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Contract Theory for Continuous Dynamical Systems using Simulation

Master Thesis Applied Mathematics

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Abstract

This report explores contract theory, which is used in product design to (mathematically) specify and relate the requirements of multi-component products. The aim is to provide a thorough and comprehensible study on assume-guarantee contracts and what it means for systems to satisfy them. In particular, the aim is to study how contracts of subsystems (components) relate to the contract of the overall system (final product). Several key aspects of this research are defining the notions of simulation and contracts, and to analyze the properties of these notions and how they relate to one another. From our analysis we were able to formulate what it means for a system to satisfy a contract and how this can be checked. Furthermore, we were able to relate contracts of subsystems to the contract of the overall interconnected system, in case of series interconnection of the subsystems.

Contents

1 Introduction and Problem Description

In this section, the use of contracts in product design will be motivated, after which previous research on the topic will be presented. With this background information, the research goals and objectives can be stated. Thereafter, the structure of the remainder of this report will be given.

1.1 Motivation

"Design is everything. Everything!", Paul Rand.
Nowadays, the products that are being designed and manufactured are more and more complex. These products often consist of many components which are manufactured in bulk in different parts of a factory or even across multiple factories. Think for example of cars: they consist of multiple components such as an engine, a battery and a braking system. These, and other components, are manufactured separately and then put together to make a car. In order for the resulting product to be a functioning car, however, all individual components must not only do their job properly, but also must work together nicely with the other components. This is where the notion of *contracts* comes in.

Contracts are (mathematical) specifications of what is required from (the interconnection of) components, i.e. it specifies the properties that the (interconnected) components are expected to have. This allows for the individual components of a multi-component product to be designed separately from one another, according to their respective contracts. Here, correct design of all the local contracts guarantees the desired global behaviour. This means that, if all designed components satisfy the requirements in their associated contracts, then the overall interconnection of these components results in a functioning product. To get back to the car example, this can be thought of as follows: if the engine, battery, braking system and all other components satisfy their respective contracts, then the overall car, resulting from interconnecting these components, should be a functioning and safe car. From this, it is easily understood that, using contracts, complex multi-component products such as cars can be analyzed by analyzing each of its components with their associated contract. That is, analyzing whether or not individual components meet the requirements as stated in their associated contracts, can simplify analysis of the product resulting from the interconnection of all these components. This will make it easier to analyze complex, multi-component products which would otherwise be nearly impossible to analyze.

In addition, using contracts to specify the properties of each component in a complex product not only simplifies the analysis done on the final product, it also allows for easy substitution of one component by an improved version of that component as long as both satisfy the same associated contract. To get back to the car example, this means that, once we have a functioning car, we can improve its design by merely replacing one component by another (with the same associated contract) instead of redesigning the entire car from scratch. One could, for example, replace only one component, such as a diesel engine, by another component, such as a hybrid or electric engine. Hence, it is clear that making use of contracts may allow for faster development and improvement of products.

Therefore, it goes without saying that it is of great importance to study contracts. In particular, there is a need for a better understanding in the construction of contracts with regards to the mathematical systems underlying complex multi-component products. More specifically, there is a need to study the properties of contracts, in order to gain knowledge on how contracts of subsystems (components) relate to the contract of the overall system (product).

1.2 Previous Research

"To understand a science, it is necessary to know its history.", Auguste Comte.
Existing methods for expressing specifications on control systems are dissipativity, as discussed by Willems (1972), and set-invariance, as introduced by Blanchini (1999). The contracts introduced in this paper provide an alternative method to represent specifications. In other studies, such as that of Benveniste et al. (2018), contracts have already been studied for discrete systems: researching if and how contracts of the overall system relate to contracts of the components (i.e. subsystem) of such a system. Inspired by this research, we define assume-guarantee contracts for *continuous* dynamical systems. Here, the assumptions and guarantees respectively represent the expected input and desired output behaviour of such systems. In other words, the contracts introduced in this paper specify the external behaviour.

In particular, when dealing with contracts, we are interested in comparing the input-output behavior of two systems and, with that, talk about equivalence of the systems. The field of Computer Science has presented powerful methods for analyzing when different systems are, in one way or another, equivalent to each other. The paper by Pappas (2003) is an example of this, in which the notion of (bi)simulation is introduced when talking about equivalence. In addition, also the field of systems and control theory has presented methods for analysis of equivalence of systems. An example of such a study can be found in the paper by van der Schaft (2004). Here, the notion of (bi)simulation is worked out for continuous dynamical systems and its properties are explored.

In this paper, we will define and use simulation as a tool for system comparison, which will allow us to investigate contracts and to define what it means for systems to satisfy a contract. Here, the emphasis will be on continuous dynamical systems, to be formalized later in Chapter 2, and comparing them to each other using contracts.

1.3 Research Goals and Objective

"Design is a funny word. Some people think design means how it looks. But of course, if you dig deeper, it's really how it works.", Steve Jobs.
In order to gain knowledge on contacts, we formulate the following research goals:

1. Define assume-guarantee contracts.
2. Analyze how to verify if a system satisfies a specific contract or not.
3. Find a way to compare contracts.
4. Investigate how one can find a contract of the interconnection of a finite number of systems.

So, first of all, we obviously need to define what assume-guarantee contracts are. Then, we would like to analyze how we can verify if a system satisfies (or, equivalently, implements) a specific contract or not. In other words, we want to be able to determine if, for example, the engine of a car satisfies its associated contract or not. Furthermore, we would then like to find a way to compare contracts. This would allow us to compare the contract of one component (e.g. a diesel engine) to that of a possibly improved version of that component (a hybrid engine), which may allow for substitution of the former component by the latter (should the contracts of both components be found to be the same). Hereto, the notion of refinement will be discussed, which is a way of comparing contracts. Lastly, we want to investigate how one can find a contract for the interconnection of a finite number of systems. This means that we want to find the global contract of the interconnection of multiple components (say, a car) from the contracts of the individual components (think of an engine, a battery, a braking system, etc.). In other words, we want to investigate what the overall contract of a multi-component product looks like. This will lead to the introduction of the notion of composition. In short, we make the following contributions in this paper: (1) we define assume-guarantee contracts for continuous dynamical systems based on the notion of simulation; (2) we define and analyze implementation of a contract; (3) we define and review refinement; (4) we define and investigate composition.

Upon reaching the goals, a theoretical basis is provided that is needed to reach the main goal, where the ultimate goal of the research is to explore the properties of contracts. The results of this research will then be beneficial to companies that work with designing and manufacturing multi-component systems which are quite complex to analyze as a whole. Here, using contracts in modular design allows for each component to be independently analyzed, where each component can be designed and modified according to its corresponding contract. The notions of contract refinement and composition will be especially useful in doing modular design and analysis, where refinement allows us to compare contracts of different components - allowing for components to be independently replaced or exchanged with other components - and composition allows us to determine if the interconnection of all components satisfies the desired external behaviour. That is, this research presents powerful results on contract theory, which can make it easier for companies to use modular design to create, modify and analyze complex, multi-component products.

1.4 Structure of this paper

The structure of this report will be as follows: In order to reach the research goals listed above, we will first need to introduce some notation and preliminaries. This will be done in Section 2.1. After this, the notion of simulation will be defined for two different types of systems. This will be done in the remainder of Section 2. Based on these system classes, the notion of contracts will be introduced in Section 3. Expanding on this research, the concept of refinement will be introduced in Section 4. What follows is analysis on contracts for series interconnected systems, which will be done in Section 5. Next, the results retrieved will be summarized, using the previously formulated research goals, and conclusions will be drawn from it in Section 6. However, we realise that the discussed approaches are based on some assumptions and therefore have limitations; we will discuss these too in Section 6. Some proofs that are not included in the main body of this paper can be found in the Appendix.

2 Simulation Relations

2.1 Notation and Preliminaries

"To solve math problems, you need to know the basic mathematics before you can start applying it.", Catherine Asaro.
In order to reach the goals set in the introduction, it is necessary to introduce some notation and preliminaries that will be used in coming sections.

The *transpose* of any matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $A^T \in \mathbb{R}^{n\times m}$.

Definition 2.1 (Canonical Projection)**.** *The canonical projection of a subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ on $\mathcal{X}_1$, denoted by $\Pi_{\mathcal{X}_1}(S)$, is defined as*

$$\Pi_{\mathcal{X}_1}(S) = \{x_1 \in \mathcal{X}_1 \mid \exists x_2 \in \mathcal{X}_2 \text{ such that } (x_1, x_2) \in S\}.$$

Furthermore, a number of theorems and lemmas that will be introduced in the following sections are based on properties of the image of matrices. The definition of this now follows.

Definition 2.2 (Image)**.** *The image of a matrix $A \in \mathbb{R}^{m\times n}$ is defined by*

$$\operatorname{im} A = \{y \in \mathbb{R}^m \mid y = Ax, x \in \mathbb{R}^n\}.$$

If for example $A = [a_1 \dots a_n]$ with column vectors a_i, $i = 1, ..., n$, then we have $\operatorname{im} A = \operatorname{span}\{a_1, \dots, a_n\}$.

2.2 Simulation

"The beauty of mathematics only shows itself to more patient followers.", Maryam Mirzakhani.
In this section, we are interested in comparing the external behavior of two systems, which is where the notion of simulation comes in. Simulation is a way of relating the external variables of one system to those of another. The analysis on this will be mainly based on the paper by van der Schaft (2004). Hereto, the focus will be on continuous dynamical systems of the form

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + F_i d_i(t), & x_i \in \mathcal{X}_i, \quad u_i \in \mathcal{U}_i, \quad d_i \in \mathcal{D}_i, \\ y_i(t) = C_i x_i(t), & y_i \in \mathcal{Y}, \end{cases} \tag{1}$$

where we have a state variable x_i, input variable u_i, output variable y_i and an additional input variable, called disturbance, d_i. Here, $\mathcal{X}_i$,$\mathcal{U}_i$, $\mathcal{D}_i$ and $\mathcal{Y}$ are finite-dimensional vector spaces. Note that these systems are in *input-output form*, where the external behavior consists of the set of all input trajectories u_i and corresponding output trajectories y_i for which there exists an internal-variable trajectory x_i (and a disturbance d_i) such that the equations (1) hold. Furthermore, note that the disturbance d_i is an independent variable which accounts for the non-determinism in the system as a result of external disturbances or unmodelled dynamics. These systems can be visualized as in Figure 1. Now, we will consider these systems in combination with a more general form of them, given by

$$\Xi_i : \begin{cases} \dot{x}_i(t) &= A_i x_i(t) + F_i d_i(t), \\ z_i(t) &= C_i x_i(t), \\ 0 &= H_i x_i(t), \end{cases} \tag{2}$$

where z_i represents a combination of the external variables u_i and y_i and is itself an external variable that interacts with the environment. Here, note that, compared to input-output systems of the form (1), these systems are in *driving variable representation*, where the external behavior consists of all output trajectories z_i for which there exist auxiliary variable trajectories x_i and d_i such that equations (2) hold. Furthermore, another difference that we note between the two classes of systems is the presence of the algebraic constraints, given by the last equation in (2). These constraints are included here as they will turn out to be useful later. Now, these systems can be visualized as in Figure 2.

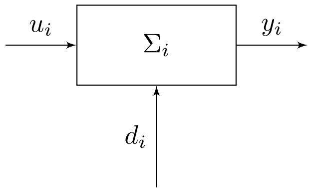

Figure 1: System Σ_i.

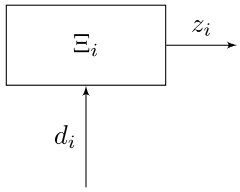

Figure 2: System Ξ_i.

2.3 Input-output systems

Having introduced the system equations of the two types of systems that will be considered, we can introduce the notion of simulation for such systems. Note that the definition of simulation is slightly different for these two types of systems. In this section, we will therefore define simulation for input-output systems of the form (1) and the next section will treat driving variable systems of the form (2).

2.3.1 Simulation

Let us start by considering input-output systems of the form (1). Then, we have the following definition of the notion of simulation:

Definition 2.3. *Consider two systems Σ_1 and Σ_2 of the form (1). A simulation relation of Σ_1 by Σ_2 is a linear subspace*

$$S \subset \mathcal{X}_1 \times \mathcal{X}_2$$

with the following property: $\forall (x_{10}, x_{20}) \in S$, $\forall u_1(\cdot) = u_2(\cdot)$ and $\forall d_1(\cdot)$, there exists $d_2(\cdot)$ such that the resulting state solution trajectories $x_1(\cdot)$ and $x_2(\cdot)$, respectively with $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$, satisfy

$$(x_1(t), x_2(t)) \in S, \qquad \forall t \geq 0, \tag{3a}$$

$$C_1 x_1(t) = C_2 x_2(t), \qquad \forall t \geq 0. \tag{3b}$$

Then, Σ_1 is said to be simulated by Σ_2, denoted by $\Sigma_1 \preceq \Sigma_2$, if the simulation relation S satisfies

$$\Pi_{\mathcal{X}_1}(S) = \mathcal{X}_1.$$

Note that property (3b) is equivalent to saying that the output of system Σ_1 is equal to that of system Σ_2. In other words, this definition says that if one system is simulated by another system, then the one system Σ_1 is mimicked by the other system Σ_2 in such a way that the (externally measurable) input-output data from the first is indistinguishable from that of the latter for all time $t \geq 0$, without having imposed a relation between the disturbance values d_1 and d_2. Or, to put it differently, simulation means that: each trajectory that can be generated by Σ_1, for a given input u, can also be generated by Σ_2. Here, it is important to note that the converse does not necessary hold, nor is it required for the notion of simulation.

Alternatively to checking if the definition is satisfied, one can check if a simulation relation satisfies the properties as stated in this lemma:

Lemma 2.4. *Let Σ_1 and Σ_2 be two systems of the form (1). A linear subspace*

$$S \subset \mathcal{X}_1 \times \mathcal{X}_2$$

is a simulation relation of Σ_1 by Σ_2 if and only if $\forall(x_1, x_2) \in S, u \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2, d_1 \in \mathcal{D}_1$, there exists $d_2 \in \mathcal{D}_2$ such that

$$1. \quad (A_1x_1 + B_1u + F_1d_1, A_2x_2 + B_2u + F_2d_2) \in S. \tag{4a}$$

$$2. \quad C_1x_1 = C_2x_2. \tag{4b}$$

Proof. Let us prove that (3a) and (4a), respectively, (3b) and (4b) are equivalent. For this, let $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ be a linear subspace.
(⇒) Take any $(x_{10}, x_{20}) \in S, u_1(\cdot) = u_2(\cdot)$ and $d_1(\cdot) \in \mathcal{D}_1$, property (3a) then implies that there exists a $d_2(\cdot) \in \mathcal{D}_2$ such that $x_1(\cdot)$ with $x_1(0) = x_{10}$ and $x_2(\cdot)$ with $x_2(0) = x_{20}$ satisfy

$$(x_1(t), x_2(t)) \in S, \quad \forall t \geq 0.$$

By linearity of S, it is then implied that

$$\left(\frac{x_1(s) - x_1(t)}{s - t}, \frac{x_2(s) - x_2(t)}{s - t}\right) \in S, \quad \forall s > t \geq 0.$$

In particular, this then implies that

$$(\dot{x}_1(t), \dot{x}_2(t)) = \lim_{s \downarrow t} \left(\frac{x_1(s) - x_1(t)}{s - t}, \frac{x_2(s) - x_2(t)}{s - t}\right) \in S, \quad \forall t \geq 0.$$

In other words, for any time instant $t \geq 0$ we found, by the system equations, that

$$(A_1x_1 + B_1u + F_1d_1, A_2x_2 + B_2u + F_2d_2) \in S,$$

with $x_i = x_i(t), u = u_i(t), d_i = d_i(t)$ for $i = 1, 2$. From this, it is then also clear that (3b) implies (4b). Hence, properties (4a) and (4b) are proved to hold.

(⇐) Conversely, take any $(x_1, x_2) \in S, u \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ and $d_1 \in \mathcal{D}_1$. Property (4a) then implies that there exists $d_2 \in \mathcal{D}_2$ such that

$$(A_1x_1 + B_1u + F_1d_1, A_2x_2 + B_2u + F_2d_2) \in S.$$

Here, for any $t \geq 0$, we have denoted $x_i = x_i(t), d_i = d_i(t), u = u_i(t)$ with $i = 1, 2$. This then clearly shows that (4b) implies (3b). Furthermore, by the system equations, it is implied that

$$(\dot{x}_1(t), \dot{x}_2(t)) \in S \tag{5}$$

for any such time instant t. Here, we note that the first Fundamental Theorem of Calculus, see Chapter 5 of (Stewart, 2016), tells us that

$$x_i(t) = \int_0^t \dot{x}_i(\hat{t})d\hat{t}.$$

This integral can then be replaced by an infinite sum in the following way

$$x_i(t) = \lim_{n \to \infty} \sum_{r=0}^{n-1} h\dot{x}_i(t_r)$$

where the time instances are defined in the following way $t_0 = 0$, ..., $t_r = rh$, ..., $t_n = nh = t$. Combining this result with equation (5), it is implied that: for any time instant $t \geq 0$, we have that

$$(x_1(t), x_2(t)) = \lim_{n \to \infty} \left(\sum_{r=0}^{n-1} h\dot{x}_i(t_r), \sum_{r=0}^{n-1} h\dot{x}_i(t_r)\right) \in S$$

by the linearity of S. In other words, properties (3a) and (3b) have been proved to hold. □

Furthermore, the notion of a simulation relation of Σ_1 by Σ_2 can be characterized algebraically as seen in the following theorem:

Theorem 2.5. *Let Σ_1 and Σ_2 be systems of the form (1). A subspace*

$$S \subset \mathcal{X}_1 \times \mathcal{X}_2$$

is a simulation relation of Σ_1 by Σ_2 if and only if

$$\operatorname{im}\begin{bmatrix} F_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \tag{6a}$$

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S + \operatorname{im}\begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \tag{6b}$$

$$\operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \tag{6c}$$

$$S \subset \ker\begin{bmatrix} C_1 & -C_2 \end{bmatrix}. \tag{6d}$$

Proof. The proof of this theorem can be found in Appendix A.1. □

Note that this theorem is very relevant as it implies that we can verify simulation through the algebraic conditions stated above, for which efficient computational tools are available. Hence, this theorem is very useful for determining whether one system is simulated by another, as we will also show with the following examples.

2.3.2 Examples

Example 2.6. *Consider the systems Σ_1 and Σ_2 of the form (1) with*

$$A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F_1 = F_2 = 0, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \textit{and} \quad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Then, using the above theorem, we can show that Σ_1 is not simulated by Σ_2. Let us show this by a proof by contradiction. Assume that Σ_1 is simulated by Σ_2, i.e. that

$$\Sigma_1 \preceq \Sigma_2.$$

By definition, this means that there must exist a simulation relation $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ that satisfies the four properties in Theorem 2.5. In particular, we must have that

$$S \subset \ker\begin{bmatrix} C_1 & -C_2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Furthermore, S must also satisfy the following inclusion

$$\operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \operatorname{im}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\} \subset S.$$

However, it is impossible for both of these inclusions to hold simultaneously, which is in contradiction with S being a simulation relation.

However, we can define a system that is very similar to Σ_2 such that Σ_1 is simulated by it. This altered system, which we will call Σ_3, has the same system matrices as Σ_2 only with

$$B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We then find, by the theorem, that S must satisfy

$$S \subset \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$\operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \subset S.$$

Here, we note that

$$S = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

satisfies both inclusions. From this, it is clear that the second property in the theorem is also satisfied since: any element $s \in S$ is of the form

$$s = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} r$$

for some $r \in \mathbb{R}$. Therefore, we find that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} r = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} r = x \in S.$$

Note that since $F_1 = F_2 = 0$, the first property of the theorem is automatically satisfied. Therefore, we have hereby proved that

$$S = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

satisfies the four properties in Theorem 2.5, by which it has been proved that S is a simulation relation of Σ_1 by Σ_3.

Now, there is one other thing we would like to note here. Namely, if a system Σ_1 is simulated by Σ_2, then that does not imply that the system Σ_2 is also simulated by Σ_1. We will show this in the following example.

Example 2.7. *Consider two systems Σ_1 and Σ_2 of the form (1) with the following system matrices*

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, F_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We will show that there is a simulation relation of Σ_1 by Σ_2, but that the converse is not true. In particular, we find that

$$S = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

satisfies all 4 properties of Theorem 2.5 and is hence a simulation relation of Σ_1 by Σ_2.

However, the converse does not hold. There is no simulation relation of Σ_2 by Σ_1, meaning that Σ_2 is not simulated by Σ_1. This can easily be shown by a proof by contradiction, similar to the previous example. When considering properties 3 and 4 of the theorem, we find that S must satisfy the following inclusion relations

$$S \subset \ker \begin{bmatrix} C_2 & -C_1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$\operatorname{im} \begin{bmatrix} B_2 \\ B_1 \end{bmatrix} = \operatorname{im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \subset S.$$

However, it is clear that there is no set S that satisfies both inclusions simultaneously, which is in contradiction with S being a simulation relation.

2.3.3 Series Interconnection

Before we can analyze more properties on the notion of simulation of one system by another, we need to define what it means to interconnect systems of the form (1). These notions of interconnection can then be used to provide properties on simulation.

