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Bézout's theorem in $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract

Bézout's theorem states that, given two projective curves, the number of intersection points is at most the product of their degrees. Moreover, we have equality if we work over an algebraically closed field. However, when looking at the intersection of two affine curves, we often find that the number of intersection points we expect by Bézout's theorem is higher than we can perceive by plotting them, even when working over an algebraically closed field. The latter shows that the hypothesis that we work over a projective space is essential to have a uniform result, i.e., one that depends only on the degrees of the curves. We can go from the affine space to the projective space by compactifying the affine space. One possible compactification of the affine plane is the projective plane $\mathbb{P}^2$. It is natural to wonder what kind of results one gets when considering other compactifications, such as $\mathbb{P}^1 \times \mathbb{P}^1$. In contrast to $\mathbb{P}^2$, where we have one line at infinity, in $\mathbb{P}^1 \times \mathbb{P}^1$ we have two lines at infinity. This will change the intersection behaviour of curves. Using divisors on surfaces we will discover a version of Bézout's theorem in $\mathbb{P}^1 \times \mathbb{P}^1$. Before this can be done, we will learn about plane curves and their intersection behaviour to prove Bézout's theorem in the classical sense. At the end of the paper, the two versions of Bézout's theorem are compared and we will shed a light on how the techniques used to find Bézout's theorem in $\mathbb{P}^1 \times \mathbb{P}^1$ can be used to find a version of Bézout's theorem on even more surfaces.

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Introduction

The fundamental theorem of algebra states that every polynomial in one variable $f(x)$ over an algebraically closed field k of degree n has exactly n roots, when counted with multiplicity. This fact admits an interpretation in terms of the so-called algebraic curves. Indeed, the graph of f, which is given by the zeros of $g(x, y) = y - f(x)$ is an algebraic curve and the fundamental theorem of algebra tells us that it intersects the algebraic curve $y = 0$ exactly n times, counting multiplicities.

We would like to generalize this result in a way that our second curve $y = 0$ could also be any curve $h(x, y) = 0$. This is where Bézout's theorem comes into play. It tells us the following:

Let f and g be two algebraic curves that do not share an irreducible component. Let m and n be the degrees of f and g respectively. Their corresponding projective curves F and G intersect in exactly mn points, assuming we work in an algebraically closed field.

This result is not always clear when graphing the curves, as we will see at the start of Chapter 2. We might find that our curves do not intersect in our field k. This is easily solved by only working in an algebraically closed field. Some curves do not intersect transversally at every point and hence what seems one intersection point might be two or even more. This can be solved by defining the intersection multiplicity well.

Most interesting is our third problem. This problem deals with curves not intersecting in the affine plane, but in the projective plane. As an example, consider the two affine curves defined by $f(x, y) = x + 1$ and $g(x, y) = x^2 - y$. According to Bézout's theorem, their projective curves should intersect at two points. However, f and g only intersect in one point in the affine plane, namely $(-1, 1)$, and hence their corresponding projective curves will also only intersect once in the affine plane. To make sure that the projective curves satisfy Bézout's theorem, we do a compactification of the affine plane, in this case the compactification $\mathbb{P}^2$. The idea is to extend the affine plane by adding points at infinity. However, two non-parallel lines should not meet at infinity, as they already satisfy Bézout's theorem. Hence we add one point at infinity for each direction in the affine space.

In Chapter 1 of this thesis we will give background information that is needed throughout the whole thesis, but more specifically to understand the concepts of Chapter 2. In this chapter we look at the theory used to prove Bézout's theorem in $\mathbb{P}^2$ and then prove it in Chapter 3. In this way we can get used to the definitions and theorems in an accessible setting.

After we have proven Bézout's theorem in $\mathbb{P}^2$, we start looking at the problem of intersecting curves more generally. In order to do this, we need to work in a specific topological space, related to algebraic varieties. We will do this in Chapter 4. We also introduce a more general way of looking at curves in Chapter 5, namely the concept of divisors.

The focus of the last two chapters will be to construct versions of Bézout's theorem in both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. We will do the former in Chapter 6 and the latter in Chapter 7. In Chapter 7 we will moreover compare the two statements of Bézout's theorem and look at how the techniques used to construct a version of Bézout in $\mathbb{P}^1 \times \mathbb{P}^1$ can be used to find a version of Bézout on other surfaces.

1 Background

1.1 Projective space

Before we can start talking about any sort of curves, we will first introduce the projective space. I'd like you to imagine that you are standing on a train track. Since you've probably been on a train, you know the rails will never meet each other, as they are parallel. However, from the position you are in now, they do seem to meet very far in the distance in a point, of which you cannot exactly pinpoint its location. This is the intuition behind the point at infinity.

Going back to the affine plane, we find ourselves in the same situation. It appears as though two parallel lines never meet anywhere. In order to get a more uniform theory on the intersection of curves, we would like that any pair of distinct lines meet in a single point. So, this means that we need to add points at infinity to the affine plane. The question that arises is: Would one such point at infinity be enough? The answer is no. Consider the following situation.

We have two parallel lines, L_1 and L_2, which intersect in a point P at infinity. Now, let there be another set of parallel lines L_1' and L_2' meeting at infinity in P'. Now suppose that L_1 and L_1' are not parallel and they meet in a point Q different from P and P'. If we had that $P = P'$, L_1 and L_1' would moreover meet in P. A consequence of Bézout's theorem is that any two lines can only meet in one point, which contradicts the fact from before. So P and P' are distinct. Hence we add one point at infinity in each direction of the affine plane [6, App. A.1].

So, the projective plane is actually the affine plane together with a set containing a point for each direction in the affine plane. The set of all points at infinity forms a projective line, where each direction of a line through the origin in affine space corresponds to a point on the projective line. In good notation we would write:

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1.$$

We will now dive deeper into how the projective plane actually works. From here on, we work over an algebraically closed field k.

Remark 1. A field k is called algebraically closed if every polynomial $f \in k[x]$ has at least one root. For example, $\mathbb{R}$ is not algebraically closed, as $x^2 + 1 = 0$ does not have a root in $\mathbb{R}$. On the other hand, $\mathbb{C}$ is algebraically closed.

Recall that the affine space denoted by $\mathbb{A}^n = \mathbb{A}^n_k$ is simply equal to k^n. We define that projective plane as:

$$\mathbb{P}^2 = (k^3 \setminus \{0\})/\sim,$$

where the equivalence relation is defined as

Definition 1.1. *[2, Not. 3.3]*

$$[x, y, z] \sim [x', y', z'] \iff \exists \lambda \in \mathbb{N} \text{ such that } x = \lambda x', y = \lambda y' \ \& \ z = \lambda z'.$$

We denote the equivalence class of the point $[x, y, z]$ by $[x : y : z] \in \mathbb{P}^2$. We call these coordinates x, y, z the homogeneous coordinates of the point $[x, y, z]$.

Looking at the projective plane geometrically, we would like to embed the affine plane $\mathbb{A}^2 = k^2$ in the vector space k^3 by adding a new coordinate equal to 1. We do this with the following map:

$$\phi\colon \mathbb{A}^2 \to k^3, \ [x, y] \to [x, y, 1].$$

To make it more visual, we will look at how this situation works in the projective line (see Figure 1). Here the blue dot represents the point $[x, y]$ and the yellow dot represents the point $[x', y']$. Given any line through the origin, except for the line $y = 0$, we map all points on that line to the point of intersection with the red line, with y-coordinate equal to 1. Indeed, when two points lie on the same line, they only differ by a factor λ and hence lie in the same equivalence class given by Definition 1.1. This works the same in the projective plane, but here our line at height one will become a plane at height one.

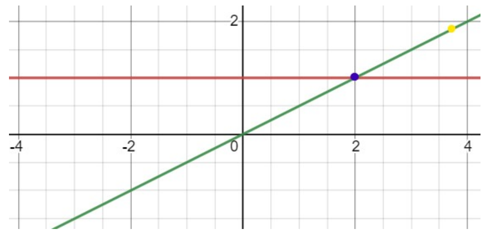


Figure 1: The projective line as a visual representation.

Let us go back to the projective plane again. Looking at the projective plane like this, we are missing the points with z-coordinate equal to 0, or the plane $z = 0$. These points represents the so-called points at infinity. So, in conclusion, in the projective plane, there are two types of points. First the ones with homogeneous coordinates $[x, y, 1]$, for any value of x and y. Second are the points at infinity with homogeneous coordinates $[x, y, 0]$.

This idea is the same as above when talking about the projective plane as the union of the affine plane with its directions. The points with homogeneous coordinates $[x, y, 1]$ are in bijection with the affine plane $\mathbb{A}^2$ and the points with homogeneous coordinates $[x, y, 0]$ are in bijection with the projective line $\mathbb{P}^1$.

1.2 Affine plane curves

Since we are not only interested in the intersection between lines, but in the intersection of any two curves, we will now define affine plane curves. They are given by the zeros of a polynomial, so in order to define them, we first need to define what such a zero set is.

Definition 1.2. *[2, Def 1.3] Given a subset $S \subset k[x_1, \ldots, x_n]$ of polynomials we call*

$$V(S) := \{P \in \mathbb{A}^n : f(P) = 0 \textit{ for all } f \in S\} \subset \mathbb{A}^n$$

the affine zero locus of S.
Such subsets $V(S)$ are called affine varieties. If S is a finite set, $S = \{f_1, \ldots, f_k\}$, we also write $V(S) = V(\{f_1, \ldots, f_k\}) = V(f_1, \ldots, f_k)$.

Definition 1.3. *[2, Rem. 3.4] An affine plane curve C over k is given by the zero locus of a polynomial in 2 variables. This is denoted as*

$$C = V(f) = \{(x, y) \in \mathbb{A}^2_k : f(x, y) = 0\}.$$

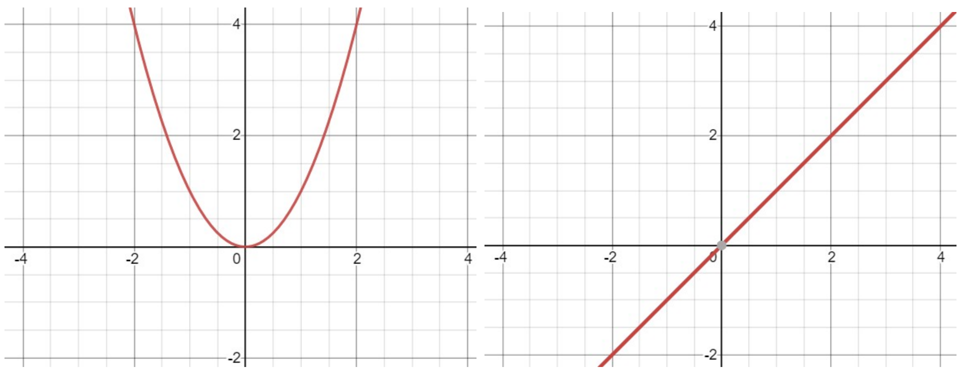
The degree of the curve C is equal to the degree of the polynomial f.

