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Irrationality proofs of e and π

Research Project Master Science Education & Comunication

May 2022

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1 Introduction

Irrationality is a property of numbers that we are all too familiar with. Rational numbers are those that can be written as the ratio between two integers. Every other number is irrational. It is rather straightforward to show a number is rational by finding its numerator and denominator. The difficulty of showing a number is irrational, however, can range from rather trivial to nearly impossible. Consider for example the square root of two. In Pythagoras' time (circa 580 - 500 B.C.), the Pythagoreans initially believed everything to be constructed by whole numbers and ratios of whole numbers (Spencer, 2022). This is reflected perhaps best in the Pythagoras' motto "All is Number", that was carved above the entrance of their school. At some point, it was discovered that the diagonal of a square with unit length had to be irrational. This discovery is most often credited to Hippasus of Metapontum, who may or may not have paid for revealing this fact with his life. The Pythagoreans tried to keep this discovery a secret. Some believe that Hippasus' death by drowning was unrelated to the fact that he revealed the secret existence of irrational numbers, while others believe he was murdered for this revelation (Clegg, 2014). Whatever the truth may be, it is safe to say that irrational numbers have their fair share of interesting history. Luckily, we currently live in more accepting time where discussing the irrationality of numbers does not result in a watery grave. With this history in mind, let's look at a proof of the irrationality of $\sqrt{2}$.

Theorem 1.1. *$\sqrt{2}$ is irrational.*

Proof. Assume $\sqrt{2} = \frac{a}{b}$ for $a, b \in \mathbb{N}$, and let the fraction be in its simplest possible form, i.e. a and b are co-prime. Then we can square the identity and rewrite the it as $2b^2 = a^2$. This means that a is an even number. So we can write a as $a = 2 \cdot c$, where c is also a natural number. We can then substitute this into our expression and simplify to get $b^2 = 2a^2$. This implies that b is even and that contradicts our assumption that a and b are co-prime. Hence our assumption that $\sqrt{2}$ is rational is false and we must conclude that $\sqrt{2}$ is irrational. $\square$

In this thesis we will focus on the constant π and e to discuss some different methods of proving irrationality, what underlying principles they use, and try to make a comparison between the constants by trying to apply methods meant for one constant on the other. Like in the example shown above, we will start our proofs by assuming our constant is rational and then reach a contradiction due to that assumption, forcing us to conclude our constant is irrational instead. We also show this result a different way, namely by the use of continued fractions. Every number can be written as a continued fraction, the definition of which will become clear in a moment. For $\sqrt{2}$ we have that

$$\sqrt{2} = 1 + \sqrt{2} - 1 = 1 + \frac{1}{1+\sqrt{2}} = 1 + \cfrac{1}{1 + \left(1 + \cfrac{1}{1+\sqrt{2}}\right)} = 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}.$$

This pattern continue, where we can keep substituting $1 + \dfrac{1}{1+\sqrt{2}}$ for $\sqrt{2}$ to obtain the continued fraction

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

This continued fraction is infinite, which means, for reasons we will see later, that $\sqrt{2}$ is irrational. In particular, simple continued fractions that are infinite are always irrational. We can use the same method to show the irrationality of a different constant, namely the golden ration.

Theorem 1.2. *The golden ration, $\varphi = \frac{1+\sqrt{5}}{2}$, is irrational.*

Proof. Since φ is a solution to $x = 1 + 1/x$, we can quickly find its continued fraction as

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \cfrac{1}{1 + \cfrac{1}{\varphi}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\varphi}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}.$$

This is infinite and hence φ is irrational. □

This lets us conclude not only that φ is irrational but also that using this method of observing the continued fraction can be used for several constants. However, it is clear that this would not be the obvious way to prove the irrationality of φ. The proof of irrationality of $\sqrt{2}$ can be applied to the square root of any prime number. In particular, we can see through that proof that $\sqrt{5}$ is irrational, and when we know this, it is straightforward that φ is also irrational.

Proof. Assume $\frac{1+\sqrt{5}}{2} = \frac{a}{b}$ for some $a, b \in \mathbb{N}$. Then we would have $\sqrt{5} = \frac{2a-b}{b}$ which would mean $\sqrt{5}$ is rational as well, which we know not to be the case. Hence, φ is irrational. □

Lastly, if we can show that a constant c is irrational, it follows immediately that $\sqrt{c}$ is also irrational. Since if $\sqrt{c}$ were irrational, we could say $\sqrt{c} = \frac{a}{b}$ and consequently $c = \frac{a^2}{b^2}$. Here, we see an example of the same constant being proven to be irrational in two different ways.

In this thesis, we will look at different proofs for the constants e and π. These proofs will be far less trivial than the ones above, especially for π. We will work out these proofs in detail, providing extra theorems and lemma's were needed. When we have explored these proofs, we will compare them and look at the similarities and differences between them. We will discuss the complexity of the methods, taking into account the mathematical tools used, the interchangeability of the methods between constants.

2 Proofs of irrationality of e

The constant e is one of the most well-known constants in mathematics and it shows up in many different fields. Its irrationality has been proven several times and in this section we will highlight and explain some of them in more detail. We will often use the expansion $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ and make the assumption that $e = a/b$ for two natural numbers a, b, to arrive at a contradiction.

Theorem 2.1. *e is irrational.*

2.1 Kifowit, repeated integration by parts

Our first proof is by Steve Kifowit from 2009. It starts by taking the definite integral of e^{-x} from 0 to 1 and applying integration by parts a variable, but finite, amount of times. This allows us to split the resulting expression in a part that consists only of numbers that are necessarily integer and a part that is strictly between 0 and 1. This then gives us the contradiction since a number cannot be integer and strictly between 0 and 1 simultaneously.

Proof. Suppose $e = a/b$ for $a, b \in \mathbb{N}$. Consider the integral

$$\int_0^1 e^{-x} \, \mathrm{d}x = 1 - \frac{1}{e}$$

and take an integer $n \geq \max\{b, e\}$. We integrate by parts. Re-imagining the integral as

$$\int_0^1 e^{-x} \cdot 1 \ \mathrm{d}x$$

and integrating by parts once gives us gives us

$$\begin{aligned} \int_0^1 e^{-x} \cdot 1 \ \mathrm{d}x &= \left[xe^{-x}\right]_0^1 - \int_0^1 -e^{-x} \cdot x \ \mathrm{d}x \\ &= \frac{1}{e} + \int_0^1 e^{-x} \cdot x \ \mathrm{d}x. \end{aligned}$$

This does not yet give us a clear general form so we integrate by parts again.

$$\begin{aligned} \int_0^1 e^{-x} \cdot 1 \ \mathrm{d}x &= \frac{1}{e} + \left[\frac{1}{2}x^2 e^{-x}\right]_0^1 - \int_0^1 -e^{-x} \cdot \frac{1}{2}x^2 \ \mathrm{d}x \\ &= \frac{1}{e}\left(1 + \frac{1}{2}\right) + \int_0^1 e^{-x} \cdot \frac{x^2}{2} \ \mathrm{d}x \end{aligned}$$

Here, we see a pattern start to emerge. When we compare the expression to the one after a single integration by parts, we see that the amount we multiply e with starts to grow and the polynomial in the integral starts to change. We apply integration by part again to see this this another iteration in this pattern.

