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Bachelor's Project

Applied Mathematics

The Modeling and Analysis of Memristive Circuits

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Abstract

Cognitive computing mimics human reasoning techniques. A way to look at the development of cognitive computing systems, is the electrical systems point of view. In order to develop these systems, electrical circuits with memristors can be studied. This paper discusses a framework for modeling electrical circuits with capacitors and memristors based on graph theory. In a classical modeling approach, this would yield a system in which both the capacitor voltages and memristor fluxes are states. However, in this report the obtained equations are partially solved analytically. This gives a reduction in the state-space dimension and provides an easier way to analyze the system. Both the stability of the nonlinear ordinary differential equation and the influence of the change of input on the state is studied.

Contents

1 Introduction

Cognitive computing is a part of artificial intelligence. Cognitive computing systems are sometimes regarded as a more human part of artificial intelligence. This part of artificial intelligence mimics human reasoning techniques. It can for example deal with uncertainties in solving problems. Furthermore, it can learn from the past from both errors and successful findings. From a theoretical point of view, cognitive computing could replace existing calculators in a lot of applications, such as analyzing emerging patterns or state critical process-centric issues in real time, but hardware requirements are still high (Coccoli, Maresca, and Stanganelli 2016).

In order to develop new cognitive computing systems, technological and theoretical innovations are needed to advance the field of cognitive computing. Self-learning materials that perform the tasks that are currently assigned to thousands of transistors and complex algorithms have to be created in a more efficient manner. Such a self-learning material can be a physical building block with intrinsic cognitive functionality via cross-linked networks at nanoscale. A way to look at the development of cognitive computing systems is the electrical systems point of view. In order to develop these systems, electrical circuits with certain properties can be studied. An electrical component of interest is the memristor. The study of memristors is motivated by the fact that material scientists are developing materials with memristive behavior. A memristor is a contraction of memory and resistor. As its name already suggests, it is a resistor with memory. This paper focuses on a circuit with at least one memristor, called memristive circuits, and its modeling and analysis.

Before being able to study these circuits, a theoretical foundation is needed. Chapter 2 shows a theoretical framework in which four topics will be discussed. Since we look at the electrical circuits from a graph theoretical perspective, graph theory is studied first. By the fact that we consider electrical circuits, physical laws are needed. The two most important laws for electrical circuits are Kirchhoff's Laws. Thereafter, memristors and capacitors are introduced. Finally the basic algebraic equations for the electrical components are studied. After defining the theoretical background information, mathematical modeling can be done. The equations needed for the mathematical modeling are given by the Kirchhoff's Laws and the Constitutive Relations. For a given electrical component, the constitutive relation gives the corresponding relation between current and voltage. In chapter three, a mathematical model is derived. After deriving this mathematical model, this model is solved partially analytically in order to obtain a reduction in state-space dimension. With this reduced model, numerical modeling is done in Chapter 4. In this chapter, first some elements of the obtained nonlinear ordinary differential equation are explained. Thereafter, the numerical implementation using MATLAB can be found. The results of this chapter are elaborated in Chapter 5, which discusses the analysis of a class of memristor-capacitor circuits. This chapter starts with the notion of stability of both linear and nonlinear systems. Thereafter, the results of the numerical implementation will be studied with respect to the stability. Finally, state trajectories resulting from a step input on the system will be studied. This paper concludes with a brief conclusion showing all the subjects explored throughout this paper. In the end, some recommendations for further research are given.

2 Theoretical framework

In order to study memristors and their (dynamic) behaviour, a theoretical framework is required. Within this section, four topics will be elaborated. The first topic studied is graph theory. Graph theory provides a background to understand electrical circuits. The next topics studied are Kirchhoff's laws. These laws are the basic laws for electrical circuits and describe the relation between voltages and currents in a circuit. Thirdly, memristors and capacitors are studied since this research focuses on electrical circuits consisting of these two basic elements. The mathematics needed can be derived from the constitutive relations of these basic elements together with the laws of Kirchhoff. First, let us start with the notion of graph theory.

2.1 Graph theory

When looking at electrical circuits from a graph theoretical perspective, a useful mathematical framework to describe the interconnection structure in an electrical circuit is provided (Shen 2019). This section will recall the basic notions of graph theory.

A finite, directed graph, $\mathcal{D}$ is defined as a pair $\mathcal{D} = (N, B)$. Here $N = \{n_1, n_2, \ldots, n_k\}$ is a set of k nodes. The set B is the set of branches, which consists of elements of the form (n_i, n_j), where $(n_i, n_j) \in B$ if there is a connection from node n_i to node n_j (Wang et al. 2009). An example of a directed graph $\mathcal{D}$ can be found in Figure 1.

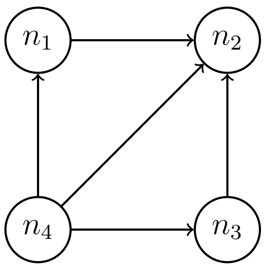

Figure 1: A directed graph with 4 nodes. The nodes are denoted by $N = \{n_1, n_2, n_3, n_4\}$ and the branches are denoted by $B = \{(n_1, n_2), (n_4, n_1), (n_3, n_2), (n_4, n_2), (n_4, n_3)\}$.

A graph is called an undirected graph, denoted by $\mathcal{G}$, if it satisfies the following property

$$(n_i, n_j) \in B \iff (n_j, n_i) \in B.$$

Note that for undirected graphs, the branches are represented as straight lines rather than arrows (Bapat 2010).

As this paper focuses on the graph theoretical perspective for electrical circuits, undirected graphs with an associated orientation will be studied. A graph can be represented using various matrices. For electrical circuits, a matrix of interest is the incidence matrix D. The incidence matrix of an undirected graph with an associated orientation, $D \in \mathbb{R}^{N \times B}$, is given by

$$D = d_{jm},$$

with d_{jm} defined as follows

$$d_{jm} = \begin{cases} 1 & \text{if } n_j \text{ is the head of } b_m, \\ -1 & \text{if } n_j \text{ is the tail of } b_m, \\ 0 & \text{otherwise} \end{cases}.$$

see, e.g., (Rahmani et al. 2009). Recall that in total there are N nodes and there are B branches. Note that for an undirected graph with an associated direction, the orientation will determine the signs of the variables in the incidence matrix. In order to define the incidence matrix for our example in Figure 1, we first need to define our branches $B = \{b_1, b_2, b_3, b_4, b_5\}$. This can be found in Figure 2.

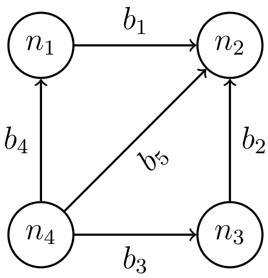

Figure 2: An undirected graph with an associated orientation with 4 nodes and 5 branches.

When studying the undirected graph with an associated orientation with 4 nodes and 5 branches, we obtain the incidence matrix $D \in \mathbb{R}^{4\times 5}$ given by

$$D = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Another matrix that plays a role in graph theoretical aspects of electrical circuits is the graph Laplacian matrix which is defined as

$$\mathcal{L} = DD^T$$

see, e.g., (Rahmani et al. 2009).
Another way to define the graph Laplacian matrix is by using the adjacency matrix and the degree matrix. The adjacency matrix $A \in \mathbb{R}^{N \times N}$ is defined as

$$A = a_{ij},$$

with a_{ij} defined as follows

$$a_{ij} = \begin{cases} 1 & \text{if } \{n_i, n_j\} \in B \\ 0 & \text{otherwise.} \end{cases}$$

see, e.g., (Bapat 1996). The degree matrix is defined by

$$Q = \begin{pmatrix} \deg(n_1) & 0 & \dots & 0 \\ 0 & \deg(n_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \deg(n_k) \end{pmatrix},$$

where $\deg(n_k)$ denotes the number of neighbors of node k. The graph Laplacian matrix then satisfies

$$\mathcal{L} = Q - A$$

see, e.g., (Camlibel 2021).

For an undirected graph, one finds that the Laplacian matrix is symmetric. When one considers a directed graph, a non-symmetric Laplacian matrix is obtained (Dong and Qiu 2014). After studying (un)directed graphs and their properties, the next section introduces Kirchhoff's laws.

2.2 Kirchhoff's Laws

Before deriving any mathematical model for a physical phenomenon, physical laws need to be studied. In order to study mechanical systems, Newton's laws are used. In this paper, in which we want to study electrical circuits, two important physical laws need to be taken into account: Kirchhoff's Voltage Law and Kirchhoff's Current Law. In short, Kirchhoff's Current Law states that the sum of electric currents flowing from a node equals zero while Kirchhoff's Voltage Law states that all voltages around a closed path sum to zero (Nise 2020).

We start by noticing that Kirchhoff's Laws follow directly from Fundamental Laws, like the Law of Conservation of Energy. From Kirchhoff's Laws and the Fundamental Laws, it can be seen that neither voltages nor currents can get lost in a circuit. Kirchhoff's Laws show a simple and powerful tool to quantitatively analyse the processes in an electrical circuit (Robbins and Miller 2012).

When one takes a look at Kirchhoff's Voltage Law from a graph theoretical perspective, the voltage law can be written as

$$\text{For } \mathbf{v} \in \mathbb{R}^m,\ \exists\, \mathbf{p} \in \mathbb{R}^n \text{ s.t. } \mathbf{v} = D^T\mathbf{p}.$$

Note that D is the incidence matrix which was defined in Section 2.1, $\mathbf{v}$ is the vector of voltages across the branches. As such, the length of this vector equals the number of branches and $\mathbf{p}$ is the vector of voltage potentials, which are defined at the nodes. The length of this vector equals the number of nodes. In the example in Figure 3, it can be seen that there are five voltages belonging to the five branches. Each branch represents a basic algebraic element such as a resistor or memristor, which will be studied in Section 2.4.

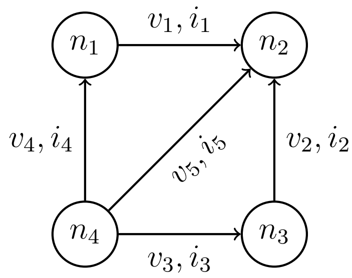

Figure 3: An undirected graph with an associated orientation with 4 Nodes and 5 Measured Voltages and Currents.