Notation. *As we go on, we will refer just to the polynomial f when talking about an affine curve. This is intended to avoid clumsy notation. The curve is still defined as in Definition 1.3 and whenever it is desirable for clarity we will use the notation from above.*

Example 1. Here our field k is the real numbers $\mathbb{R}$. The affine curve that you can see in Figure 2a is determined by the polynomial $f(x, y) = y - x^2$ and the affine curve that you can see in Figure 2b is determined by the polynomial $g(x, y) = y - x$.

Definition 1.4. *[7, Def 5.1.1] A polynomial $f \in k[x, y]$ is called irreducible if it only allows trivial factorisations. In other words, no matter how you factorise the polynomial, one of the elements of the factorisation must be a unit in $k[x, y]$.*

Example 2. Let $f(x) = x^2 + 1$. The polynomial f is irreducible over $\mathbb{R}$, as it is a polynomial of degree 2 without any roots. However, in $\mathbb{C}[x]$, we have the decomposition $x^2 + 1 = (x - i)(x + i)$. Since $x - i$



(a) The parabola (b) The identity line

Figure 2: Two examples of affine plane curves.

and $x + i$ are not units in $\mathbb{C}[x]$, f is not irreducible over $\mathbb{C}$.

Definition 1.5. *[2, Def 1.5] Let $f(x, y)$ be a polynomial and $C = V(f)$ its corresponding affine curve.*

- *The curve C is irreducible if $f(x, y)$ is an irreducible polynomial in $k[x, y]$.*
- *If $f = f_1^{a_1} \cdot \ldots f_n^{a_n}$ is the irreducible decomposition of f as a polynomial, we also call this the irreducible decomposition of the curve C. The curves $f_1, \ldots, f_n$ are called the irreducible components of f and $a_1, \ldots a_n$ their multiplicities.*
- *The curve C is called reduced if all its irreducible components have multiplicity 1.*

1.3 Properties of affine plane curves

In this section we will treat several properties of zero loci of affine plane curves.

Remark 2. [2, Rem. 1.4] For two polynomials $f, g \in k[x, y]$ we have

a. $V(f) \cup V(g) = V(fg)$,

b. $V(f) \cap V(g) = V(f, g)$.

Lemma 1.6. *[2, Lem 1.11] Let f be an affine curve.*

a. If k is algebraically closed, $V(f)$ is infinite.

b. If k is infinite, then $\mathbb{A}^2 \backslash V(f)$ is infinite.

Proof. First note that when k is algebraically closed, it is moreover infinite. If $k = \{c_1, \ldots c_n\}$ was finite, then the polynomial $g = \prod_{i=1}^{n}(x - c_i) + 1$ would have no zero and k would no longer be algebraically closed. The opposite is not true.

As f is a nonconstant polynomial, it has positive degree in at least one of the variables x or y. Without loss of generality we may assume this is x, so let $f = a_n x^n + \cdots + a_0$ for some $a_0, \ldots, a_n \in k[y]$, with $n > 0$ and $a_n \neq 0$.

As $a_n \in k[y]$ is nonzero, it has finitely many zeros. Hence there are infinitely many $y \in k$ such that $a_n(y) \neq 0$. For each such y, the polynomial $f(x, y)$ is nonconstant in x.

In case a this means that there is an $x \in k$ such that $f(x, y) = 0$, hence giving infinitely many solutions. In case b this means that there is an $x \in k$ with $f(x, y) \neq 0$, as $f(x, y)$ has only finitely many solutions. $\square$

Proposition 1.7. *[2, Prop 1.12] If two affine curves f and g have no common component, i.e. their decompositions into irreducible factors share no common factor, then their intersection $V(f, g)$ is finite.*

Proof. By assumption, f and g are coprime in $k[x, y]$. We want to argue that f and g are also coprime in $k(x)[y]$. Suppose this is not the case, and f and g have a common component in $k(x)[y]$. This would mean, that after clearing denominators we have

$$af = Hf' \qquad\qquad ag = Hg',$$

for some $H, f', g' \in k[x, y]$ of positive y-degree and nonzero $a \in k[x]$. But this means that every irreducible factor of a must divide either H or both f' and g' in $k[x, y]$. Replacing H or both f' and g' by these quotients we get

$$f = Hf' \qquad\qquad g = Hg',$$

where $H, f', g' \in k[x, y]$, of positive y-degree. However, this would mean that f and g are not coprime in $k[x, y]$, which is a contradiction.

Since $k(x)[y]$ is a principal ideal domain and f and g are coprime, we can write 1 as a linear combination of f and g with coefficients in $k(x)[y]$ by Bézout's lemma. So we have that $1 = \frac{D}{C}f + \frac{E}{C}g$ and after clearing denominators we get $C = Df + Eg$ for some $D, E \in k[x, y]$ and a nonzero $C \in k[x]$.

Then, for any $P \in V(f, g)$ we have that

$$C(P) = D(P)f(P) + E(P)g(P) = D(P) \cdot 0 + E(P) \cdot 0 = 0,$$

so the x-coordinate of all points $P \in V(f, g)$ is restricted to the finitely many zeros of C. This process is similar for y, so when multiplying the two results together we still find only finitely many points in the intersection of f and g. □

Notation. *If it is clear from the context what is meant, we will write*

a. $P \in f$ *instead of* $P \in V(f)$*, i.e.* $f(P) = 0$*.*

b. $f \cap g$ *instead of* $V(f, g)$*.*

1.4 Projective plane curves

If we want to define projective plane curves, we must keep something in mind. Since any point in the projective plane can be represented by different homogeneous triples, we would like to only work with functions that give the same output, no matter the factor λ. In other words, we only want polynomials $F(X, Y, Z)$ such that when $F(a, b, c) = 0$, also $F(\lambda a, \lambda b, \lambda c) = 0$. We call these polynomials homogeneous. This gives us the following definition.

Definition 1.8. *[6, Sect. A.2] Let* $F \in k[X, Y, Z]$ *be a polynomial. We say that* F *is a homogeneous polynomial of degree* $d \in \mathbb{Z}_{>0}$ *if* F *is of the form*

$$F = \sum_{i \geq 0, j \geq 0} F_{ij} X^i Y^j Z^{d-i-j},$$

where F_{ij} *are coefficients in* k*. Moreover, a homogeneous polynomial satisfies*

$$F(\lambda X, \lambda Y, \lambda Z) = \lambda^d F(X, Y, Z).$$

We are now ready to define a projective plane curve.

Definition 1.9. *[2, Def 3.8] For a subset* $S \subset k[x_1, \ldots, x_n, x_{n+1}]$ *of homogeneous polynomials we call*

$$V(S) := \{P \in \mathbb{P}^n : F(P) = 0 \textit{ for all } F \in S\}$$

its projective zero locus.
Subsets of $\mathbb{P}^n$ *of this form are called projective varieties.*

Definition 1.10. *[6, Sect. A.2] A projective plane curve D over k is given by the zero locus of a nonconstant homogeneous polynomial in 3 variables. This is denoted as*

$$D = V(F) = \{(X, Y, Z) \in \mathbb{P}^2_k : F(X, Y, Z) = 0\}.$$

The degree of the curve D is the degree of the polynomial F.

Notation. *Similar to affine curves, we will refer to the homogeneous polynomial $F(X, Y, Z)$ when talking about a projective curve. Moreover, if not indicated else, lower case letters will from now on refer to affine curves and capital letters will refer to projective curves. If clarity is needed, it will be stated which type of curves we are talking about.*

Remark 3. The results from Remark 2, Lemma 1.6 and Proposition 1.7 also hold for homogeneous polynomials and projective curves.

Example 3. Given below are two curves D_1 and D_2 in $k[x, y]$, both of degree 2.

$$D_1 \colon X^2 + 2XY + YZ + 4Z^2 \qquad\qquad D_2 \colon X^2 + 2XY + YZ + 4Z^2 + 8X.$$

We would like to determine if these curves are projective curves. As you can see, in D_1, all powers add up to 2, so we expect it to be homogeneous. In D_2, our last term only has a power of 1, so it should not be homogeneous. Let us check this: Take two points $[1 : 1 : 1]$ and $[2 : 2 : 2]$, which are in the same class in $\mathbb{P}^2$. We have that

$$D_1(1, 1, 1) = 1 + 2 + 1 + 4 = 8 \qquad D_2(1, 1, 1) = 1 + 2 + 1 + 4 + 8 = 16$$
$$D_1(2, 2, 2) = 4 + 8 + 4 + 16 = 32 = 2^2 \cdot 8 \quad D_2(2, 2, 2) = 4 + 8 + 4 + 16 + 16 = 48 \neq 2^2 \cdot 16 = 64.$$

Indeed, we see that the polynomial given by the curve D_1 is a homogeneous polynomial and the polynomial given by curve D_2 is not a homogeneous polynomial. Hence we must conclude that the curve D_1 is a projective curve and that the curve D_2 is not a projective curve.

Definition 1.11. *We define irreducible/reducible/reduced projective curves as well as irreducible components and their multiplicities in the same way as for affine curves in Definition 1.5.*

Lemma 1.12. *[2, Ex. 3.11] Every homogeneous polynomial in two variables over an algebraically closed field k can be written as a product of linear polynomials.*

1.5 Homogenization

The question we would like to answer now: How are affine and projective curves related to each other? The processes of transforming an affine curve into a projective curve and vice versa are called homogenization and dehomogenization respectively.

Definition 1.13. *[2, Const. 3.13] The homogenization of a polynomial*

$$f(x, y) = \sum_{i+j \leq d} a_{ij} x^i y^j$$

of degree d is defined to be

$$f^h(X, Y, Z) = F(X, Y, Z) := \sum_{i+j \leq d} a_{ij} X^i Y^j Z^{d-i-j}.$$

Indeed, following Definition 1.8, the resulting polynomial is indeed homogeneous.

