeled by l with the generalized coordinate $\tilde{\xi}^k$ corresponding to the internal vertex labeled by k, because actually no internal vertex has a generalized coordinate $\tilde{\xi}$ corresponding to it.

With the selection rules at hand, we rewrite the expectation value of the observable $\mathcal{O}_{f,g;u,v}$, with the help of the Taylor expansion of the exponential, and we recall that $\langle\cdot\rangle_0$ means the expectation value with respect to the Gaussian integral $e^{\frac{i}{\hbar}S_0}$, so that, similar to how we rewrote the expectation value in section 7.2, we get

$$\begin{aligned} \langle \mathcal{O}_{f,g;u,v} \rangle &= \frac{\int \mathrm{D}\xi \, \mathrm{D}\eta \; e^{\frac{i}{\hbar}(S_0+S_1)} \mathcal{O}_{f,g;0,1}(\mathbf{x})}{\int D\xi \; D\eta \; e^{\frac{i}{\hbar}(S_0+S_1)}} \\ &= \frac{\left\langle e^{\frac{i}{\hbar}S_1} \mathcal{O}_{f,g;0,1}(\mathbf{x}) \right\rangle_0}{\left\langle e^{\frac{i}{\hbar}S_1} \right\rangle_0} \\ &= \frac{1}{Z} \left\langle \sum_{m=0}^{\infty} \frac{i^m}{\hbar^m m!} \int \mathrm{D}\xi \, \mathrm{D}\eta \; (S_1)^m \mathcal{O}_{f,g;0,1} \right\rangle_0 \\ &= \frac{1}{Z} \sum_{m=0}^{\infty} \frac{i^m}{\hbar^m m!} \int \mathrm{D}\xi \, \mathrm{D}\eta \; \langle (S_1)^m \mathcal{O}_{f,g;0,1} \rangle_0 \, , \end{aligned} \tag{197}$$

where $Z = \left\langle \int D\xi \; D\eta \; e^{\frac{i}{\hbar}S_1} \right\rangle_0$ and only contains vacuum diagrams, that will cancel out all vacuum diagrams in the numerator.

Remark 7.12. Observe that indeed in Equation (197), we can see indeed that the calculation does not require S_1 to be written in terms of superfields.

We now pick out all terms in Equation (197) up to and including order $\hbar^3$. So, we are left with

$$\frac{1}{Z} \sum_{m=0}^{3} \frac{i^m}{\hbar^m m!} \int \mathrm{D}\xi \, \mathrm{D}\eta \; \langle (S_1)^m \mathcal{O}_{f,g;0,1} \rangle_0 \, . \tag{198}$$

Now, this gives a sum of four terms. Within each term, we put the Taylor expansions of $S_1(X = x + \tilde{\xi}, \tilde{\eta})$, $f(X = x + \tilde{\xi}(0))$, $g(X = x + \tilde{\xi}(1))$, only up to and including order $\hbar^m$ for each respective order. This is, because we do not want to consider double edge diagrams. If, in diagram with m internal vertices, more than m arrows come in at a vertex, the diagram is a double edge diagram and will thus vanish because of selection rule 7.1. Here, we do not want to discuss the factor $\frac{1}{Z}$, but we keep in mind that it actually is present in the expression for $\mathcal{O}_{f,g;0,1}$, knowing that it will divide out all the vacuum diagrams, as explained in selection rule 7.3. Thus, Equation (198) without the factor $\frac{1}{Z}$. and only with the Taylor expansions up to and including the order of the number of internal vertices in that term (higher orders will only contribute double edge diagrams), is