For the example in Figure 3, from a graph theoretical perspective, the following incidence matrix is given

$$D = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix},$$

From a graph theoretical perspective, the physical formula for the Kirchhoff's Current Law is given by

$$\text{For } \mathbf{i} \in \mathbb{R}^m, \quad D\mathbf{i} = 0,$$

where D is again the incidence matrix and $\mathbf{i}$ is the vector consisting of all n currents measured (Bretschneider and De Weille 2018). In the example in Figure 3, it can be seen that also five currents belong to the five branches. But what happens with Kirchhoff's Current Law when there is another current incoming? Take for example four grounded capacitors, an example of such an undirected graph with an associated orientation can be found in Figure 4.

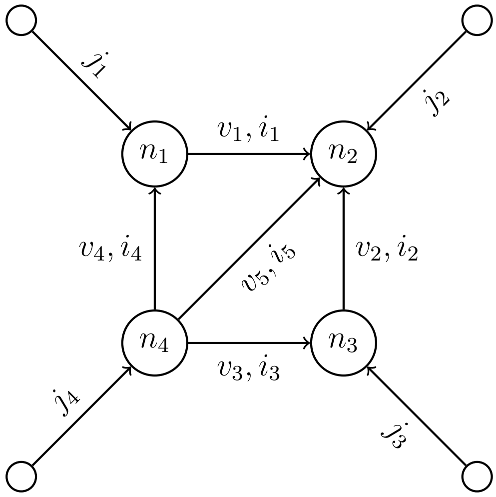

Figure 4: An undirected graph with an associated orientation with 4 Nodes and 4 Incoming Currents.

Note that for each node n_k, there is an incoming current j_k and hence it can be seen that the Kirchhoff's Current Law in such a case is given as

$$\text{For } \mathbf{i} \in \mathbb{R}^m \text{ and for } \mathbf{j} \in \mathbb{R}^n, \quad D\mathbf{i} = \mathbf{j},$$

with D and $\mathbf{i}$ as above and $\mathbf{j}$ the vector consisting of all n incoming currents.

In this section, we concluded that Kirchhoff's Laws consist of two different laws, namely the Kirchhoff's Current Law and the Kirchhoff's Voltage Law. In the next section, both memristors and capacitors are studied.

2.3 Memristors and capacitors

There are four basic algebraic elements for electrical circuits; resistors, memristors, capacitors and inductors. The mathematical representation for all four elements can be found in Section 2.4. As the focus of this paper is mainly on memristor-capacitor circuits, this section will give more insight in what these two elements actually are and show their role within an electrical circuit.

A capacitor is a basic algebraic element that stores electrical energy in an electric field. The capacitor is also know as a condensor or condensator. The effect of a capacitor is known as the capacitance. In order to understand the capacitor, an analogy of a water tank can be used. In this analogy, the flow of electric current is compared to the flow of water out of a tank. The water tank represents the capacitor and it will be charged by a battery, a pump, to fill it up. The amount of water in the tank represents the amount of charge in the capacitor. The height of the water above a certain reference point can be compared to the voltage to which the capacitor is pumped up by the battery. Finally, the capacitance is represented by the area of the tank. Note that a tall skinny tank can contain the same amount of water as a shallow flat tank, but it holds it at a higher pressure. This can be compared to a capacitor; a tall, skinny capacitor has a higher voltage and lower capacitance than a shallow, flat capacitor. Another thing to notice is, when a capacitor is charged by a battery, then one electrode of the capacitor will be positively charged, while the other one will correspondingly charged negatively which can be seen in Figure 5.

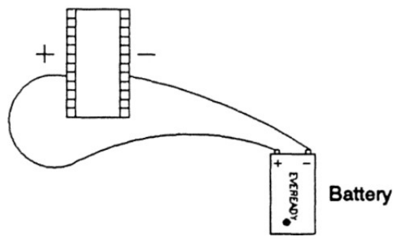

Figure 5: A Capacitor Charged by a Battery.

The water flow analogy can also be used to understand a resistor. Note that this knowledge is needed in order to understand how memristors work since a memristor is a resistor with memory. When the valve in Figure 6 is opened, the water inside runs out. Note that the valve is a resistor and a switch (Kaiser 2012). If the resistance is high and hence the opening of the valve is small, the water runs slowly while when the resistance is low, the water runs more freely. To conclude, the capacitor is the tank, the battery is the pump and the resistor is given by the valve. This whole analogy can be found in Figure 6 (Brophy and Schwartz 1998).

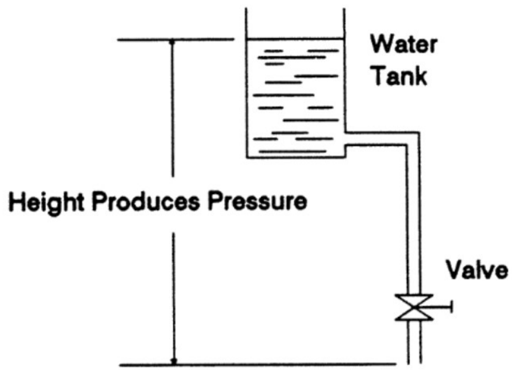

Figure 6: The Water Flow Analogy depicted for the capacitor, resistor and memristor.

The name memristor is a contraction of memory resistor. As its name already suggests, it can be regarded as a class of two-terminal resistive device. It shows the behavior of a non-linear resistor and it shows volatile or non-volatile memory properties. Volatile memory is memory that is lost when power is cut off, while non-volatile memory remains stored in case the power is cut off (Pham, Volos, and Kapitaniak 2021). Focussing on the water flow analogy, also a memristor can be represented. The representation of the memristor is given by a sand filter, which is used in water-purification plants. When contaminated water flows through the sand filter, sediment clogs the pores of the filter. By this fact, the resistance gradually increases. When this process is reversed, the sediment is flushed out and hence the resistance is reduced. Note that the process described is different than the valve described above since for the sand filter, the direction of the flow controls the state of the device. At any given instant the resistance of the sand filter is the same in both directions. At that point, the memristor, too, is symmetric (Hayes 2011a). Although the water flow analogy gives some intuition in how a component works, a more mathematical framework is needed. This mathematical framework can be found in the next section.

2.4 Basic algebraic elements

For a given electrical component, the corresponding relation between current and voltage can be given, this is sometimes referred to as a constitutive relation. These equations can be combined and rewritten in order to obtain mathematical models of these basic algebraic elements. The four basic algebraic elements are the resistor, the capacitor, the inductor and the memristor. These elements can be described by Constitutive Relations, abbreviated as CR, that take the form of a mathematical model.

First, the equations for the four basic algebraic elements will be explored. Let us start with some notation: v_k represents the voltage across a component on a certain branch k, i_k denotes the current through the component on the branch k. Furthermore, the charge on the branch k is denoted by q_k, while the flux on a branch k is given by the symbol φ_k. When the time derivative of the charge is taken, the current through the component is obtained, i.e. $\frac{dq_k(t)}{dt} = i_k(t)$, while the time derivative of the flux is denoted by the voltage across a component, so $\frac{d\varphi_k(t)}{dt} = v_k(t)$ (W. K. Chen 2004).

Let us now explore the equations for the components. First, start with the resistor. For the resistor, the constitutive relation is given by Ohm's Law, i.e.,

$$v_k(t) = Ri_k(t).$$

The constant R given in Ohm (Ω) denotes the resistance. Note that this is the explicit form. In implicit form, one can write

$$f_R(v_k(t), i_k(t)) = v_k(t) - Ri_k(t) = 0.$$

Next is the capacitor. The constitutive relation in explicit form is given by

$$q_k(t) = Cv_k(t),$$

where the constant C is called the capacitance. This equation can be rewritten in terms of $v_k(t)$ and $i_k(t)$. Observe that by taking the derivative of both sides of this equation yields

$$\begin{aligned}\frac{dq_k(t)}{dt} &= \frac{d(Cv_k(t))}{dt}\\ &= C\frac{d(v_k(t))}{dt}.\end{aligned}$$

Note that $\frac{dq_k(t)}{dt} = i_k(t)$ and thus

$$i_k(t) = C\frac{d(v_k(t))}{dt}.$$

The next basic element is the inductor. The equation for the inductor reads,

$$\varphi_k(t) = Li_k(t).$$

The goal is again to rewrite this in a form consisting of $v_k(t)$ and $i_k(t)$. By taking the derivative with respect to time, one obtains

$$\begin{aligned}\frac{d\varphi(t)}{dt} &= \frac{d(Li(t))}{dt}\\ &= L\frac{di(t))}{dt}.\end{aligned}$$

Note that $\frac{d\varphi_k(t)}{dt} = v_k(t)$ and hence,

$$v_k(t) = L\frac{d(i_k(t))}{dt}.$$

The constant L is called the inductance (Corinto, Forti, and Chua 2020).

The final basic algebraic element that is taken into consideration is the memristor. The equation is given as,

$$\varphi_k(t) = Mq_k(t).$$

After deriving this in a similar fashion as the resistor, the following is obtained

$$v_k(t) = Mi_k(t).$$

The constant M is named the memristance (Hayes 2011b). From this, it can be seen that linear memristor is just a resistor where the resistance R is replaced by the memristance M. Therefore, memristors are always regarded as nonlinear elements.

In order to describe the nonlinear relation between the flux and the charge for the memristor, either q can be given by some nonlinear function g which depends on φ or φ is denoted as some nonlinear function h which depends on q. These nonlinear functions are respectively called flux-controlled and charge-controlled representations. These representations are given by:

$$\begin{cases} q = g(\varphi) \\ \varphi = h(q) \end{cases}$$

When one describes a memristor, only one of these functions is needed. The nonlinear relation chosen throughout this paper is $q = g(\varphi)$. In the next chapter the mathematical modelling together with partially solving the corresponding equations analytically will be shown.

3 Mathematical modeling

Within this section the network consisting of at least one memristor and at least one capacitor is modelled. The parallel and series circuits of a memristor and a capacitor are foundational building blocks for realistic memristive circuits and hence the composite characteristic of memristor-capacitor, abbreviated as MC, circuits have been studied due to their wide applications (L. Chen et al. 2019). After the mathematical modeling has been done, the dynamics of the network can be analyzed. The main question arises within learning via inputs. How does collected memory change due to external input, or phrased more mathematically, how does the state $\mathbf{x}$ change when the input $\mathbf{u}$ changes?