When we are given a homogeneous polynomial, we would like to be able to go back to a non-homogeneous polynomial again. This process is called dehomogenization. We can dehomogenize with respect to every variable, depending on what we are trying to achieve.

Definition 1.14. *[2, Const. 3.13] The dehomogenization with respect to the coordinate Z of a homo-*

geneous polynomial

$$F(X,Y,Z) = \sum_{i+j+k=d} a_{ijk} X^i Y^j Z^k$$

of degree d is defined to be

$$F^i(x,y) = f(x,y) := F(X,Y,1) = \sum_{i+j+k=d} a_{ijk} x^i y^j.$$

Example 4. Let $C : x^2 - y^2 - 1 = f(x,y) = 0$. If we want to transform into a projective curve, we apply the process of homogenization:

$$D\colon X^2Y^0Z^{(2-2-0)} - X^0Y^2Z^{(2-0-2)} - X^0Y^0Z^{(2-0-0)} = X^2 - Y^2 - Z^2 = F(X,Y,Z) = 0.$$

If we would now apply the process of dehomogenization as described in Definition 1.14, we would get:

$$C\colon f(x,y) = F(X,Y,1) = x^2 - y^2 - 1,$$

which is indeed the curve we started with.

In Definitions 1.13 and 1.14 above we have written the notations f^h and F^i for the process of homogenization and dehomogenization respectively. These notations will be used in the rest of the paper when referring to the homogenization or dehomogenization of a specific curve. In what follows, the reader will find formal definitions of these notions.

Definition 1.15. *[2, Rem. 3.15] For a projective curve D, its affine set of points is*

$$\begin{aligned} V_p(D) \cap \mathbb{A}^2 &= V_a(F(X,Y,1)) \\ &= V_a(F^i). \end{aligned}$$

So we will call $F(X,Y,1) = F^i$ the affine part of $F(X,Y,Z)$.

Notation. *[2, Not 2.18] For a polynomial $f \in k[x,y]$ of degree d and $i = 0,\dots,d$, we define the degree i-part of f to be the sum of all term of f of degree i. We can then write $f = f_0 + \cdots + f_d$, so all f_i are homogeneous. We call f_0 the constant part of f, f_1 the linear part of f and f_d the leading part of f.*

Definition 1.16. *[2, Rem 3.15] Given an affine curve f, we call the curve $f^h = F$ its projective closure. By Definition 1.13 it is a projective curve whose affine part is f, but does not contain the line at infinity as a component.*

However, F may contain points at infinity. If $f = f_0 + \cdots + f_d$ is the decomposition into homogeneous parts, we have $F = z^d f_0 + z^{d-1} f_1 + \cdots + z f_{d-1} + f_d$ and hence $F(X,Y,0) = f_d$. So the points at infinity of f are given by the projective zero locus of the leading part of f.

2 Intersection multiplicities

Now that we know how curves are defined, we are interested in the intersection properties of curves. Let us first start with a general example. Let C_1 and C_2 be two affine curves of degree d_1 and d_2 respectively. The curves are given by the polynomials:

$$C_1 \colon f_1(x,y) = 0 \qquad\qquad \deg(C_1) = d_1,$$
$$C_2 \colon f_2(x,y) = 0 \qquad\qquad \deg(C_2) = d_2.$$

Our question is now: At how many points do these curves intersect? The points in the intersection $C_1 \cap C_2$ are the solution to

$$f_1(x,y) = f_2(x,y) = 0. \tag{1}$$

We now interpret f_1 as a polynomial in the variable y with coordinates in $k[x]$. As f_1 has degree d_1, we expect d_1 roots $y_1, \ldots, y_d$. We can now substitute each of these roots in our polynomial f_2, to find d_1 equations for x, namely

$$f_2(x,y_1) = 0,\ f_2(x,y_2) = 0,\ \ \ldots\ ,\ f_2(x,y_{d_1}) = 0.$$

These polynomials are all polynomials in the variable x. Since they all have degree d_2, each polynomial has d_2 roots. So we should get $d_1 d_2$ pairs that satisfy Equation 1. Hence we expect that $\#(C_1 \cap C_2) = d_1 d_2$.

However, although that is true in some cases, there are also many situations in which we seem to run into problems. We will now treat examples of different cases where the result from above doesn't necessarily hold.

Examples 5 to 7 are based on examples from appendix A.3 from Silverman [6].

Example 5 (Curves only intersect in complex space). Let $f(x,y) = x + y + 2$ and $g(x,y) = x^2 + y^2 - 1$. From the reasoning above we expect these curves to have 2 intersection points, but in Figure 3a we can see that they seem to not intersect at all. Indeed, when solving the equation $f(x,y) = g(x,y) = 0$, we find the solutions

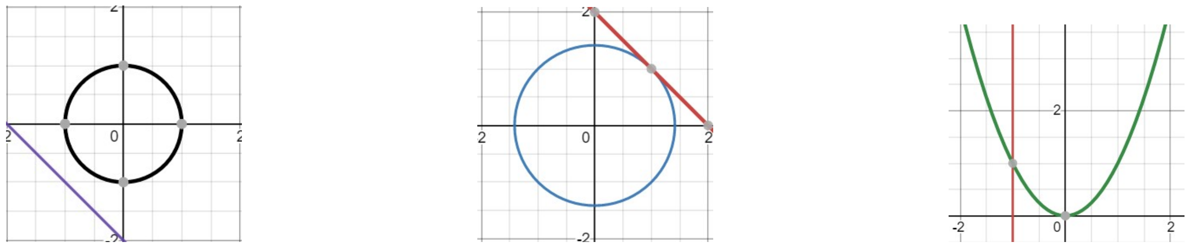
$$\left(-1 + \frac{\sqrt{2}}{2}i, -1 - \frac{\sqrt{2}}{2}i\right), \left(-1 - \frac{\sqrt{2}}{2}i, -1 + \frac{\sqrt{2}}{2}i\right),$$

which lie in the complex plane. Hence we always need to work in an algebraically closed field.

Example 6 (Curves intersect twice in the same point). Let $h = x + y - 2$ and $k = x^2 + y^2 - 2$. We again expect 2 intersection points, but Figure 3b shows us that they only intersect at one point. This means we need to find a way to count intersection points properly.

Example 7 (Curves intersect in points at infinity). Let $m = x + 1$ and $n = x^2 - y$. Again, we expect $\#(m \cap n) = 2$, but looking at Figure 3c we again see one point of intersection. However, these curves also intersect in a point at infinity, so we need to find a way to also take these points into account.

In the rest of this section, we will explain in detail how the intersection multiplicity is defined, discuss several properties of it and give an algorithm on how to calculate the intersection multiplicity explicitly. We will then relate these concepts to tangents and multiplicities of curves. Since the intersection at a given point is a local property, most of the definitions and other statements in this chapter will only be treated in the affine case. If needed, we will explain how to translate these concepts into projective space.



(a) Intersection of the curves f and g.

(b) Intersection of the curves h and k.

(c) Intersection of the curves m and n.

Figure 3: Examples of missing intersection points.

2.1 Intersection multiplicity and its properties

Definition 2.1. *[2, Def 2.1] Let P be a point in the affine plane $\mathbb{A}^2$.*

1. *We define the local ring of $\mathbb{A}^2$ at P as*

$$\mathscr{O}_P := \mathscr{O}_{\mathbb{A}^2,P} = \left\{ \frac{f}{g} : f, g \in k[x,y] \text{ with } g(P) \neq 0 \right\} \subset k(x,y).$$

2. *We can define a well-defined ring homomorphism*

$$\phi_P : \mathscr{O}_P \to k, \ \phi_P\left(\frac{f}{g}\right) = \frac{f(P)}{g(P)}$$

called the evaluation map. Its kernel is given by:

$$M_P := M_{\mathbb{A}^2,P} := \left\{ \frac{f}{g} : f, g \in k[x,y] \text{ with } f(P) = 0 \text{ and } G(P) \neq 0 \right\}.$$

Definition 2.2. *For any point $P = (x_0, y_0, 1) \in \mathbb{P}^2$ there is an isomorphism*

$$\mathscr{O}_{\mathbb{P}^2,(x_0,y_0,1)} \to \mathscr{O}_{\mathbb{A}^2,(x_0,y_0)}, \ \frac{f}{g} \mapsto \frac{f^i}{g^i},$$

where f^i, g^i denote the dehomogenization of the curves as defined in Definition 1.15.
Hence the local ring in projective space is defined the same as in affine space. However, f and g are required to be homogeneous and of the same degree. Moreover, note that here $\mathscr{O}_P \subset k(x,y,z)$ instead of $k(x,y)$.

Definition 2.3. *[7, Def 2.2.8] Let $f_1, f_2, \dots f_n \in k[x,y]$. The ideal generated by $f_1, f_2, \dots f_n$ is defined as*

$$\langle f_1, f_2, \dots f_n \rangle := \{x_1 f_1 + x_2 f_2 + \dots + x_n f_n : x_1, x_2, \dots x_n \in k[x,y]\}.$$

Definition 2.4. *[2, Def 2.3] For a point $P \in \mathbb{A}^2$ and two curves f and g we define the intersection multiplicity of f and g at P as*

$$I_P(f,g) := \dim(\mathscr{O}_P / \langle f, g \rangle),$$

where $\langle f, g \rangle$ is defined as in Definition 2.3 above. Here dim denotes the dimension as a k-vector space.

Example 8. Let $f = y - x^2$ and $g = y + x^2$. Our only point of intersection is $P = (0, 0)$. Then

$$\begin{aligned}\mathscr{O}_{(0,0)} &= \left\{\frac{h}{l} : h, l \in k[x,y] : l(0,0) \neq 0\right\} \\ &= \left\{\frac{h}{l} : h, l \in k[x,y] : l = c + l'\right\},\end{aligned}$$

where $c \neq 0$ and l any curve with $\deg(l') \geq 1$ and $l'(0) \neq -c$. Moreover,

$$\begin{aligned}\langle f, g\rangle &= \langle y - x^2, y + x^2\rangle = \langle y - x^2, y + x^2 + (y - x^2)\rangle \\ &= \langle y - x^2, 2y\rangle = \langle y - x^2, y\rangle = \langle -x^2, y\rangle = \langle x^2, y\rangle.\end{aligned}$$

So, in $\mathscr{O}_{(0,0)}/\langle f, g\rangle$ we send y to 0 and x^2 to 0. Hence any term of y and any term of x with a power higher than or equal to 2 will vanish. We conclude that a basis for this space is $\{1, x\}$. Hence the intersection multiplicity of f and g at $(0, 0)$ is equal to 2.