$$
\begin{aligned}
&\sum_{n=0}^{3} \frac{i^n}{\hbar^n n!} \left\langle (S_1)^n \mathcal{O}_{f,g;0,1} \right\rangle_0 = \langle f(x)g(x) \rangle_0 \\
&+ \frac{i}{\hbar 1!} \int_{\mathbb{H}} \int \mathrm{d}^2\theta \left\langle P^{i_1 j_1} \tilde{\eta}_{i_1} \tilde{\eta}_{j_1} \left(f(x)g(x) + \frac{1}{1!0!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot g(x) + \frac{1}{0!1!} f(x) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) \right.\right. \\
&+ \left.\left. \frac{1}{1!1!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) \right) \right\rangle_0 \\
&+ \frac{i^2}{\hbar^2 2!} \int_{\mathbb{H}} \int \mathrm{d}^2\theta \left\langle P^{i_1 j_1} P^{i_2 j_2} \tilde{\eta}_{i_1} \tilde{\eta}_{j_1} \tilde{\eta}_{i_2} \tilde{\eta}_{j_2} \left(f(x)g(x) + \frac{1}{1!0!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot g(x) + \frac{1}{0!1!} f(x) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) \right.\right. \\
&+ \frac{1}{2!0!} \tilde{\xi}^{k_1 k_2} \partial_{k_1 k_2}(f(x)) \cdot g(x) + \frac{1}{1!1!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) + \frac{1}{0!2!} f(x) \cdot \tilde{\xi}^{l_1 l_2} \partial_{l_1 l_2}(g(x)) \\
&+ \left.\left. \frac{1}{2!1!} \tilde{\xi}^{k_1 k_2} \partial_{k_1 k_2}(f(x)) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) + \frac{1}{1!2!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot \tilde{\xi}^{l_1 l_2} \partial_{l_1 l_2}(g(x)) + \frac{1}{2!2!} \tilde{\xi}^{k_1 k_2} \partial_{k_1 k_2}(f(x)) \cdot \tilde{\xi}^{l_1 l_2} \partial_{l_1 l_2}(g(x)) \right) \right\rangle_0 \\
&+ \frac{i^3}{\hbar^3 3!} \int_{\mathbb{H}} \int \mathrm{d}^2\theta \left\langle P^{i_1 j_1} P^{i_2 j_2} P^{i_3 j_3} \tilde{\eta}_{i_1} \tilde{\eta}_{j_1} \tilde{\eta}_{i_2} \tilde{\eta}_{j_2} \tilde{\eta}_{i_3} \tilde{\eta}_{j_3} \left(f(x)g(x) + \frac{1}{1!0!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot g(x) + \frac{1}{0!1!} f(x) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) \right.\right. \\
&+ \frac{1}{2!0!} \tilde{\xi}^{k_1 k_2} \partial_{k_1 k_2}(f(x)) \cdot g(x) + \frac{1}{1!1!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) + \frac{1}{0!2!} f(x) \cdot \tilde{\xi}^{l_1 l_2} \partial_{l_1 l_2}(g(x)) \\
&+ \frac{1}{3!0!} \tilde{\xi}^{k_1 k_2 k_3} \partial_{k_1 k_2 k_3}(f(x)) \cdot g(x) + \frac{1}{2!1!} \tilde{\xi}^{k_1 k_2} \partial_{k_1 k_2}(f(x)) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) + \frac{1}{1!2!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot \tilde{\xi}^{l_1 l_2} \partial_{l_1 l_2}(g(x)) \\
&+ \frac{1}{0!3!} f(x) \cdot \tilde{\xi}^{l_1 l_2 l_3} \partial_{l_1 l_2 l_3}(g(x)) + \frac{1}{3!1!} \tilde{\xi}^{k_1 k_2 k_3} \partial_{k_1 k_2 k_3}(f(x)) \cdot \tilde{\xi}^{l_1} \partial_{l_1}(g(x)) + \frac{1}{2!2!} \tilde{\xi}^{k_1 k_2} \partial_{k_1 k_2}(f(x)) \cdot \tilde{\xi}^{l_1 l_2} \partial_{l_1 l_2}(g(x)) \\
&+ \frac{1}{1!3!} \tilde{\xi}^{k_1} \partial_{k_1}(f(x)) \cdot \tilde{\xi}^{l_1 l_2 l_3} \partial_{l_1 l_2 l_3}(g(x)) + \frac{1}{3!2!} \tilde{\xi}^{k_1 k_2 k_3} \partial_{k_1 k_2 k_3}(f(x)) \cdot \tilde{\xi}^{l_1 l_2} \partial_{l_1 l_2}(g(x)) \\
&+ \left.\left. \frac{1}{2!3!} \tilde{\xi}^{k_1 k_2} \partial_{k_1 k_2}(f(x)) \cdot \tilde{\xi}^{l_1 l_2 l_3} \partial_{l_1 l_2 l_3}(g(x)) + \frac{1}{3!3!} \tilde{\xi}^{k_1 k_2 k_3} \partial_{k_1 k_2 k_3}(f(x)) \cdot \tilde{\xi}^{l_1 l_2 l_3} \partial_{l_1 l_2 l_3}(g(x)) \right) \right\rangle_0 ,
\end{aligned}
\tag{199}
$$

where we recall that all Poisson structure coefficients depend on the generalized coordinates: $P^{i_x j_x} = P^{i_x j_x}(x + \tilde{\xi}(v_x)) = \sum_{|Y| \geq 0}^{\infty} \xi^Y \partial_Y P^{i_x j_x}$ as in Equation (191). Also, we implicitly use that all ξs that are in the expansion of f and labelled by some k_i depend on zero: $\tilde{\xi}^{k_i} = \tilde{\xi}^{k_i}(0)$, similarly all ξs that are in the expansion of g and are labelled by some l_i depend on one: $\tilde{\xi}^{l_i} = \tilde{\xi}^{l_i}(1)$.

The expectation values in Equation (199) can be calculated using Wick's theorem. The contraction gives the sum of partial differential operators acting on $f(x)$ and $g(x)$, which we represent by the Feynman diagrams. The weight of the Feynman diagrams are obtained by integrating the Green functions over $\int_{\mathbb{H}} \int \mathrm{d}\theta$, and taking into account the combinatorial factor of the different contraction contributing to the same Feynman diagrams. The weight will be equal to the Kontsevich weights, and will come down to Equation (195).

We will spend no more time on the weights. From now on we will only illustrate which contractions amount to which Feynman diagrams. And then explain, using the selection rules, which diagrams will not be present in $f \star_K g$ eventually.

First observe that many products of terms in the contractions in Equation (199) will already have dropped out, because they have an unequal amount of $\tilde{\xi}$s and $\tilde{\eta}$s. To further simplify our task, we will only draw the non-double edge non-tadpole diagrams. All non-double edge non-tadpole Feynman diagrams up to and including order $\hbar^3$ are drawn in Figure 15.

We will now list which contractions in Equation (199) will produce which Feynman diagrams in Figure 15, where we first write the diagram numbers and then the corresponding contraction term.