3.1 Derivation of equations

In this section, the main goal is to model an electrical circuit consisting of memristors and capacitors. In the end, something of the form $\dot{x} = f(x, u)$ should be obtained where x represents the state and u represents the input. Note that x will also contain the state of the memristor and thus represents memory. The input u is described by current and voltage. Recall that by Kirchhoff's Laws we have the following two equations

$$\begin{aligned} \mathbf{j} &= D\mathbf{i} && \text{Kirchhoff's Current Law (KCL) and} \\ D^T\mathbf{p} &= \mathbf{v} && \text{Kirchhoff's Voltage Law (KVL)} \end{aligned}$$

Furthermore, remember from Section 2.2, the meaning for the following symbols was given. Notice that n represents the number of nodes and that m represents the number of branches.

Symbol	Meaning
$\mathbf{j}$	n incoming currents
$\mathbf{i}$	m measured currents
$\mathbf{p}$	n measured potentials
$\mathbf{v}$	m measured voltages
D	Incidence matrix

From Section 2.4, for the memristor, the following memristor law chosen is $q = g(\varphi)$. Furthermore note that the derivative of the charge is equal to the current and the derivative of the flux is equal to the voltage. Hence,

$$\begin{cases} \frac{dq}{dt} = i \\ \frac{d\varphi}{dt} = v. \end{cases}$$

Before doing the modeling, an important remark has to be made. We consider a very specific class of circuits. We study circuits with at least one memristor and a grounded capacitor attached to each node. An example of such a circuit can be found in Figure 7.

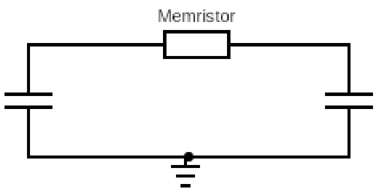

Figure 7: Circuit with one memristor and grounded capacitors attached to each node.

For these type of circuits, the following equation for the capacitor is derived,

$$C\frac{dp}{dt} = -j, \quad \text{with} \quad C \quad \text{representing the capacitance.}$$

This equation holds since we consider circuits with grounded capacitors.

Note that by KCL we have, $j = Di$. Furthermore, by the equation for the capacitor we have $-C\frac{dp}{dt} = j$ and it is given that $\frac{dq}{dt} = i$. By substitution the following can be obtained,

$$C\frac{dp}{dt} = -D\frac{dq}{dt}.$$

By the Chain Rule, $\frac{dq}{dt}$ can be split up in the multiplication of two derivatives, $\frac{\partial q}{\partial \varphi}\frac{\partial \varphi}{\partial t}$. It is already known that $\frac{\partial \varphi}{\partial t} = v = D^T p$ by Kirchhoff's Voltage law. Therefore,

$$\begin{aligned} C\frac{dp}{dt} &= -D\frac{\partial q}{\partial \varphi}\frac{\partial \varphi}{\partial t} \\ &= -D\frac{\partial q}{\partial \varphi}D^T p \implies \\ \frac{dp}{dt} &= -C^{-1}D\frac{\partial g(\varphi)}{\partial \varphi}D^T p. \end{aligned}$$

This is just a nonlinear expression since the representation for the memristor is given by $q = g(\varphi)$. Hence, this gives something of the form $\dot{x} = h(x)$ and thus, the system can be represented in the following manner,

$$\begin{aligned} x &= \begin{bmatrix} p \\ \varphi \end{bmatrix} \\ \dot{x} &= \begin{bmatrix} \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -C^{-1}D\frac{\partial g(\varphi)}{\partial t}D^T & 0 \\ D^T & 0 \end{bmatrix}\begin{bmatrix} p \\ \varphi \end{bmatrix}. \end{aligned}$$

Note that the equation above is a derivation for a circuit without an input u. Next, let us have a look at an circuit with an input u. An example of such an input is when a current source is added to a circuit. An example of such a circuit can be found in Figure 8.

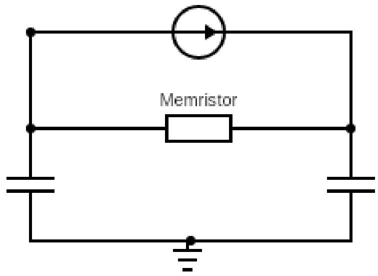

Figure 8: Circuit with one memristor, one current source and grounded capacitors attached to each node.

The next model that will be considered is a model where an external input u is added. In the end, a system of the form $\dot{x} = f(x, u)$ can be derived. Note that we have by Kirchhoff's Current Law that $j = Di$. When an input is added, the incidence matrix can be split up in two parts, namely $D = [D_m \quad D_s]$. D_m collects the memristor branches, whereas D_s collects the branches on which sources are located. Furthermore, the measured currents, $\mathbf{i}$ can be split up in two parts. First the measured currents i_m are associated to the branches on which memristors are located, whereas i_s are the currents associated to the branches on which sources are located. Hence the KCL can therefore be defined as $j = D_m i_m + D_s i_s$. Furthermore, the measured voltages, $\mathbf{v}$ can be split up in two parts as well. First, the voltages associated to the branches on which memristors are located are given by v_m. The voltages associated to the branches on which the sources are located are given by v_s. Hence the Kirchhoff's Voltage Law, $\exists\, p$ s.t. $v = D^T p$, can be rewritten as,

$$\begin{bmatrix} v_m \\ v_s \end{bmatrix} = \begin{bmatrix} D_m^T \\ D_s^T \end{bmatrix} p.$$

for some p. A similar reasoning as the derivation for the mathematical model without external input can be implemented in order to derive the mathematical model for the electrical circuit with external input. Note that we can use the following relations; $j = D_m i_m + D_s i_s$ by Kirchhoff's Current Law and $-C\frac{dp}{dt} = j$ by the equation derived for the capacitor and finally $\frac{dq_m}{dt} = i_m$.
Note that after substituting the three last-mentioned relations in the KCL, $j = D_m i_m + D_s i_s$, leads to

$$C\frac{dp}{dt} = -D_m\frac{dq_m}{dt} - D_s\frac{dq_s}{dt}.$$

By the chain rule, $\frac{dq_m}{dt}$ can be split up as $\frac{dq_m}{dt} = \frac{\partial q_m}{\partial \varphi_m}\frac{\partial \varphi_m}{\partial t}$.
Note that from Section 2.4 $i_s(t) = \frac{dq_s}{dt}$.

Since $\frac{\partial \varphi_m}{\partial t} = v_m = D_m^T p$, we can obtain our final equation. Namely,

$$\begin{aligned} C\frac{dp}{dt} &= -D_m \frac{\partial q_m}{\partial \varphi_m}\frac{\partial \varphi_m}{\partial t} - D_s i_s \\ &= -D_m \frac{\partial q_m}{\partial \varphi_m} D_m^T p - D_s i_s. \end{aligned}$$

By assuming that C is invertible and using the memristor law,

$$\dot{p} = -C^{-1}D_m \frac{\partial g(\varphi_m)}{\partial \varphi_m} D_m^T p - C^{-1}D_s i_s.$$

Note that this gives something of the form $\dot{x} = f(x, u)$, where the system can be represented in the following manner where the input u therefore is chosen as i_s,

$$\begin{aligned} x &= \begin{bmatrix} p \\ \varphi \end{bmatrix} \\ \dot{x} &= \begin{bmatrix} \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -C^{-1}D_m \frac{\partial g(\varphi_m)}{\partial t} D_m^T & 0 \\ D_m^T & 0 \end{bmatrix} \begin{bmatrix} p \\ \varphi \end{bmatrix} + \begin{bmatrix} -C^{-1}D_s \\ 0 \end{bmatrix} i_s. \end{aligned}$$

3.2 Analytical solutions

In the previous section, two models were obtained, namely $\dot{x} = h(x)$ and $\dot{x} = f(x, u)$. In this section, it is shown that after partially solving these two models analytically, a reduction in state-space dimension will be obtained. These models only involve the state φ_m. After solving the model obtained in the previous section partially analytically, these reduced models will be analyzed.

From the derivation of equations where no external inputs are considered, $\dot{x} = h(x)$, the following analytical solution can be computed.
Note that the equation derived in the previous section is given as,

$$C\frac{dp}{dt} = -D\frac{dq}{dt}.$$

By integrating this expression over an interval from a starting time $\tau = 0$ till a certain time $\tau = t$, the following can be obtained

$$\begin{gathered} \int_0^t C\frac{dp}{dt}dt = \int_0^t -D\frac{dq}{dt}dt \\ C(p(t) - p(0)) = -D(q(t) - q(0)) \end{gathered}$$

By the memristor law, we have that $q = g(\varphi)$, so

$$\begin{aligned} C(p(t) - p(0)) &= -D(q(t) - q(0)) \\ &= -D(g(\varphi(t)) - g(\varphi(0))). \end{aligned}$$

By assuming that C is invertible, we obtain

$$\begin{aligned} p(t) - p(0) = -C^{-1}D(g(\varphi(t)) - g(\varphi(0))) &\implies \\ p(t) = p(0) - C^{-1}D(g(\varphi(t)) - g(\varphi(0))). \end{aligned}$$

Recall that $\frac{d\varphi}{dt} = D^T p$, which leads to

$$\frac{d\varphi}{dt} = D^T p(0) - D^T C^{-1} D(g(\varphi(t)) - g(\varphi(0))),$$

which is the partially analytical solution to our ordinary differential equation with no external input. This reduces to a model that only involves the state φ.

Note that the analytical solution of $\dot{x} = f(x,u)$, for the electrical circuit with external inputs can be computed in a very similar fashion. Although there is no physical charge for the branches associated to the current source, we use the property $\frac{dq_s}{dt} = i_s$ as derived above in order to simplify our equation. In this case, we start from,

$$C\frac{dp}{dt} = -D_m\frac{dq_m}{dt} - D_s\frac{dq_s}{dt}$$

by Kirchhoff's Current Law. By taking the integrals from a time $\tau = 0$ till a certain time $\tau = t$, we have,

$$C\int_0^t \frac{dp}{dt}dt = -D_m\int_0^t \frac{dq_m}{dt}\,dt - D_s\int_0^t \frac{dq_s}{dt}\,dt,$$

which leads to,

$$C(p(t) - p(0)) = -D_m(q_m(t) - q_m(0)) - D_s(q_s(t) - q_s(0))$$

After again using the memristor law $q = g(\varphi)$,

$$C(p(t) - p(0)) = -D_m(g(\varphi(t)) - g(\varphi(0))) - D_s(q_s(t) - q_s(0)).$$

Again, by assuming C is invertible,

$$\begin{aligned} p(t) - p(0) = -C^{-1}D_m(g(\varphi(t)) - g(\varphi(0))) - C^{-1}D_s(q_s(t) - q_s(0)) \implies \\ p(t) = p(0) - C^{-1}D_m(g(\varphi(t)) - g(\varphi(0))) - C^{-1}D_s(q_s(t) - q_s(0)). \end{aligned}$$

By substituting the Kirchhoff's Voltage Law, $\frac{d\varphi_m}{dt} = D_m^T p$,

$$\frac{d\varphi_m}{dt} = D_m^T p(0) - D_m^T C^{-1} D_m(g(\varphi_m(t)) - g(\varphi_m(0))) - D_m^T C^{-1} D_s(q_s(t) - q_s(0)).$$

Note that in this case, we obtain indeed the solution in the form $\dot{x} = f(x,u)$ with the state $x = \varphi_m$ and the input $u = q_s$.

