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The CM class number one problem for sextic fields with Galois group $(C_2)^3 \rtimes C_3$

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Preface

A *complex multiplication field* (CM-field) of degree $2g$ is a totally imaginary quadratic extension of a totally real field of degree g over $\mathbb{Q}$. An abelian variety of dimension g has complex multiplication if its endomorphism ring contains an order of a CM-field of degree $2g$. Let K be a CM-field of degree $2g$ with maximal order $\mathcal{O}_K$ By the first main theorem of CM ([ST61, Main Theorem 1]) a principally polarized simple abelian variety of dimension g that has CM by $\mathcal{O}_K$ is defined over the Hilbert class field of a *reflex field* of K. An implication of this theorem is that if a principally polarized simple abelian variety with CM by $\mathcal{O}_K$ for primitive (K, Φ) is defined over a reflex field of K, then the CM class group is trivial ([Kıl16, Corollary 1.5.7]). The CM class number one problem asks to determine *CM-pairs* (K, Φ), where K is a CM field and Φ is a primitive CM-type of K, with a trivial CM class group.

When $g = 1$ the problem becomes the usual class number one problem, which was solved by Heegner [Hee52], Baker [Bak67] and Stark [Sta67]. For $g = 2$ the problem was solved by [KS18]. For $g = 3$ all possible CM-fields are given in [Dod84, Section 5.1.1]. For sextic CM-fields containing an imaginary quadratic subfield, the problem was solved by Kılıçer [Kıl16].

In this thesis we discuss the problem for sextic CM-fields that do not contain an imaginary quadratic subfield and that have a Galois closure of degree 24 over the rational numbers. Chapter 1 of this thesis contains preliminaries on CM-fields, Representation Theory and L-functions. In Chapter 2 we examine the subfield structures of a sextic CM-field K whose Galois closure L has Galois group $(C_2)^3 \rtimes C_3$ and its reflex field for each primitive CM type Φ. We give relations between the discriminants and relative class numbers of such a sextic CM field and its reflex field. We prove sufficient conditions for such fields to be of CM class number one and show that there exist finitely many CM class number one fields of this form. We give algorithms for computing these fields in SageMath and provide some examples. In Chapter 3 we assume that K is a CM class number one field and find restrictions on K and give an expression for h_K^* depending on the number of primes that ramify in K/K_+. We give a full ramification table for fields K and K^r as subfields of L. Furthermore, we find a bound for d_K/d_{K_+} and prove finiteness in Theorem 3.3.6. Finally we list all CM class number one fields K where $d_{K_+^r} \leq 10^8$ and $d_k \leq 10^4$ such that at most 2 primes are ramifying in $k/\mathbb{Q}$.

List of Notation

$\mathcal{O}_K$	Ring of integers of number field K
d_K	Discriminant of K
I_K	Group of fractional ideals of K
P_K	Group of principal fractional ideals of K
Cl_K	Class group I_K/P_K of K
h_K	Class number of K
$\mathcal{O}_K^\times$	Unit group of K
$\mathrm{Reg}(K)$	Regulator of K
$r_1(K), r_2(K)$	Number of real embeddings, respectively pairs of complex embeddings of K
W_K	Group of roots of unity of K
μ_K	Cardinality of W_K
$N_{K/F}$	Ideal norm of number field extension K/F
KL	Smallest field containing both fields K, L
K^H	Subfield of a field K Galois over $\mathbb{Q}$ fixed by subgroup $H \subset \mathrm{Gal}(K/\mathbb{Q})$
K_+	Maximal totally real subfield of a CM-field K
h_K^*	Relative class number $\frac{h_K}{h_{K_+}}$ of a CM-field K
(K, Φ)	CM-pair with K a CM-field and Φ a CM type of K
(K^r, Φ^r)	Reflex pair of (K, Φ) with K^r a reflex field and Φ^r the reflex type of Φ
N_Φ	Type norm map for a CM-type Φ of CM-field K
Q_K	Hasse unit index $[\mathcal{O}_K^\times : W_K \mathcal{O}_{K_+}^\times]$ for a CM-field K

Contents

1. Preliminaries

1.1. Complex multiplication

We introduce elementary concepts and results from complex multiplication theory, which are mostly due to Shimura and Taniyama [ST61]. For the structure of the section we follow [Mil06, Chapter 1], [Lan83, Section 1.2 and 1.5] and [Str10, Chapter 1.2 and 1.3].

1.1.1. CM fields and CM types

Let K be a number field.

Definition 1.1.1. The field K is *totally real* if $\varphi(K) \subset \mathbb{R}$ for all embeddings $\varphi : K \to \mathbb{C}$. It is *totally imaginary* if $\varphi(K) \not\subset \mathbb{R}$ for all φ.

Write $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[X]/(f(X))$ with $\alpha \in \mathbb{C}$ and minimal polynomial $f(X)$ of α. Then K is totally real if all roots of $f(X)$ are real and totally imaginary if none of the roots of $f(X)$ are real. This is equivalent to Definition 1.1.1.

Definition 1.1.2. A number field K is a *CM-field* if it is a totally imaginary quadratic extension of a totally real number field.

We denote the totally real subfield of a CM-field K by K_+. Here K_+ is the maximal totally real subfield of K. Definition 1.1.2 implies that if $n := [K_+ : \mathbb{Q}]$, then $[K : \mathbb{Q}] = 2n$.

Let $\bar{\cdot}$ denote the regular complex conjugation. For every embedding $\varphi : K \to \mathbb{C}$ of K and for every $a \in K$ we have $(\bar{\cdot} \circ \varphi)(a) = \overline{\varphi(a)}$. We give an alternative definition of a CM-field in Proposition 1.1.3.

Proposition 1.1.3. *A number field K is a CM-field if and only if K is not totally real and there exists a nontrivial automorphism ρ_K of K such that for all embeddings φ of K we have $\varphi \circ \rho_K = \bar{\cdot} \circ \varphi$.*

Proof. Assume K is a CM-field. Because $[K : K_+] = 2$ and K is Galois over K_+, there exists a unique nontrivial automorphism ρ of K that has order 2 and fixes K_+. Then ρ is complex conjugation on K and $\rho_K := \rho$ satisfies the desired condition.
Conversely, assume there exists a nontrivial $\rho_K \in \mathrm{Aut}(K)$ such that for all embeddings φ of K we have $\varphi \circ \rho_K = \bar{\cdot} \circ \varphi$. Then $(\varphi \circ \rho_K) \circ \rho_K = \bar{\cdot} \circ (\bar{\cdot} \circ \varphi) = \varphi$, so ρ_K has order 2. Take F to be the fixed field of ρ_K, then K is an imaginary quadratic extension of F. Moreover, F is fixed by complex conjugation and therefore totally real. So K is a CM-field with $K_+ := F$. □

We call the automorphism ρ_K in Proposition 1.1.4 the complex conjugation on K.

Corollary 1.1.4. *A finite composite KM of two CM-fields K, M is a CM-field.*

Proof. All complex embeddings of KM are induced from those of K, M. Since K, M are CM-fields, by Proposition 1.1.3 each embedding on K, M commutes with complex conjugation on K, M. Let ρ_{KM} be the automorphism on KM induced from complex conjugation on K, M. We have $\varphi \circ \rho_{KM} = \bar{\cdot} \circ \varphi$ for all embeddings $\varphi : KM \to \mathbb{C}$, hence KM is a CM-field. □

Corollary 1.1.5. *Let K be a non-Galois CM-field with Galois closure L, then L is a CM-field.*

Proof. Since L is the Galois closure of K, it is the smallest Galois extension of K. Moreover, it is the smallest extension containing the Galois conjugates K_σ of K. This makes L the composite field of K and its Galois conjugates, so L is a CM-field by Corollary 1.1.4. □

For K a number field with Galois closure L, define $\mathrm{Hom}(K, L)$ to be the group of embeddings of K with values in L.

Definition 1.1.6. Let K be a CM-field with Galois closure L. A *CM-type* Φ of K is a subset of $\mathrm{Hom}(K, L)$ such that $\mathrm{Hom}(K, L) = \Phi \sqcup \overline{\Phi}$. The pair (K, Φ) is called a *CM-pair*.

For every complex conjugate pair $\{\varphi, \overline{\varphi}\} \subset \mathrm{Hom}(K, L)$ a CM-type Φ of K contains precisely one embedding of said pair. That is, no two elements in Φ are complex conjugates of each other.

Definition 1.1.7. Two CM-types Φ_1, Φ_2 of a CM-field K are called *equivalent* if there exists $\sigma \in \mathrm{Aut}(K)$ such that $\sigma\Phi_1 = \Phi_2$.

Let (K, Φ) be a CM-pair such that $K \subset M$ is an extension of CM-fields and let N be the Galois closure of M. Define

$$\Phi_M := \{\varphi \in \mathrm{Hom}(M, N) : \varphi|_K = \psi \text{ for some } \psi \in \Phi\}. \tag{1.1}$$

Here Φ_M is a CM-type of M, because for every $\varphi \in \Phi_M$ we have $\overline{\varphi} \notin \Phi_M$.

Definition 1.1.8. The set Φ_M as in (1.1) is called the CM-type of K *induced* by Φ.

Definition 1.1.9. A CM-type of a CM-field K is called *primitive* if it is not induced from a CM-type of a CM-subfield of K.

Proposition 1.1.10. *[Mil06, Proposition 1.9] Let (K, Φ) be a CM-pair. There exists a unique primitive CM-pair (F, Φ_F) such that K/F is a field extension and Φ is induced from Φ_F. Moreover, for M a CM-field extension of K such that M is Galois over $\mathbb{Q}$, let Φ_M be the CM-type of M induced from Φ_F. Then F is the fixed field of*

$$\{\sigma \in \mathrm{Gal}(M/\mathbb{Q}) : \Phi_M\sigma = \Phi_M\}.$$

Proof. First let $K/\mathbb{Q}$ be Galois and define $F \coloneqq K^H$ where $H \coloneqq \{\sigma \in \mathrm{Gal}(K/\mathbb{Q}) : \Phi\sigma = \Phi\}$. We show that F is a CM-field and that $\Phi|_F$ is a CM-type on F. By Proposition 1.1.3 there exists $\rho_K \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\Phi\rho_K = \overline{\Phi}$. Since $\Phi \cap \overline{\Phi}_K = \emptyset$ we have $\rho_K \notin H$.
Next we show that for every $\varphi \in \mathrm{Hom}(F, \overline{F})$ also $\varphi \circ \rho_K|_F = \overline{\varphi}$. Let $\sigma \in H$, then

$$\Phi(\rho_K \circ \sigma \circ \rho_K) = \overline{\Phi}(\sigma \circ \rho_K) = \overline{\Phi}\rho_K = \Phi.$$

So $\rho_K \circ \sigma \circ \rho_K \in H$, hence for all $x \in F$ we have $(\sigma \circ \rho_K)|_F(x) = \rho_K|_F(x)$. This shows that F is invariant under H, so indeed $\varphi \circ \rho_K|_F = \overline{\varphi}$ for every embedding φ of F. This makes F a CM-field. Now suppose that $\Phi|_F$ is not a CM-type of F. Then there exist $\varphi, \psi \in \Phi$ such that $\psi|_F = \overline{\varphi}|_F$. Because F is fixed by H we have

$$\psi^{-1}\overline{\varphi} \in H \iff \overline{\varphi} \in \psi H \subset \Phi_K.$$

However $\varphi \in \Phi_K$, so this is a contradiction and hence $\Phi|_F$ is a CM-type of F.
Left to show is that F is a primitive CM-field. Suppose there exists another CM-field F' such that $F' \subset K$ with CM-type $\Phi|_{F'}$. Then Φ is induced by $\Phi_{F'}$. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that σ fixes F', so $\Phi_{F'} \circ \sigma = \Phi_{F'}$. Since Φ is induced by $\Phi_{F'}$, this implies that $\Phi \circ \sigma = \Phi$ hence $\sigma \in H$. It follows that $F' \supset F$ which makes F primitive.
The above proves the proposition for the case where K is Galois, hence $M = K$. For the second part assume K is not Galois and M is its Galois closure. Then there exists a CM-type Φ_M of M such that $\Phi_M|_K = \Phi$. By the above there exists a primitive CM-field $F \subset K$ with a CM-type Φ_F that induces Φ_M, hence $\Phi_F = \Phi_M|_F$. □

Corollary 1.1.11. *[Mil06, Corollary 1.10] Let M/K be an extension of CM-fields where M is Galois over $\mathbb{Q}$. Then (K, Φ) is primitive if and only if there exists Φ_M induced from Φ such that*

$$\{\sigma \in \mathrm{Gal}(M/\mathbb{Q}) : \Phi_M\sigma = \Phi_M\}$$

fixes K.

Proof. Assume that (K, Φ) is primitive and let M/K be an extension of fields. Define Φ_M as in Definition 1.1.8, then Φ_M is the CM-type of M induced by Φ. From Proposition 1.1.10 it follows that K is fixed by the given subgroup. Conversely, for (M, Φ_M) and (K, Φ) it was shown in the proof of Proposition 1.1.10 that K must be primitive. □

Proposition 1.1.12. *Let K be a CM-field and $\overline{K}$ a Galois closure of K. Take $\sigma \in \mathrm{Aut}(K)$ and let Φ be a CM-type of K with values in $\overline{K}$. Then $\sigma\Phi$ is a CM-type K.*

Proof. Suppose not, then there exist $\varphi, \psi \in \Phi$ such that $\overline{\sigma \circ \varphi} = \sigma \circ \psi \in \sigma\Phi$. Since K is a CM-field, by Proposition 1.1.3 there exists $\rho_K \in \mathrm{Aut}(K)$ such that

$$\overline{\sigma \circ \varphi} = (\sigma \circ \varphi) \circ \rho_K = \sigma \circ (\varphi \circ \rho_K) = \sigma \circ \overline{\varphi} \in \sigma\overline{\Phi}.$$

Then $\sigma \circ \overline{\varphi} = \sigma \circ \psi$, and because σ is an automorphism it follows that $\psi = \overline{\varphi} \in \overline{\Phi}$. This is a contradiction, so $\sigma\Phi$ is a CM-type on K. □

1.1.2. Reflex fields

Let K be a CM-field and let $L \coloneqq \overline{K}$ be the Galois closure of K. Then L is a CM-field by Corollary 1.1.5. Let Φ be a CM-type of K and Φ_L a CM-type of L induced from Φ. That is,

$$\Phi_L = \{\sigma \in \mathrm{Gal}(L/\mathbb{Q}) : \sigma|_K \in \Phi\}.$$

The elements of Φ_L are automorphisms of L, so we can define Φ_L^{-1} to be the set of inverses of elements in Φ_L. Then Φ_L^{-1} is a CM-type of L by [Lan83, Theorem 5.1(ii)]. By Proposition 1.1.10 there exists a unique primitive CM-pair (K^r, Φ^r) such that $K^r \subset L$ is an extension of fields and Φ_L^{-1} is induced from Φ^r.

Definition 1.1.13. A CM-pair (K^r, Φ^r) as described above is called the *reflex pair* of a CM-pair (K, Φ). Here K^r is its *reflex field* and Φ^r its *reflex type.*

Proposition 1.1.14. *Let (K, Φ) and (L, Φ_L) be CM-pairs such that L is the Galois closure of K and Φ_L is induced from Φ. Then*

$$\mathrm{Gal}(L/K^r) = \{\sigma \in \mathrm{Gal}(L/\mathbb{Q}) : \sigma\Phi_L = \Phi_L\}.$$

Proof. Recall that (K^r, Φ^r_K) is a primitive CM-pair that induces (L, Φ_L^{-1}). For any $\varphi \in \Phi_L$ and $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ we have

$$\varphi^{-1} \circ \sigma = \varphi^{-1} \iff \varphi^{-1} \circ \sigma \circ \varphi = \mathrm{id}_L \iff \sigma \circ \varphi = \varphi.$$

Combine this with the result of Proposition 1.1.10 to obtain

$$\mathrm{Gal}(L/K^r) = \{\sigma \in \mathrm{Gal}(L/\mathbb{Q}) : \Phi_L^{-1}\sigma = \Phi_L^{-1}\} = \{\sigma \in \mathrm{Gal}(L/\mathbb{Q}) : \sigma\Phi_L = \Phi_L\}.$$

□

Proposition 1.1.15. *[ST61, paragraph above Proposition 29] Let (K, Φ) be a CM-pair with reflex pair (K^r, Φ^r). Then (K^r, Φ^r) has a reflex pair (K^{rr}, Φ^{rr}), where $K^{rr} \subseteq K$ and Φ is induced by primitive type Φ^{rr}. If Φ is primitive, then $K^{rr} = K$ and $\Phi^{rr} = \Phi$.*

Proof. From the definition of a reflex field it follows immediately that $K^{rr} \subset K$ and $\Phi|_{K^{rr}} = \Phi^{rr}$. Assume that Φ is primitive. Because Φ^{rr} is primitive by the definition of the reflex and Φ^{rr} induces Φ, it follows that $\Phi = \Phi^{rr}$ and hence $K = K^{rr}$. □

Lemma 1.1.16. *Let K be a CM-field and let Φ_1, Φ_2 equivalent CM-types of K. Then (K, Φ_1) and (K, Φ_2) correspond to the same reflex fields.*

Proof. There exists $\sigma \in \mathrm{Aut}(K)$ such that $\Phi_1 = \Phi_2\sigma$. Let L be the Galois closure of K and let $\Phi_{L,1}, \Phi_{L,2}$ be CM-types of L induced from respectively Φ_1, Φ_2. Let K_1^r, K_2^r denote the reflex fields corresponding to $(K, \Phi_1), (K, \Phi_2)$. It suffices to show that K_1^r and K_2^r are fixed by the same elements in $\mathrm{Gal}(L/\mathbb{Q})$. Let $\tau \in \mathrm{Gal}(L/K_1^r)$, then

$$\tau|_K \Phi_1 = \Phi_1 \iff \tau|_K \Phi_2\sigma = \Phi_2\sigma \iff \tau|_K \Phi_2 = \Phi_2,$$

hence $\tau \in \mathrm{Gal}(L/K_2^r)$. This argument is reversible, so $\mathrm{Gal}(L/K_1^r) = \mathrm{Gal}(L/K_2^r)$. □

1.1.3. Type norm and CM-class group

Let (K, Φ) be a CM-pair with reflex pair (K^r, Φ^r) and let L be the Galois closure of K. Define a map

$$N_\Phi : K \to K^r, \quad x \mapsto \prod_{\varphi \in \Phi} \varphi(x). \tag{1.2}$$

By Proposition 1.1.14, $K^r \subset L$ is fixed by $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that $\sigma|_K \Phi = \Phi$. For such σ we have

$$\sigma|_K \circ \Big(\prod_{\varphi \in \Phi} \varphi \Big) = \prod_{\varphi \in \Phi} \sigma|_K \circ \varphi = \prod_{\varphi \in \Phi} \varphi.$$

It follows that $\sigma(\prod_{\varphi\in\Phi} \varphi(x)) = \prod_{\varphi\in\Phi} \varphi(x)$ for all $x \in K$, so the image of N_Φ lies in K^r (see also [ST61, Proposition 29]).

Definition 1.1.17. The map N_Φ defined as (1.2) is the *CM-type norm* of (K, Φ).

For any number field F, let I_F denote the group of fractional ideals of its ring of integers $\mathcal{O}_F$ and let Cl_F denote the class group of fractional ideals of F.

Lemma 1.1.18. *[Str10, Lemma 8.3] Let (K, Φ) be a CM-pair with reflex (K^r, Φ^r) and Galois closure L of K. The type norm N_Φ induces the following homomorphisms:*

$$\begin{aligned} N_\Phi : & I_K \to I_{K^r}, \quad \mathfrak{a} \mapsto \mathfrak{a}'; \\ N_\Phi : & \mathrm{Cl}_K \to \mathrm{Cl}_{K^r}, \quad [\mathfrak{a}] \mapsto [\mathfrak{a}']. \end{aligned}$$

Here $\mathfrak{a}'\mathcal{O}_L := \prod_{\varphi\in\Phi} \varphi(\mathfrak{a})\mathcal{O}_L$.

Proof. The image of N_Φ on I_K lies in I_{K^r} by the second statement in [ST61, Proposition 29]. The image of N_Φ on Cl_K lies in Cl_{K^r} by the first result and because N_Φ as in (1.2) maps from $K^\times$ to $K^{r,\times}$. □

For $\nu\mathcal{O}_K \in P_K$ with $\nu \in K^\times$,

$$N_\Phi(\nu\mathcal{O}_K) = N_\Phi(\nu)\mathcal{O}_{K^r} \in P_{K^r}.$$

So we have that $N_{\Phi^r}(P_{K^r}) \subset P_K$.

Lemma 1.1.19. *Let (K, Φ) be a CM-pair, (K^r, Φ) its reflex pair and let $N_{K/\mathbb{Q}}$ denote the norm map for K as a number field over $\mathbb{Q}$. Then for $x \in K^\times$ and $\mathfrak{a} \in I_K$,*

$$\begin{aligned} N_\Phi(x)\overline{N_\Phi(x)} &= N_{K/\mathbb{Q}}(x) \in \mathbb{Q}, \\ N_\Phi(\mathfrak{a})\overline{N_\Phi(\mathfrak{a})} &= N_{K/\mathbb{Q}}(\mathfrak{a})\mathcal{O}_{K^r} \in I_{K^r}. \end{aligned}$$

Proof. Let L such that $K \subset L$ be Galois over $\mathbb{Q}$. Complex conjugation on K commutes with every embedding in $\mathrm{Hom}(K, L)$, so for all $x \in K^\times$ we have

$$N_\Phi(x)\overline{N_\Phi(x)} = \prod_{\varphi\in\Phi} \varphi(x)\overline{\varphi}(x) = \Big(\prod_{\varphi\in\Phi}\varphi(x)\Big)\Big(\prod_{\psi\in\overline{\Phi}}\psi(x)\Big) = \prod_{\varphi\in\mathrm{Hom}(K,L)} \varphi(x) = N_{K/\mathbb{Q}}(x).$$

Similarly it follows that $N_\Phi(\mathfrak{a})\overline{N_\Phi(\mathfrak{a})} = N_{K/\mathbb{Q}}(\mathfrak{a})\mathcal{O}_{K^r}$ for all $\mathfrak{a} \in I_K$. □

Let (K, Φ) be a CM-pair and (K^r, Φ^r) its reflex pair. Define

$$I_0(\Phi^r) \coloneqq \{\mathfrak{a} \in I_{K^r} : N_{\Phi^r}(\mathfrak{a}) = (\alpha) \text{ for some } \alpha \in K^\times \text{ such that } \alpha\overline{\alpha} \in \mathbb{Q}\}.$$

Proposition 1.1.20. *$I_0(\Phi^r)$ is a subgroup of I_{K^r}.*

Proof. The ring $\mathcal{O}_{K^r} \in I_{K^r}$ is an element of $I_0(\Phi^r)$. Let $\mathfrak{a}, \mathfrak{b} \in I_0(\Phi^r)$, where $N_{\Phi^r}(\mathfrak{a}) = (\alpha)$ and $N_{\Phi^r}(\mathfrak{b}) = (\beta)$ for $\alpha, \beta \in K^{r\times}$ such that $\alpha\overline{\alpha}, \beta\overline{\beta} \in \mathbb{Q}$. Then

$$N_{\Phi^r}(\mathfrak{a}\mathfrak{b}) = N_{\Phi^r}(\mathfrak{a})N_{\Phi^r}(\mathfrak{b}) = (\alpha)(\beta) = (\alpha\beta).$$

Then $\mathfrak{a}\mathfrak{b} \in I_0(\Phi^r)$, so $I_0(\Phi^r)$ is closed under the group action of I_{K^r}. For $\mathfrak{a} \in I_0(\Phi^r)$ with $N_{\Phi^r}(\mathfrak{a}) = (\alpha)$ and $\alpha\overline{\alpha} \in \mathbb{Q}$ there exists $\mathfrak{c} \in I_{K^r}$ such that $\mathfrak{a}\mathfrak{c} = \mathcal{O}_{K^r}$. Then

$$N_{\Phi^r}(\mathfrak{a}\mathfrak{c}) = N_{\Phi^r}(\mathfrak{a})N_{\Phi^r}(\mathfrak{c}) = N_{\Phi^r}(\mathcal{O}_{K^r}) = \mathcal{O}_K = (1).$$

Therefore $N_{\Phi^r}(\mathfrak{c}) = (\alpha^{-1})$ with $\alpha^{-1} \in K^\times$ and $\alpha^{-1}\overline{\alpha^{-1}} = (\alpha\overline{\alpha})^{-1} \in \mathbb{Q}$. So $\mathfrak{c} \in I_0(\Phi^r)$. This makes $I_0(\Phi^r)$ a subgroup of I_{K^r}. □

Definition 1.1.21. Let (K, Φ) be a CM-pair with reflex (K^r, Φ^r). The quotient group $I_{K^r}/I_0(\Phi^r)$ is called the *CM-class group* of (K, Φ). The *CM-class number* of (K, Φ) is the cardinality of $I_{K^r}/I_0(\Phi^r)$.

