university of groningen

faculty of science and engineering

mathematics and applied mathematics

Studying the Nature of the Hopf Bifurcation of the Lorenz-96 Model

Bachelor's Project Mathematics

March 2022

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Abstract

In this thesis, we study theory on the Hopf bifurcation, and apply this theory to the Lorenz-96 model. We consider the system in four dimensions, and determine whether the bifurcation is supercritical or subcritical using center manifold reduction and normal form analysis. We compute the first Lyapunov coefficient to be negative, meaning that the Hopf bifurcation is supercritical, and results in a stable limit cycle.

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1 Introduction and Historical Context

Given the incredibly dynamic world we live in, it is difficult to believe that there was a time when people only possessed the mathematical tools to describe it in terms of static qualities. For scientists to make predictions, they often needed to assume linearity of variables, which impacted the accuracy of their work in such a nonlinear world. It was only in the last 400 years when scientists and mathematicians seriously began studying the world with respect to time. Fermat, Descartes, and most notably Newton and Leibniz were among them, and together their contributions pioneered the field of Calculus. This major breakthrough in mathematics paved the way for a number of new fields and discoveries.

Dynamical systems evolved from Calculus in order to study the evolution of systems in time. The field originated during Poincaré's study of the well-known 'three body problem' in the field of Celestial Mechanics, and since then, it has been used to help make sense the complex world around us. Dynamical systems can be seen everywhere on chemical, biological, physical and social levels, and this is largely our motivation in analysing these systems. When a dynamical system undergoes a large qualitative change as its parameter value is varied, this is known as a bifurcation. A particular bifurcation which results in the birth or death of a limit cycle is known as a Hopf bifurcation. When the Hopf bifurcation of a system results in a stable limit cycle we call the bifurcation supercritical, and subcritical otherwise. Stable limit cycles cause self-sustained oscillations, and small perturbations from their trajectory do not affect their long term behaviour. Hopf bifurcations are present in many physical systems including the firing of neurons in nervous systems, oscillations in autocatalytic chemical reactions, oscillations in fish populations and epidemic models of disease, and so studying them is of both theoretical and practical importance [2].

A particular dynamical system of interest is the Lorenz-96 Model, which was used by American Mathematician and Meteorologist Edward Lorenz to study weather prediction. This system provides a simplified model to understand challenges in long term weather forecasting, and is still used today in testing data assimilation techniques [3]. The goal of this thesis is to analyse this model by ultimately determining whether the bifurcation is supercritical or subcritical, in order to determine whether or not the limit cycle is be stable. In two dimensions, normal form analysis can be used. However, in

higher dimensions, it will be necessary to first restrict the system to a family of smooth two-dimensional invariant manifolds near the origin, as prescribed by the Center Manifold Theorem. We will consider the system in four dimensions, and use the theory from the main reference for this thesis "Elements of Applied Bifurcation Theory" by Kuznetsov [4], to compute the center manifold of the system, in order to determine the nature of the bifurcation. Our purpose in doing so is not only to understand this particular system better, but also to develop general methods which in turn be used to examine other dynamical systems.

2 Preliminaries

Firstly, we will begin by reminding the reader of the basic definitions of 'dynamical system' and 'orbit'. As previously noted, the majority of definitions and theorems in the paper come from the main reference [4].

Definition 2.1. A **dynamical system** is a triple $\{T, X, \varphi^t\}$ where T is a time set, X is a state space and $\varphi^t : X \to X$ is a family of evolution operators parametrised by $t \in T$ and satisfying $\varphi^0 = \text{id}$ and $\varphi^{t+s} = \varphi^t \circ \varphi^s$.

Definition 2.2. An **orbit** starting at x_0 is a subset of the state space X, $Orb(x_0) = \{x \in X : x = \varphi^t x_0, \forall t \in T \text{ where } \varphi^t x_0 \text{ is defined}\}$.

Now, in order to study compare dynamical systems, we will need to introduce some terminology. Recall that a **homeomorphism** is a continuous, invertible map between topological spaces where the inverse is also continuous. With this said, we can now define topologically equivalent dynamical systems.

Definition 2.3. Dynamical systems $\{T, \mathbb{R}^n, \varphi^t\}$ and $\{T, \mathbb{R}^n, \psi^t\}$ are called **topologically equivalent** if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ which maps the orbits of the first system to the orbits of the second system, and preserves the direction of time.

Often dynamical systems are not topologically equivalent, however they could be equivalent near an equilibrium. Hence, we give the following definition.

Definition 2.4. A dynamical system $\{T, \mathbb{R}^n, \varphi^t\}$ is **locally topologically equivalent** to a dynamical system $\{T, \mathbb{R}^n, \psi^t\}$ near an equilibrium y_0 if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ that is

1. defined in a small neighbourhood $U \subset \mathbb{R}^n$ of x_0;
2. satisfies $y_0 = h(x_0)$
3. maps orbits of the first system in U to orbits of the second system in $V = f(U) \subset \mathbb{R}^n$, preserving the direction of time.

This leads to the concept of a bifurcation, which we will study throughout this thesis. It is clear that when we vary the parameters of a system, the phase portrait will change. The extent to which the phase portrait changes determines whether a bifurcation is present.

Definition 2.5. A **bifurcation** is the appearance of a topologically nonequivalent phase portrait under variation of a parameters.

The particular bifurcation that we will focus on is a Hopf bifurcation. An important concept in its definition is a limit cycle.

Definition 2.6. A **limit cycle** is a periodic solution of a system of differential equations which has the additional property that at least one other trajectory spirals into it.

Now that we have understood the important foundational concepts, we can move on to studying the particular bifurcation of interest, the "Andronov-Hopf bifurcation".

3 The Hopf Bifurcation

We begin by developing theory about 2-dimensional systems, and later we will extend this to include higher-dimensional systems. One instance of a bifurcation occurring is when when the stability of an equilibrium point changes. The stability of the equilibrium point depends on the eigenvalues of the Jacobian of the linerised system evaluated at the equilibrium point. When $\mathrm{Re}(\lambda_{1,2}) < 0$, the equilibrium is said to be **stable**, and **unstable** otherwise. Qualitatively, this means that the solutions near the point either converge toward it or diverge away from it. There are a number of types of bifurcations, but here we define our particular bifurcation of interest.

Definition 3.1. A **Hopf bifurcation** refers to a limit cycle surrounding an equilibrium point either arising or disappearing as a parameter crosses a critical value.

Definition 3.2. We call a Hopf bifurcation **supercritical** if the system changes from having a stable equilibrium point to having an unstable equilibrium point along with a stable limit cycle, or vice versa. We call a Hopf bifurcation **subcritical** if the system goes from having an unstable equilibrium point to having a stable equilibrium point along with an unstable limit cycle, or vice versa.

In order to identify Hopf bifurcations within systems, we need to define the following terminology about equilibrium points.

Definition 3.3. We call a equilibrium point **hyperbolic** when both eigenvalues of the Jacobian matrix (of the system evaluated at the point) have non-zero real parts. A hyperbolic equilibrium is referred to as a **focus** if both eigenvalues are complex (conjugates) [1].