After deriving the analytical solution, the simpler model only involves the state φ_m. After obtaining these easier models, numerical modeling will be done to study the equation $\dot{\varphi}_m = f(\varphi_m, q_s)$. In the next section, first conditions will be described to the different parts of the function $f(\varphi_m, q_s)$, next the function will be numerically implemented with the use of build-in functions of MATLAB and finally analysis of the results will be done.

4 Numerical modeling

In order to study the behavior of the electrical circuits with memristors and capacitors, numerical methods can be used. From Section 3.2, it can be seen that the ordinary differential equation that will be studied is the following:

$$\frac{d\varphi_m}{dt} = D_m^T p(0) - D_m^T C^{-1} D_m(g(\varphi_m(t)) - g(\varphi_m(0))) - D_m^T C^{-1} D_s(q_s(t) - q_s(0))$$

Therefore, we are interested in the change of flux over the memristor over time. Note that this change depends on the incidence matrices, namely the one of the memristor D_m and the one of the current sources D_s. Furthermore it depends on the constitutive relation for the memristor which will be discussed in this section. Furthermore, the matrix C represents the capacitances and finally q_s represents the charge across the current source. In this section, first the choice for the variables on which the ODE depends will be discussed, next a numerical method will be applied and finally a mathematical foundation for the obtained results will be shown.

4.1 Variables

Capacitance Matrix C
Note that the capacitance matrix is a diagonal matrix containing the capacitances on the diagonal. Each capacitance is measured on a conductor. For example, for the first capacitor, we have a capacitance α and this is denoted on position (1,1) in the capacitance matrix. The second conductor had a capacitance β and this is denoted on position (2,2) in the capacitance matrix. This procedure is repeated till all capacitances are contained in the matrix C. Note that for simplicity within this paper, all capacitances are set to 1 and hence C is equal to the identity matrix.

Memristor Constitutive Relation
In order to study the behavior of the memristor, the function of the memristor has to be determined. For the memristor we want a certain function that has two properties. First of all this function must be nonlinear. Furthermore, this function must be monotone meaning that it is either decreasing or increasing (Abbott 2012), so the function must look like the function in Figure 9.

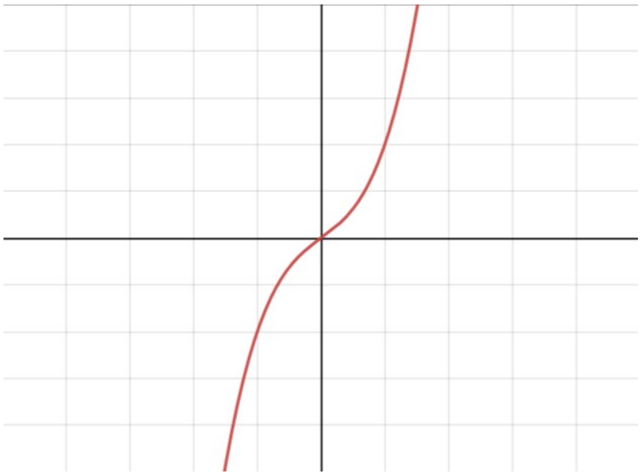

Figure 9: The function chosen for the memristor should be a function that looks like this figure.

The function chosen within the code is

$$g_k(\varphi_k) = \varphi_k + \varphi_k^3.$$

If the electrical circuit consists of more than one memristor, the function $g_k(\varphi_k)$ depends on different values for φ_k, since each φ_k denotes the flux of a certain memristor and hence $g(\varphi)$ can be denoted as,

$$g(\varphi) = \begin{bmatrix} g_1(\varphi_1) \\ g_2(\varphi_2) \\ \vdots \\ g_l(\varphi_l) \end{bmatrix},$$

where l denotes the total number of branches containing a memristor.

Function of the charge

The function of the charge across the current source is a step function. It is 0 till a certain time τ and will increase to a certain height A which represents the charge when the time τ is reached. This can be mathematically rewritten as,

$$\begin{cases} q_s(t) = 0 & t < \tau \\ q_s(t) = A & t \geq \tau. \end{cases}$$

4.2 Numerical implementation

After denoting the properties of the capacitance matrix and the two functions, three examples will be numerically implemented. The first circuit is depicted in the following representation:

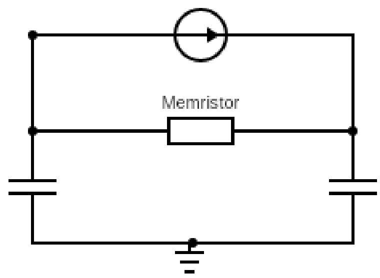

Figure 10: The first circuit containing two capacitors, one current source and one memristor where the direction is clockwise.

Note that for this circuit we have the following matrices:

$$D_m = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad D_s = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Furthermore, we set $\tau = 5$ and $A = 1$ for the step function. Our initial conditions are $\varphi_m(0) = 0$ and $q_s(0) = 0$. MATLAB code used can be found in Section 8. From this code, the solution to the dynamics is given in Figure 11. Note that from this figure the voltage across the memristor, $\dot{\varphi}_m$, and the solution, denoted by the flux, φ_m is obtained,

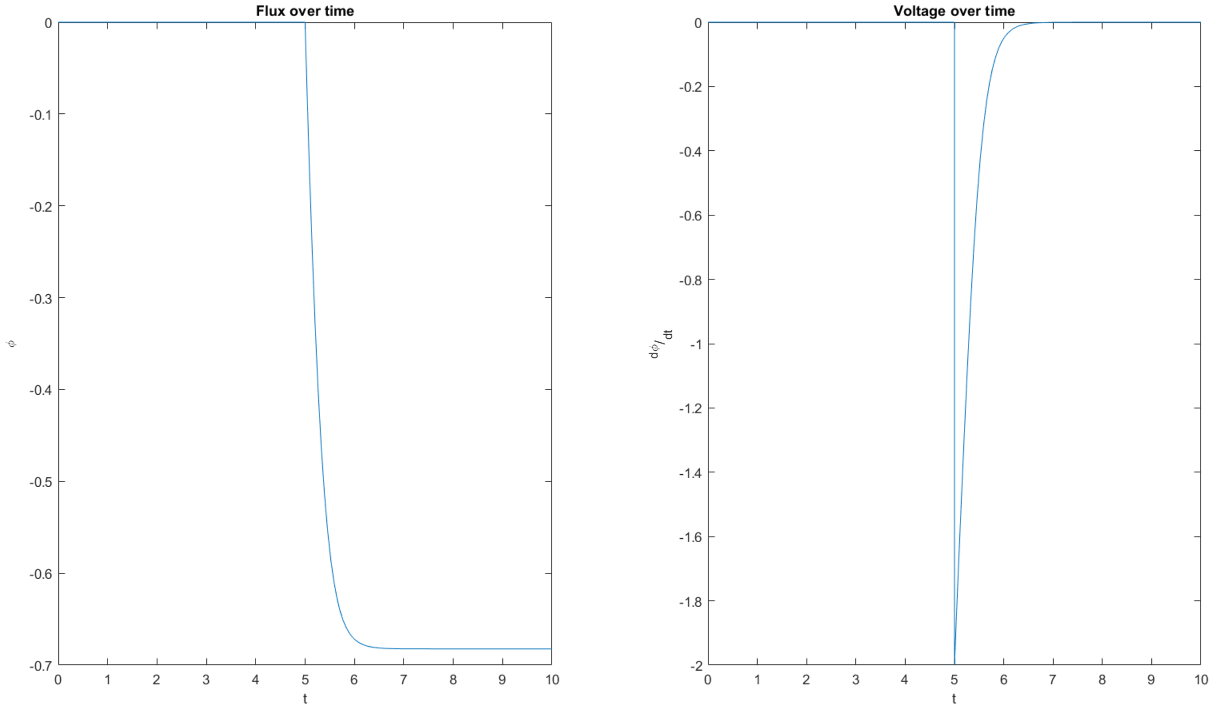

Figure 11: The flux and voltage of one memristor, two capacitors and one current source.

From this figure it can be seen that when the time is equal to 5, the flux is not constant anymore. This can be explained from the fact that, when $t \geq 5$, $q_s = 1$ instead of 0. Note that this means that the charge, q_s, was zero all the time and hence we first considered no input. But when we reached time $\tau = 5$, the input causes a decrease of the flux which can be explained from the way the input matrix B is defined. Note that we have a system of the form $\dot{\varphi}_m = Ag(\varphi_m)+Bq_s$. From Section 3, the input matrix B is given as $B = -D_m^T C^{-1} D_s$, with D_m, D_s and C as above. From this example, the input matrix B reads,

$$\begin{aligned} B &= -D_m^T C^{-1} D_s \\ &= -2. \end{aligned}$$

Since $B = -2$, when there is an positive input q_s, Bq_s will be negative. This can be seen in the figure as a decrease of the flux. Note that the voltage is the derivative of the flux. This can be seen from the figure as well. Till time $t = 5$, there is no change and hence the derivative is zero. When $t \geq 5$, first the decrease of flux is increasing which can be seen by the fact that the voltage is negative and decreasing. Next, the decrease flux is decreasing, from the figure, the voltage is still negative but increasing. Around $t = 7$, the flux is stable around $\varphi_m = 0.68$ and hence there is no change and thus the derivative is equal to 0.

The second circuit has two memristors instead of one and contains three capacitors instead of two.

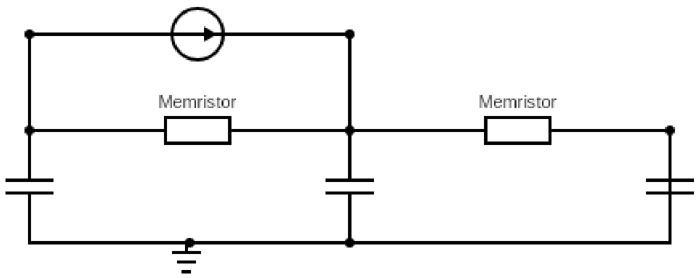

Figure 12: The second circuit containing three capacitors, one current source and two memristors where the direction is clockwise.