$$\begin{aligned} \int_0^1 e^{-x} \cdot 1 \ \mathrm{d}x &= \frac{1}{e}\left(1 + \frac{1}{2}\right) + \left[\frac{x^3}{3!}e^{-x}\right]_0^1 - \int_0^1 -e^{-x} \cdot \frac{x^3}{3!} \ \mathrm{d}x \\ &= \frac{1}{e}\left(1 + \frac{1}{2!} + \frac{1}{3!}\right) + \int_0^1 e^{-x} \cdot \frac{x^3}{3!} \ \mathrm{d}x \end{aligned}$$

Here, the pattern is especially clear. We rewrite the terms after the after the $\frac{1}{e}$ term to make clear that the terms are the reciprocals of factorials. From this, we can see that applying integration by parts n many times gives us

$$1 - \frac{1}{e} = \frac{1}{e}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}\right) + \int_0^1 \frac{x^n}{n!} e^{-x}\,\mathrm{d}x.$$

Multiplying by $n!e$ and isolating the integral gives us

$$n!(e-1) - n!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}\right) = e\int_0^1 x^n e^{-x}\,\mathrm{d}x.$$

Because of our choice of n, the left hand reduces to an integer. However the right hand side is strictly between zero and one:

$$0 < e\int_0^1 x^n e^{-x}\,\mathrm{d}x \le e\int_0^1 x^n\,\mathrm{d}x = \frac{e}{n+1} < 1.$$

Hence, we have reached a contradiction and so e is irrational (Kifowit, 2009). $\square$

We see here that manipulating the identity given by the integral allows us to separate a term that we know is an integer and equate it to something we can make arbitrarily small to force a contradiction. We do this here by choosing n specifically to be larger than e in advance. This condition could also have been omitted and replaced with an argument at the end of the proof that states that the fraction of e over $n+1$ can be made smaller than 1 for n sufficiently large. However, because we already know how large n has to be we can include this earlier. It might otherwise seem odd at first to require n to be larger than b and e, since for an integer to be larger than e only means it has to be larger than 2. And since n is also required to be larger than b this seems redundant at first, since b will most definitely be larger than 2. For if it wasn't, then $2e = 5.4366$ would be natural, which it clearly is not. The proof concludes with a contradiction that the quantity we reasoned to be an integer under the assumption that e is irrational has to be in the interval $(0,1)$. This is a conclusion we will see more often, though not always, when discussing proofs of irrationality of e.

2.2 Proofs from THE BOOK I

The proof in Proofs from THE BOOK (Aigner and Zeigler, 2005) is again a proof by contradiction where we manipulate the expansion of e and separate the integer terms and are left with a term that is necessarily in the interval $(0,1)$. This is a clear contradiction which proves the theorem.

Proof. Suppose $e = a/b$ for $a, b \in \mathbb{N}$
Then, $n!be = n!a$ for all $n \geq 0$. On the right hand side, we again have an integer. While on the left hand side, we have

$$n!b\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right) = n!a,$$

which we can rewrite as

$$n!b\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + n!b\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right) = n!a.$$

While the first term reduces to an integer, the second part falls strictly between zero and one. For this, notice that

$$
\begin{aligned}
&n!b\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots\right)\\
&=b\left(\frac{1}{(n+1)}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots\right)\\
&<b\left(\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\cdots\right)\\
&=b\cdot\frac{1}{n}.
\end{aligned}
$$

The last equality is due to the geometric series which tells us that

$$\sum_{k=1}^{\infty}\frac{1}{r^k}=\frac{1}{r-1},\ \text{for } r>1.$$

We use $r = n + 1$. For n sufficiently large (greater than b), this is clearly less than one. So we have reached a contradiction. □

This proof is rather short an straightforward. The equality $n!be = n!a$ follows directly from the assumption that $e = a/b$. Since we specify at the start of the proof that the equality holds for *all* values of n greater or equal to 0, we can comfortably let n be greater than b at the end of the proof. The contradiction in this proof doesn't arise because the equality $n!bx = n!a$ is false when $x = a/b$ but because when we use e in place of x we have to take into account the special properties of the number e that then show it cannot be expressed as the ration between two natural numbers.

2.3 MathOnline and variations

An adaptation of the proof from MathOnline revolves around multiplying the expansion of e by the factorial of b, where b is again the assumed denominator of e as a fraction of natural numbers. This time, we look at the difference between two integers, called $\mathcal{M}$ and apply different bounds on b to see that it cannot be larger than 1. We will explore two different ways that a suitable upper bound on $\mathcal{M}$ can be attained. One assumes b is at least 1, which is trivially true, and the second assumes b is at least 2, which can be seen quite easily as $2e \approx 7.389$ which is clearly not a natural number, so $b \neq 2$. If we increase the minimum value of b, we can also lower the upper bound of $\mathcal{M}$ (MathOnline, 2017).

Proof. Suppose $e = a/b$ for $a, b \in \mathbb{N}$
Consider

$$
\begin{aligned}
b!e =&b!\sum_{n=0}^{\infty}\frac{1}{n!}\\
=&b!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots\right)\\
=&b!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{b!}\right)+b!\left(\frac{1}{(b+1)!}+\frac{1}{(b+2)!}+\frac{1}{(b+3)!}+\cdots\right)
\end{aligned}
$$

Here, the first term on the right hand side is an finite sum of integers, while the second is an infinite sum of numbers less than one. Clearly, the first term is an integer while the second is clearly greater than zero. Since $b!e$ is also an integer, we define a new number $\mathcal{M}$ as

$$
\begin{aligned}
\mathcal{M} =& b!e - b!\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{b!}\right)\\
=& b!\left(\frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \frac{1}{(b+3)!} + \cdots\right)\\
=& \frac{1}{(b+1)} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots .
\end{aligned}
$$

This is similar to the previous proof, where we now have to show that $\mathcal{M} < 1$. We could use the geometric series again, but for now we will show two different methods of obtaining this upper bound. First, we will bound the denominator of each term by a single power of b. We then substitute 2 for b to get an upper bound. This is because $\mathcal{M}$ decreases as b increases. For the second method, we immediately substitute 1 for b to obtain an upper bound.

Method 1 Using $b \geq 2$

$$
\begin{aligned}
\mathcal{M} =& \frac{1}{(b+1)} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \ldots\\
<& \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots\\
\leq& \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\\
=& \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\\
=& 1
\end{aligned}
$$

Since $\mathcal{M}$ gets smaller as b gets larger, we can upper bound it by the lowest value of b. This assumes that $b \neq 1$, which means we would have to show that e isn't an integer. This is clear.

Method 2 Using $b \geq 1$

$$
\begin{aligned}
\mathcal{M} =& \frac{1}{(b+1)} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots\\
\leq& \frac{1}{(1+1)} + \frac{1}{(1+1)(1+2)} + \frac{1}{(1+1)(1+2)(1+3)} + \cdots\\
=& \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots\\
=& \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\\
=& e - 2\\
\approx& 0.71828
\end{aligned}
$$

Here, we use again that $\mathcal{M}$ decreases as b increases. We use $b \geq 1$, if we instead use $b \geq 2$ as we did in the proof above, we get a lower upper bound. Namely,

$$\begin{aligned}
\mathcal{M} <& \frac{1}{(2+1)} + \frac{1}{(2+1)(2+2)} + \frac{1}{(2+1)(2+2)(2+3)} + \cdots \\
=& \frac{1}{3} + \frac{1}{3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots \\
=& \frac{2}{2} \cdot \frac{1}{3} + \frac{2}{2} \cdot \frac{1}{3 \cdot 4} + \frac{2}{2} \cdot \frac{1}{3 \cdot 4 \cdot 5} + \cdots \\
=& \frac{2}{3!} + \frac{2}{4!} + \frac{2}{5!} + \cdots \\
=& 2 \cdot (e - 2.5) \\
\approx& 0.4366
\end{aligned}$$

Both methods (and sub-methods) produce a sufficient upper bound for $\mathcal{M}$. The fact that $\mathcal{M} > 0$ follows directly from the definition of $\mathcal{M}$ as a sum of positive numbers. □

This proof uses the identity of e as a series most directly and cuts it up in two parts, one integer and one non-integer and less than 1. The multiplication with $b!$ only serves to makes the left hand side, $b!e$, an integer. Multiplying just by b would suffice in this, the factorial allows us to also make sure all the terms in the series up to $b!/b!$ are integers as well. In theory, we wouldn't have to multiply by $b!$, just by the smallest integer that is a multiple of b and a multiple of exclusively factorials. For example, if $b = 12$, then 24 divides 12 and the first five factorials, and 24 is a lot smaller than $12! = 479001600$. Finding an expression for this, however, would require more explanation and more complicated notation and doesn't make the argument any stronger. Therefore, we can take a larger number than necessary and simply multiply by $b!$ to keep the proof simple to understand.