Remark 3.1. We can characterise Hopf bifurcations by an equilibrium point switching from a stable to an unstable focus (or vice versa) as the parameter value changes - In other words, when the parameter value is such that the eigenvalues of the Jacobian matrix evaluated at the equilibrium are purely imaginary.

This is a **local bifurcation**, meaning that the bifurcation can be detected in arbitrarily small neighbourhoods of the bifurcating equilibrium. We will illustrate this theory on the Hopf bifurcation in the following example.

Example 3.1. Consider the following system

$$\begin{cases} x = ax - y - x(x^2 + y^2) \\ y = x + ay - y(x^2 + y^2) \end{cases} \tag{1}$$

for some parameter $a \in \mathbb{R}$.

We can see that at $(x, y) = (0, 0)$ we have $\dot{x} = \dot{y} = 0$ and hence this is an equilibrium point. We can put the system into polar coordinates, by introducing the complex

variable $z = x + iy$. Using (1), we can compute

$$\begin{aligned}
\dot{z} &= \dot{x} + i\dot{y} \\
&= ax - y - x(x^2 + y^2) + i[x + ay - y(x^2 + y^2)] \\
&= a(x + iy) + i(x + iy) - (x + iy)(x^2 + y^2) \\
&= (a + i)(x + iy) - (x + iy)(x^2 + y^2) \\
&= (a + i)z - z|z|^2. \qquad (2)
\end{aligned}$$

Let $z = re^{i\theta}$. Using this substitution, along with the chain rule and (2) we get the equations

$$\begin{aligned}
\dot{z} &= \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta} \\
&= (a + i)re^{i\theta} - r^3 e^{i\theta}\dot{r}e^{i\theta} \\
&= re^{i\theta}(a + i - r^2) \qquad (3)
\end{aligned}$$

If we equate the real and imaginary parts of (3) we get the polar form of (1) to be

$$\begin{cases} r = ar - r^3 \\ \dot{\theta} = 1 \end{cases} \qquad (4)$$

This form is much easier to analyse. We can see that the origin is the only equilibrium point for the system, since this is the only point for which it is possible that $\dot{r} = 0$ and $\dot{\theta} \neq 0$. Using the definition of Hopf bifurcation, we would like to find the parameter value at which a limit cycle either appears or disappears, and the stability conditions are satisfied.

(i) For $a \leq 0$, we have $\dot{r} < 0$, $\forall r > 0$, and so the origin is a stable equilibrium as the solutions will spiral towards it.
(ii) For $a > 0$, we can describe the phase portrait by considering different values of r. When $0 < r < \sqrt{a}$, we get $\dot{r} > 0$ and so the origin is an unstable equilibrium. When $r = \sqrt{a}$, we get $\dot{r} = 0$ and hence the circle $r = \sqrt{a}$ is a solution. This solution has period 2π since we have that the angular frequency is one radian per second, and hence after 2π units of time the solutions will repeat. Finally, when $r > \sqrt{a}$ we get that $\dot{r} < 0$ and hence solutions will spiral toward the circular solution. We therefore call this a

(a) $a < 0$ (b) $a > 0$

Figure 1: Supercritical Hopf Bifurcation [4].

stable limit cycle.

We have noticed that the system goes from having a stable equilibrium, to having an unstable equilibrium along with a stable limit cycle. Hence, this is a supercritical Hopf bifurcation. This is shown in figure 1.

We could also use our characterisation of a Hopf bifurcation using eigenvalues. We can linearise the original system around the equilibrium point as

$$\dot{X} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} X,$$

and then calculate the eigenvalues to be $a \pm i$. From this we can easily see that the bifurcation should be at $a = 0$ as the eigenvalues go from being a stable focus to an unstable focus as a passes through zero.

Example 3.2. We can modify example (3.1) to become a subcritical Hopf bifurcation. Consider the system with a difference in signs

$$\begin{cases} \dot{x} = ax - y + x(x^2 + y^2) \\ \dot{y} = x + ay + y(x^2 + y^2). \end{cases} \tag{5}$$

Similarly to the first example, we compute the polar form of (5) to be

(a) $a > 0$ (b) $a < 0$

Figure 2: Subcritical Hopf Bifurcation [4].

$$\begin{cases} \dot{r} = a + r^3 \\ \dot{\theta} = 1. \end{cases} \tag{6}$$

In this case, we see that
(i) For $a \geq 0$ we have that $\dot{r} > 0$ and so the origin is an unstable equilibrium.
(ii) For $a < 0$, we get $\dot{r} < 0$ for $0 < r < \sqrt[3]{a}$, meaning the origin is a stable equilibrium. Moreover, $\dot{r} = 0$ for $r = \sqrt[3]{a}$ and $\dot{r} > 0$ for $r > \sqrt[3]{a}$. This represents an **unstable limit cycle** with radius $r = \sqrt[3]{a}$ and period 2π.

Clearly this satisfies the definition of a subcritical Hopf bifurcation as the equilibrium point changes from being unstable to being stable, and an unstable limit cycle emerges.

4 The Normal Form of the Hopf Bifurcation in Two Dimensions

In example 3.1 and 3.2, it was the case that converting the systems into polar coordinates simplified them significantly to the point where they were easy to analyse for different parameter values. This is not always possible to do, and so we would like to find a way of determining the nature of systems in a methodical way.

Example 3.1 and 3.2 (continued). We can write system (1) and (5) in matrix-

vector form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pm (x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

These systems take on a very particular form, which is important for studying Hopf bifurcations. We say that they are **normal forms**. For many purposes, higher order systems can also be viewed in this form, as we have the following lemma.

Lemma 4.1. *The system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pm (x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} + O(||x||^4) \tag{8}$$

is locally topologically equivalent near the origin to system (7).

This means that we are often able to work with simplified systems and disregard the higher order terms. The normal form is useful in determining the nature of Hopf bifurcations. Notice that the only difference between the two formulas in (7) is the sign before the nonlinear terms. Recall that changing this sign changed the Hopf bifurcation from being supercritical to subcritical. In general, this coefficient does determine whether the system is supercritical or subcritical. Moreover, it is true that, under certain conditions, all systems with a Hopf bifurcation are topologically equivalent to a normal form. We will spend this section proving the following theorem.

Theorem 4.2 (Topological normal form for the Hopf bifurcation)**.** *Any generic two-dimensional, one-parameter system*

$$\dot{x} = f(x, a),$$

having at $a = 0$ *the equilibrium* $x = 0$ *with eigenvalues*

$$\lambda_{1,2}(0) = \pm i\omega_0,\ \omega_0 > 0,$$

is locally topologically equivalent near the origin to one of the following normal forms:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Notice that if we can show that any two-dimensional system undergoing a Hopf bifurcation can be transformed into either the form (7), with the addition of higher order terms (degree four or more), then we can use theorem (4.2) to remove the higher order terms. It is therefore sufficient to prove the following theorem.