Note that for this circuit we have the following matrices:

$$D_m = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad D_s = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, we set $\tau = 5$ and $A = 1$ for the input function. The initial conditions are again defined as $\varphi_m(0) = 0$ and $q_s(0) = 0$. This gives the following result.

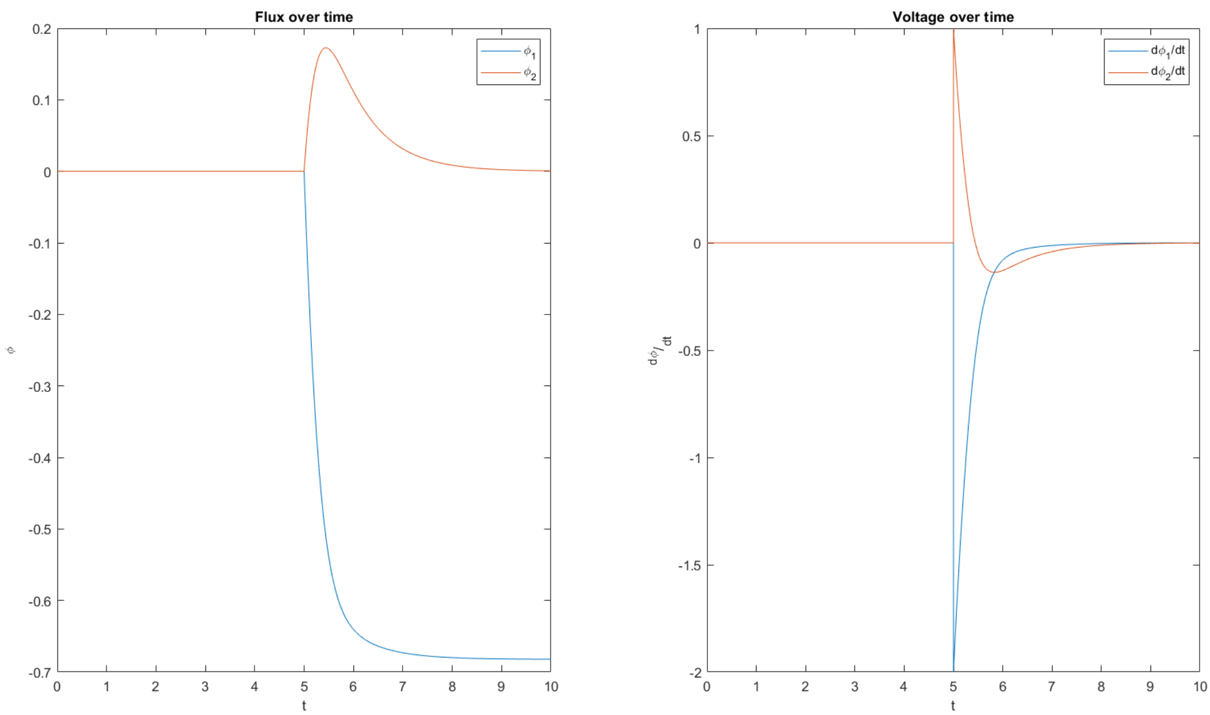

Figure 13: The flux and voltage of two memristors, three capacitors and one current source.

Again, after time $t = 5$, the flux is not constant anymore. Let us again consider the matrix B. In this case,

$$\begin{aligned} B &= -D_m^T C^{-1} D_s \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \end{aligned}$$

This matrix B explains the behavior of the flux in the figure above. When the time $t = 5$ is reached, the input is taken into account. In this case,

$$\begin{aligned} Bq_s &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} q_s \\ &= \begin{bmatrix} -2q_s \\ 1q_s \end{bmatrix}. \end{aligned}$$

For this second memristor, the input is given as q_s. The input is positive if q_s is positive. This can be seen in the figure as an increase of flux when the input is considered. The behavior of the voltage can be explained in a similar fashion as in the previous example.

The third and final circuit consists of three memristors and contains four capacitors. Still, there is one current source within this circuit.

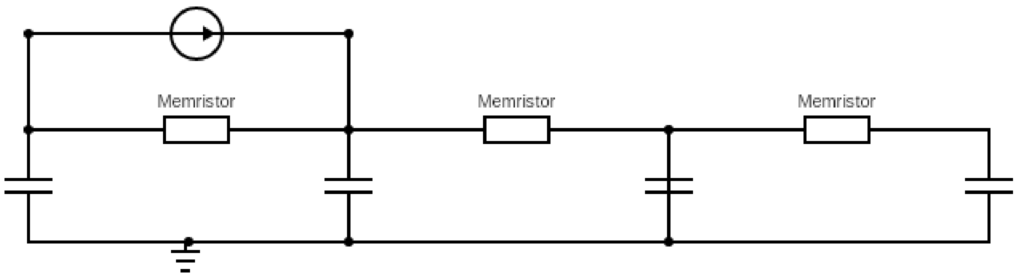

Figure 14: The third circuit containing four capacitors, one current source and three memristors where the direction is clockwise.

Note that for this circuit we have the following matrices:

$$D_m = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad D_s = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the input function, the chosen value for τ is five and for A it is one. The initial conditions are $\varphi_m(0) = 0$ and $q_s(0) = 0$. This gives the following result.

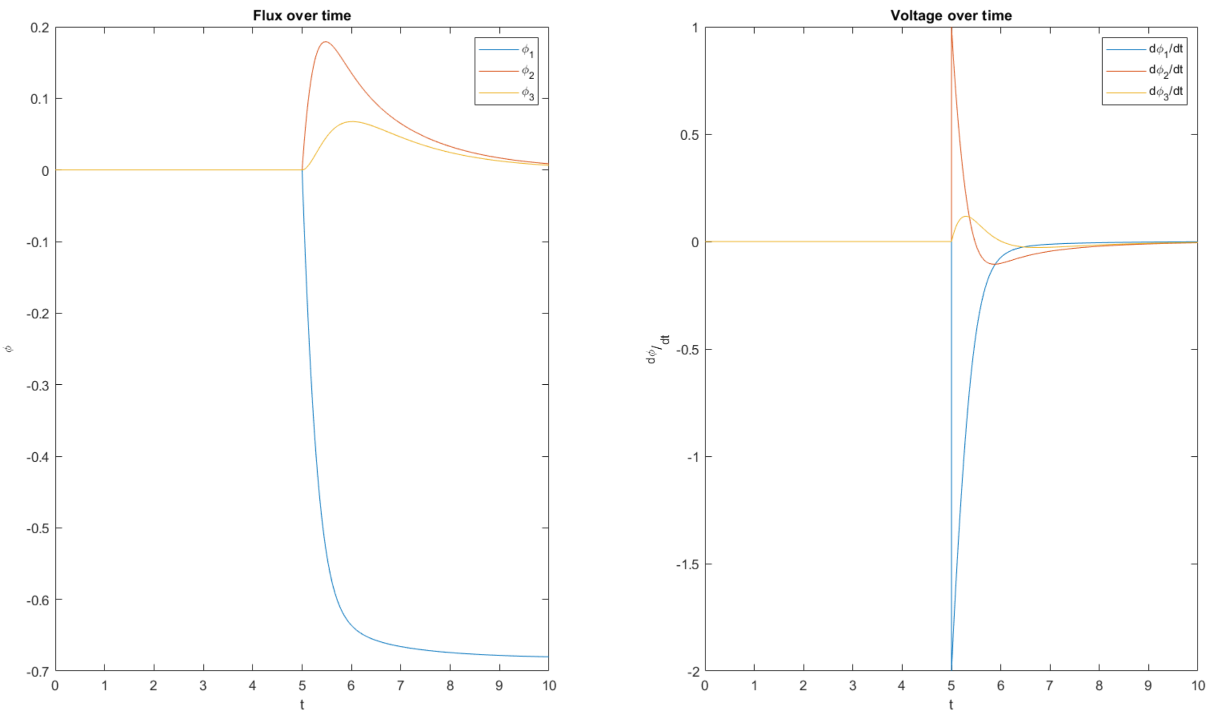

Figure 15: The flux and voltage of three memristors, four capacitors and one current source.

Note that in the third case, the input matrix B is given as,

$$\begin{aligned} B &= -D_m^T C^{-1} D_s \\ &= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

In order to study the input, we look again at Bq_s, which is in this case,

$$\begin{aligned} Bq_s &= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} q_s \\ &= \begin{bmatrix} -2q_s \\ 1q_s \\ 0 \end{bmatrix}. \end{aligned}$$

For this third memristor, there is on immediate effect of the input. From the figure it can be seen that the flux is changing a little bit. This can be explained from the fact, when $t \geq 5$, also the state matrix A plays a role. Note that the current source is only attached to the first two nodes. Hence, the input matrix B does not affect this graph.

If more memristors will be added in the way that is done in Figure 12 and 14, the new fluxes all behave in a similar fashion as φ_3 as can be seen in Figure 15. When new sources are added the fluxes will change. In this chapter, numerical results and the explanation of these results were discussed. The next chapter will elaborate on the explanation and analysis of the results.

5 Analysis

In this chapter, first the general notion of stability for linear systems will be given. After defining stability for linear systems it will be expanded to defining stability for nonlinear systems. After presenting the definition the stability, it will be applied to the examples in the previous section. Finally, the impact of the change of the input q_s on the state φ_m will be studied.

5.1 Stability

The first topic that will be studied is the stability of the nonlinear system. If a system is stable, a small perturbation does not have a too big influence on the system. Stability is the first requirement for control systems. Before being able to define the theorem for the stability, some definitions will be given.
First of all, note that we consider systems that are defined in the following way,

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

where x represents the state, u the input and y the output. The matrix A is called the system matrix, B is called the input matrix, C is the output matrix and finally D is the feedthrough matrix. The spectrum of A, denoted by $\sigma(A)$ consists of all eigenvalues of A, where each eigenvalue is denoted by λ_i, with $i = 1, ..., n$, meaning that there are in total n eigenvalues. The following notion of stability for linear systems can be defined (Besselink 2020).

Definition 5.1. A linear system is stable if for all eigenvalues in the spectrum of A, the real part of these eigenvalues is located in the left-hand side of the complex plane, i.e. $\forall\ \lambda_i \in \sigma(A)$, $\mathrm{Re}(\lambda_i) \leq 0$ for $i = 1, ..., n$.