After we have separated the integer and non-integer part of $n!b$ and factoring out the $b!$ term, we are left with showing that $\mathcal{M}$ is less than 1. As we have seen, this can be done is multiply ways, each yielding a different valid bound. When we look at the second method we used to show that $\mathcal{M}$ is less than 1, we can see that increasing the value of b gives us an increasingly smaller upper bound for $\mathcal{M}$.

In the end, this proof also relies on showing that a quantity which is reasoned to be an integer, based on the assumption that $e = a/b$, is contained in the interval $(0, 1)$.

2.4 Sondow, nested intervals

This proof takes a different approach than the ones we have seen before. Instead, we construct set of nested closed intervals around e. We will call these intervals I_n. We define

$$I_n = \left[\sum_{i=0}^{n} \frac{1}{i!}, \sum_{i=0}^{n} \frac{1}{i!} + \frac{1}{n!} \right].$$

This means the first three intervals are

$$\begin{aligned}
I_1 &= \left[\frac{2}{1!}, \frac{3}{1!} \right] = [2, 3], \\
I_2 &= \left[\frac{5}{2!}, \frac{6}{2!} \right] = [2.5, 3], \\
I_3 &= \left[\frac{16}{3!}, \frac{17}{3!} \right] = [2.666\ldots, 2.8333\ldots].
\end{aligned}$$

We will show that the intervals each contain e and that the intersection of all intervals contains *only* e. Then, we will argue that e cannot be on the boundary of any I_n. After this, we will reason that this means e cannot be rational.

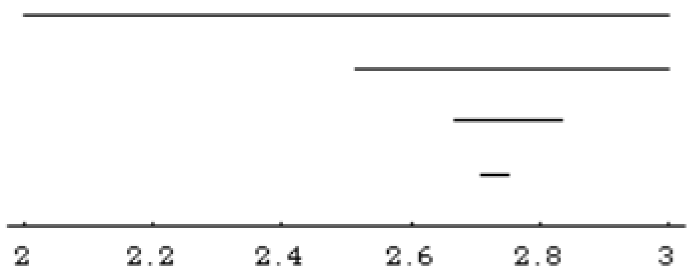

Figure 1: The intervals I_1, I_2, I_3, and I_4.

Lemma 2.2. $e \in I_n$ for every $n \in \mathbb{N}$.

We will show this by showing that e is greater than the left bound of I_n, and less than the right bound.

Proof. For the left bound, since the left bound of I_n is the n-th partial sum in the expansion of e which contains strictly positive terms, it follows immediately that

$$\sum_{i=0}^{n} \frac{1}{i!} < e.$$

For the right endpoint, note that

$$\sum_{i=1}^{\infty} \frac{n!}{(n+i)!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

is decreasing as n increases and is equal to $e - 2$ for $n = 1$, hence

$$\sum_{i=1}^{\infty} \frac{n!}{(n+i)!} \leq e - 2 < 1$$

for all values of $n \geq 1$. If we multiply by $\frac{1}{n!}$, we find that

$$\sum_{i=1}^{\infty} \frac{1}{(n+i)!} = \sum_{i=n+1}^{\infty} \frac{1}{i!} < \frac{1}{n!}.$$

This means that

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} < \sum_{i=0}^{n} \frac{1}{i!} + \frac{1}{n!}.$$

Pairing the results for each endpoint we can conclude that

$$\sum_{i=0}^{n} \frac{1}{i!} < e = \sum_{i=0}^{\infty} \frac{1}{i!} < \sum_{i=0}^{n} \frac{1}{i!} + \frac{1}{n!}.$$

$\square$

Lemma 2.3. $I_n = \left[\frac{a_n}{n!}, \frac{a_n+1}{n!}\right]$, where a_n is a natural number depending on n.

Proof. This follows immediately from the definition of I_n. The left bound is equal to $\sum_{i=1}^{n} \frac{1}{i!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ and we can rewrite each term in this sum to have $n!$ as the denominator by multiplying both sides of the i-th faction by each number from 1 to n except i. The difference between the left and right bound is immediate from the definition. $\square$

Knowing this, we also want to show that no two intervals share a boundary. The way we have currently defined I_n does not rule out. To complete this proof, we need to see that, except for $n = 2$, no interval shares a boundary with the previous interval.

Lemma 2.4. For $n \geq 2$, I_n does not share a boundary with I_m with $m > n$.

Proof. The definition of I_n forces the left boundary to be different for each interval, specifically to be higher than the previous. Also by the definition, the right boundary is given by $\sum_{i=0}^{n} \frac{1}{i!} + \frac{1}{n!}$ meaning that the right boundary of I_n is equal the to right boundary of I_{n+1} if and only if $\sum_{i=0}^{n} \frac{1}{i!} + \frac{1}{n!} = \sum_{i=0}^{n+1} \frac{1}{i!} + \frac{1}{(n+1)!}$. In this case, $\frac{1}{n!} = \frac{2}{(n+1)!}$, and this is only true if $n = 1$. $\square$

We wil now look at the intersection of all these intervals and show that in contains the element e and nothing else. This will then allow us to prove that e cannot be a fraction with $n!$ as denominator, after which we are only one step away from completing the proof.

Lemma 2.5. $\bigcap_{n=1}^{\infty} I_n = \{e\}$.

Proof. Since e is in each interval I_n, by lemma 2.2, it must also be in the intersection. To see why it cannot contain any other element, suppose some constant $\alpha \in \mathbb{R}\backslash\{e\}$ exists in this intersection, then it must be in each of the intervals. Let $d = |\alpha - e|$ be the distance from e to α. By construction, the length of interval I_n is equal to $\frac{1}{n!}$ and this length gets arbitrarily small for a large enough value of n. This means it will eventually become smaller than d and so this element cannot exist in all intervals and therefore is not in the intersection. $\square$

Proof. We now know that for $n \geq 2$, each subinterval I_n lies strictly within the endpoints of I_{n-1} (lemma 2.4) which are $\frac{a}{n!}$ and $\frac{a+1}{n!}$. This means that e cannot be on the boundary of any of the intervals. Suppose now that $e = \frac{b}{n!}$ for some $b \in \mathbb{N}$. Then we $\frac{b}{n!} \in \left[\frac{a_n}{n!}, \frac{a_n+1}{n!}\right]$. And since b cannot be equal to a_n or $a_n + 1$, it has be between a_n and $a_n + 1$ but there clearly aren't any whole numbers between a_n and $a_n + 1$, so e cannot be a fraction with a factorial as denominator. However, any fraction with denominator n can be written as

$$\frac{m}{n} = \frac{(n-1)!m}{n!}$$

for any pair of integers m and n. So this is equivalent to e not being a fraction with denominator n, where n is any integer. Hence, e cannot be a fraction (Sondow, 2006). $\square$

Originally, the intervals I_n were defined as dividing I_{n-1} into n equally long closed sub-intervals and choosing I_n as the sub-interval that contains e. This, however, would make proving things like lemma 2.3 a more complicated. That is why we instead defined this way. It ultimately didn't take anything away from the proof or any of the arguments and uncluttered the proof.