Theorem 4.3. *Suppose we have a two-dimensional system*

$$\dot{x} = f(x, a), \quad x \in \mathbb{R}^2, a \in \mathbb{R}, \tag{9}$$

where f is smooth, and for sufficiently small $|a|$, there is an equilibrium $x = 0$ with eigenvalues

$$\lambda_{1,2}(a) = \mu(a) \pm i\omega(a),$$

where $\lambda(0), \omega(0) = \omega_0 > 0$.

If the following conditions are met

(A.1) The first Lyapunov coefficient (to be defined in the proof) satisfies $l_1(0) \neq 0$;
(A.2) $\dot{\mu}(0) \neq 0$,

then, there are invertible coordinate and parameter changes and a time reparameterisation which transforms (8) into

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(||y||^4). \tag{10}$$

Since the proof is too long, we will include a sketch of it.

Sketch of proof:

Consider the system

$$\dot{x} = f(x, a), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \quad a \in \mathbb{R},$$

where f is smooth and has $a = 0$ at the equilibrium $x = 0$ and the eigenvalues of the Jacobian are $\lambda_{1,2} = \pm i\omega_0$, for $\omega_0 > 0$. The Implicit Function Theorem guarantees

that the system has a unique equilibrium $x_0(a)$ in a neighbourhood of the origin for sufficiently small $|a|$ as $\lambda \neq 0$. We can perform a coordinate shift which places the equilibrium at the origin, and hence we can assume that when $|a|$ is sufficiently small, that $x = 0$ is the corresponding equilibrium point. Therefore, we can write the system as

$$\dot{x} = A(a)x + F(x, a), \tag{11}$$

where F is a smooth vector function whose components have Taylor expansions in x starting with at least quadratic terms. The eigenvalues of the Jacobian, $A(a)$, will be

$$\lambda_{1,2}(a) = \frac{1}{2}\left(\operatorname{Tr} A(a) \pm \sqrt{\operatorname{Tr}^2 A(a) - 4\det A(a)}\right), \quad \operatorname{Tr} A(0) = 0, \quad \det A(0) = \omega_0^2 > 0$$

This can be written as

$$\lambda_{1,2}(a) = \mu(a) \pm i\omega(a), \quad \mu(0) = 0, \quad \omega(0) = \omega_0 > 0.$$

for small $|a|$.

Lemma 4.4. *System* (11) *can be written for sufficiently small* $|a|$ *as*

$$\dot{z} = \lambda(a)z + g(z, \overline{z}, a), \tag{12}$$

where z *is a complex variable and* $g = O(|z|^2)$ *is a smooth function of* $(z, \overline{z}, a)$.

Proof. Let $q(a) \in \mathbb{C}^2$ be an eigenvector of $A(a)$ corresponding to $\lambda(a)$ and let $p(a) \in \mathbb{C}^2$ be an eigenvector of $A(a)^T$ corresponding to its eigenvalue $\overline{\lambda(a)}$. Choose the eigenvectors p and q such that $\langle p, q\rangle = \overline{p_1}q_1 + \overline{p_2}q_2 = 1$. Note that if the eigenvalues are specified, any vector $x \in \mathbb{R}^2$ can be uniquely represented as

$$x = zq(a) + \overline{zq}(a) \tag{13}$$

for small a, for some complex variable z. We can calculate that $\langle p(a), \overline{q}(a)\rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the complex inner product in $\mathbb{C}^2$ satisfying $\langle p, q\rangle = \overline{p}_1 q_1 + \overline{p}_2 q_2$. Therefore

$z = \langle p(a), x\rangle$. This variable satisfies

$$\dot{z} = \lambda(a)z + \langle p(a), F(zq(a) + \bar{z}\bar{q}(a), a)\rangle,$$

which clearly is of the form (12). $\square$

Remark: We can write g as a formal Taylor series in two complex variables z and $\bar{z}$ as follows

$$g(z, \bar{z}, a) = \sum_{2 \leq k+l} \frac{1}{k!l!} g_{kl}(a) z^k \bar{z}^l \Bigg|_{z=0},$$

for $2 \leq k + l \quad k, l = 0, 1, \ldots,$

where

$$g_{kl}(a) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \langle p(a), F(zq(a) + \bar{z}\bar{q}(a), a)\rangle$$

Now suppose that at $a = 0$ the function F from equation (11) can be written as

$$F(x, 0) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(||x||^4),$$

where $B(x, y)$ and $C(x, y, u)$ are symmetric multilinear vector functions of $x, y, u \in \mathbb{R}^2$. We can write the coordinates as

$$B_i(x, y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_i(\xi, 0)}{\partial \xi_j \partial \xi_k}\Bigg|_{\xi=0} x_j y_k, \quad i = 1, 2, \ldots,$$

$$C_i(x, y, u) = \sum_{i,j,k=1}^{2} \frac{\partial^2 F_i(\xi, 0)}{\partial \xi_j \partial \xi_k \partial \xi_l}\Bigg|_{\xi=0} x_j y_k u_l \quad i = 1, 2, \ldots.$$

The Taylor coefficients g_{kl} are then given by

$$g_{20} = \langle p, B(q, q)\rangle, \quad g_{11} = \langle p, B(q, \bar{q})\rangle, \quad g_{02} = \langle p, B(\bar{q}, \bar{q})\rangle, \quad g_{21} = \langle p, C(q, q, \bar{q})\rangle.$$

Using this notation along with two transformations, we can write equation (12) as follows:

Lemma 4.5 (**Poincaré normal form for the Hopf bifurcation**). *The equation*

$$\dot{z} = \lambda z + \sum_{2 \leq k+l \leq 3} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l + O(|z|^4), \tag{14}$$

where λ and g are defined as before, can be transformed by an invertible parameter-dependent change of complex coordinate, smoothly depending on the parameter,

$$z = w + \frac{h_{20}}{2} w^2 + h_{11} w\bar{w} + \frac{h_{02}}{2} \bar{w}^2 + \frac{h_{30}}{6} w^3 + \frac{h_{12}}{2} w\bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3$$

for all sufficiently small $|a|$, into an equation with only the resonant cubic term:

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + O(|w|^4).$$

Sketch of proof. First, we can perform the transformation

$$z = w + \frac{h_{20}}{3} w^2 + h_{11} w\bar{w} + \frac{h_{02}}{2} \bar{w}^2, \tag{15}$$

with

$$h_{20} = \frac{g_{20}}{\lambda},\ h_{11} = \frac{g_{11}}{\bar{\lambda}},\ h_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda} \tag{16}$$

This will annihilate all the quadratic terms and changes the coefficients of the cubic terms. Then, we can perform the transformation

$$z = w + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2 \bar{w} + \frac{h_{12}}{2} w\bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3 \tag{17}$$

which annihilates all of the cubic terms except for the resonant cubic term. We are therefore left with the equation

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + O(|w|^4).$$

□

We can then transform the Poincaré normal form to the normal forms in equation (7).

Lemma 4.6. *Consider the equation*

$$\frac{dw}{dt} = (\mu(a) + i\omega(a))w + c_1(a) w|w|^2 + O(|w|^4),$$

where $\mu(0) = 0$ *and* $\omega(0) = \omega_0 > 0$.