$$
\begin{aligned}
0 &: \langle f(x)g(x)\rangle_0 \\
1 &: \left\langle P^{i_1j_1}\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\tilde{\xi}^{k_1}\partial_{k_1}(f(x))\cdot\tilde{\xi}^{l_1}\partial_{l_1}(g(x))\right\rangle_0 \\
2.1 &: \left\langle P^{i_1j_1}\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}P^{i_2j_2}\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))\cdot\tilde{\xi}^{l_1l_2}\partial_{l_1l_2}(g(x))\right\rangle_0 \\
2.2 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}P^{i_2j_2}\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))\cdot\tilde{\xi}^{l_1}\partial_{l_1}(g(x))\right\rangle_0 \\
2.3 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{k_1}\partial_{k_1}(f(x))\cdot\tilde{\xi}^{l_1}\partial_{l_1}(g(x))\right\rangle_0 \\
2.4 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))\right\rangle_0 \\
3.1 &: \left\langle P^{i_1j_1}\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot P^{i_2j_2}\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2k_3}\partial_{k_1k_2k_3}(f(x))\tilde{\xi}^{l_1l_2l_3}\partial_{l_1l_2l_3}(g(x))\right\rangle_0 \\
3.2 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1m_2}\partial_{m_1m_2}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1}\partial_{k_1}(f(x))\tilde{\xi}^{l_1}\partial_{l_1}(g(x))\right\rangle_0 \\
3.3, 3.8, 3.9 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))\tilde{\xi}^{l_1l_2}\partial_{l_1l_2}(g(x))\right\rangle_0 \\
3.4, 3.7, 3.20, 3.22 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))\tilde{\xi}^{l_1}\partial_{l_1}(g(x))\right\rangle_0 \\
3.5 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot P^{i_2j_2}\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2k_3}\partial_{k_1k_2k_3}(f(x))\tilde{\xi}^{l_1}\partial_{l_1}(g(x))\right\rangle_0 \\
3.6 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))\tilde{\xi}^{l_1l_2}\partial_{l_1l_2}(g(x))\right\rangle_0 \\
3.10 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1m_2}\partial_{m_1m_2}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{p_1}\partial_{p_1}(P^{i_3j_3})\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1}\partial_{k_1}(f(x))g(x)\right\rangle_0 \\
3.11 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1m_2}\partial_{m_1m_2}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))g(x)\right\rangle_0 \\
3.12 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{p_1}\partial_{p_1}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2k_3}\partial_{k_1k_2k_3}(f(x))g(x)\right\rangle_0 \\
3.13, 3.14 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1m_2}\partial_{m_1m_2}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{p_1}\partial_{p_1}(P^{i_3j_3})\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial_{k_1k_2}(f(x))g(x)\right\rangle_0 \\
3.15 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}(P^{i_3j_3})\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2k_3}\partial_{k_1k_2k_3}(f(x))g(x)\right\rangle_0 \\
3.16 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1m_2}\partial_{m_1m_2}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{p_1p_2}\partial_{p_1p_2}(P^{i_3j_3})\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}f(x)g(x)\right\rangle_0 \\
3.17 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot P^{i_2j_2}\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}P^{i_3j_3}\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial k_1k_2(f(x))\tilde{\xi}^{l_1l_2}\partial l_1l_2(g(x))\right\rangle_0 \\
3.18, 3.21 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{p_1}\partial_{p_1}(P^{i_3j_3})\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial k_1k_2(f(x))\tilde{\xi}^{l_1}\partial l_1(g(x))\right\rangle_0 \\
3.19 &: \left\langle \tilde{\xi}^{n_1n_2}\partial_{n_1n_2}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1}\partial_{m_1}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{p_1}\partial_{p_1}(P^{i_3j_3})\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1k_2}\partial k_1(f(x))\tilde{\xi}^{l_1}\partial l_1(g(x))\right\rangle_0 \\
3.23, 3.24 &: \left\langle \tilde{\xi}^{n_1}\partial_{n_1}(P^{i_1j_1})\tilde{\eta}_{i_1}\tilde{\eta}_{j_1}\cdot\tilde{\xi}^{m_1m_2}\partial_{m_1m_2}(P^{i_2j_2})\tilde{\eta}_{i_2}\tilde{\eta}_{j_2}\tilde{\xi}^{p_1}\partial_{p_1}(P^{i_3j_3})\tilde{\eta}_{i_3}\tilde{\eta}_{j_3}\tilde{\xi}^{k_1}\partial k_1(f(x))\tilde{\xi}^{l_1}\partial l_1(g(x))\right\rangle_0
\end{aligned}
$$

As we can see, some diagrams come from the same contraction term. For example, Figures 3.3, 3.8 and 3.9 comes frome the same contraction term and also 3.18 and 3.21 come from the same contraction term, but they are different because different pairings of ξs and ηs within contraction term result in different Feynman diagrams. This also happened in case of the non-associative star product (cf. Equation (175), see page 75).

We have now come to the most exciting result of this whole thesis. Namely, we can conclude which Feynman diagrams in Figure 15 on page 90 drop out, due to the selection rules, and thus how we arrive from the large set of Feynman diagrams in the path integral calculation to the Kontsevich oriented graphs in the expansion of $f \star_K g(\mathbf{x}) \mod \hbar^4$ given in Equation 105.

In Table 1 on page 91, we have summarized which Feynman diagrams do not appear in the final expression of $f \star_K g(\mathbf{x})$. After selection rule 7.5, we are left with all diagrams that appear in the Kontsevich star product for general Poisson structures, as expressed in Equation (105). After selection rule 7.7, we are left with all diagrams that appear in the Kontsevich star product for affine Poisson structures, i.e. only diagrams with at most incoming arrow at internal vertices. After selection rule 7.8, we are left with the diagrams that appear in the Groenewold-Moyal