Because $I_{K^r}/I_0(\Phi^r) \cong \mathrm{Cl}_{K^r}/(I_0(K^r)/P_{K^r})$ and the cardinality of Cl_{K^r} is finite, the CM class number of K is also finite.

1.2. Representation theory of finite groups

We will give preliminaries of representation theory of finite groups, mainly following [Ser77, Chapters 1-3]. We will use Proposition 1.2.14 and Corollary 1.2.15 to prove the Dedekind zeta function relation in Proposition 2.4.2.

Let G be a finite group.

Definition 1.2.1. A *representation* of G over $\mathbb{C}$ is a vector space V over $\mathbb{C}$ together with a group homomorphism $\tau : G \to \mathrm{GL}(V)$. Here $\mathrm{GL}(V)$ is the group of automorphisms on V.

In the rest of the section we will assume that V is finite dimensional.

Definition 1.2.2. Let $\tau : G \to \mathrm{GL}(V)$ be a representation and define $\deg(\tau) \coloneqq \dim_{\mathbb{C}}(V)$. Then $\deg(\tau)$ is called the *degree* of τ.

A representation of degree 1 is of the form $\tau : G \to \mathbb{C}^\times$. If $\tau(s) = 1$ for all $s \in G$, we call τ the *trivial representation*. For a finite group G the trivial representation always exists.

Let $\tau : G \to \mathrm{GL}(V)$ be a representation of G and let $W \subset V$ be a subspace of V such that W is invariant under the action of G. That is, if $w \in W$ then for every $s \in G$ also $\tau(s)(w) \in W$. Then $\tau|_W : G \to \mathrm{GL}(W)$ is also a representation of G.

Definition 1.2.3. Let $W \subset V$ and $\tau|_W$ be as defined above; then W is called a *subrepresentation* of V.

Definition 1.2.4. Let $\tau : G \to \mathrm{GL}(V)$ be a representation of G such that $V \neq \{0\}$. Then V is called *irreducible* if the only subrepresentations of V are $\{0\}$ and V itself.

Corollary 1.2.5. *[Ser77, Theorem 2] Every representation is the direct sum of irreducible representations.*

Let $\tau : G \to \mathrm{GL}(V)$ be a representation of G with $\dim_{\mathbb{C}}(V) = n < \infty$. Then for every $A := (a_{ij}) \in \mathrm{GL}(V)$ we define its trace as $\mathrm{Tr}(A) := \sum_{i=1}^{n} a_{ii}$.

Definition 1.2.6. Given a representation $\tau : G \to \mathrm{GL}(V)$, the map

$$\chi_\tau : G \to \mathbb{C}, \quad s \mapsto \mathrm{Tr}(\tau(s))$$

is the *character* of τ.

For $s \in G$ and χ the character of some representation of G, let $\overline{\chi(s)}$ denote the complex conjugate of $\chi(s)$ in $\mathbb{C}$.

Proposition 1.2.7. *[Ser77, Proposition 1] Let τ be a representation of degree n and let χ be the character of τ. Then the following hold:*

1. $\chi(1) = n$;
2. $\overline{\chi(s)} = \chi(s^{-1})$ *for* $s \in G$;
3. $\chi(s) = \chi(t^{-1}st)$ *for* $s, t \in G$.

For representations $\tau_1 : G \to \mathrm{GL}(V_1), \tau_2 : G \to \mathrm{GL}(V_2)$ of G with respective characters χ_1, χ_2, the character of $V_1 \oplus V_2$ is given by $\chi_1 + \chi_2$ (see [Ser77, Proposition 2]).

Theorem 1.2.8 (Schur's Lemma)**.** *[Ser77, Proposition 4] Let $\tau_1 : G \to \mathrm{GL}(V_1)$ and $\tau_2 : G \to \mathrm{GL}(V_2)$ be irreducible representations of G. Define a linear map $f : V_1 \to V_2$ such that for all $s \in G$, $\tau_2(s) \circ f = f \circ \tau_1(s)$. Then*

1. *If τ_1, τ_2 are not isomorphic, then f is the zero-map.*
2. *If $V_1 = V_2$ and $\tau_1 = \tau_2$, then f is a scalar multiple of the identity map.*

Consider functions $\varphi, \psi : G \to \mathbb{C}$ and let g denote the order of G. Let $\overline{\psi(s)}$ denote the complex conjugate of $\psi(s)$ and define

$$\langle \varphi, \psi \rangle := \frac{1}{g} \sum_{s \in G} \varphi(s)\overline{\psi(s)}.$$

The following lemmata give some results about isomorphic and irreducible representations.

Lemma 1.2.9. *[Ser77, Theorem 5] Let χ be the character of a representation $\tau : G \to \mathrm{GL}(V)$. Then $\langle \chi, \chi \rangle = 1$ if and only if V is irreducible.*

Lemma 1.2.10. *[Ser77, Theorem 3] Let τ_1, τ_2 be non-isomorphic representations with respective characters χ_1, χ_2. Then $\langle \chi_1, \chi_2 \rangle = 0$.*

Lemma 1.2.11. *[Ser77, Theorem 7] The number of irreducible representations of G up to isomorphism is equal to the number of conjugacy classes of G.*

Let $H \subset G$ be a proper subgroup and let V be a representation of G. Choose a subspace $W \subset V$ such that W invariant is under the action of H. That is, for any $w \in W$ we have that $\tau(t)(w) \in W$ for all $t \in H$. Define

$$S \coloneqq \{s \in G : sH \in G/H\}$$

to be the set of representatives of the left cosets of H in G.

Definition 1.2.12. Let $H \subset G$ be a subgroup as above. A representation $\tilde{\tau}$ of G in V is induced from the representation τ of H in W if

$$V = \bigoplus_{s \in S} \tau(s)W.$$

Uniqueness and existence of induced representations are proven in [Ser77, Theorem 11]. Let $(V, \tilde{\tau})$ be the induced representation of G of the representation (W, τ) of subgroup $H \subset G$. Let h denote the order of H and g the order of G.

Theorem 1.2.13. *[FH94, (3.18)] Let $R \coloneqq \{rH : r \in G\}$ be the set of left cosets of H in G. For any $s \in G$ the character $\tilde{\chi}$ of $\tilde{\tau}$ is defined as*

$$\tilde{\chi}(s) = \frac{1}{h} \sum_{\substack{t \in G, \tau \in R \\ t\tau = \tau}} \chi(t^{-1}st).$$

Proposition 1.2.14. *[FH94, Exercise 3.19(a)] Let C be a conjugacy class of G and let D_i be the conjugacy classes of $H \cap C$ for $i = 1, ..., r$. Then the formula in Theorem 1.2.13 is equivalent to*

$$\tilde{\chi}(C) = \frac{g}{h} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi(D_i).$$

Proof. Choose arbitrary $s \in G$ and let C be the conjugacy class of G containing s. Recall that S is the set of representatives of the cosets of H in G. Let the elements of H be given by $x_1, ..., x_h$ and define $T_j \coloneqq \{rx_j : r \in S\}$, it follows that $G = \sqcup_{i=1}^{h} T_i$. Then

$$\begin{aligned}
\tilde{\chi}(C) &= \frac{1}{|C|} \sum_{c \in C} \tilde{\chi}(c) \\
&= \frac{1}{|C|} \sum_{c \in C} \frac{1}{h} \sum_{\substack{t \in G, \tau \in R \\ t\tau = \tau}} \chi(t^{-1}ct) \\
&= \frac{1}{|C|h} \sum_{c \in C} \sum_{i=1}^{h} \sum_{t \in T_i} \chi(t^{-1}ct) \\
&= \frac{1}{|C|h} \sum_{c \in C} \sum_{t \in G} \chi(t^{-1}ct) \\
&= \frac{1}{|C|h} \sum_{i=1}^{r} g\chi(D_i)|D_i| \\
&= \frac{g}{h} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi(D_i).
\end{aligned}$$

□

Corollary 1.2.15. *[FH94, Exercise 3.19(b)] Let C be a conjugacy class of G. If τ is the trivial representation of H, the character of induced representation $\tilde{\tau}$ of τ is given by*

$$\tilde{\chi}(C) = \frac{[G : H]}{|C|} \cdot |C \cap H|.$$

Proof. Here τ is the trivial representation, so $\chi(x) = 1$ for every $x \in H$.
Because $C \cap H = \sqcup_{i=1}^{r} D_i$ this gives

$$\sum_{i=1}^{r} |D_i| \chi(D_i) = \sum_{i=1}^{r} |D_i| = |C \cap H|.$$

The result then follows from Proposition 1.2.14. □

1.3. Dirichlet L-functions

We discuss preliminaries about Dirichlet L-functions. We conclude in (1.3) that for a normal Galois field K the Dedekind zeta function is a product of Artin L-functions. This result will be used to prove the Dedekind zeta function relation in Proposition 2.4.2.

Let $n \in \mathbb{Z}_{\geq 1}$ and define a multiplicative homomorphism $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$. The values of χ in $\mathbb{C}^\times$ are roots of unity because χ is a homomorphism on a finite cyclic group.

Definition 1.3.1. A homomorphism of the form χ is called a *Dirichlet character.*

For every $m \in \mathbb{Z}_{\geq 0}$ such that n divides m, χ induces a character $\tilde{\chi} : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ via the natural map $\varphi : (\mathbb{Z}/m\mathbb{Z})^\times \to (\mathbb{Z}/n\mathbb{Z})^\times, a \mod m \mapsto a \mod n$ such that $\tilde{\chi} = \chi \circ \varphi$.

Definition 1.3.2. If χ is not induced by any character of modulus k such that $k \mid n$ then χ is a *primitive character.*

Definition 1.3.3. Let $d_1, ..., d_r$ be the divisors of n such that $\chi_i : (\mathbb{Z}/d_i\mathbb{Z})^\times \to \mathbb{C}^\times$ induces χ for $i = 1, ..., r$. Then $f_\chi := \gcd(d_1, ..., d_r)$ is called the *conductor* of χ.

Extend a Dirichlet character $\mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ with conductor f to $\chi_1 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$, where $\chi_1(\overline{\mathfrak{a}}) = 0$ if $\overline{\mathfrak{a}} \notin (\mathbb{Z}/n\mathbb{Z})^\times$. Define $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi = \chi_1 \circ \pi$, where $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the canonical homomorphism.

Definition 1.3.4. The *Dirichlet L-series* of χ is a function of the form

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s},$$

with s a complex variable such that $\mathrm{Re}(s) > 1$.

Let χ_0 be a principal (trivial) character. Then the corresponding L-function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$

is the Riemann zeta function.

Given a finite group of Dirichlet characters we can describe an associated number field as follows (see also [Was82, Page 20-21]). Since $(\mathbb{Z}/m\mathbb{Z})^\times \cong \mathrm{Gal}(\zeta_m/\mathbb{Q})$ for $m \in \mathbb{Z}_{\geq 1}$, a Dirichlet character mod m is a function on $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, where ζ_m is the primitive m-th root of unity. Let $X = \{\chi_1, \ldots, \chi_r\}$ be such a finite group with respective conductors $f_1, \ldots, f_r$ and define $n := \mathrm{lcm}(f_1, \ldots, f_r)$. Let H be the subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ that is isomorphic to $\cap_{i=1}^r \ker(\chi_i) \subset (\mathbb{Z}/n\mathbb{Z})^\times$. We call the fixed field $K := \mathbb{Q}(\zeta_n)^H$ the field associated to X. We have that $X \cong \mathrm{Gal}(K/\mathbb{Q})$.

Theorem 1.3.5. *[Was82, Theorem 4.3] Let X be a finite group of Dirichlet characters and let K be its associated number field. Then*

$$\zeta_K(s) = \prod_{\chi \in X} L(s, \chi).$$

Let K be a Galois number field. We will show that the Dedekind zeta-function $\zeta_K(s)$ of K is a product of L-functions ranging over the irreducible characters of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

We look at the Frobenius substitution following [Cog, Section 2.1(b)]. Let M/K be an extension of number fields, let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime of K and $\mathfrak{P} \subset \mathcal{O}_M$ a prime above $\mathfrak{p}$. Define $\kappa_\mathfrak{p} := \mathcal{O}_K/\mathfrak{p}, \kappa_\mathfrak{P} := \mathcal{O}_M/\mathfrak{P}$ denote the residue fields of respectively $\mathfrak{p}, \mathfrak{P}$.
Let $D_\mathfrak{P}, I_\mathfrak{P} \subset \mathrm{Gal}(M/K)$ denote the decomposition and inertia groups of $\mathfrak{P}$. We have the following exact sequence:

$$1 \to I_\mathfrak{P} \to D_\mathfrak{P} \to \mathrm{Gal}(\kappa_\mathfrak{P}/\kappa_\mathfrak{p}) \to 1.$$

For $\mathfrak{P}$ unramified in M/K there exists $\varphi_{\mathfrak{P}/\mathfrak{p}} \in \mathrm{Gal}(\kappa_\mathfrak{P}/\kappa_\mathfrak{p})$ defined by $\varphi_{\mathfrak{P}/\mathfrak{p}}(x) := x^{N(\mathfrak{p})}$ for all $x \in \mathcal{O}_M$.

Definition 1.3.6. For $\mathfrak{P}$ and $\varphi_{\mathfrak{P}/\mathfrak{p}}$ as defined above $\varphi_{\mathfrak{P}/\mathfrak{p}}$ is the *Frobenius automorphism* attached to prime $\mathfrak{P}$.

If $\mathfrak{P}$ is unramified in M/K the inertia group $I_{\mathfrak{P}}$ with respect to $\mathfrak{P}$ is trivial.

Definition 1.3.7. Let M/K be a Galois extension of number fields, $G := \mathrm{Gal}(M/K)$ and (V, τ) a representation of G with character χ. Let P denote the set of unramified primes of K. Then the *Artin L-function* of χ is defined as

$$L(s, \chi, M/K) := \prod_{\mathfrak{p} \in P} \det\bigl(\mathrm{Id} - N(\mathfrak{p})^{-s} \tau(\varphi_{\mathfrak{P}/\mathfrak{p}})\bigr)^{-1}.$$

Theorem 1.3.8 (Artin)**.** *[blo, Proposition 29(1)] Let M/K be a Galois number field extension and let χ, χ_1, χ_2 be characters of representations of* $\mathrm{Gal}(M/K)$ *such that $\chi := \chi_1 + \chi_2$. Then*

$$L(s, \chi, M/K) = L(s, \chi_1, M/K) L(s, \chi_2, M/K).$$

Let K be Galois over $\mathbb{Q}$ and χ_0 the character of the trivial representation of $\mathrm{Gal}(K/\mathbb{Q})$. Define X as the set of characters of irreducible representations of $\mathrm{Gal}(K/\mathbb{Q})$. Then by [blo, Remark 13] we have

$$\zeta_K(s) = L(s, \chi_0, K/\mathbb{Q}) = \prod_{\chi \in X} L(s, \chi, K/\mathbb{Q}). \tag{1.3}$$

2. Sextic CM-fields K with degree 24 Galois closure

We discuss the structure of the sextic CM-fields K with Galois closure L such that $\mathrm{Gal}(L/\mathbb{Q}) \cong (C_2)^3 \rtimes C_3$ in Section 2.1. Then in Section 2.2 we look at the reflex fields K^r of K and the corresponding reflex CM-types. We prove in Lemma 2.2.3 that the isomorphism class of the CM-class of K does not depend on the CM-type Φ. In Section 2.3 we discuss the subfields of K_1^r and compute its reflex fields. In Section 2.4 we derive discriminant and class number relations between the discriminants of K, K^r and their subfields.

2.1. Subfields of K

Let K be a sextic CM-field with Galois closure L such that $\mathrm{Gal}(L/\mathbb{Q}) \cong (C_2)^3 \rtimes C_3$. Represent the factors of the semidirect product as follows:

$$C_3 = \langle x : x^3 = 1\rangle, \ \ (C_2)^3 = \langle a, b, c : a^2 = b^2 = c^2 = 1, ab = ba, bc = cb, ac = ca\rangle.$$

In the remainder of the document we write $G := \mathrm{Gal}(L/\mathbb{Q})$. Denote by a, b, c, x the elements in G that correspond to respectively $(a, 1), (b, 1), (c, 1), (1, x) \in (C_2)^3 \rtimes C_3$. Then G has representation

$$\begin{aligned} G = \langle a, b, c, x : & a^2 = b^2 = c^2 = x^3 = 1, ab = ba, ac = ca, bc = cb, \\ & x^{-1}ax = c, x^{-1}bx = a, x^{-1}cx = b\rangle. \end{aligned}$$

Proposition 2.1.1. *The maximal totally real subfield K^+ of K is Galois over $\mathbb{Q}$.*

Proof. We have $[K : \mathbb{Q}] = 6$ and $|G|= 24$, so $[L : \mathbb{Q}] = 24$ and therefore $[L : K] = 4$. Let $L' \subset L$ be the Galois closure of K_+ over $\mathbb{Q}$. This gives the following exact sequence of groups:

$$1 \to \mathrm{Gal}(L/L') \hookrightarrow G \twoheadrightarrow \mathrm{Gal}(L'/\mathbb{Q}) \to 1$$

Let $\delta_0 \in K_+$ be square-free and strictly greater than 0, such that $K = K_+(\sqrt{-\delta_0})$. Following the proof of lemma 2.2 in [BCL+14]: $\delta_0 \in K_+$ implies $\mathbb{Q}(\delta_0) \subset K_+$, so $[\mathbb{Q}(\delta_0) : \mathbb{Q}]$ divides 3. Then there are 3 conjugates $\sqrt{-\delta_0}, \sqrt{-\delta_1}, \sqrt{-\delta_2}$ under the action of $\mathrm{Gal}(L'/\mathbb{Q})$. This implies that $L = L'(\sqrt{-\delta_0}, \sqrt{-\delta_1}, \sqrt{-\delta_2})$. It follows that $[L : L'] = 8 = [L : K_+]$, so $L' = K_+$ and $K_+/\mathbb{Q}$ is a Galois extension. □

The proof of Proposition 2.1.1 gives

$$L = K_+(\sqrt{-\delta_0}, \sqrt{-\delta_1}, \sqrt{-\delta_2}).$$

Moreover, by [Bak20, Lemma 3.3] we have $K_+(\sqrt{-\delta_0}) \cong K_+(\sqrt{-\delta_1}) \cong K_+(\sqrt{-\delta_2})$. The corresponding field lattices are shown in Figure 1.

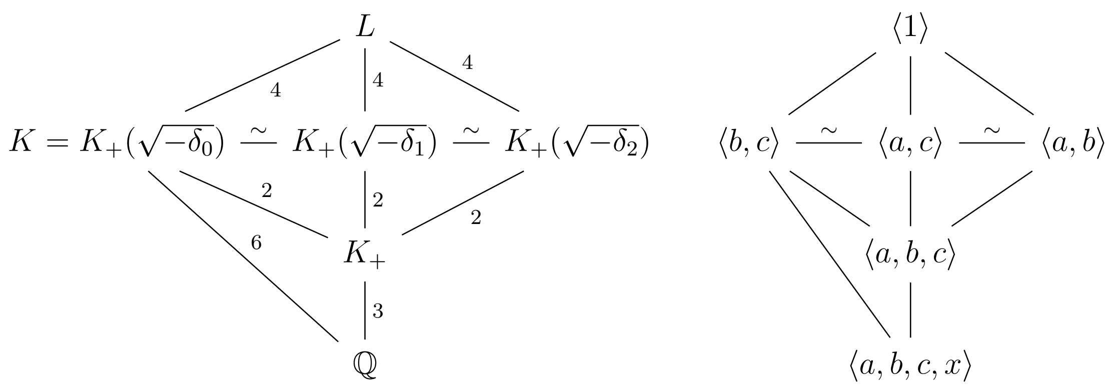

Figure 1: The (incomplete) lattices of L containing K and its isomorphic fields.

Proposition 2.1.2. *The element $abc \in G$ is complex conjugation on L.*

Proof. By [Bak20, Lemma 3.10] we have $Z(G) = \langle abc \rangle$, where $\mathrm{ord}(abc) = 2$. Since L is a CM-field by Corollary 1.1.5 and hence totally imaginary, complex conjugation is nontrivial and the result follows. □

2.2. Reflex types of K

Let K, L and G be as in Section 2.1. Because L is Galois over $\mathbb{Q}$, the complex embeddings of L correspond one-to-one to the elements of G. By Proposition 2.1.2 the complex conjugation on L is $\bar{\cdot} = abc$. We will use notation $\rho \coloneqq abc$. By [Bak20, Proposition 3.5] the embeddings $K \hookrightarrow L$ are

$$\mathrm{Hom}(K, L) = \{1|_K, x|_K, x^2|_K, \rho|_K, \rho x|_K, \rho x^2|_K\}.$$

The 8 CM-types of K are subsets of $\mathrm{Hom}(K, L)$. Under the equivalence relation defined in Definition 1.1.7, they form 4 equivalence classes:

$$\begin{aligned}
\Phi_1 &= \{1|_K, x|_K, x^2|_K\} \sim \{\rho|_K, \rho x|_K, \rho x^2|_K\} = \overline{\Phi}_5 = \overline{\Phi}_1\\
\Phi_2 &= \{\rho|_K, x|_K, x^2|_K\} \sim \{1|_K, \rho x|_K, \rho x^2|_K\} = \overline{\Phi}_6 = \overline{\Phi}_2\\
\Phi_3 &= \{1|_K, \rho x|_K, x^2|_K\} \sim \{\rho|_K, x|_K, \rho x^2|_K\} = \overline{\Phi}_7 = \overline{\Phi}_3\\
\Phi_4 &= \{1|_K, x|_K, \rho x^2|_K\} \sim \{\rho|_K, \rho x|_K, x^2|_K\} = \overline{\Phi}_8 = \overline{\Phi}_4
\end{aligned}$$

The following proposition gives that all CM-types of K are primitive.

Proposition 2.2.1. *Let K be a sextic CM-field that does not contain an imaginary quadratic subfield. Then all CM-types of K are primitive.*

Proof. Let (K, Φ_K) be a CM-pair and suppose Φ_K is not primitive. Then K contains a proper subfield F that is CM and has a CM-type Φ_F that induces Φ_K. Because K is sextic, F must be an imaginary quadratic field. This contradicts the assumption, so all CM-types of K must be primitive. □

Denote by L^H the subfield of L fixed by a subgroup $H \subset G$. For a CM-pair (K, Φ_i) of K, let (K_i^r, Φ_i^r) denote the corresponding reflex pair.