Let us start by defining what it means to interconnect systems Σ_i of the form (1). For this, note that the *series interconnection* of any two linear systems Σ_i and Σ_j, denoted by $\Sigma_i \times \Sigma_j$, is given by setting the output of the the first system equal to the latter, i.e. $y_i = u_j$. Here, it is assumed that also $Y_i = U_j$. The concept of series interconnection is illustrated in Figure 3. This interconnection clearly gives the state-space realization

$$\Sigma_i \times \Sigma_j : \begin{cases} \begin{bmatrix} \dot{x}_i \\ \dot{x}_j \end{bmatrix} &= \begin{bmatrix} A_i & 0 \\ B_j C_i & A_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i + \begin{bmatrix} F_i & 0 \\ 0 & F_j \end{bmatrix} \begin{bmatrix} d_i \\ d_j \end{bmatrix}, \\ y_j &= \begin{bmatrix} 0 & C_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}. \end{cases}$$

If we introduce the following notation

$$x_{ij} = \begin{bmatrix} x_i \\ x_j \end{bmatrix}, \; A_{ij} = \begin{bmatrix} A_i & 0 \\ B_j C_i & A_j \end{bmatrix}, \; B_{ij} = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \; u_{ij} = u_i, \; F_{ij} = \begin{bmatrix} F_i & 0 \\ 0 & F_j \end{bmatrix}, \; d_{ij} = \begin{bmatrix} d_i \\ d_j \end{bmatrix},$$
$$y_{ij} = y_j, \text{ and } C_{ij} = \begin{bmatrix} 0 & C_j \end{bmatrix},$$

then the interconnected system equations can be written more compactly as

$$\Sigma_i \times \Sigma_j : \begin{cases} \dot{x}_{ij} &= A_{ij} x_{ij} + B_{ij} u_{ij} + F_{ij} d_{ij}, \\ y_{ij} &= C_{ij} x_{ij}. \end{cases} \tag{7}$$

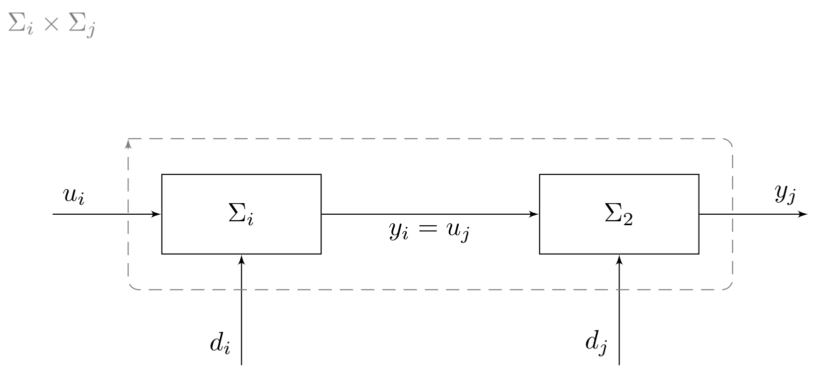

Figure 3: Interconnected system $\Sigma_i \times \Sigma_j$.

Now, using the notion of series interconnection, it is easily proved that we have the following properties for systems of the form (1):

Lemma 2.8. *Consider systems Σ_1, Σ_2, Σ_3 and Σ_4 of the form* (1)*. Then, the following properties hold:*

1. $\Sigma_1 \preceq \Sigma_1$ *for any* Σ_1. (8a)
2. *If* $\Sigma_1 \preceq \Sigma_2$ *and* $\Sigma_2 \preceq \Sigma_3$, *then* $\Sigma_1 \preceq \Sigma_3$. (8b)
3. *If* $\Sigma_1 \preceq \Sigma_3$ *and* $\Sigma_2 \preceq \Sigma_4$, *then* $\Sigma_1 \times \Sigma_2 \preceq \Sigma_3 \times \Sigma_4$. (8c)

Proof. The proof of this can be found in Appendix A.2. □

2.4 Driving variable systems

In this section, similar analysis will be done as in the previous section, only now for driving variable systems of the form (2).

2.4.1 Simulation

Let us consider the more general systems of the form (2). Due to the algebraic constraints, $0 = H_i x_i$, the definition of simulation must be altered slightly for these generalized systems. To this end, the *consistent subspace* for system Ξ_i is defined as follows

$$\mathcal{V}_{\mathcal{X}_i} = \{x_i(0) \mid \exists d_i(\cdot) \text{ for which the resulting } x_i(\cdot) \text{ satisfies } H_i x_i(t) = 0 \; \forall t \geq 0\}.$$

Hence, the consistent subspace is the set of all initial conditions for which the resulting trajectory satisfies the constraints for all time. From (Besselink et al., 2019) we then know that this space is the largest subspace $\mathcal{V}_{\mathcal{X}_i} \subset \mathcal{X}_i$ such that

$$A_i \mathcal{V}_{\mathcal{X}_i} \subset \mathcal{V}_{\mathcal{X}_i} + \operatorname{im} F_i \quad \text{and} \quad \mathcal{V}_{\mathcal{X}_i} \subset \ker H_i.$$

Furthermore, from this definition it is clear that, in case H_i is a zero matrix, then $\mathcal{V}_{\mathcal{X}_i} = \mathcal{X}_i$. Using this definition of a consistent subspace, the definition of simulation for systems Ξ_i becomes:

Definition 2.9. *Consider systems Ξ_1 and Ξ_2 of the form* (2)*. A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ satisfying*

$$\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}, \quad i = 1, 2$$

is a simulation relation of Ξ_1 by Ξ_2 if it has the following property: $\forall (x_{10}, x_{20}) \in S$ and $\forall d_1(\cdot)$ such that the resulting state solution trajectory $x_1(\cdot)$ with $x_1(0) = x_{10}$ satisfies

$$x_1(t) \in \mathcal{V}_{\mathcal{X}_1}, \quad \forall t \geq 0,$$

there exists $d_2(\cdot)$ such that the resulting state solution trajectory $x_2(\cdot)$ with $x_2(0) = x_{20}$, satisfies

$$(x_1(t), x_2(t)) \in S, \qquad \forall t \geq 0, \tag{9a}$$

$$C_1 x_1(t) = C_2 x_2(t), \quad \forall t \geq 0. \tag{9b}$$

Then, Ξ_1 is said to be simulated by Ξ_2, denoted by $\Xi_1 \preceq \Xi_2$, if the simulation relation S satisfies

$$\Pi_{\mathcal{X}_1}(S) = \mathcal{V}_{\mathcal{X}_1}.$$

Similar to what has been seen previously in Lemma 2.4, the notion of a simulation relation can now be characterized algebraically as follows:

Lemma 2.10. *Let Ξ_1 and Ξ_2 be two systems with constraints, of the form* (2)*. A linear subspace*

$$S \subset \mathcal{X}_1 \times \mathcal{X}_2$$

satisfying $\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}, i = 1, 2$ is a simulation relation of Ξ_1 by Ξ_2 if and only if $\forall (x_1, x_2) \in S$

1. *$\forall d_1 \in \mathcal{D}_1$ such that $A_1 x_1 + F_1 d_1 \in \mathcal{V}_{\mathcal{X}_1}$, there exists $d_2 \in \mathcal{D}_2$ such that $A_2 x_2 + F_2 d_2 \in \mathcal{V}_{\mathcal{X}_2}$ and $(A_1 x_1 + F_1 d_1, A_2 x_2 + F_2 d_2) \in S$.* (10a)
2. *$C_1 x_1 = C_2 x_2$.* (10b)

Proof. The proof of this follows by a similar reasoning to that of Lemma 2.4 and is therefore not repeated here. □

In addition, the following theorem can be used to check if a subspace S is in fact a simulation relation of Ξ_1 by Ξ_2. Note that this theorem can also be found in (Besselink et al., 2019, Thm. 6) and is similar to Theorem 2.5.

Theorem 2.11. *Let Ξ_1 and Ξ_2 be systems of the form* (2)*. A subspace*

$$S \subset \mathcal{X}_1 \times \mathcal{X}_2$$

is a simulation relation of Ξ_1 by Ξ_2 satisfying $\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}, i = 1, 2$ if and only if

$$\begin{bmatrix} \operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1} \\ 0 \end{bmatrix} \subset S + \begin{bmatrix} 0 \\ \operatorname{im} F_2 \end{bmatrix}, \tag{11a}$$

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S + \operatorname{im} \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \tag{11b}$$

$$S \subset \ker \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2 \end{bmatrix}. \tag{11c}$$

Proof. The proof of this is similar to that of Theorem 2.5 and can be found in Appendix A.3. □

2.4.2 Examples

Similar to what was noted for Theorem 2.5, this theorem is useful for determining whether or not one system Ξ_1 is simulated by another systems Ξ_2. Applications of this theorem will be shown in the following examples.

Example 2.12. *Consider the systems Ξ_1 and Ξ_2 of the form* (2) *given by*

$$\Xi_1 : \begin{cases} \dot{x}_1(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_1(t), \\ z_1(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_1(t), \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_1(t), \end{cases} \quad \text{and} \quad \Xi_2 : \begin{cases} \dot{x}_2(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_2(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} d_2(t), \\ z_2(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_2(t), \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_2(t). \end{cases}$$

Let us assume that $\Xi_1 \preceq \Xi_2$ holds. Then, by the previously stated theorem, we will find that this leads to a contradiction, which would imply that the assumption is wrong and that Ξ_1 is not simulated by Ξ_2. So, assuming that Ξ_1 is simulated by Ξ_2, Theorem 2.11 tells us that there must exist a simulation relation

$$S \subset \mathcal{X}_1 \times \mathcal{X}_2$$

satisfying $\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}$, $i = 1, 2$, *and satisfying properties* (11a), (11b) *and* (11c). *In particular, property* (11c) *tells us that*

$$S \subset \ker \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2 \end{bmatrix} = \ker \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Since $\Xi_1 \preceq \Xi_2$ *holds by assumption, we hence know that*

$$\mathcal{V}_{\mathcal{X}_1} = \Pi_{\mathcal{X}_1}(S) = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Furthermore, we note that

$$\operatorname{im} F_1 = \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Together, the latter two imply that

$$\operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

In addition, it is clear that, since $F_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *we have that*

$$\operatorname{im} F_2 = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

regardless of what $\mathcal{V}_{\mathcal{X}_2}$ *looks like. Hence,* S *is required to simultaneously satisfy*

$$S \subset \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \textit{and} \quad \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subset S.$$

However, it is clear that no set S *can satisfy both these inclusions at once, which is in contradiction with* S *being a simulation relation. Therefore, we have proved, by proof by contradiction, that the initial assumption does not hold, i.e.* Ξ_1 *is not simulated by* Ξ_2.

However, if we alter the system equations of Ξ_2 *slightly then we can attain that* Ξ_1 *is simulated by* Ξ_2. *Namely, consider the same system equations as before only with*

$$F_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We then have that

$$\operatorname{im} F_1 = \operatorname{im} F_2 = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

On the systems considered, we furthermore claim the following:

Claim: $\mathcal{V}_{\mathcal{X}_1} = \mathcal{V}_{\mathcal{X}_2} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Proof. *(*$\subset$*)* Let us first prove that

$$\mathcal{V}_{\mathcal{X}_i} \subset \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

for $i = 1, 2$. This is easily proved since we know, by definition, that

$$\mathcal{V}_{\mathcal{X}_i} \subset \ker H_i = \ker \begin{bmatrix} 1 & -1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(⊃) Let us now show the reverse inclusion. Hereto, take any $x \in \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. This means that x is of the form

$$x = \begin{bmatrix}1\\1\end{bmatrix} r$$

for some $r \in \mathbb{R}$. Now, set $x_i(0) = x$ and $d_i(t) = 0$ for all $t \geq 0$. By the system equations we then know that

$$\dot{x}_i(t) = A_i x_i(t), \quad \forall t \geq 0.$$

From (Trentelman et al., 2012, Ch.3) we then know that

$$x_i(t) = e^{A_i t} x_i(0).$$

Since A_i is the identity matrix, the above is easily found to be given by

$$x_i(t) = \begin{bmatrix}e^t & 0\\0 & e^t\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} r \quad = \begin{bmatrix}e^t\\e^t\end{bmatrix} r.$$

It is then clear that $x_i(t) \in \ker H_i$ for all $t \geq 0$. By definition, this then implies that

$$x = x_i(0) \in \mathcal{V}_{\mathcal{X}_i}.$$

With this, the claim has been proved. ■

With this, it is clear that

$$\operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1} = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} \quad \text{and} \quad \operatorname{im} F_2 = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

If we now take

$$S = \operatorname{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

then we know that

$$\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}$$

for $i = 1, 2$, and that property (11c) is satisfied by previous analysis. In addition, it is clear that property (11a) is satisfied since we have that: for any $x = \begin{bmatrix}1\\1\end{bmatrix} r \in \operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1}$, where r is any element in $\mathbb{R}$, there exists $\hat{x} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} r \in S$ and $\bar{x} = -\begin{bmatrix}1\\1\end{bmatrix} r \in \operatorname{im} F_2$ such that

$$\begin{bmatrix}x\\0\\0\end{bmatrix} = \hat{x} + \begin{bmatrix}0\\0\\\bar{x}\end{bmatrix},$$

which implies that

$$\begin{bmatrix}\operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1}\\0\end{bmatrix} \subset S + \begin{bmatrix}0\\\operatorname{im} F_2\end{bmatrix}.$$

Lastly, we can also prove that property (11b) is satisfied for this S. Namely, take any $\hat{x} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} r \in S$ with any $r \in \mathbb{R}$, then we find that

$$\begin{bmatrix}A_1 & 0\\0 & A_2\end{bmatrix}\hat{x} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} r = \hat{x} \in S.$$

In other words, this proves that property (11b) is satisfied. With this, we have proved by Theorem 2.11 that

$$\Xi_1 \preceq \Xi_2.$$

Another thing we can take away from the theorem is that: if $\Xi_1 \preceq \Xi_2$ then that does not imply that also $\Xi_2 \preceq \Xi_1$. In fact, this can be shown by the following example.

Example 2.13. *Consider the systems Ξ_1 and Ξ_2 given by*

$$\Xi_1 : \begin{cases} \dot{x}_1(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} d_1(t), \\ z_1(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_1(t), \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_1(t), \end{cases} \quad \text{and} \quad \Xi_2 : \begin{cases} \dot{x}_2(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_2(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_2(t), \\ z_2(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_2(t), \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_2(t). \end{cases}$$

Similar to what is seen in the previous example, one can prove that

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

satisfies

$$\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}$$

for $i = 1, 2$, and it satisfies properties (11a), (11b) and (11c). Hence, Theorem 2.11 implies that

$$\Xi_1 \preceq \Xi_2.$$

The converse, however, is not true. Hereto, note that if it were true then by Theorem 2.11 there must exist a simulation relation

$$S \subset \mathcal{X}_2 \times \mathcal{X}_1$$

satisfying both properties (11a) and (11c), i.e.

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subset S \quad \text{and} \quad S \subset \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

It is clear that no S satisfying both properties exists. Therefore, Ξ_2 is not simulated by Ξ_1, even though the converse is true.

2.4.3 Interconnection by External Variables

In order to analyze and state more properties on the notion of simulation, we must first look at a new type of interconnection. If we study systems Ξ_i of the form (2), then the system equations of the *interconnection by external variables* can be found to be given by

$$\Xi_1 \otimes \Xi_2 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \\ z_{12} &= \frac{1}{2} \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ 0 &= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{cases} \tag{12}$$

Here, we defined the external variable z_{12} by $z_{12} = z_1 = z_2$, where it is assumed that $Z_1 = Z_2$. Furthermore, the last row of the constraints in the above equation specifies the requirement that $z_1 = z_2$. This interconnection is depicted in Figure 4.

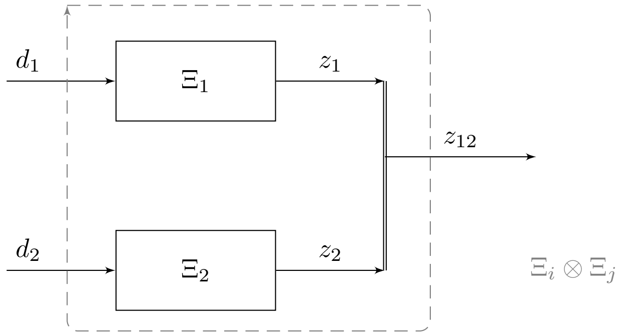

Figure 4: Interconnected system $\Xi_1 \otimes \Xi_2$.

For these type of interconnections between systems, the following properties are found:

Lemma 2.14. *Consider the systems Ξ_1, Ξ_2, Ξ_3 and Ξ_4 of the form (2). The interconnection by external variables for these systems are then as given in (12), which satisfies the following properties:*

$$\begin{aligned} &1. \quad \Xi_1 \otimes \Xi_2 \preceq \Xi_i, \quad \text{for } i = 1, 2. && \text{(13a)}\\ &2. \quad \text{If } \Xi \preceq \Xi_i, \text{ for } i = 1, 2 \text{ then } \Xi \preceq \Xi_1 \otimes \Xi_2. && \text{(13b)}\\ &3. \quad \text{If } \Xi_1 \preceq \Xi_3 \text{ and } \Xi_2 \preceq \Xi_4, \text{ then } \Xi_1 \otimes \Xi_2 \preceq \Xi_3 \otimes \Xi_4. && \text{(13c)} \end{aligned}$$

Proof. The proof of this lemma can be found in the proofs of Theorems 3 and 4 in (Besselink et al., 2019). □

In addition, also the following properties on the notion of simulation can be stated.

Lemma 2.15. *Consider systems Ξ_1, Ξ_2 and Ξ_3 of the form (2). Then, the following properties hold:*

$$\begin{aligned} &1. \quad \Xi_1 \preceq \Xi_1 \text{ for any } \Xi_1. && \text{(14a)}\\ &2. \quad \text{If } \Xi_1 \preceq \Xi_2 \text{ and } \Xi_2 \preceq \Xi_3, \text{ then } \Xi_1 \preceq \Xi_3. && \text{(14b)} \end{aligned}$$

Proof. The proof of this is similar to that of properties 1 and 2 of Lemma 2.8, see Appendix A.4. □

Note that it is also possible to interconnect systems Ξ_i of the form (2) with different z_i, i.e. where z_1 and z_2 are different combinations of the external variables u_i, y_i, $i = 1, 2$. In particular, it is possible to connect systems Ξ_i with $z_i = u_i$, together with $z_j = \begin{bmatrix} u_j \\ y_j \end{bmatrix}$. Instead of requiring that $z_i = z_j$, as we would for interconnecting systems Ξ with external variables of a similar type, we now require the following on the external variables

$$u_i = u_j$$

for such interconnections. In particular, the system equations of such interconnections are given by:

$$\Xi_i \otimes \Xi_j : \begin{cases} \begin{bmatrix} \dot{x}_i \\ \dot{x}_j \end{bmatrix} &= \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} + \begin{bmatrix} F_i & 0 \\ 0 & F_j \end{bmatrix} \begin{bmatrix} d_i \\ d_j \end{bmatrix}, \\ \begin{bmatrix} u_i \\ y_j \end{bmatrix} &= \begin{bmatrix} C_i & 0 \\ 0 & C_j^y \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}, \\ 0 &= \begin{bmatrix} H_i & 0 \\ 0 & H_j \\ C_i & -C_j^u \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}. \end{cases}$$

Hereto, note that the external variable of this interconnection is precisely that of Ξ_j. For this interconnection of two slightly different types of systems Ξ, we can formulate similar properties as seen in Lemma 2.14. These properties are stated in the following lemma.

Lemma 2.16. *Consider systems Ξ_1 and Ξ_2 of the form (2) with $z_1 = u_1$ and $z_2 = \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}$. Then, the following property holds:*

$$\Xi_1 \otimes \Xi_2 \preceq \Xi_2.$$

Proof. The proof of this statement is similar to that of property 1 in Theorem 3 in (Besselink et al., 2019), for $i = 2$. In particular, proving that the first condition in Lemma is satisfied follows the same steps as seen there. Hereto, note that we consider the linear subspace

$$S = \{(x_1, x_2, \bar{x}_2) \mid \bar{x}_2 = x_2,\ (x_1, x_2) \in \mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_2}\} \subset \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_2.$$

The only difference is in proving that the second condition in Lemma 2.10 is satisfied, which goes as follows: take any $x = (x_1, x_2, \bar{x}_2) \in S$, then we know that $(x_1, x_2) \in \Pi_{\mathcal{X}_1 \times \mathcal{X}_2}(S) \subset \mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_2}$. By property of $\mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_2}$ and by the system equations of $\Xi_1 \otimes \Xi_2$, we furthermore know that

$$\mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_2} \subset \ker \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2^u \end{bmatrix}.$$

In particular, this means that $C_1 x_1 = C_2^u x_2$. Using this, we find that: for any $x = (x_1, x_2, \bar{x}_2) \in S$, we have that

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2^y \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C_1 x_1 \\ C_2^y x_2 \end{bmatrix} = \begin{bmatrix} C_2^u x_2 \\ C_2^y x_2 \end{bmatrix} = C_2 x_2,$$

finalizing the proof. □

Note that we can also define the series interconnection of systems Ξ_i of the form (2), where $z_i = u_i$, together with systems Σ_j of the form (1). Namely, this gives the following system equations of the interconnection:

$$\Xi_i \times \Sigma_j : \begin{cases} \begin{bmatrix} \dot{x}_i \\ \dot{x}_j \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ B_j C_i & A_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} + \begin{bmatrix} F_i & 0 \\ 0 & F_j \end{bmatrix} \begin{bmatrix} d_i \\ d_j \end{bmatrix}, \\ \begin{bmatrix} u_i \\ y_j \end{bmatrix} = \begin{bmatrix} C_i & 0 \\ 0 & C_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}, \\ 0 = \begin{bmatrix} H_i & 0 \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}. \end{cases}$$

Using this series interconnection of the two different types of systems, two more important properties on these systems can be given, the first of which is formulated in the following lemma.