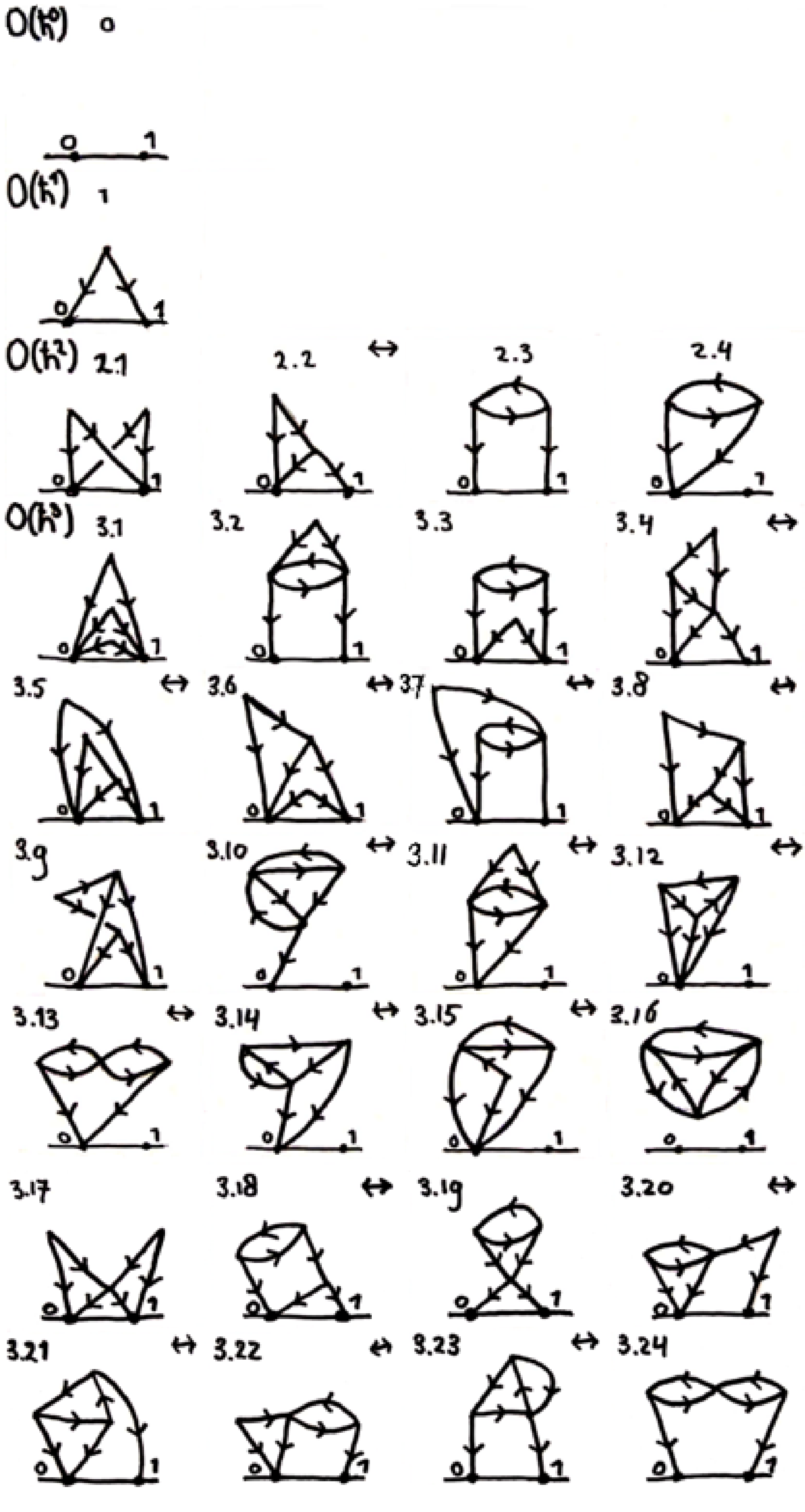


Figure 15: Non-double edge non-tadpole Feynman diagrams that are in principle present in the expansion of $\langle \mathcal{O}_{f,g;0,1} \rangle$ up to and including order $\hbar^3$. The diagrams have a diagram number either straight above the diagram or in the top left corner (from diagram 3.2 onwards). The symbol $\leftrightarrow$ in the top right corner of diagram indicates that such a diagram has a mirror-reflection (over the sinks $0 \leftrightarrow 1$). Only the sinks are labeled by 0 and 1, we did leave out the labeling of the internal vertices and of the edges.

# Selection Rule	Name Selection Rule	Diagrams in Figure 15
7.1	Double edge	n.a.
7.2	Tadpole	n.a.
7.3	Vacuum	3.16
7.4	Lame	2.4, 3.10-3.15
7.5	Zero diagram	3.9
7.6	Zero weight	3.17-3.24
7.7	Affine Poisson	3.2, 3.4, 3.7
7.8	Symplectic Poisson	2.2, 2.3, 3.3, 3.5, 3.6, 3.8

Table 1: Selection rules together with the respective Feynman diagrams in Figure 15 that do not appear in the Kontsevich star product.

star product, i.e. the star product for the symplectic case.

To summarize, we have seen explicitly illustrated which Feynman diagrams, that appear in the path integral calculation of the expectation value $\langle \mathcal{O}_{f,g;0,1} \rangle (\mathbf{x})$ mod $\hbar^4$, eventually appear in the Kontsevich star product $f \star_K g(\mathbf{x})$. We formulated eight selection rules that reduce the big set of Feynman diagrams to the smaller set of Kontsevich graphs. Also, we have explained how formulas in the path integral calculation should be related to the Feynman diagrams. In Figure 15, we have give drawn all non-double edge non-tadpole Feynman diagrams that are in principle present up to and including order $\hbar^3$. In Table 1, we have listed the selection rules. Alongside with the selection rules, we have listed the Feynman diagrams that are disappearing due to the respective selection rules.

8 Summary

In this thesis, we studied different methods to go from a classical to a quantum theory. Eventually, we had a closer look at deformation quantization. We illustrated the relation between the Kontsevich graphs appearing in the Kontsevich star product (with a central role in the theory of deformation quantization) and Feynman diagrams appearing in a path integral calculation within the Poisson sigma model found by Cattaneo and Felder.

The starting point for the classical theory was Hamiltonian mechanics on phase space. This was described via a commutative algebra of functions and a Poisson bracket governing the time evolution of functions on phase space.

Then, we first looked at what we call geometric quantization. Central in geometric quantization is the idea of Dirac, namely that the commutative algebra of functions on phase space is replaced by the non-commutative algebra of operators on Hilbert space. In addition, Dirac replaced the Poisson bracket acting on functions q and p, i.e. $\{q, p\}$, by a multiple of the anti-commutator acting on the quantized versions of q and p, i.e. $\frac{1}{i\hbar}[\hat{q}, \hat{p}]$.

The quantization proposed by Dirac has two problems. First of all, the problem of ordering. The more general problem formulated by Groenewold and Van Hove is: no quantization that sends the classical Poisson bracket on phase space to the anti-commutator on Hilbert space generates a one-to-one correspondence between the algebra of functions on phase space and the operators on Hilbert space.