2.5 Diao, decreasing sequence of natural numbers

This proof requires little to no calculus and relies on the identity instead of the one we have used so far. Recall that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 1 + \frac{1}{1}\left(1 + \frac{1}{2}\left(1 + \frac{1}{3}\left(1 + \cdots\right)\right)\right).$$

Define a sequence (x_n) as

$$x_n = \sum_{i=1}^{\infty} \frac{(n-1)!}{(n+i)!} = \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \cdots$$

We claim the following two properties:
1. $x_n = \frac{1}{n}(1 + x_{n+1})$.
2. $x_n > x_{n+1} > 0$ for all $n \in \mathbb{N}$.

Property 1 can be shown with a straightforward computation:

$$\begin{aligned} x_n &= \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \cdots \\ &= \frac{1}{n}\left(1 + \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\right) \\ &= \frac{1}{n}\Big(1 + x_{n+1}\Big). \end{aligned}$$

For property 2, we compare x_n to x_{n+1}:

$$\begin{aligned} x_n &= \quad \frac{1}{n} \quad + \quad \frac{1}{n(n+1)} \quad + \quad \frac{1}{n(n+1)(n+2)} \quad + \cdots \\ x_{n+1} &= \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots. \end{aligned}$$

We can see that in x_{n+1}, the denominators are smaller than the ones of the respective fractions in x_n. Because $e = 1 + x_1$, if e is irrational, then so is x_1. Pair this with property 1 and we see that if e is rational then so is each x_n. Lets assume this is the case and write

$$x_n = \frac{p_n}{q_n},$$

where p_n and q_n are natural numbers that are relatively prime. Using property 1 and substituting this, we see that

$$\frac{p_n}{q_n} = \frac{1}{n}\left(1 + \frac{p_{n+1}}{q_{n+1}}\right).$$

We then isolate x_{n+1} and write it in terms of p_n, q_n, and n and get

$$\frac{p_{n+1}}{q_{n+1}} = \frac{np_n - q_n}{q_n}.$$

The fraction $\frac{p_{n+1}}{q_{n+1}}$ is in its simplest form. If q_n were smaller than q_{n+1}, then $np_n - q_n$ would have to be less than p_{n+1}, since the fractions are equal, and in that case, we would have a fraction with a smaller numerator and denominator, which is not possible. Hence, $q_n \geq q_{n+1}$. When we combine this with property 3, which tells us that $\frac{p_n}{q_n} > \frac{p_{n+1}}{q_{n+1}}$, we can conclude that $p_n > p_{n+1}$ meaning p_n is a decreasing sequence of natural numbers, which is impossible. Hence we have arrived at a contradiction and so e must be irrational (Diao, 2012).

Here, we see another proof that does not rely on showing an integer quantity is also in the interval $(0,1)$. Instead, we arrive at a contradiction by constructing a sequence of natural numbers that is strictly decreasing. This proof also leans less heavily on the series expansion of e and the assumption that e is rational. Instead, it relies more heavily on the sequence x_n and its properties.

2.6 Proofs from THE BOOK II

To close the section of proofs of irrationality of e, we will provide a proof that takes a completely different route from the ones we have seen before, as well as show a result that dwarfs the ones we have seen so far. We will show that any rational power of e is irrational, making the irrationality of e a specific case of this statement.

Theorem 2.6. *e^r is irrational for every $r \in \mathbb{Q} \setminus \{0\}$.*

We will construct a specific family of polynomials, f_n that will have a number of properties that will keep important values as integers. We then sum the derivatives of these polynomials and integrate over them. The result should then also be an integer but we will show that it is simultaneously in the interval $(0,1)$, giving rise to a contradiction.
For any fixed $n \in \mathbb{N}$, define

$$f_n(x) = \frac{x^n(1-x)^n}{n!}.$$

While this is one way to write $f_n(x)$, it is also a polynomial of the form

$$f_n(x) = \frac{1}{n!}\sum_{i=0}^{2n} c_i x^i,$$

where c_i are integers.

Lemma 2.7. The k-th derivative of $f_n(x)$ has an integer value if $x = 0$ or $x = 1$, for all natural numbers k.

Proof. We will prove this for two separate cases.

First, in case $0 \leq k < n$. In this case, $f_n^{(k)}$ still has an x^{n-k} term and an $(1-x)^{n-k}$ term, which means that $f_n^{(k)}(0) = 0$ and $f_n^{(k)}(1) = 0$.

Second, in the case that $n \leq k \leq 2n$, we can see from the polynomial expression that $f_n^{(k)}(0) = \frac{k!}{n!}c_k$ and since $k \geq n$, $\frac{k!}{n!}$ is an integer, $f_n^{(k)}(0)$ is the product of two integers and thus an integer. We know that $f(x) = f(1-x)$ so the k-th derivative satisfies $f_n^{(k)}(1-x) = (-1)^k f_n^{(k)}(x)$. With this, we can see that $f_n^{(k)}(1) = (-1)^k f_n^{(k)}(0)$ which is an integer.
It suffices to show that e^r is irrational for $r \in \mathbb{N}$ since if $e^{\frac{p}{q}}$ were rational then $\left(e^{\frac{p}{q}}\right)^q = e^p$ would be rational too.
Define

$$F_n(x) = r^{2n} f_n(x) - r^{2n-1} f_n'(x) + r^{2n-2} f_n''(x) - \cdots - r f^{(2n-1)}(x) + f_n^{(2n)}(x),$$

which has derivative

$$F_n'(x) = r^{2n} f_n'(x) - r^{2n-1} f_n''(x) + r^{2n-2} f_n'''(x) - \cdots - f_n^{(2n)}(x).$$

We know from lemma 2.7 that $f_n^{(k)}(0)$ and $f_n^{(k)}(1)$ are integers for all $k \geq 0$. Since r and all of its powers are integers, it follows that $F_n(0)$ and $F_n(1)$ are integers as well. With this, we can see that

$$F_n'(x) = -rF_n(x) + f_n^{(2n+1)}(x).$$

Next, we will differentiate the function $e^{rx}F_n(x)$ which, by the previous result, gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{rx}F_n(x) =& re^{rx}F_n(x) + e^{rx}F_n'(x)\\ =& re^{rx}F_n(x) - re^{rx}F_n(x) + r^{2n+1}e^{rx}f_n(x)\\ =& r^{2n+1}e^{rx}f_n(x)\end{aligned}$$

Now assume $e^r = \frac{a}{b}$ for two natural numbers a and b and define a new quantity, $\mathcal{N}$ as

$$\mathcal{N} = b\int_0^1 r^{2n+1}e^{rx}f_n(x)\,\mathrm{d}x$$

where b is the natural number we assumed to be the denominator of e^r. We can evaluate $\mathcal{N}$ to be

$$\begin{aligned}\mathcal{N} =& b\int_0^1 r^{2n+1}e^{rx}f_n(x)\,\mathrm{d}x\\ =& b\left[e^{rx}F_n(x)\right]_0^1\\ =& b\left[e^{r\cdot 1}F_n(1) - e^{r\cdot 0}F_n(0)\right]\\ =& b\left[\frac{a}{b}F_n(1) - F_n(0)\right]\\ =& aF_n(1) - bF_n(0)\end{aligned}$$