Note that the equation studied is the following,

$$\frac{d\varphi_m}{dt} = D_m^T p(0) - D_m^T C^{-1} D_m(g(\varphi_m(t)) - g(\varphi_m(0))) - D_m^T C^{-1} D_s(q_s(t) - q_s(0)).$$

This equation is nonlinear since the ordinary differential equation depends on the nonlinear function $g(\varphi_m)$. Hence, in order to say something about stability, a linearization of the system around equilibrium points should be considered.

In order to find equilibrium points, the equation $\dot{\bar{x}} = f(\bar{x}, \bar{u}) = (0, 0)$ needs to be studied (Kulakowski, Gardner, and Shearer 2007). First note the case where no input is considered, so $\bar{u} = 0$. In this case the equation that needs to be solved is $\dot{\bar{x}} = f(\bar{x}, 0) = 0$. The equation considered is,

$$\frac{d\varphi_m}{dt} = D_m^T p(0) - D_m^T C^{-1} D_m(g(\varphi_m(t)) - g(\varphi_m(0))) - D_m^T C^{-1} D_s(q_s(t) - q_s(0)).$$

When no input is considered, $u = q_s(t) = 0$ and hence the equation can be simplified to

$$\frac{d\varphi_m}{dt} = D_m^T p(0) - D_m^T C^{-1} D_m(g(\varphi_m(t)) - g(\varphi_m(0))).$$

In order to find the equilibrium points, we need to solve

$$D_m^T p(0) - D_m^T C^{-1} D_m(g(\overline{\varphi_m}) - g(\varphi_m(0))) = 0.$$

Note that this is obtained if

$$D_m^T p(0) = D_m^T C^{-1} D_m (g(\overline{\varphi_m}) - g(\varphi_m(0))).$$

In order to find the equilibrium points, first the initial conditions need to be defined. For $p(0)$, the condition chosen is $p(0) = 0$. Also for $\varphi_m(0)$, the initial condition is equal to zero. Now, the build-in function $fzero$ of MATLAB can be used. This can be found in Section 8. The equilibrium point found after using this function is the point $f(\bar{x}, \bar{u}) = (0, 0)$.

After determining the equilibrium point, the following definition for a nonlinear system will be applied (Nise 2020).

Definition 5.2. The equilibrium point $(\bar{x}, \bar{u})$ of a nonlinear system is asymptotically stable if for all eigenvalues in the spectrum of $\overline{A}$, the real part of these eigenvalues is located in the left-hand side of the complex plane, i.e. $\forall\ \lambda_i \in \sigma(\overline{A})$, $\mathrm{Re}(\lambda_i) < 0$ for $i = 1, ..., n$.

Note that $\overline{A}$ denotes the linearization of the system matrix around the equilibrium point and hence it is given as (Besselink 2020),

$$\begin{aligned} \overline{A} &= \frac{\partial f}{\partial x}(x, u)|_{(\bar{x}, \bar{u})} \\ &= \frac{\partial f}{\partial x}(x, u)|_{(0,0)}. \end{aligned}$$

With the initial conditions given, $p(0) = 0$ and $\varphi_m(0) = 0$, and the fact that the system is chosen to be without input, the following simplified equation can be obtained,

$$f(\varphi_m, 0) = -D_m^T C^{-1} D_m g(\varphi_m).$$

From this matrix $\overline{A}$ can be defined,

$$\begin{aligned} \overline{A} &= \frac{\partial(-D_m^T C^{-1} D_m g(\varphi_m))}{\partial \varphi_m}|_{\varphi_m = 0} \\ &= -D_m^T C^{-1} D_m \frac{\partial g(\varphi_m)}{\partial \varphi_m}|_{\varphi_m = 0}. \end{aligned}$$

Note that from Section 4.1, if the electrical circuit consists of at least one memristor, the following function for $g(\varphi_m(t))$ is defined

$$g(\varphi_m(t)) = \begin{bmatrix} g_1(\varphi_1(t)) \\ g_2(\varphi_2(t)) \\ \vdots \\ g_l(\varphi_l(t)) \end{bmatrix} = \begin{bmatrix} \varphi_1(t) + \varphi_1(t)^3 \\ \varphi_2(t) + \varphi_2(t)^3 \\ \vdots \\ \varphi_l(t) + \varphi_l(t)^3 \end{bmatrix}.$$

Where l denotes the total number of branches containing memristors. By taking the partial derivatives, we need to consider the Jacobian and hence obtain,

$$\frac{\partial g(\varphi_m)}{\partial \varphi_m} = \begin{bmatrix} 1 + 3\varphi_1(t)^2 & 0 & \dots & 0 \\ 0 & 1 + 3\varphi_2(t)^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 + 3\varphi_l(t)^2 \end{bmatrix}$$

If the derivative is now evaluated at the equilibrium point $\overline{\varphi_m} = 0$,

$$\frac{\partial g(\varphi_m)}{\partial \varphi_m}|_{\varphi_m=0} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I$$

Now, the matrix $\overline{A}$ can be denoted in its final terms,

$$\begin{aligned} \overline{A} &= -D_m^T C^{-1} D_m \frac{\partial g(\varphi_m)}{\partial \varphi_m}|_{\varphi_m=0} \\ &= -D_m^T C^{-1} D_m I \\ &= -D_m^T C^{-1} D_m. \end{aligned}$$

5.2 Stability in numerical examples

From the previous section, it can be concluded that a nonlinear system is asymptotically stable around an equilibrium point if for all eigenvalues in the spectrum of $\overline{A}$, the real part of these eigenvalues is located in the left-hand side of the complex plane. From the previous section it is found that $\overline{A} = -D_m^T C^{-1} D_m$. Note that for the first example, the following incidence matrix for the memristor and the following capacitance matrix is given

$$D_m = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the following is obtained for $\overline{A}$,

$$\begin{aligned} \overline{A} &= -D_m^T C^{-1} D_m \\ &= -\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= -2. \end{aligned}$$

The only eigenvalue of this matrix is $\lambda = -2$, since the real part is on the left-hand side of the complex plane, the solution is asymptotically stable around the equilibrium point $(0, 0)$.

Next, consider the second example with two memristors. The following incidence and capacitance matrix are given:

$$D_m = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case, for $\overline{A}$, the following can be derived,

$$\begin{aligned} \overline{A} &= -D_m^T C^{-1} D_m \\ &= -\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}. \end{aligned}$$

In order to study the eigenvalues, we want to solve the equation $\det(\bar{A}-\lambda I)=0$. After rewriting,

$$\begin{aligned}\det(\bar{A}-\lambda I) &= \det\left(\begin{bmatrix}-2-\lambda & 1\\ 1 & -2-\lambda\end{bmatrix}\right)\\ &= \lambda^2+4\lambda+3\\ &= (\lambda+1)(\lambda+3).\end{aligned}$$

From $\det(\bar{A}-\lambda I)=(\lambda+1)(\lambda+3)=0$, it can be concluded that the eigenvalues of $\overline{A}$ are $\lambda_1=-1$ and $\lambda_2=-3$. Since these are both on the left-hand side, also this system is asymptotically stable around the equilibrium point $(0,0)$.

The final example with three memristors has the following incidence and capacitance matrices:

$$D_m=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix} \qquad C=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

For $\overline{A}$, we have,

$$\begin{aligned}\overline{A} &= -D_m^TC^{-1}D_m\\ &= \begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}\\ &= \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}.\end{aligned}$$

In order to find the eigenvalues,the equation $\det(\bar{A}-\lambda I)=0$ can be solved. Another way to find the eigenvalues is to use the build-in function of MATLAB called eig(A) which returns the eigenvalues of a certain matrix A. In particular, we compute $\text{eig}(-D_m^TC^{-1}D_m)$ which gives,

$$\text{eig}\left(\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\right).$$

For this equation, three eigenvalues are found. Namely $\lambda_1=-0.5858$, $\lambda_2=-2$ and finally, $\lambda_3=-3.4142$. As all these eigenvalues are in the left-hand side of the complex plane, also this system is asymptotically stable around the equilibrium point $(0,0)$.

5.3 Change of charge

The next topic that will be studied is the influence of the change over the current source. We are interested in the change of flux over the memristor(s) when this charge changes, i.e. when q_s changes, what is the influence on φ_m? Note that the charge is chosen as the input in the system. Since there is a system of the form $\dot{\varphi}_m=Ag(\varphi_m)+Bq_s$, the change of the solution can be explained by two matrices. First, due to the state matrix A, various memristors will interact. Furthermore, the way B is defined will explain the change of the solution. When one looks at Section 4, it can be seen that $B=-D_m^TC^{-1}D_s$.

Consider the third example, for which we have the 4 nodes which can be seen in Figure 16.

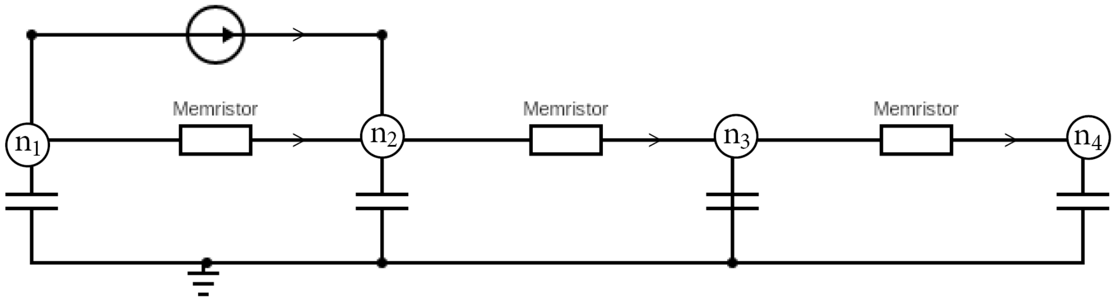

Figure 16: The Third Example Given with the 4 Nodes Belonging To This Circuit.

For this example, the following incidence matrices are given,

$$D_m = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad D_s = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case, B is equal to,

$$\begin{aligned} B &= -D_m^T C^{-1} D_s \\ &= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

The result of interest is the change of the flux over the memristors, φ_m when the capacitor voltages, q_s change. Let us first consider an increase of the charge of the source. Assume at $t = 5$, the charge q_s is equal to 5 instead of 1.