Theorem 2.2.2. *[Bak20, Theorem 3.9] The reflex pairs of K are*

$$\begin{aligned}
(K_1^r, \Phi_1^r) &= (L^{\langle x\rangle}, \langle b, c\rangle) & (K_5^r, \Phi_5^r) &= (L^{\langle x\rangle}, \rho\langle b, c\rangle)\\
(K_2^r, \Phi_2^r) &= (L^{\langle xac\rangle}, \rho\langle b, c\rangle) & (K_6^r, \Phi_6^r) &= (L^{\langle xac\rangle}, \langle b, c\rangle)\\
(K_3^r, \Phi_3^r) &= (L^{\langle xab\rangle}, \langle b, c\rangle) & (K_7^r, \Phi_7^r) &= (L^{\langle xab\rangle}, \rho\langle b, c\rangle)\\
(K_4^r, \Phi_4^r) &= (L^{\langle xbc\rangle}, \langle b, c\rangle) & (K_8^r, \Phi_8^r) &= (L^{\langle xbc\rangle}, \rho\langle b, c\rangle).
\end{aligned}$$

Moreover, the reflex fields in these reflex pairs are isomorphic.

By Theorem 2.2.2 the isomorphism class of the CM class of (K, Φ) does not depend on the reflex field K^r. In Lemma 2.2.3 we prove that it is also independent of Φ and hence Φ^r. As a consequence, in the following chapters we only need to prove results for one reflex pair (K^r, Φ^r) of K and they be true for all of them.

Lemma 2.2.3. *Let K be a sextic CM-field with Galois group $(C_2)^3 \rtimes C_3$. Then for a CM-type Φ of K, the isomorphism class of the CM-class of (K, Φ) does not depend on Φ.*

Proof. We follow the proof of [KS18, Lemma 2.4]. Let L be the Galois closure of K and let $\rho \in \mathrm{Gal}(L/\mathbb{Q})$ be the complex conjugation on L. Let Φ, Ψ be arbitrary CM-types of K with corresponding reflex pairs $(K_\Phi^r, \Phi^r), (K_\Psi^r, \Psi^r)$. By Theorem 2.2.2 there exists an isomorphism $\varphi : K_\Phi^r \to K_\Psi^r$ that induces isomorphism $\varphi : I_{K_\Phi^r} \to I_{K_\Psi^r}$. Restricting φ to subgroup $I_0(K_\Phi^r) \leq I_{K_\Phi^r}$ gives

$$\varphi|_{I_0(\Phi^r)} \colon I_0(\Phi^r) \to I_{K_\Psi^r}.$$

By the proof of [Bak20, Theorem 3.9] we have that $\varphi \in \{1, a, b, c\}$ such that $\varphi(K_\Phi^r) = K_\Psi^r$. Because a, b, c all have order 2 in G, we have $\varphi^{-1} = \varphi$. We will take two particular reflex fields K_Φ^r, K_Ψ^r such that $\varphi K_\Phi^r = K_\Psi^r$ and show that $\Psi^r = \Phi^r \circ \varphi$. Let $K_\Phi^r = L^{\langle x\rangle}, K_\Psi^r = L^{\langle xac\rangle}$ with respective CM-types $\langle b, c\rangle|_{L^{\langle x\rangle}}$ and $\rho\langle b, c\rangle|_{L^{\langle xac\rangle}}$, where ρ is complex conjugation on L. Then $\varphi = a$ and for every $\alpha \in L^{\langle x\rangle}$ we have $a(\alpha) \in L^{\langle xac\rangle}$. Let $\beta \in K_\Psi^r$, then we have:

$$\begin{aligned}
&\rho bc|_{L^{\langle xac\rangle}}(\beta) = a|_{L^{\langle xac\rangle}}(\beta) = a(\beta);\\
&\rho b|_{L^{\langle xac\rangle}}(\beta) = ac|_{L^{\langle xac\rangle}}(\beta) = ca|_{L^{\langle xac\rangle}}(\beta) = c|_{L^{\langle x\rangle}} a(\beta);\\
&\rho c|_{L^{\langle xac\rangle}}(\beta) = ab|_{L^{\langle xac\rangle}}(\beta) = ba|_{L^{\langle xac\rangle}}(\beta) = b|_{L^{\langle x\rangle}} a(\beta);\\
&\rho(\beta) = abc|_{L^{\langle xac\rangle}}(\beta) = bc|_{L^{\langle x\rangle}} a(\beta).
\end{aligned}$$

The above gives $\rho\langle b, c\rangle|_{L^{\langle xac\rangle}} = \langle b, c\rangle|_{L^{\langle x\rangle}} \circ a$. For all pairs of reflex fields K_Φ^r, K_Ψ^r in Theorem 2.2.2 the proof is similar to the above, so in general $\Psi^r = \Phi^r \circ \varphi$. It then follows that

$$N_{\Psi^r} = N_{\Phi^r} \circ \varphi. \tag{2.1}$$

From (2.1) we conclude that the image of $\varphi|_{I_0(\Phi^r)}$ lies in $I_0(\Psi^r)$, so

$$\varphi|_{I_0(\Phi^r)} \colon I_0(\Phi^r) \to I_0(\Psi^r).$$

We will now prove that $\varphi|_{I_0(\Phi^r)}$ is an isomorphism. It is an injective group homomorphism because it is the restricted map of φ. To show that it is surjective, let $\mathfrak{a} \in I_0(\Psi^r)$. By definition of the CM class group there exists $\alpha \in K^\times$ such that $\alpha\overline{\alpha} \in \mathbb{Q}$ and $N_{\Psi^r}(\mathfrak{a}) = \alpha\mathcal{O}_K$. Then by (2.1) we have

$$N_{\Psi^r}(\mathfrak{a}) = N_{\Phi^r} \circ \varphi^{-1}(\mathfrak{a}) = \alpha\mathcal{O}_K.$$

Since φ is an isomorphism and therefore surjective, for every $\mathfrak{a} \in I_0(\Psi^r)$ there exists $\mathfrak{b} := \varphi^{-1}(\mathfrak{a}) \in I_0(\Phi^r)$ such that $\varphi|_{I_0(\Phi^r)}(\mathfrak{b}) = \mathfrak{a}$. Then $I_0(\Phi^r)$ and $I_0(\Psi^r)$ are isomorphic and the result follows. □

2.3. Subfields of the reflex fields of K

By Theorem 2.2.2 all reflex fields are isomorphic and by Lemma 2.2.3 the isomorphism class of the CM-class of (K, Φ) is independent of the choice of Φ. Therefore we can find all CM class number one fields by only looking at one CM-pair of Theorem 2.2.2. All results in this section and in Chapter 3 are given for (K_1^r, Φ_1^r) and will then also hold for the other reflex pairs. In the remainder of the document we use notation $\Phi := \Phi_1$, $K_r := K_1^r$ and $\Phi^r := \Phi_1^r$.

By Proposition 2.1.2 complex conjugation on L is abc, so complex conjugation on $K^r \subset L$ is $abc|_{K^r}$. We will write abc as shorthand notation for $abc|_{K^r}$. The field K^r is fixed by non-normal subgroup $\langle x \rangle \subset G$, so K^r is not Galois over $\mathbb{Q}$. By Galois theory any subgroups of G that contain $\langle x \rangle$ fix subfields of K^r. For instance, the subgroup $\langle x, abc \rangle \subset G$ fixes the maximal totally real field K_+^r of K^r. This is not a normal subgroup in G, so K_+^r is not Galois over $\mathbb{Q}$. The normal subgroup $\langle x, ab, ac \rangle$ fixes a subfield $k \subset K^r$, where k is an imaginary quadratic field. Since K^r is of degree 8 over $\mathbb{Q}$ it follows that $K^r = kK_+^r$. The subfield lattices of K and K^r as subfields of L are in Figure 2.

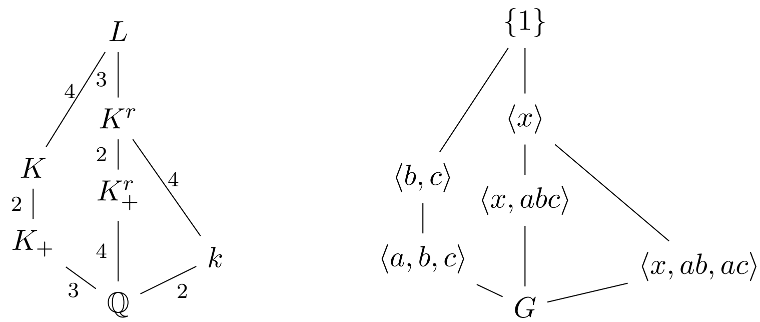

Figure 2: Lattices containing all subfields of K and K^r.

The field L is the normal closure of both K and K^r, so it is a CM-field by Corollary 1.1.5. The maximal totally real field L_+ of L is fixed by $\langle abc \rangle \subset G$, and we have $[L_+ : \mathbb{Q}] = 12$. Then $K_+, K_+^r \subset L_+$ with $[L_+ : K_+] = 4$ and $[L_+ : K_+^r] = 3$. By Galois theory there exist intermediate fields $L_+ \supset N_i \supset K_+$ fixed by respective subgroups $\langle a, bc \rangle, \langle b, ac \rangle, \langle c, ab \rangle \subset \langle abc \rangle$ such that $[N_i : K_+] = 2$ for $i = 0, 1, 2$. Then there exist $\alpha_i \in K_+ \setminus K_+^2$ such that $N_i := K_+(\sqrt{\alpha_i})$. Recall from Section 2.1 that

$$K = K_+(\sqrt{-\delta_0}) \cong K_+(\sqrt{-\delta_1}) \cong K_+(\sqrt{-\delta_2}),$$

where $K_+(\sqrt{-\delta_1}), K_+(\sqrt{-\delta_2})$ are the sextic CM-fields fixed by respectively $\langle a,c\rangle, \langle a,b\rangle$. Write $k = \mathbb{Q}(\sqrt{-m})$ for some $m \in \mathbb{Q} \setminus \mathbb{Q}^2$, then $\delta_i = m\alpha_i$ for $i = 0,1,2$ (see [LLO99, Section 1.4]). In particular, for $\alpha \coloneqq \alpha_0$ we have $K = K_+(\sqrt{-m\alpha})$. Since $K_+(\sqrt{\alpha_0})$ is fixed by $\langle b, ac\rangle$ we have that $\alpha_0 = \delta_0\delta_2$. Similarly, $\alpha_1 = \delta_1\delta_2$ and $\alpha_2 = \delta_0\delta_1$. The subfield structure of the intermediate fields in L_+/K_+^r is shown in Figure 3.

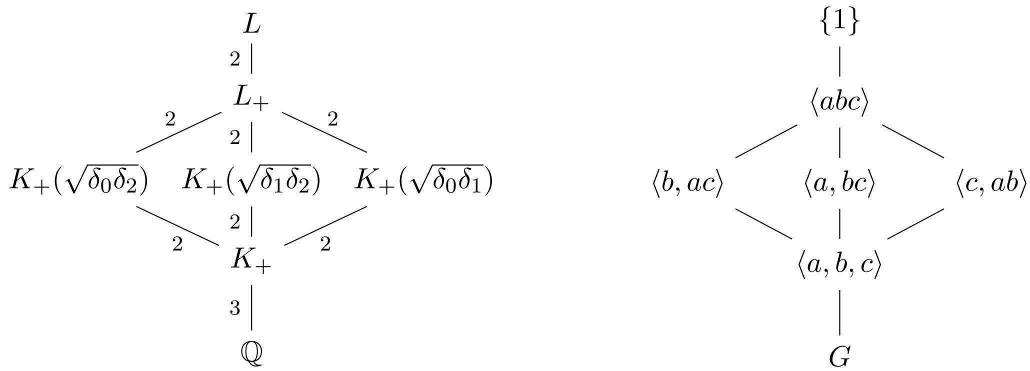

Figure 3: Lattices for the intermediate fields of L_+/K_+.

To make Algorithm 3.4.5 more efficient, we compute the reflex fields of K^r and show that there exists only one sextic reflex field of K^r up to isomorphism. Then we can terminate the algorithm when one sextic CM class number one field has been found.

The embeddings from K^r into L up to equivalence are $\langle a,b,c\rangle|_{K^r} \coloneqq \mathrm{Gal}(L/\mathbb{Q})/\langle x\rangle$. When it is clear from the context that we mean embeddings of K^r into L, we write $\langle a,b,c\rangle$ for notation purposes. In that case, let $abc \coloneqq abc|_{K^r}$ denote complex conjugation on K^r.

Proposition 2.3.1. *The automorphism group of K^r is $\langle abc\rangle$.*

Proof. Since K^r is not Galois over $\mathbb{Q}$ and $[K^r : \mathbb{Q}] = 8$ we have $|\mathrm{Aut}(K^r)| < 8$. The order of abc is 2 and $\langle abc\rangle \subset \mathrm{Aut}(K^r)$. Moreover, all elements in $\mathrm{Aut}(K^r)$ have order 1 or 2, so $|\mathrm{Aut}(K^r)| \in \{2,4\}$. Suppose $|\mathrm{Aut}(K^r)| = 4$. Then $K^{r\langle abc\rangle} = k$, but k is an imaginary quadratic field and cannot be fixed under complex conjugation. So $|\mathrm{Aut}(K^r)| = 2$ and we must have $\mathrm{Aut}(K^r) = \langle abc\rangle$. ☐

By Proposition 2.3.1 there exist 8 equivalence classes of CM-types that are of the form $\{\underline{\Phi}_i, \overline{\underline{\Phi}_i}\}$. The 16 CM-types of K^r form 8 equivalence classes that are given by

$$\underline{\Phi}_1 = \{1,a,b,c\} \sim \{abc, bc, ac, ab\} = \overline{\underline{\Phi}}_1$$
$$\underline{\Phi}_2 = \{abc,a,b,c\} \sim \{1, bc, ac, ab\} = \overline{\underline{\Phi}}_2$$
$$\underline{\Phi}_3 = \{1,bc,b,c\} \sim \{abc, a, ac, ab\} = \overline{\underline{\Phi}}_3$$
$$\underline{\Phi}_4 = \{1,a,ac,c\} \sim \{abc, bc, b, ab\} = \overline{\underline{\Phi}}_4$$
$$\underline{\Phi}_5 = \{1,a,b,ab\} \sim \{abc, bc, ac, c\} = \overline{\underline{\Phi}}_5$$
$$\underline{\Phi}_6 = \{abc,bc,b,c\} \sim \{1, a, ac, ab\} = \overline{\underline{\Phi}}_6$$
$$\underline{\Phi}_7 = \{abc,a,ac,c\} \sim \{1, bc, b, ab\} = \overline{\underline{\Phi}}_7$$
$$\underline{\Phi}_8 = \{abc,a,b,ab\} \sim \{1, bc, ac, c\} = \overline{\underline{\Phi}}_8$$

Not all CM-types of K^r are primitive because $k \subset K^r$ is an imaginary quadratic subfield, hence a CM-field. So one of the equivalence classes above consists of CM-types that are induced from the CM-types of k.

Proposition 2.3.2. *Let k be the imaginary quadratic subfield of K^r. The pairs $\{\underline{\Phi}_i, \overline{\underline{\Phi}}_i\}$ give the following reflex fields of K^r:*

- $\{\underline{\Phi}_2, \overline{\underline{\Phi}}_2\}$ *corresponds to k;*
- $\{\underline{\Phi}_3, \overline{\underline{\Phi}}_3\}, \{\underline{\Phi}_4, \overline{\underline{\Phi}}_4\}, \{\underline{\Phi}_5, \overline{\underline{\Phi}}_5\}$ *correspond to respectively $K, K_+(\sqrt{-\delta_1}), K_+(\sqrt{-\delta_2})$;*
- $\{\underline{\Phi}_1, \overline{\underline{\Phi}}_1\}, \{\underline{\Phi}_6, \overline{\underline{\Phi}}_6\}, \{\underline{\Phi}_7, \overline{\underline{\Phi}}_7\}, \{\underline{\Phi}_8, \overline{\underline{\Phi}}_8\}$ *correspond to K^r up to isomorphism.*

Proof. Since equivalent CM-types give the same reflex field, it suffices to find the reflex fields corresponding to CM-types $\underline{\Phi}_1, ..., \underline{\Phi}_8$. In the notation of the previous sections we have that $\underline{\Phi}_3 = \Phi^r = \langle b, c\rangle$, so the reflex pair corresponding to $(K^r, \underline{\Phi}_3)$ is (K, Φ). For the remainder of the CM-types we compute the reflex fields one by one.

Let $\underline{\Phi}_{4,L}$ be the CM-type of L induced by $\underline{\Phi}_4$. Then

$$\underline{\Phi}_{4,L} = \underline{\Phi}_4 \operatorname{Gal}(L/K^r) = \{1, a, c, ac, x, ax, cx, acx, cx, x^2, ax^2, cx^2, acx^2\}.$$

Computing the set $S \subset G$ such that $\sigma\underline{\Phi}_{4,L} = \underline{\Phi}_{4,L}$ for all $\sigma \in S$ gives $S = \langle a, b\rangle$, which fixes the sextic field $K_+(\sqrt{-\delta_1})$. From similar computations it follows that $\underline{\Phi}_5$ corresponds to sextic field $K_+(\sqrt{-\delta_2})$, fixed by $\langle a, c\rangle$.

Similar computations for the remaining CM-types give that $\underline{\Phi}_1, \underline{\Phi}_6, \underline{\Phi}_7, \underline{\Phi}_8$ correspond to K^r up to isomorphism. Finally CM-type $\underline{\Phi}_2$ corresponds to the imaginary quadratic field k fixed by $\langle x, ab, ac\rangle$. This covers all CM-types, so all sextic reflex fields of K^r are isomorphic. □

The following corollary of Proposition 2.3.2 is applied in step 6 of Algorithm 3.4.5.

Corollary 2.3.3. *All sextic reflex fields of K^r are isomorphic.*

Proof. By Proposition 2.3.2 the sextic reflex fields of K^r are $K, K_+(\sqrt{-\delta_1})$ and $K_+(\sqrt{-\delta_2})$. From Section 2.1 we know that that these are all isomorphic, so the result follows. □

2.4. Discriminant and relative class number relations

Given a CM-field F with maximal totally real subfield F_+ the relative class number of F is defined as $h_F^* = h_F/h_{F_+}$. We derive a relation between the discriminants of K, K^r and k in Proposition 2.4.3 and a relation between their relative class numbers in Theorem 2.4.5.

In the following lemma we look at the number of roots of unity of K and K^r.

Lemma 2.4.1. *Let K be a sextic CM that does not contain an imaginary quadratic subfield. Let K^r be a reflex field of K with imaginary quadratic subfield $k = \mathbb{Q}(\sqrt{-m}) \subset K^r$ and denote by μ_K, μ_{K^r}, μ_k the number of roots of unity of each respective field. Then*

$$\mu_K = 2 \quad \text{and} \quad \mu_{K^r} \in \{2, 4, 6\}.$$

In particular, if $m \neq 1, 3$ then $\mu_{K^r} = 2$. Moreover, $\mu_{K^r} = \mu_k$.

Proof. For K, suppose there exists an n-th root of unity with $n > 2$. Then $\mathbb{Q}(\zeta_n) \subsetneq K$, excluding equality because K is non-normal. Then we must have $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] \in \{1, 2, 3\}$, reducing to $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] \in \{2, 3\}$ because $n > 2$. Then $n \in \{3, 4, 6\}$, so either $\sqrt{-1}$ or $\sqrt{-3}$ is an element in K. This is a contradiction because K does not contain an imaginary quadratic field, so $n \leq 2$ and therefore $\mu_K = 2$.

For K^r, let $\zeta_n \in K^r$ with $n > 2$. Since $[K^r : \mathbb{Q}] = 8$ and K^r is non-normal, we must have $\mathbb{Q}(\zeta_n) \subsetneq K^r$ with $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] \in \{2, 4\}$. This gives $n \in \{3, 4, 5, 6, 8, 10, 12\}$. Because K^r does not contain a degree 4 imaginary subfield, we can reduce this to $n \in \{3, 4, 6\}$. We have

$$\begin{aligned} \mu_{\mathbb{Q}(\zeta_3)} &= \mu_{\mathbb{Q}(\sqrt{-3})} = 6, \\ \mu_{\mathbb{Q}(\zeta_4)} &= \mu_{\mathbb{Q}(\sqrt{-1})} = 4, \\ \mu_{\mathbb{Q}(\zeta_6)} &= \mu_{\mathbb{Q}(\sqrt{-3})} = 6. \end{aligned}$$

This gives $\mu_{K^r} \in \{4, 6\}$. Including the case when $n \leq 2$ then gives $\mu_{K^r} \in \{2, 4, 6\}$.

For the statements in the last line, recall that $K^r = kK^r_+$. Since K^r_+ is a totally real field we have $\mu_{K^r_+} = 2$. Since k is a totally imaginary quadratic field and K^r is a totally imaginary quadratic extension of K^r_+, it follows that $\mu_{K^r} = \mu_k$. If $k = \mathbb{Q}(\sqrt{-m})$ with $m \notin \{1, 3\}$, then $\mu_{K^r} = \mu_k = 2$. □

For a number field F, let $\zeta_F(s)$ denote the Dedekind zeta function of F. We use shorthand notation ζ_F.

Proposition 2.4.2. *Let K be a sextic CM-field whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$. Then*

$$\frac{\zeta_{K^r}}{\zeta_{K^r_+}} = \frac{\zeta_K}{\zeta_{K_+}} \frac{\zeta_k}{\zeta_{\mathbb{Q}}}.$$

Proof. The conjugacy classes of elements in $G := \mathrm{Gal}(L/\mathbb{Q})$ are given by

$$\begin{aligned} C(1) &= \{1\} \\ C(abc) &= \{abc\} \\ C(x^2) &= \{x^2, abx^2, acx^2, bcx^2\} \\ C(ab) &= \{ab, ac, bc\} \\ C(abcx^2) &= \{abcx^2, ax^2, bx^2, cx^2\} \\ C(a) &= \{a, b, c\} \\ C(x) &= \{x, abx, acx, bcx\} \\ C(abcx) &= \{abcx, ax, bx, cx\}. \end{aligned}$$

By Lemma 1.2.11 the number of irreducible characters of G is equal to the number of conjugacy classes, which is 8. Let $\psi_0, \dots, \psi_7$ denote these irreducible characters.
The corresponding character table below was computed using the preliminaries in Section

1.2.

	$C(1)$	$C(abc)$	$C(x^2)$	$C(ab)$	$C(abcx^2)$	$C(a)$	$C(x)$	$C(abcx)$
ψ_0	1	1	1	1	1	1	1	1
ψ_1	1	-1	1	1	-1	-1	1	-1
ψ_2	1	-1	$-\zeta_3 - 1$	1	$\zeta_3 + 1$	-1	ζ_3	$-\zeta_3$
ψ_3	1	-1	ζ_3	1	$-\zeta_3$	-1	$-\zeta_3 - 1$	$\zeta_3 + 1$
ψ_4	1	1	$-\zeta_3 - 1$	1	$-\zeta_3 - 1$	1	ζ_3	ζ_3
ψ_5	1	1	ζ_3	1	ζ_3	1	$-\zeta_3 - 1$	$-\zeta_3 - 1$
ψ_6	3	-3	0	-1	0	1	0	0
ψ_7	3	3	0	-1	0	-1	0	0

For H a subgroup of G let L^H be the subfield of L fixed by H. Denote by χ_{L^H} the character on H induced by the character of the trivial representation on G. Define $A := L^{\langle ab,ac\rangle}$, which is a cyclic sextic CM-field with $\mathrm{Gal}(A/\mathbb{Q}) = \langle xabc\rangle$ and $A_+ = K_+$. Using Corollary 1.2.15 we find the characters χ_{L^H} with respect to subgroups $H \subset G$. The results are given in the following table.