Remark 4. [2, Rem. 2.4] The intersection multiplicity has the following properties:

a. It is symmetric, i.e. we have that

$$I_P(f, g) = I_P(g, f)$$

for all f and g.

b. For all f, g and h we have that $\langle f, g + fh\rangle = \langle f, g\rangle$ and hence

$$I_P(f, g + fh) = I_P(f, g).$$

Lemma 2.5. *[2, Lem. 2.5] Let $P \in \mathbb{A}^2$ and let f and g be two curves. We have*

a. $I_P(f, g) \geq 1$ if and only if $P \in f \cap g$.

b. $I_P(f, g) = 1$ if and only if $\langle f, g\rangle = M_P$ in $\mathscr{O}_P$.

Proof. First assume that $f(P) \neq 0$. Then f is a unit in $\mathscr{O}_P$ and hence $\langle f, g\rangle = \mathscr{O}_P$, i.e. $I_P(f, g) = 0$, as $\dim(\mathscr{O}_P/\mathscr{O}_P) = 0$. Moreover, we have that $P \notin f$ and $f \neq M_p$, which proves both (a) and (b). The reasoning for $g(P) \neq 0$ is similar.

So now assume that $f(P) = g(P) = 0$, i.e. $P \in f \cap g$. The evaluation map at P (Definition 2.1.2) induces a well defined and surjective map $\psi : \mathscr{O}_P/\langle f, g\rangle \to k$. Since $\dim(k) \geq 1$ we also have that $\dim(\mathscr{O}_P/\langle f, g\rangle) \geq 1$, proving part a. We have that $I_P(f, g) = 1$ if and only if the map ψ is an isomorphism, i.e. if it is moreover injective and hence its kernel is $\{0\}$. This then implies that the kernel of the evaluation map is precisely $\langle f, g\rangle$, proving part b. $\square$

Corollary 2.6. *[2, Exercise 2.7] Let f and g be two curves through a point $P \in \mathbb{A}^2$. If f and g have a common component through P, we have that $I_P(f, g) = \infty$.*

Before we can define more properties of the intersection multiplicity, we first need to explain the following construction.

We say that a sequence

$$0 \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to 0$$

of linear maps between vector spaces (where 0 denotes the zero vector space) is exact if the following hold:

1. $\ker(\varphi) = 0$,
2. $\mathrm{im}(\varphi) = \ker(\psi)$,
3. $\mathrm{im}(\psi) = W$.

These properties imply that

$$\dim(U) + \dim(V) = \dim(W).$$

Proposition 2.7. *[2, Prop 2.5] Let $P \in \mathbb{A}^2$ and let f, g and h be three curves.*

1. *If f and g have no common component through P, then there is an exact sequence*

$$0 \to \mathscr{O}_P/\langle f, h\rangle \xrightarrow{g} \mathscr{O}_P/\langle f, gh\rangle \xrightarrow{\pi} \mathscr{O}_P/\langle f, g\rangle \to 0,$$

where π is the natural map.

2. *$I_P(f, gh) = I_P(f, g) + I_P(f, h)$.*

Now, using property 2 of Proposition 2.7 and the properties discussed before in Remark 4 and Lemma 2.5, we will now propose an algorithm to calculate the intersection multiplicity.

Algorithm 2.8. *Let f and g be any two curves with $f(0,0) = g(0,0) = 0$. If $f(x,y)$ and $g(x,y)$ intersect in $P = (x_0, y_0)$, we shift coordinates to $x' = x - x_0$ and $y' = y - y_0$. We then have two cases:*

1. *If f and g both contain a monomial independent of y, we will write*

$$\begin{aligned} f &= ax^m + (\textit{other terms}) \\ g &= bx^n + (\textit{other terms}) \end{aligned}$$

for some $a, b \in K^*$ and $m, n \in \mathbb{N}$, where without loss of generality we have $m \geq n$. We then set

$$f' := f - \frac{a}{b}x^{m-n}g$$

By Remark 4b we then have that $I_0(f', g) = I_0(f, g)$. As $f'(0) = g(0) = 0$, we can repeat the algorithm recursively.

2. *If, however, f or g does not contain a monomial independent of y, without loss of generality take f, we can write $f = yf'$ and we find by Proposition 2.7.2 that*

$$I_0(f, g) = I_0(y, g) + I_0(f', g)$$

To calculate $I_0(y, g)$, note that by Remark 4b we can remove all multiples of y from g, so we replace g by $g(x, 0)$. Then we can take out the lowest power of x, i.e. the multiplicity m of 0 and write $g(x, 0) = x^m h$, with h nonzero at the origin. Now we find that

$$\begin{aligned} I_0(y, g) &= I_0(y, x^m h) \\ &= I_0(y, x^m) + I_0(y, h) \\ &= mI_0(y, x) + I_0(y, h) \\ &= m \end{aligned}$$

In other words, it is equal to the lowest power of x in a term of g independent of y. For $I_0(f', g)$, if f' does not vanish at 0 then $I_0(f', g) = 0$ by Lemma 2.5a. If $f'(0) = 0$ we repeat the algorithm now computing $I_P(f', g)$.

Example 9. Let us get back to our example from before with the curves $f = y - x^2$ and $g = y + x^2$. Since $f(0,0) = g(0,0) = 0$, we can use the algorithm immediately. We have a monomial independent of y in both f and g, so we start by using strategy 1 from Algorithm 2.8.

$$\begin{aligned} f' &= -x^2 + y + (x^2 + y) \\ &= 2y \end{aligned}$$

Since f' does not contain a monomial independent of y anymore, we now apply strategy 2 to split up

2 and y to find

$$\begin{aligned} I_0(f,g) &= I_0(f',g) \\ &= I_0(y,g) + I_0(2,g) \\ &= 2. \end{aligned}$$

As expected, we get the same answer as in Example 8.

Let us look at another example.

Example 10. Let $f = 3y^2 - x^4$ and $g = x^2 - 5y^3 + y^4$. They have an intersection point at $P = (0,0)$, so we will be calculating $I_0(f,g)$. Then

$$\begin{aligned} f &= -1 \cdot x^4 + 3y^2 \\ g &= 1 \cdot x^2 - 5y^3 + y^4. \end{aligned}$$

So we get

$$\begin{aligned} f' &= -x^4 + 3y^2 + x^2(x^2 - 5y^3 + y^4) \\ &= 3y^2 - 5x^2y^3 + x^2y^4 \\ &= y^2(3 - 5x^2y + x^2y^2) \\ &= y^2 \cdot f''. \end{aligned}$$

This then gives us

$$\begin{aligned} I_0(f,g) &= I_0(f',g) \\ &= I_0(y^2,g) + I_0(f'',g)^* \\ &= 2I_0(y,g) \\ &= 4. \end{aligned}$$

$^* = 0$ as $f''(0,0) = 3 \neq 0$.

Now, suppose we calculated the intersection multiplicity of two curves in the affine plane and found that it is not equal to the product of their respective degrees. Then we need to transform these affine curves into projective curves using the process of homogenization (Definition 1.13). Then we find all the intersection points (x_0, y_0, z_0) in the projective plane. We then have 2 cases

1. $P = (x_0 : y_0 : 1)$, i.e it lies the in the affine part of $\mathbb{P}^2$. Via the isomorphism from Definition 2.2 we see that $\langle F, G\rangle = \langle F^i, G^i\rangle$ and hence we have that $I_P(F,G) = I_{(x_0,y_0)}(F^i, G^i)$.
2. $P = (x_0 : y_0 : 0)$. In this case, we choose a different nonzero coordinate to define the line at infinity and proceed as in 1.

Example 11. Now that we have the knowledge from above, let us finish Examples 8 and 9. So, we have $f = y - x^2$ and $g = y + x^2$. Their respective projective curves are $F = YZ - X^2$ and $G = YZ + X^2$. We have one point in the affine part, $P = (0 : 0 : 1)$, so that will give us intersection multiplicity 2 as seen in Example 9. For the other point, $Q = (0 : 1 : 0)$ we choose a different coordinate to define the line at infinity. As y is the only nonzero coordinate here, we take it to be the line at infinity and hence x and z are our affine coordinates. This gives us

$$\begin{aligned} I_Q(F,G) &= I_Q(F(Y=1), G(Y=1)) \\ &= I_Q(z - x^2, z + x^2) \\ &= 2, \end{aligned}$$

as it is the same calculation as for P, but with z instead of y. So, we find that the total intersection multiplicity is equal to 4.

We would like to have an easier way to determine if the intersection multiplicity is equal to one. For this, recall Notation 1.5.

Proposition 2.9. *[2, Prop 2.19] Let f and g be two curves through the origin. Then $I_0(f, g) = 1$ if and only if the linear parts f_1 and g_1 are linearly independent.*

2.2 Tangents and multiplicities

We could also look at the problem of intersection curves more geometrically. In this section we will look at different properties of curves, such as their multiplicity and tangents at different points. From this information we can deduce if a curve is singular or regular. Based on this information we will discover some interesting properties of the intersection number.

Definition 2.10. *[2, Def 2.20] Let f be an affine curve.*

a. The smallest $m \in \mathbb{N}$ for which the homogeneous part f_m (Notation 1.5) is nonzero is called the multiplicity $m_0(f)$ of f at the origin. Any linear factor of f_m is called a tangent to f at the origin.

b. For a general point $P = (x_0, y_0) \in \mathbb{A}^2$, we define the multiplicity $m_P(f)$ and tangent at P by first shifting coordinates to $x' = x - x_0$ and $y' = y - y_0$. Then apply a at the new origin (x', y').

Definition 2.11. *Let F be a projective curve.*

a. Let $P = (x_0, y_0, 1)$ be a point on F in the affine part $\mathbb{A}^2$. In this case we define the multiplicity $m_P(F)$ to be $m_{(x_0,y_0)}(F^i)$ as in Definition 2.10 above. A tangent to F at P is given by the projective closure of a tangent to F^i at (x_0, y_0).

b. Let P be a point not lying the affine part $\mathbb{A}^2$. In this case we choose a different coordinate for the line at infinity. Then multiplicity and tangents are defined as in a.