Suppose $\mu'(0) \neq 0$ *and* $\mathrm{Re}(c_1) \neq 0$. *Then, the equation can be transformed by a parameter-dependent linear coordinate transformation, a time rescaling, and a nonlinear time reparametrisation into an equation of the form*

$$\frac{du}{d\theta} = (\beta + i)u + su|u|^2 + O(|u|^4) \tag{18}$$

where u *is a new complex coordinate, and* θ, β *are the new time and parameter, and* $s =$*sign* $\mathrm{Re}(c_1(0)) = \pm 1$.

Sketch of proof. Firstly, we can scale time linearly by introducing the new time $\tau = \omega(a)t$. Then our equation is transformed to

$$\frac{dw}{d\tau} = (\beta + i)w + d_1(\beta)w|w|^2 + O(|w|^4),$$

where

$$\beta = \beta(a) = \frac{\mu(a)}{\omega(a)},\ d_1(\beta) = \frac{c_1(a(\beta))}{\omega(a(\beta))}.$$

We then perform nonlinear time reparametrisation by introducing the new time $\theta = \theta(\tau, \beta)$, where $d\theta = (1 + e_1(\beta)|w|^2)d\tau$ with $e_1(\beta) = \mathrm{Im}(d_1(\beta))$ We obtain

$$\frac{dw}{d\theta} = (\beta + i)w + l_1(\beta)w|w|^2 + O(|w|^4),$$

where $l_1(\beta) = \mathrm{Re}(d_1(\beta)) - \beta e_1(\beta)$ is called the **first Lyapunov coefficient.** Note that

$$l_1(0) = \frac{\mathrm{Re}(c_1(0))}{\omega(0)} \tag{19}$$

$$= \frac{1}{2\omega_0^2}\,\mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{21}). \tag{20}$$

Finally, we can introduce a new complex variable u by the formula

$$w = \frac{u}{\sqrt{|l_1(\beta)|}},$$

and so the equation takes the required form

$$\frac{du}{d\theta} = (\beta + i)u + \frac{l_1(\beta)}{|l_1(\beta)|}u|u|^2 + O(|u|^4) \tag{21}$$

$$= (\beta + i)u + su|u|^2 + O(|u|^4), \tag{22}$$

with $s = sign(l_1(0)) = \mathrm{Re}(c_1(0))$. □

Now, finally, we notice that if we write equation (18) in real form with $s = -1$ then we get the desired equation (9), and hence the proof is complete.

■

As mentioned at the beginning of the section, the reason that the normal form is important is because it determines the nature of the bifurcation. Now that we have developed the necessary terminology, we can state the following theorem.

Theorem 4.7. *Consider the normal form of a dynamical system given by*

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \sigma(y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $\sigma = sign\, l_1(0) = \pm 1$. If $\sigma = 1$, then the Hopf bifurcation is subcritical, and if $\sigma = -1$, then the Hopf bifurcation is supercritical. □

Using the theorems in this section, we now know how to determine the nature of the Hopf bifurcation of any continuous-time dynamical system in two dimensions. Let's apply this to an example.

Example 4.1. Consider the following system

$$\begin{cases} \dot{u} = (v - 1) - (u - 1)^3 + a(u - 1) \\ \dot{v} = -(u - 1), \end{cases} \tag{23}$$

for $a \in \mathbb{R}$ a parameter. By setting the left hand sides of the equations equal to zero, we calculate that the only equilibrium point of the system lies at $(u, v) = (1, 1)$. The Jacobian of the system evaluated at the equilibrium is

$$A(a)|_{(1,1)} = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix},$$

and hence we can calculate the eigenvalues to be

$$\begin{aligned} \lambda_{1,2}(a) &= \frac{1}{2}\operatorname{Tr} A(a) \pm \frac{1}{2}\sqrt{\operatorname{Tr} A(a) - 4\det A(a)} \\ &= \frac{1}{2}\operatorname{Tr} A(a) \pm \frac{1}{2}\sqrt{4\det A(a) - \operatorname{Tr} A(a)}\ i \\ &= \mu(a) \pm i\omega(a). \end{aligned}$$

In order for a Hopf bifurcation to take place the eigenvalues need to be purely imaginary as a passes a critical value, a_0. Notice that

$$\mu(a_0) = 0 \implies a_0 = 0;$$

$$\omega(a_0) = \sqrt{4\det A(a)} = 1.$$

Therefore at the critical value $a_0 = 0$, the equilibrium $(1,1)$ has eigenvalues

$$\lambda_{1,2}(a_0) = \pm i.$$

Now we apply theorem (4.3) to check whether it is possible to write the system in normal form. To check (A2) we calculate $\mu(a) = \frac{a}{2} \implies \dot{\mu}(a_0) = \frac{1}{2} \neq 0$. Next, we need to check condition (A1), which is that $l_1(0) \neq 0$. We will therefore compute the first Lyapunov coefficient. First, we need to perform a coordinate shift so that the equilibrium lies at the origin. Use the change of variables

$$x = u - 1, \quad y = v - 1.$$

Then we can convert equation (23) to

$$\begin{cases} \dot{x} = y - x^3 + ax \\ \dot{y} = -x. \end{cases} \tag{24}$$

By grouping the terms of the same degree together, we can write this in matrix form

as

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} &= \begin{pmatrix}y+ax\\ -x\end{pmatrix} + \begin{pmatrix}-x^3\\ 0\end{pmatrix}\\ &= \begin{pmatrix}a & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \frac{1}{6}C\left(\begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}x\\ y\end{pmatrix}\right)\\ &= A(a)\begin{pmatrix}x\\ y\end{pmatrix} + \frac{1}{6}C\left(\begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}x\\ y\end{pmatrix}, \begin{pmatrix}x\\ y\end{pmatrix}\right),\end{aligned}$$

where C is a multilinear function satisfying $C(\xi,\eta,\zeta) = \begin{pmatrix}-6\xi_1\eta_1\zeta_1\\ 0\end{pmatrix}$.

We now calculate

$$A(a_0) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad A^T = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We would like to choose eigenvectors p and q that correspond to

$$Aq = i\omega q = iq, \quad A^T p = -i\omega p = -ip.$$

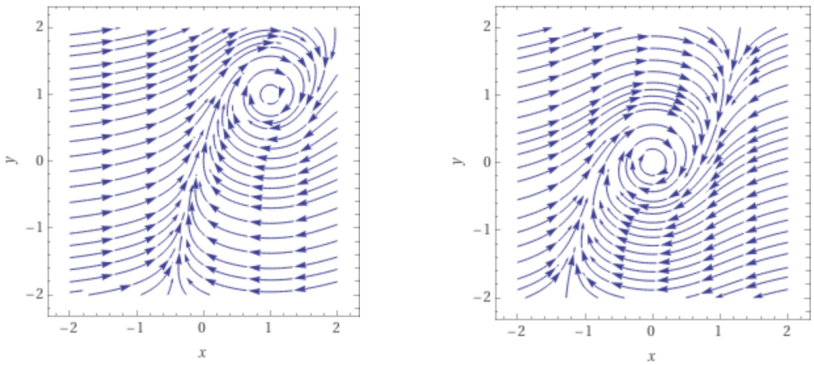
In other words, q is the eigenvector of A corresponding to eigenvalue i and p is the eigenvector of A^T corresponding to the eigenvalue $-i$. We need to choose the eigenvectors such that $\langle p, q\rangle = 1$. Therefore, we have

$$q = p = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\ 1\end{pmatrix}.$$

Now we are able to calculate the coefficients. Note that since there are no quadratic terms in our system, $g_{11} = g_{20} = 0$. On the other hand,

$$\begin{aligned}g_{21} = \langle p, C(q,q,\bar{q})\rangle &= \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\ 1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-6(-i)(-i)(i)\\ 0\end{pmatrix}\right\rangle\\ &= -\frac{3}{2}.\end{aligned}$$

Finally we can calculate



(a) System (23). (b) System (24).