Lemma 2.17. *Consider systems Ξ_1 and Ξ_2 of the form (2) with $z_i = u_i$ for $i = 1, 2$, and consider systems Σ_1 and Σ_2 of the form (1). Then, the following property holds:*

$$\text{If } \Xi_1 \preceq \Xi_2 \text{ and } \Sigma_1 \preceq \Sigma_2, \text{ then } \Xi_1 \times \Sigma_1 \preceq \Xi_2 \times \Sigma_2. \tag{15a}$$

Proof. The proof of this is similar to that of property 3 in Lemma 2.8. See Appendix A.5 for the proof of this. □

The second important property on interconnecting the two types of systems is specified in the now following lemma.

Lemma 2.18. *Consider systems* Ξ_1 *and* Ξ_2 *of the form* (2) *with* $z_1 = u_1$ *and* $z_2 = \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}$. *In addition, consider system* Σ *of the form* (1). *Then, the following property holds:*

$$\textit{If } \Xi_1 \times \Sigma \preceq \Xi_2, \textit{ then } \Xi_1 \times \Sigma \preceq \Xi_1 \otimes \Xi_2. \tag{16a}$$

Proof. The proof of this can be found in Appendix A.6. □

In this chapter, we have defined the notion of simulation for input-output systems and driving variable systems. This notion provides a way of comparing the external behaviour of two systems and expresses if one system can be mimicked by the other, in the sense that the external trajectories of the one system can be matched by that of the other. In particular, this notion presents a way in which one can compare two systems, which will turn out to be very useful when we will introduce and characterize assume-guarantee contracts in the next chapter.

3 Contract Theory: Compatibility and Consistency

3.1 Contract Theory

"The study of mathematics, like the Nile, begins in minuteness but ends in magnificence.", Charles Caleb Colton.

Now that we know what it means for one system to be simulated by another, let us connect this to the notion of contracts. Hereto, concepts from the book by Benveniste et al. (2018) will be used to formulate some definitions. Recall that we consider continuous dynamical systems of the form

$$\Sigma : \begin{cases} \dot{x}_\Sigma &= A_\Sigma x_\Sigma + B_\Sigma u_\Sigma + F_\Sigma d_\Sigma, \quad x_\Sigma \in \mathcal{X}_\Sigma, \quad u_\Sigma \in \mathcal{U}_\Sigma, \quad d_\Sigma \in \mathcal{D}_\Sigma, \\ y_\Sigma &= C_\Sigma x_\Sigma, \quad y_\Sigma \in \mathcal{Y}, \end{cases} \tag{17}$$

with state variable x_Σ, input variable u_Σ, output variable y_Σ and disturbance d_Σ. Note that the variables' time-dependency is omitted for simplicity. This system can be visualized as in Figure 5. Here, the system is an *open* system, meaning that its inputs u are provided by another system or by the external world. This other system or exterior world is called the *environment* of the system Σ. An environment E for the system Σ is defined to be a system of the form

$$E : \begin{cases} \dot{x}_E = A_E x_E + F_E d_E, \\ u_E = C_E x_E, \\ 0 \ \ = H_E x_E, \end{cases} \tag{18}$$

as depicted in Figure 6. Here, the system Σ can operate in interconnection with an environment E, where the two interact through the external variable u_Σ. In particular, an environment E generates outputs u_E that can serve as inputs u_Σ to the open system Σ, meaning that an environment E is responsible for providing input trajectories to the open system Σ. Furthermore, it is clear that the environment E is a system of the form (2) with $z_E = u_E$. Therefore, all the theory that was found for systems Ξ_i of the form (2), applies to environments E.

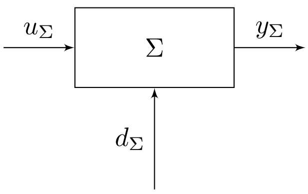

Figure 5: System Σ.

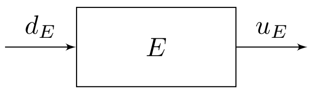

Figure 6: Environment E.

3.2 Series Interconnection

Since an environment E provides the inputs to the open system Σ, let us explain what the series interconnection of an environment E with the system Σ looks like. Similar to what has been defined in the previous chapter, the series interconnection of E and Σ, denoted by $E \times \Sigma$ is given by setting the output of the environment, u_E, equal to the input of the system, u_Σ. This gives the following:

$$E \times \Sigma : \begin{cases} \begin{bmatrix} \dot{x}_E \\ \dot{x}_\Sigma \end{bmatrix} &= \begin{bmatrix} A_E & 0 \\ B_\Sigma C_E & A_\Sigma \end{bmatrix} \begin{bmatrix} x_E \\ x_\Sigma \end{bmatrix} + \begin{bmatrix} F_E & 0 \\ 0 & F_\Sigma \end{bmatrix} \begin{bmatrix} d_E \\ d_\Sigma \end{bmatrix}, \\ \begin{bmatrix} u_E \\ y_\Sigma \end{bmatrix} &= \begin{bmatrix} C_E & 0 \\ 0 & C_\Sigma \end{bmatrix} \begin{bmatrix} x_E \\ x_\Sigma \end{bmatrix}, \\ 0 &= \begin{bmatrix} H_E & 0 \end{bmatrix} \begin{bmatrix} x_E \\ x_\Sigma \end{bmatrix}. \end{cases} \tag{19}$$

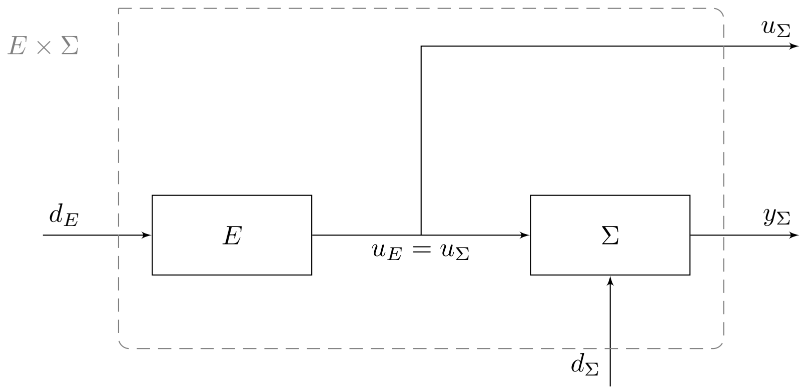

Figure 7: Interconnected system $E \times \Sigma$.

So, the series interconnection of an environment E with constraints and a system Σ without constraints, is found to be a system with constraints. An illustration is given in Figure 7. Now, we want to provide specifications on the system Σ, which will be formalized as the notion of a contract. In particular, we want to specify the inputs that are allowed to be fed to our system Σ and the corresponding outputs that we then expect. In other words, we are interested in the input-output trajectories and the properties of the interconnected system $E \times \Sigma$. To this end, we need to specify a couple of things. Firstly, it needs to be specified what type of environments are allowed to interconnect with the system Σ. In particular, it needs to be specified in which environments the system Σ can operate, i.e. we need to specify what properties the output trajectories of the environment, u_E, are expected to satisfy. These properties are represented by the so-called *assumptions*. Here, the assumptions A are given by a system of a form similar to that of the system E, namely:

$$A : \begin{cases} \dot{x}_A &= A_A x_A + F_A d_A, \\ u_A &= C_A x_A, \\ 0 &= H_A x_A. \end{cases} \tag{20}$$

Secondly, we need to specify what properties the input-output trajectories (u, y) from the interconnected system are expected to have. The *guarantees* represent these properties and are given by a system of a similar form as the system $E \times \Sigma$, namely:

$$G : \begin{cases} \dot{x}_G &= A_G x_G + F_G d_G, \\ \begin{bmatrix} u_G \\ y_G \end{bmatrix} &= C_G x_G, \\ 0 &= H_G x_G. \end{cases} \tag{21}$$

Here, note that all these systems, $E, E \times \Sigma, A$ and G, are systems of the form (2). Therefore, all properties of systems Ξ_i, as discussed in the previous chapter, in particular apply to these systems.

This leads to the notion of a contract:

Definition 3.1. *A contract $\mathcal{C}$ is given as a pair $\mathcal{C} = (A, G)$ of assumptions A and guarantees G.*

Hence, a contract specifies what assumptions are being made about the environments E the system Σ can operate in, and what guarantees the interconnected system $E \times \Sigma$ then provides. In other words, the assumptions lead to a class of possible environments, whereas the guarantees lead to a class of possible systems Σ.

3.3 Compatible Contracts

We can now define the following for contracts, using the previously defined notion of simulation:

Definition 3.2. 1. *An environment E is compatible with contract $\mathcal{C} = (A, G)$ if E is simulated by A, i.e. $E \preceq A$. A contract is called incompatible if it has no such environment.*

2. *The system Σ implements contract $\mathcal{C} = (A, G)$ if the interconnected system $E \times \Sigma$ is simulated by G, i.e. $E \times \Sigma \preceq G$, for any environment E that is compatible with $\mathcal{C}$.*

In this second part of the definition, we are asked to show that the interconnected system $E \times \Sigma$ is simulated by G for *any* environment E that is compatible with $\mathcal{C}$. Now, we would, of course, like to check if a system implements a contract, without having to check if the definition holds for all environments E that are compatible with $\mathcal{C}$. Here, we note that, as stated before, the assumptions A can be considered as the class of possible environments, meaning that A is a set of all environments in which the system can operate. Intuitively, one might then think that it is enough to check whether or not the system Σ is such that $A \times \Sigma \preceq G$ holds, in order for the system Σ to implement the contract. This is indeed found to be true, as is formalized and proved in the following theorem. Note that this theorem is similar to lemma 5 in (Besselink et al., 2019), although in a slightly different setting.

Theorem 3.3. *A system Σ implements contract $\mathcal{C} = (A, G)$ if and only if the interconnected system $A \times \Sigma$ is simulated by G, i.e. $A \times \Sigma \preceq G$.*

Proof. *($\Rightarrow$)* Assume that system Σ implements contract $\mathcal{C} = (A, G)$. By definition this implies that, for any environment E that is compatible with the contract, we have that

$$E \times \Sigma \preceq G.$$

By property 1 of Lemma 2.15 we know that $A \preceq A$. So, in particular, we know that A is compatible with $\mathcal{C}$ and hence

$$A \times \Sigma \preceq G.$$

($\Leftarrow$) Conversely, assume that $A \times \Sigma \preceq G$ holds. Let E be any compatible environment of contract $\mathcal{C} = (A, G)$, then

$$E \preceq A.$$

Lemma 2.8 furthermore tells us that

$$\Sigma \preceq \Sigma.$$

The above two relations together give, by Lemma 2.17, that

$$E \times \Sigma \preceq A \times \Sigma.$$

Combining this resulting simulation with the assumption that $A \times \Sigma \preceq G$, property 2 of Lemma 2.15 implies that

$$E \times \Sigma \preceq G.$$

Hence, it has been proved that $E \times \Sigma \preceq G$ for all environments E that are compatible with $\mathcal{C}$. Therefore, Σ implements contract $\mathcal{C}$ by definition. □

Note that, using Theorem 2.11, the theorem above leads to conditions for contract implementation that can be verified numerically, as stated in the following corollary.

Corollary 3.4. *A system Σ implements contract $\mathcal{C} = (A, G)$ if and only if there exists a subspace $S \subset \mathcal{X}_A \times \mathcal{X}_\Sigma \times \mathcal{X}_G$ satisfying*

$$\Pi_{\mathcal{X}_A \times \mathcal{X}_\Sigma}(S) = \mathcal{V}_{\mathcal{X}_A} \times \mathcal{X}_\Sigma, \quad \Pi_{\mathcal{X}_G}(S) \subset \mathcal{V}_{\mathcal{X}_G},$$

and, additionally, satisfying the following properties

$$\begin{bmatrix} \mathrm{im}(F_A) \cap \mathcal{V}_{\mathcal{X}_A} \\ \mathrm{im}(F_\Sigma) \cap \mathcal{X}_\Sigma \\ 0 \end{bmatrix} \subset S + \begin{bmatrix} 0 \\ 0 \\ \mathrm{im}\, F_G \end{bmatrix}, \tag{22a}$$

$$\begin{bmatrix} A_A & 0 & 0 \\ B_\Sigma C_A & A_\Sigma & 0 \\ 0 & 0 & A_G \end{bmatrix} S \subset S + \mathrm{im} \begin{bmatrix} F_A & 0 & 0 \\ 0 & F_\Sigma & 0 \\ 0 & 0 & F_G \end{bmatrix}, \tag{22b}$$

$$S \subset \ker \begin{bmatrix} H_A & 0 & 0 \\ 0 & 0 & H_G \\ C_A & 0 & -C_G^u \\ 0 & C_\Sigma & -C_G^y \end{bmatrix}. \tag{22c}$$

Proof. The proof of this follows directly from applying Theorem 2.11 to the result in Theorem 3.3, together with the proofs of the claims in Appendix A.5 that tell us that

$$\mathcal{V}_{\mathcal{X}_A \times \mathcal{X}_\Sigma} = \mathcal{V}_{\mathcal{X}_A} \times \mathcal{X}_\Sigma.$$

□

3.4 Consistent Contracts

There is one more definition on contracts that we are interested in, namely:

Definition 3.5. *A contract $\mathcal{C} = (A, G)$ is called consistent if there exists at least one implementation Σ of $\mathcal{C}$. Conversely, when a contract has no implementation, then it is called inconsistent.*

Theorem 3.3 can be used to check if a system Σ implements the contract and can hence be used to say that the system is consistent. However, one may not always be able to find an implementation as there need not exist one. The following theorem gives us conditions to check whether or not a contract is consistent, which in return, by the previous Theorem, tells us if an implementation of the contract exists or not. See also lemma 3 in (Shali et al., 2021) for a similar result in a different setting.

Theorem 3.6. *A contract $\mathcal{C} = (A, G)$ is consistent only if A is simulated by G^u, i.e. $A \preceq G^u$. Here, G^u denotes system G in which we are only concerned about the generated variables u_G. This system is given by the following equations*

$$G^u : \begin{cases} \dot{x}_G &= A_G x_G + F_G d_G, \\ u_G &= C_G^u x_G, \\ 0 &= H_G x_G. \end{cases} \tag{23}$$

Proof. The proof of this can be found in Appendix A.7. □

This theorem can be used to imply inconsistency of a contract $\mathcal{C} = (A, G)$ in the following way: namely, if A is not simulated by G^u then the negation of the statement in the theorem tells us that contract $\mathcal{C}$ is not consistent. Checking for a contract if A is simulated by G^u can hence tell us that the contract considered is not consistent.

3.5 Examples

An applications of this theorem will be shown in the following example.

Example 3.7. *Let us consider the contract $\mathcal{C} = (A, G)$ with*

$$A : \begin{cases} \dot{x}_A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_A + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_A, \\ u_A &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_A, \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_A, \end{cases} \quad \text{and} \quad G : \begin{cases} \dot{x}_G &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x_G + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} d_G, \\ \begin{bmatrix} u_G \\ y_G \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_G, \\ 0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} x_G. \end{cases}$$

We claim here that this contract is not consistent, which will be shown using the negation of the previously stated theorem. Hereto, assume that $A \preceq G^u$ holds and let us show that this leads to a contradiction, therefore implying that the assumption is wrong. Assuming that $A \preceq G^u$, Theorem 2.11 implies that there exists a simulation relation

$$S \subset \mathcal{X}_A \times \mathcal{X}_G$$

which satisfies $\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}$, $i = A, G$ as well as properties (11a), (11b) and (11c). In particular, the latter property tells us that we must have that

$$S \subset \ker \begin{bmatrix} H_A & 0 \\ 0 & H_G \\ C_A & -C_{G^u} \end{bmatrix} = \ker \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

In order to check if property (11a) holds, we note that

$$\operatorname{im} F_A = \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \operatorname{im} F_G = \operatorname{im} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Furthermore, as proved in Example 2.12, we know that

$$\mathcal{V}_{\mathcal{X}_A} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

With this, we have found that

$$\operatorname{im}(F_A) \cap \mathcal{V}_{\mathcal{X}_A} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \operatorname{im} F_G = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For properties (11a) and (11c) to both hold, we must hence have that

$$\operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subset S + \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad S \subset \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

It is clear that there does not exist a set S satisfying both inclusions simultaneously, which contradicts with S being a simulation relation of A by G^u. Therefore, we found that A is not simulated by G^u. By the negation of Theorem 3.6, it is hence implied that contract $\mathcal{C} = (A, G)$ is not consistent.

The previous example therefore shows how Theorem 3.6 can be used to show that a contract is inconsistent. Next, we will give an example of how Theorem 3.3 can be used to show that a contract is consistent.

Example 3.8. *For this example, we note that slightly changing the system equations of G in the previous example can lead to a contract $\mathcal{C} = (A, G)$ that is consistent. Hereto, let us consider the same system equations as in the previous example only with*

$$F_G = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

From this, it is clear that we have that

$$\operatorname{im} F_G = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We claim that the contract $\mathcal{C} = (A, G)$ is now consistent. This will be shown by proving that the system

$$\Sigma : \begin{cases} \dot{x}_\Sigma(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_\Sigma(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\Sigma(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_\Sigma(t), \\ y_\Sigma(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_\Sigma(t), \end{cases}$$

is in fact an implementation of it. In order to show this, we must first of all state the system equations of the series interconnection of A with this system. The system equations of this interconnection are easily found to be given by

$$A \times \Sigma : \begin{cases} \begin{bmatrix} \dot{x}_A(t) \\ \dot{x}_\Sigma(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_\Sigma(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_A(t) \\ d_\Sigma(t) \end{bmatrix}, \\ \begin{bmatrix} u_A(t) \\ y_\Sigma(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_\Sigma(t) \end{bmatrix}, \\ 0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_\Sigma(t) \end{bmatrix}. \end{cases}$$

Similar to what has been proved for $\mathcal{V}_{\mathcal{X}_G}$ in the previous example, it can be shown that

$$\mathcal{V}_{\mathcal{X}_{A\times\Sigma}} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

From this, it is easily found that

$$\operatorname{im}(F_{\mathcal{X}_{A\times\Sigma}}) \cap \mathcal{V}_{\mathcal{X}_{A\times\Sigma}} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Let us now use Theorem 2.11 to show that this interconnection is simulated by G which, by Theorem 3.3, implies that the contract is consistent. Here, one can easily verify that

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

satisfies the three conditions in Theorem 2.11. In addition, note that we have the following:

Claim: $\mathcal{V}_{\mathcal{X}_G} = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$.

Proof. *(⊂)* By definition, it is known that

$$\mathcal{V}_{\mathcal{X}_G} \subset \ker H_G \quad = \ker \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix} \quad = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}.$$

(⊃) To show the reverse inclusion, take any

$$x \in \operatorname{span}\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}.$$

We then know that x is of the form $x = \begin{bmatrix}a & a & b & c\end{bmatrix}^T$ for some $a, b, c \in \mathbb{R}$. Consecutively, set $x_G(0) = x$ and $d_G(t) = 0$ for all $t \geq 0$. The system equations then imply that

$$\dot{x}_G(t) = A_G x_G(t), \quad \forall t \geq 0.$$

In return, this implies that

$$x_G(t) = e^{A_G t} x_G(0).$$

This can easily be computed and is found to be given by

$$x_G(t) = \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^t & 0 \\ 0 & te^t & 0 & e^t \end{bmatrix} \begin{bmatrix}a\\a\\b\\c\end{bmatrix} \quad = \begin{bmatrix} ae^t \\ ae^t \\ be^t \\ ate^t + ce^t \end{bmatrix}.$$

From this, it is clear that $H_G x_G(t) = 0$ for all $t \geq 0$. By definition, this then implies that

$$x = x_G(0) \in \mathcal{V}_{\mathcal{X}_G},$$

with which the claim has been proved. ■

With this claim, it is clear that the set S is such that

$$\Pi_{\mathcal{X}_i}(S) = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\} = \mathcal{V}_{\mathcal{X}_i} \tag{24}$$

for $i = A \times \Sigma$ and $i = G$. With this, it has been proved that $A \times \Sigma \preceq G$ and that we hence have that the contract $\mathcal{C} = (A, G)$ is consistent.

In this chapter, we introduced assume-guarantee contracts as a pair of assumptions, specifying the input trajectories that are allowed to be fed to an input-output system by the environment in which it operates, and guarantees, specifying the input-output trajectories that we then expect from the interconnection of the input-output system with its environment. In other words, contracts give specifications on the external behaviour of a system. Here, the notion of simulation plays an important role in determining whether or not there exists an implementation of a contract.

4 Contract Refinement

"What keeps me going are my learnings, which I would rather call my 'experience,' and my urge to explore.", Sushant Singh Rajput.
Having discussed the notion of simulation for two types of systems, let us continue discussing some other concept of contract theory. In particular, let us discuss the notion of refinement. Here, the book by Benveniste et al. (2018) will once more be used as the main source of inspiration.