And we know that a, b, $F_n(0)$, and $F_n(1)$ are all integers by definition and by lemma 2.7, hence $\mathcal{N}$ has to be an integer. Recall that $f_n(x) = \frac{x^n(1-x)^n}{n!}$. From this we can see easily that $0 < f_n(x) < \frac{1}{n!}$ for $0 < x < 1$ with $f_n(x) = 0$ if $x = 0$ or $x = 1$. We can use his inequality to bound $\mathcal{N}$, namely

$$0 < \mathcal{N} = b\int_0^1 r^{2n+1}e^{rx}f_n(x)\,\mathrm{d}x < br^{2n+1}e^r\frac{1}{n!} = \frac{ar^{2n+1}}{n!} < 1.$$

Here, the last inequality is if n is large enough. This gives us a contradiction, namely that $\mathcal{N}$ is both and integer and in the interval $(0, 1)$. Therefore, our assumption that e^r is rational must be false, so e^r must be irrational. □

2.7 Euler, continued fraction

One way of expressing a (real) number is by means of a continued fraction. The continued fraction of a real number r is typically of the form.

$$r = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}},$$

where each a_i and b_i is an integer although there are more than one way to generalise the structure of a continued fraction. In this representation, if $b_i = 1$ for all values of i then we call the continued fraction a *simple* continued fraction. For example, in 1737, Leonhard Euler gave the simple continued fraction for $(e-1)/2$, namely

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cdots}}}}$$

(Roegel, 2020; Euler, 1748). We will use this continued fraction to prove the irrationality of e. For this, we will use a lemma by Lambert that was used to originally prove that π is irrational (Lambert, 1761). The outline of that proof will be discussed later. As mention in the introduction, a number is irrational if and only if its simple continued fraction is infinite. To see this, we will first introduce some notation. For any real number x we can write $x = n + u$ where $n \in \mathbb{N}$ and $0 \leq u < 1$ and moreover this representation is unique. If $u = 0$ then x is an integer. If instead $u > 0$, then $1/u > 1$. We can use this to decompose x and construct a sequence u_i and n_i. For any real x we can then write $x = n_1 + u_1$ and if $u_1 > 0$ then $1/u_1 = n_2 + u_2$ and we can start forming the continued fraction

$$x = n_1 + \frac{1}{n_2 + u_2}.$$

Theorem 2.8. *The simple continued fraction of a real number is finite if and only if that number is rational.*

Proof. We will first prove the statement from right to left. Let $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ with $b > 0$. We can then write

$$\frac{a}{b} = \frac{a_0}{b_0} = n_0 + \frac{a_0 - n_0 b_0}{b_0} := n_0 + \frac{a_1}{b_0},$$

where n_0 is a natural number such that $b_0 > a_0$, or zero if it is already the case that $b_0 > a_0$. We can then repeat this process and write

$$n_0 + \frac{a_1}{b_0} = \frac{1}{\left(\frac{b_0}{a_1}\right)} = n_0 + \cfrac{1}{n_1 + \cfrac{b_0 - n_1 a_1}{a_1}} = n_0 + \cfrac{1}{n_1 + \cfrac{b_1}{a_1}}$$

Since $b_0 > a_0$, we have $n_1 \geq 1$ and we can repeat this process and keep adding layers, introducing a new n_i, a_i, and b_i each time. We define n_i each time through division with remainder, we have that $a_i > a_{i+1}$ and $b_i > b_{i+1}$ we have two decreasing sequences of natural numbers so they converge to 1. As soon as one of them reaches 1, the algorithm stops and we have a finite continued simple fraction. The proof from left to right is rather straightforward. If a continued fraction is finite, we can repeatedly reduce the fraction to one as a ratio of only two numbers. In conclusion, a continued fraction is finite exactly when it is rational. □

It is important here that we construct the continued to be simple. Otherwise, we can easily construct a counter example that seemingly contradicts the theorem. Consider for example

$$\frac{7}{4} = 1 + \cfrac{6}{4 + \cfrac{11}{1 + \cfrac{7}{4}}}.$$

Here, we deliberately didn't construct a *simple* continued fraction and have ended up with sequences a_i and b_i that weren't decreasing and we can extend this continued fraction to be infinite by reiterating $7/4$ indefinitely. So it is necessary for the theorem that the continued fraction is simple. We will later see an example where this requirement can be somewhat relaxed, however.

We can see in our example for e that the continued fraction has a pattern where each next n_i is 4 larger than the previous. Since the continued fraction is infinite, we can apply theorem 2.8 and conclude that e is irrational. Like mentioned before, we will generalize theorem 2.8 to allow for a larger class of continued fractions. Let a general continued fraction be of the form

$$\varphi = \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}},$$

where we have a_n, $b_n \in \mathbb{N}$ we have the following theorem.

Theorem 2.9. *If $1 + b_n \leq a_n$ for all $n \in \mathbb{N}$, and $1 + b_n < a_n$ infinitely many times, then φ is irrational.*

This theorem is proven by contradiction by constructing a strictly decreasing sequence of natural numbers. In our notation omit the a_0 term in our representation so we can also write

$$\varphi = \frac{b_1}{a_1 - p_1}, \qquad p_n = \frac{b_{n+1}}{a_{n+1} - p_{n+1}}.$$

Omitting a_0 can be justified by reasoning that it does not change the rationality of φ so it does not matter for our purposes.

Proof. Assume instead that $\varphi = \frac{\lambda_1}{\lambda_0}$ for two natural numbers λ_0 and λ_1. We then have

$$\frac{\lambda_1}{\lambda_0} = \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}}.$$

Because of our assumption, this will be less than 1, or equivalently $\lambda_0 > \lambda_1$. If we use the notation involving p_n, we get the equation

$$\frac{\lambda_1}{\lambda_0} = \frac{b_1}{a_1 - p_1}.$$

We can rewrite this to

$$p_1 = \frac{a_1\lambda_1 - b_1\lambda_0}{\lambda_1} < 1.$$

This means we can write p_1 as $p_1 = \frac{\lambda_2}{\lambda_1}$. Continuing this, we obtain a strictly decreasing sequence of positive integers. $\lambda_0 > \lambda_1 > \lambda_2 > \ldots$, which is not possible. The contradiction means that if $1 + b_n \leq a_n$ for all n, and an inequality infinitely many times, then φ is irrational. $\square$

We can see that if our continued fraction is simple, then $b_n = -1$ for all n so the condition $1 + b_n \leq a_n$ is always satisfied as the left hand side vanishes and $a_n > 0$. The only condition left is for the strict inequality to happen infinitely many times, which only means we need our simple continued fraction to be infinite and so we have reached theorem 2.8. Since, our continued fraction for $(e-1)/2$ is simple, the term b_n is equal to 1 for all $n \in \mathbb{N}$ and we can see that $a_n = 2 + 4n$ for all $n \in \mathbb{N} \setminus \{1\}$ and $a_1 = 1$. This means that $1 + b_n = 2$ and so $1 + b_n \leq a_n$ for $n \geq 2$. Therefore, by theorem 2.9 we can conclude that $2/(e-1) - 1$ is irrational, and equivalently, e is irrational. We also note that the criterion $1 + b_n \leq a_n$ for all $n \in \mathbb{N}$ can be relaxed slightly. As long as there exists an $N \in \mathbb{N}$ such that all the criterion holds for all $n \geq N$, the conclusion holds as we can use finitely many transformations (addition, subtraction, multiplication, division) by non-zero rational numbers on our original number to be equal to a continued fraction that satisfies the original theorem, and none of these transformations change the (ir)rationality of our number.