Figure 3: The phase portraits of the system and transformed system.

$$l_1(a_0) = \frac{1}{2\omega_0^2}\,\mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{21}) = -\frac{3}{4}.$$

Therefore (A1) is satisfied as $l_1(0) \neq 0$. Therefore, system (23) is locally topologically equivalent to a normal form. Moreover, by theorem (4.7), since $\mathrm{sign}(l_1(0)) = -1$, the Hopf bifurcation is supercritical which results in a stable limit cycle.

The phase portraits of equations (23) and (24) are given in figure 3. These portraits agree with our result, as we can see the stable equilibrium points with stable limit cycles surrounding them.

5 Hopf Bifurcations in n-dimensions

We have developed theory to determine the nature of Hopf bifurcations in two-dimensional systems. In particular, this method involves using the two-dimensional normal form, and so it is not possible to simply extend the theorems to higher dimensions. However, in this section we will lay out how to restrict the system to a family of smooth two-dimensional invariant manifolds near the origin. The system restricted to the manifold is two dimensional, and hence we can compute the normal form in the usual manner. We will formulate the theorems that allow us to do so, but we will not prove them as they are too lengthy. We consider the **critical case**, where the parameters of the system are fixed at their bifurcation values. Arguably the most

important theorem is as follows.

Theorem 5.1 (**The Center Manifold Theorem**). *Consider a continuous-time dynamical system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{25}$$

where f is sufficiently smooth and $f(0) = 0$. Assume that the equilibrium of the system is not hyperbolic, and denote the number of eigenvalues of the Jacobian matrix evaluated at the equilibrium with $\mathrm{Re}(\lambda) = 0$, counting multiplicities, as n_0. Let T^c denote the generalised eigenspace of A corresponding to the union of the n_0 eigenvalues on the imaginary axis (i.e. T^c is the direct sum of each individual eigenspace). Let φ^t be the flow associated with (25).

Then, there is a locally defined smooth n_0-dimensional invariant manifold $W^c_{loc}(0)$ of (25) that is tangent to T^c at $x = 0$.

Moreover, there is a neighbourhood U of $x_0 = 0$ such that if $\varphi^t x \in U$ for all $t \geq 0$ ($t \leq 0$), then $\varphi^t x \to W^c_{loc}(0)$ for $t \to +\infty$ ($t \to -\infty$). □

Definition 5.1. The manifold $W^c_{loc}(0)$ is called the **center manifold**.

The eigenvalues with $\mathrm{Re}(\lambda) = 0$ as well as the generalised eigenspace, T^c are often called **critical**. We denote the number of eigenvalues (including multiplicities) with $\mathrm{Re}(\lambda) > 0$ as n_+ and the number of eigenvalues with $\mathrm{Re}(\lambda) < 0$ as n_-.

Remark 5.1. The second statement in the theorem implies that orbits near the equilibrium for $t \geq 0$ or $t \leq 0$ tend to W^c in the corresponding time direction. Moreover W^c is not necessarily unique, which does not matter for applications.

Now that we have set out theorems verifying the existence of a center manifold of n-dimensional systems, we will now lay out a method for computing it, along with the subsequent first Lyapunov coefficient computation. We will use eigenvectors corresponding to the critical values of A and A^T to project the system into the critical eigenspace and its complement. Suppose that (25) can be written as

$$\dot{x} = Ax + F(x), \quad x \in \mathbb{R}^n, \tag{26}$$

where $F(x) = O(||x||^2)$ is a smooth function.

Assume that the system undergoes a Hopf bifurcation, and A has a pair of complex eigenvalues, $\lambda_{1,2} = \pm i\omega_0,\ \omega_0 > 0$, which are the only eigenvalues with $\mathrm{Re}(\lambda) = 0$. Let $q \in \mathbb{C}^n$ be the eigenvector corresponding to λ_1. We therefore have

$$Aq = i\omega_0 q, \quad A\bar{q} = -i\omega_0 \bar{q}.$$

Let $p \in \mathbb{C}^n$ be the adjoint eigenvector such that

$$A^T p = -i\omega_0 p, \quad A^T \bar{p} = i\omega_0 \bar{p}, \quad \langle p, q\rangle = 1.$$

The critical **real** eigenspace T^c corresponding to $\pm i w_0$ is real and two-dimensional, and it is spanned by $\{\mathrm{Re}(q), \mathrm{Im}(q)\}$. The eigenspace T^{su} corresponding to all other eigenvalues of A, is real and $(n-2)$-dimensional. To determine whether a vector is in T^{su} we can use the following lemma.

Lemma 5.2. *A vector $y \in \mathbb{R}^n$ is in T^{su} if and only if $\langle p, y\rangle = 0$.* □

We can therefore decompose any $x \in \mathbb{R}^n$ as

$$x = zq + \bar{z}\bar{q} + y, \quad \text{for } z \in \mathbb{C},\ zq + \bar{z}\bar{q} \in T^c,\ y \in T^{su}.$$

The complex variable z is a coordinate on T^c, and therefore we have

$$\begin{cases} z = \langle p, x\rangle, \\ y = x - \langle p, x\rangle - \langle \bar{p}, x\rangle \bar{q}. \end{cases} \tag{27}$$

with $\langle p, \bar{q}\rangle = 0$. In these coordinates, (26) takes on the form

$$\begin{cases} \dot{z} = i\omega_0 z + \langle p, F(zq + \bar{z}\bar{q} + y)\rangle, \\ \dot{y} = Ay + F(zq + \bar{z}\bar{q} + y) - \langle p, F(zq, \bar{z}\bar{q} + y)\rangle q - \langle \bar{p}, F(zq + \bar{z}\bar{q} + y)\rangle \bar{q}. \end{cases} \tag{28}$$