Recall from the introduction that the notion of refinement is crucial in doing modular design and analysis. In particular, refinement allows us to compare contracts of different components. The following definition formalizes what it means for one contract to refine another.

Definition 4.1. *Consider two contracts $\mathcal{C}_1$ and $\mathcal{C}_2$. Contract $\mathcal{C}_1$ refines contract $\mathcal{C}_2$, denoted by $\mathcal{C}_1 \preceq \mathcal{C}_2$, if*

1. *any environment that is compatible with $\mathcal{C}_2$ is also compatible with $\mathcal{C}_1$.*
2. *any implementation of $\mathcal{C}_1$ is an implementation of $\mathcal{C}_2$.*

This definition hence tells us that a contract $\mathcal{C}_1$ refines contract $\mathcal{C}_2$ if it enlarges the class of compatible environments, but reduces the possible implementations. It is not always easy to check this, so we will state a theorem that gives sufficient conditions which, upon satisfaction, imply that one contract refines another.

However, before we can state this theorem, we must first see what the interconnection by external variables between the assumptions A and the guarantees G, denoted by $A \otimes G$, looks like. The theorem will heavily rely on this interconnection, and is therefore useful to state for future reference. Now, using what we have seen previously, the system equations of the assumptions and the guarantees are coupled to each other by requiring the following on the external variables

$$u_A = u_G.$$

The system equations of the interconnection by external variables are then found to be given by

$$A \otimes G : \begin{cases} \begin{bmatrix} \dot{x}_A \\ \dot{x}_G \end{bmatrix} &= \begin{bmatrix} A_A & 0 \\ 0 & A_G \end{bmatrix} \begin{bmatrix} x_A \\ x_G \end{bmatrix} + \begin{bmatrix} F_A & 0 \\ 0 & F_G \end{bmatrix} \begin{bmatrix} d_A \\ d_G \end{bmatrix}, \\ \begin{bmatrix} u_G \\ y_G \end{bmatrix} &= \begin{bmatrix} C_A & 0 \\ 0 & C_G^y \end{bmatrix} \begin{bmatrix} x_A \\ x_G \end{bmatrix}, \\ 0 &= \begin{bmatrix} C_A & -C_G^u \\ H_A & 0 \\ 0 & H_G \end{bmatrix} \begin{bmatrix} x_A \\ x_G \end{bmatrix}. \end{cases} \tag{25}$$

Note that the last equation specifies the requirement that $u_A = u_G$, see also the illustration in Figure 8. Using the system equations of the interconnection $A \otimes G$, we can state and prove the following theorem.

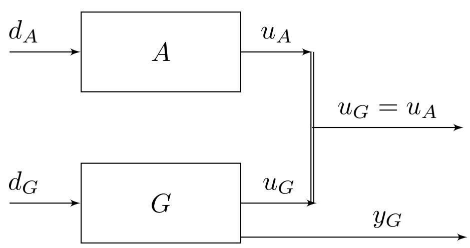

Figure 8: Interconnected system $A \otimes G$.

Theorem 4.2. *Consider contracts $\mathcal{C}_1 = (A_1, G_1)$ and $\mathcal{C}_2 = (A_2, G_2)$. If contract $\mathcal{C}_1$ is consistent, then it refines contract $\mathcal{C}_2$ if*

$$1. \quad A_2 \preceq A_1. \tag{26a}$$
$$2. \quad A_2 \otimes G_1 \preceq G_2. \tag{26b}$$

In this case, $\mathcal{C}_2$ is consistent.

Proof. Assume that $\mathcal{C}_1$ is consistent, $A_2 \preceq A_1$ and $A_2 \otimes G_1 \preceq G_2$. Since $\mathcal{C}_1$ is consistent by assumption, we know that there exists an implementation Σ of $\mathcal{C}_1$. In particular, this implies that

$$A_1 \times \Sigma \preceq G_1.$$

In addition, property 1 of Lemma 2.8 implies that $\Sigma \preceq \Sigma$ and, by assumption, we know that $A_2 \preceq A_1$. Lemma 2.17 then implies that

$$A_2 \times \Sigma \preceq A_1 \times \Sigma.$$

Combining this with the refinement relation that was found previously, this implies that

$$A_2 \times \Sigma \preceq G_1.$$

By Lemma 2.18, it is then implied that

$$A_2 \times \Sigma \preceq A_2 \otimes G_1.$$

Combining this together with the assumption that $A_2 \otimes G_1 \preceq G_2$, we find that

$$A_2 \times \Sigma \preceq G_2.$$

In other words, this proves that Σ implements $\mathcal{C}_2$ and that hence $\mathcal{C}_2$ is consistent. The second property of Definition 4.1 is therefore proved to hold.

Let us now also prove the first property of the definition, i.e. that any compatible environment of $\mathcal{C}_2$ is also a compatible environment of $\mathcal{C}_1$. To prove this, let us assume that E is a compatible environment of $\mathcal{C}_2$, i.e.

$$E \preceq A_2.$$

In addition, it is assumed that

$$A_2 \preceq A_1.$$

Together, these two imply by Lemma 2.15 that

$$E \preceq A_1.$$

With this, it has been proved that any compatible environment of $\mathcal{C}_2$ is also a compatible environment of $\mathcal{C}_1$. This then concludes the proof of the theorem. □

An example of how this theorem can be applied in practice, can be found below.

Example 4.3. *Consider the contracts $\mathcal{C}_1 = (A_1, G_1)$ and $\mathcal{C}_2 = (A_2, G_2)$ where we have*

$$A_1 : \begin{cases} \dot{x}_{A_1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_{A_1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_{A_1}, \\ u_{A_1} &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_{A_1}, \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_{A_1}, \end{cases} \qquad G_1 : \begin{cases} \dot{x}_{G_1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x_{G_1} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} d_{G_1}, \\ \begin{bmatrix} u_{G_1} \\ y_{G_1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_{G_1}, \\ 0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} x_{G_1}, \end{cases}$$

$$G_2 : \begin{cases} \dot{x}_{G_2} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} x_{G_2} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} d_{G_2}, \\ \begin{bmatrix} u_{G_2} \\ y_{G_2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x_{G_2}, \\ 0 &= \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} x_{G_2}, \end{cases}$$

and where A_2 has the exact same system equations as A_1. From Example 3.8 we then know that the contract $\mathcal{C}_1$ is consistent. Furthermore, it is easily found that the interconnection by external variables $A_2 \otimes G_1$ has the same system equations as G_2. In other words, systems A_1 and A_2 are the same, and systems $A_2 \otimes G_1$ and G_2 are also the same. By property 1 of Lemma 2.15 we know that this implies that

$$A_2 \preceq A_1 \quad \text{and} \quad A_2 \otimes G_1 \preceq G_2.$$

By Theorem 4.2 it is then implied that

$$\mathcal{C}_1 \trianglelefteq \mathcal{C}_2$$

and that the contract $\mathcal{C}_2$ is consistent. In particular, this means that the system Σ given in Example 3.8 should implement the contract $\mathcal{C}_2$. Indeed, it can be shown that the space

$$S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

satisfies all properties stated in Theorem 2.11 and is hence a simulation relation of $A_2 \times \Sigma$ by G_2,.

In this chapter, we have defined refinement. This is an important concept in contract theory that allows us to compare contracts with one another. In particular, it provides a way to determine if one component in a multi-component product can be substituted by another component. Hereto note that, from the definition of refinement, it follows that one component can be replaced by another if the contract corresponding to the latter refines that of the former. That is, once we have a functioning product, we can improve its design by merely replacing one component - satisfying contract $\mathcal{C}$ - by another component - satisfying contract $\tilde{\mathcal{C}}$ - if we have that $\tilde{\mathcal{C}} \trianglelefteq \mathcal{C}$, instead of redesigning the entire product from scratch. Contract refinement may hence allow for faster development and improvement of products.

5 Contract Composition

"To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science.", Albert Einstein.
Now, we are also interested in the concept of contract composition. More specifically, we want to relate the contracts for subsystems to the contract of the overall series interconnection of these subsystems.

5.1 Series Composition - Two Systems

Let us start by analysing series compositions, i.e. the series interconnection of systems Σ. Hereto, recall that the series composition $\Sigma_1 \times \Sigma_2$ is given by

$$\Sigma_{12} := \Sigma_1 \times \Sigma_2 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \\ y_2 &= \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{cases}$$

Now, considering contracts $\mathcal{C}_i = (A_i, G_i)$ for $i = 1, 2$, we are interested in analyzing what a contract looks like that implements the series composition $\Sigma_1 \times \Sigma_2$ of any implementations Σ_i of the contracts $\mathcal{C}_i = (A_i, G_i)$. In other words, we are interested in finding a contract $\mathcal{C} = (A, G)$, given contracts $\mathcal{C}_1$ and $\mathcal{C}_2$, such that

$$A \times \Sigma_1 \times \Sigma_2 \preceq G$$

for any systems Σ_i that implement the contracts $\mathcal{C}_i = (A_i, G_i)$ for $i = 1, 2$, see also Figure 9. Using this depiction, we can formulate some other properties that the contract $\mathcal{C} = (A, G)$ should have. Namely, we see that we need for the inputs u_A to be valid inputs to the system Σ_1, meaning that any u_A that can be generated by A should be an acceptable input for the system Σ_1. This can be written as the requirement that

$$A \preceq A_1.$$

In addition, we need the outputs y_{Σ_1} to be acceptable input for the system Σ_2, which translates to the requirement that

$$(A \times \Sigma_1)^y \preceq A_2$$

for any implementation Σ_1 of the contract $\mathcal{C}_1 = (A_1, G_1)$. Here, the subscript y denotes that the output variable is the only external variable considered of the interconnection.

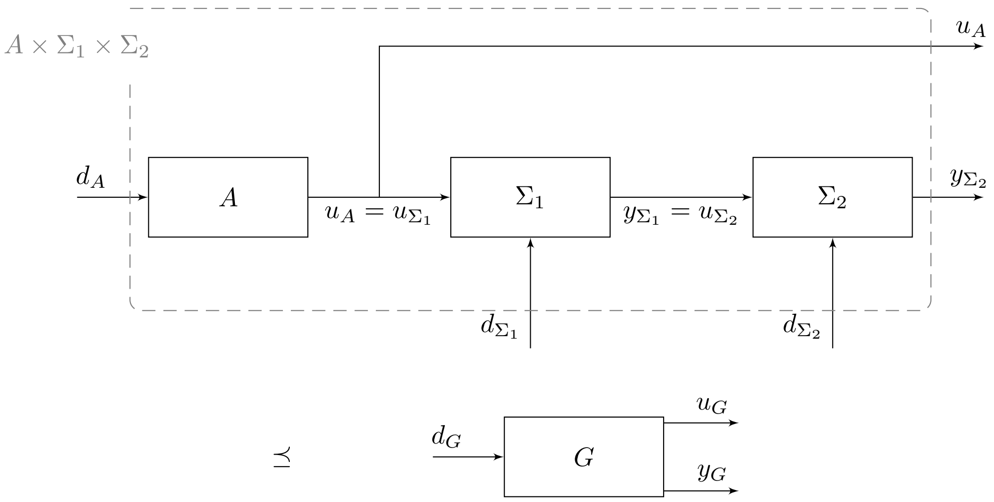

Figure 9: Depiction of $A \times \Sigma_1 \times \Sigma_2 \preceq G$.

In short, the desired properties on $\mathcal{C} = (A, G)$ are: for any implementations Σ_i of the contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, 2$, we want that

$$\begin{aligned} &1. \quad A \times \Sigma_1 \times \Sigma_2 \preceq G, && (27a)\\ &2. \quad A \preceq A_1, && (27b)\\ &3. \quad (A \times \Sigma_1)^y \preceq A_2. && (27c)\end{aligned}$$

Taking inspiration from Theorem 1 in (Saoud et al., 2018), we then formulate the following lemma:

Lemma 5.1. *Consider consistent contracts $\mathcal{C}_i = (A_i, G_i)$ for $i = 1, 2$, which are such that $G_1^y \preceq A_2$. Let systems Σ_i be any implementations of the contracts $\mathcal{C}_i = (A_i, G_i)$. Then, a contract $\mathcal{C} = (A, G)$ that satisfies*

- $A \preceq A_1$,
- $G_{12} \preceq G$,

has the three desired properties (27a), (27b) and (27c). Here, we define

$$G_{12} : \begin{cases} \begin{bmatrix} \dot{x}_{G_1} \\ \dot{x}_{G_2} \end{bmatrix} = \begin{bmatrix} A_{G_1} & 0 \\ 0 & A_{G_2} \end{bmatrix} \begin{bmatrix} x_{G_1} \\ x_{G_2} \end{bmatrix} + \begin{bmatrix} F_{G_1} & 0 \\ 0 & F_{G_2} \end{bmatrix} \begin{bmatrix} d_{G_1} \\ d_{G_2} \end{bmatrix}, \\ \begin{bmatrix} u_{G_1} \\ y_{G_2} \end{bmatrix} = \begin{bmatrix} C^u_{G_1} & 0 \\ 0 & C^y_{G_2} \end{bmatrix} \begin{bmatrix} x_{G_1} \\ x_{G_2} \end{bmatrix}, \\ \quad 0 = \begin{bmatrix} H_{G_1} & 0 \\ 0 & H_{G_2} \\ C^y_{G_1} & -C^u_{G_2} \end{bmatrix} \begin{bmatrix} x_{G_1} \\ x_{G_2} \end{bmatrix}. \end{cases} \tag{28}$$

Proof. Consider consistent contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, 2$, together with any implementations Σ_i. Assume that the contracts satisfy $G_1^y \preceq A_2$ and let $\mathcal{C} = (A, G)$ be any contract such that $A \preceq A_1$ and $G_{12} \preceq G$. Then, in order to show that the first desired property is satisfied, we claim the following:

Claim: $A_1 \times \Sigma_{12} \preceq G_{12}$.

Proof. The proof of this can be found in Appendix A.8. ∎

From the assumption that $A \preceq A_1$ and by Lemma 2.17, we then find that

$$A \times \Sigma_{12} \preceq A_1 \times \Sigma_{12}.$$

Using the claim and the assumption that $G_{12} \preceq G$, it then follows that

$$A \times \Sigma_{12} \preceq G,$$

by property 2 of Lemma 2.15. With this, the first desired property is proved to hold. Furthermore, note that the second desired property is trivially satisfied by the first condition on the contract $\mathcal{C} = (A, G)$ stated in the lemma. The third property can easily be proved to also hold. The proof of this goes as follows: since we have $A \preceq A_1$ by assumption, we know by Lemma 2.17 that

$$A \times \Sigma_1 \preceq A_1 \times \Sigma_1.$$

In addition, since Σ_1 is assumed to be an implementation of the contract $\mathcal{C}_1 = (A_1, G_1)$, we know that

$$A_1 \times \Sigma_1 \preceq G_1.$$

It then follows that

$$(A \times \Sigma_1)^y \preceq G_1^y.$$

Since it is assumed that $G_1^y \preceq A_2$, we hereby find that

$$(A \times \Sigma_1)^y \preceq A_2$$

and hence the third property also holds. □

From this lemma, we note that: if $G_1^y \preceq A_2$ holds, then a contract $\mathcal{C} = (A, G)$ that satisfies the three desired properties exists. In particular, one possible contract is given by taking

$$A = A_1 \quad \text{and} \quad G = G_{12}.$$

This then gives rise to the following corollary:

Corollary 5.2. *Consider consistent contracts $\mathcal{C}_i = (A_i, G_i)$, for $i = 1, 2$, together with any of their implementations Σ_i. Then, a contract $\mathcal{C} = (A, G)$ exists that satisfies the three desired properties (27a), (27b) and (27c) if $G_1^y \preceq A_2$.*

With this, it is also clear that the following result holds:

Corollary 5.3. *If two contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, 2$, are consistent and satisfy $G_1^y \preceq A_2$, then any contract $\mathcal{C} = (A, G)$ with $A \preceq A_1$ and $G_{12} \preceq G$ is also consistent.*

Relating this to refinement as was introduced in the previous chapter, we can even prove that the contract $\mathcal{C} = (A_1, G_{12})$ is a refinement of any other contract satisfying Lemma 5.1. So, in a way, the contract $\mathcal{C} = (A_1, G_{12})$ is the best we can do with respect to the series interconnection of any two systems Σ_1, Σ_2 implementing, respectively, contracts $\mathcal{C}_1 = (A_1, G_1)$ and $\mathcal{C}_2 = (A_2, G_2)$. This is formalized in the following lemma:

Lemma 5.4. *Consider consistent contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, 2$, with $G_1^y \preceq A_2$ and let $\mathcal{C}' = (A', G')$ be any contract that satisfies the conditions of Lemma 5.1. Then, the contract $\mathcal{C} = (A_1, G_{12})$ refines $\mathcal{C}' = (A', G')$, i.e. $\mathcal{C} \trianglelefteq \mathcal{C}'$.*

Proof. Assume that $\mathcal{C}_i = (A_i, G_i)$, $i = 1, 2$, with $G_1^y \preceq A_2$ are consistent contracts and let $\mathcal{C}' = (A', G')$ be any contract that satisfies the conditions of Lemma 5.1. By Theorem 4.2 we know that the contract $\mathcal{C} = (A_1, G_{12})$ refines contract $\mathcal{C}' = (A', G')$ if

$$1. \quad A' \preceq A_1. \tag{29a}$$

$$2. \quad A' \otimes G_{12} \preceq G'. \tag{29b}$$

The first property trivially holds since $\mathcal{C}' = (A', G')$ is assumed to satisfy the conditions of Lemma 5.1. The second property is easily proved to hold as well. The proof of this goes as follows: by Lemma 2.16, we know that

$$A' \otimes G_{12} \preceq G_{12}.$$

In addition, since $\mathcal{C}' = (A', G')$ is assumed to satisfy the conditions stated in Lemma 5.1, we know that

$$G_{12} \preceq G'.$$

Together, by Lemma 2.15, these two statements imply that

$$A' \otimes G_{12} \preceq G',$$

with which property 2 stated above is proved to hold. With this, it has been proved that the contract $\mathcal{C} = (A_1, G_{12})$ refines $\mathcal{C}' = (A', G')$. □

5.2 Series Composition - Generalized

So far, we have considered the interconnection of only two systems satisfying certain contracts. However, the theory can easily be generalized to the case where we have $n \geq 2$ systems. In this case, the three desired properties on $\mathcal{C} = (A, G)$ need to be slightly rewritten. Namely, they now read as follows: for any implementations Σ_i of the contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, ..., n$,

1. the contract $\mathcal{C} = (A, G)$ must have $\Sigma_{1,n} := \Sigma_1 \times ... \times \Sigma_n$ as an implementation :
$$A \times \Sigma_{1,n} \preceq G. \tag{30a}$$
2. the inputs u_A need to be valid inputs for the system Σ_1 :
$$A \preceq A_1. \tag{30b}$$
3. the outputs, y_{Σ_i}, of the system Σ_i need to be acceptable inputs , $u_{\Sigma_{i+1}}$, to the system Σ_{i+1} :
$$(A \times \Sigma_{1,i})^y \preceq A_{i+1}, \text{ for } i = 1, ..., n-1. \tag{30c}$$

The generalization of Lemma 5.1 then reads as follows:

Lemma 5.5. *Consider n consistent contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, ..., n$, which are such that $G^y_{1,j} \preceq A_{j+1}$ for $j = 1, ..., n-1$. Let systems Σ_i be any implementations of the contracts $\mathcal{C}_i = (A_i, G_i)$. Then, a contract $\mathcal{C} = (A, G)$ that satisfies*

- $A \preceq A_1$,
- $G_{1,n} \preceq G$,

has the three desired properties (30a), (30b) and (30c). Here, we define

$$G_{1,i} : \begin{cases} \begin{bmatrix} \dot{x}_{G_1} \\ \vdots \\ \dot{x}_{G_i} \end{bmatrix} = \begin{bmatrix} A_{G_1} & & \\ & \ddots & \\ & & A_{G_i} \end{bmatrix} \begin{bmatrix} x_{G_1} \\ \vdots \\ x_{G_i} \end{bmatrix} + \begin{bmatrix} F_{G_1} & & \\ & \ddots & \\ & & F_{G_i} \end{bmatrix} \begin{bmatrix} d_{G_1} \\ \vdots \\ d_{G_i} \end{bmatrix}, \\ \begin{bmatrix} u_{G_1} \\ y_{G_i} \end{bmatrix} = \begin{bmatrix} C^u_{G_1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & C^y_{G_i} \end{bmatrix} \begin{bmatrix} x_{G_1} \\ \vdots \\ x_{G_i} \end{bmatrix}, \\ 0 = \left[\begin{array}{cccc} H_{G_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & H_{G_i} \\ \hline C^y_{G_1} & -C^u_{G_2} & & \\ & \ddots & \ddots & \\ & & C^y_{G_{i-1}} & -C^u_{G_i} \end{array}\right] \begin{bmatrix} x_{G_1} \\ \vdots \\ x_{G_i} \end{bmatrix}. \end{cases} \tag{31}$$

Proof. Assume that we have n consistent contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, ..., n$, which are such that $G^y_{1,j} \preceq A_{j+1}$ for $j = 1, ..., n-1$. Furthermore, let systems Σ_i be any implementations of the contracts $\mathcal{C}_i = (A_i, G_i)$ and let $\mathcal{C} = (A, G)$ be any contract that satisfies $A \preceq A_1$ and $G_{1,n} \preceq G$. Let us now prove that each of the three desired properties (30a), (30b) and (30c) holds.