2.8 Discussion

Having seen a number of proofs of irrationality of e and powers thereof, we can note a number of similarities between them. To start, all of them are proofs by contradiction, assuming e or a power of e is equal to a fraction $\frac{a}{b}$ and showing this can't be the case. In the first few proofs, we saw how different identities involving e were manipulated to allows us to collect integer terms which would equal an expression that was then shown to be strictly between 0 and 1, leading to a contradiction. In the final proof, we saw the use of a family of functions, f_n, which we will visit again when discussing proofs of irrationality of π and powers thereof. It is no surprise that this proof is more complicated than the ones before it, since it proves a far stronger statement.

3 Proofs of irrationality of π

The irrationality of π is more difficult to show than the irrationality of e. While there are many infinite sums that result in a power of π divided by a natural number, we can't assume $\pi = a/b$, multiply by a variation of $b!$ and shuffle some terms around.

Theorem 3.1. *π is irrational.*

3.1 Original proof

The first proof of the irrationality of π is credited to Johann Heinrich Lambert who showed the result in 1761 in his work *Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques* (Lambert, 1761). He first showed that the function $\tan(x)$ can be written as the continued fraction

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cdots}}}}}.$$

Then, he showed that if x is non-zero and rational then this expression is irrational. Since $\tan(\pi/4) = 1$, it follows that $\pi/4$ is irrational, so then also π is irrational. Following is a brief outline of the proof that non-zero rational values of x give irrational values for this expression. Recall theorem 2.9 which stated that, for a continued fraction given by

$$\varphi = \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}},$$

if $1 + b_n \leq a_n$ for all $n \in \mathbb{N}$, and $1 + b_n < a_n$ infinitely many times, then φ is irrational, where we noted that the condition "for all $n \in \mathbb{N}$ can be relaxed to "for all $n \geq N$ for some $N \in \mathbb{N}$. Knowing this, we can prove that π is irrational by substituting $x = \pi/4$ into Lambert's continued fraction for $\tan(x)$. We assume that $\pi/4$ is rational, equivalent to saying π is rational, and show that this leads to a contradiction with theorem 2.9. Assume $\pi/4 = p/q$ for $p, q \in \mathbb{N}$. We write

$$\tan\left(\frac{\pi}{4}\right) = 1 = \cfrac{\frac{p}{q}}{1 - \cfrac{\frac{p^2}{q^2}}{3 - \cfrac{\frac{p^2}{q^2}}{5 - \cdots}}}.$$

And this can be rewritten to

$$\tan\left(\frac{\pi}{4}\right) = 1 = \cfrac{p}{q - \cfrac{p^2}{3q - \cfrac{p^2}{5q - \cdots}}}.$$

For a sufficiently large value of n the term $(2n-1)q > p^2 + 1$ meaning that the continued fraction from that point on is irrational according to theorem 2.9. Since this happens after a finite amount of ‘steps’, the entire continued fraction must be irrational. But this clearly cannot be the case, since the continued fraction equal 1. This contradictions leads us to conclude that $\pi/4$ must be irrational (Schipperus, 2014).

The finer details of this proof, such as showing the continued fraction for $\tan(x)$, are rather complicated. Later on, we will see a simplified proof by Laczkovich that follows the original proof, and the steps omitted here, closely.

3.2 Continued fractions

We have seen earlier that continued fractions can be used to show irrationality. Recall theorem 2.9 which stated that for a continued fraction given by

$$\varphi = \cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}},$$

φ is irrational if $1 + b_n \leq a_n$ for all $n > N \in \mathbb{N}$, for some fixed $N \in \mathbb{N}$ and $1 + b_n < a_n$ infinitely many times. We can also apply this to show the irrationality of π. We have the following continued fraction involving π:

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cfrac{11^2}{2 + \cdots}}}}}}.$$

This fraction is not simple so we cannot apply theorem 2.8. We can see that all our valued of b_n are negative and decreasing and $a_n = 2$ for all n. So the conditions of the theorem are met and we can conclude that $\frac{4}{\pi}$ is irrational and thus that π is irrational. If we want to try to apply theorem 2.8, we have to find a simple continued fraction containing π. We can apply the algorithm in section 2.7. People have of course computed the simple continued fraction of π to great length and we know the first values are given by

$$\pi = \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}.$$

This fraction does not have a known pattern so we cannot tell if it satisfies the theorem 2.9 or theorem 2.8. Since the continued fraction is simple, the theorems are equivalent. So using this method we cannot conclude whether or not π is irrational (Weisstein, 2022). Of course, knowing that π is irrational and using either theorem 2.9 or 2.8 we can conclude that the contingued fraction must be irrational, but we cannot use this method to prove the irrationality of π.

3.3 Proofs from THE BOOK III

Continuing in using the ideas from "Proofs from THE BOOK", we use the same family of functions $f_n(x)$ as we did in section 2.6. We will use a different function $F_n(x)$ where we will use π^{2n} instead of r^{2n}. We will then again define an integer $\mathcal{N}$ which will be used to arrive at a contradiction by showing it is simultaneously an integer and in the interval $(0, 1)$. In this proof, we will take a detour by showing that π^2 is irrational, which is a stronger statement than saying π is irrational, since if π were rational, so would π^2.

Theorem 3.2. *π^2 is irrational.*

We will construct a function F_n similar to the proofs we have seen before. We will define f_n the same way we did in section 2.6 and define F_n as

$$F_n(x) = b^n \left(\pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \pi^{(2n-4)} f_n^{(4)}(x) - \cdots \pm f_n^{(2n)}(x)\right).$$

We know from lemma 2.7 that $f_n^{(k)}(0)$ and $f_n^{(k)}(1)$ are integers for all integer values of k. And since each power of π is a fraction with denominator b^m with $m \leq n$, the multiplication with b^n makes all the powers of π a product of powers of a and b which are integers. Hence, $F_n(0)$ and $F_n(1)$ are integers. Similarly to the derivative of F_n in section 2.6, we now have that the second derivative of $F_n(x)$ satisfies

$$F_n''(x) = -\pi^2 F_n(x) + b^n \pi^{2n+2} f_n(x).$$

We compute the derivative of $F_n'(x)\sin(\pi x) - \pi F_n(x)\cos(\pi x)$, which will be easier given the above result and will help us in an integration we perform later.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[F_n'(x)\sin(\pi x) - \pi F_n(x)\cos(\pi x)\right] =& F_n''(x)\sin(\pi x) + \pi^2 F_n(x)\sin(\pi x) \\
=& \left(F_n''(x) + \pi^2 F_n(x)\right)\sin(\pi x) \\
=& b^n \pi^{2n+2} f_n(x)\sin(\pi x) \\
=& \pi^2 a^n f_n(x)\sin(\pi x).
\end{aligned}$$

We now define the quatity $\mathcal{N}$ as

$$\mathcal{N} = \pi a^n \int_0^1 f_n(x)\sin(\pi x)\,\mathrm{d}x.$$

We evaluate this to find

$$\begin{aligned}
\mathcal{N} =& \pi a^n \int_0^1 f_n(x)\sin(\pi x)\,\mathrm{d}x \\
=& \left[\frac{1}{\pi}F_n'(x)\sin(\pi x) - F_n(x)\cos(\pi x)\right]_0^1 \\
=& \frac{1}{\pi}F_n'(1)\sin(\pi) - F_n(1)\cos(\pi) - \frac{1}{\pi}F_n'(0)\sin(0) + F_n(0)\cos(0) \\
=& F_n(0) + F_n(1).
\end{aligned}$$