Although this system is $(n+2)$-dimensional, there are two real constraints imposed on y. We can use the Taylor expansions of z, $\bar{z}$ and y to convert this system to the form

$$\begin{cases} \dot{z} = i\omega_0 z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} + \langle G_{10}, y\rangle z + \langle G_{01}, y\rangle \bar{z} + \dots, \\ \dot{y} = Ay + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \dots, \end{cases} \tag{29}$$

where $G_{20}, G_{11}, G_{02}, G_{21} \in \mathbb{C}$ and $G_{01}, G_{10}, H_{ij} \in \mathbb{C}^n$ can be computed as

$$G_{ij} = \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \langle p, F(zq + \bar{z}\bar{q}) \rangle \Big|_{z=0}, \quad i + j \geq 2,$$

$$\overline{G}_{10,i} = \frac{\partial^2}{\partial y_i \partial z} \langle p, F(zq + \bar{z}\bar{q} + y) \rangle \Big|_{z=0,\ y=0}, \quad i = 1, 2, \ldots n,$$

$$\overline{G}_{01,i} = \frac{\partial^2}{\partial y_i \partial \bar{z}} \langle p, F(zq + \bar{z}\bar{q} + y) \rangle \Big|_{z=0,\ y=0}, \quad i = 1, 2, \ldots n,$$

$$H_{ij} = \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} F(zq + \bar{z}\bar{q}) \Big|_{z=0} - G_{ij} q - \overline{G}_{ji} \bar{q}, \quad i + j = 2.$$

The center manifold now has the representation

$$y = V(z, \bar{z}) = \frac{1}{2} w_{20} z^2 + w_{11} z\bar{z} + \frac{1}{2} w_{02} \bar{z}^2 + O(|z|^3),$$

where $\langle p, w_{ij} \rangle = 0$. The vectors $w_{ij} \in \mathbb{C}^n$ can be found from the following equations

$$\begin{cases} (2i\omega_0 I - A) w_{20} & = H_{20}, \\ -A w_{11} & = H_{11}, \\ (-2i\omega_0 I - A) w_{02} & = H_{02}. \end{cases}$$

The matrices on the left hand sides of the equations are invertible because 0 and $\pm 2i\omega_0$ are not eigenvalues of A, and so we get

$$\begin{cases} w_{20} & = (2i\omega_0 I - A)^{-1} H_{20}, \\ w_{11} & = (-A)^{-1} H_{11}, \\ w_{02} & = (-2i\omega_0 I - A)^{-1} H_{02}. \end{cases}$$

and so the equations have unique solutions. We can therefore write the restriction of (28) to its center manifold up to cubic terms as

$$\begin{aligned} \dot{z} = i\omega_0 z + \frac{1}{2} G_{20} z^2 + G_{11} z\bar{z} + \frac{1}{2} G_{02} \bar{z}^2 + \\ \frac{1}{2}(G_{21} - 2\langle G_{10}, A^{-1} H_{11} \rangle + \langle G_{01}, (2iw_0 I - A)^{-1} H_{20} \rangle) z^2 \bar{z} + \ldots. \end{aligned} \tag{30}$$

This gives the restricted system directly in the complex form suitable for the Lyapunov coefficient computations.

An easier method for computations is by writing $F(x)$ in terms of multilinear functions $B(x,y)$ and $C(x,y,z)$

$$F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(||x||^4). \tag{31}$$

We can write

$$\langle G_{10}, y\rangle = \langle p, B(q,y)\rangle,\ \langle G_{01}, y\rangle = \langle p, B(\bar{q},y)\rangle,$$

and then the restricted equation (30) takes the form

$$\begin{aligned}\dot{z} &= i\omega_0 z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 \\ &+ \frac{1}{2}(G_{21} - 2\langle p, B(q, A^-1H_{11})\rangle + \langle p, B(\bar{q}, (2iw_0I - A)^{-1}H_{20})\rangle)z^2\bar{z} + \dots.\end{aligned} \tag{32}$$

with

$$\begin{aligned}G_{20} &= \langle p, B(q,q)\rangle \\ G_{11} &= \langle p, B(q,\bar{q})\rangle \\ G_{02} &= \langle p, B(\bar{q},\bar{q})\rangle \\ G_{21} &= \langle p, C(q,q,\bar{q})\rangle,\end{aligned} \tag{33}$$

and

$$\begin{cases} H_{20} = B(q,q) - \langle p, B(q,q)\rangle q - \langle \bar{p}, B(q,q)\rangle \bar{q}, \\ H_{11} = B(q,\bar{q}) - \langle p, B(q,\bar{q})\rangle q - \langle \bar{q}, B(q,\bar{q})\rangle \bar{q}. \end{cases} \tag{34}$$

We can then substitute (44) and (34) into (32) and use the identities

$$A^{-1}q = \frac{1}{i\omega_0}q, \quad A^{-1}\bar{q} = -\frac{1}{i\omega_0}\bar{q}, \quad (2i\omega_0 I - A)^{-1}q = \frac{1}{i\omega_0}q, \quad (2i\omega_0 I - A)^{-1}\bar{q} = \frac{1}{3i\omega_0}\bar{q}.$$

Then equation (32) becomes

$$\dot{z} = i\omega_0 z + \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \dots$$

where

$$g_{20} = \langle p, B(q,q)\rangle, \quad g_{11} = \langle p, B(q,\bar{q})\rangle,$$

and

$$\begin{aligned} g_{21} = &\langle p, C(q,q,\bar{q})\rangle && (35)\\ &- 2\langle p, B(q, A^{-1}B(q,\bar{q})))\rangle + \langle p, B(\bar{q}, (2i\omega_0 I - A)^{-1}B(q,q))\rangle && (36)\\ &+ \frac{1}{i\omega_0}\langle p, B(q,q)\rangle\langle p, B(q,\bar{q})\rangle && (37)\\ &- \frac{2}{i\omega_0}|\langle p, B(q,\bar{q})\rangle|^2 - \frac{1}{3i\omega_0}|\langle p, B(\bar{q},\bar{q})\rangle|^2 && (38) \end{aligned}$$

Since the terms in the fourth line are purely imaginary and the third line contains the same scalar products as in the product $g_{20}g_{11}$, we can use formula (20) from section 4,

$$l_1(0) = \frac{1}{2\omega_0^2}\,\mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{21}),$$

to calculate the invariant expression

$$\begin{aligned} l_1(0) =& \frac{1}{2w_0}\,\mathrm{Re}[\langle p, C(q,q,\bar{q}\rangle - 2\langle p, B(q, A^{-1}B(q,\bar{q}))\rangle \\ &+ \langle p, B(\bar{q}, (2i\omega_0 I - A)^{-1}B(q,q))\rangle]. \end{aligned} \tag{39}$$

6 Center Manifold Reduction of the Lorenz-96 Model

We will now implement the method in the previous section to determine the nature of the Hopf bifurcation of the Lorenz-96 model.

Definition 6.1. The **Lorenz-96 model** is governed by the equation

$$\dot{y}_i = y_{i-1}(y_{i+1} - y_{i-2}) - y_i + \alpha,$$

for $i \in \{0, ..., n-1\}$, $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. We take the indices modulo n, (i.e) $y_{i+n} = y_i$,

and we call n the **dimension** of the system. The variable α is the **forcing parameter**. Both of these variables are free parameters [5].