Proof of (30a). By proof by induction, we can show that $A_1 \times \Sigma_{1,i} \preceq G_{1,i}$ for any natural number $i \geq 1$. The proof of this goes as follows:

- Base case ($i = 1$): by assumption we know that Σ_1 is any implementations of the contract $\mathcal{C}_1 = (A_1, G_1)$. Therefore, we know that
$$A_1 \times \Sigma_1 \preceq G_1.$$

Here, note that, by definition, $G_{1,1} = G_1$ and $\Sigma_{1,1} = \Sigma_1$. This then proves that $A_1 \times \Sigma_{1,i} \preceq G_{1,i}$ holds for the base case, $i = 1$.

- Induction step: let us assume that $A_1 \times \Sigma_{1,i-1} \preceq G_{1,i-1}$ holds and use this to prove that then also $A_1 \times \Sigma_{1,i} \preceq G_{1,i}$ holds. In particular, this latter assumption tells us that $\Sigma_{1,i-1}$ is an implementation of the contract $(A_1, G_{1,i-1})$. Furthermore, we know, by the other assumptions, that Σ_i implements contract $\mathcal{C}_i = (A_i, G_i)$ and that $G^y_{1,i-1} \preceq A_i$. Lemma 5.1 then tells us that $\Sigma_{1,i} = \Sigma_{1,i-1} \times \Sigma_i$ is an implementation of the contract $(A_1, G_{1,i})$. In other words, we found that
$$A_1 \times \Sigma_{1,i} \preceq G_{1,i},$$
as was desired.

Hence, by proof by induction, we have shown that $A_1 \times \Sigma_{1,i} \preceq G_{1,i}$ holds for any natural number $i \geq 1$. In particular, this holds for $i = n$, where $n \geq 2$ is the number of contracts we consider. Since $A \preceq A_1$ and $G_{1,n} \preceq G$ by assumption, it is then implied that
$$A \times \Sigma_{1,n} \preceq G_{1,n}.$$
Therefore, property (30a) is proved to hold.

Proof of (30b). This property is trivially satisfied since we have, by assumption, that $A \preceq A_1$.

Proof of (30c). From the proof of property (30a), we know that $A_1 \times \Sigma_{1,i} \preceq G_{1,i}$ for any natural number $i \geq i$. In particular, this implies that
$$(A \times \Sigma_{1,i})^y \preceq G^y_{1,i}$$
for $i = 1, ..., n-1$. Now, by assumption, we know that $G^y_{1,i} \preceq A_{i+1}$ for any such $i = 1, ..., n-1$. Therefore, we find that
$$(A \times \Sigma_{1,i})^y \preceq A_{i+1}$$
for $i = 1, ..., n-1$, with which property (30c) is proved to hold.

This then concludes the proof of the lemma. □

Similar to before, we note that: if $G^y_{1,j} \preceq A_{j+1}$ holds for $j = 1, ..., n-1$, then there exists at least one contract $\mathcal{C} = (A, G)$ that satisfies the three desired properties. Namely, one possible contract is $\mathcal{C} = (A, G)$ where
$$A = A_1 \quad \text{and} \quad G = G_{1,n}.$$
This then gives rise to the following corollary:

Corollary 5.6. *Consider $n \geq 2$ consistent contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, ..., n$, together with any of their implementations Σ_i. Then, a contract $\mathcal{C} = (A, G)$ exists that satisfies the three desired properties* (27a), (27b) *and* (27c) *if $G^y_{1,j} \preceq A_{j+1}$ for all $j = 1, ..., n-1$.*

5.3 Example

Let us see an example of how the lemma on series composition of two systems, i.e. Lemma 5.1, can be used.

Example 5.7. *Consider the contract $\mathcal{C}_1 = (A_1, G_1)$ with*

$$A_1 : \begin{cases} \dot{x}_{A_1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_{A_1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_{A_1}, \\ u_{A_1} &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_{A_1}, \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_{A_1}, \end{cases} \quad \text{and} \quad G_1 : \begin{cases} \dot{x}_{G_1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x_{G_1} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} d_{G_1}, \\ \begin{bmatrix} u_{G_1} \\ y_{G_1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_{G_1}, \\ 0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} x_{G_1}. \end{cases}$$

This contract is consistent since we know, by Example 3.8, that it is implemented by system Σ_1 given by

$$\Sigma_1 : \begin{cases} \dot{x}_{\Sigma_1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_{\Sigma_1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{\Sigma_1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_{\Sigma_1}, \\ y_{\Sigma_1} &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_{\Sigma_1}. \end{cases}$$

In addition, let us consider the following contract $\mathcal{C}_2 = (A_2, G_2)$, where

$$A_2 : \begin{cases} \dot{x}_{A_2} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_{A_2} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_{A_2}, \\ u_{A_2} &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_{A_2}, \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_{A_2}, \end{cases} \quad \text{and} \quad G_2 : \begin{cases} \dot{x}_{G_2} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_{G_2} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} d_{G_2}, \\ \begin{bmatrix} u_{G_2} \\ y_{G_2} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_{G_2}, \\ 0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} x_{G_2}. \end{cases}$$

This contract is consistent too. In particular, it is clear that Σ_2, defined by

$$\Sigma_2 : \begin{cases} \dot{x}_{\Sigma_2} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_{\Sigma_2} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{\Sigma_2} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_{\Sigma_2}, \\ y_{\Sigma_2} &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_{\Sigma_2}, \end{cases}$$

is an implementation of this contract, since $A_2 \times \Sigma_2$ has the same system equations as G_2, for which we then clearly have that

$$A_2 \times \Sigma_2 \preceq G_2.$$

Now, let us use Theorem 2.11 to show that $G_1^y \preceq A_2$. Hereto, note that we know from Example 3.8, respectively Example 2.12, that

$$\operatorname{im}(F_{G_1}) \cap \mathcal{V}_{\mathcal{X}_{G_1}} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \operatorname{im} F_{A_2} = \Pi_{\mathcal{X}_{A_2}}(S_2) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Using this, it can easily be shown that

$$S_{12} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

satisfies all conditions in Theorem 2.11. This then proves that $G_1^y \preceq A_2$ which, by Corollary 5.2, implies that there exists a contract $\mathcal{C} = (A, G)$ that satisfies the three desired properties (27a), (27b) *and* (27c)*. In particular, one such possible contract* $\mathcal{C} = (A, G)$ *is given by*

$$A: \begin{cases} \dot{x}_A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_A + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d_A, \\ u_A &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_A, \\ 0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} x_A, \end{cases} \quad \textit{and} \quad G: \begin{cases} \dot{x}_G &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x_{G_1} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} d_{G_1}, \\ \begin{bmatrix} u_{G_1} \\ y_{G_1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_{G_1}, \\ 0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} x_{G_1}. \end{cases}$$

In this chapter, we have defined and discussed the notion of composition. This concept is of great importance in contract theory as it provides a way to determine the overall contract of a multi-component product, based on the contracts of its individual components. Here, we treated the case in which the product was formed by the series interconnection of a finite number of components. Furthermore, contract composition allows us to determine whether or not the interconnection of all components satisfies the desired external behaviour, making it a very useful notion in modular design and analysis of multi-component products.

6 Conclusion

In this section, we will summarize the results and we will draw conclusions from these results. In particular, we will look at the research goals stated in the introduction. After this, we will discuss the limitations of the results in this paper and we will discuss what further research can be done regarding the further development of this theory.

6.1 Summary of Results and Conclusion

"Each problem that I solved became a rule, which served afterwards to solve other problems.", Rene Descartes.
As was mentioned in the introduction, the purpose of this report is to address the following research goals:

1. Define assume-guarantee contracts.
2. Analyze how to verify if a system satisfies a specific contract or not.
3. Find a way to compare contracts.
4. Investigate how one can find a contract of the interconnection of a finite number of systems.

In our research, we managed to reach all of these goals. Hereto, we first needed to introduce the two types of systems that we studied and what it means for such systems to be simulated by another. Recall that simulation is a notion that allows us to compare the behaviour of two systems. This was done in Chapter 2. Next, in the beginning of Chapter 3, we defined contracts as a pair $\mathcal{C} = (A, G)$ of assumptions A and guarantees G. With this definition, we were able to define what it means for a system to satisfy such a contract, which we referred to as implementing the contract. In particular, we found that a system Σ implements a contract $\mathcal{C} = (A, G)$ if and only if the interconnected system $A \times \Sigma$ is simulated by G. Here, we used the previously introduced notion of simulation. Furthermore, we found that we could write this condition more numerically as conditions on a subspace S. Thus, in order to check if a system implements a contract, it suffices to find a subspace S that satisfies some numerical conditions. Here, we would like to stress that these conditions can be checked efficiently. With this, we have met the first and second research goal.

The third research goal was then achieved in Chapter 4. In particular, we defined the notion of refinement as a way to compare contracts. Here, the definition of refinement can be understood as follows: a contract $\mathcal{C}_1$ refines contract $\mathcal{C}_2$ if it enlarges the class of compatible environments, but reduces the possible implementations. We were able to rewrite these conditions as conditions on the assumptions and guarantees of the two contracts. Note that this is a powerful result as it tells us that, contrary to the definition, we need not find implementations of the two contracts. Instead, in order to check refinement, we only need to check conditions on the two contracts themselves.

Lastly, the fourth research goal was reached in Chapter 5. There, we looked into contracts for series compositions. It was found that if we have a finite number of local contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, ..., n$, satisfying some assumptions, and any of their implementations, then the contract $\mathcal{C} = (A_1, G_{1,n})$ implements the series composition of the n implementations. In fact, we even found that this global contract refines any other contract that has the series composition as an implementation. This result is again quite powerful since we are able to find the global contract, using only information on the local contracts. In other words, the global contract can be found independently from any implementations of the local contracts. Instead, it suffices to check conditions on the assumptions and guarantees of the local contracts.

Altogether, refinement proved to be an important concept for modular design, as it provides a way to determine if one component can be substituted by another. In particular, it was found that that one component can be replaced by another if the contract corresponding to the latter refines that of the former. Another important concept for modular design is the notion of composition. It provides a way to determine the (global) contract of a multi-component product, based on the (local) contracts of its individual components. More specifically, it allows us to determine if the interconnection of all components satisfies the desired external behaviour.

In conclusion, having achieved the research goals, a theoretical basis on contracts is provided that can be beneficial to companies that work with designing and manufacturing multi-component systems. Using contracts in modular design allows for easier analysis, design and improvement of complex, multi-component products. In particular, the notion of refinement allows for components to be independently replaced or exchanged with other components, whereas the notion of composition allows us to determine if the final product satisfies the desired external behaviour. That is, this research presents powerful results on contract theory, which can make it easier for companies to use modular design to create, modify and analyze complex, multi-component products.

6.2 Discussion

"The aim of argument, or of discussion, should not be victory, but progress.", Joseph Joubert.
Having met the research goals, we believe that the reader is in a position to use the results on contract theory that are presented in this paper. The results do not come without warning as to their applicability however. In the research conducted here, we chose to work with different types of systems: the assumptions A and guarantees G were chosen to be constrained systems, whereas systems Σ were taken to be unconstrained systems. Here, we deliberately chose the pair of systems that form the contract to be of a different type than the system implementing the contract, as we thought it would be most useful for application. In practice, we would like the mathematical equations describing our system Σ to be as much simplified as possible, to make it easier to work with them. This is why we chose the system equations to be unconstrained and of a general linear form that is widely used. On the other hand, we can imagine that the corresponding contracts contain some constraints that, for example, guarantee that the outputs are as desired. Hence, we deliberately chose the two different types of system equations in our research. However, it is possible that one deals with systems Σ (or contracts) that are of a different form, in which one does (not) have constraints. So, while the research done here is useful, it can be improved.

In particular, the research conducted here can serve as a basis for further research and application, where work expanding on this research could look into different types of systems Σ and/or contract pairs. We also suggest further research to explore how contracts for subsystems relate to the contract of the parallel and/or feedback interconnection of these subsystems. In addition to this, we also encourage to research what the overall contract would look like when a finite number of subsystems are interconnected in different ways: a combination of series, parallel and feedback interconnections. Furthermore, it would be interesting to look into questions such as: how can we guarantee contract satisfaction of a closed-loop system? Or, how can we design local contracts given a global contract? This would be very helpful and would allow for more applications, i.e. a wider range of systems and products for which contracts can be used in modular design.

A Appendix

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity.", Alan Turing.
In this section, proofs to previously stated results can be found.

A.1 Proof of Theorem 2.5

The proof of the theorem will be given in multiple parts. Firstly, we note the following: by Lemma 2.4 it is known that property (3a) is equivalent to (4a). This equivalence will now be used to prove that, in return, property (3a) is equivalent to properties (6a), (6b) and (6c).

Proof of (3a) $\iff$ (6a), (6b) *and* (6c). Here, we use that it was previously proved that (3a) is equivalent to (4a), which is in return equivalent to saying that: for any $(x_1, x_2) \in S, u \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ and $d_1 \in \mathcal{D}_1$, there exists $d_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 \in S. \tag{32}$$

This matrix-vector form will prove useful in the remainder of this proof.

($\Rightarrow$) Assume the above inclusion holds, then the following is found:

- In particular, let $u = 0$ and $(x_1, x_2) = (0, 0)$, which is an element in S since S is a linear subspace. Then, property (32) reads as follows: for all $d_1 \in \mathcal{D}_1$ there exists $d_2 \in \mathcal{D}_2$ such that
$$\begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 \in S.$$
From this, it immediately follows that
$$\operatorname{im} \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 \\ F_2 \end{bmatrix}.$$
Hence, property (6a) holds.

- Instead, let $u = 0$ and $d_1 = 0$, property (32) then implies that: for all $(x_1, x_2) \in S$ there exists $d_2 \in \mathcal{D}_2$ such that
$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 \in S.$$
Therefore, it follows that
$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S + \operatorname{im} \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$
since (x_1, x_2) is an arbitrary element in S. Hence, property (6b) is implied.

- Now, let $(x_1, x_2) = (0, 0)$ and $d_1 = 0$, then property (32) implies that: for all $u \in \mathcal{U}$ there exists $d_2 \in \mathcal{D}_2$ such that
$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 \in S.$$
This then implies that
$$\operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$
with which property (6c) is proved to hold.

Hence, it is proved that property (3a) implies that properties (6a), (6b) and (6c) hold.

(⇐) Take any $(x_1, x_2) \in S, u \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ and $d_1 \in \mathcal{D}_1$, and assume that properties (6a), (6b) and (6c) hold. Note that it is sufficient to prove that these properties imply that

$$(A_1x_1 + B_1u + F_1d_1, A_2x_2 + B_2u + F_2d_2) \in S$$

since this has previously been proved to be equivalent to property (3a). As seen before, the matrix-vector representation of the latter is given by

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 \in S.$$

Now, let us study the properties in more detail, starting with property (6a). By definition, this property implies that there exists $(\tilde{x}_1, \tilde{x}_2) \in S$ and $\bar{d}_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \bar{d}_2.$$

Secondly, property (6b) implies that there exists $(\underline{x}_1, \underline{x}_2) \in S$ and $\underline{d}_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \underline{d}_2.$$

Lastly, property (6c) implies that there also exists $(\hat{x}_1, \hat{x}_2) \in S$ and $\hat{d}_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \hat{d}_2.$$

Together, these give that

$$\begin{aligned} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \underline{d}_2 + \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \hat{d}_2 + \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \\ + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \bar{d}_2 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2. \end{aligned}$$

Now, define $\check{x}_i = \underline{x}_i + \hat{x}_i + \bar{x}_i, i = 1, 2$, then by linearity of subspace S we know that $(\check{x}_1, \check{x}_2) \in S$. Furthermore, let us choose a particular d_2, namely $d_2 = -\underline{d}_2 - \hat{d}_2 - \bar{d}_2$. Note that since $\mathcal{D}_2$ is a linear space and since $\underline{d}_2, \hat{d}_2, \bar{d}_2 \in \mathcal{D}_2$, then also $d_2 \in \mathcal{D}_2$. This then gives that:

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 = \begin{bmatrix} \check{x}_1 \\ \check{x}_2 \end{bmatrix} \in S.$$

In other words, it has now been proved that properties (6a), (6b) and (6c) imply that: if we take any $(x_1, x_2) \in S, u \in \mathcal{U}$ and $d_1 \in \mathcal{D}_1$, then there exists a $d_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 \in S$$

as desired. This then concludes the proof that properties (6a), (6b) and (6c) imply that property (3a) holds.

With this, it has been proved that property (3a) is equivalent to properties (6a), (6b) and (6c).

Proof of (3b) $\iff$ (6d). Now, note that property (3b) is equivalent to property (6d). The proof of this goes as follows: take any $(x_1, x_2) \in S$, then

$$\begin{aligned} C_1x_1 = C_2x_2 &\iff C_1x_1 - C_2x_2 = 0 \\ &\iff \begin{bmatrix} C_1 & -C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \end{aligned}$$

Since the above holds for any arbitrary $(x_1, x_2) \in S$, this is in return equivalent to

$$S \subset \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix}$$

by definition of the kernel. This latter statement is precisely what is stated in property (6d). Hence, we have proved that properties (3b) and (6d) are equivalent.

A.2 Proof of Lemma 2.8

The properties will be proved separately, where we do not write the time dependency explicitly for readability here.

Proof of 1. This is immediate with the choice

$$S = \{(x_1, x_1) \mid x_1 \in \mathcal{X}_1\}. \qquad \blacksquare$$

Proof of 2. Secondly, let us prove the second statement. For this, assume that

$$\Sigma_1 \preceq \Sigma_2 \text{ and } \Sigma_2 \preceq \Sigma_3$$

for any systems Σ_1, Σ_2 and Σ_3. In other words, there exists a simulation relation of Σ_1 by Σ_2, say $S_{12} \subset \mathcal{X}_1 \times \mathcal{X}_2$, and a simulation relation of Σ_2 by Σ_3, called $S_{23} \subset \mathcal{X}_2 \times \mathcal{X}_3$, satisfying Definition 2.3. From this, let us define another linear subspace as

$$S_{13} = \{(x_1, x_3) \mid \exists x_2 \text{ such that } (x_1, x_2) \in S_{12}, (x_2, x_3) \in S_{23}\}.$$

Take any $(x_1, x_3) \in S_{13}$, $u \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_3$ and $d_1 \in \mathcal{D}_1$. In particular, this gives us that there exists x_2 such that $(x_1, x_2) \in S_{12}, (x_2, x_3) \in S_{23}$ since $(x_1, x_3) \in S_{13}$. Now, let us again use Lemma 2.4. By definition of S_{12} being a simulation relation, this lemma implies that there exists $d_2 \in \mathcal{D}_2$ such that

$$(A_1x_1 + B_1u + F_1d_1, A_2x_2 + B_2u + F_2d_2) \in S_{12}. \tag{33}$$

Furthermore, for that d_2 we know, by definition of S_{23} being a simulation relation and by the lemma, that there exists $d_3 \in \mathcal{D}_3$ such that

$$(A_2x_2 + B_2u + F_2d_2, A_3x_3 + B_3u + F_3d_3) \in S_{23}. \tag{34}$$

By definition of S_{13}, equations (33) and (34) together imply that: $\forall (x_1, x_3) \in S_{13}, u \in \mathcal{U}, d_1 \in \mathcal{D}_1$, there exists $d_3 \in \mathcal{D}_3$ such that

$$(A_1x_1 + B_1u + F_1d_1, A_3x_3 + B_3u + F_3d_3) \in S_{13}.$$

Hence, property (3a) is proved to hold. In addition, it is clear that, by definition of S_{13}, also property (3b) holds since: for any $(x_1, x_3) \in S_{13}$ there exists x_2 such that $(x_1, x_2) \in S_{12}, (x_2, x_3) \in S_{23}$. Since S_{12} and S_{23} are simulation relations, we then know that

$$C_1x_1 = C_2x_2 \quad \text{ and } \quad C_2x_2 = C_3x_3.$$

Hence, we know that

$$C_1x_1 = C_3x_3,$$

with which property (3b) is proved. Lastly, it can easily be proved that from

$$\Pi_{\mathcal{X}_1}(S_{12}) = \mathcal{X}_1 \quad \text{ and } \quad \Pi_{\mathcal{X}_2}(S_{23}) = \mathcal{X}_2$$

it follows that

$$\Pi_{\mathcal{X}_1}(S_{13}) = \mathcal{X}_1.$$

Here, it is obvious that $\Pi_{\mathcal{X}_1}(S_{13}) \subset \mathcal{X}_1$, by definition of the canonical projection. Therefore, the only thing that is left to show is that $\mathcal{X}_1 \subset \Pi_{\mathcal{X}_1}(S_{13})$. To prove the inclusion, we need to show that: for any $x_1 \in \mathcal{X}_1$, there exists $x_3 \in \mathcal{X}_3$ such that $(x_1, x_3) \in S_{13}$. Therefore, take any $x_1 \in \mathcal{X}_1$. Since S_{12} is a simulation relation, we know that $\Pi_{\mathcal{X}_1}(S_{12}) = \mathcal{X}_1$. Hence, we find that there exists an $x_2 \in \mathcal{X}_2$ such that

$$(x_1, x_2) \in S_{12}.$$

Since S_{23} is a simulation relation, we also know that $\Pi_{\mathcal{X}_2}(S_{23}) = \mathcal{X}_2$. Therefore, it is implied that there exists an $x_3 \in \mathcal{X}_3$ such that

$$(x_2, x_3) \in S_{23}.$$

So, we found that $(x_1, x_2) \in S_{12}$ and $(x_2, x_3) \in S_{23}$ which, by definition of S_{13}, implies that

$$(x_1, x_3) \in S_{13}.$$

Thus, with this, it is proved that Σ_1 is simulated by Σ_3, i.e. $\Sigma_1 \preceq \Sigma_3$, if $\Sigma_1 \preceq \Sigma_2$ and $\Sigma_2 \preceq \Sigma_3$. ■