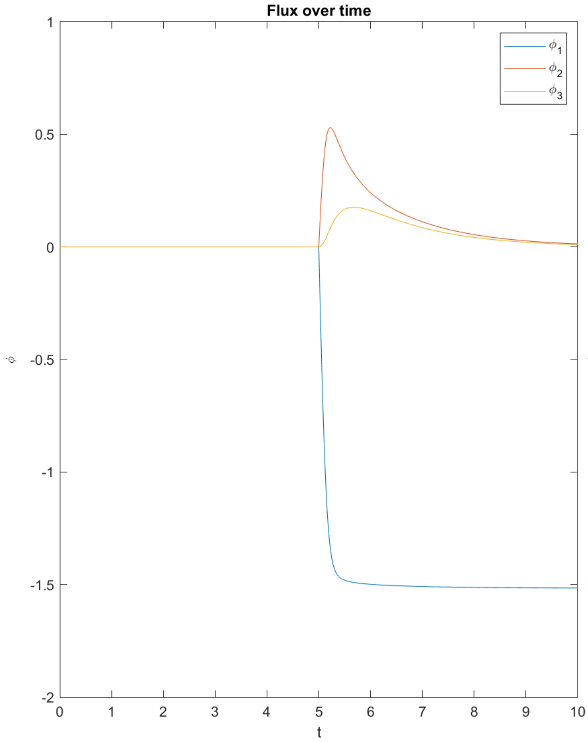

Figure 17: Flux with $q_s = 5$

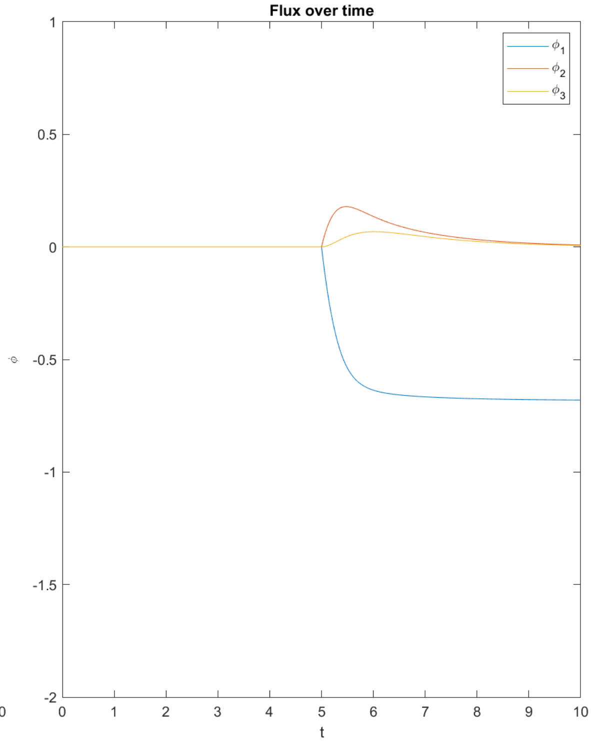

Figure 18: Flux with $q_s = 1$

From the figure above it can be seen that when the charge increases and so does the input, the flux has a larger amplitude in comparison to when one looks at the original charge. This can be explained from the fact that

$$\begin{aligned} Bq_s &= -D_m^T C^{-1} D_s q_s \\ &= \begin{bmatrix} -2q_s \\ q_s \\ 0 \end{bmatrix}. \end{aligned}$$

And hence, when q_s increases to 5, one gets the matrix $\begin{bmatrix} -10 \\ 5 \\ 0 \end{bmatrix}$ in stead of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

The next change to be considered is a decrease of the charge of the source. Let us take for example $q_s = 0.1$ instead of $q_s = 1$. The following figure is obtained:

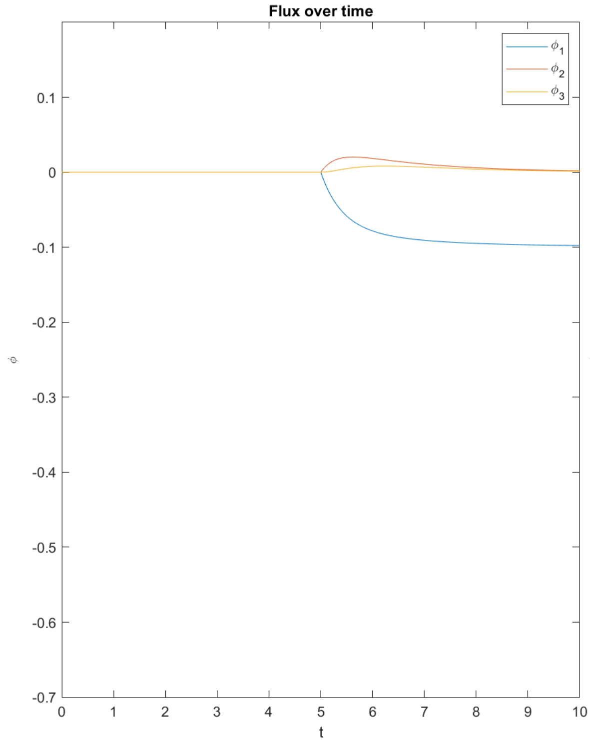

Figure 19: Flux with $q_s = 0.1$

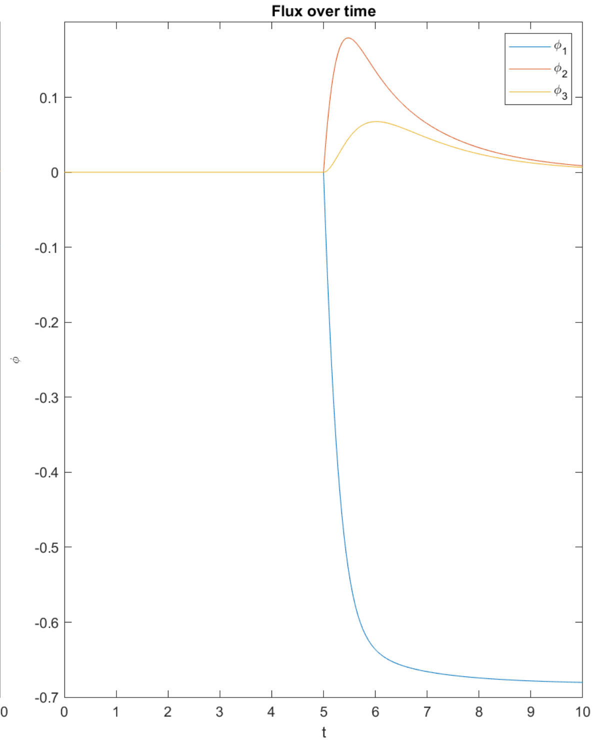

Figure 20: Flux with $q_s = 1$

From the figure, when the charge decreases, so does the influence of the charge on the flux. It can be explained in a similar fashion as above as we now chose $q_s = 0.1$, the following matrix for Bq_s is obtained $\begin{bmatrix} -0.2 \\ 0.1 \\ 0 \end{bmatrix}$ instead of $Bq_s = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

Finally, consider a negative charge, so let $q_s = -1$. Then the following graphs are obtained:

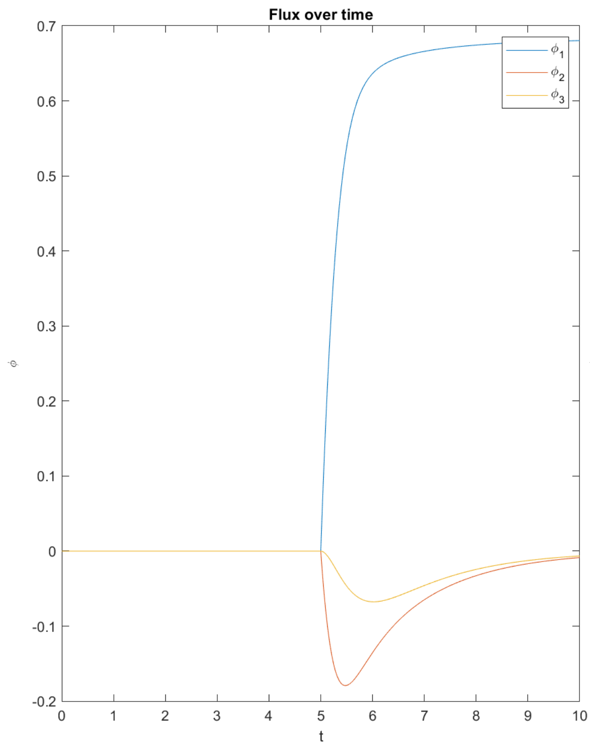

Figure 21: Flux with $q_s = -1$

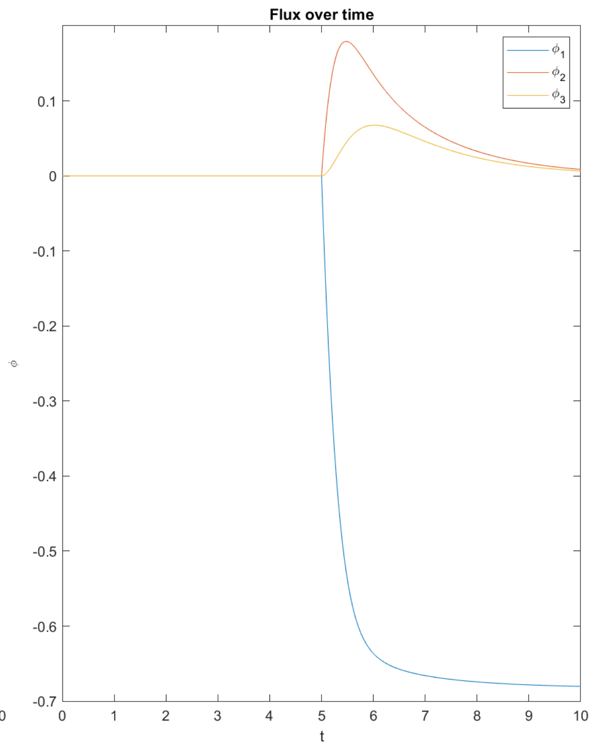

Figure 22: Flux with $q_s = 1$

As one can mention, there is now a reflection in the x-axis. When considering the vector Bq_s this can be explained since there is a change of signs for each element of the vector. So now we have, $Bq_s = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ instead of $Bq_s = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

In this chapter, some analytical properties were derived. First of all, the definition of asymptotically stability was studied, next the stability of the memristor-capacitor networks of Chapter 4 was analyzed and finally the change of the state when the input changes has been considered. The next chapter will give a brief conclusion on all topics studied throughout this research.