	$C(1)$	$C(abc)$	$C(x^2)$	$C(ab)$	$C(abcx^2)$	$C(a)$	$C(x)$	$C(abcx)$
χ_L	24	0	0	0	0	0	0	0
χ_{L_+}	12	12	0	0	0	0	0	0
χ_K	6	0	0	2	0	4	0	0
χ_{K_+}	3	3	0	3	0	3	0	0
χ_{K^r}	8	0	2	0	0	0	2	0
$\chi_{K^r_+}$	4	4	1	0	1	0	1	1
χ_k	2	0	2	2	0	0	2	0
χ_A	6	0	0	6	0	0	0	0
$\chi_{\mathbb{Q}}$	1	1	1	1	1	1	1	1

Let $L_{\psi_i} := L(s, \psi_i, L/\mathbb{Q})$ be the Artin L-function corresponding to ψ_i. Then the character tables give the following relations and factorisations of Dedekind zeta functions (see Theorem 1.3.8 and (1.3)):

$$\begin{aligned}
\chi_L &= \psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + 3\psi_6 + 3\psi_7 & \zeta_L &= L_{\psi_0}L_{\psi_1}L_{\psi_2}L_{\psi_3}L_{\psi_4}L_{\psi_5}L_{\psi_6}^3L_{\psi_7}^3\\
\chi_{L_+} &= \psi_0 + \psi_4 + \psi_5 + 3\psi_7 & \zeta_{L_+} &= L_{\psi_0}L_{\psi_4}L_{\psi_5}L_{\psi_7}^3\\
\chi_K &= \psi_0 + \psi_4 + \psi_5 + \psi_6 & \zeta_K &= L_{\psi_0}L_{\psi_4}L_{\psi_5}L_{\psi_6}\\
\chi_{K_+} &= \psi_0 + \psi_4 + \psi_5 & \zeta_{K_+} &= L_{\psi_0}L_{\psi_4}L_{\psi_5}\\
\chi_{K^r} &= \psi_0 + \psi_1 + \psi_6 + \psi_7 & \zeta_{K^r} &= L_{\psi_0}L_{\psi_1}L_{\psi_6}L_{\psi_7}\\
\chi_{K^r_+} &= \psi_0 + \psi_7 & \zeta_{K^r_+} &= L_{\psi_0}L_{\psi_7}\\
\chi_k &= \psi_0 + \psi_1 & \zeta_k &= L_{\psi_0}L_{\psi_1}\\
\chi_A &= \psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 & \zeta_A &= L_{\psi_0}L_{\psi_1}L_{\psi_2}L_{\psi_3}L_{\psi_4}L_{\psi_5}\\
\chi_{\mathbb{Q}} &= \psi_0 & \zeta_{\mathbb{Q}} &= L_{\psi_0}.
\end{aligned}$$

From these relations we conclude

$$\frac{\zeta_{K^r}}{\zeta_{K_+^r}} = \frac{\zeta_K}{\zeta_{K_+}}\frac{\zeta_k}{\zeta_{\mathbb{Q}}}.$$

□

Let F be a number field with discriminant d_F. We denote by $r_1(F)$ the number of real embeddings from F to $\mathbb{C}$ and by $r_2(F)$ the number of pairs of complex embeddings from F to $\mathbb{C}$. Let $\Gamma(s)$ be the Gamma function. The completed zeta function is defined as

$$\Lambda_F(s) := |d_F|^{s/2}(\pi^{-s/2}\Gamma(s/2))^{r_1(F)}(2(2\pi)^s\Gamma(s))^{r_2(F)}\zeta_F(s), \tag{2.2}$$

where $\Lambda_F(s)$ satisfies the functional equation $\Lambda_F(s) = \Lambda_F(1-s)$.

Proposition 2.4.3. *Let K be a sextic CM-field whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$ and let k be the imaginary quadratic subfield of the reflex field K^r of K. Then*

$$\frac{d_K}{d_{K_+}}d_k = \frac{d_{K^r}}{d_{K_+^r}}.$$

Proof. The proof is similar to the proof of [Lou94a, Theorem A]. Let $D := \frac{d_{K^r}d_{K_+}}{d_{K_+^r}d_Kd_k}$ and define the function

$$f(s) := \left(\frac{\Lambda_{K^r}(s)\Lambda_{K_+}(s)\Lambda_{\mathbb{Q}}(s)}{\Lambda_{K_+^r}(s)\Lambda_K(s)\Lambda_k(s)}\right)^2,$$

where $\Lambda_F(s)$ is defined as in (2.2) for any number field F. Then (2.2) gives

$$f(s) = D^s(\pi^{-s/2}\Gamma(s/2))^{2m}(2(2\pi)^{-s}\Gamma(s))^{2n}\left(\frac{\zeta_{K^r}(s)\zeta_{K_+}(s)\zeta_{\mathbb{Q}}(s)}{\zeta_{K_+^r}(s)\zeta_K(s)\zeta_k(s)}\right)^2, \tag{2.3}$$

where

$$m := r_1(K^r) + r_1(K_+) + r_1(\mathbb{Q}) - r_1(K_+^r) - r_1(K) - r_1(k), \tag{2.4}$$
$$n := r_2(K^r) + r_2(K_+) + r_2(\mathbb{Q}) - r_2(K_+^r) - r_2(K) - r_2(k). \tag{2.5}$$

Moreover, by the functional equation for (2.2) gives $f(s) = f(1-s)$. By Proposition 2.4.2 we have

$$\frac{\zeta_{K^r}(s)\zeta_{K_+}(s)\zeta_{\mathbb{Q}}(s)}{\zeta_{K_+^r}(s)\zeta_K(s)\zeta_k(s)} = 1,$$

so $m + 2n = 0$. For any CM-field F we have $r_2(F) = [F : \mathbb{Q}]/2$ and $r_1(F_+) = [F_+ : \mathbb{Q}]$, while $r_1(F) = r_2(F_+) = 0$. Plugging this into (2.4) and (2.5) for K, K^r, k gives $m = n = 0$. Then (2.3) gives $f(s) = D^s$. From the relation $f(s) = f(1-s)$ we find that $D^s = D^{1-s}$, so $D = 1$ and the discriminant relation follows. □

Let F be a number field with $r_1(F)$ real embeddings and $r_2(F)$ pairs of complex embeddings. Let R_F denote the regulator of F and $\mathrm{Res}(\zeta_F)$ the residue of Dedekind zeta function ζ_F. Then the analytic class number formula gives

$$h_F = \frac{\mu_K \sqrt{|d_F|}\,\mathrm{Res}(\zeta_F)}{2^{r_1}(2\pi)^{r_2} R_F}. \tag{2.6}$$

Furthermore, if F is a CM-field we define the Hasse unit index of F as

$$Q_F \coloneqq [\mathcal{O}_F^\times : W_F \mathcal{O}_{F_+}^\times],$$

where $Q_F \in \{1, 2\}$ ([Lem95, Proposition 1(a)]). The following proposition gives a relation between the regulator and the Hasse unit index of a CM-field.

Proposition 2.4.4 (Proposition 4.16, [Was82])**.** *Let F be a CM-field with $2n \coloneqq [F : \mathbb{Q}]$. Then*

$$\frac{R_F}{R_{F_+}} = \frac{2^{n-1}}{Q_{K^r}}.$$

Theorem 2.4.5. *Let K be a sextic CM-field whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$ and let k be the imaginary quadratic subfield of the reflex field K^r of K. Then*

$$2h^*_{K^r} Q_K = h^*_K h_k Q_{K^r}. \tag{2.7}$$

Proof. The relation for the Dedekind zeta functions of K, K^r and their subfields in Proposition 2.4.2 extends to their residues. Combining the zeta function relation with (2.6) for these fields gives

$$\frac{(2\pi)^4 h_{K^r} \mu_{K^r_+} \sqrt{|d_{K^r_+}|}}{2^4 h_{K^r_+} \mu_{K^r} \sqrt{|d_{K^r}|}} \frac{R_{K^r}}{R_{K^r_+}} = \frac{(2\pi)^4 h_K h_k \mu_{K_+} \mu_{\mathbb{Q}} \sqrt{|d_{K_+} d_{\mathbb{Q}}|}}{2^4 h_{K_+} h_{\mathbb{Q}} \mu_K \mu_k \sqrt{|d_K d_k|}} \frac{R_K R_k}{R_{K_+} R_{\mathbb{Q}}}. \tag{2.8}$$

The fields $\mathbb{Q}, K_+, K^r_+$ are totally real, so $\mu_{\mathbb{Q}} = \mu_{K_+} = \mu_{K^r_+} = 2$. Moreover $\mu_K = 2$ and $\mu_{K^r} = \mu_k$ by Lemma 2.4.1. Because k is an imaginary quadratic field, $\mathcal{O}_k^\times = W_k$. Since $\mathcal{O}_{\mathbb{Q}}^\times = \{\pm 1\}$ this gives $Q_k = [\mathcal{O}_k^\times : W_k \mathcal{O}_{\mathbb{Q}}^\times] = 1$. By Proposition 2.4.4 we then have

$$\frac{R_K}{R_{K_+}} = \frac{2^2}{Q_K}, \qquad \frac{R_{K^r}}{R_{K^r_+}} = \frac{2^3}{Q_{K^r}}, \qquad \frac{R_k}{R_{\mathbb{Q}}} = 1.$$

Plugging the above into (2.8) gives

$$\frac{2h_{K^r} \sqrt{|d_{K^r_+}|}}{Q_{K^r} h_{K^r_+} \sqrt{|d_{K^r}|}} = \frac{h_K h_k \sqrt{|d_{K_+} d_{\mathbb{Q}}|}}{Q_K \sqrt{|d_K d_k|}}. \tag{2.9}$$

By applying Proposition 2.4.3 to (2.9) we get $2h^*_{K^r} Q_K = h^*_K h_k Q_{K^r}$. □

3. The CM class number one problem

The CM class number one problem for CM-fields K of degree $2g$ consists of determining all primitive CM-pairs (K, Φ) that correspond to principally polarized absolutely simple abelian varieties of dimension $2g$ defined over the corresponding reflex field K^r. Let Φ^r be the corresponding reflex CM-type and recall that

$$I_0(\Phi^r) := \{\mathfrak{a} \in I_K : N_{\Phi^r}(\mathfrak{a}) = (\alpha) \text{ for some } \alpha \in K^* \text{ such that } \alpha\overline{\alpha} \in \mathbb{Q}\}.$$

An equivalent way to describe the CM class number one problem is as follows.

Definition 3.0.1. The *CM class number one problem* for CM-fields K of degree $2g$ asks to determine the primitive CM-pairs (K, Φ) such that $I_{K^r} = I_0(\Phi^r)$.

In this chapter we give sufficient conditions such that K is a CM class number one field. In Section 3.1 we give the full decomposition table of primes in K^r and K as subfields of normal field L. In Section 3.2 we give in Proposition 3.2.9 an expression for h_K^* depending only on t_K when K is a CM class number one field. We then prove in Section 3.3 that there exist finitely many CM class number one sextic fields K (Theorem 3.3.6), by bounding the discriminant quotient d_K/d_{K_+} assuming K has CM class number one. In Section 3.4 we list the CM class number one fields K where $d_{K_+^r} \leq 10^9$ and $d_k \leq 10^4$ such that $t_k \leq 2$.

3.1. Decompositions of primes in $L/\mathbb{Q}$

We give the full ramification table of K^r and K as the subfields of Galois CM-field L in Table 3.1. From this table we make several observations. Lemma 3.1.1 provides a criterion that excludes several ramification cases in Table 3.1 if K is a CM class number one field.

Lemma 3.1.1. *Let K be a sextic CM field whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$. Assume $I_{K^r} = I_0(\Phi^r)$, then there exist no primes $\mathfrak{p}_{K^r} \in I_{K^r}$ above a prime number p satisfying $N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r}) = p$ and $N_{\Phi^r}(\mathfrak{p}_{K^r})^2 = p\mathcal{O}_K$.*

Proof. Suppose there is a prime $\mathfrak{p}_{K^r}$ such that $N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r}) = p$ and $N_{\Phi^r}(\mathfrak{p}_{K^r})^2 = p\mathcal{O}_K$. By the assumption $I_{K^r} = I_0(\Phi^r)$ there exists $\alpha \in K^\times$ such that $N_{\Phi^r}(\mathfrak{p}_{K^r}) = \alpha\mathcal{O}_K$ and $\alpha\overline{\alpha} \in \mathbb{Q}$. Then

$$\alpha\overline{\alpha}\mathcal{O}_K = N_{\Phi^r}(\mathfrak{p}_{K^r})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r})} = N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r}) = p\mathcal{O}_K$$

and hence $\alpha\overline{\alpha} = p$. Moreover,

$$\alpha^2\mathcal{O}_K = N_{\Phi^r}(\mathfrak{p}_{K^r})^2 = p\mathcal{O}_K.$$

This gives $u \cdot \alpha^2 = p = \alpha\overline{\alpha}$ with $u \in \mathcal{O}_K^\times$ and hence $\frac{\overline{\alpha}}{\alpha} = u$. Since $\frac{\overline{\alpha}}{\alpha}\frac{\alpha}{\overline{\alpha}} = 1$, the unit u is a root of unity. Since $W_K = \{\pm 1\}$ by Lemma 2.4.1, we get $\alpha = \pm\overline{\alpha}$. This gives $\alpha^2 = \pm p$, hence $\alpha = \sqrt{\pm p}$. This gives a contradiction because K does not contain a quadratic field, so these cases can not happen. □

Case	D	E	Decomp. in L	Decomp. in K^r	Decomp. in K_+^r	Decomp. in k	Decomp. in K	Decomp. in K_+	$N_{\Phi^r}(\mathfrak{p}_{K^r,1})$	CM
1	G	G	$\mathfrak{p}_{L,1}^{24}$	$\mathfrak{p}_{K^r,1}^{8}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{6}$	$\mathfrak{p}_{K_+,1}^{3}$	$\mathfrak{p}_{K,1}^{3}$	×
2	G	$\langle ab, ac, x\rangle$	$\mathfrak{p}_{L,1}^{12}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}^{6}$	$\mathfrak{p}_{K_+,1}^{3}$	$p\mathcal{O}_K$	✓
3	G	$\langle a, b, c\rangle$	$\mathfrak{p}_{L,1}^{8}$	$\mathfrak{p}_{K^r,1}^{8}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}$	$\mathfrak{p}_{K_+,1}$	$\mathfrak{p}_{K,1}$	×
4	G	$\langle ab, ac\rangle$	$\mathfrak{p}_{L,1}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}^{2}$	$\mathfrak{p}_{K_+,1}$	$p\mathcal{O}_K$	✓
5	$\langle ab, ac, x\rangle$	$\langle ab, ac, x\rangle$	$\mathfrak{p}_{L,1}^{12}\mathfrak{p}_{L,abc}^{12}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,abc}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,abc}$	$\mathfrak{p}_{K,1}^{6}$	$\mathfrak{p}_{K_+,1}^{3}$	$\mathfrak{p}_{K,1}^{3}$	×
6	$\langle ab, ac, x\rangle$	$\langle ab, ac\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,abc}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,abc}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,abc}$	$\mathfrak{p}_{K,1}^{2}$	$\mathfrak{p}_{K_+,1}$	$\mathfrak{p}_{K,1}$	×
7	$\langle a, b, c\rangle$	$\langle a, b, c\rangle$	$\mathfrak{p}_{L,1}^{8}\mathfrak{p}_{L,x}^{8}\mathfrak{p}_{L,x^2}^{8}$	$\mathfrak{p}_{K^r,1}^{8}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	×
8	$\langle a, b, c\rangle$	$\langle a, bc\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,x^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{2}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
9	$\langle a, b, c\rangle$	$\langle b, ac\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,x^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{2}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
10	$\langle a, b, c\rangle$	$\langle c, ab\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,x^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{2}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
11	$\langle a, b, c\rangle$	$\langle a, b\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,x^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
12	$\langle a, b, c\rangle$	$\langle a, c\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,x^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
13	$\langle a, b, c\rangle$	$\langle b, c\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,x^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
14	$\langle a, b, c\rangle$	$\langle ab, bc\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,x^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
15a	$\langle abcx\rangle$	$\langle abcx\rangle$	$\mathfrak{p}_{L,1}^{6}\mathfrak{p}_{L,a}^{6}\mathfrak{p}_{L,b}^{6}\mathfrak{p}_{L,c}^{6}$	$\mathfrak{p}_{K^r,1}^{2}\mathfrak{p}_{K^r,b}^{6}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,b}^{3}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{6}$	$\mathfrak{p}_{K_+,1}^{3}$	$\mathfrak{p}_{K,1}^{3}$	×
15b	$\langle ax\rangle$	$\langle ax\rangle$	$\mathfrak{p}_{L,1}^{6}\mathfrak{p}_{L,a}^{6}\mathfrak{p}_{L,b}^{6}\mathfrak{p}_{L,c}^{6}$	$\mathfrak{p}_{K^r,1}^{6}\mathfrak{p}_{K^r,b}^{2}$	$\mathfrak{p}_{K_+^r,1}^{3}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{6}$	$\mathfrak{p}_{K_+,1}^{3}$	$\mathfrak{p}_{K,1}^{3}$	×
15c	$\langle bx\rangle$	$\langle bx\rangle$	$\mathfrak{p}_{L,1}^{6}\mathfrak{p}_{L,a}^{6}\mathfrak{p}_{L,b}^{6}\mathfrak{p}_{L,c}^{6}$	$\mathfrak{p}_{K^r,1}^{6}\mathfrak{p}_{K^r,c}^{2}$	$\mathfrak{p}_{K_+^r,1}^{3}\mathfrak{p}_{K_+^r,c}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{6}$	$\mathfrak{p}_{K_+,1}^{3}$	$\mathfrak{p}_{K,1}^{3}$	×
15d	$\langle cx\rangle$	$\langle cx\rangle$	$\mathfrak{p}_{L,1}^{6}\mathfrak{p}_{L,a}^{6}\mathfrak{p}_{L,b}^{6}\mathfrak{p}_{L,c}^{6}$	$\mathfrak{p}_{K^r,1}^{6}\mathfrak{p}_{K^r,a}^{2}$	$\mathfrak{p}_{K_+^r,1}^{3}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{6}$	$\mathfrak{p}_{K_+,1}^{3}$	$\mathfrak{p}_{K,1}^{3}$	×
16a	$\langle abcx\rangle$	$\langle x\rangle$	$\mathfrak{p}_{L,1}^{3}\mathfrak{p}_{L,a}^{3}\mathfrak{p}_{L,b}^{3}\mathfrak{p}_{L,c}^{3}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}^{3}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}^{3}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}^{3}$	$\mathfrak{p}_{K_+,1}^{3}$	$p\mathcal{O}_K$	✓
16b	$\langle ax\rangle$	$\langle bcx\rangle$	$\mathfrak{p}_{L,1}^{3}\mathfrak{p}_{L,a}^{3}\mathfrak{p}_{L,b}^{3}\mathfrak{p}_{L,c}^{3}$	$\mathfrak{p}_{K^r,1}^{3}\mathfrak{p}_{K^r,b}$	$\mathfrak{p}_{K_+^r,1}^{3}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}^{3}$	$\mathfrak{p}_{K_+,1}^{3}$	$p\mathcal{O}_K$	✓
16c	$\langle bx\rangle$	$\langle acx\rangle$	$\mathfrak{p}_{L,1}^{3}\mathfrak{p}_{L,a}^{3}\mathfrak{p}_{L,b}^{3}\mathfrak{p}_{L,c}^{3}$	$\mathfrak{p}_{K^r,1}^{3}\mathfrak{p}_{K^r,c}$	$\mathfrak{p}_{K_+^r,1}^{3}\mathfrak{p}_{K_+^r,c}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}^{3}$	$\mathfrak{p}_{K_+,1}^{3}$	$p\mathcal{O}_K$	✓
16d	$\langle cx\rangle$	$\langle abx\rangle$	$\mathfrak{p}_{L,1}^{3}\mathfrak{p}_{L,a}^{3}\mathfrak{p}_{L,b}^{3}\mathfrak{p}_{L,c}^{3}$	$\mathfrak{p}_{K^r,1}^{3}\mathfrak{p}_{K^r,a}$	$\mathfrak{p}_{K_+^r,1}^{3}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}^{3}$	$\mathfrak{p}_{K_+,1}^{3}$	$p\mathcal{O}_K$	✓
17a	$\langle abcx\rangle$	$\langle abc\rangle$	$\mathfrak{p}_{L,1}^{2}\mathfrak{p}_{L,a}^{2}\mathfrak{p}_{L,b}^{2}\mathfrak{p}_{L,c}^{2}$	$\mathfrak{p}_{K^r,1}^{2}\mathfrak{p}_{K^r,b}^{2}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}$	$\mathfrak{p}_{K_+,1}$	$p\mathcal{O}_K$	✓
17b	$\langle ax\rangle$	$\langle abc\rangle$	$\mathfrak{p}_{L,1}^{2}\mathfrak{p}_{L,a}^{2}\mathfrak{p}_{L,b}^{2}\mathfrak{p}_{L,c}^{2}$	$\mathfrak{p}_{K^r,1}^{2}\mathfrak{p}_{K^r,b}^{2}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}$	$\mathfrak{p}_{K_+,1}$	$p\mathcal{O}_K$	✓
17c	$\langle bx\rangle$	$\langle abc\rangle$	$\mathfrak{p}_{L,1}^{2}\mathfrak{p}_{L,a}^{2}\mathfrak{p}_{L,b}^{2}\mathfrak{p}_{L,c}^{2}$	$\mathfrak{p}_{K^r,1}^{2}\mathfrak{p}_{K^r,c}^{2}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,c}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}$	$\mathfrak{p}_{K_+,1}$	$p\mathcal{O}_K$	✓
17d	$\langle cx\rangle$	$\langle abc\rangle$	$\mathfrak{p}_{L,1}^{2}\mathfrak{p}_{L,a}^{2}\mathfrak{p}_{L,b}^{2}\mathfrak{p}_{L,c}^{2}$	$\mathfrak{p}_{K^r,1}^{2}\mathfrak{p}_{K^r,a}^{2}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}$	$\mathfrak{p}_{K_+,1}$	$p\mathcal{O}_K$	✓
18	$\langle ab, bc\rangle$	$\langle ab, bc\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,a}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,ax}^{4}\mathfrak{p}_{L,x^2}^{4}\mathfrak{p}_{L,ax^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,a}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	×
19a	$\langle a, bc\rangle$	$\langle a, bc\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,b}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,bx}^{4}\mathfrak{p}_{L,x^2}^{4}\mathfrak{p}_{L,bx^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,b}^{4}$	$\mathfrak{p}_{K_+^r,1}^{2}\mathfrak{p}_{K_+^r,b}^{2}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	×
19b	$\langle b, ac\rangle$	$\langle b, ac\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,c}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,cx}^{4}\mathfrak{p}_{L,x^2}^{4}\mathfrak{p}_{L,cx^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,c}^{4}$	$\mathfrak{p}_{K_+^r,1}^{2}\mathfrak{p}_{K_+^r,c}^{2}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	×
19c	$\langle c, ab\rangle$	$\langle c, ab\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,a}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,ax}^{4}\mathfrak{p}_{L,x^2}^{4}\mathfrak{p}_{L,ax^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,a}^{4}$	$\mathfrak{p}_{K_+^r,1}^{2}\mathfrak{p}_{K_+^r,a}^{2}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	×
20a	$\langle a, b\rangle$	$\langle a, b\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,c}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,cx}^{4}\mathfrak{p}_{L,x^2}^{4}\mathfrak{p}_{L,cx^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,c}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
20b	$\langle a, c\rangle$	$\langle a, c\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,b}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,bx}^{4}\mathfrak{p}_{L,x^2}^{4}\mathfrak{p}_{L,bx^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,b}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}^{2}\mathfrak{p}_{K,x}\mathfrak{p}_{K,bx}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
20c	$\langle b, c\rangle$	$\langle b, c\rangle$	$\mathfrak{p}_{L,1}^{4}\mathfrak{p}_{L,a}^{4}\mathfrak{p}_{L,x}^{4}\mathfrak{p}_{L,ax}^{4}\mathfrak{p}_{L,x^2}^{4}\mathfrak{p}_{L,ax^2}^{4}$	$\mathfrak{p}_{K^r,1}^{4}\mathfrak{p}_{K^r,a}^{4}$	$\mathfrak{p}_{K_+^r,1}^{4}$	$\mathfrak{p}_{k,1}^{2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}_{K,x}^{2}\mathfrak{p}_{K,x^2}^{2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓

Case	D	E	Decomp. in L	Decomp. in K^r	Decomp. in K^r_+	Decomp. in k	Decomp. in K	Decomp. in K_+	$N_{\Phi^r}(\mathfrak{p}_{K^r,1})$	CM
$21a$	$\langle ab, bc\rangle$	$\langle ab\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{x^2}\mathfrak{p}^2_{ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$21b$	$\langle ab, bc\rangle$	$\langle ac\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{x^2}\mathfrak{p}^2_{ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$21c$	$\langle ab, bc\rangle$	$\langle bc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{x^2}\mathfrak{p}^2_{ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$22a$	$\langle a, bc\rangle$	$\langle a\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,bx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,b}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,b}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$22b$	$\langle b, ac\rangle$	$\langle b\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,cx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,c}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,c}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$22c$	$\langle c, ab\rangle$	$\langle c\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,a}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$23a$	$\langle a, bc\rangle$	$\langle bc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,bx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,b}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,b}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$23b$	$\langle b, ac\rangle$	$\langle ac\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,cx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,c}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,c}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$23c$	$\langle c, ab\rangle$	$\langle ab\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,a}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$24a$	$\langle a, bc\rangle$	$\langle abc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,bx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,b}$	$\mathfrak{p}_{K^r_+,1}\mathfrak{p}_{K^r_+,b}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$24b$	$\langle b, ac\rangle$	$\langle abc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}_{K^r_+,1}\mathfrak{p}_{K^r_+,a}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$24c$	$\langle c, ab\rangle$	$\langle abc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,cx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,c}$	$\mathfrak{p}_{K^r_+,1}\mathfrak{p}_{K^r_+,c}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
$25a$	$\langle a, b\rangle$	$\langle a\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,cx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,c}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}^2_{K,x^2}$	✓
$25b$	$\langle a, b\rangle$	$\langle b\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,cx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,c}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	✓
$25c$	$\langle a, c\rangle$	$\langle a\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,bx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,b}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,bx}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}$	✓
$25d$	$\langle a, c\rangle$	$\langle c\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,bx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,b}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,bx}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,bx^2}$	✓
$25e$	$\langle b, c\rangle$	$\langle b\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}$	✓
$25f$	$\langle b, c\rangle$	$\langle c\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}^2_{K,x^2}$	✓
$26a$	$\langle a, b\rangle$	$\langle ab\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,cx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,c}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	✓
$26b$	$\langle a, c\rangle$	$\langle ac\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,bx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,b}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,bx}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	✓
$26c$	$\langle b, c\rangle$	$\langle bc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}$	$\mathfrak{p}^2_{K^r_+,1}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	✓
$27a$	$\langle x\rangle$	$\langle x\rangle$	$\mathfrak{p}^3_{L,1}\mathfrak{p}^3_{L,a}\mathfrak{p}^3_{L,b}\mathfrak{p}^3_{L,c}\mathfrak{p}^3_{L,ab}\mathfrak{p}^3_{L,ac}\mathfrak{p}^3_{L,bc}\mathfrak{p}^3_{L,abc}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}^3_{K^r,a}\mathfrak{p}^3_{K^r,bc}\mathfrak{p}_{K^r,abc}$	$\mathfrak{p}_{K^r_+,1}\mathfrak{p}^3_{K^r_+,a}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}^3_{K,1}\mathfrak{p}^3_{K,a}$	$\mathfrak{p}^3_{K_+,1}$	$\mathfrak{p}^3_{K,1}$	✓
$27b$	$\langle abx\rangle$	$\langle abx\rangle$	$\mathfrak{p}^3_{L,1}\mathfrak{p}^3_{L,a}\mathfrak{p}^3_{L,b}\mathfrak{p}^3_{L,c}\mathfrak{p}^3_{L,ab}\mathfrak{p}^3_{L,ac}\mathfrak{p}^3_{L,bc}\mathfrak{p}^3_{L,abc}$	$\mathfrak{p}^3_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,bc}\mathfrak{p}^3_{K^r,abc}$	$\mathfrak{p}^3_{K^r_+,1}\mathfrak{p}_{K^r_+,a}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}^3_{K,1}\mathfrak{p}^3_{K,abc}$	$\mathfrak{p}^3_{K_+,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,abc}$	✓
$27c$	$\langle acx\rangle$	$\langle acx\rangle$	$\mathfrak{p}^3_{L,1}\mathfrak{p}^3_{L,a}\mathfrak{p}^3_{L,b}\mathfrak{p}^3_{L,c}\mathfrak{p}^3_{L,ab}\mathfrak{p}^3_{L,ac}\mathfrak{p}^3_{L,bc}\mathfrak{p}^3_{L,abc}$	$\mathfrak{p}^3_{K^r,1}\mathfrak{p}_{K^r,c}\mathfrak{p}_{K^r,ab}\mathfrak{p}^3_{K^r,abc}$	$\mathfrak{p}^3_{K^r_+,1}\mathfrak{p}_{K^r_+,c}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,c}$	$\mathfrak{p}^3_{K,1}\mathfrak{p}^3_{K,abc}$	$\mathfrak{p}^3_{K_+,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,abc}$	✓
$27d$	$\langle bcx\rangle$	$\langle bcx\rangle$	$\mathfrak{p}^3_{L,1}\mathfrak{p}^3_{L,a}\mathfrak{p}^3_{L,b}\mathfrak{p}^3_{L,c}\mathfrak{p}^3_{L,ab}\mathfrak{p}^3_{L,ac}\mathfrak{p}^3_{L,bc}\mathfrak{p}^3_{L,abc}$	$\mathfrak{p}^3_{K^r,1}\mathfrak{p}_{K^r,b}\mathfrak{p}_{K^r,ac}\mathfrak{p}^3_{K^r,abc}$	$\mathfrak{p}^3_{K^r_+,1}\mathfrak{p}_{K^r_+,b}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,b}$	$\mathfrak{p}^3_{K,1}\mathfrak{p}^3_{K,abc}$	$\mathfrak{p}^3_{K_+,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,abc}$	✓
$28a$	$\langle a\rangle$	$\langle a\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,bc}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,bcx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,bx^2}\mathfrak{p}^2_{L,cx^2}\mathfrak{p}^2_{L,bcx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,b}\mathfrak{p}^2_{K^r,c}\mathfrak{p}^2_{K^r,bc}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,b}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,cx}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
$28b$	$\langle b\rangle$	$\langle b\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,ac}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,acx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}\mathfrak{p}^2_{L,cx^2}\mathfrak{p}^2_{L,acx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}\mathfrak{p}^2_{K^r,c}\mathfrak{p}^2_{K^r,ac}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,a}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,ax^2}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
$28c$	$\langle c\rangle$	$\langle c\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,ab}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,abx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}\mathfrak{p}^2_{L,bx^2}\mathfrak{p}^2_{L,abx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}\mathfrak{p}^2_{K^r,b}\mathfrak{p}^2_{K^r,ab}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,c}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,c}\mathfrak{p}_{K,x}\mathfrak{p}_{K,cx}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
$29a$	$\langle ab\rangle$	$\langle ab\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,ac}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,cx}\mathfrak{p}^2_{L,acx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}\mathfrak{p}^2_{L,cx^2}\mathfrak{p}^2_{L,acx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}\mathfrak{p}^2_{K^r,c}\mathfrak{p}^2_{K^r,ac}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,c}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,c}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
$29b$	$\langle bc\rangle$	$\langle bc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,ab}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,abx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}\mathfrak{p}^2_{L,bx^2}\mathfrak{p}^2_{L,abx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}\mathfrak{p}^2_{K^r,b}\mathfrak{p}^2_{K^r,ab}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,a}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
$29c$	$\langle ac\rangle$	$\langle ac\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,ab}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,abx}\mathfrak{p}^2_{L,x^2}\mathfrak{p}^2_{L,ax^2}\mathfrak{p}^2_{L,bx^2}\mathfrak{p}^2_{L,abx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}\mathfrak{p}^2_{K^r,b}\mathfrak{p}^2_{K^r,ab}$	$\mathfrak{p}^2_{K^r_+,1}\mathfrak{p}^2_{K^r_+,b}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,b}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,bx}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
30	$\langle abc\rangle$	$\langle abc\rangle$	$\mathfrak{p}^2_{L,1}\mathfrak{p}^2_{L,a}\mathfrak{p}^2_{L,b}\mathfrak{p}^2_{L,c}\mathfrak{p}^2_{L,x}\mathfrak{p}^2_{L,ax}\mathfrak{p}^2_{L,bx}\mathfrak{p}^2_{L,cx}\mathfrak{p}_{L,x^2}\mathfrak{p}^2_{L,ax^2}\mathfrak{p}^2_{L,bx^2}\mathfrak{p}^2_{L,cx^2}$	$\mathfrak{p}^2_{K^r,1}\mathfrak{p}^2_{K^r,a}\mathfrak{p}^2_{K^r,b}\mathfrak{p}^2_{K^r,c}$	$\mathfrak{p}_{K^r_+,1}\mathfrak{p}_{K^r_+,a}\mathfrak{p}_{K^r_+,b}\mathfrak{p}_{K^r_+,c}$	$\mathfrak{p}^2_{k,1}$	$\mathfrak{p}^2_{K,1}\mathfrak{p}^2_{K,x}\mathfrak{p}^2_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	×

Case	D	E	Decomp. in L	Decomp. in K^r	Decomp. in K_+^r	Decomp. in k	Decomp. in K	Decomp. in K_+	$N_{\Phi^r}(\mathfrak{p}_{K^r,1})$	CM
$31a$	$\langle ax\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,b}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}$	$\mathfrak{p}_{K_+,1}$	$p^3\mathcal{O}_K$	✓
$31b$	$\langle bx\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,c}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,c}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}$	$\mathfrak{p}_{K_+,1}$	$p^3\mathcal{O}_K$	✓
$31c$	$\langle cx\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}$	$\mathfrak{p}_{K_+,1}$	$p^3\mathcal{O}_K$	✓
$31d$	$\langle abcx\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}$	$\mathfrak{p}_{K_+,1}$	$p^3\mathcal{O}_K$	✓
$32a$	$\langle x\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}\mathfrak{p}_{L,abc}\mathfrak{p}_{L,ab}\mathfrak{p}_{L,ac}\mathfrak{p}_{L,bc}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,abc}\mathfrak{p}_{K^r,bc}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}$	$\mathfrak{p}_{K_+,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}^2$	✓
$32b$	$\langle abx\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}\mathfrak{p}_{L,abc}\mathfrak{p}_{L,ab}\mathfrak{p}_{L,ac}\mathfrak{p}_{L,bc}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,abc}\mathfrak{p}_{K^r,bc}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}$	$\mathfrak{p}_{K_+,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}^2$	✓
$32c$	$\langle bcx\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}\mathfrak{p}_{L,abc}\mathfrak{p}_{L,ab}\mathfrak{p}_{L,ac}\mathfrak{p}_{L,bc}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,b}\mathfrak{p}_{K^r,abc}\mathfrak{p}_{K^r,ac}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,b}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}$	$\mathfrak{p}_{K_+,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}^2$	✓
$32d$	$\langle acx\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}\mathfrak{p}_{L,abc}\mathfrak{p}_{L,ab}\mathfrak{p}_{L,ac}\mathfrak{p}_{L,bc}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,c}\mathfrak{p}_{K^r,abc}\mathfrak{p}_{K^r,ab}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,c}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,c}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}$	$\mathfrak{p}_{K_+,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}^2$	✓
$33a$	$\langle a\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}\mathfrak{p}_{L,bc}\mathfrak{p}_{L,x}\mathfrak{p}_{L,bx}\mathfrak{p}_{L,cx}\mathfrak{p}_{L,bcx}\mathfrak{p}_{L,x^2}\mathfrak{p}_{L,bx^2}\mathfrak{p}_{L,cx^2}\mathfrak{p}_{L,bcx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,b}\mathfrak{p}_{K^r,c}\mathfrak{p}_{K^r,bc}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,cx}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}^2$	✓
$33b$	$\langle b\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,c}\mathfrak{p}_{L,ac}\mathfrak{p}_{L,x}\mathfrak{p}_{L,ax}\mathfrak{p}_{L,cx}\mathfrak{p}_{L,acx}\mathfrak{p}_{L,x^2}\mathfrak{p}_{L,ax^2}\mathfrak{p}_{L,cx^2}\mathfrak{p}_{L,acx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,c}\mathfrak{p}_{K^r,ac}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}_{K,x}\mathfrak{p}_{K,ax^2}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}^2$	✓
$33c$	$\langle c\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,ab}\mathfrak{p}_{L,x}\mathfrak{p}_{L,ax}\mathfrak{p}_{L,bx}\mathfrak{p}_{L,abx}\mathfrak{p}_{L,x^2}\mathfrak{p}_{L,ax^2}\mathfrak{p}_{L,bx^2}\mathfrak{p}_{L,abx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,b}\mathfrak{p}_{K^r,ab}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}_{K,x}\mathfrak{p}_{K,ax}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}$	✓
$34a$	$\langle ab\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,c}\mathfrak{p}_{L,ac}\mathfrak{p}_{L,x}\mathfrak{p}_{L,ax}\mathfrak{p}_{L,cx}\mathfrak{p}_{L,acx}\mathfrak{p}_{L,x^2}\mathfrak{p}_{L,ax^2}\mathfrak{p}_{L,cx^2}\mathfrak{p}_{L,acx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,c}\mathfrak{p}_{K^r,ac}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}^2$	✓
$34b$	$\langle bc\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,a}\mathfrak{p}_{L,b}\mathfrak{p}_{L,ab}\mathfrak{p}_{L,x}\mathfrak{p}_{L,ax}\mathfrak{p}_{L,bx}\mathfrak{p}_{L,abx}\mathfrak{p}_{L,x^2}\mathfrak{p}_{L,ax^2}\mathfrak{p}_{L,bx^2}\mathfrak{p}_{L,abx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,b}\mathfrak{p}_{K^r,ab}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,a}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓
$34c$	$\langle ac\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,b}\mathfrak{p}_{L,c}\mathfrak{p}_{L,bc}\mathfrak{p}_{L,x}\mathfrak{p}_{L,bx}\mathfrak{p}_{L,cx}\mathfrak{p}_{L,bcx}\mathfrak{p}_{L,x^2}\mathfrak{p}_{L,bx^2}\mathfrak{p}_{L,cx^2}\mathfrak{p}_{L,bcx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,b}\mathfrak{p}_{K^r,c}\mathfrak{p}_{K^r,bc}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,b}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,b}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,ax}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}$	✓
35	$\langle abc\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\mathfrak{p}_{L,ab}\mathfrak{p}_{L,ac}\mathfrak{p}_{L,bc}\mathfrak{p}_{L,x}\mathfrak{p}_{L,abx}\mathfrak{p}_{L,acx}\mathfrak{p}_{L,bcx}\mathfrak{p}_{L,x^2}\mathfrak{p}_{L,abx^2}\mathfrak{p}_{L,acx^2}\mathfrak{p}_{L,bcx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,ab}\mathfrak{p}_{K^r,ac}\mathfrak{p}_{K^r,bc}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,ab}\mathfrak{p}_{K_+^r,ac}\mathfrak{p}_{K_+^r,bc}$	$\mathfrak{p}_{k,1}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$p\mathcal{O}_K$	✓
36	$\langle e\rangle$	$\langle e\rangle$	$\mathfrak{p}_{L,1}\cdots\mathfrak{p}_{L,abcx^2}$	$\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,a}\mathfrak{p}_{K^r,b}\mathfrak{p}_{K^r,c}\mathfrak{p}_{K^r,abc}\mathfrak{p}_{K^r,ab}\mathfrak{p}_{K^r,ac}\mathfrak{p}_{K^r,bc}$	$\mathfrak{p}_{K_+^r,1}\mathfrak{p}_{K_+^r,a}\mathfrak{p}_{K_+^r,b}\mathfrak{p}_{K_+^r,c}$	$\mathfrak{p}_{k,1}\mathfrak{p}_{k,a}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,a}\mathfrak{p}_{K,ax}\mathfrak{p}_{K,ax^2}$	$\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2}$	$\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}$	✓

Table 3.1: The complete prime decompositions for K and K^r as subfields of L with $\mathrm{Gal}(L/\mathbb{Q}) \cong (C_2)^3 \times C_3$ and the image of $\mathfrak{p}_{K^r}$ under the type norm N_{Φ^r}. In the last column $\times$ indicates that the decomposition cannot occur if K is CM-class number one as a consequence of Corollary 3.1.2; otherwise we write ✓.

The following corollary follows from Lemma 3.1.1 and allows us to eliminate fields K^r in steps 1, 3, 4 of Algorithm 3.4.2.

Corollary 3.1.2. *Let K be as in Lemma 3.1.1 and assume $I_{K^r} = I_0(\Phi^r)$, then cases* 1–3, 5–8, 16*a*–16*d*, 19*a*–21 *in Table 3.1 do not occur.*

Proof. This follows from checking every case in Table 3.1.1 and applying Lemma 3.1.1. □

For a CM-field F, let t_F denote the number of primes in F_+ that ramify in F/F_+. The statements in the following Lemma are immediate consequences of Table 3.1.

Lemma 3.1.3. *Let p be a prime number. If there exists a prime $\mathfrak{p}_{K_+^r}$ of K_+^r above prime number p such that $\mathfrak{p}_{K_+^r}$ ramifies in K^r/K_+^r, then*

(i) p ramifies in $k/\mathbb{Q}$;

(ii) the primes above p in $\mathcal{O}_{K_+}$ are all ramified in K/K_+.

Furthermore,

(iii) if p ramifies in $k/\mathbb{Q}$, then there is at least one prime above p in $\mathcal{O}_{K_+}$ that ramifies in K/K_+;

(iv) if $I_{K^r} = I_0(\Phi^r)$, then d_k and d_{K_+} have no prime factors in common.

Proof. All statements follow from Table 3.1. □

3.2. Relative class numbers

We give several properties of the relative class number h_K^* of K and sufficient conditions on h_K^* such that K has CM class number one. Most notably we prove in Corollary 3.2.9 that if K is of CM class number one, then $h_K^* \in \{2^{t_K}, 2^{t_K-1}\}$. Furthermore, Corollary 3.2.7 provides a criterion to exclude fields K^r of the list of fields left to check in Algorithm 3.4.2.

Let F be a CM-field and let $\bar{\cdot}$ denote complex conjugation on F. Define $H_F \coloneqq \mathrm{Gal}(F/F_+)$ and

$$I_F^{H_F} \coloneqq \{\mathfrak{a} \in I_F : \mathfrak{a} = \overline{\mathfrak{a}}\}, \quad P_F^{H_F} \coloneqq P_F \cap I_F^{H_F}.$$

The following lemmata give an expression for the relative class number h_F^*.

Lemma 3.2.1. *[Kıl16, Lemma 2.2.1] Let F be a CM-field and let t_F be the number of primes in F_+ that ramify in F. Then $h_F^* = 2^{t_F} \frac{[I_F : I_F^{H_F} P_F]}{[P_F^{H_F} : P_{F_+}]}$.*

Recall that $Q_F = [\mathcal{O}_F^\times : W_F \mathcal{O}_{F_+}^\times]$.

Lemma 3.2.2. *[Kıl16, Lemma 2.2.2] If $Q_F = 1$ then $[P_F^{H_F} : P_{F_+}] = 2$. Moreover, we have $h_F^* = 2^{t_F-1}[I_F : I_F^{H_F} P_F]$.*

Proposition 3.2.3. *[BL02, Proposition 1] Let F be a sextic CM-field, then $t_F \geq 1$.*

Let K be a sextic CM-field with Galois closure L such that $\mathrm{Gal}(L/\mathbb{Q}) \cong (C_2)^3 \rtimes C_3$. Proposition 3.2.3 implies that $d_K/d_{K_+}^2 > 1$. Furthermore, we have the following result.

Lemma 3.2.4. *Let F be a CM-field such that $W_F = \{\pm 1\}$. Then $[P_F^{H_F} : P_{F_+}] \in \{1, 2\}$.*

Proof. Define the map

$$\lambda : P_F^{H_F} \to W_F, \quad \alpha\mathcal{O}_F \mapsto \frac{\alpha}{\overline{\alpha}}.$$

Then λ is a group homomorphism. By Lemma 2.4.1 we have $W_F = 2$. For $(\alpha) \in \ker(\lambda)$ we have $\alpha = \overline{\alpha}$ and hence $\alpha \in F_+$. Then $(\alpha) \in P_{F_+}$. This argument is reversible, so $\ker(\lambda) = P_{F_+}$. Then

$$P_F^{H_F}/P_{F_+} \cong \mathrm{im}(\lambda) \subset W_F.$$

We have that $W_F = \{\pm 1\}$, so $[P_F^{H_F} : P_{F_+}] \in \{1, 2\}$. □

We know from Lemma 2.4.1 that $W_K = \{\pm 1\}$, so Lemma 3.2.4 gives

$$[P_K^{H_K} : P_{K_+}] \in \{1, 2\}. \tag{3.1}$$

We will now prove several statements about the relative class number of K under the condition $I_{K^r} = I_0(\Phi^r)$.

Proposition 3.2.5. *Let K be a non-normal sextic CM-field with Galois group $(C_2)^3 \rtimes C_3$. Assume $I_{K^r} = I_0(\Phi^r)$, then the following hold:*

(i) For every $\mathfrak{a} \in I_K$, $[\mathfrak{a}]^2 = [\overline{\mathfrak{a}}]^2$ in Cl_K.

(ii) There exists $u \in \mathbb{Z}_{\geq 0}$ such that $h_K^* = 2^{t_K + u}$.

Proof. (i) Let $\mathfrak{a} \in I_K$, then

$$N_{\Phi^r} N_{\Phi}(\mathfrak{a}) = \frac{\mathfrak{a}^2}{\overline{\mathfrak{a}}^2} N_{K/\mathbb{Q}}(\mathfrak{a})^2 \in I_K.$$

Since $I_{K^r} = I_0(\Phi^r)$, there exists $\lambda \in K^\times$ with $\lambda\overline{\lambda} \in \mathbb{Q}$ such that

$$\frac{\mathfrak{a}^2}{\overline{\mathfrak{a}}^2} N_{K/\mathbb{Q}}(\mathfrak{a})^2 = \lambda\mathcal{O}_K.$$

Then $\frac{\mathfrak{a}^2}{\overline{\mathfrak{a}}^2} = \lambda N_{K/\mathbb{Q}}(\mathfrak{a})^{-2}\mathcal{O}_K \in P_K$, so $[\mathfrak{a}]^2 = [\overline{\mathfrak{a}}]^2 \in \mathrm{Cl}_K$.

(ii) For this part we use ideas from the proof of [Kıl16, Lemma 2.2.3]. By Lemma 3.2.1 we have

$$h_K^* = 2^{t_K} \frac{[I_K : I_K^{H_K} P_K]}{[P_K^{H_K} : P_{K_+}]}, \tag{3.2}$$

Let $\alpha := N_{K/\mathbb{Q}}(\mathfrak{a})^2 \in \mathbb{Q}$. Then by part (i) there exists $\lambda \in K^\times$ with $\lambda\overline{\lambda} \in \mathbb{Q}$ such that

$$\alpha\frac{\mathfrak{a}^2}{\overline{\mathfrak{a}}^2} = \lambda\mathcal{O}_K.$$

Moreover,

$$\lambda\overline{\lambda}\mathcal{O}_K = \alpha\overline{\alpha}\frac{\mathfrak{a}^2\overline{\mathfrak{a}^2}}{\mathfrak{a}^2\overline{\mathfrak{a}^2}} = \alpha\overline{\alpha}\mathcal{O}_K.$$

This gives that $\lambda\overline{\lambda} = \alpha\overline{\alpha} \in \mathbb{Q}$. Define $\delta := \frac{\lambda}{\alpha}$, so we have $\frac{\mathfrak{a}^2}{\overline{\mathfrak{a}^2}} = \delta\mathcal{O}_K$. Then

$$\delta\overline{\delta} = \frac{\lambda\overline{\lambda}}{\alpha\overline{\alpha}} = 1.$$

Then by Hilbert's theorem 90 there exists $\gamma \in K^*$ such that $\delta = \frac{\overline{\gamma}}{\gamma}$. This gives

$$\mathfrak{a}^2 = \overline{\gamma\mathfrak{a}^2}\left(\frac{1}{\gamma}\mathcal{O}_K\right) \in I_K^{H_K}P_K.$$

For every $\mathfrak{a} \in I_K$ we have that $\mathfrak{a}^2 \in I_K^{H_K}P_K$, so $[I_K : I_K^{H_K}P_K] = 2^{u_0}$ for some $u_0 \in \mathbb{Z}_{\geq 0}$. Moreover we have $[P_K^{H_K} : P_{K_+}] \in \{1, 2\}$ by (3.1). Then

$$\frac{[I_K : I_K^{H_K}P_K]}{[P_K^{H_K} : P_{K_+}]} = 2^u,$$

where $u \in \{u_0, u_0 - 1\}$. Combining this with (3.2) gives $h_K^* = 2^{t_K+u}$ with $u \geq -1$. □

The following proposition is a generalisation of [Hor92, Theorem 1].