Proof of 3. Lastly, let us show that the third property holds. Hereto, assume that $\Sigma_1 \preceq \Sigma_3$ and $\Sigma_2 \preceq \Sigma_4$. By Definition 2.3, there exist simulation relation S_{13} of Σ_1 by Σ_3 and simulation relation S_{24} of Σ_2 by Σ_4. Let us then consider the following linear subspace

$$S = \{(x_1, x_2, x_3, x_4) \mid (x_1, x_3) \in S_{13}, (x_2, x_4) \in S_{24}\}.$$

It will be shown that this is a simulation relation of $\Sigma_1 \times \Sigma_2$ by $\Sigma_3 \times \Sigma_4$. For this, note that the series interconnection of Σ_1 and Σ_2 has as input u_1 and takes $u_2 = y_1 = C_1 x_1$, where it is assumed that $\mathcal{Y}_1 = \mathcal{U}_2$. Similarly, note that the series interconnection of Σ_3 and Σ_4 has as input u_3 and takes $u_4 = y_3 = C_3 x_3$, where it is assumed that $\mathcal{Y}_3 = \mathcal{U}_4$. So, to prove the claim, take any $(x_1, x_2, x_3, x_4) \in S, u \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_3, d_1 \in \mathcal{D}_1$ and $d_2 \in \mathcal{D}_2$. Then, since $\Sigma_1 \preceq \Sigma_3$, we know by Lemma 2.4 that: for such $(x_1, x_3) \in S_{13}, u \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_3$ and $d_1 \in \mathcal{D}_1$, there exists $d_3 \in \mathcal{D}_3$ such that

$$(A_1 x_1 + B_1 u + F_1 d_1, A_3 x_3 + B_3 u + F_3 d_3) \in S_{13}.$$

In addition, the lemma also tells us that

$$u_2 = C_1 x_1 = C_3 x_3 = u_4.$$

Furthermore, since we also have that $\Sigma_2 \preceq \Sigma_4$ by assumption, Lemma 2.4 implies that: for our variables $(x_2, x_4) \in S_{24}, u_2 = C_1 x_1 = C_3 x_3 = u_4 \in \mathcal{U}_2 \cap \mathcal{U}_4$ and $d_2 \in \mathcal{D}_2$, there exists $d_4 \in \mathcal{D}_4$ such that

$$(A_2 x_2 + B_2 C_1 x_1 + F_2 d_2, A_4 x_4 + B_4 C_3 x_3 + F_4 d_4) \in S_{24}.$$

By definition of S, these statements together imply that

$$(A_1 x_1 + B_1 u + F_1 d_1, A_2 x_2 + B_2 C_1 x_1 + F_2 d_2, A_3 x_3 + B_3 u + F_3 d_3, A_4 x_4 + B_4 C_3 x_3 + F_4 d_4) \in S.$$

Using the previously introduced notation for x_{ij}, d_{ij}, etc, this can be rewritten as

$$(A_{12} x_{12} + B_{12} u + F_{12} d_{12}, A_{34} x_{34} + B_{34} u + F_{34} d_{34}) \in S.$$

In other words, this proves that property (3a) is satisfied. By definition of S, it is then clear that also property (3b) holds since: for any $(x_1, x_2, x_3, x_4) \in S$, we have that $(x_1, x_3) \in S_{13}$ and $(x_2, x_4) \in S_{24}$. Since S_{13} and S_{24} are simulation relations, we know that

$$C_1 x_1 = C_3 x_3 \quad \text{and} \quad C_2 x_2 = C_4 x_4.$$

Hence, using the previously introduced notation, we in particular know that

$$C_{12} x_{12} = C_2 x_2 = C_4 x_4 = C_{34} x_{34}$$

with which property (3b) is proved. In addition, it is clear that from

$$\Pi_{\mathcal{X}_1}(S_{13}) = \mathcal{X}_1 \quad \text{and} \quad \Pi_{\mathcal{X}_2}(S_{24}) = \mathcal{X}_2$$

it follows that

$$\Pi_{\mathcal{X}_1\times\mathcal{X}_2}(S) = \mathcal{X}_1 \times \mathcal{X}_2.$$

Here, it is clear that $\Pi_{\mathcal{X}_1\times\mathcal{X}_2}(S) \subset \mathcal{X}_1 \times \mathcal{X}_2$. Now, to prove that the converse inclusion also holds, we need to show that: for any $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$, there exists $(x_3, x_4) \in \mathcal{X}_3 \times \mathcal{X}_4$ such that $(x_1, x_2, x_3, x_4) \in S$. So, let us take any $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. Then, since S_{13} and S_{24} are simulation relations, we know that

$$\Pi_{\mathcal{X}_1}(S_{13}) = \mathcal{X}_1 \quad \text{and} \quad \Pi_{\mathcal{X}_2}(S_{24}) = \mathcal{X}_2.$$

In particular, this implies that there exist x_3 and x_4 such that

$$(x_1, x_3) \in S_{13} \quad \text{and} \quad (x_2, x_4) \in S_{24}.$$

By definition of S, this means that

$$(x_1, x_2, x_3, x_4) \in S.$$

Thus, with this, it is proved that $\Sigma_1 \times \Sigma_2$ is simulated by $\Sigma_3 \times \Sigma_4$, i.e. $\Sigma_1 \times \Sigma_2 \preceq \Sigma_3 \times \Sigma_4$, if $\Sigma_1 \preceq \Sigma_3$ and $\Sigma_2 \preceq \Sigma_4$.

■

With this, all properties have been proven, which concludes our proof of the lemma.

A.3 Proof of Theorem 2.11

(⇒) Assume that S is a simulation relation of Ξ_1 by Ξ_2, satisfying

$$\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}, \quad i = 1, 2.$$

By Lemma 2.10, we then know that: for any $(x_1, x_2) \in S$ and $d_1 \in \mathcal{D}_1$ such that $A_1x_1 + F_1d_1 \in \mathcal{V}_{\mathcal{X}_1}$, there exists $d_2 \in \mathcal{D}_2$ such that $A_2x_2 + F_2d_2 \in \mathcal{V}_{\mathcal{X}_2}$ and

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2 \in S,$$
$$C_1x_1 = C_2x_2.$$

Using this, we find the following:

- In particular, the first property tells us that: if we take $(x_1, x_2) = (0, 0) \in S$ and any $d_1 \in \mathcal{D}_1$, then there exists $d_2 \in \mathcal{D}_2$ such that

  $$\begin{bmatrix} F_1 \\ 0 \end{bmatrix} d_1 = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ F_2 \end{bmatrix} d_2$$

  for some $(\bar{x}_1, \bar{x}_2) \in S$. Here, note that, by definition of the canonical projection, we have that

  $$F_1d_1 \in \Pi_{\mathcal{X}_1}(S) \subset \mathcal{V}_{\mathcal{X}_1},$$

  where the latter inclusion follows by assumption. In addition, it is clear that

  $$F_id_i \in \operatorname{im} F_i, \quad i = 1, 2$$

  by definition of the image. Therefore, we know that

  $$F_1d_1 \in \operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1} \quad \text{and} \quad F_2d_2 \in \operatorname{im} F_2.$$

  From the above, it then follows

  $$\begin{bmatrix} \operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1} \\ 0 \end{bmatrix} \subset S + \begin{bmatrix} 0 \\ \operatorname{im} F_2 \end{bmatrix}.$$

  Hence, property (11a) is proved to hold.

- Furthermore, the second property tells us that, for any $(x_1, x_2) \in S$, we have that

$$\begin{bmatrix} C_1 & -C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

By the system equations of $\Xi_i, i = 1, 2$ we also know that we must have

$$\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Together, we hence find that

$$\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From this, it follows that

$$S \subset \ker \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2 \end{bmatrix}$$

which implies that property (11c) holds.

Therefore, it has been proved that property (9a) implies properties (11a) and (11b), and that property (9b) implies property (11c).

(⇐) Conversely, assume that properties (11a), (11b) and (11c) hold. We will then show that a linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ satisfying these properties satisfies the conditions stated in Lemma 2.10. The proof of this is given in multiple parts and goes as follows:

- Properties (11b) and (11c) respectively imply that

$$A_i \Pi_{\mathcal{X}_i}(S) \subset \Pi_{\mathcal{X}_i}(S) + \operatorname{im} F_i \quad \text{and} \quad \Pi_{\mathcal{X}_i}(S) \subset \ker H_i$$

for $i = 1, 2$. Since $\mathcal{V}_{\mathcal{X}_i}$ is known to be the **largest** space in $\mathcal{X}_i$ satisfying the above inclusions, we know that

$$\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}, \quad i = 1, 2.$$

So, this proves that the space S satisfies

$$\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}, \quad i = 1, 2$$

as required in Lemma 2.10.

- Let us now show that the first property of the lemma is satisfied. Hereto, take any $(x_1, x_2) \in S$ and $\hat{d}_1 \in \mathcal{D}_1$. Property (11b) then implies that there exist $(\hat{x}_1, \hat{x}_2) \in S$ such that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} - \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix}.$$

for some $\hat{d}_2 \in \mathcal{D}_2$. In particular, this implies that

$$A_1 x_1 + F_1 \hat{d}_1 = \hat{x}_1 \in \Pi_{\mathcal{X}_1}(S) \subset \mathcal{V}_{\mathcal{X}_1}$$

where the inclusions follow by definition and by the previous analysis.

Now, take any $d_1 \in \mathcal{D}_1$ such that

$$A_1 x_1 + F_1 d_1 \in \mathcal{V}_{\mathcal{X}_1}.$$

This then implies that

$$F_1(d_1 - \hat{d}_1) = A_1 x_1 + F_1 d_1 - \hat{x}_1 \in \mathcal{V}_{\mathcal{X}_1}.$$

In addition, we clearly have that

$$F_1(d_1 - \hat{d}_1) \in \operatorname{im} F_1.$$

Therefore, we know that

$$F_1(d_1 - \hat{d}_1) \in \operatorname{im}(F_1) \cap \mathcal{V}_{\mathcal{X}_1}.$$

By property (11a) we then know that there exist $(\bar{x}_1, \bar{x}_2) \in S$ and $\bar{d}_2$ such that $F_2 \bar{d}_2 \in \operatorname{im} F_2$ and

$$\begin{bmatrix} F_1(d_1 - \hat{d}_1) \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ F_2 \bar{d}_2 \end{bmatrix}.$$

Let us then choose a specific d_2, namely $d_2 = \hat{d}_2 + \bar{d}_2$, for which we find that

$$\begin{aligned} A_2 x_2 + F_2 d_2 &= A_2 x_2 + F_2(\hat{d}_2 + \bar{d}_2) \\ &= \hat{x}_2 + \bar{x}_2. \end{aligned} \tag{35}$$

In addition, we previously found that

$$\begin{aligned} A_1 x_1 + F_1 d_1 &= \hat{x}_1 + F_1(d_1 - \hat{d}_1) \\ &= \hat{x}_1 + \bar{x}_1. \end{aligned} \tag{36}$$

Together, these two equalities imply that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 + \bar{x}_1 \\ \hat{x}_2 + \bar{x}_2 \end{bmatrix} \in S$$

where the latter follows by linearity of S. With this, it has been proved that: for all $(x_1, x_2) \in S$ and $d_1 \in \mathcal{D}_1$ such that

$$A_1 x_1 + F_1 d_1 \in \mathcal{V}_{\mathcal{X}_1},$$

there exists d_2 such that

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in S.$$

Note here that d_2 is such that

$$A_2 x_2 + F_2 d_2 \in \mathcal{V}_{\mathcal{X}_2}$$

since

$$A_2 x_2 + F_2 d_2 \in \Pi_{\mathcal{X}_2}(S) \subset \mathcal{V}_{\mathcal{X}_2}$$

where the latter follows from previous analysis. In other words, with this, property (9a) is proved to hold.

- It is clear that also property (9b) holds. The proof of this goes as follows: by property (11c) we have that

$$\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \\ C_1 & -C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

for all $(x_1, x_2) \in S$. In particular, this implies that

$$C_1 x_1 = C_2 x_2$$

with which we have proved that property (9b) indeed holds.

Hence, it has been proved that a linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ satisfying properties (11a), (11b) and (11c), also satisfies

$$\Pi_{\mathcal{X}_i}(S) \subset \mathcal{V}_{\mathcal{X}_i}, \quad i = 1, 2$$

and is a simulation relation of Ξ_1 by Ξ_2. This concludes the proof of the theorem.

A.4 Proof of Lemma 2.15

Note that the proof of this is similar to that of Lemma 2.8. Here, the two properties will be proved separately.

Proof of 1. This follows immediately from choosing

$$S = \{(x_1, x_1) \mid x_1 \in \mathcal{V}_{\mathcal{X}_1}\}. \qquad \blacksquare$$

Proof of 2. Let us prove the second statement. For this, assume that for any systems Ξ_1, Ξ_2 and Ξ_3 we have the following

$$\Xi_1 \preceq \Xi_2 \text{ and } \Xi_2 \preceq \Xi_3.$$

In other words, there exists a simulation relation $S_{12} \subset \mathcal{X}_1 \times \mathcal{X}_2$ of Ξ_1 by Ξ_2 and a simulation relation $S_{23} \subset \mathcal{X}_2 \times \mathcal{X}_3$ of Ξ_2 by Ξ_3, both satisfying Definition 2.9. From this, define linear subspace

$$S_{13} = \{(x_1, x_3) \mid \exists x_2 \text{ such that } (x_1, x_2) \in S_{12}, (x_2, x_3) \in S_{23}\}.$$

Using that S_{12} and S_{23} are simulation relations, it can be found that subspace S_{13} satisfies that

$$\Pi_{\mathcal{X}_1}(S_{13}) = \mathcal{V}_{\mathcal{X}_1} \quad \text{and} \quad \Pi_{\mathcal{X}_3}(S_{13}) \subset \mathcal{V}_{\mathcal{X}_3}.$$

The proof of this is done in three parts and goes as follows:

1. **Claim:** $\Pi_{\mathcal{X}_1}(S_{13}) \subset \mathcal{V}_{\mathcal{X}_1}$.

   Proof. Take any $x_1 \in \Pi_{\mathcal{X}_1}(S_{13})$. Then, by definition, there exists an x_3 such that $(x_1, x_3) \in S_{13}$. In particular, by the definition of S_{13}, this means that there exists x_2 such that $(x_1, x_2) \in S_{12}$. Since S_{12} is a simulation relation, we then know that

   $$x_1 \in \Pi_{\mathcal{X}_1}(S_{12}) = \mathcal{V}_{\mathcal{X}_1}.$$

   Since x_1 was taken to be any element in $\mathcal{X}_1(S_{13})$, the claim has been proven. $\blacksquare$

2. **Claim:** $\mathcal{V}_{\mathcal{X}_1} \subset \Pi_{\mathcal{X}_1}(S_{13})$.

   Proof. Take any $x_1 \in \mathcal{V}_{\mathcal{X}_1}$. Since S_{12} is a simulation relation of Ξ_1 by Ξ_2, we then know that

   $$x_1 \in \mathcal{V}_{\mathcal{X}_1} = \Pi_{\mathcal{X}_1}(S_{12}).$$

   By definition, this means that there exists an x_2 such that $(x_1, x_2) \in S_{12}$. Here, we can use that S_{12} and S_{23} are simulation relations, which respectively imply that

   $$\Pi_{\mathcal{X}_2}(S_{12}) \subset \mathcal{V}_{\mathcal{X}_2} \quad \text{and} \quad \mathcal{V}_{\mathcal{X}_2} = \Pi_{\mathcal{X}_2}(S_{23}).$$

   By definition, this means that there exists an x_3 such that $(x_2, x_3) \in S_{23}$. By definition of S_{13}, we have then found that $(x_1, x_3) \in S_{13}$. So, in other words, it is found that

   $$x_1 \in \Pi_{\mathcal{X}_1}(S_{13}).$$

   Therefore, since x_1 is any element in $\mathcal{V}_{\mathcal{X}_1}$, we have proved the claim. $\blacksquare$

3. **Claim:** $\Pi_{\mathcal{X}_3}(S_{13}) \subset \mathcal{V}_{\mathcal{X}_3}$.

Proof. Take any $x_3 \in \Pi_{\mathcal{X}_3}(S_{13})$. By definition we then know that there exists an x_1 such that $(x_1, x_3) \in S_{13}$. In particular, the definition of S_{13} implies that there exists x_2 such that

$$(x_2, x_3) \in S_{23}.$$

Since S_{23} is a simulation relation, we then know that

$$x_3 \in \Pi_{\mathcal{X}_3}(S_{23}) \subset \mathcal{V}_{\mathcal{X}_3}.$$

Since x_3 was taken to be any element in $\mathcal{X}_1(S_{13})$, the claim has hereby been proved. ∎

Here, note that the first two claims together imply that

$$\Pi_{\mathcal{X}_1}(S_{13}) = \mathcal{V}_{\mathcal{X}_1}.$$

Now, take any $(x_1, x_3) \in S_{13}$ and $d_1 \in \mathcal{D}_1$ then, by definition of S_{13}, there exists x_2 such that $(x_1, x_2) \in S_{12}, (x_2, x_3) \in S_{23}$. Since S_{12} and S_{23} are simulation relations, Lemma 2.10 implies that there exists $d_2 \in \mathcal{D}_2$ and there exists $d_3 \in \mathcal{D}_3$ such that

$$(A_1x_1 + F_1d_1, A_2x_2 + F_2d_2) \in S_{12} \quad \text{and} \quad (A_2x_2 + F_2d_2, A_3x_3 + F_3d_3) \in S_{23}.$$

By definition of S_{13}, these equations together imply that: $\forall (x_1, x_3) \in S_{13}, d_1 \in \mathcal{D}_1$, there exists $d_3 \in \mathcal{D}_3$ such that

$$(A_1x_1 + F_1d_1, A_3x_3 + F_3d_3) \in S_{13}.$$

Hence, the first property in Lemma 2.10, i.e. property (10a), is proved to hold. In addition, it is clear that, by definition of S_{13}, also the second property (10b) of the lemma holds since: for any $(x_1, x_3) \in S_{13}$ there exists x_2 such that $(x_1, x_2) \in S_{12}, (x_2, x_3) \in S_{23}$. We then know that

$$C_1x_1 = C_2x_2 \quad \text{and} \quad C_2x_2 = C_3x_3$$

by definition of S_{12} and S_{23} being simulation relations. Hence, it is found that

$$C_1x_1 = C_3x_3,$$

with which property (10a) is proved. ∎

A.5 Proof of Lemma 2.17

Assume that

$$\Xi_1 \preceq \Xi_2 \quad \text{and} \quad \Sigma_3 \preceq \Sigma_4.$$

We then know that there exist, respectively, a simulation relation $S_{12} \subset \mathcal{X}_1 \times \mathcal{X}_2$ of Ξ_1 by Ξ_2 and a simulation relation $S_{34} \subset \mathcal{X}_3 \times \mathcal{X}_4$ satisfying the properties in Lemma 2.10. From that, define the following linear subspace

$$S = \{(x_1, x_3, x_2, x_4) \mid (x_1, x_2) \in S_{12}, (x_3, x_4) \in S_{34}\}.$$

Take any $(x_1, x_3, x_2, x_4) \in S, u \in \mathcal{U}, d_1 \in \mathcal{D}_1$ and $d_3 \in \mathcal{D}_3$, then from the simulation relations S_{12} and S_{34} we know that: there exists $d_2 \in \mathcal{D}_2$ and $d_4 \in \mathcal{D}_4$ such that

$$(A_1x_1 + F_1d_1, A_3x_3 + B_3u + F_3d_3, A_2x_2 + F_2d_2, A_4x_4 + B_4u + F_4d_4) \in S$$

by definition of S. Now, note that the series interconnection of Ξ_i and Σ_{i+2} for $i = 1, 2$ is obtained by setting $u_i = u_{i+2}$, and is given by the following system equations

$$\Xi_i \times \Sigma_{i+2} : \begin{cases} \begin{bmatrix} \dot{x}_i \\ \dot{x}_{i+2} \end{bmatrix} &= \begin{bmatrix} A_i & 0 \\ B_{i+2}C_i & A_{i+2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix} + \begin{bmatrix} F_i & 0 \\ 0 & F_{i+2} \end{bmatrix} \begin{bmatrix} d_i \\ d_{i+2} \end{bmatrix}, \\ \begin{bmatrix} u_i \\ y_{i+2} \end{bmatrix} &= \begin{bmatrix} C_i & 0 \\ 0 & C_{i+2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix}, \\ 0 &= \begin{bmatrix} H_i & 0 \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix}. \end{cases}$$

In particular, we thus have $u_3 = u_1$ for the series interconnection of Ξ_1 and Σ_3, and we have $u_4 = u_2$ for the series interconnection of Ξ_2 and Σ_4. Furthermore, note that we have $u = u_3$ and $u = u_4$ since $u \in \mathcal{U} = \mathcal{U}_3 \cap \mathcal{U}_4$. Together with the system equations, this implies that

$$B_3 u = B_3 u_3 = B_3 u_1 = B_3 C_1 x_1 \quad \text{and} \quad B_4 u = B_4 u_4 = B_4 u_2 = B_4 C_2 x_2.$$

The first property of Lemma 2.10, see also (10a), is hence satisfied.