In contrast to Dirac's quantization, the Wigner-Weyl quantization establishes a one-to-one relation between phase space distributions and operators on Hilbert space. Phase distributions are quasi-probability distributions. There are multiple quantizations like the Wigner-Weyl quantization. Every quantization comes with its own phase space distribution and has its own (dis-)advantages when doing calculations on quantum systems.

Based on the Groenewold-van Hove theorem and the one-to-one relation established by Wigner and Weyl, Groenewold proposed to deform the Poisson bracket on phase space. The deformed Poisson bracket, called the Groenewold-Moyal bracket, allows for a consistent quantization (via the Wigner-Weyl quantization scheme). The Groenewold-Moyal bracket is defined through the Groenewold-Moyal star product. The Groenewold-Moyal star product is a deformation (in parameter $\hbar$) of the usual commutative product on phase space.

After the seminal papers by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer, the idea of deforming the commutative algebra of functions instead of replacing it by a non-commutative algebra of operators, was generalized from only symplectic Poisson manifolds (phase space) to general Poisson manifolds. The study of deforming the commutative algebra of classical functions (or observables) is called deformation quantization. Deformation quantization is seen as an alternative to geometric quantization (Dirac quantization and phase space quantization). The deformed products in deformation quantization are no longer commutative, but they are associative.

After a lot of different contributions, Kontsevich, in 1999 (published in 2003), claimed and proved that for any algebra $\mathcal{A}$ of smooth functions on a finite-dimensional differentiable manifold any Poisson bracket on $\mathcal{A}$ lifts to an associative formal deformation on the respective deformed algebra $A[[t]]$. This deformed product is known as the Kontsevich star product.

Kontsevich star product is a formal sum (usually expanded in $\hbar$) of linear partial differential operators that can act on functions in the (deformed) algebra. The partial differential operators are represented by the Kontsevich oriented graphs. In Kontsevich graphs, Poisson structure coefficients are placed at internal vertices, functions where the star product is acting on are placed at the sinks of the graph and a directed edge represents a derivative with respect to the target vertex.

Every Kontsevich graph has its own associated weight. This weight is calculated through through integration of the angles of the Kontsevich graph. The angles of the Kontsevich graph are determined by embedding the Kontsevich graph into the hyperbolic plane.

We noted that the Kontsevich star product is not unique: there are other star products, with a different distribution of and a different calculation of the weights of graphs, that also are formal associative deformations of the algebra of functions on any Poisson manifold.

In 2000, Cattaneo and Felder published a paper explaining how the Kontsevich formula can also be calculated by the use of a path integral formulation for a Poisson-sigma model [CF]. Their search for a path integral theory had been stimulated by Kontsevich's ideas that the Kontsevich graphs looked like Feynman diagrams present in path integral methods for quantum field theory. Whether this is the only functional integral theory that reproduces the Kontsevich star product (with these specific weights), is still open for research.

In Chapter 6, we revised the relevant theory to understand the most important details of the path integral calculation of the Kontsevich star product. This theory include: Gaussian integration (around a critical point), perturbed Gaussian integration, Wick's theory, normal ordering prescriptions and some examples about the relation between the Wick contractions and Feynman diagrams.

In Chapter 7, we worked out three examples of path integral calculations to obtain star products. We followed the lecture notes written by Cattaneo [CKTB] and added explanations, illustrations and calculations.

First, we calculated the Groenewold-Moyal star product via the path integral approach, where the action is quadratic. We explained the rules for drawing the Feynman diagrams based on the Wick contractions in that theory. Since the action is quadratic, internal vertices of the Feynman diagrams can not be a target vertex for other internal vertices.

Secondly, we calculated calculated the weights of the Feynman diagrams in the expansion of $f \,\hat{\star}_{a,b;u,v}\, g(q,0) \mod \hbar^3$, the non-associative star product between of f and g, via a path integral method. This calculation served as bridge to go from the Groenewold-Moyal star product to the Kontsevich star product for affine Poisson structures. Whereas the Groenewold-Moyal star product was calculated via a quadratic action in the path integral, the non-associative star product is calculated via a quadratically perturbed action in the path integral. Due to the perturbation, internal vertices can be a target vertex of other internal vertices. Moreover, we have seen the importance of several rules, or mechanisms, that reduce the set of Feynman diagrams that eventually appear in the non-associative star product.

Lastly, we illustrated the calculation of the Kontsevich star product for affine Poisson manifolds. We summarized the most important details about the Poisson sigma model (from [CF] and [CKTB]) to eventually continue where Cattaneo stopped in Chapter 14 of [CKTB].

Namely, we formulated the theorem that the path integral calculation in the Poisson sigma model (via BRST-method) would give the Kontsevich star product for affine Poisson manifolds (see Theorem 7.9 on 82), i.e. the Kontsevich star product is equal to the expectation value of $\mathcal{O}_{f,g;0,1}$,

$$f \star_K g(\mathbf{x}) = \langle \mathcal{O}_{f,g:0,1} \rangle (\mathbf{x}) = \frac{\int D\xi \; D\eta \; e^{\frac{i}{\hbar}(S_0+S_1)} \mathcal{O}_{f,g;0,1}(\mathbf{x})}{\int D\xi \; D\eta \; e^{\frac{i}{\hbar}(S_0+S_1)}}. \tag{200}$$

We wrote out all relevant terms in the expansion $\langle \mathcal{O}_{g,;0,1} \rangle$ up to and including order $\hbar^3$. We explicitly explained which Feynman diagrams are obtained from which Wick's contractions in the path integral calculation. We drew all (30) non-double edge non-tadpole Feynman diagrams that are in principle present in the expansion of $\langle \mathcal{O}_{g,;0,1} \rangle \mod \hbar^4$ (we also included the graphs for general Poisson manifolds). Moreover, we explained how the large set of Feynman diagrams in the path integral calculation is reduced via eight selection rules to the correct set of Kontsevich graphs which eventually appear in the star product expansion for general, affine and symplectic Poisson manifolds.

So, we illustrated the relation between Feynman diagrams in quantum field theory and Kontsevich graphs in deformation quantization.