Proposition 3.2.6. *[Oka00, Theorem 1] Let F_0, F be two CM-fields such that $F_0 \subset F$. Then $h_{F_0}^* \mid 4h_F^*$.*

The following corollary of of Proposition 3.2.5(ii) and Proposition 3.2.6 provides a criterion that we use to eliminate fields in step 5 of Algorithm 3.4.2.

Corollary 3.2.7. *Assume $I_{K^r} = I_0(\Phi^r)$, then $4h_{K^r}^*/h_k$ is a power of 2.*

Proof. Proposition 3.2.6 implies $h_k \mid 4h_{K^r}^*$. From (2.7) we have

$$\frac{4h_{K^r}^*}{h_k} = 2h_K^*\frac{Q_{K^r}}{Q_K}.$$

Here h_K^* is a power of 2 by Proposition 3.2.5(ii). Moreover $Q_K, Q_{K^r} \in \{1, 2\}$, so $4h_{K^r}^*/h_k$ is a power of 2. □

In the proof of Proposition 3.2.5(i) we showed that every element in the quotient $I_K/I_K^H P_K$ has order 1 or 2. Assuming $I_{K^r} = I_0(\Phi^r)$ we prove $I_K = I_K^H P_K$ by looking at all possible ramifications of primes in $K/\mathbb{Q}$. We then use this to conclude $h_K^* \in \{2^{t_K}, 2^{t_K-1}\}$ in Corollary 3.2.9.

Proposition 3.2.8. *Let K be a non-normal sextic CM-field with Galois group $(C_2)^3 \rtimes C_3$. Assume $I_{K^r} = I_0(\Phi^r)$, then $I_K = I_K^H P_K$.*

Proof. In the proof of Proposition 3.2.5(ii) we showed the following:

$$\text{for any ideal } \mathfrak{a} \in I_K \text{ we have } \frac{\mathfrak{a}^2}{\overline{\mathfrak{a}}^2} = \delta\mathcal{O}_K \text{ for some } \delta \in K^\times \text{ with } \delta\overline{\delta} = 1. \qquad (3.3)$$

If we can prove that for every $\mathfrak{a} \in I_K$ there exists $\beta \in K^\times$ such that $\frac{\mathfrak{a}}{\overline{\mathfrak{a}}} = \beta\mathcal{O}_K$ and $\beta\overline{\beta} = 1$, then by the same argument as in the proof of the proposition we have $\mathfrak{a} = \overline{\gamma\mathfrak{a}}\left(\frac{1}{\gamma}\right) \in I_K^H P_K$ for some $\gamma \in K^\times$. Because we chose $\mathfrak{a}$ arbitrarily, it then follows that $I_K = I_0(\Phi^r)$.
Claim. For any prime ideal $\mathfrak{p}_K \in I_K$ we have

$$\frac{\mathfrak{p}_K}{\overline{\mathfrak{p}_K}} = \beta\mathcal{O}_K$$

for some $\beta \in K^\times$ with $\beta\overline{\beta} = 1$.
Proof of the claim. Let $\mathfrak{p}_{K_+} \subset \mathcal{O}_{K_+}$ with $p := \mathfrak{p}_{K_+} \cap \mathbb{Z}$ and let $\mathfrak{p}_K \subset \mathcal{O}_K$ be such that $\mathfrak{p}_K \cap \mathcal{O}_{K_+} = \mathfrak{p}_{K_+}$. If $\mathfrak{p}_{K_+}$ is inert or ramified in $\mathcal{O}_K$ then $\mathfrak{p}_K$ is fixed under complex conjugation, hence $\mathfrak{p}_K \in I_K^H P_K$. In the remainder of the proof we will consider the case where $\mathfrak{p}_{K_+}\mathcal{O}_K = \mathfrak{p}_K\overline{\mathfrak{p}_K}$ such that $\mathfrak{p}_K\overline{\mathfrak{p}_K}$ and show that then also $\mathfrak{p}_K \in I_K^H P_K$. We split this case into subcases with respect to the decomposition of p in $\mathcal{O}_{K_+}$. We will prove the result for each decomposition separately by going over cases in Table 3.1 that are possible when assuming $I_{K^r} = I_0(\Phi^r)$.

(i) p is inert in $K_+/\mathbb{Q}$: We have $\mathfrak{p}_{K_+}\mathcal{O}_K = \mathfrak{p}_K\overline{\mathfrak{p}}_K = p\mathcal{O}_K$, which only happens in case 32 of the table. Write $\overline{\mathfrak{p}}_K := \mathfrak{p}_{K,a}$. There exists $\alpha \in K^\times$ such that $\alpha\overline{\alpha} \in \mathbb{Q}$ and

$$N_{\Phi^r}(\mathfrak{p}_{K^r}) = \mathfrak{p}_K\overline{\mathfrak{p}}_K^2 = p\overline{\mathfrak{p}}_K = \alpha\mathcal{O}_K.$$

This gives $\overline{\mathfrak{p}}_K = \frac{\alpha}{p}\mathcal{O}_K$, hence

$$\mathfrak{p}_K = (p\mathcal{O}_K)\overline{\mathfrak{p}}_K^{-1} = \frac{p^2}{\alpha}\mathcal{O}_K.$$

Then

$$\frac{\mathfrak{p}_K}{\overline{\mathfrak{p}}_K} = \frac{p^3}{\alpha^2}\mathcal{O}_K,$$

where $\alpha\overline{\alpha} = (p\overline{\mathfrak{p}}_K)(p\mathfrak{p}_K) = p^3$. Let $\beta := \frac{p^3}{\alpha^2}$, then $\beta\overline{\beta} = \frac{p^6}{p^6} = 1$ which proves the claim.

(ii) p ramifies in $K_+/\mathbb{Q}$: Then $\mathfrak{p}_{K_+}^3\mathcal{O}_K = \mathfrak{p}_K^3\overline{\mathfrak{p}}_K^3 = p\mathcal{O}_K$, which is case 27 in the table. For 27a $N_{\Phi^r}(\mathfrak{p}_{K^r})$ decomposes differently than for 27b-d, so we will look at these separately.

(27a) Assuming $I_{K^r} = I_0(\Phi^r)$ there exists $\alpha \in K^\times$ such that $\alpha\overline{\alpha} \in \mathbb{Q}$ and

$$N_{\Phi^r}(\mathfrak{p}_{K^r}) = \mathfrak{p}_K^3 = \alpha\mathcal{O}_K.$$

This implies $\overline{\mathfrak{p}}_K^3 = \frac{p}{\alpha}\mathcal{O}_K$. By (3.3) there exists $\delta \in K^\times$ such that

$$\frac{\mathfrak{p}_K}{\overline{\mathfrak{p}}_K} = \frac{\mathfrak{p}_K^3}{\overline{\mathfrak{p}}_K^3}\frac{\overline{\mathfrak{p}}_K^2}{\mathfrak{p}_K^2} = \frac{\mathfrak{p}_K^3}{\overline{\mathfrak{p}}_K^3}\frac{1}{\delta} = \frac{\alpha^2}{p\delta}\mathcal{O}_K.$$

This gives

$$\frac{\mathfrak{p}_K}{\overline{\mathfrak{p}}_K} = \beta\mathcal{O}_K,$$

where $\beta = \frac{\alpha^2}{p\delta}$ and

$$\beta\overline{\beta} = \frac{\alpha^2\overline{\alpha}^2}{p^2\delta\overline{\delta}} = 1.$$

(27b-d) We have

$$N_{\Phi^r}(\mathfrak{p}_{K^r}) = \mathfrak{p}_K\overline{\mathfrak{p}}_K^2 = \alpha\mathcal{O}_K.$$

By (3.3) there exists $\delta \in K^\times$ such that $\frac{\mathfrak{p}_K^2}{\overline{\mathfrak{p}}_K^2} = \delta\mathcal{O}_K$ and $\delta\overline{\delta} = 1$. Then

$$\frac{\mathfrak{p}_K}{\overline{\mathfrak{p}}_K} = \frac{\mathfrak{p}_K}{\overline{\mathfrak{p}}_K}\frac{\mathfrak{p}_K^2}{\mathfrak{p}_K^2} = \frac{\mathfrak{p}_K}{\overline{\mathfrak{p}}_K}\frac{\overline{\mathfrak{p}}_K^2\delta}{\mathfrak{p}_K^2} = \frac{\alpha\delta}{\overline{\alpha}}\mathcal{O}_K.$$

This gives

$$\frac{\mathfrak{p}_K}{\overline{\mathfrak{p}}_K} = \beta\mathcal{O}_K,$$

where $\beta = \frac{\alpha\delta}{\overline{\alpha}}$ and $\beta\overline{\beta} = \frac{\alpha\overline{\alpha}\delta\overline{\delta}}{\overline{\alpha}\alpha} = 1$.

(iii) p splits in $K_+/\mathbb{Q}$: Because K_+ is Galois over $\mathbb{Q}$, the only case where p splits is given by $\mathfrak{p}_{K_+,1}\mathfrak{p}_{K_+,x}\mathfrak{p}_{K_+,x^2} = p\mathcal{O}_{K_+}$. The possible decompositions are given in cases 20, 25, 26, 28, 29, 33, 34 and 36 of Table 3.1. For each of these we prove the claim for the subcase a. The other subcases follow identically.

(20) We prove the claim for 20a. We have $\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2} = p\mathcal{O}_K$ and

$$N_{\Phi^r}(\overline{\mathfrak{p}}_{K^r,1}) = \mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,cx^2} = \alpha\mathcal{O}_K.$$

for some $\alpha \in K^\times$. Write $\overline{\mathfrak{p}}_{K,x^2} \coloneqq \mathfrak{p}_{K,cx^2}$, then

$$\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \frac{\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}}{(\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\overline{\mathfrak{p}}_{K,x^2})^2}.$$

We have

$$N_{\Phi^r}(\mathfrak{p}_{K^r})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r})} = N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r}) = p^{f(\mathfrak{p}_{K^r,1}/p)} = p,$$

so $\alpha\overline{\alpha} = p$. Then $\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta\mathcal{O}_K$ where $\beta = \frac{p}{\alpha^2}$ with

$$\beta\overline{\beta} = \frac{p^2}{\alpha^2\overline{\alpha}^2} = 1.$$

(25) We have $\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2} = p\mathcal{O}_K$ and $N_{\Phi^r}(\overline{\mathfrak{p}}_{K^r,1}) = \mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\mathfrak{p}_{K,cx^2}^2 = \alpha\mathcal{O}_K$ for some $\alpha \in K^\times$. Write $\overline{\mathfrak{p}}_{K,x^2} \coloneqq \mathfrak{p}_{K,cx^2}$, then for δ as in (3.3) we have

$$\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \delta\frac{\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,x^2}} = \delta\frac{\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}} = \frac{p}{N_{\Phi^r}(\mathfrak{p}_{K^r,1})}\mathcal{O}_K.$$

Then

$$N_{\Phi^r}(\mathfrak{p}_{K^r,1})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r,1})} = N_{K^r}/\mathbb{Q}(\mathfrak{p}_{K^r}) = p^{f(\mathfrak{p}_{K^r,1}/p)} = p^2.$$

This gives $\alpha\overline{\alpha} = p^2$, so

$$\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta\mathcal{O}_K$$

where $\beta \coloneqq \frac{p}{\alpha}$ with $\beta\overline{\beta} = \frac{p^2}{\alpha\overline{\alpha}} = 1$.

(28) We have $\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\mathfrak{p}_{K,cx}\mathfrak{p}_{K,x^2}\mathfrak{p}_{K,cx^2} = p\mathcal{O}_K$. There exist $\alpha_1, \alpha_2 \in K^\times$ such that

$$N_{\Phi^r}(\mathfrak{p}_{K^r,1}) = \mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2} = \alpha_1\mathcal{O}_K, \quad N_{\Phi^r}(\mathfrak{p}_{K^r,c}) = \mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,cx^2} = \alpha_2\mathcal{O}_K.$$

Write $\overline{\mathfrak{p}}_{K,x} \coloneqq \mathfrak{p}_{K,cx}$ and $\overline{\mathfrak{p}}_{K,x^2} \coloneqq \mathfrak{p}_{K,cx}^2$. For δ as in (3.3) we have

$$\frac{\mathfrak{p}_{K,x}}{\overline{\mathfrak{p}}_{K,x}} = \delta\frac{\overline{\mathfrak{p}}_{K,x}}{\mathfrak{p}_{K,x}} = \delta\frac{p}{\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}} = \delta\frac{p}{\alpha_1\alpha_2}\mathcal{O}_K.$$

We have

$$\begin{aligned} N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,c})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,c})} &= N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,c}) \\ &= p^{f(\mathfrak{p}_{K^r,1}/p)}p^{f(\mathfrak{p}_{K^r,c},p)} \\ &= p^2, \end{aligned}$$

so $\alpha_1\alpha_2\overline{\alpha_1\alpha_2} = p^2$. Then $\frac{\mathfrak{p}_{K,x}}{\overline{\mathfrak{p}}_{K,x}} = \beta\mathcal{O}_K$ with $\beta \coloneqq \delta p(N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,c}))^{-1}$ gives $\beta\overline{\beta} = 1$. Similarly for δ' as in (3.3) we have

$$\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta'\mathcal{O}_K,$$

where $\beta' \coloneqq \delta' p(N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,b}))^{-1}$ and we have $\beta\overline{\beta} = 1$.

(29) We have $\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}_{K,x^2}} = p\mathcal{O}_K$ and $N_{\Phi^r}(\mathfrak{p}_{K^r,1}) = \mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2} = \alpha\mathcal{O}_K$ for $\alpha \in K^\times$. Write $\overline{\mathfrak{p}}_{K,x^2} = \mathfrak{p}_{K,cx^2}$, then for δ as in (3.3) we get

$$\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \delta\frac{\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,x^2}} = \delta\frac{\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}^2} = \frac{p\delta}{\alpha^2}\mathcal{O}_K.$$

We have

$$N_{\Phi^r}(\mathfrak{p}_{K^r,1})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r,1})} = N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r,1}) = p^f(\mathfrak{p}_{K^r,1}/p) = p,$$

so $\alpha\overline{\alpha} = p$. Then $\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta\mathcal{O}_K$, where $\beta \coloneqq \frac{\delta\alpha}{p}$ with

$$\beta\overline{\beta} = \frac{\alpha\overline{\alpha}}{p^2} = 1.$$

(33) We have $\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\overline{\mathfrak{p}}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2} = p\mathcal{O}_K$ and $N_{\Phi^r}(\mathfrak{p}_{K^r,1}) = \mathfrak{p}_{K,1}\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}^2 = \alpha\mathcal{O}_K$. Write $\overline{\mathfrak{p}}_{K,x} \coloneqq \mathfrak{p}_{K,cx}$ and $\overline{\mathfrak{p}}_{K,x^2} \coloneqq \mathfrak{p}_{K,cx^2}$. For δ, δ' as in (3.3) we have

$$\frac{\mathfrak{p}_{K,x}^2}{\overline{\mathfrak{p}}_{K,x}^2} = \delta\mathcal{O}_K, \quad \frac{\mathfrak{p}_{K,x^2}^2}{\overline{\mathfrak{p}}_{K,x^2}^2} = \delta'\mathcal{O}_K.$$

Then we have

$$\frac{\mathfrak{p}_{K,x}}{\overline{\mathfrak{p}}_{K,x}} = \delta\frac{\overline{\mathfrak{p}}_{K,x}}{\mathfrak{p}_{K,x}} = \delta\frac{\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\overline{\mathfrak{p}}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}} = \delta\delta'\frac{p}{\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}^2\mathfrak{p}_{K,x^2}^2} = \frac{p\delta\delta'}{\alpha}\mathcal{O}_K.$$

Since $f(\mathfrak{p}_{K^r,1}/p) = 2$, we have

$$N_{\Phi^r}(\mathfrak{p}_{K^r})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r})} = N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r}) = p,$$

so $\alpha\overline{\alpha} = p$. Then $\frac{\mathfrak{p}_{K,x}}{\overline{\mathfrak{p}}_{K,x}} = \beta\mathcal{O}_K$, where $\beta \coloneqq \frac{p\delta\delta'}{\alpha}$ with

$$\beta\overline{\beta} = \frac{p^2}{\alpha\overline{\alpha}} = 1.$$

Similar to the above we also have

$$\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \frac{p\delta\delta'}{\alpha}\mathcal{O}_K,$$

hence $\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta\mathcal{O}_K$.

(34) We have $\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2} = p\mathcal{O}_K$ and $N_{\Phi^r}(\mathfrak{p}_{K^r}) = \mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}^2 = \alpha\mathcal{O}_K$. Write $\overline{\mathfrak{p}}_{K,x^2} \coloneqq \mathfrak{p}_{K,cx^2}$, then

$$\frac{\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,x^2}} = \frac{\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2}^2} = \frac{p}{\alpha}\mathcal{O}_K.$$

Since $f(\mathfrak{p}_{K^r}/p) = 2$, we have

$$N_{\Phi^r}(\mathfrak{p}_{K^r})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r})} = N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r}) = p^2,$$

hence $\alpha\overline{\alpha} = p^2$. For δ as in (3.3) we have $\frac{\mathfrak{p}_{K,x}^2}{\overline{\mathfrak{p}}_{K,x^2}} = \delta\mathcal{O}_K$. Then $\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta\mathcal{O}_K$, where $\beta \coloneqq \frac{\delta\alpha}{p}$ with

$$\beta\overline{\beta} = \frac{\alpha\overline{\alpha}}{p^2} = 1.$$

(36) We have $\mathfrak{p}_{K,1}\overline{\mathfrak{p}}_{K,1}\mathfrak{p}_{K,x}\overline{\mathfrak{p}}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2} = p\mathcal{O}_K$ and there exist $\alpha_1, \alpha_2 \in K^\times$ such that

$$N_{\Phi^r}(\mathfrak{p}_{K^r,1}) = \mathfrak{p}_{K,1}\mathfrak{p}_{K,x}\mathfrak{p}_{K,x^2} = \alpha_1\mathcal{O}_K, \quad N_{\Phi^r}(\mathfrak{p}_{K^r,bc}) = \mathfrak{p}_{K,1}\overline{\mathfrak{p}}_{K,x}\overline{\mathfrak{p}}_{K,x^2} = \alpha_2\mathcal{O}_K.$$

Write $\overline{\mathfrak{p}}_{K,1} = \mathfrak{p}_{K,a}, \overline{\mathfrak{p}}_{K,x} = \mathfrak{p}_{K,ax}, \overline{\mathfrak{p}}_{K,x^2} = \mathfrak{p}_{K,ax^2}$, then

$$\frac{\overline{\mathfrak{p}}_{K,1}}{\mathfrak{p}_{K,1}} = \frac{\mathfrak{p}_{K,1}\overline{\mathfrak{p}}_{K,1}\mathfrak{p}_{K,x}\overline{\mathfrak{p}}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}}{\mathfrak{p}_{K,1}^2\mathfrak{p}_{K,x}\overline{\mathfrak{p}}_{K,x}\mathfrak{p}_{K,x^2}\overline{\mathfrak{p}}_{K,x^2}} = \frac{p}{\alpha_1\alpha_2}\mathcal{O}_K.$$

Since $f(\mathfrak{p}_{K^r,1}/p) = f(\mathfrak{p}_{K^r,bc}/p) = 1$, we have

$$N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,bc})\overline{N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,bc})} = N_{K^r/\mathbb{Q}}(\mathfrak{p}_{K^r,1}\mathfrak{p}_{K^r,bc}) = p^2.$$

For δ as in (3.3) we have $\frac{\mathfrak{p}_{K,x}^2}{\overline{\mathfrak{p}}_{K,x}^2} = \delta\mathcal{O}_K$. Then $\frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta\mathcal{O}_K$, where $\beta := \frac{\delta\alpha}{p}$ with

$$\beta\overline{\beta} = \frac{\alpha_1\alpha_2\overline{\alpha_1\alpha_2}}{p^2} = 1.$$

Similarly we have δ', δ'' as in (3.3) such that

$$\frac{\mathfrak{p}_{K,x}}{\overline{\mathfrak{p}}_{K,x}} = \beta'\mathcal{O}_K, \quad \frac{\mathfrak{p}_{K,x^2}}{\overline{\mathfrak{p}}_{K,x^2}} = \beta''\mathcal{O}_K,$$

where $\beta' := \delta' p(N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,ac}))^{-1}$ and $\beta'' := \delta'' p(N_{\Phi^r}(\mathfrak{p}_{K^r,1})N_{\Phi^r}(\mathfrak{p}_{K^r,ab}))^{-1}$. In the same way as above we can show $\beta'\overline{\beta'} = \beta''\overline{\beta''} = 1$.

□

Corollary 3.2.9. *Let K be as in Proposition 3.2.8. Assume $I_{K^r} = I_0(\Phi^r)$, then $h_K^* = 2^{t_K - \delta_K}$ for $\delta_K \in \{0, 1\}$.*

Proof. Assuming $I_{K^r} = I_0(\Phi^r)$ we have by Proposition 3.2.8 that $[I_K : I_K^{H_K} P_K] = 1$. Then by Lemma 3.2.1,

$$h_K^* = \frac{2^{t_K}}{[P_K^{H_K} : P_{K_+}]}. \tag{3.4}$$

By (3.1), $\delta_K = 0$ if $[P_K^{H_K} : P_{K_+}] = 1$ and $\delta_K = 1$ if $[P_K^{H_K} : P_{K_+}] = 2$. □

We will now give a few additional results regarding the relation between $h_{K^r}^*$ and h_k. By Lemmata 3.2.1 and 3.2.2 we have

$$h_{K^r}^* = 2^{t_{K^r}} \frac{[I_{K^r} : I_{K^r}^{H_{K^r}} P_{K^r}]}{[P_{K^r}^{H_{K^r}} : P_{K_+^r}]}, \quad h_k = 2^{t_k - 1}[I_k : I_k^{H_k} P_k]. \tag{3.5}$$

The second equality in (3.5) follows because $Q_k = [\mathcal{O}_k^\times : W_k\mathcal{O}_\mathbb{Q}^\times] = [W_k : W_k] = 1$. In Proposition 3.2.11 we give a relation between $h_{K^r}^*$ and h_k.

Lemma 3.2.10. *[Was82, Theorem 5, Appendix 3] Let M/F be an extension of number fields that contains no unramified subextensions M/F_0 with $F_0 \neq F$. Then the norm map $N_{M/F}$: $\mathrm{Cl}_M \to \mathrm{Cl}_F$ is surjective and therefore h_F divides h_M.*

Proposition 3.2.11. *Let K^r be a CM field of degree 8 with Galois group $(C_2)^3 \rtimes C_3$. Let $k \subset K^r$ be the imaginary quadratic subfield of K^r. Then $[I_k : I_k^{H_k} P_k]$ divides $[I_{K^r} : I_{K^r}^{H_{K^r}} P_{K^r}]$.*

Proof. Define $N_{K^r/k} : I_{K^r} \longrightarrow I_k/I_k^{H_k}P_k$ such that $\mathfrak{a} \mapsto N_{K^r/k}(\mathfrak{a}) \cdot I_k^{H_k}P_k$. This is a surjective map, since $I_k^{H_k}P_k \supset P_k$ and the norm map $N_{K^r/k} : \mathrm{Cl}_{K^r} \to \mathrm{Cl}_k$ is surjective by Lemma 3.2.10. We will show $I_{K^r}^{H_{K^r}}P_{K^r} \subset \mathrm{Ker}(N_{K^r/k})$.