6 Conclusion

Throughout this paper memristive circuits and their modeling, solutions and analysis have been studied. First, the theoretical framework was given. Within this framework, first some graph theory was discussed. Memristive circuits can be represented as undirected graphs with an associated orientation. From these graphs, incidence matrices can be derived. These incidence matrices can be used in order to derive the mathematics needed to study the properties of the circuits. After defining incidence matrices, Kirchhoff's Laws were given. First the general notion of these laws has been shown and thereafter, the mathematical representation, consisting of incidence matrices, has been stated. After stating Kirchhoff's Laws, memristors and capacitors were discussed since the focus of this paper was on memristor-capacitor circuits. Thereafter, the constitutive relations for the resistor and capacitor were given. The theoretical framework concludes with the fact that a linear memristor is nothing else than a linear resistor and hence in order to study the memristor, a nonlinear approach should be taken. Therefore, the flux-controlled memristor law was obtained.

After deriving the theoretical framework, mathematical modeling has been done. First the modeling for the system without an input u is shown. After modeling a circuit without external input, next the circuit with external input is considered. After obtaining the models, the ordinary differential equations are solved partially in order to obtain a reduction in the state-space dimension. This has been shown for both the circuits; with and without external input.

When these models and its partially analytical solution has been obtained, a numerical implementation was considered. First, the capacitance matrix, the function for the memristor and the function for the charge needed to be defined. The capacitance matrix is a diagonal matrix with the capacitances on its diagonal. The function for the memristor must be monotone and nonlinear and the function for the charge is a step function. By using ODE45, the build-in function of MATLAB, the ODE is solved for three examples. These three circuits consists of one current source and of one, two or three memristors. Both the state matrix A and the input matrix explain the behaviour of the three graphs of both the flux and the voltage.

After obtaining the numerical results, a further explanation of the obtained results is shown in the analysis section. First the definition of stability is investigated. The equilibrium point $(\bar{x}, \bar{u})$ of a nonlinear system is asymptotically stable if the real part of the eigenvalues in the spectrum of $\overline{A}$ lie in the left-hand side of the complex plane. Where $\overline{A}$ denotes the linearization of the system matrix around the equilibrium point. By analytical derivation and making use of the found equilibrium point, it has been shown that for all these three examples the equilibrium point $(0, 0)$ of the nonlinear systems is asymptotically stable. Finally, the change of charge is studied. When the charge considered is higher, the flux has a larger amplitude than when one compares it with the original charge. On the other side, when the charge considered is lower, the flux has a smaller amplitude compared to the original charge. The last change considered is to add a negative charge in stead of a positive charge. From this, a reflection in the x-axis was obtained. All these three changes has been explained from the way the input matrix B is defined. With this section, the report is concluded.

After denoting all the results, some questions still arise. One question within the study of capacitor-memristor networks that for example arises is: What can be said about equilibrium points when one changes the values for A or the initial conditions p_0 and q_0? Another topic of interest are electrical circuits with more current sources. This paper focuses on electrical circuits with at least one memristor but with only one current source. What can be said about the modeling and analysis when more current sources are added? Note that throughout this research only a very specific type of circuits are studied, namely the circuits containing of memristors with a capacitor attached to each node. What can be said about the modeling and analysis of other types of circuits and how can these circuits be used in order to study the field of cognitive computing? These questions can be studied in further research.

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8 Appendix

8.1 Functions

Function of the memristor
gflux.m

```
function gphi = gflux(phi)
gphi = phi+(phi).^(3);
end
```

Ordinary Differential Equation
ODEphi.m

```
%Input
%Dm              Incidence matrix of memristor
%Ds              Incidence matrix of source
%phi             Function of flux over memristor
%gflux           Monotone, nonlinear function depending on phi
%C               Capacitance matrix
%qs              Stepfunction of charge by the source

%Initial conditions
%p0              Potential at t=0
%phi0            Flux at at t=0

%OUTPUT
%f               Change of flux with respect tot time

function f=ODEphi(Dm, Ds, C, p0, phi, phi0, qs, qs0)
f=-Dm'*C'*Dm*gflux(phi)+Dm'*p0+Dm'*C'*Dm*gflux(phi0)-Dm'*C'*Ds*(qs-qs0);
end
```

Ordinary Differential Equation without input
ODEphiwithoutinput.m

```
function f=ODEphiwithoutinput(phi)
Dm = [1; -1];        % Incidence matrix of the memristor
C = [1 0; 0 1];      % Capacitance matrix
phi0 = 0;            % Flux at t=0
p0 = [0;0];          % Potential at t=0
f=-Dm'*C'*Dm*gflux(phi)+Dm'*p0+Dm'*C'*Dm*gflux(phi0);
end
```

8.2 ODE solvers

Solve ODE with one memristor

ODEsolver.m

```
close all
%DEFINING PARAMETERS
A = 1;                  % Pulse at t=tau
tau = 5;                % Time where pulse occurs
Dm = [1;-1];            % Incidence matrix of the memristor
Ds = [1;-1];            % Incidence matrix of the source
C = [1 0; 0 1];         % Matrix consisting of capacitances


%INITIAL CONDITIONS
p0 = [0;0];             % Potential at t=0
phi0 = 0;               % Flux at t=0

%Solving ODE for t<5
qs = 0;
tspan = [0 5];
[t1,phi1] = ode45(@(t,phi) ODEphi(Dm, Ds, C, p0, phi,
    phi0, qs, qs0), tspan, phi0);

%Solving ODE for t>=5
qs = 1;
tspan = [5 10];
[t2,phi2] = ode45(@(t,phi) ODEphi(Dm, Ds, C, p0, phi,
    phi0, qs, qs0), tspan, phi0);

%Combining both ODE's
phi = [phi1; phi2];
t = [t1; t2];

%Defining voltages for t<5
qs=0;
y1 = ODEphi(Dm, Ds, C, p0, phi1, phi0,qs,qs0);
%Defining voltages for t=>5
qs=1;
y2 = ODEphi(Dm, Ds, C, p0, phi2, phi0,qs,qs0);

%Combining both voltages
y = [y1; y2];

%Plot solutions
figure;
subplot(1,2,1)
plot(t,phi)
title('Flux over time')
xlabel('t')
ylabel('^\phi')
subplot(1,2,2)
```

```
plot(t,y)
title('Voltage over time')
xlabel('t')
ylabel('^{d\phi}/_{dt}')

%Finding roots
fun = @ODEphiwithoutinput;
x = fzero(fun,phi0);
```

Solve ODE with two memristors

ODEsolver2D.m

```
close all
%DEFINING PARAMETERS
A = 1;                              % Pulse at t=tau
tau = 5;                            % Time where pulse occurs
Dm = [1 0; -1 1; 0 -1];             % Incidence matrix of the memristor
Ds = [1;-1; 0];                     % Incidence matrix of the source
C = [1 0 0 ; 0 1 0; 0 0 1]; %Matrix consisting of capacitances

%INITIAL CONDITIONS
p0 = [0;0;0];                       % Potential at t=0
phi0 = [0;0];                       % Flux at t=0

%Solving ODE for t<5
qs = 0;
tspan = [0 5];
[t1,phi1] = ode45(@(t,phi) ODEphi(Dm, Ds, C, p0, phi, phi0, qs, qs0), tspan, phi0);

%Solving ODE for t>=5
qs = 1;
tspan = [5 10];
[t2,phi2] = ode45(@(t,phi) ODEphi(Dm, Ds, C, p0, phi, phi0, qs, qs0), tspan, phi0);

%Combining both ODE's
phi = [phi1; phi2];
t = [t1; t2];

%Defining voltages for t<5
qs=0;
y1 = ODEphi(Dm, Ds, C, p0, phi1', phi0,qs,qs0);
%Defining voltages for t=>5
qs=1;
y2 = ODEphi(Dm, Ds, C, p0, phi2', phi0,qs,qs0);

%Combining both voltages
y = [y1, y2];
```

```
%Plot solutions
figure;
subplot(1,2,1)
plot(t,phi(:,1))
hold on
plot(t,phi(:,2))
legend('\phi_1','\phi_2')
title('Flux over time')
xlabel('t')
ylabel('^\phi')
subplot(1,2,2)
plot(t,y(1,:))
hold on
plot(t,y(2,:))
legend('{d\phi_1}/{dt}', '{d\phi_2}/{dt}')
title('Voltage over time')
xlabel('t')
ylabel('^{d\phi}/_{dt}')
```

Solve ODE with three memristors

ODEsolver3D.m

```
close all
%DEFINING PARAMETERS
A = 1;                                    % Pulse at t=tau
tau = 5;                                  % Time where pulse occurs
Dm = [1 0 0; -1 1 0; 0 -1 0; 0 0 -1];     % Incidence matrix of the memristor
Ds = [1;-1; 0 ;0];                        % Incidence matrix of the source
C = [1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1]; % Matrix consisting of capacitances

%INITIAL CONDITIONS
p0 = [0;0;0;0];           % Potential at t=0
phi0 = [0;0;0];           % Flux at t=0

%Solving ODE for t<5
qs = 0;
tspan = [0 5];
[t1,phi1] = ode45(@(t,phi) ODEphi(Dm, Ds, C, p0, phi, phi0, qs, qs0), tspan, phi0);

%Solving ODE for t>=5
qs = 1;
tspan = [5 10];
[t2,phi2] = ode45(@(t,phi) ODEphi(Dm, Ds, C, p0, phi, phi0, qs, qs0), tspan, phi0);

```

```
%Combining both ODE's
phi = [phi1; phi2];
t = [t1; t2];

%Defining voltages for t<5
qs=0;
y1 = ODEphi(Dm, Ds, C, p0, phi1', phi0,qs,qs0);
%Defining voltages for t=>5
qs=1;
y2 = ODEphi(Dm, Ds, C, p0, phi2', phi0,qs,qs0);

%Combining both voltages
y = [y1, y2];

%Plot solutions
figure;
subplot(1,2,1)
plot(t,phi(:,1))
hold on
plot(t,phi(:,2))
hold on
plot(t,phi(:,3))
legend('\phi_1','\phi_2','\phi_3')
title('Flux over time')
xlabel('t')
ylabel('^\phi')
subplot(1,2,2)
plot(t,y(1,:))
hold on
plot(t,y(2,:))
hold on
plot(t,y(3,:))
legend('{d\phi_1}/{dt}', '{d\phi_2}/{dt}', '{d\phi_3}/{dt}')
title('Voltage over time')
xlabel('t')
ylabel('^{d\phi}/_{dt}')
```