To use the theory from the previous section, we will apply a coordinate transformation $x_i = y_i - \alpha$. We then get the equation

$$\dot{x}_i = (x_{i-1} - \alpha)(x_{i+1} - x_{i-2}) - x_i.$$

We aim to analytically determine whether there Hopf bifurcation in this system is supercritical or subcritical. We will consider the system in four dimensions, and so we can write it as

$$\begin{cases} \dot{x}_0 = (x_3 - \alpha)(x_1 - x_2) - x_0 \\ \dot{x}_1 = (x_0 - \alpha)(x_2 - x_3) - x_1 \\ \dot{x}_2 = (x_1 - \alpha)(x_3 - x_0) - x_2 \\ \dot{x}_3 = (x_2 - \alpha)(x_0 - x_1) - x_3. \end{cases} \tag{40}$$

The Jacobian, $A(\alpha)$ of the system is

$$A(\alpha) = \begin{pmatrix} -1 & x_3 - \alpha & -x_3 + \alpha & x_1 - x_2 \\ x_2 - x_3 & -1 & x_0 - \alpha & -x_0 + \alpha \\ -x_1 + \alpha & x_3 - x_0 & -1 & x_1 - \alpha \\ x_2 - \alpha & -x_2 + \alpha & x_0 - x_1 & -1 \end{pmatrix},$$

and so we can evaluate the Jacobian at the equilibrium $(0, 0, 0, 0)$, and write the system in the form

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & -\alpha & \alpha & 0 \\ 0 & -1 & -\alpha & \alpha \\ \alpha & 0 & -1 & -\alpha \\ -\alpha & \alpha & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_3 x_1 - x_3 x_2 \\ x_0 x_2 - x_0 x_3 \\ x_1 x_3 - x_1 x_0 \\ x_2 x_0 - x_2 x_1 \end{pmatrix} \tag{41}$$

$$\implies \dot{x} = A(\alpha)|_{(0,0,0,0)} x + F(x),$$

where $F(x) = O(||x||^2)$ is a smooth function.

Now notice that at $\alpha = -1$, the Jacobian has eigenvalues

$$\lambda_{1,2} = \pm i, \quad \lambda_{3,4} = -1, -3.$$

Since the system has one pair of purely imaginary eigenvalues, we can conclude that it undergoes a Hopf bifurcation at $\alpha = -1$. Note that if we did not apply the transformation initially, the value at which the system undergoes the bifurcation would be at $\alpha = 1$. Similarly to example (4.1), we would like to find $p, q \in \mathbb{C}^n$ such that

$$Aq = iq, \quad A^T p = -ip, \quad \langle p, q \rangle = 1.$$

So q is the eigenvector of A corresponding to $\lambda_1 = i$, and p is the eigenvector of A^T corresponding to $-\lambda_1$. We can calculate that

$$q = p = \frac{1}{2} \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}.$$

Recall that T^c is the real generalised linear eigenspace, spanned by $\{\mathrm{Re}(q), \mathrm{Im}(q)\}$, and made up of vectors of the form $zq + \bar{z}\bar{q}$, $z \in \mathbb{C}$. Due to lemma (5.2), we can write any $x \in \mathbb{R}^n$ as $x = zq + \bar{z}\bar{q} + y$, for $y \in T^{su}$, since T^c and T^{su} are orthogonal. As mentioned in the previous section, we introduce the coordinates

$$\begin{cases} z = \langle p, x \rangle, \\ y = x - \langle p, x \rangle - \langle \bar{p}, x \rangle \bar{q}. \end{cases} \tag{42}$$

We can now write $F(x)$ in terms of multilinear functions $B(x, y)$ and $C(x, y, z)$: $F(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x)$ with

$$B\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \right) = \begin{pmatrix} x_3 y_1 - x_3 y_2 + y_3 x_1 - y_3 x_2 \\ x_0 y_2 - x_0 y_3 + y_0 x_2 - y_0 x_3 \\ x_1 y_3 - x_1 y_0 + y_1 x_3 - y_1 x_0 \\ x_2 y_0 - x_2 y_1 + y_2 x_0 - y_2 \end{pmatrix}$$

and $C(x, y, z) = 0$.

The equation restricted to the center manifold then takes the form

$$\begin{aligned}\dot{z} &= iz + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 \\ &+ \frac{1}{2}(G_{21} - 2\langle p, B(q, A^{-1}H_{11})\rangle + \langle p, B(\bar{q}, (2iI - A)^{-1}H_{20})\rangle)z^2\bar{z} + \dots.\end{aligned} \tag{43}$$

with

$$\begin{aligned}G_{20} &= \langle p, B(q,q)\rangle = 0 \\ G_{11} &= \langle p, B(q,\bar{q})\rangle = 0 \\ G_{02} &= \langle p, B(\bar{q},\bar{q})\rangle = 0 \\ G_{21} &= \langle p, C(q,q,\bar{q})\rangle = 0,\end{aligned} \tag{44}$$

and

$$\begin{aligned}H_{20} &= B(q,q) - \langle p, B(q,q)\rangle q - \langle \bar{p}, B(q,q)\rangle \bar{q}, \\ H_{11} &= B(q,\bar{q}) - \langle p, B(q,\bar{q})\rangle q - \langle \bar{q}, B(q,\bar{q})\rangle \bar{q}\end{aligned}$$

$$\Longrightarrow \quad H_{20} = \tfrac{1}{2}\begin{pmatrix} -1+i \\ 1-i \\ -1+i \\ 1-i \end{pmatrix}, \quad H_{11} = -\tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then equation (43) becomes

$$\dot{z} = iz + \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \dots$$

where

$$g_{20} = \langle p, B(q,q)\rangle = 0, \quad g_{11} = \langle p, B(q,\bar{q})\rangle = 0, \quad g_{02} = \langle p, B(\bar{q},\bar{q})\rangle = 0,$$

and

$$
\begin{aligned}
g_{21} &= \langle p, C(q,q,\bar{q})\rangle && (45)\\
&- 2\langle p, B(q, A^{-1}B(q,\bar{q})))\rangle + \langle p, B(\bar{q}, (2iI - A)^{-1}B(q,q))\rangle && (46)\\
&+ \frac{1}{i}\langle p, B(q,q)\rangle\langle p, B(q,\bar{q})\rangle && (47)\\
&- \frac{2}{i}|\langle p, B(q,\bar{q})\rangle|^2 - \frac{1}{3i}|\langle p, B(\bar{q},\bar{q})\rangle|^2 && (48)\\
&= -\frac{16}{13} - \frac{11}{13}i. && (49)
\end{aligned}
$$

Hence, (43) takes the form

$$
\dot{z} = iz - (\frac{8}{13} + \frac{11}{26}i)z^2\bar{z}. \tag{50}
$$

Finally, we compute the first Lyapunov coefficient to be

$$
l_1(0) = \frac{1}{2}\,\mathrm{Re}(ig_{20}g_{11} + g_{21}) = \frac{1}{2}\,\mathrm{Re}(g_{21}) = -0.6154.
$$

Since the $l_1(0) < 0$ we know that the Hopf bifurcation in the Lorenz-96 model is supercritical, and results in a stable limit cycle.