Since $F_n(0)$ and $F_n(1)$ are integers, $\mathcal{N}$ is an integer as well. However, since $\mathcal{N}$ is defined as an integral of a non-negative function that is only equal to 0 on it's boundary, it must be positive. Furthermore, since $f_n(x) < \frac{1}{n!}$ for $0 < x < 1$ and $0 < \sin(\pi x) < 1$, we have

$$0 < \mathcal{N} = \pi \int_0^1 a^n f_n(x) \sin(\pi x) \, \mathrm{d}x < \frac{\pi a^n}{n!}$$

And this can be made arbitrarily small for n arbitrarily large. Hence we can make it smaller than 1 and so $\mathcal{N}$ is both an integer and in the interval $(0, 1)$, which is not possible. So π^2 is irrational.

3.4 Niven

Here, we will use a technique similar to the ones we have seen from Proofs from THE BOOK. We use a more specific family of functions f_n which allows for an extra property, namely that $f_n(\pi) = f_n(0)$. This property is necessary for the proof to be completed. Assume that $\pi = a/b$ for $a, b \in \mathbb{N}$ and define the following polynomials:

$$\begin{aligned} f_n(x) =& \frac{x^n \left(a - bx\right)^n}{n!}, \\ F_n(x) =& f_n(x) - f_n^{(2)}(x) + f_n^{(4)}(x) - f_n^{(6)}(x) + \cdots + (-1)^n \, f_n^{(2n)}(x). \end{aligned}$$

These polynomials will be our tools to show the irrationality of π and are similar to the ones we first saw in section 2.6. Due to the way they are constructed, some necessary properties regarding their derivatives can be used to get the result we want. First, lets expand f_n as a polynomial to make some generalisations. Recall that we can write

$$f_n(x) = \frac{1}{n!} \sum_{i=0}^{2n} c_i x^i.$$

Note that these c_i may be different from the ones in Proofs from THE BOOK. Since $(a - bx)^n$ expands into a polynomial of degree n, f_n has degree $2n$. Since a and b are natural numbers, all c_i must be natural numbers. Furthermore, since we multiply by x^n, we must have that $c_i = 0$ for all $i < n$. So we can write

$$f_n(x) = \frac{1}{n!} \sum_{i=n}^{2n} c_i x^i.$$

For the j-th derivative of f_n we have the general form

$$f_n^{(j)}(x) = \frac{1}{n!} \sum_{i=n}^{2n} j! c_i x^{i-j}.$$

From this, we can see that for $j < n$, we have $f_n^{(j)}(0) = 0$. For $j \geq n$, the fraction $\frac{j!}{n!}$ is strictly greater than 1 and, moreover, a natural number. Together with the knowledge that a_i is a natural number, this means that in these cases $f^{(j)}(0)$ is a natural number, as all the terms with a power of x vanish and the constant reduced to a natural number. We want some of the properties of $f_n(0)$ to also be true for $f_n(\pi)$. This is why $f_n(x)$ is set up in such a way that it gives the same values for π as for 0, which means we can show the derivatives of $f_n(\pi)$ are also integers.

Lemma 3.3. $f_n(x) = f_n(a/b - x)$.

Proof. We substitute $\frac{a}{b} - x$ for x

$$\begin{aligned} n! f_n\left(\frac{a}{b} - x\right) &= \left(\frac{a}{b} - x\right)^n \cdot \left(a - b\left(\frac{a}{b} - x\right)\right)^n \\ &= \left(\left(\frac{a}{b} - x\right) bx\right)^n \\ &= x^n \left(a - bx\right)^n \\ &= n! f_n(x). \end{aligned}$$

This concludes the proof that $f_n(x) = f_n(a/b - x)$. $\square$

With this, we can conclude that $f_n(x)$ and all of its derivatives have a natural number as their value at $x = 0$ or $x = \pi$. Using the fundamental theorem of calculus, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F_n'(x)\sin(x) - F_n'(x)\cos(x)\right] = F_n''(x)\sin(x) + F_n(x)\sin(x) = f_n(x)\sin(x)$$

as well as

$$\int_0^\pi f_n(x)\sin(x)\,\mathrm{d}x = \left[F_n'(x)\sin(x) - F_n'(x)\cos(x)\right]_0^\pi = F_n(\pi) + F_n(0).$$

Recall that $F_n(x) = f_n(x) - f_n^{(2)}(x) + f_n^{(4)}(x) - f_n^{(6)}(x) + \cdots + (-1)^n f_n^{(2n)}(x)$ and that $f_n^{(j)}(0) = f_n^{(j)}(\pi)$ and that these values are integers. Hence, $F(\pi) + F(0)$ is an integer. However, for $0 < x < \pi$, we have that

$$0 < f_n(x)\sin(x) < \frac{\pi^n a^n}{n!}$$

which we can make arbitrarily small by making n arbitrarily large. This means that we can get an arbitrarily small result when integrating over it from 0 to π. Hence, π cannot be rational (Niven, 1947).

3.5 Laczkovich

Miklós Laczkovich wrote a paper in 1997 discussing the proof by Lambert of the irrationality of π from 1761 and provides a somewhat different, simplified proof. It involves a family of functions, $f_k(x)$, and a theorem which tells us that if x^2 is rational, then $f_k(x) \neq 0$ and $f_{k+1}(x)/f_k(x)$ is irrational (Laczkovich, 1997). In this section, we will define and discuss f_k and the aforementioned theorem, after which we will see some straightforward proofs that follow quickly from the theorem.
Define the family of functions

$$\begin{aligned} f_k(x) &= 1 - \frac{x^2}{k} + \frac{x^4}{k(k+1)\cdot 2!} - \frac{x^6}{k(k+1)(k+2)\cdot 3!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\prod\limits_{i=0}^{n-1}(k+i)\cdot n!}. \end{aligned}$$

For convenience we define $\prod\limits_{i=0}^{-1}(k+i) = 1$.

Lemma 3.4. The series f_k converges absolutely for all $x \in \mathbb{R}$.

Proof. By the ratio test, a series converges absolutely if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, where a_n is the n-th term of the series. Because a_n is given by

$$a_n = (-1)^n \frac{x^{2n}}{k(k+1) \cdot \ldots \cdot (k+n-1) \cdot n!}$$

we can find that L is given by

$$\begin{aligned} L &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty} \left|-\frac{x^2}{(n+1)(k+n)}\right| \\ &= \lim_{n\to\infty} \frac{x^2}{(n+1)|k+n|}. \end{aligned}$$

Clearly, this tends to 0 as n tends to infinity so we can conclude that $L < 1$. □

The way f_k is defined is very deliberate. Which becomes apparent when we recall the Taylor expansion of $\cos(x)$ and $\sin(x)$:

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \end{aligned} \qquad \begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

With this in mind, it becomes clear that f_k is related to the sin and cos functions and that with the right choice of k we can get f_k to be a variation of one of these functions. Knowing this, we will look at a few examples of f_k for particular values of k. First, consider the case where $k = 1/2$. First we compute $\prod_{i=0}^{n-1} (k+i)$ for $k = 1/2$.