By definition of S, it is clear that also property (10b) is satisfied. The proof of this is as follows: since S_{12} and S_{34} are simulation relations, we know that

$$C_1 x_1 = C_2 x_2 \quad \text{and} \quad C_3 x_3 = C_4 x_4.$$

This implies that

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ 0 & C_4 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix},$$

with which property (10b) is proved. Furthermore, since S_{12} and S_{34} are simulation relations, it can be shown that

$$\Pi_{\mathcal{X}_1 \times \mathcal{X}_3}(S) = \mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_3} \quad \text{and} \quad \Pi_{\mathcal{X}_2 \times \mathcal{X}_4}(S) \subset \mathcal{V}_{\mathcal{X}_2 \times \mathcal{X}_4}.$$

Note that the proof of this follows shortly. Now, assuming the above holds, one can additionally show that

$$\begin{bmatrix} A_1 & 0 \\ B_3 C_1 & A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_3 \end{bmatrix} \in \mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_3} \quad \text{and} \quad \begin{bmatrix} A_2 & 0 \\ B_4 C_2 & A_4 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} F_2 & 0 \\ 0 & F_4 \end{bmatrix} \begin{bmatrix} d_2 \\ d_4 \end{bmatrix} \in \mathcal{V}_{\mathcal{X}_2 \times \mathcal{X}_4}.$$

Now, let us prove that we indeed have the following

$$\Pi_{\mathcal{X}_1 \times \mathcal{X}_3}(S) = \mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_3} \quad \text{and} \quad \Pi_{\mathcal{X}_2 \times \mathcal{X}_4}(S) \subset \mathcal{V}_{\mathcal{X}_2 \times \mathcal{X}_4}.$$

Hereto, it is clear that $\Pi_{\mathcal{X}_1 \times \mathcal{X}_3}(S) \subset \mathcal{V}_{\mathcal{X}_1} \times \mathcal{X}_3$ and $\Pi_{\mathcal{X}_2 \times \mathcal{X}_4}(S) \subset \mathcal{V}_{\mathcal{X}_2} \times \mathcal{X}_4$. Using this, the proof goes as follows:

1. **Claim:** $\mathcal{V}_{\mathcal{X}_1} \times \mathcal{X}_3 \subset \Pi_{\mathcal{X}_1 \times \mathcal{X}_3}(S)$.

   Proof. Take any $(x_1, x_3) \in \mathcal{V}_{\mathcal{X}_1} \times \mathcal{X}_3$. Then, since S_{12} and S_{34} are simulation relations, it is implied that
   $$x_1 \in \mathcal{V}_{\mathcal{X}_1} = \Pi_{\mathcal{X}_1}(S_{12}) \quad \text{and} \quad x_3 \in \mathcal{X}_3 = \Pi_{\mathcal{X}_3}(S_{34}).$$
   In particular, this means that there exist x_2 and x_4 such that
   $$(x_1, x_2) \in S_{12} \quad \text{and} \quad (x_3, x_4) \in S_{34}.$$
   By definition of S, this implies that
   $$(x_1, x_3, x_2, x_4) \in S,$$
   which proves the claim. ■

2. **Claim:** $\mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2} \subset \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}, \quad$ for $i = 1, 2$.

   Proof. For $i = 1, 2$, take any $(x_i, x_{i+2}) \in \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2}$. In particular, this means that
   $$x_i \in \mathcal{V}_{\mathcal{X}_i} \quad \text{and} \quad x_{i+2} \in \mathcal{X}_{i+2}.$$
   Here, note that since Σ_3 and Σ_4 are unconstrained systems, we know that
   $$\mathcal{X}_{i+2} = \mathcal{V}_{\mathcal{X}_{i+2}}.$$

Furthermore, we know that the consistent subspaces are respectively such that

$$A_i \mathcal{V}_{\mathcal{X}_i} \subset \mathcal{V}_{\mathcal{X}_i} + \operatorname{im} F_i \quad \text{and} \quad \mathcal{V}_{\mathcal{X}_i} \subset \ker H_i;$$
$$A_{i+2} \mathcal{V}_{\mathcal{X}_{i+2}} \subset \mathcal{V}_{\mathcal{X}_{i+2}} + \operatorname{im} F_{i+2} \quad \text{and} \quad \mathcal{V}_{\mathcal{X}_{i+2}} \subset \ker H_{i+2}.$$

Hence, for our x_i, x_{i+2} we know that there exist $\tilde{x}_i \in \mathcal{V}_{\mathcal{X}_i}, \tilde{x}_{i+2} \in \mathcal{X}_{i+2}, d_i \in \mathcal{D}_i$ and $d_{i+2} \in \mathcal{D}_{i+2}$ such that

$$\begin{aligned} A_i x_i &= \tilde{x}_i + F_i d_i, \\ H_i x_i &= 0, \\ \begin{bmatrix} B_{i+2} C_i & A_{i+2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix} &= \tilde{x}_{i+2} + F_{i+2} d_{i+2}. \end{aligned}$$

Here, we used that Σ_3 and Σ_4 are unconstrained systems, therefore H_{i+2} is the zero matrix. In addition, we used that $u_{i+2} = u_i = C_i x_i$ for the interconnection of systems Ξ_i and Σ_{i+2}. These equations together tell us that

$$\begin{bmatrix} A_i & 0 \\ B_{i+2} C_i & A_{i+2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix} = \begin{bmatrix} \tilde{x}_i \\ \tilde{x}_{i+2} \end{bmatrix} + \begin{bmatrix} F_i & 0 \\ 0 & F_{i+2} \end{bmatrix} \begin{bmatrix} d_i \\ d_{i+2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} H_i & 0 \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix} = 0.$$

In fact, since we took any $(x_i, x_{i+2}) \in \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2}$, this implies that

$$\begin{bmatrix} A_i & 0 \\ B_{i+2} C_i & A_{i+2} \end{bmatrix} \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2} \subset \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2} + \operatorname{im} \begin{bmatrix} F_i & 0 \\ 0 & F_{i+2} \end{bmatrix} \quad \text{and} \quad \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2} \subset \ker \begin{bmatrix} H_i & 0 \end{bmatrix}.$$

By definition, we know that $\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}$ is the **largest** subspace such that the above inclusions hold. Therefore, we must have that

$$\mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2} \subset \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}. \qquad \blacksquare$$

3. **Claim:** $\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}} \subset \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2}, \quad$ for $i = 1, 2$.

Proof. In order to prove the claim, let us show that

$$\Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) \subset \mathcal{V}_{\mathcal{X}_i}$$

for $i = 1, 2$. If this holds, then the claim holds as well. Namely, assuming the above inclusion holds, we find that: if we take any $(x_i, x_{i+2}) \in \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}$ then, by definition, we have that

$$x_i \in \Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) \subset \mathcal{V}_{\mathcal{X}_i} \quad \text{and} \quad x_{i+2} \in \Pi_{\mathcal{X}_{i+2}}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) \subset \mathcal{X}_{i+2}.$$

Note that the latter inclusion is trivial by definition of the canonical projection. With this, it has been found that

$$(x_i, x_{i+2}) \in \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2}.$$

This then proves that

$$\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}} \subset \mathcal{V}_{\mathcal{X}_i} \times \mathcal{X}_{i+2}.$$

So, let us now prove that

$$\Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) \subset \mathcal{V}_{\mathcal{X}_i}$$

from which the claim follows. Hereto, take any $x_i \in \Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}})$. By definition of the canonical projection, we know that there exists an x_{i+2} such that

$$(x_i, x_{i+2}) \in \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}.$$

As was used to prove the previous claim, we know that the consistent subspace $\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}$ satisfies the following

$$\begin{bmatrix} A_i & 0 \\ B_{i+2} C_i & A_{i+2} \end{bmatrix} \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}} \subset \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}} + \operatorname{im} \begin{bmatrix} F_i & 0 \\ 0 & F_{i+2} \end{bmatrix} \quad \text{and} \quad \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}} \subset \ker \begin{bmatrix} H_i & 0 \end{bmatrix}.$$

This means that there exist $(\tilde{x}_i, \tilde{x}_{i+2}) \in \mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}, d_i \in \mathcal{D}_i$ and $d_{i+2} \in \mathcal{D}_{i+2}$ such that

$$\begin{bmatrix} A_i & 0 \\ B_{i+2}C_i & A_{i+2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix} = \begin{bmatrix} \tilde{x}_i \\ \tilde{x}_{i+2} \end{bmatrix} + \begin{bmatrix} F_i & 0 \\ 0 & F_{i+2} \end{bmatrix} \begin{bmatrix} d_i \\ d_{i+2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} H_i & 0 \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+2} \end{bmatrix} = 0.$$

In particular, this implies that

$$A_i x_i = \tilde{x}_i + F_i d_i \quad \text{and} \quad H_i x_i = 0$$

with $\tilde{x}_i \in \Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}})$. Since we took any $x_i \in \Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}})$, it is then implied that

$$A_i \Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) \subset \Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) + \operatorname{im} F_i \quad \text{and} \quad \Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) \subset \ker H_i.$$

Since $\mathcal{V}_{\mathcal{X}_i}$ is, by definition, the **largest** subspace that satisfies the above inclusion, we know that the following must hold

$$\Pi_{\mathcal{X}_i}(\mathcal{V}_{\mathcal{X}_i \times \mathcal{X}_{i+2}}) \subset \mathcal{V}_{\mathcal{X}_i}.$$

As shown before, this then in return implies that the claim holds. ■

Note that claims together imply that

$$\Pi_{\mathcal{X}_1 \times \mathcal{X}_3}(S) = \mathcal{V}_{\mathcal{X}_1 \times \mathcal{X}_3} \quad \text{and} \quad \Pi_{\mathcal{X}_2 \times \mathcal{X}_4}(S) \subset \mathcal{V}_{\mathcal{X}_2 \times \mathcal{X}_4}$$

as was desired. With this, it has been proved that $\Xi_1 \times \Sigma_3$ is simulated by $\Xi_2 \times \Sigma_4$, i.e. $\Xi_1 \times \Sigma_3 \preceq \Xi_2 \times \Sigma_4$, if $\Xi_1 \preceq \Xi_2$ and $\Sigma_3 \preceq \Sigma_4$.

A.6 Proof of Lemma 2.18

Let us prove the claim. For this, assume that Ξ_1 and Ξ_2 are systems of the form (2) with $z_1 = u_1$ and $z_2 = \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}$. In addition, consider system Σ of the form (1) and let these systems be such that

$$\Xi_1 \times \Sigma \preceq \Xi_2.$$

By definition, there then exists a simulation relation of $\Xi_1 \times \Sigma$ by Ξ_2, say $\tilde{S} \subset \mathcal{X}_{\Xi_1} \times \mathcal{X}_{\Sigma} \times \mathcal{X}_{\Xi_2}$. From this, let us define another linear subspace

$$S = \{(x_1, x_\Sigma, \tilde{x}_1, x_2) \mid (x_1, x_\Sigma, x_2) \in \tilde{S}, \tilde{x}_1 = x_1\}.$$

Take any $(x_1, x_\Sigma, \tilde{x}_1, x_2) \in S$ and $(d_1, d_\Sigma) \in \mathcal{D}_1 \times D_\Sigma$ such that

$$\begin{bmatrix} A_1 & 0 \\ B_\Sigma C_1 & A_\Sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_\Sigma \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_\Sigma \end{bmatrix} \begin{bmatrix} d_1 \\ d_\Sigma \end{bmatrix} \in \mathcal{V}_{\mathcal{X}_{\Xi_1 \times \Sigma}}.$$

Then, since $\tilde{S}$ is a simulation relation, this implies that there exists $d_2 \in \mathcal{D}_2$ such that $A_2 x_2 + F_2 d_2 \in \mathcal{V}_{\mathcal{X}_2}$ and

$$\left(\begin{bmatrix} A_1 & 0 \\ B_\Sigma C_1 & A_\Sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_\Sigma \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & F_\Sigma \end{bmatrix} \begin{bmatrix} d_1 \\ d_\Sigma \end{bmatrix}, A_2 x_2 + F_2 d_2 \right) \in \tilde{S}.$$

Rewritten this gives that

$$(A_1 x_1 + F_1 d_1, B_\Sigma C_1 x_1 + A_\Sigma x_\Sigma + F_\Sigma d_\Sigma, A_2 x_2 + F_2 d_2) \in \tilde{S}.$$

Now, choose $(\tilde{d}_1, \tilde{d}_2) = (d_1, d_2)$, then, by definition of S and by the above, we know that: $\forall (x_1, x_\Sigma, \tilde{x}_1, x_2) \in S$ and $(d_1, d_\Sigma) \in \mathcal{D}_1 \times D_\Sigma$, there exists $(\tilde{d}_1, \tilde{d}_2) = (d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2$ such that

$$(A_1 x_1 + F_1 d_1, B_\Sigma C_1 x_1 + A_\Sigma x_\Sigma + F_\Sigma d_\Sigma, A_1 \tilde{x}_1 + F_1 \tilde{d}_1, A_2 x_2 + F_2 \tilde{d}_2) \in S.$$

Hence, property (10a) is proved to hold. In addition, it is clear that also property (10b) holds since $\tilde{S}$ being a simulation relation implies that

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_\Sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_\Sigma \end{bmatrix} = \begin{bmatrix} C_2^u \\ C_2^y \end{bmatrix} x_2.$$

In particular, this implies that

$$C_\Sigma x_\Sigma = C_2^y x_2.$$

Furthermore, $\tilde{x}_1 = x_1$ implies that $C_1 \tilde{x}_1 = C_1 x_1$. Together, the two equalities imply that

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_\Sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_\Sigma \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2^y \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ x_2 \end{bmatrix},$$

with which property (10b) is proved. Lastly, note that it can be shown that

$$\Pi_{\mathcal{X}_{\Xi_1} \times \mathcal{X}_\Sigma}(S) = \mathcal{V}_{\mathcal{X}_{\Xi_1}} \times \mathcal{X}_\Sigma \quad \text{and} \quad \Pi_{\mathcal{X}_{\Xi_1} \times \mathcal{X}_{\Xi_2}}(S) \subset \mathcal{V}_{\mathcal{X}_{\Xi_1}} \times \mathcal{V}_{\mathcal{X}_{\Xi_2}}$$

hold since $\tilde{S}$ is a simulation relation. The proof of this follows similar steps as the proofs of the claims in A.5. With this, it has been proved, by Lemma 2.10, that $\Xi_1 \times \Sigma$ is simulated by $\Xi_1 \otimes \Xi_2$, i.e. $\Xi_1 \times \Sigma \preceq \Xi_1 \otimes \Xi_2$, if $\Xi_1 \times \Sigma \preceq \Xi_2$.

A.7 Proof of Theorem 3.6

Assume that contract $\mathcal{C} = (A, G)$ is consistent. This assumption implies that there exists an implementation Σ of the contract. By Theorem 3.3 we then know that

$$A \times \Sigma \preceq G$$

for the implementation. Hence, there exists a simulation relation $\tilde{S} \subset \mathcal{X}_A \times \mathcal{X}_\Sigma \times \mathcal{X}_G$ which satisfies the conditions in Lemma 2.10. From the space $\tilde{S}$, let us define the following linear subspace

$$S = \{(x_A, x_G) \mid \exists x_\Sigma \text{ such that } (x_A, x_\Sigma, x_G) \in \tilde{S}\}.$$

Since $\tilde{S}$ is a simulation relation, we in particular now know that

$$C_A x_A = C_G^u x_G.$$

In addition, we know that: for any $(x_A, x_G) \in S$ and $d_A \in \mathcal{D}_A$, there exists $d_G \in \mathcal{D}_G$ such that

$$(A_A x_A + F_A d_A \ , \ A_G x_G + F_G d_G) \in S.$$

Furthermore, note that we can prove that $\Pi_{\mathcal{X}_A}(S) = \mathcal{V}_{\mathcal{X}_A}$ and that $\Pi_{\mathcal{X}_G}(S) \subset \mathcal{V}_{\mathcal{X}_G}$. In particular, this then shows that

$$A_A x_A + F_A d_A \in \mathcal{V}_{\mathcal{X}_A} \quad \text{and} \quad A_G x_G + F_G d_G \in \mathcal{V}_{\mathcal{X}_G},$$

as desired. Now, let us prove 2 separate claims that together provide the full proof that

$$\Pi_{\mathcal{X}_A}(S) = \mathcal{V}_{\mathcal{X}_A} \quad \text{and} \quad \Pi_{\mathcal{X}_G}(S) \subset \mathcal{V}_{\mathcal{X}_G}.$$

1. **Claim:** $\Pi_{\mathcal{X}_A}(S) \subset \mathcal{V}_{\mathcal{X}_A}$.

   Proof. Take any $x_A \in \Pi_{\mathcal{X}_A}(S)$. Then, by definition, this means that there must exist a x_G such that $(x_A, x_G) \in S$. By definition of S, we then know that there exist an x_Σ such that

   $$(x_A, x_\Sigma) \in \Pi_{\mathcal{X}_A \times \mathcal{X}_\Sigma}(\tilde{S}) \quad = \mathcal{V}_{\mathcal{X}_A \times \mathcal{X}_\Sigma} \quad = \mathcal{V}_{\mathcal{X}_A} \times \mathcal{X}_\Sigma.$$

   Here, the former equality follows from $\tilde{S}$ being a simulation relation and the latter equality follows from the claims in Appendix A.5. Therefore, it has been proved that

   $$\Pi_{\mathcal{X}_A}(S) \subset \mathcal{V}_{\mathcal{X}_A}. \qquad \blacksquare$$

   Note that we can similarly prove that

   $$\Pi_{\mathcal{X}_G}(S) \subset \mathcal{V}_{\mathcal{X}_G}.$$

2. **Claim:** $\mathcal{V}_{\mathcal{X}_A} \subset \Pi_{\mathcal{X}_A}(S)$.

Proof. In this proof, we reverse the analysis done in the proof of the previous claim. For this, take any $x_A \in \mathcal{V}_{\mathcal{X}_A}$. For any x_Σ we then know that

$$(x_A, x_\Sigma) \in \mathcal{V}_{\mathcal{X}_A} \times \mathcal{X}_\Sigma = \Pi_{\mathcal{X}_A \times \mathcal{X}_\Sigma}(\tilde{S}).$$

In return, it is then implied that there exists x_G such that

$$(x_A, x_\Sigma, x_G) \in \tilde{S}.$$

By definition of S, this means that

$$(x_A, x_G) \in S$$

which implies that $x_A \in \Pi_{\mathcal{X}_A}(S)$. In other words, we have hereby proved that

$$\mathcal{V}_{\mathcal{X}_A} \subset \Pi_{\mathcal{X}_A}(S).$$

■

Now that it has been proved that $\Pi_{\mathcal{X}_A}(S) = \mathcal{V}_{\mathcal{X}_A}$ and that $\Pi_{\mathcal{X}_G}(S) \subset \mathcal{V}_{\mathcal{X}_G}$, we have proved (by Lemma 2.10) that A is simulated by G^u, which concludes the proof.