Looking at multiplicities of curves is most interesting when $m_P(f) = 1$. Namely, in this case there is a nonzero local linear approximation of f around P.

Definition 2.12. *[2, Def 2.22] Let f be a curve.*

a. We call a point P on f smooth if we have $m_P(f) = 1$. In this case f has a unique tangent at P, denoted $T_P f$. In case $P = (0, 0)$, we have $T_0 f = f_1$, the linear part of f.

b. If P is not smooth, i.e. $m_P(f) \geq 1$, we call P a singular point or singularity of f. In particular, in the case that $m_P(f) = 2$ and f has 2 tangents at P, we call P a node.

c. If all points P on f are smooth, f is called a smooth curve. Otherwise, f is called singular.

Definition 2.13. *Using the definitions of multiplicity and tangents from Definition 2.11, we define the concepts of a smooth point/curve and singular points/curves in the exact way as in the affine case.*

Example 12. Let us look at three different curves and discuss their multiplicities and tangents at $P = (0, 0)$.

1. Let $f : y - x^2$. The curve f has a linear term, so $m_0(f) = 1$. By Definition 2.12a we have that $T_0 f = y$.
2. Let $f = y^2 - x^2 - x^3$. f has a lowest term of degree 2, so $m_0(f) = 2$. To see if $(0, 0)$ is a node, note that $y^2 - x^2 = (y - x)(y + x)$, so we have two tangents, $y - x$ and $y + x$. So $(0, 0)$ is indeed a node.
3. Let $f = y^2 - x^3$. Again, $m_0(f) = 2$. However, here the quadratic term does not split into factors and hence we have only 1 tangent of multiplicity 2. So we conclude that $(0, 0)$ is not a node.

Checking if a curve f is smooth using Definition 2.10 is very tedious, especially since you have to change coordinates every time. Let us now propose a criterion that makes finding singular points easier.

Proposition 2.14. *[2, Prop 2.26] Let $P = (x_0, y_0)$ be a point on a curve f. Then*

a. P is a singular point of f if and only if $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0$.

b. If P is a smooth point of f the tangent to f at P is given by

$$T_P f = \frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0).$$

Example 13. Let $f = y^2 - x^3 - x^2$. We would like to find its singular points using Proposition 2.14. We first need to calculate the partial derivatives:

$$\frac{\partial f}{\partial x}(x, y) = -3x^2 - 2x = -x(3x + 2) \qquad \frac{\partial f}{\partial y}(x, y) = 2y$$

The common zeros are $(0, 0)$ and $(-\frac{2}{3}, 0)$. Note that $f(-\frac{2}{3}, 0) \neq 0$, so we conclude that $P = (0, 0)$ is our only singular point.

Using the information above, we can now reformulate Proposition 2.9.

Proposition 2.15. *[2, Prop 2.24] Let $P \in f \cap g$. Then $I_P(f, g) = 1$ if and only if P is a smooth point of both f and g, and $T_P f \neq T_P g$. In this case we say that f and g intersect transversally at P.*

In projective space, we have a similar construction.

Proposition 2.16. *[2, Prop 3.25] Let P be a point on a curve F. Then*

a. P is a singular point of F if and only if

$$\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0.$$

b. If P is a smooth point of F then the tangent to F at P is given by

$$T_P F = \frac{\partial F}{\partial X}(P)X + \frac{\partial F}{\partial Y}(P)Y + \frac{\partial F}{\partial Z}(P)Z.$$

3 Bézout's theorem

We've now acquired enough information to state and prove Bézout's theorem in projective space.

Theorem 3.1. *Let F and G be two projective curves with no common component over an infinite field k. Then*

$$\sum_{P \in F \cap G} I_P(F,G) \leq \deg(F) \cdot \deg(G). \tag{2}$$

If k is moreover algebraically closed, equality holds.

To prove this theorem, we first need the results of the following lemmas:

Lemma 3.2. *Let F and G be two affine curves without a common component. There is a ring homomorphism*

$$\varphi\colon k[x,y]/\langle F,G\rangle \longrightarrow \prod_{P \in F \cap G} \mathscr{O}_P/\langle F,G\rangle$$

that sends the class of a polynomial $f \in k[x,y]$ to the class of $f \in \mathscr{O}_P$ in each factor $\mathscr{O}_P/\langle F,G\rangle$. This homomorphism satisfies the following:

1. *φ is surjective*
2. *If k is algebraically closed, φ is an isomorphism, i.e. φ is moreover injective.*

Then, using the rank-nullity theorem we find that

$$\sum_P I_P(F,G) \leq dim\big(k[x,y]/\langle F,G\rangle\big)$$

with equality if k is algebraically closed.

Lemma 3.3. *Let F and G be two affine curves without a common component, of degrees $m := \deg(F)$ and $n := \deg(G)$. We have that*

a. $\dim\big(k[x,y]/\langle F,G\rangle\big) \leq m \cdot n$,

b. If the leading parts F_m and G_n of F and G have no common component, then equality holds in a.

Using these lemmas, we can now prove Bézout's theorem 3.1.

Proof. First, recall that here F and G are projective curves. By Lemma 1.6b, there is a point Q in the affine part of $\mathbb{P}^2$ that lies neither on F nor G, i.e. $Q \notin F^i \cup G^i$. Moreover, as k is infinite but $V(F,G)$ is finite by Proposition 1.7, we can pick a line L through Q that does not intersect any point $P \in V(F,G)$.

We then do a coordinate transformation, so L is our new line at infinity. This ensures that neither F nor G contains the line at infinity as a component, so $\deg(F) = \deg(F^i)$ and $\deg(G) = \deg(G^i)$. Moreover, we have that all intersection points of F and G lie in the affine part and hence they are also intersection points of the curves F^i and G^i.

This construction ensures that we can use the lemmas proposed earlier in this section. We have

$$\sum_{P \in F \cap G} I_P(F,G) = \sum_{P \in F^i \cap G^i} I_P(F^i,G^i) \leq dim\big(k[x,y]/\langle F^i,G^i\rangle\big) \leq \deg(F^i) \cdot \deg(G^i) = \deg(F) \cdot \deg(G)$$

where we used Lemma 3.2 is step 2 and Lemma 3.3a in step 3.

Now, if k is algebraically closed, we have equality in step 2 by Lemma 3.2. The equality in step 3 is not as straightforward. We know that the leading parts of F^i and G^i are homogeneous polynomials in two variables by definition and hence they are a product of linear factors by Lemma 1.12. By Definition 1.16 these factors correspond to the points at infinity. By our choice of L there are no such common points, so the leading parts of F^i and G^i do not have a common component. Then, by Lemma 3.3b equality also holds in step 3. $\square$

Example 14. We can also use Bézout's theorem to calculate intersection multiplicities more easily. Let us go back to our example with $f = y - x^2$ and $g = y + x^2$ once more. According to Bézout's theorem the total sum of the intersection multiplicities is equal to 4. Once we have calculated that the intersection multiplicity of the point $(0, 0)$ in the affine plane is equal to 2 and determined that there is only one point at infinity, by Bézout's theorem that point has multiplicity $4 - 2 = 2$. This will save us an extra calculation, for which we otherwise would have needed Algorithm 2.8.

4 The Zariski topology

In this chapter we will introduce more generally the topics treated before in Section 1.2 and the beginning of Section 2.1. We will first define what closed sets in our space are and from there we will give several definitions in order to define the local ring at the end of this chapter.

4.1 Closed sets

Definition 4.1. *A topological space $T = (X, \mathscr{T})$ consists of a nonempty set X together with a fixed family $\mathscr{T}$ of subsets of X satisfying*

1. *$X, \emptyset \in \mathscr{T}$,*
2. *The intersection of a finite collection of sets in $\mathscr{T}$ is in $\mathscr{T}$,*
3. *The union of any collection of sets in $\mathscr{T}$ is in $\mathscr{T}$.*

The family $\mathscr{T}$ is called a topology for X. The members of $\mathscr{T}$ are called the open sets of $\mathscr{T}$. The closed sets are then the sets that are a complement of an open set. For example, let U be an open set in $T = (X, \mathscr{T})$. Then X/U is a closed set.

Definition 4.2. *[2, Def 1.3] Let $S \subset k[x_1, \ldots, x_n]$ be a subset of polynomials. Then*

$$V(S) := \{P \in \mathbb{A}^n : f(P) = 0 \text{ for all } f \in S\} \subset \mathbb{A}^n$$

denotes the common zeros of the elements of S, which we call the the affine zero locus of S.

We say that a subset $Y \subset \mathbb{A}^n$ is an affine variety if there exists $S \subset k[x_1, \ldots, x_n]$ such that $Y = V(S)$.

We would now like to prove the following about affine varieties. Let $S, R \subset k[x_1, \ldots, x_n]$. Then

1. The intersection of an arbitrary collection of affine varieties is a variety.
2. The union of a finite collection of affine varieties is a variety.
3. $\emptyset$ and $\mathbf{A}^n$ an affine variety.

Proof. 1. In order to show this we will show that $V(S \cup R) = V(S) \cap V(R)$.
$\subset$ Let $P \in V(S \cup R)$. Then for all $f \in S \cup R, f(P) = 0$. So, moreover, for all $f \in S$ and for all $f \in R$, $f(P) = 0$. This precisely means that $P \in V(S) \cap V(R)$.

$\supset$ Let $P \in V(S) \cap V(R)$. Suppose $g \in S \cup R$. If $g \in S$, then $g(P) = 0$ since $P \in V(S)$. Similarly if $g \in R$. Hence $P \in V(S \cup R)$.

2. In order to show this we will show that $V(SR) = V(S) \cup V(R)$.
$\subset$ Let $P \in V(SR)$. Suppose $P \notin V(S) \cup V(R)$, then there exist $f \in S$ and $g \in R$ such that $f(P) \neq 0$ and $g(P) \neq 0$. If we take $h = fg$, then $h(P) \neq 0$. But $h \in SR$, which is a contradiction, so $P \in V(S) \cup V(R)$.

$\supset$ Let $P \in V(S) \cup V(R)$. Again, if $h = fg \in SR$, either $f(P) = 0$ or $g(P) = 0$, so $h(P) = 0$. Hence $P \in V(SR)$.