Let $\mathfrak{a} \in I_{K^r}^{H_{K^r}}P_{K^r}$ be arbitrary, then there exist $\mathfrak{b} \in I_{K^r}^{H_{K^r}}$ and $\beta \in K^{r\times}$ such that $\mathfrak{a} = \mathfrak{b}\beta$. This gives

$$N_{K^r/k}(\mathfrak{a}) = N_{K^r/k}(\mathfrak{b})N_{K^r/k}(\beta),$$

such that $N_{K^r/k}(\mathfrak{b}) = N_{K^r/k}(\overline{\mathfrak{b}}) = \overline{N_{K^r/k}(\mathfrak{b})}$ and $N_{K^r/k}(\beta) \in k^\times$. So $N_{K^r/k}(\mathfrak{a}) \in I_k^{H_k}P_k$ and hence $\mathfrak{a} \in \mathrm{Ker}(N_{K^r/k})$. So we have $I_{K^r}^{H_{K^r}}P_{K^r} \subset \mathrm{Ker}(N_{K^r/k}) \subset I_{K^r}$ and by the isomorphism theorem we have

$$[I_k : I_k^{H_k}P_k] = [I_{K^r} : \mathrm{Ker}(N_{K^r/k})] \Big| [I_{K^r} : I_{K^r}^{H_{K^r}}P_{K^r}].$$

□

A consequence of Proposition 3.2.11 is that, if K is a CM class number one field, the odd prime factors in $h^*_{K^r}$ are precisely those in h_k.

3.3. A bound for d_K/d_{K_+}

Assuming that K is a CM class number one field we give an upper bound for d_K/d_{K_+}. In Theorem 3.3.6 we conclude that there exist finitely many non-normal sextic CM class number one fields with Galois group $(C_2)^3 \rtimes C_3$.

For a number field F, let f_F denote the conductor of F.

Theorem 3.3.1. *[BL99, Theorem 4] Let K be a sextic CM-field with Galois closure L such that* $\mathrm{Gal}(L/\mathbb{Q}) \cong (C_2)^3 \rtimes C_3$*. Define $\epsilon_K := 1 - (6\pi e^{1/24}/d_K^6)$. Then*

$$h^*_K \geq \epsilon_K \frac{\sqrt{|d_K/d_{K_+}|}}{e^{1/8}\pi^3(\log(f_{K_+}) + 0.05)^2 \log(d_L)}.$$

Moreover, $d_L \leq d_K^{12}$. Therefore h^*_K goes to infinity and there are only finitely many fields K of this form of a given class number.

Theorem 3.3.2. *Let K be a sextic CM-field whose Galois closure L has Galois group $(C_2)^3 \rtimes C_3$. Assume $d_K \geq 6 \cdot 10^7$, then*

$$h^*_K \geq \frac{\sqrt{|d_K/d_{K_+}|}}{213(\log(|d_K/3d_{K_+}|) + 0.05)^2 \log(|d_K/3d_{K_+}|)}. \tag{3.6}$$

Proof. Consider the bound for h^*_K given in Theorem 3.3.1. For $d_K \geq 6 \cdot 10^7$ we have that $\epsilon_K \geq 0.99$. Moreover $f_{K_+} = \sqrt{d_{K_+}}$ and $d_L \leq d_K^{12}$, so the bound becomes

$$h^*_K \geq \frac{0.99\sqrt{|d_K/d_{K_+}|}}{12e^{1/8}\pi^3(\frac{1}{2}\log(d_{K_+}) + 0.05)^2 \log(|d_K|)}. \tag{3.7}$$

There is at least one prime ramifying in K/K_+ by Proposition 3.2.3, so $d_K/3d_{K_+} \geq d_{K_+}$ and $d_K \leq d_K^2/3d_{K_+}^2$. Combining this with (3.7) gives

$$h_K^* \geq \frac{0.99\sqrt{|d_K/d_{K_+}|}}{24e^{1/8}\pi^3(\frac{1}{2}\log(|d_K/3d_{K_+}|) + 0.05)^2 \log(|d_K/3d_{K_+}|)}.$$

Since $0.99/6e^{1/8}\pi^3 > 1/213$ the result follows. □

Lemma 3.3.3. *For a real number $D \geq 1$ and integer $t \geq 1$ define*

$$f(D) := \frac{\sqrt{D}}{213(\log(D/3) + 0.05)^2 \log(D/3)} \text{ and } g(t) = 2^{w-t} f(49\Delta_t),$$

where Δ_t is the product of the first t primes and $w \in \{0, 1\}$. Then f is monotonically increasing for $D > 263$ and $g(t)$ is monotonically increasing for $t \geq 12$.

Proof. We follow the proof of [KS18, Lemma 3.11]. The derivative of $f(D)$ is strictly positive for $D > 263$, so $f(D)$ is monotonically increasing for $D > 263$. Define

$$h(t) = 2^{w-t} f(49\Delta_{\lceil t/2\rceil}\Delta_{\lfloor t/2\rfloor}).$$

We have that $\Delta_4 = 210$ and $\Delta_5 = 2310$, so $f(49\Delta_t) \geq f(49\Delta_{\lceil t/2\rceil}\Delta_{\lfloor t/2\rfloor})$ for all $t \geq 6$ and hence $g(t) \geq h(t)$ for all $t \geq 6$. We will now show that $h(t)$ increases monotonically for $t \geq 12$. We have that $\Delta_{t+1} = p_{t+1}\Delta_t$, where p_{t+1} is the prime factor in position $t+1$ in the product Δ_{t+1}. Then

$$\frac{\Delta_{\lceil (t+1)/2\rceil}\Delta_{\lfloor (t+1)/2\rfloor}}{\Delta_{\lceil t/2\rceil}\Delta_{\lfloor t/2\rfloor}} = \frac{\Delta_{\lceil (t+1)/2\rceil}}{\Delta_{\lfloor t/2\rfloor}} = p_{\lceil (t+1)/2\rceil}.$$

Let $D_t := \Delta_{\lceil t/2\rceil}\Delta_{\lfloor t/2\rfloor}$ for all $t \geq 1$, then this gives

$$\frac{h(t+1)}{h(t)} = \frac{\sqrt{p_{\lceil (t+1)/2\rceil}}}{2} \frac{(\frac{1}{2}\log(49D_t/3) + 0.05)^2 \log(49D_t/3)}{(\frac{1}{2}\log(49D_{t+1}/3) + 0.05)^2 \log(49D_{t+1}/3)}. \tag{3.8}$$

Similar to the proof of [KS18, Lemma 3.11], by Bertrand's Postulate we have for $s \geq 4$ that $p_{s+1} \leq 2p_s$. This gives for $s \geq 4$

$$p_{s+1}^4 < 2^4 p_s^4 < 2^6 p_s^2 p_{s-1}^2 < 49\Delta_s^2$$

Then for all $t \geq 8$,

$$p_{\lceil (t+1)/2\rceil}^4 < 49\Delta_{\lfloor t/2\rfloor}^2 \leq 49D_t.$$

This implies

$$\log\big(p_{\lceil (t+1)/2\rceil}/3\big) < \frac{1}{4}\log(49D_t/3) < (\sqrt{2} - 1)\log(49D_t/3). \tag{3.9}$$

We have that $D_{t+1} = D_t p_{\lceil (t+1)/2 \rceil}$, so from (3.9) we obtain

$$\log(49D_{t+1}/3) = \log(49D_t/3) + \log\bigl(p_{\lceil (t+1)/2 \rceil}/3\bigr) < \sqrt{2}\log(49D_t/3). \tag{3.10}$$

Then we have

$$\frac{1}{2}\log(49D_t/3) + \frac{1}{2}\log\bigl(p_{\lceil (t+1)/2 \rceil}/3\bigr) + 0.05 < \frac{1}{2}(\sqrt{2}\log(49D_t/3) + 0.05). \tag{3.11}$$

Applying (3.11) to the bound in (3.8) gives

$$\frac{h(t+1)}{h(t)} > \frac{\sqrt{p_{\lceil (t+1)/2 \rceil}}}{4}.$$

This gives that $h(t)$ increases for $t \geq 12$, hence $g(t)$ increases monotonically for $t \geq 12$. □

The following results are consequences of Theorem 3.3.2.

Corollary 3.3.4. *Let K be as in Theorem 3.3.2 and assume $I_{K^r} = I_0(\Phi^r)$. Then $t_K \leq 20$ and $d_K/d_{K_+} \leq 5.6 \cdot 10^{26}$.*

Proof. We follow the proof of [KS18, Proposition 3.13]. The smallest possible discriminant for a totally real cyclic cubic field is 49 (see [Voi]). Let t_K be the number of primes ramifying in K/K_+, then for $t_K \geq 4$ we have

$$d_K/d_{K_+} \geq d_{K_+}\Delta_{t_K} \geq 49\Delta_{t_K} \geq 49\Delta_{\lceil t_K/2 \rceil}\Delta_{\lfloor t_K/2 \rfloor} > 263.$$

By Corollary 3.2.9 we have $h_K^* = 2^{t_K - \delta_K}$ for $\delta_K \in \{0, 1\}$, so take $f(D), g(t)$ as in Lemma 3.3.3. These were shown to be monotonically increasing for $D > 263$ and $t \geq 12$. Then Lemma 3.3.3 implies

$$2^{t_K - \delta_K} \geq f(d_K/d_{K_+}) \geq f(49\Delta_{t_K})$$

and hence $g(t_K) \leq 1$ for $t_K \geq 1$. However from Lemma 3.3.3 it also follows that $g(t_K)$ increases monotonically for $t_K \geq 12$. For both $\delta_K = 0$ and $\delta_K = 1$ we have that $g(t_K) \leq 1$ for $t_K \geq 20$. Then the restriction $f(d_K/d_{K_+}) \leq 2^{t_K - \delta_K}$ gives $d_K/d_{K_+} \leq 5.6 \cdot 10^{26}$. □

Remark 3.3.5. When K is a non-Galois quartic CM field with CM class number one, in [KS18, Lemma 3.25] Kılıçer-Streng found a bound for $d_{K^r}/d_{K^r_+}$. They showed that the ramified primes $\mathfrak{p}$ in K^r/K^r_+ contribute to $d_{K^r}/d_{K^r_+}$ with p^2, where $p := \mathfrak{p} \cap \mathbb{Z}$. So in the quartic case the right hand side of the lower bound for $h^*_{K^r}$ grows with the product of primes numbers. However, in our case the right hand side of (3.6) grows with the *square root* of the product of the prime numbers p, such that at least one of the primes lying above p in K_+ ramifies in K. Therefore, our discriminant bound is much larger compare to the bound in the quartic case.

Moreover, when K is a sextic CM field containing an imaginary quadratic field with CM class number one, in [Kıl16, Proposition 3.4.1] Kılıçer showed that the imaginary quadratic field contained in K has class number one. This gives $t_K \leq 3$ and hence a small discriminant bound. If K is a sextic CM class number one field with degree 24 Galois closure, the imaginary quadratic field $k \subset K^r$ does not always have class number one. See Table 3.2 for counter-examples. For this reason we deal with a larger bound for t_K and hence a much larger bound for d_K/d_{K_+}, as given in Theorem 3.3.2.

Theorem 3.3.6. *There exist finitely many sextic CM class number one fields whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$.*

Proof. Under the assumptions $I_{K^r} = I_0(\Phi^r)$ and h_{K_+}, we have $d_K/d_{K_+} \leq 5.6 \cdot 10^{26}$ by Corollary 3.3.4. By Proposition 3.2.3 at least one prime of K_+ ramifies in K/K_+, so

$$3d_{K_+} \leq d_K/d_{K_+} \leq 5.6 \cdot 10^{26}.$$

Then $d_{K_+} \leq 5.6 \cdot 10^{26}/3$ and this gives

$$d_K \leq 5.6 \cdot 10^{26} \cdot d_{K_+} \leq (5.6 \cdot 10^{26})^2/3.$$

This implies that there exist finitely many CM class number one fields K. □

Unfortunately the bound for d_K/d_{K_+} in Corollary 3.3.4 is too large to compute all CM class number one fields K up to this bound. In Section 3.4 we compute all CM class number one fields K that give $d_{K_+^r} \leq 10^9$ and $d_k \leq 10^4$ such that $t_k \leq 2$.

Remark 3.3.7. It is possible to give a bound for d_K/d_{K_+} because if K is CM class number one we showed that h_K^* is a power of 2 that depends only on t_K. For $h_{K^r}^*$ this is not the case, see the examples in Table 3.2. We could therefore not compute a bound for $d_{K^r}/d_{K_+^r}$ in the same way without imposing further restrictions. In Appendix B we construct a lower bound for $h_{K^r}^*$.

3.4. Listing CM class number one fields

We compute all CM class number one fields K for $d_{K_+^r} \leq 10^8$ and $d_k \leq 10^4$ such that $t_k \leq 2$. The totally real quartic field K_+^r is a non-normal primitive field with Galois closure L_+ such that $\mathrm{Gal}(L_+/\mathbb{Q}) \cong A_4$. We construct the fields K_+^r up to discriminant bound 10^9 using the method in [CDO02, Section 3] (available in PARI/GP [PAR21] via the function `nflist`). We obtain all fields k such that $d_k \leq 10^4$ and $t_k \leq 2$ from [LMF21], where the list is complete for $d_k \leq 2 \cdot 10^6$.

In Algorithm 3.4.2 we construct K^r and simultaneously eliminate fields that will not give $I_{K^r} = I_0(\Phi^r)$. For all resulting fields K^r we find sextic fields K that are CM class number one in Algorithm 3.4.5. Both algorithms implemented in SageMath [Sag20].

The following lemma gives a criterion to eliminate fields K^r that do not give a CM class number one field K.

Lemma 3.4.1. *Let K be a non-normal sextic CM-field not containing an imaginary quadratic field. Let K^r be the reflex field of K such that $I_{K^r} = I_0(\Phi^r)$. If a prime number p splits completely in $K^r/\mathbb{Q}$, then $p \geq \dfrac{\sqrt[3]{d_K/d_{K_+}^2}}{4}$.*

Proof. The result is a special case of [Lou94b, Theorem D] and the proof is an adaptation of the proof of [Kıl16, Lemma 2.3.17] to this specific case. Let p split completely in $K^r/\mathbb{Q}$ and let $\mathfrak{p}_{K^r}$ be a prime of K^r above p. Under the assumption $I_{K^r} = I_0(\Phi^r)$ there exists $\mu \in K^*$ such that $N_{\Phi^r}(\mathfrak{p}_{K^r}) = (\mu)$ and $\mu\overline{\mu} = p$. Since K does not contain any quadratic field we have $\sqrt{\pm p} \in K$, so $\mu \neq \overline{\mu}$. There exists positive $\beta \in \mathcal{O}_{K_+} \setminus \mathcal{O}_{K_+}^2$ such that $K = K_+(\sqrt{-\beta})$, hence $\mu = \frac{a+b\beta}{2}$ for some $a, b \in \mathcal{O}_{K_+}$. This gives $\mathcal{O}_K \supset \mathcal{O}_{K_+}[\mu]$. Moreover we have the discriminant ideal $\delta(K/K_+) = (\mu - \overline{\mu})^2$ for K/K_+. Then

$$d_K/d_{K_+}^2 = N_{K_+/\mathbb{Q}}(\delta(K/K_+)) \leq N_{K_+/\mathbb{Q}}((\mu - \overline{\mu})^2).$$

On the other hand $\mu\overline{\mu} = p$ implies $\phi((\mu - \overline{\mu})^2) \leq 4p$ for all embeddings $\phi : K_+ \to \mathbb{R}$. Because K_+ is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(K_+/\mathbb{Q}) = \langle x \rangle$, this gives

$$d_K/d_{K_+}^2 = N_{K_+/\mathbb{Q}}((\mu - \overline{\mu})^2) = (\mu - \overline{\mu})^2 x((\mu - \overline{\mu})^2) x^2((\mu - \overline{\mu})^2) \leq 4^3 p^3.$$

□

Algorithm 3.4.2. *Constructing K^r from K_+^r and k such that $d_{K_+^r} \leq 10^8$, $d_k \leq 10^4$ and $t_k \leq 2$:*
Input: Primitive totally real non-normal quartic field K_+^r whose Galois closure has Galois group A_4, imaginary quadratic field k.
Output: octic CM-field $K^r = kK_+^r$ whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$.

Step 1 *For all prime numbers p dividing both d_k and $d_{K_+^r}$, check if it decomposes as in cases 1, 3, 5, 15 of Table 3.1. If it does, end the algorithm.*

Step 2 *Construct K^r by extending K_+^r with k.*

Step 3 *For all p dividing $d_{K_+^r}$ and not d_k, check if p decomposes as in cases 5, 6, 18 of Table 3.1. If it does, end the algorithm.*

Step 4 *For all p dividing both $d_{K_+^r}$ and d_k, check if p decomposes as in case 19 of Table 3.1. If it does, end the algorithm.*

Step 5 *If $4h_{K^r}^*/h_k$ is a power of 2, store K^r.*

Proof. Step 1, 3 and 4 only eliminate fields K^r that do not give a CM class number one field by Corollary 3.1.2. Step 3 eliminates only such fields by Lemma 3.4.1. If the ratio $4h_{K^r}^*/h_k$ is not a power of 2 in step 5, then K^r will also not give a CM class number one field K by Corollary 3.2.7. □

Remark 3.4.3. In our computations we assumed the General Riemann Hypothesis for step 5 of Algorithm 3.4.2. This way the class numbers could be computed using the Bach bound instead of the Minkowski bound.

Remark 3.4.4. In Algorithm 3.4.5 steps 1 and 3 will eliminate some fields, while step 4 will eliminate very few fields. The elimination by step 5 is by far the most effective, in particular for the case when $d_{K_+^r} \leq 10^8, d_k \leq 10^4$ such that $t_k \leq 2$. However the computation of class numbers makes this step slower than steps 1,3 and 4, even when assuming the GRH (see Remark 3.4.3). For this reason we execute this elimination step last.

Algorithm 3.4.5. *Find CM class number one fields K from K^r, if they exist:*
Input: octic CM-field K^r whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$.
Output: sextic field K whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$ such that K has CM class number one, if it exists.

Step 1 *Compute the set of CM-types Φ^r of K^r up to equivalence classes with functions* `CM_Field`, `CM_types` *from [Str].*

Step 2 *Fix a CM-type Φ^r found in step 1 and compute the reflex field K of Φ^r with function* `reflex_field` *from [Str]. If $[K : \mathbb{Q}] \neq 6$ repeat step 2, otherwise continue.*

Step 3 *If there exists a prime number p up to the bound $\sqrt[3]{d_K/d_{K_+}^2}/4$ that splits completely in K^r, go back to step 2. Otherwise continue.*

Step 4 *Compute the generators of the class group of K^r and find a representative prime ideal $\mathfrak{p}$ for each generator.*

Step 5 *If for all $\mathfrak{p}$ in step 4 there exists α such that $N_{\Phi^r}(\mathfrak{p}) = (\alpha)$ and $\alpha\overline{\alpha} \in \mathbb{Q}$ with function* `a_to_mu` *from [Str], store K.*

Proof. It suffices to only consider CM-types Φ^r of K^r up to equivalence class in step 1, because equivalent CM-types correspond to the same reflex fields by Lemma 1.1.16. In step 5 we check if K^r is the reflex field of a CM class number one field K by testing if $I_{K^r} = I_0(\Phi^r)$. After a CM class number one field K of degree 6 has been found in step 6, all other degree 6 reflex fields of K^r are isomorphic to K by Corollary 2.3.3. This implies that if we find one sextic field K, we have found all for the given field K^r and we can terminate the algorithm. □

Remark 3.4.6. We run Algorithm 3.4.2 only for fields with $d_k \leq 10^4$ and $t_k \leq 2$, so by Lemma we get $\sqrt[3]{d_K/d_{K_+}^2}/4 \leq 5$. However in Corollary 3.3.4 the absolute bound is $d_K/d_{K_+} \leq 5.6 \cdot 10^{26}$, where $t_K \leq 20$. The discriminant of k is much larger in theory, so then eliminating fields using step 3 may be useful.

By repeating Algorithm 3.4.5 for all fields K^r constructed with Algorithm 3.4.2, we find all sextic CM class number one fields K with $d_{K_+^r} \leq 10^8$ and $d_k \leq 10^4$ such that $t_k \leq 2$. The computed fields are given in Table 3.2.

$p(x)$	h_K	h_{K_+}	h_{K^r}	$h_{K^r_+}$	h_k	$\left\|d_K/d_{K_+}^2\right\|$	$d_{K^r}/d_{K^r_+}^2$	$\|d_k\|$
$x^6 - x^5 - 6x^4 + 20x^3 + 33x^2 - 15x + 9$	1	1	1	1	1	3	1	3
$x^6 - x^5 - x^4 + 2x^3 - 3x^2 - 9x + 27$	1	1	1	1	1	2^3	1	2^3
$x^6 - 2x^5 - 2x^4 + 11x^3 - 6x^2 - 18x + 27$	1	1	1	1	1	11	1	11
$x^6 - x^5 + 3x^3 - 4x + 8$	1	1	3	1	3	31	1	31
$x^6 - x^5 + 2x^4 - 5x^3 + 4x^2 - 4x + 8$	1	1	5	1	5	47	1	47
$x^6 - 3x^5 + 9x^4 - 13x^3 + 15x^2 - 9x + 3$	1	1	7	1	7	71	1	71
$x^6 - 2x^5 + 3x^4 - 3x^3 + 6x^2 - 8x + 8$	1	1	5	1	5	79	1	79
$x^6 - 2x^5 + 5x^4 - 7x^3 + 10x^2 - 8x + 8$	1	1	11	1	11	167	1	167
$x^6 + 9x^4 + 24x^2 + 19$	2	1	2	1	1	$2^6 \cdot 19$	1	19
$x^6 - 2x^5 + 3x^4 + x^3 + 8x^2 + 9x + 7$	1	1	3	1	3	83	1	83
$x^6 - x^5 + 4x^4 - 3x^3 + 8x^2 - 4x + 8$	1	1	15	1	15	239	1	239
$x^6 - 3x^5 + 10x^4 - 15x^3 + 21x^2 - 14x + 7$	1	1	7	1	7	251	1	251
$x^6 + 3x^4 - x^3 + 6x^2 + 8$	1	1	9	1	9	199	1	199
$x^6 + 12x^4 + 41x^2 + 43$	2	1	2	1	1	$2^6 \cdot 43$	1	43
$x^6 - 2x^5 + 7x^4 - 12x^3 + 21x^2 - 15x + 13$	1	1	3	1	3	379	1	379
$x^6 - 2x^5 + 7x^4 - 5x^3 + 14x^2 - x + 13$	1	1	9	1	9	491	1	491
$x^6 - x^5 + 6x^4 - 2x^3 + 16x^2 - 4x + 13$	1	1	3	1	3	547	1	547
$x^6 - 2x^5 + 9x^4 - 10x^3 + 23x^2 - 9x + 29$	1	1	5	1	5	1051	1	1051
$x^6 - 3x^5 + 9x^4 - 12x^3 + 15x^2 - 12x + 19$	1	1	3	1	3	379	1	379
$x^6 - 3x^5 + 9x^4 - 12x^3 + 24x^2 - 21x + 19$	1	1	5	1	5	523	1	523
$x^6 - 3x^5 + 12x^4 - 19x^3 + 33x^2 - 24x + 17$	1	1	5	1	5	739	1	739

Table 3.2: Defining polynomials for all sextic CM class number one fields K whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$, such that $d_{K^r_+} \leq 10^8$, $d_k \leq 10^4$ and $t_k \leq 2$.