A.8 Proof of claim in Lemma 5.1

In order to prove the claim, consider consistent contracts $\mathcal{C}_i = (A_i, G_i)$, $i = 1, 2$, with any of their implementations Σ_i. In addition, assume that the contracts satisfy $G_1^y \preceq A_2$. Theorem 2.11 then tells us that $A_1 \times \Sigma_{12} \preceq G_{12}$ if there exists a subspace $S \subset \mathcal{X}_{A_1} \times \mathcal{X}_{\Sigma_{12}} \times \mathcal{X}_{G_{12}}$ satisfying

$$\begin{bmatrix} \operatorname{im}(F_{A_1}) \cap \mathcal{V}_{\mathcal{X}_{A_1}} \\ \operatorname{im}(F_{\Sigma_1}) \cap \mathcal{X}_{\Sigma_1} \\ \operatorname{im}(F_{\Sigma_2}) \cap \mathcal{X}_{\Sigma_2} \\ 0 \\ 0 \end{bmatrix} \subset S + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \operatorname{im} F_{G_1} \\ \operatorname{im} F_{G_2} \end{bmatrix}, \tag{37a}$$

$$\begin{bmatrix} A_{A_1} & 0 & 0 & 0 & 0 \\ B_{\Sigma_1} C_{A_1} & A_{\Sigma_1} & 0 & 0 & 0 \\ 0 & B_{\Sigma_2} C_{\Sigma_1} & A_{\Sigma_2} & 0 & 0 \\ 0 & 0 & 0 & A_{G_1} & 0 \\ 0 & 0 & 0 & 0 & A_{G_2} \end{bmatrix} S \subset S + \operatorname{im} \begin{bmatrix} F_{A_1} & 0 & 0 & 0 & 0 \\ 0 & F_{\Sigma_1} & 0 & 0 & 0 \\ 0 & 0 & F_{\Sigma_2} & 0 & 0 \\ 0 & 0 & 0 & F_{G_1} & 0 \\ 0 & 0 & 0 & 0 & F_{G_2} \end{bmatrix}, \tag{37b}$$

$$S \subset \ker \begin{bmatrix} H_{A_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{G_1} & 0 \\ 0 & 0 & 0 & 0 & H_{G_2} \\ 0 & 0 & 0 & C_{G_1}^y & -C_{G_2}^u \\ C_{A_1} & 0 & 0 & -C_{G_1}^u & 0 \\ 0 & 0 & C_{\Sigma_2} & 0 & -C_{G_2}^y \end{bmatrix}. \tag{37c}$$

In other words, we need to prove that these properties are satisfied. Hereto, note that Σ_i is assumed to implement contract $\mathcal{C}_i$, $i = 1, 2$. By Corollary 3.4, we hence know that there exist subspaces $S_i \subset \mathcal{X}_{A_i} \times \mathcal{X}_{\Sigma_i} \times \mathcal{X}_{G_i}$ satisfying

$$\Pi_{\mathcal{X}_{A_i} \times \mathcal{X}_{\Sigma_i}}(S_i) \subset \mathcal{V}_{\mathcal{X}_{A_i} \times \mathcal{X}_{\Sigma_i}} = \mathcal{V}_{\mathcal{X}_{A_i}} \times \mathcal{X}_{\Sigma_i}, \quad \Pi_{\mathcal{X}_{G_i}}(S_i) \subset \mathcal{V}_{\mathcal{X}_{G_i}},$$

and, additionally, satisfying the following properties

$$\begin{bmatrix} \operatorname{im}(F_{A_i}) \cap \mathcal{V}_{\mathcal{X}_{A_i}} \\ \operatorname{im}(F_{\Sigma_i}) \cap \mathcal{X}_{\Sigma_i} \\ 0 \end{bmatrix} \subset S_i + \begin{bmatrix} 0 \\ 0 \\ \operatorname{im} F_{G_i} \end{bmatrix}, \tag{38a}$$

$$\begin{bmatrix} A_{A_i} & 0 & 0 \\ B_{\Sigma_i} C_{A_i} & A_{\Sigma_i} & 0 \\ 0 & 0 & A_{G_i} \end{bmatrix} S_i \subset S_i + \operatorname{im} \begin{bmatrix} F_{A_i} & 0 & 0 \\ 0 & F_{\Sigma_i} & 0 \\ 0 & 0 & F_{G_i} \end{bmatrix}, \tag{38b}$$

$$S_i \subset \ker \begin{bmatrix} H_{A_i} & 0 & 0 \\ 0 & 0 & H_{G_i} \\ C_{A_i} & 0 & -C^u_{G_i} \\ 0 & C_{\Sigma_i} & -C^y_{G_i} \end{bmatrix}. \tag{38c}$$

Since $G_1^y \preceq A_2$ hold by assumption, we furthermore know that there exists a subspace $T \subset \mathcal{X}_{G_1} \times \mathcal{X}_{A_2}$ satisfying

$$\Pi_{\mathcal{X}_{G_1}}(T) \subset \mathcal{V}_{\mathcal{X}_{G_1}}, \quad \Pi_{\mathcal{X}_{A_2}}(T) \subset \mathcal{V}_{\mathcal{X}_{A_2}},$$

as well as satisfying the following properties

$$\begin{bmatrix} \operatorname{im}(F_{G_1}) \cap \mathcal{V}_{\mathcal{X}_{G_1}} \\ 0 \end{bmatrix} \subset T + \begin{bmatrix} 0 \\ \operatorname{im} F_{A_2} \end{bmatrix}, \tag{39a}$$

$$\begin{bmatrix} A_{G_1} & 0 \\ 0 & A_{A_2} \end{bmatrix} T \subset T + \operatorname{im} \begin{bmatrix} F_{G_1} & 0 \\ 0 & F_{A_2} \end{bmatrix}, \tag{39b}$$

$$T \subset \ker \begin{bmatrix} H_{G_1} & 0 \\ 0 & H_{A_2} \\ C^y_{G_1} & -C_{A_2} \end{bmatrix}. \tag{39c}$$

From this, let us define

$$\begin{aligned} S = \{(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, x_{G_1}, x_{G_2}) \mid (x_{A_1}, x_{\Sigma_1}, x_{G_1}) \in S_1, \\ \exists x_{A_2} \text{ such that } (x_{G_1}, x_{A_2}) \in T \text{ and } (x_{A_2}, x_{\Sigma_2}, x_{G_2}) \in S_2\}. \end{aligned} \tag{40}$$

Now, let us show that properties (37a), (37b) and (37c) are satisfied by this choice of S.

Proof of (37a). Take any

$$(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, 0, 0) \in \begin{bmatrix} \operatorname{im}(F_{A_1}) \cap \mathcal{V}_{\mathcal{X}_{A_1}} \\ \operatorname{im}(F_{\Sigma_1}) \cap \mathcal{X}_{\Sigma_1} \\ \operatorname{im}(F_{\Sigma_2}) \cap \mathcal{X}_{\Sigma_2} \\ 0 \\ 0 \end{bmatrix}.$$

In particular, this means that

$$x_{A_1} \in \operatorname{im}(F_{A_1}) \cap \mathcal{V}_{\mathcal{X}_{A_1}} \quad \text{and} \quad x_{\Sigma_i} \in \operatorname{im}(F_{\Sigma_i}) \cap \mathcal{X}_{\Sigma_i}, \quad i = 1, 2.$$

Property (38a) then implies that there exist $(\bar{x}_{A_1}, \bar{x}_{\Sigma_1}, \bar{x}_{G_1}) \in S_1$ and $\hat{x}_{G_1} \in \operatorname{im} F_{G_1}$ such that

$$x_{A_1} = \bar{x}_{A_1}, \quad x_{\Sigma_1} = \bar{x}_{\Sigma_1} \quad \text{and} \quad 0 = \bar{x}_{G_1} + \hat{x}_{G_1}.$$

Now, choose x_{A_2} such that $(\bar{x}_{G_1}, x_{A_2}) \in T$ and $(x_{A_2}, x_{\Sigma_2}, x_{G_2}) \in S_2$, then property (38a) also implies that there exist $(\bar{x}_{A_2}, \bar{x}_{\Sigma_2}, \bar{x}_{G_2}) \in S_2$ and $\hat{x}_{G_2} \in \operatorname{im} F_{G_2}$ such that

$$x_{A_2} = \bar{x}_{A_2}, \quad x_{\Sigma_2} = \bar{x}_{\Sigma_2} \quad \text{and} \quad 0 = \bar{x}_{G_2} + \hat{x}_{G_2}.$$

In particular, we now know that

$$\begin{aligned}(\bar{x}_{A_i}, \bar{x}_{\Sigma_i}, \bar{x}_{G_i}) &\in S_i,\\ (\bar{x}_{G_1}, \bar{x}_{A_2}) = (\bar{x}_{G_1}, x_{A_2}) &\in T,\\ \hat{x}_{G_i} &\in \operatorname{im} F_{G_i},\end{aligned}$$

for $i = 1, 2$. This means, by definition of S, that: for any

$$(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, 0, 0) \in \begin{bmatrix} \operatorname{im}(F_{A_1}) \cap \mathcal{V}_{\mathcal{X}_{A_1}} \\ \operatorname{im}(F_{\Sigma_1}) \cap \mathcal{X}_{\Sigma_1} \\ \operatorname{im}(F_{\Sigma_2}) \cap \mathcal{X}_{\Sigma_2} \\ 0 \\ 0 \end{bmatrix},$$

we are able to find

$$(\bar{x}_{A_1}, \bar{x}_{\Sigma_1}, \bar{x}_{\Sigma_2}, \bar{x}_{G_1}, \bar{x}_{G_2}) \in S \quad \text{and} \quad (0, 0, 0, \hat{x}_{G_1}, \hat{x}_{G_2}) \in \begin{bmatrix} 0 \\ 0 \\ 0 \\ \operatorname{im} F_{G_1} \\ \operatorname{im} F_{G_2} \end{bmatrix}$$

such that

$$(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, 0, 0) = (\bar{x}_{A_1}, \bar{x}_{\Sigma_1}, \bar{x}_{\Sigma_2}, \bar{x}_{G_1}, \bar{x}_{G_2}) + (0, 0, 0, \hat{x}_{G_1}, \hat{x}_{G_2}).$$

With this, property (37a) is proved to hold.

Proof of (37b). Take any $(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, x_{G_1}, x_{G_2}) \in S$. By definition of S, we know that $(x_{A_1}, x_{\Sigma_1}, x_{G_1}) \in S_1$, and that there exists an element x_{A_2} such that $(x_{G_1}, x_{A_2}) \in T$ and $(x_{A_2}, x_{\Sigma_2}, x_{G_2}) \in S_2$. Furthermore, take any d_{Σ_1} and d_{Σ_2}, together with any d_{A_1} such that

$$A_{A_1} x_{A_1} + F_{A_1} d_{A_1} \in \mathcal{V}_{\mathcal{X}_{A_1}}.$$

Note that, since S_1 is such that $\Pi_{\mathcal{X}_{A_1} \times \mathcal{X}_{\Sigma_1}}(S_1) \subset \mathcal{V}_{\mathcal{X}_{A_1}} \times \mathcal{X}_{\Sigma_1}$, we know by the first line of property (38b) that such d_{A_1} exists. Let us now use Lemma 2.10 a number of times to show that (37b) holds.

- Firstly, since S_1 is a simulation relation, we know by Lemma 2.10 that: for $(x_{A_1}, x_{\Sigma_1}, x_{G_1}) \in S_1$ and d_{A_1}, d_{Σ_1}, there exists a d_{G_1} such that $A_{G_1} x_{G_1} + F_{G_1} d_{G_1} \in \mathcal{V}_{\mathcal{X}_{G_1}}$ and

  $$\left(\begin{bmatrix} A_{A_1} & 0 \\ B_{\Sigma_1} C_{A_1} & A_{\Sigma_1} \end{bmatrix} \begin{bmatrix} x_{A_1} \\ x_{\Sigma_1} \end{bmatrix} + \begin{bmatrix} F_{A_1} & 0 \\ 0 & F_{\Sigma_1} \end{bmatrix} \begin{bmatrix} d_{A_1} \\ d_{\Sigma_1} \end{bmatrix}, \ A_{G_1} x_{G_1} + F_{G_1} d_{G_1} \right) \in S_1.$$

  Hereto, note that

  $$\begin{bmatrix} A_{A_1} & 0 \\ B_{\Sigma_1} C_{A_1} & A_{\Sigma_1} \end{bmatrix} \begin{bmatrix} x_{A_1} \\ x_{\Sigma_1} \end{bmatrix} + \begin{bmatrix} F_{A_1} & 0 \\ 0 & F_{\Sigma_1} \end{bmatrix} \begin{bmatrix} d_{A_1} \\ d_{\Sigma_1} \end{bmatrix} \in \mathcal{V}_{\mathcal{X}_{A_1}} \times \mathcal{X}_{\Sigma_1}$$

  since we took any d_{A_1} such that $A_{A_1} x_{A_1} + F_{A_1} d_{A_1} \in \mathcal{V}_{\mathcal{X}_{A_1}}$. Now, let us define

  $$\bar{x}_{A_1} := A_{A_1} x_{A_1} + F_{A_1} d_{A_1}, \quad \bar{x}_{\Sigma_1} := B_{\Sigma_1} C_{A_1} x_{A_1} + A_{\Sigma_1} x_{\Sigma_1} + F_{\Sigma_1} d_{\Sigma_1} \quad \text{and} \quad \bar{x}_{G_1} := A_{G_1} x_{G_1} + F_{G_1} d_{G_1}.$$

  Then, from the above analysis, it is clear that

  $$\begin{bmatrix} A_{A_1} & 0 & 0 \\ B_{\Sigma_1} C_{A_1} & A_{\Sigma_1} & 0 \\ 0 & 0 & A_{G_1} \end{bmatrix} \begin{bmatrix} x_{A_1} \\ x_{\Sigma_1} \\ x_{G_1} \end{bmatrix} = \begin{bmatrix} \bar{x}_{A_1} \\ \bar{x}_{\Sigma_1} \\ \bar{x}_{G_1} \end{bmatrix} + \begin{bmatrix} F_{A_1} & 0 & 0 \\ 0 & F_{\Sigma_1} & 0 \\ 0 & 0 & F_{G_1} \end{bmatrix} \begin{bmatrix} -d_{A_1} \\ -d_{\Sigma_1} \\ -d_{G_1} \end{bmatrix}$$

  where $(\bar{x}_{A_1}, \bar{x}_{\Sigma_1}, \bar{x}_{G_1}) \in S_1$ by definition.

- Secondly, since T is also a simulation relation, we know by Lemma 2.10 that: for $(x_{G_1}, x_{A_2}) \in T$ and for the d_{G_1} found in the previous step, which satisfies $A_{G_1}x_{G_1} + F_{G_1}d_{G_1} \in \mathcal{V}_{\mathcal{X}_{G_1}}$, we have that there exists a d_{A_2} such that $A_{A_2}x_{A_2} + F_{A_2}d_{A_2} \in \mathcal{V}_{\mathcal{X}_{A_2}}$ and

$$(A_{G_1}x_{G_1} + F_{G_1}d_{G_1} \ , \ A_{A_2}x_{A_2} + F_{A_2}d_{A_2}) \in T.$$

From this, we define

$$\bar{x}_{A_2} := A_{A_2}x_{A_2} + F_{A_2}d_{A_2}$$

and note that this implies that $(\bar{x}_{G_1}, \bar{x}_{A_2}) \in T$.

- Lastly, since S_2 is a simulation as well, we know by Lemma 2.10 that: for $(x_{A_2}, x_{\Sigma_2}, x_{G_2}) \in S_2$, d_{Σ_2} and d_{A_2} from the previous step, which satisfies $A_{A_2}x_{A_2} + F_{A_2}d_{A_2} \in \mathcal{V}_{\mathcal{X}_{A_2}}$, there exists a d_{G_2} such that $A_{G_2}x_{G_2} + F_{G_2}d_{G_2} \in \mathcal{V}_{\mathcal{X}_{G_2}}$ and

$$\left(\begin{bmatrix} A_{A_2} & 0 \\ B_{\Sigma_2}C_{A_2} & A_{\Sigma_2} \end{bmatrix} \begin{bmatrix} x_{A_2} \\ x_{\Sigma_2} \end{bmatrix} + \begin{bmatrix} F_{A_2} & 0 \\ 0 & F_{\Sigma_2} \end{bmatrix} \begin{bmatrix} d_{A_2} \\ d_{\Sigma_2} \end{bmatrix} \ , \ A_{G_2}x_{G_2} + F_{G_2}d_{G_2} \right) \in S_2.$$

Note that

$$\begin{bmatrix} A_{A_2} & 0 \\ B_{\Sigma_2}C_{A_2} & A_{\Sigma_2} \end{bmatrix} \begin{bmatrix} x_{A_2} \\ x_{\Sigma_2} \end{bmatrix} + \begin{bmatrix} F_{A_2} & 0 \\ 0 & F_{\Sigma_2} \end{bmatrix} \begin{bmatrix} d_{A_2} \\ d_{\Sigma_2} \end{bmatrix} \in \mathcal{V}_{\mathcal{X}_{A_2}} \times \mathcal{X}_{\Sigma_2}$$

since d_{A_2} is such that $A_{A_2}x_{A_2} + F_{A_2}d_{A_2} \in \mathcal{V}_{\mathcal{X}_{A_2}}$. In addition, note that $B_{\Sigma_2}C_{A_2}x_{A_2} = B_{\Sigma_2}C_{\Sigma_1}x_{\Sigma_1}$. The proof of this goes as follows: by (38c) and (39c) we, respectively, know that

$$C_{\Sigma_1}x_{\Sigma_1} = C^y_{G_1}x_{G_1} \quad \text{and} \quad C^y_{G_1}x_{G_1} = C_{A_2}x_{A_2}.$$

Together, these equations imply that $C_{\Sigma_1}x_{\Sigma_1} = C_{A_2}x_{A_2}$, and hence that

$$B_{\Sigma_2}C_{A_2}x_{A_2} = B_{\Sigma_2}C_{\Sigma_1}x_{\Sigma_1}.$$

Using this, we define the following

$$\bar{x}_{\Sigma_2} := B_{\Sigma_2}C_{\Sigma_1}x_{\Sigma_1} + A_{\Sigma_2}x_{\Sigma_2} + F_{\Sigma_2}d_{\Sigma_2} \quad \text{and} \quad \bar{x}_{G_2} := A_{G_2}x_{G_2} + F_{G_2}d_{G_2}.$$

From the above analysis, it is then clear that

$$\begin{bmatrix} B_{\Sigma_2}C_{\Sigma_1} & A_{\Sigma_2} & 0 \\ 0 & 0 & A_{G_2} \end{bmatrix} \begin{bmatrix} x_{\Sigma_1} \\ x_{\Sigma_2} \\ x_{G_2} \end{bmatrix} = \begin{bmatrix} \bar{x}_{\Sigma_2} \\ \bar{x}_{G_2} \end{bmatrix} + \begin{bmatrix} F_{\Sigma_2} & 0 \\ 0 & F_{G_2} \end{bmatrix} \begin{bmatrix} -d_{\Sigma_2} \\ -d_{G_2} \end{bmatrix}$$

where $(\bar{x}_{A_2}, \bar{x}_{\Sigma_2}, \bar{x}_{G_2}) \in S_2$.

With this, we have proved that: for any $(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, x_{G_1}, x_{G_2}) \in S$, we can find

$$(\bar{x}_{A_1}, \bar{x}_{\Sigma_1}, \bar{x}_{\Sigma_2}, \bar{x}_{G_1}, \bar{x}_{G_2}) \in S \quad \text{and} \quad (\bar{d}_{A_1}, \bar{d}_{\Sigma_1}, \bar{d}_{\Sigma_2}, \bar{d}_{G_1}, \bar{d}_{G_2}) = -(d_{A_1}, d_{\Sigma_1}, d_{\Sigma_2}, d_{G_1}, d_{G_2})$$

such that

$$\begin{bmatrix} A_{A_1} & 0 & 0 & 0 & 0 \\ B_{\Sigma_1}C_{A_1} & A_{\Sigma_1} & 0 & 0 & 0 \\ 0 & B_{\Sigma_2}C_{\Sigma_1} & A_{\Sigma_2} & 0 & 0 \\ 0 & 0 & 0 & A_{G_1} & 0 \\ 0 & 0 & 0 & 0 & A_{G_2} \end{bmatrix} \begin{bmatrix} x_{A_1} \\ x_{\Sigma_1} \\ x_{\Sigma_2} \\ x_{G_1} \\ x_{G_2} \end{bmatrix} = \begin{bmatrix} \bar{x}_{A_1} \\ \bar{x}_{\Sigma_1} \\ \bar{x}_{\Sigma_2} \\ \bar{x}_{G_1} \\ \bar{x}_{G_2} \end{bmatrix} + \begin{bmatrix} F_{A_1} & 0 & 0 & 0 & 0 \\ 0 & F_{\Sigma_1} & 0 & 0 & 0 \\ 0 & 0 & F_{\Sigma_2} & 0 & 0 \\ 0 & 0 & 0 & F_{G_1} & 0 \\ 0 & 0 & 0 & 0 & F_{G_2} \end{bmatrix} \begin{bmatrix} \bar{d}_{A_1} \\ \bar{d}_{\Sigma_1} \\ \bar{d}_{\Sigma_2} \\ \bar{d}_{G_1} \\ \bar{d}_{G_2} \end{bmatrix}.$$

This then proves that property (37b) holds.

Proof of (37c). Take any $(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, x_{G_1}, x_{G_2}) \in S$. By definition of S, we know that $(x_{A_1}, x_{\Sigma_1}, x_{G_1}) \in S_1$, and that there exists an element x_{A_2} such that $(x_{G_1}, x_{A_2}) \in T$ and $(x_{A_2}, x_{\Sigma_2}, x_{G_2}) \in S_2$. For these elements in S_i $(i = 1, 2)$ and T, we know by properties (38c) and (39c) that

$$
\begin{aligned}
H_{A_i} x_{A_i} &= 0, \\
H_{G_i} x_{G_i} &= 0, \\
C_{A_i} x_{A_i} &= C^u_{G_i} x_{G_i}, \\
C_{\Sigma_i} x_{\Sigma_i} &= C^y_{G_i} x_{G_i}, \\
C^y_{G_1} x_{G_1} &= C_{A_2} x_{A_2}.
\end{aligned}
$$

Together, these equations imply that

$$
(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, x_{G_1}, x_{G_2}) \in \ker \begin{bmatrix} H_{A_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{G_1} & 0 \\ 0 & 0 & 0 & 0 & H_{G_2} \\ 0 & 0 & 0 & C^y_{G_1} & -C^u_{G_2} \\ C_{A_1} & 0 & 0 & -C^u_{G_1} & 0 \\ 0 & 0 & C_{\Sigma_2} & 0 & -C^y_{G_2} \end{bmatrix}.
$$

Since we took any $(x_{A_1}, x_{\Sigma_1}, x_{\Sigma_2}, x_{G_1}, x_{G_2}) \in S$, this proofs that property (37c) holds.

With this, we have shown that our choice of S is such that all properties in Theorem 2.11 are satisfied. Hence, it has been proved that

$$
A_1 \times \Sigma_1 \times \Sigma_2 \preceq G_{12}.
$$

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