3. In order to show this we will first show $V(\emptyset) = \mathbb{A}^2$ and then $V(k[x_1, \ldots, x_n]) = \emptyset$.
If $S = \emptyset$, there do not exist $f \in S$. Hence for all $P \in \mathbb{A}^2$, we have $f(P) = 0$.

Suppose $f, g \in k[x_1, \ldots, x_n]$ and that f and g are coprime. Then by Proposition 1.7 $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) = 0$ has finitely many solutions. Then there must exist another function $h \in k[x_1, \ldots, x_n]$ such that $h(P) \neq 0$ for all $P \in V(f, g)$. Hence $V(f, g, h) = \emptyset$ already. Since $f, g, h \in k[x_1, \ldots, x_n]$ the affine locus of $k[x_1, \ldots, x_n]$ will certainly be empty as well.

$\square$

The reasoning above shows that the collection of affine varieties forms a topology on $\mathbb{A}^n$ where the closed sets are the affine varieties. Namely, as stated in the definition of a topological space, the closed sets are the complements of the open sets and the statements we proved above are exactly the complements of the statements in Definition 4.1 and hence are true for closed sets. We will call this topology the Zariski topology and its closed sets are the Zariski closed sets. In the rest of this text we will use the notions of affine varieties, and Zariski closed sets interchangeably, depending on which better suits the theory.

4.2 Local ring

Definition 4.3. *[3, p. 18] Let X be a Zariski closed set in the affine space $\mathbb{A}^n$ over the field k. We define the ideal of X to be the ideal*

$$I(X) = \{f \in k[x_1, \dots, x_n] : f \equiv 0 \text{ on } X\}$$

of functions vanishing on X. We then call

$$A(X) = k[x_1, \dots, x_n]/I(X)$$

the coordinate ring of X.

Example 15. If X is a point, then $A(X) = k$.

Example 16. If $X = \mathbb{A}^n$, then $I(X) = 0$ and $A(X) = k[x_1, \dots, x_n]$.

Now we know what a coordinate ring is, we can define regular functions.

Definition 4.4. *[5, p. 25] A function φ defined on a Zariski closed set X is called regular at a point $P \in X$ if it can be written as $\varphi = \frac{f}{g}$ for some $f, g \in A(X)$ with $g(P) \neq 0$. The function φ is called regular if it is regular at every point of X. In the affine space regular functions are polynomials.*

Definition 4.5. *[5, p. 83] Let X be a closed set and let $P \in X$. The local ring of X at P is defined as*

$$\mathscr{O}_{X,P} = \left\{ \frac{f}{g} : f, g \in A(X) \text{ with } g(P) \neq 0 \right\},$$

i.e. the set of all regular functions on X at P.

Let the pair (f, g) denote the regular function $\frac{f}{g} \in \mathscr{O}_{X,P}$. The operations on (f, g) are defined as

$$\begin{aligned} (f, g) + (f', g') &= (fg' + gf', gg'), \\ (f, g)(f', g') &= (ff', gg'). \end{aligned}$$

We identify pairs according to the rule

$$(f, g) = (f', g') \iff \exists h \in \mathscr{O}_{X,P} \text{ with } h(P) \neq 0 \text{ such that } h(fg' - f'g) = 0.$$

Indeed, according to Definition 1.1.1 of [7] this construction defines a ring.

5 Intersection of divisors

Before we can define divisors, we first need some background definitions. From now on, to clean up notation, we will refer only to a variety when we actually mean an affine variety. Recall from last chapter, these sets are always assumed to be closed.

Definition 5.1. *[5, p. 34] Let X be a variety. We say that X is reducible if there are proper closed subsets X_1, X_2 such that $X = X_1 \cup X_2$. Otherwise, we say that X is irreducible.*

Definition 5.2. *Let X be a variety. Then $Y \subset X$ is a subvariety of X if Y is a subset of X and Y is an affine variety itself.*

For example, with our definition, every variety is a subset of the variety $\mathbb{A}^n$.

Definition 5.3. *[5, p. 67] Let X be a variety and let $Y \subset X$ be a subvariety. The codimension of Y in X is equal to* $\dim(X) - \dim(Y)$.

In this section we will often be looking at subvarieties with codimension 1, so in this case the dimension of Y will be one less than the dimension of X. These type of subvarieties are also called hypersurfaces.

5.1 Divisors and their properties

The contents from Definition 5.4 until Corollary 5.6 are based on pages 147-150 from Shafarevich [5].

Definition 5.4. *Let X be an irreducible variety. A divisor D on X is given by a finite linear combination*

$$D = \sum_i m_i C_i, \tag{3}$$

where the C_i are closed hypersurfaces of X and the m_i integers.

Remark 5. The following definitions are associated to a divisor.

1. If all $m_i = 0$, we say $D = 0$.
2. If all $m_i \geq 0$ and some $m_i > 0$ we say $D > 0$ and we call D effective.
3. We call $\cup_i C_i$ the support of D, denoted $\mathrm{supp}(D)$.
4. We call $\sum_i m_i$ the degree of D.

Let D and D' be two divisors on X with the same support, denoted

$$D = m_1 C_1 + \cdots + m_r C_r \qquad\qquad D' = m_1' C_1 + \cdots + m_r' C_r.$$

We define the operation:

$$D + D' = (m_1 + m_1')C_1 + \cdots + (m_r + m_r')C_r. \tag{4}$$

With this operation, the set of all divisors on X form a group. We denote this group by $\mathrm{Div}(X)$. Our goal now is to find a map taking a nonzero function $f \in k[x_1, \ldots, x_n]$ to its divisor $\mathrm{div}(f)$. First, assume that X is nonsingular in codimension 1, i.e. the set of singular points of X has codimension greater than or equal to 2. Let $C \subset X$ be a hypersurface defined by some regular equation $\{g = 0\}$, i.e. the ideal of C is generated by g, notated $I(C) = \langle g \rangle$. We then define the order $\mathrm{ord}_C(f)$ of f along the hypersurface C to be the largest absolute integer k such that $f \in \langle g^k \rangle$, but $f \notin \langle g^{k+1} \rangle$ [5, p. 148].

Example 17. If $X = \mathbb{A}^1$, then $\mathrm{ord}_C(f)$ is the order of a zero or a pole of f at a point. Let

$$f = \frac{(x-1)^2(x-2)}{x^7}.$$

Since we define the order of f along the hypersurfaces of $\mathbb{A}^1$, we are mostly interested in the hypersurfaces

1. $C_1 = 1,\ I(C_1) = \langle x - 1 \rangle$,

2. $C_2 = 2,\ I(C_2) = \langle x-2 \rangle$,

3. $C_3 = 0,\ I(C_3) = \langle x \rangle$.

Indeed, for each of these hypersurfaces, if we choose k one higher than their corresponding order in f, we find that

1. $f \notin \langle (x-1)^3 \rangle$,
2. $f \notin \langle (x-2)^2 \rangle$,
3. $f \notin \langle x^{-8} \rangle$.

So indeed, when working in $\mathbb{A}^1$, $\mathrm{ord}_C(f)$ is equal to the order of a zero or a pole of f at the point corresponding to the hypersurface C.

Given $f, g \in k(X)$, we have

$$\mathrm{ord}_{\mathrm{C}}(f \cdot g) = \mathrm{ord}_{\mathrm{C}}(f) + \mathrm{ord}_{\mathrm{C}}(g)$$
$$\mathrm{ord}_{\mathrm{C}}(f + g) \geq \min\{\mathrm{ord}_{\mathrm{C}}(f), \mathrm{ord}_{\mathrm{C}}(g)\}$$

If f is a rational function, so $f = \frac{m}{n}$, with $m, n \in A(X)$, we define the order of f to be

$$\mathrm{ord}_{\mathrm{C}}(f) = \mathrm{ord}_{\mathrm{C}}(m) - \mathrm{ord}_{\mathrm{C}}(n) \tag{5}$$

If $\mathrm{ord}_{\mathrm{C}}(f) = k > 0$, we say that f has a zero of order k at C. If $\mathrm{ord}_{\mathrm{C}}(f) = k < 0$, we say that f has a pole of order k at C.

The divisor of f is then given by

$$\mathrm{div}(f) = \sum_C \mathrm{ord}_{\mathrm{C}}(f) C, \tag{6}$$

where we sum over all hypersurfaces C of X. A divisor of the form $D = \mathrm{div}(f)$ is called a principal divisor.

Example 18. Coming back to Example 17, we can now write f in the form of Equation 3. Recall,

$$f = \frac{(x-1)^2(x-2)}{x^7}.$$

So, f has order

1. 2 along the hypersurface 1,
2. 1 along the hypersurface 2,
3. -7 along the hypersurface 0,
4. 0 along every other hypersurface.

Hence we can write

$$\mathrm{div}(f) = 2[1] + 1[2] - 7[0],$$

where the square brackets denote the hypersurfaces.

Definition 5.5. *If* $\mathrm{div}(f) = \sum_i k_i C_i$, *then the divisors*

$$\mathrm{div}_0(f) = \sum_{\{i : k_i > 0\}} k_i C_i \qquad \qquad \mathrm{div}_\infty(f) = \sum_{\{i : k_i < 0\}} -k_i C_i$$

are called respectively the divisor of zeros and the divisor of poles of f.

Corollary 5.6. *The divisor of* f *has the following properties:*

a. $\mathrm{div}_0(f), \mathrm{div}_\infty(f) \geq 0$, *b.* $\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f)$,

c. $\operatorname{div}(f_1 \cdot f_2) = \operatorname{div}(f_1) + \operatorname{div}(f_2)$,

d. $\operatorname{div}(f) = 0 \Rightarrow f$ *is constant,*

e. If $\operatorname{div}(f) \geq 0$, *then* f *is regular.*

Proposition 5.7. *[1, Ch. 8 Prop. 1] Let* f *be a regular function. The degree of the divisor of* f *is equal to zero.*

Proof. Since f is regular, we can write $f = \frac{g}{h}$, with g and h nonzero and of the same degree m. Using equation 5, we find that

$$\begin{aligned}
\operatorname{div}(f) &= \sum_C \operatorname{ord}_C(f)C \\
&= \sum_C \big((\operatorname{ord}_C(g) - \operatorname{ord}_C(h))C\big) \\
&= \sum_C \big(\operatorname{ord}_C(g)C - \operatorname{ord}_C(h)C\big) \\
&= \sum_C \operatorname{ord}_C(g)C - \sum_C \operatorname{ord}_C(h)C \\
&= \operatorname{div}(g) - \operatorname{div}(h).
\end{aligned}$$

So then

$$\deg\big(\operatorname{div}(f)\big) = \deg\big(\operatorname{div}(g)\big) - \deg\big(\operatorname{div}(h)\big).$$

By Definition 5.8 and Remark 6 below, $\operatorname{div}(g)$ and $\operatorname{div}(h)$ are linearly equivalent and thus have the same degree. Hence

$$\deg(\operatorname{div}(f)) = 0.$$

□

5.2 Intersection theory

In this next part, we are going to introduce the intersection theory for divisors.