8.1 Outlook

After this thesis, there still remain many open problems. Here, we list some topics that can be studied in more detail.

First of all, we and Cattaneo [CKTB] have studied and illustrated details of the path integral theory to deform quantize affine Poisson manifolds. This illustration needed the BRST-formalism. A detailed illustration in case of general Poisson manifolds, which needs the BV-formalism, is not yet given.

Secondly, it would be interesting to search for a family of Poisson sigma models that interpolate between the harmonic and logarithmic propagators.

Thirdly, it has not yet been explained how the gauge transformations of star products can be given in terms of diffeomorphisms in the Poisson sigma model. One could describe the deformation cohomology of star products via the Poisson sigma model.

Fourthly, in the example of coherent state optics (mentioned in Chapter 4), the physical sense of the path integral techniques and the expansion of the expectation value of observables can be investigated.

Fifthly, it is remarkable that the Kontsevich star product is associative. The requirement of associativity ensures that the deformed bracket is a Poisson bracket at lowest order. The phenomenology of associativity in Nature and the physics of the mechanisms of associativity are still very open for exploration.

Sixthly, one can study extensions of Cattaneo and Felder's Poisson sigma model. For example, extensions from finite-dimensional geometries to field theories. Another possible extension, it to study not only (string) theories with dimensions one (the circle S^1) and two (the upper-half plane $\mathbb{H}$), but higher dimensional hidden brane spaces, and graphs embedded into higher dimensional branes.

Seventhly, we want to mention two problems open for improvement regarding the weights of Kontsevich graphs in the star product. One can try to invent new ways and formulas to constrain (by (non-)linear relations) the weights of Kontsevich graphs and the factorization mechanism for associativity of the star product. Also, one can try to improve the cost efficiency of the calculation of the weights of Kontsevich graphs.

This concludes our list of open problems. For a review and a summary of open problems in semi-classics and quantization, we advise to have a look at [KM].

Appendices

A Path integral theory for Grassman variables

A theory of integration, in particular Gaussian integration, is well suited to be developed also for **Grassmann variables**. In the section 7.3, we discuss the path integral calculation of the Kontsevich star product. In that calculation, we use Grassman variables (in the form of (anti-)ghosts). However, we do not need to use it very explicitly. Nevertheless, we give some background on Grassman variables. The theory here is based on Chapter 11 in [CKTB].

We introduce some algebraic structures:

- V: a vector space
- SV^*: the symmetric algebra, i.e., the algebra of polynomials
- ΛV^*: the exterior algebra, the *odd* counterpart of SV^*, which is regarded as the algebra of functions on the odd vectors space ΠV
- $\Lambda^{\text{top}} V^*$: an exterior algebra, with chosen basis and orientation, identified with $\mathbb{R}$
- $\int_{\Pi V} : \Lambda V^* \to \mathbb{R}$, the integral on ΠV defined through composition and isomorphism, namely $\Lambda V^* \to \Lambda^{\text{top}} V^* \to \mathbb{R}$.

Let now B be an endomorphism of V. We may regard B as an element of $V^* \otimes V$ and so as a function on $\Pi V^* \times \Pi V := \Pi(V^* \oplus V)$. Up to a sign, there is a natural identifiation of $\Lambda^{\text{top}}(V^* \oplus V)$ with $\mathbb{R}$. In this way, up to sign which we fix to agree with this formula, we have

$$\int_{\Pi V^* \times \Pi V} e^{\mathrm{B}} = \det \mathrm{B}.$$

For nondegenerate B, the expectation of a function f on $\Pi V^* \times \Pi V$ is given by

$$\langle\, f \,\rangle_0 \;:=\; \frac{\int_{\Pi V^* \times \Pi V} e^{\mathrm{B}} f}{\int_{\Pi V^* \times \Pi V} e^{\mathrm{B}}}.$$

Let us now choose a basis $\{e_i\}$ of V and denote by $\overline{e}^i$ its dual basis. Then $\Lambda(V^* \oplus V)$ may be identified with the Grassmann algebra generated by the anticommuting "coordinate functions" $\overline{e}^i$ and e_j. Functions on $\Pi V^* \times \Pi V$ are then linear combinations of monomials in $e_{j_1}, \ldots, e_{j_r}, \overline{e}^{i_1}, \ldots, \overline{e}^{i_s}$. With the endomorphism B we associate the function $\langle \overline{e},\, \mathrm{B}e \rangle \;=\; \overline{e}^j \mathrm{B}^i_j e_i$, where $\langle\ ,\ \rangle$ denotes the canonical pairing between V^* and V. The above formulas can then be rewritten as

$$\int e^{\langle \overline{e}, \mathrm{B}e \rangle} = \det \mathrm{B},$$

$$\Big\langle\, e_{j_1} \,\ldots\, e_{j_r}\, \overline{e}^{i_1} \ldots\, \overline{e}^{i_s} \,\Big\rangle_0 \;=\; \frac{\int e^{\langle \overline{e}, \mathrm{B}e \rangle} e_{j_1} \,\ldots\, e_{j_r}\, \overline{e}^{i_1} \ldots\, \overline{e}^{i_s}}{\int e^{\overline{e}^j \mathrm{B}^i_j e_i}}.$$

A **vector field on ΠV** is a graded derivation of the algebra ΛV^*. Namely, we say that

- an endomorphism X of ΛV^* is *a vector field of degree* $|X|$ if

$$X(fg) \;=\; X(f)g + (-1)^{|X|r} f X(g), \qquad \forall f \in \Lambda^r V^*,\ \forall g \in \Lambda V^*,\ \forall r.$$

- an endomorphism $f \mapsto (f)X$ of ΛV^* is *a right vector field X of degree $|X|$* if it satisfies

$$X(fg) \;=\; f(g)\overleftarrow{X} + (-1)^{|X|s}(f)\overleftarrow{X}g, \qquad \forall f \in \Lambda V^*,\ \forall g \in \Lambda^s V^*,\ \forall s.$$

The vector space of all vector fields on ΠV may be identified with $\Lambda V^* \otimes V$, elements of V being constant vector fields. Integration has the natural property that

$$\int_{\Pi V} X(f) = 0, \qquad \forall f,$$

if X is a constant vector field. In general, one defines the divergence div X of X by the formula

$$\int_{\Pi V} X(f) = \int_{\Pi V} \operatorname{div} X\ f, \qquad \forall f.$$

References

[AK] Arnold V.I., Khesin B. A. (1998) Topological Methods in Hydrodynamics. New York: Springer-Verlag (Applied Mathematical Sciences), Vol. 125, 376 pages.