Now that we have determined the nature of the Hopf bifurcation, we can demonstrate our findings by qualitatively analysing the system. We can convert equation (50) back to Cartesian coordinates in order to plot a phase portrait and time series. We get

$$
\begin{cases}
\dot{x} = \frac{1}{13}\big(11x^2y - 8x(x^2-y^2) - \frac{11}{2}y(x^2-y^2) - 16xy^2 - \frac{1}{13}y\big)\\
\dot{y} = \frac{1}{13}\big(13x - \frac{11}{2}x(x^2-y^2) + 8y(x^2-y^2) - 11xy^2 - 16x^2y\big).
\end{cases} \tag{51}
$$

Figure 4 shows the phase portrait and time series plots of (51). Moving through the initial conditions in anticlockwise order, we see from the time series that each successive trajectory is $\frac{1}{2}\pi$ out of phase with the previous trajectory. This suggests a radial symmetry, with each of these trajectories (equidistant from the origin), approaching the origin at the same rate. We can clearly see from the phase portrait that we have a stable limit cycle, which confirms our result from the first Lyapunov coefficient calculation.

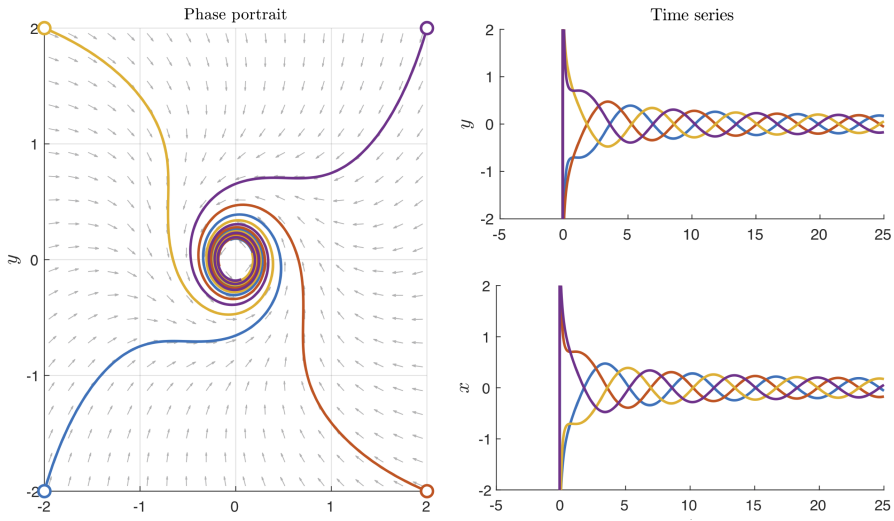


Figure 4: The phase portrait and time series of equation (51) plotted for the following initial conditions:

	$(x_0, y_0) = (-2, 2)$
	$(x_0, y_0) = (2, 2)$
	$(x_0, y_0) = (2, -2)$
	$(x_0, y_0) = (-2, -2)$

7 Conclusion

Since dynamical systems are such an integral part of the world on chemical, biological, physical and social levels, it is important to develop general methods that work to analyse a range of these systems. In this thesis, we focused primarily on Hopf bifurcations, which are commonly observed in various physical and biological systems. We aimed to find a method to determine whether a Hopf bifurcation of a dynamical system is supercritical or subcritical, resulting in a stable or unstable limit cycle respectively. We used the main reference [4], along with other material to do so. We started off in two dimensions, and proved that any two-dimensional dynamical system with a Hopf bifurcation is topologically equivalent to a normal form, which can then be easily categorised as supercritical or subcritical, using the first Lyapunov coefficient. We illustrated this by transforming a system into its normal form. Next, we developed a method for n-dimensional systems, by using center manifold reduction to restrict the system to a family of smooth two-dimensional invariant manifolds near the origin, and then computing the first Lyapunov coefficient to determine the nature of the bifurca-

tion. We illustrated this method by using the four-dimensional Lorenz-96 model as an example. We computed the first Lyapunov coefficient to be negative, meaning that it has a supercritical Hopf bifurcation, and hence a stable limit cycle. The methods outlined in this thesis can easily be applied to other systems with a Hopf bifurcation.

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A Matlab Code

1. Inner product

```
% Computes complex inner product
function z = ip(p,q)
z = p' * q;
end
```

2. Bilinear function

```
% The bilinear function B(x,y)
function z = bilinear(x,y,n)

    z = zeros(n,1);

    for j=0:n-1
        a = mod(j+3,n)+1;
        b = mod(j+1,n)+1;
        c = mod(j+2,n)+1;

        z(j+1) = x(a)*(y(b)-y(c)) + y(a)*(x(b)-x(c));
```

```
    end

end
```

3. Coefficients

```
% This code computes the relevant coefficients and equations in Section 6

A = [-1 1 -1 0 ; 0 -1 1 -1 ; -1 0 -1 1 ; 1 -1 0 -1 ];
q = 0.5*[1i ; -1 ; -1i ; 1 ]; % Eigenvector corresponding to eigenvalue i
syms z;

% G_20, G_11, G_02
G_20 = ip(q,bilinear(q,q,4))
G_11 = ip(q,bilinear(q,conj(q),4))
G_02 = ip(q,bilinear(conj(q),conj(q),4))

% H_20, H_11
H_20 = bilinear(q,q,4) - ip(q,bilinear(q,q,4))*q
- ip(conj(q),bilinear(q,q,4))*conj(q)
H_11 = bilinear(q,conj(q),4) - ip(q,bilinear(q,conj(q),4))*q -
ip(conj(q),bilinear(q,conj(q),4))*conj(q)

% g_20
g_20 = ip(q,bilinear(q,q,4))

% g_11
g_11 = ip(q,bilinear(q,conj(q),4))

% g_02
g_02 = ip(q,bilinear(conj(q),conj(q),4))

% g_21
a = -2*ip(q,bilinear(q,inv(A)*bilinear(q,conj(q),4),4));
b = ip(q,bilinear(conj(q),inv(2*1i*eye(4)-A)*bilinear(q,q,4),4));
c = 1/(1i) * ip(q,bilinear(q,q,4)) * ip(q,bilinear(q,conj(q),4));
```

```
d = -2/(1i) * (ip(q,bilinear(q,conj(q),4)))^2
- 1/(3i) * (ip(q,bilinear(conj(q),conj(q),4)))^2;

g_21 = a + b + c + d

% Compute restricted equations
restricted1 = 1i*z + 0.5*g_20*z^2 + g_11*z*conj(z) +
0.5*g_02*(conj(z))^2 + 0.5*g_21*z^2*(conj(z))

syms x y real
g = x+1i*y;
restricted2 = subs(restricted1,z,g);
xdot = real(restricted2)
ydot = imag(restricted2)

% l1(0)
l = 0.5 * real(1i*g_20*g_11 + g_21)

% check l1(0)
x = inv(A)*bilinear(q, conj(q), 4);
l1 = -2*ip(q,bilinear(q, x, 4))

y = inv(2*1i*eye(4)-A)*bilinear(q, q, 4);
l2 = ip(q,bilinear(conj(q), y, 4))

lcheck = real(l1+l2) / (2)
```