Definition 5.8. *[4, Sect. 1.6] Let* X *be an affine variety. We say two divisors* D *and* D' *on* X *are linearly equivalent, denoted* $D \sim D'$, *if* $D - D'$, *as defined in Equation 4, is a principal divisor, i.e.*

$$D \sim D' \iff D - D' = \operatorname{div}(f)$$

for some function $f \in k(X)$.

Remark 6. The equivalence relation preserves degrees. In other words, if D and D' are linearly equivalent to each other, they must have the same degree.

Definition 5.9. *[4, Sect. 1.6] We define the class group of* X, *and denote it by* $\mathrm{Cl(X)}$, *to be the quotient group of the divisors group of* X *by the equivalence relation defined above. That is,*

$$\mathrm{Cl(X)} = \mathrm{Div(X)}/\sim.$$

Let us look at some examples of class groups for specific surfaces X.

Example 19. [5, Ex. 3.1, p. 150] Let $X = \mathbb{A}^n$. We know that every hypersurface $C \subset \mathbb{A}^n$ is defined by a single polynomial g, so $I(C) = \langle g \rangle$, where $g \in k[x_1, \ldots, x_n]$.

Hence $C = \operatorname{div}(g)$, so every prime divisor is principal. Given a divisor $D = \sum_i C_i$, it is equal to the divisor $\operatorname{div}(\prod_i g_i)$.

So, every divisor D in $\mathbb{A}^n$ is principal and it is given by the divisor of the product of the generators of the hypersurfaces in D. As every divisor is principal, the difference between any two divisors is principal as well and thus all divisors are linearly equivalent. This means that any divisor is equal to 0 in this class group and hence

$$Cl(\mathbb{A}^n) = 0.$$

Example 20. Let $X = \mathbb{P}^2$. Let L be a line. Consider the sequence

$$0 \to \mathbb{Z} \xrightarrow{\varphi} \mathrm{Cl}(\mathbb{P}^2) \xrightarrow{\psi} \mathrm{Cl}(\mathbb{A}^2) \to 0,$$

where φ maps an integer m to mL and ψ maps a divisor D to $D \cap \mathbb{A}^2$. We would like to show that this sequence is exact. Recall from Section 2.1 that we need to prove that

1. $\ker(\varphi) = 0$,
2. $\mathrm{im}(\varphi) = \ker(\psi)$,
3. $\mathrm{im}(\psi) = \mathrm{Cl}(\mathbb{A}^2)$.

Since $\mathrm{Cl}(\mathbb{A}^2) = 0$, fact 3 follows immediately.

For fact 2, notice that $\psi(D) = 0$ if and only if $D \cap \mathbb{A}^2 = 0$, i.e if $D = nL$, since $\mathbb{P}^2 = \mathbb{A}^2 \cup L$. The image of φ is exactly all multiples of the line L, so $\langle L \rangle$. We now show these sets are equal.
$\subset$ Let $kL \in \mathrm{im}(\varphi)$. Then $kL \cap \mathbb{A}^2 = 0$, so $kL \in \ker(\psi)$.
$\supset$ Let $nL \in \ker(\psi)$. Since nL is a multiple of the line L, we find that $nL \in \mathrm{im}(\varphi)$.

Now we are left to show that φ is injective. Suppose there exists an integer m such that $mL = 0$. Since we are working in $\mathrm{Cl}(\mathbb{P}^2)$, this means that $mL \sim \mathrm{div}(f)$ for some function f. Since the equivalence relation preserves degrees, and in Proposition 5.7 we have seen that $\deg(\mathrm{div}(f)) = 0$, we must have that $\deg(mL) = 0$ is well. But this is an absurd, since $\deg(mL) = m\deg(L) = m$. So we can only have an equivalence if $m = 0$. This means that the kernel of φ is trivial and hence φ is injective.

So, since this sequence is exact, we have that

$$\begin{aligned} \dim(\mathbb{Z}) + \dim(\mathrm{Cl}(\mathbb{P}^2)) &= \dim(\mathrm{Cl}(\mathbb{A}^2)) = 0 \\ \dim(\mathbb{Z}) &= -\dim(\mathrm{Cl}(\mathbb{P}^2)) = \dim(\mathrm{Cl}(\mathbb{P}^2)). \end{aligned}$$

So it follows that

$$\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}.$$

Moreover, from the exact sequence it follows that the generator of this group is the line L we chose at the beginning of the example. Since this line is arbitrary, we take this line to be the line at infinity (which is actually arbitrary as well, as we saw in the proof of Bézout's theorem in Chapter 3).

Example 21. [4, Lem 2.7] Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. We have that $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{Cl}(\mathbb{P}^1) \times \mathrm{Cl}(\mathbb{P}^1)$. We proved above that $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}$, but more generally, we have that $\mathrm{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}$ for any integer $n \geq 1$. So we find that $\mathrm{Cl}(\mathbb{P}^1) \simeq \mathbb{Z}$. If we combine these two facts, we find that

$$\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{Cl}(\mathbb{P}^1) \times \mathrm{Cl}(\mathbb{P}^1) \simeq \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2.$$

Similar to $\mathbb{P}^2$, the generators of this group are the lines at infinity, of which we now have two, denoted L_1 and L_2. We will explain more thoroughly what these lines exactly are in Section 7.1.
Alternatively, one can show that

$$\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$$

as follows. Let D be a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ and consider the affine open set

$$U = \mathbb{P}^1 \times \mathbb{P}^1 \setminus (L_1 \cup L_2).$$

Then $U \simeq \mathbb{A}^2$ and hence $\mathrm{Cl}(U) = 0$. This means that D restricted to U is the divisor of a rational function φ. Thus, in $\mathbb{P}^1 \times \mathbb{P}^1$, we can write any divisor D as

$$D = div(\varphi) + aL_1 + bL_2,$$

for some $a, b \in \mathbb{Z}$. To see that L_1 and L_2 are not linearly equivalent, notice that L_1 and L_2 intersect transversally while having zero self-intersection. This means that $L_1 \circ L_2 = 1$ and $L_1 \circ L_1 = 0$. If it were the case that $L_1 \sim L_2$, by Theorem 5.10.3 below we would have that $L_1 \circ L_2 = 0$. This contradicts the fact that $L_1 \circ L_2 = 1$, showing that L_1 and L_2 are not linearly equivalent.

Theorem 5.10. *[4, p.26] The intersection number of two divisors D_1, D_2 is defined via a map* $\mathrm{Div(X)} \times \mathrm{Div(X)} \to \mathbb{Z}$, *$(D_1, D_2) \to D_1 \circ D_2$, which satisfies the following:*

1. *It is symmetric, so $D_1 \circ D_2 = D_2 \circ D_1$,*
2. *It is bilinear in each factor, so $D_1 \circ (mD_2 + nD_3) = m(D_1 \circ D_2) + n(D_1 \circ D_3)$ and $(pD_1 + qD_2) \circ D_3 = p(D_1 \circ D_3) + q(D_2 \circ D_3)$,*
3. *$D_1 \circ D_2$ depends on D_1 and D_2 only up to linear equivalence, so*

$$D_1 \sim D_1' \iff D_1 \circ D_2 = D_1' \circ D_2,$$

4. *If D_1 and D_2 are two effective divisors without common components, then their intersection number $D_1 \circ D_2$ is defined as*

$$D_1 \circ D_2 = \sum_P (D_1 \circ D_2)_P,$$

where we sum over all points of intersection $P \in D_1 \cap D_2$. The local intersection number $(D_1 \circ D_2)_P$ is defined as

$$(D_1 \circ D_2)_P = \dim_k \mathscr{O}_{X,P} / \langle f_1, f_2 \rangle.$$

Here $\mathscr{O}_{X,P}$ is the local ring of $P \in X$ as defined in Definition 4.5 and f_1, f_2 are the local equations of D_1 and D_2 near P respectively.

6 Revisiting Bézout

In this chapter we will deduce Bézout's theorem in $\mathbb{P}^2$ using the concepts of Chapter 5. In this process we will learn techniques that are useful in constructing the version of Bézout's theorem in $\mathbb{P}^1 \times \mathbb{P}^1$.

In Example 20 we have seen that $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}$. Consider now a map from $\mathrm{Cl}(\mathbb{P}^2)$ to $\mathbb{Z}$ that sends a divisor D to its degree. Recall, if $D = \sum_i m_i C_i$, then $\deg(D) = \sum_i m_i$.

As we have seen in Example 20, $\mathrm{Cl}(\mathbb{P}^2)$ is generated by a line L. Since $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}$, we can say that $\mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z} \cdot L$. Hence any divisor can be written as

$$D = mL + \mathrm{div}(f),$$

where m is an integer and f is some function. In $\mathrm{Cl}(\mathbb{P}^2)$, $\mathrm{div}(f) = 0$ for all f, so we find that $D = mL$. In fact, by the map we discussed above, this integer m is equal to the degree of the divisor. So we can write a divisor of degree d as dL.

Now let D be a divisor of degree d and let E be a divisor of degree e. By the reasoning above, to find their intersection number, we must find the intersection number of dL and eL. So their intersection number $dL \circ eL$ is given by

$$dL \circ eL = de(L \circ L),$$

since we can pull d and e out in front by property 2 of Theorem 5.10. We now only have to compute the intersection number of a line with itself.

As the class group of $\mathbb{P}^2$ is generated by any line, any two lines must be linearly equivalent. By property 3 of Theorem 5.10, to calculate the intersection number of a line L with itself, we can also take any line L' equivalent to L and calculate $L^2 = L \circ L = L \circ L'$.

Since all these lines are arbitrary, let us take two explicit lines to calculate L^2. Take $L : \{x = 0\}$ and $L' : \{y = 0\}$. We know these lines intersect transversally and they intersect precisely in $P = \{(0 : 0 : 1)\}$, so $L \circ L' = 1$.