[ARTW] Alekseev A., Rossi C.A., Torossian C., Willwacher T. (2016) Logarithms and deformation quantization. Invent. math., 206:1–28. DOI 10.1007/s00222-016-0647-7.

[BBG] Bertelson M., Bieliavsky P., Gutt S. (1998) Parametrizing equivalence classes of invariant star products. Lett. Math. Phys., 46(4):339–345.

[BCG] Bertelson M., Cahen M., Gutt S. (1997) Equivalence of star products. Class. Quantum Grav., 14:A93–A107.

[BRS] Becchi C., Rouet A., Stora R. (1975) Renormalization of the abelian Higgs–Kibble model. Commun. Math. Phys. 42, 127.

[Ber] Berezin, F. A. (1987). Introduction to superanalysis. Edited by A. A. Kirillov. Dordrecht. (Mathematical physics and applied mathematics, 9).

[BM] Berra-Montiel J. (2020) Star product representation of coherent state path integrals. Eur. Phys. J. Plus 135, 906. https://doi.org/10.1140/epjp/s13360-020-00930-7

[BoeWaa] Boer D., Waalkens H. (2017) The Groenewold product, In: Dutch Journal of Physics. 1 https://www.ntvn.nl/magazines/2017-1/The%20Groenewold%20product_1.html

[BBK] Bouisaghouane A., Buring R., Kiselev A. (2017). The Kontsevich tetrahedral flow revisited. Journal of geometry and physics, 119, 272-285. https://doi.org/10.1016/j.geomphys.2017.04.014

[BFFLS] Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D. (1978) Deformation theory and quantization. I, II. Deformations of symplectic structures, Ann. Phys. 111:1, 61–110, 111–151. doi:10.1016/0003-4916(78)90224-5

[BurKis16] Buring R., Kiselev A.V. (2016) On The Kontsevich $\star-$product Associativity Mechanism, Physics of Particles and Nuclei Letters 14 (2), 403-407. arXiv:1602.09036v1 [math.QA]

[BurKis19] Buring R., Kiselev A.V. (2019) The Expansion $\star$ Mod $\bar{o}(\hbar^4)$ and computer-assisted proof schemes in the Kontsevich deformation quantization, Experimental Mathematics, 76 pages. DOI: 10.1080/10586458.2019.1680463. arXiv:1702.00681v4 [math.CO]

[BV] Batalin I., Vilkovisky G. (1981) Gauge algebra and quantization, Phys. Lett. 102 B, 27; (1983) Quantization of gauge theories with linearly dependent generators, Phys. Rev. D29, 2567.

[CKTB] Cattaneo A., Keller B., Torossian C., Bruguières A. (2005) Déformation, quantification, théorie de Lie. Panoramas et Synthèses 20, Soc. Math. de France, Paris.

[CF] Cattaneo A., Felder G. (2000) A Path Integral Approach to the Kontsevich Quantization Formula. Comm. Math. Phys. 212, 591–611. https://doi.org/10.1007/s002200000.

[CI] Cattaneo A., Indelicato D. (2004) Formality and Star Product. DOI:10.5167/uzh-21691. arXiv:math/0403135 [math.QA]

[Chl] Chloup L. (1997) Star products on the algebra of polynomials on the dual of a semi-simple Lie algebra. Acad. Roy. Belg. Bull. Cl. Sci. 8 (7-12), 263-269.

[Del] Deligne P. (1995) Déformations de l'Algèbre des Fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte. Selecta Math. N. S., 1, 667–697.

[D] Dirac P.A.M. (1964) "Lectures on Quantum Mechanics," Belfer Graduate School of Sciences Monograph Series No. 2, Yeshiva Univ., New York.

[DNF] Dubrovin B. A., Fomenko A. T., Novikov S. P. (1992) Modern geometry–methods and applications. 2nd edn. New York: Springer-Verlag (Graduate texts in mathematics), 432 pages.

[Esp] Esposito C. (2015) Formality theory. From Poisson structures to deformation quantization. SpringerVerlag, Heidelberg, Berlin.

[Fed94] Fedosov B. (1994) A simple Geometrical Construction of Deformation Quantization. Journal of Differential Geometry 40, 213–238.

[Fed96] Fedosov B.V. (1996) Deformation quantization and Index theory. Wiley-VCH.

[Fey] Feynman R.P. (1949) Space-time approach to quantum electrodynamics. Physical Review, 76 (6), 769-789.

[Ger] Gerstenhaber M. (1964) On the Deformation of rings and algebras. Annals of Mathematics, 79 (1), 59-103.

[GT] Gitman D. M., Tyutin I. V. (1990) Quantization of fields with constraints. Springer Ser. Nucl. Part. Phys., Springer-Verlag, Berlin.

[Got96] Gotay M.J. (1996) Obstructions to quantization, J. Nonlin. Sci. **6**, 469, doi:10.1007/BF02440163 [arXiv:math-ph/9809011 [math-ph]].

[Got99] Gotay M.J. (1999) On the Groenewold-van Hove Theorem on for $\mathbb{R}^{2n}$, Journal of Mathematical Physics 40, 2107-2116. https://doi.org/10.1063/1.532854

[Groe] Groenewold H. J. (1946) On the principles of elementary quantum mechanics. Physica, 12(7), 405–460. https://doi.org/10.1016/S0031-8914(46)80059-4.