Summary and discussion

A *complex multiplication field* (CM-field) of degree $2g$ is a totally imaginary quadratic extension of a totally real field of degree g over $\mathbb{Q}$. An abelian variety of dimension g has complex multiplication if its endomorphism ring contains an order of a CM-field. Let K be a CM-field of degree $2g$ over $\mathbb{Q}$ and let L be its Galois closure. A *CM-type* Φ of K is a subset of $\mathrm{Hom}(K, L)$ of cardinality g that contains no embeddings that are complex conjugates. Let (K, Φ) be a *CM-pair*, to which we associate a *reflex field* K^r and a *reflex CM-type* Φ^r. By the first main theorem of CM ([ST61, Main Theorem 1]) a principally polarized simple abelian variety of dimension g that has CM by the maximal order $\mathcal{O}_K$ in K is defined over the Hilbert class field of K^r. An implication of this theorem is that the CM class group $I_{K^r} \ / \ I_0(\Phi^r)$ is trivial ([Kıl16, Corollary 1.5.7]).

The CM class number one problem asks to determine CM pairs (K, Φ) such that K has a trivial CM class group. For $g = 1$ the problem reduces to the usual class number one problem, which was solved by Heegner [Hee52], Baker [Bak67] and Stark [Sta67]. For $g = 2$ (CM-fields of degree 4) the problem was solved by Kılıçer-Streng. For $g = 3$ it was solved for sextic CM-fields that contain an imaginary quadratic subfield by Kılıçer [Kıl16]. For $g = 6$ Somoza [Hen19] solved the problem for the specific case of CM-fields of degree 12 that contain $\mathbb{Q}(\zeta_5)$, where ζ_5 is the primitive 5th root of unity.

In this thesis we discuss the CM class number one problem for sextic CM-fields K whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$. We give relations between the discriminants and relative class numbers of K, its reflex field K^r and their subfields. We also give a full ramification table for prime fractional ideals of K and K^r as subfields of L. Furthermore, we give a bound for d_K/d_{K_+} and prove that there exist finitely many CM class number one fields K in Theorem 3.3.6. We give algorithms for computing these fields in SageMath and list all CM class number one fields K such that $d_{K_+^r} \leq 10^8, d_k \leq 10^4$ and $t_k \leq 2$.

With the bounds $d_K/d_{K_+} \leq 5.6 \cdot 10^{26}$ and $t_K \leq 20$ in Corollary 3.3.4 we were able to prove the finiteness result in Theorem 3.3.6. However, these bounds are too large to compute all CM class number one fields K. For non-Galois quartic CM-fields with CM class number one, Kılıçer-Streng [KS18, Lemma 3.25] showed that the ramified primes K^r/K_+^r contribute to $d_{K^r}/d_{K_+^r}$ with squares of prime numbers. For this reason in their case the lower bound for $h_{K^r}^*$ grows with the product of primes, while in our case the right hand side of (3.6) grows with the square root of the product prime numbers. Furthermore, for a CM class number one sextic CM-field K containing an imaginary quadratic subfield, this subfield has class number one by [Kıl16, Proposition 3.4.1]. This gives $t_K \leq 3$ and hence a small discriminant bound. In our case this is not true for $k \subset K^r$, see Table 3.2. This gives a much larger discriminant bound compared to the non-Galois quartic case or the case where a sextic CM-field contains an imaginary quadratic subfield.

In Chapter 3 we focused on results for K and h_K^*, assuming $I_{K^r} = I_0(\Phi^r)$. The main result is that $h_K^* \in \{2^{t_K}, 2^{t_K-1}\}$, see Theorem 3.2.9. It is not the case that $h_{K^r}^*$ is also a power of two if K has CM class number one, see Table 3.2. For non-Galois quartic CM-fields K not only is $h_{K^r}^*$ a power of two by [KS18, Proposition 3.1], but we also have $t_K = t_{K^r}$ by

[KS18, Corollary 3.10]. Because this is not the case for sextic CM-fields with Galois closure of degree 24, we could not give sufficient conditions on $h^*_{K^r}$ such that K has CM class number one without imposing severe restrictions. Therefore it was not possible to give a bound for $d_{K^r}/d_{K^r_+}$. Assuming the GRH we give a lower bound for $h^*_{K^r}$ in Appendix B.

The overview of prime decompositions in Table 3.1 could be used in other types of research. For instance, in [GL10] the authors give a bound on the denominators of Igusa class polynomials of genus 2 curves with CM on non-Galois quartic CM-fields. Using the table this could be extended to genus 3 curves with CM on sextic CM-fields with Galois closure of degree 24.

Appendices

In Appendix A we assume that the narrow class number of the maximal totally real subfield K_+ of K is odd. Under this additional condition we determine the structure of Cl_K and prove that $h_K^* = 2^{t_K - 1}$. In Appendix B we construct a lower bound for the relative class number of octic CM-fields, assuming the GRH.

A. Narrow class number of the totally real subfield

Let K be a sextic CM-field whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$. We assume that the the narrow class number of K_+ is odd and give the class group structure of K. We also prove that, if K is a CM class number one field, $h_K^* = 2^{t_K - 1}$.

Definition A.1. Let F be a number field and let $P_F^+ \leq P_F$ be the group of principal fractional ideals of F with a totally positive generator. Then the narrow class group and narrow class number of F are defined as

$$\mathrm{Cl}_F^+ \coloneqq I_F / P_F^+, \quad h_F^+ \coloneqq |\mathrm{Cl}_F^+|.$$

Lemma A.2. *Let F be a totally real number field such that $n \coloneqq [F : \mathbb{Q}]$ and let $\mathcal{O}_F^{\times +}$ contain all totally positive elements of unit group $\mathcal{O}_F^\times$. Then the following are equivalent:*

(i) $P_F = P_F^+$;

(ii) $\mathrm{sign} : \mathcal{O}_F^\times \to (\mathbb{Z}/2\mathbb{Z})^n$ *is surjective;*

(iii) $\mathcal{O}_F^{\times,+} = (\mathcal{O}_F^\times)^2$.

Proof. $(ii) \implies (i)$: Let $\alpha\mathcal{O}_F \in P_F$ with $\alpha \in \mathcal{O}_F$. By surjectivity of the map sign there exists $\beta \in \mathcal{O}_F^\times$ such that $\mathrm{sign}(\alpha) = \mathrm{sign}(\beta)$. Then $\mathrm{sign}(\alpha\beta) = 1$, hence $\alpha\beta$ is totally positive. Since β is a unit, $(\alpha\beta) = (\alpha)$ and it is generated by a totally positive element.
$(i) \implies (ii)$: Assume $P_F = P_F^+$ and extend the sign map to $\mathcal{O}_F$. Then $(\alpha) \in P_F$ for arbitrary $\alpha \in \mathcal{O}_F$. There exists $\beta \in \mathcal{O}_F^+$ such that $(\alpha) = (\beta)$, so $\alpha = u\beta$ for some $u \in \mathcal{O}_F^\times$. Because β is totally positive, $\mathrm{sign}(\beta) = (1, \cdots, 1)$. Since sign is a homomorphism, we have $\mathrm{sign}(\alpha) = \mathrm{sign}(u)\,\mathrm{sign}(\beta) = \mathrm{sign}(u)$. Because α was chosen arbitrarily, for every $i \in \{\pm 1\}^n$ there exists $u_\alpha \in \mathcal{O}_F$ such that $\mathrm{sign}(\alpha) = \mathrm{sign}(u_\alpha) = i$. This makes sign surjective.
$(ii) \implies (iii)$: The map sign is a homomorphism with $\ker(\mathrm{sign}) = \mathcal{O}_F^{\times,+}$. If sign is surjective then $\mathcal{O}_F^\times / \mathcal{O}_F^{\times,+} \cong \{\pm 1\}^n$. This gives $[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}] = 2^m$ and because

$$(\mathcal{O}_F^\times)^2 \subset \mathcal{O}_F^{\times,+} \subset \mathcal{O}_F^\times,$$

the result follows.
$(iii) \implies (ii)$: From Dirichlet's unit theorem we know that $\mathcal{O}_F^\times = W_F \times \varepsilon_1\mathbb{Z} \times \cdots \times \varepsilon_{n-1}\mathbb{Z}$, where the ε_i are fundamental units. Because F is totally real we have $W_F \cong \mathbb{Z}/2\mathbb{Z}$. Because $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$ contains all elements of order 2 in $\mathcal{O}_F^\times$, $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \cong (\mathbb{Z}/2\mathbb{Z})^n$. Because $\mathcal{O}_F^{\times,+} = (\mathcal{O}_F^\times)^2$ this gives $\mathcal{O}_F^\times/\mathcal{O}_F^{\times,+} \cong (\mathbb{Z}/2\mathbb{Z})^n$. So sign is a surjective homomorphism, because $\ker(\mathrm{sign}) = \mathcal{O}_F^{\times,+}$. □

Proposition A.3. *For any number field F of degree g over $\mathbb{Q}$, we have*

$$h_F^+ = [\mathcal{O}_F^{\times,+} : (\mathcal{O}_F^\times)^2] h_F.$$

Moreover, every non-trivial element in $\mathcal{O}_F^{\times,+}/(\mathcal{O}_F^\times)^2$ is of order 2.

Proof. We have surjective homomorphism

$$\nu : \mathrm{Cl}(F)^+ \to \mathrm{Cl}(F)$$

with kernel P_F/P_F^+. So it follows that $h_F^+ = [P_F : P_F^+]$. If $(\alpha) \in P_F$ for some $\alpha \in \mathcal{O}_F$, then $(\alpha^2) \in P_F^+$ since $\alpha^2 \gg 0$. So every non-trivial element in P_F/P_F^+ is of order 2. By Lemma A.2 have $[P_F : P_F^+] = [\mathcal{O}_F^{\times,+} : \mathcal{O}_F^{\times 2}]$, so the statement follows. □

Corollary A.4. *If F is a totally real cubic number field, then either $h_F = h_F^+$ or $h_F^+ = 4h_F$.*

Proof. First assume h_F^+ is odd. Since $[\mathcal{O}_F^{\times,+} : (\mathcal{O}_F^\times)^2]$ is always a power of 2, it follows from Proposition A.3 that $h_F^+ = h_F$. Now assume h_F^+ is even and $h_F \neq h_F^+$. Then we have $[\mathcal{O}_F^{\times,+} : (\mathcal{O}_F^\times)^2] = 2^l$ for $l \geq 1$. The quotient group $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$ contains all elements of order 2 in $\mathcal{O}_F^\times$. Because F is a totally real cubic field, we have $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2 \cong (\mathbb{Z}/2\mathbb{Z})^3$. This gives

$$[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}][\mathcal{O}_F^{\times,+} : (\mathcal{O}_F^\times)^2] = 2^3.$$

Here $[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}] = 2$, because for every $\eta \in \mathcal{O}_F^\times$ we have $\eta^2 \in \mathcal{O}_F^{\times,+}$. Then the above gives $[\mathcal{O}_F^{\times,+} : (\mathcal{O}_F^\times)^2] = 4$ and the result follows. □

Lemma A.5. *[LLO99, Page 395] Let K be a sextic CM-field that does not contain an imaginary quadratic subfield. If $h_{K_+}^+$ is odd, then $Q_K = 1$.*

Proof. By Lemma 2.4.1 we have $\mu_K = 2$. Let $K = K_+(\sqrt{\alpha})$ for some totally negative $\alpha \in \mathcal{O}_{K_+}$ such that $\alpha\mathcal{O}_K = \mathfrak{a}^2$ for some $\mathfrak{a} \in I_K$. Suppose we are in case (i)-2(a) of [Lem95, Theorem 1], with $\mathfrak{a}$ principal and $Q_K = 2$. Then there exists $\beta \in \mathcal{O}_K^\times$ such that $\mathfrak{a} = \beta\mathcal{O}_K$. Then $(\alpha) = \mathfrak{a}^2 = (\beta^2)$, hence $\alpha = u\beta^2$ for some totally negative $u \in \mathcal{O}_{K_+}^\times$. Then

$$K = K_+(\sqrt{u\beta^2}) = K_+(\sqrt{u}).$$

But since $h_{K_+}^+$ is odd, we have that $-u = v^2$ for some $v \in \mathcal{O}_{K_+}^\times$. Then

$$K = K_+(\sqrt{-v^2}) = K_+(\sqrt{-1}).$$

Because K does not contain an imaginary quadratic subfield, this is a contradiction. So we must be in case (i)-2(a) and $Q_K = 1$. □

Proposition A.6. *Assume $I_{K^r} = I_0(\Phi^r)$ and let $h_{K_+}^+$ be odd. Then $h_K^* = 2^{t_K - 1}$.*

Proof. Lemma A.5 gives that $Q_K = 1$, so by Lemma 3.2.2 we have $[P_K^{H_K} : P_{K_+}] = 2$. Then by (the proof of) Corollary 3.2.9 we have $h_K^* = 2^{t_K - \delta_K}$, with $\delta_K = 1$ if $[P_K^{H_K} : P_{K_+}] = 2$. □

We will now determine the structure of Cl_K. For a CM-field F with maximal totally real subfield F_+, let $i_{F/F_+} : \mathrm{Cl}_{F_+} \to \mathrm{Cl}_F$ be the natural map between class groups. That is, for a prime $\mathfrak{p} \in I_{F_+}$ it gives $i_{F/F_+}([\mathfrak{p}]) = [\mathfrak{p}\mathcal{O}_F]$.

Lemma A.7. *Let K be a sextic CM-field that does not contain an imaginary quadratic subfield. Then i_{K/K_+} is injective.*

Proof. We combine the proofs of [Lou94b, Statement 2.(1) and (2)] and adapt these to sextic CM-fields that do not contain an imaginary quadratic subfield. Let $\mathfrak{p} \in I_{K_+}$ be a prime in the kernel of i_{K/K_+} such that $\mathfrak{p}\mathcal{O}_K = (\gamma)$ for some $\gamma \in K^\times$. Then $\gamma\mathcal{O}_K = \overline{\gamma}\mathcal{O}_K$, so there exists $\eta \in \mathcal{O}_K^\times$ such that $\overline{\gamma} = \eta\gamma$. Then

$$\eta\overline{\eta} = \frac{\gamma\overline{\gamma}}{\gamma\overline{\gamma}} = 1,$$

so we have $\eta \in W_K$. By Lemma 2.4.1 we have $W_K = \{\pm 1\}$. First assume $\eta = 1$, then $\gamma = \overline{\gamma}$ and hence $\gamma \in K_+$. This implies $\mathfrak{p} = \gamma\mathcal{O}_{K_+} \in P_{K_+}$, so $i_{K/K_+}([\mathfrak{p}]) = [\mathcal{O}_K]$ is the trivial element in Cl_K.
Now assume $\eta = -1$, then $\overline{\gamma} = -\gamma$. Then we have

$$\gamma^2 = -\gamma\overline{\gamma} = -N_{K/K_+}(\gamma) \in K_+.$$

From Section 2.1 we have $K = K_+(\sqrt{-\delta_0})$ for $\delta_0 \coloneqq m\alpha_0$, where $m \in \mathbb{Z}_{>0}$ and $\alpha_0 \in K_+$. Define $\beta \coloneqq \gamma\sqrt{-\delta}$, then

$$\overline{\beta} = -\overline{\gamma}\sqrt{-\delta} = \gamma\sqrt{-\delta} = \beta,$$

so $\beta \in K_+$. We have $\beta^2 = -\delta\gamma^2$, so $\delta = -\frac{\beta^2}{\gamma^2} \in K_+$.

It follows that $\delta\mathcal{O}_{K_+} = \frac{\beta^2}{\gamma^2}\mathcal{O}_{K_+}$ is a square ideal. Then $N_{K_+/\mathbb{Q}}(\delta)$ is a square in $\mathbb{Q}$, so K is normal over $\mathbb{Q}$. This is a contradiction, hence $\eta = 1$. Then the kernel of i_{K/K_+} is trivial and the result follows. □

For Cl_K the class group of a number field K, the 2-class group $\mathrm{Cl}_K[2]$ of Cl_K is its subgroup containing all elements of at most order 2. The 2-rank of Cl_K is the order of $\mathrm{Cl}_K[2]$.

Proposition A.8. *Let K be a sextic CM-field that does not contain an imaginary quadratic subfield. Assume that $h^+_{K_+}$ is odd, then the 2-rank of Cl_K is $t_K - 1$.*

Proof. Since $h^+_{K_+}$ is odd, by Proposition A.3 we must have that h_{K_+} is odd as well and $\mathcal{O}^{\times +}_{K_+} = \mathcal{O}^{\times 2}_{K_+}$. Since we have

$$\mathcal{O}^{\times 2}_{K_+} \subset \mathcal{O}^{\times +}_{K_+} \cap N_{K/K_+}(K^\times) \subset \mathcal{O}^{\times +}_{K_+},$$

also $\mathcal{O}^{+}_{K_+} \cap N_{K/K_+}(K^\times) = \mathcal{O}^{2}_{K_+}$. Moreover, by Lemma A.7 the map $i(K/K_+)$ is injective and by Lemma A.5 we have $Q_K = 1$. Then by [Lou96, Proposition 9] the 2-rank of Cl_K is $t_K - 1$. □

Theorem A.9. *Let K be a sextic CM-field whose Galois closure has Galois group $(C_2)^3 \rtimes C_3$ and assume $I_{K^r} = I_0(\Phi^r)$. Let $h^+_{K_+}$ be odd. Then $\mathrm{Cl}_K \cong \mathrm{Cl}_{K_+} \times (\mathbb{Z}/2\mathbb{Z})^{t_K - 1}$, where Cl_{K_+} is a product of odd order subgroups.*

Proof. By Lemma A.7 we have that Cl_{K_+} is isomorphic to a subgroup of Cl_K, which we will simply denote by Cl_{K_+}. If $h^+_{K_+}$ is odd, then h_{K_+} is odd. This gives

$$\mathrm{Cl}_K[2] \cap \mathrm{Cl}_{K_+} = \{[\mathcal{O}_K]\}$$

and hence

$$\mathrm{Cl}_{K_+} \oplus \mathrm{Cl}_K[2] \subset \mathrm{Cl}_K \, .$$

By Proposition A.8 the 2-rank of Cl_K is $t_K - 1$, so $\mathrm{Cl}_K[2] \cong (\mathbb{Z}/2\mathbb{Z})^{t_K-1}$ and $\mathrm{Cl}_K[2]$ is isomorphic to a subgroup of $\mathrm{Cl}_K/\mathrm{Cl}_{K_+}$. On the other hand we have h^*_K Proposition A.6, so

$$\mathrm{Cl}_K/\mathrm{Cl}_{K_+} \cong \mathrm{Cl}_K[2] \cong (\mathbb{Z}/2\mathbb{Z})^{t_K-1}.$$

□

B. Lower bound for the relative class number of octic CM-fields

We give a lower bound for the relative class number of a non-normal octic CM-field F with Galois group $(C_2)^3 \rtimes C_3$ or $(C_2)^3 \rtimes S_3$.

Define $H_F := \mathrm{Gal}(F/F_+)$ and

$$I_F^{H_F} := \{\mathfrak{a} \in I_F : \mathfrak{a} = \overline{\mathfrak{a}}\}, \quad P_F^{H_F} := P_F \cap I_F^{H_F}.$$

By Lemma 3.2.2 we have

$$h^*_F = 2^{t_F} \frac{[I_F : I_F^{H_F} P_F]}{[P_F^{H_F} : P_{F_+}]}.$$

Assuming the generalised Riemann Hypothesis (GRH) we construct a lower bound for h^*_F and use this to obtain a bound for d_{F_+}. Let $\mathrm{Res}(\zeta_F), \mathrm{Res}(\zeta_{F_+})$ denote the residues of respectively $\zeta_F(s)$ and $\zeta_{F_+}(s)$ at $s = 1$. We have the following class number identities:

$$h_F = \frac{\mu_F \mathrm{Res}(\zeta_F)\sqrt{d_F}}{2^4\pi^4 R_F}, \quad h_{F_+} = \frac{\mathrm{Res}(\zeta_{F_+})\sqrt{d_{F_+}}}{2^3 R_{F_+}} \tag{12}$$

Proposition B.1. *[Lou11, Theorem 1] Let F be a totally real number field of degree 4 over $\mathbb{Q}$. Then*

$$Res(\zeta_F) \leq \frac{1}{48} \log^3(d_F).$$

Theorem B.2 (J.Oesterlé)**.** *Let F be a number field not equal to $\mathbb{Q}$. Assume the Riemann Hypothesis for ζ_F, then*

$$Res(\zeta_F) \geq \frac{e^{-3/2}}{\sqrt{\log(|d_F|)}} \exp\left\{ \frac{-1}{\sqrt{\log(|d_F|)}} \right\}.$$

Theorem B.3. *Let F be an octic CM-field, then*

$$h_F^* \geq \frac{\mu_F}{146} \frac{\sqrt{d_F/d_{F_+}}}{\sqrt{\log(d_F)}\log^3(d_{F_+})} \exp\left\{\frac{-1}{\sqrt{\log(d_F)}}\right\}.$$

Proof. By Proposition 2.4.4 we have

$$\frac{R_F}{R_{F_+}} = \frac{2^3}{Q_F}. \tag{13}$$

Then

$$h_F^* = \frac{\mu_F R_{F_+}}{2\pi^4 R_F} \frac{Res(\zeta_F)}{Res(\zeta_{F_+})} \sqrt{\frac{d_F}{d_{F_+}}} = \frac{\mu_F Q_F}{16\pi^4} \frac{Res(\zeta_F)}{Res(\zeta_{F_+})} \sqrt{\frac{d_F}{d_{F_+}}}.$$

Assume the Riemann Hypothesis for ζ_F. Applying Proposition B.1 to $\mathrm{Res}(F_+)$ and Theorem B.2 to $\mathrm{Res}(F)$ gives the lower bound

$$h_F^* \geq \frac{\mu_F}{146} \frac{\sqrt{d_F/d_{F_+}}}{\sqrt{\log(d_F)}\log^3(d_{F_+})} \exp\left\{\frac{-1}{\sqrt{\log(d_F)}}\right\}.$$

□

In Corollary B.4 we give a lower bound for h_F^* that only depends on d_F/d_{F_+}.

Corollary B.4. *Let F be as in Theorem B.3. Assume the Riemann Hypothesis for ζ_F, then*

$$h_F^* \geq \frac{\sqrt{d_F/d_{F_+}}}{73\sqrt{2\log\left(d_F/d_{F_+}\right)}\log^3(d_F/d_{F_+})} \exp\left\{\frac{-1}{\sqrt{\log\left(d_F/d_{F_+}\right)}}\right\}$$

Proof. We have that $d_{F_+} \leq d_F/d_{F_+}$ and $d_F \leq d_F^2/d_{F_+}^2$, hence

$$\frac{1}{\sqrt{\log(d_F)}} \geq \frac{1}{\sqrt{2\log\left(d_F/d_{F_+}\right)}}.$$

Moreover $d_F/d_{F_+} \leq d_F$, so

$$\exp\left\{\frac{-1}{\sqrt{\log\left(d_F/d_{F_+}\right)}}\right\} \geq \exp\left\{\frac{-1}{\sqrt{\log(d_F)}}\right\}.$$

Applying these to the bound in Theorem B.3 gives

$$h_F^* \geq \frac{\mu_F\sqrt{d_F/d_{F_+}}}{146\sqrt{2\log\left(d_F/d_{F_+}\right)}\log^3(d_F/d_{F_+})} \exp\left\{\frac{-1}{\sqrt{\log\left(d_F/d_{F_+}\right)}}\right\}.$$

□

Remark B.5. If K^r is the octic reflex field of a sextic CM-field K with Galois closure of degree 24 over $\mathbb{Q}$ and we assume $I_{K^r} = I_0(\Phi^r)$, then $h_{K^r}^*$ is not generally a power of 2. For instance, see Table 3.2. For this reason we could not use the lower bound in Corollary B.4 to give a bound on $d_{K^r}/d_{K_+^r}$ and t_{K^r} in Chapter 3.

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