Hence the intersection number of two divisors is equal to the product of their degrees. And indeed, this is exactly what Bézout's theorem tells us.

7 Bézout in $\mathbb{P}^1 \times \mathbb{P}^1$

7.1 Construction of $\mathbb{P}^1 \times \mathbb{P}^1$

Since our goal is to prove a version of Bézout's theorem in $\mathbb{P}^1 \times \mathbb{P}^1$, we should first take a look at what $\mathbb{P}^1 \times \mathbb{P}^1$ actually is. Instead of compactifying $\mathbb{A}^2$ directly, we split it up into two copies of $\mathbb{A}^1$. We then compactify those separately and then consider their product to get $\mathbb{P}^1 \times \mathbb{P}^1$.

In order to compactify $\mathbb{A}^1$, we need to add a point at infinity. This point has homogeneous coordinates $[0:1]$. So we find that

$$\mathbb{P}^1 = \mathbb{A}^1 \cup [0:1].$$

What one must realise is that every point in the first copy of $\mathbb{P}^1$ corresponds to a line in $\mathbb{P}^1 \times \mathbb{P}^1$, so also our point at infinity. This works the same for every point in the second copy of $\mathbb{P}^1$.

So our first line at infinity is precise the line

$$L_1 = [0:1] \times \mathbb{P}^1.$$

Similarly, our second line at infinity is the line

$$L_2 = \mathbb{P}^1 \times [0:1].$$

These two lines are precisely the lines that generate $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$. Note that the point $[0:1]$ is not a special point. In fact, we could have taken any point $[a:b]$ in $\mathbb{P}^1$ and they would have generated $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$ as well.

7.2 Homogenization in $\mathbb{P}^1 \times \mathbb{P}^1$

If we move from affine space to projective space, we 'add' a new coordinate z in order to homogenize the polynomial given by the curve f. This homogenization can be formulated as

$$F(X,Y,Z) = z^d \cdot f\left(\frac{x}{z}, \frac{y}{z}\right),$$

where d is the degree of f.

In $\mathbb{P}^1$ x $\mathbb{P}^1$ this works a bit different. We do not homogenize with respect to one variable, but with respect to two. In essence, we look at the x-coordinate and y-coordinate separately. We then also have two different degrees, the x-degree d_1 and the y-degree d_2. We homogenize x with respect to z and y with respect to w. This gives us the following formula

$$F(X,Z,Y,W) = z^{d_2} w^{d_1} \cdot f\left(\frac{x}{w}, \frac{y}{z}\right).$$

Let us now look at how this formula works in action

Example 22. Let $f(x,y) = x^2 - y + 1$. The x-degree is equal to two, and the y-degree is equal to 1. Applying our formula from above we get

$$\begin{aligned} F(X,Z,Y,W) &= z^1 w^2 \cdot \left(\left(\frac{x}{w}\right)^2 - \frac{y}{z} + 1\right) \\ &= zx^2 - w^2 y + zw^2. \end{aligned}$$

You can see that the total degree of our polynomial F is one higher than the degree of our polynomial f. In general we have that

$$\deg(F) = \deg(f_x) + \deg(f_y),$$

where $\deg(f_x)$ denotes the x-degree of f and $\deg(f_y)$ denotes the y-degree of f.

7.3 A version of Bézout for $\mathbb{P}^1 \times \mathbb{P}^1$

Using the ideas from Chapter 6 we will now propose a version of Bézout's theorem in $\mathbb{P}^1 \times \mathbb{P}^1$.

As we showed in Example 21, the generators of $\mathbb{P}^1 \times \mathbb{P}^1$ are two distinct lines L_1 and L_2. Hence any divisor D in $\mathbb{P}^1 \times \mathbb{P}^1$ is of the form

$$D = m_1 L_1 + m_2 L_2 + \operatorname{div}(f),$$

with m_1, m_2 integers and again $\operatorname{div}(f) = 0$, since we are working in the class group.

Let $D = m_1 L_1 + m_2 L_2$ and $D' = m'_1 L_1 + m'_2 L_2$ be two divisors in $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$. Their intersection number is given by

$$\begin{aligned}
D \circ D' &= (m_1 L_1 + m_2 L2) \circ (m'_1 L_1 + m'_2 L_2) && (7)\\
&= (m_1 m'_1) L_1 \circ L_1 + (m_1 m'_2) L_1 \circ L_2 + (m_2 m'_1) L_2 \circ L_1 + (m_2 m'_2) L_2 \circ L_2 && (8)\\
&= (m_1 m'_1) L_1 \circ L_1 + (m_2 m'_2) L_2 \circ L_2 + (m_1 m'_2 + m'_1 m_2) L_1 \circ L_2. && (9)
\end{aligned}$$

So, in order to find the intersection number of two divisors in $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$ we need to find

$$L_1 \circ L_1, \ L_2 \circ L_2, \ L_1 \circ L_2.$$

Let us first look at the cases $L_1 \circ L_1$ and $L_2 \circ L_2$, as they are similar. We will now treat the case $L_1 \circ L_1$.

Similarly to what we did in Chapter 6, we can look for a line that is equivalent to $L_1 = [0:1] \times \mathbb{P}^1$. As we've seen in Section 7.1, we can take any point to define that line. For example, let us take $L'_1 = [1:0] \times \mathbb{P}^1$. No matter what point $[x, y]$ we choose in the second copy of $\mathbb{P}^1$, there does not exist a point with both coordinates $[0:1:x:y]$ and coordinates $[1:0:x:y]$. Hence they do not intersect in any point and thus

$$L_1 \circ L_1 = 0.$$

The reasoning for $L_2 \circ L_2$ works exactly the same, but $\mathbb{P}^1$ and $[0:1]$ are swapped there.

We are left showing $L_1 \circ L_2$. So, with the same reasoning as above, we need to find a point that has both homogeneous coordinates equal to $[0:1:x:y]$, and homogeneous coordinates equal to $[w:z:0:1]$. This only works if we choose precisely

$$x = 0 \qquad\qquad y = 1 \qquad\qquad w = 0 \qquad\qquad z = 1,$$

so we get precisely one point P with homogeneous coordinates $[0:1:0:1]$. Hence we find that

$$L_1 \circ L_2 = 1.$$

Coming back to what we said in Equation 7, we then find

$$\begin{aligned}
D \circ D' &= (m_1 m'_1) L_1 \circ L_1 + (m_2 m'_2) L_2 \circ L_2 + (m_1 m'_2 + m'_1 m_2) L_1 \circ L_2\\
&= (m_1 m'_1) \cdot 0 + (m_2 m'_2) \cdot 0 + (m_1 m'_2 + m'_1 m_2) \cdot 1\\
&= m_1 m'_2 + m'_1 m_2.
\end{aligned}$$

Our next goal is to find out what these m_1, m_2, m'_1 and m'_2 are. Similarly to Bézout's theorem in $\mathbb{P}^2$, it has something to do with the degree of the curve.

Consider a map $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ that sends a divisor D to its bidegree (d_1, d_2). As $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by two lines, the d_1 and d_2 correspond exactly to the integers m_1 and m_2. So any divisor in $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$ can be written as

$$D = d_1 L_1 + d_2 L_2.$$

Hence, Bézout's theorem in $\mathbb{P}^1 \times \mathbb{P}^1$ tells us the following

Theorem 7.1. *Let D and E be two effective divisors without common components over an infinite, algebraically closed field k. The divisors D and E have bidegree (d_1, d_2) and (e_1, e_2) respectively. Then*

$$D \circ E = \sum_{P \in D \cap E} (D \circ E)_P = d_1 e_2 + e_1 d_2. \tag{10}$$

7.4 Comparison of Bézout in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

Now that we know what Bézout's theorem tells us in both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, we can compare them. For example, given two curves, is the intersection number in $\mathbb{P}^1 \times \mathbb{P}^1$ equal to, higher than or lower than in $\mathbb{P}^2$. It turns out that there is not one exclusive answer and it depends on the degree of the curves. Let us look at some examples to see if there is still a certain pattern.

Example 23. Let us go back to the example we have first seen in Example 8. We have $f = y - x^2$ and $g = y + x^2$.

In $\mathbb{P}^2$ we found that the intersection of f and g is equal to 4. In $\mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\deg(f_x) = 2 \text{ and } \deg(f_y) = 1,$$
$$\deg(g_x) = 2 \text{ and } \deg(g_y) = 1.$$

So the intersection number of f and g is $2 \cdot 1 + 1 \cdot 2 = 4$. We see that in this case, the intersection numbers are equal.

Example 24. Now let $f = y - x^2$ and $g = y^2 + x$.

In $\mathbb{P}^2$ the intersection number is still 4. In $\mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\deg(f_x) = 2 \text{ and } \deg(f_y) = 1,$$
$$\deg(g_x) = 1 \text{ and } \deg(g_y) = 2.$$

So the intersection number of f and g is $2 \cdot 2 + 1 \cdot 1 = 5$, which is greater than in $\mathbb{P}^2$.

Example 25. Let $f = y^4 - x$ and $g = y^4 + x$.

In $\mathbb{P}^2$ the intersection number is $4 \cdot 4 = 16$. In $\mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\deg(f_x) = 1 \text{ and } \deg(f_y) = 4,$$
$$\deg(g_x) = 1 \text{ and } \deg(g_y) = 4.$$

So the intersection number of f and g is $1 \cdot 4 + 4 \cdot 1$, which is less than in $\mathbb{P}^2$.

As you can see, in general, there is not a certain pattern between the intersection numbers of curves in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

7.5 Further study

In this thesis we have focused on a version of Bézout's theorem for $\mathbb{P}^1 \times \mathbb{P}^1$ specifically. However, using the methods explained in Section 5.2 and 7.3 we could be able to find a version of Bézout's theorem for many more surfaces.

More precisely, we saw that in order to produce a version of Bézout's theorem it was enough to know the generators of the class group and how they intersect among themselves. Thus to generalize this to other smooth projective surfaces, it is enough to obtain this information. This is not always an easy task. I conclude this text with two references for cases where the data described above is well-known. For example, for the class group of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, one can look at Example 3.3 on page 150 of Shafarevich [5]. An explanation of the class group of the cubic surface $\mathbb{P}^3$ can be found in the example in Section 1.6 of Reid [4].

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