[HT] Henneaux M., Teitelboim C. (1992) Quantization of gauge systems. Princeton University Press. https://doi.org/10.1515/9780691213866

[HZ] Hillery M., Zubairy M. S. (1982) Path-integral approach to problems in quantum optics. Physical Review A, 26(1), 451–460. https://doi.org/10.1103/PhysRevA.26.451

[I] Ikeda N. (1994) Two-dimensional gravity and nonlinear gauge theory, Annals Phys. **235**, 435-464. doi:10.1006/aphy.1994.1104 [arXiv:hep-th/9312059 [hep-th]].

[Kai] Kaiser D. (2005) Physics and Feynman's diagrams. American Scientist. Volume 93, 156-165.

[Kar] Karabegov A. (1998) Cohomological classification of deformation quantisations with separation of variables, Lett. Math. Phys. 43, 347–357.

[KM] Karasev M.V., Maslov V.P. (1991) Nonlinear Poisson Brackets. Geometry and Quantization, Nauka, Moscow; English transl. (1993), Ser. Translations of Mathematical Monographs, Vol. 119, Amer. Math. Soc., Providence, RI.

[Kir] Kirillov A.A. (2000) Lectures on the orbit method. American Mathematical Society. Graduate Studies in Mathematics, Volume 64, 1-408.

[Kis12] Kiselev A.V. (2012) The twelve lectures in the (non)commutative geometry of differential equations, IHÉS/M/12/13, 140 pages.

[Kis20] Kiselev A.V. (2020). Deformation quantization, graph complex and number theory, course read at Mastermath, NL. (Lecture notes, 102 pages.)

[Kont97] Kontsevich M. (1997) Formality conjecture. In D. Sternheimer et al., editors, Deformation theory and symplectic geometry, pages 139–156. Kluwer.

[Kont99] Kontsevich M. (1997) Operads and Motives in Deformation Quantization. Letter in Mathematical Physics, 48: 35-72.

[Kont03] Kontsevich M. (2003) Deformation quantization of Poisson manifolds. I, Lett. Math. Phys. 66:3, 157–216. arXiv:q-alg/9709040.

[KW] Khesin B. A., Wendt R. (2009) The Geometry of Infinite-Dimensional Groups. New York: Springer-Verlag (Ergebnisse der Mathematik und ihrer Grenzgebiete book series), Vol. 51, 304 pages. ISBN: 978-3-540-77262-0.

[LPV] Laurent-Gengoux C., Pichereau A., Vanhaecke P. (2013) Poisson Structures. Springer-Verlag Berlin Heidelberg, Volume 347, 464 pages.

[Lee95] Lee H.W. (1995) Theory and applications of the quantum phase-space distribution functions. Physics Reports, 259:147–211.

[Lee03] Lee J. M. (2003) Introduction to smooth manifolds. Grad. Texts in Math. 218, Springer–Verlag, NY.

[Lei] Lein M. (2010) Weyl quantization and semiclassics (lecture notes). arXiv:1009.0444v1 [math-ph].

[NT] Nest R., Tsygan B. (1995) Algebraic index theorem for families. Advances in Math., 113:151–205.

[NV] Neroslavsky O.M., Vlasov A.T. (1981). Sur les déformations de l'algèbre des fonctions d'une variété symplectique, C. R. Acad. Sc. Paris, 292, 71-73.

[NieTol] Niederle J., Tolar J. (1979) Quantization as mapping and as deformation. Czech. J. Phys. B 29, 1358-1368.

[R] Rattazzi R. (2011) The Path Integral approach to Quantum Mechanics, Lecture Notes for Quantum Mechanics IV.

[RW] Rossi C.A., Willwacher T. (2014) P. Etingof's conjecture about Drinfeld associators. arXiv:1404.2047 [math.QA].

[RouDu] Roubtsov V., Dutykh D. (2020) Poisson and Symplectic structures, Hamiltonian action, momentum and reduction. arXiv:2003.14173v1 [math.DG]

[Sch] Schwartz Matthew D. (2014) Quantum Field Theory and the Standard Model, 1st ed., Cambridge University Press.

[Sternh] Sternheimer D. (2007) Some reflections on mathematicians' views of quantization. J Math Sci 141, 1494–1505. https://doi.org/10.1007/s10958-007-0054-0.

[ThM] Thorton S.T., Marion J.B. (2005) Classical Dynamics of particles and systems, 5th ed., Thomson/Cole - Thomson Learning.

[Tyu] Tyutin I.V. (1975). Gauge invariance in field theory and statistical physics in operator formulation (in Russian), Lebedev Institute preprint N39.

[VH] Hove L., van (1951). Sur certaines representations unitaires d'un groupe infini de transformations. Mém. Acad. Roy. Belgique. Cl. Sci. 26 (6), 1-102.

[Vey] Vey J. (1975). Déformation du crochet de Poisson sur une variété symplectique. Commentarii Mathematici Helvetici, 50 (1), 421-454.

[Wey] Weyl H. (1927) Quantenmechanik und gruppentheorie. Zeitschrift für Physik, 46, 1–46.

[Wez] Wezeman R. (2014) Weyl quantization and Wigner distributions on phase space. Bacherlor's Thesis Physics, University of Groningen, Groningen, Netherlands. URI: http://fse.studenttheses.ub.rug.nl/id/eprint/11920

[Wig] Wigner E.P. (1932) On the quantum correction for thermodynamic equilibrium. Physics Review, 40, 749–759.

[W] Wright, M. (2006) Green function or green's function?. Nature Phys 2, 646. https://doi.org/10.1038/nphys411

[Zac] Zachos C. (1999) Geometrical Evaluation of Star Products. Journal of Mathematical Physics, 41(7) 5129, DOI: 10.1063/1.533395 / arXiv:hep-th/9912238 (hep-th).

[ZJ] Zinn-Justin J., Path Integrals in Quantum Mechanics, Oxford University Press, New York, 2005.