$$\prod_{i=0}^{n-1} \left(\frac{1}{2} + i\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \ldots \cdot \frac{2n-1}{2} = \frac{(2n-1)!!}{2^n}.$$

Here, $n!!$ denotes the double factorial which is defined for a natural number n as

$$n!! = \begin{cases} n \cdot (n-2) \cdot (n-4) \cdot \ldots \cdot 2 & \text{for } n \text{ even}, \\ n \cdot (n-2) \cdot (n-4) \cdot \ldots \cdot 1 & \text{for } n \text{ odd}. \end{cases}$$

We will use the double factorial again to rewrite $n!$. Namely, if we multiply each term in $n!$ by 2, we get a product of every even term up to $2n$, i.e. $2^n n! = (2n)!!$. This means we can write $\prod_{i=0}^{n-1} \left(\frac{1}{2} + i\right) \cdot n!$ as

$$\prod_{i=0}^{n-1} \left(\frac{1}{2} + i\right) \cdot n! = \frac{(2n-1)!!}{2^n} \frac{(2n)!!}{2^n} = \frac{(2n)!}{4^n}.$$

With this, we can make the necessary substitutions and write the expression for $f_{1/2}$. Keeping in mind the aforementioned Taylor expansion we can see that

$$f_{1/2}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\prod\limits_{i=0}^{n-1} \left(\frac{1}{2} + i\right) \cdot n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n} \cdot 4^n}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \cos(2x).$$

We can get a similar result for $k = 3/2$ with some quick steps. First, note that

$$\prod_{i=0}^{n-1} \left(\frac{3}{2} + i\right) = \prod_{i=0}^{n-1} \left(\frac{1}{2} + i + 1\right) = \prod_{i=1}^{n} \left(\frac{1}{2} + i\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \ldots \cdot \frac{2n+1}{2} = \frac{(2n+1)!!}{2^n}.$$

And consequently $\prod\limits_{i=0}^{n-1} \left(\frac{3}{2} + i\right) \cdot n!$ is equal to

$$\prod_{i=0}^{n-1} \left(\frac{3}{2} + i\right) \cdot n! = \frac{(2n+1)!!}{2^n} \frac{(2n)!!}{2^n} = \frac{(2n+1)!}{4^n}.$$

Resulting in the following identity for $f_{3/2}$:

$$f_{3/2}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\prod\limits_{i=0}^{n-1} \left(\frac{3}{2} + i\right) \cdot n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n} \cdot 4^n}{(2n+1)!} \cdot \frac{2x}{2x} = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} \cdot \frac{1}{2x} = \frac{\sin(2x)}{2x}.$$

Now that we have a grasp of what f_k is, we want to show a lemma about the relationship between f_{k+2}, f_{n+1}, and f_n which will be instrumental in proving the theorem we will discus later.

Lemma 3.5. For all $x \in \mathbb{R}$ and $k \in \mathbb{Q} \setminus \{0, -1, -2, \ldots\}$ we have $\frac{x^2}{k(k+1)} f_{k+2}(x) = f_{k+1}(x) - f_k(x)$.

Proof. From the definition of f_k we get that f_{k+1} is given by

$$f_{k+1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\prod\limits_{i=0}^{n-1} (k + 1 + i) \cdot n!}.$$

We first subtract the two series component-wise in closed form after which we will compare it to the series for $\frac{x^2}{k(k+1)} f_{k+2}(x)$.

$$\begin{aligned} f_{k+1}(x) - f_k(x) &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\prod\limits_{i=0}^{n-1} (k+1+i)} - \frac{1}{\prod\limits_{i=0}^{n-1} (k+i)} \right) \cdot \frac{x^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{k}{\prod\limits_{i=0}^{n} (k+i)} - \frac{k+n}{\prod\limits_{i=0}^{n} (k+i)} \right) \cdot \frac{x^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{-n}{\prod\limits_{i=0}^{n} (k+i)} \cdot \frac{x^{2n}}{n!}. \end{aligned}$$

The $-n$ cancels out if we reduce the $n!$ to $(n-1)!$ and change the parity of the $(-1)^n$ term. We have to remember that the term for $n = 0$ vanishes, so we will start counting at $n = 1$. This means that we are left with

$$f_{k+1}(x) - f_k(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{\prod\limits_{i=0}^{n} (k+i) \cdot (n-1)!}.$$

Next, we will look at the series of $\frac{x^2}{k(k+1)} f_{k+2}(x)$ and its closed form and compare it to the one for $f_{k+1}(x) - f_k(x)$ that we have seen just now. From the definition, we immediately get

$$\begin{aligned} f_{k+2}(x) &= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{\prod\limits_{i=0}^{m-1} (k+2+i) \cdot m!} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{\prod\limits_{i=2}^{m+1} (k+i) \cdot m!}. \end{aligned}$$

We use the variable m instead of n because we want to make a substitution later. For now, we will multiply by the fraction $\frac{x^2}{k(k+1)}$ to get

$$\frac{x^2}{k(k+1)} f_{k+2}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+2}}{\prod\limits_{i=0}^{m+1} (k+i) \cdot m!}.$$

Lastly, we will make the aforementioned substitution, namely $n = m + 1$ to get

$$\frac{x^2}{k(k+1)} f_{k+2}(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{\prod\limits_{i=0}^{n} (k+i) \cdot (n-1)!}.$$

□

This lemma will be used later to aid our reasoning in the proof of the theorem and later to inductively show a result that is necessary for the theorem. Before proceeding to the theorem and its proof, we first have to visit one more lemma.

Lemma 3.6. For all $x \in \mathbb{R}$ we have $\lim\limits_{k\to\infty} f_k(x) = 1$

Proof. We know that $\lim\limits_{n\to\infty} \frac{x^{2n}}{n!} = 0$. This means there exists some natural number N such that $\left|\frac{x^{2n}}{n!}\right| \leq N$ for every $n \in \mathbb{N}$. Therefore, if $k > 1$ then

$$
\begin{aligned}
\left|f_k(x)-1\right| &= \left|\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n}}{\prod\limits_{i=0}^{n-1}(k+i)\cdot n!}\right| \\
&\leq \sum_{n=1}^{\infty}\left|\frac{1}{\prod\limits_{i=0}^{n-1}(k+i)}\frac{x^{2n}}{n!}\right| \\
&\leq \sum_{n=1}^{\infty}\frac{1}{k^n}N \\
&= \frac{N}{k-1}.
\end{aligned}
$$

This clearly tends to zero as k tends to infinity. □

With these lemmas in mind, we can move on to the theorem that will help us prove that π^2 is irrational. After the theorem and the proof, the proof of the irrationality of π^2, and subsequently of π, will follow immediately.

Theorem 3.7. *If $x \neq 0$ and x^2 is rational, then $f_k(x) \neq 0$ and ${}^{f_{k+1}(x)}/_{f_k(x)}$ is irrational for every $k \in \mathbb{Q} \setminus \{0, -1, -2, -3, \ldots\}$.*

Proof. Let x be a non-zero real number such that x^2 is rational and let $k \in \mathbb{Q} \setminus \{0, -1, -2, -3, \ldots\}$ be fixed. Assume that for those given $f_k(x) = 0$ or $\frac{f_{k+1}(x)}{f_k(x)}$ is rational. In either case, $f_{k+1}(x)$ and $f_k(x)$ are multiples of the same number, say $f_k(x) = ay$ and $f_{k+1}(x) = by$ for two integers a and b. We allow a or b to be zero and we allow y to be any real number except zero because if y were equal to zero, then $f_k(x) = f_{k+1}(x)$ and lemma 3.5 would imply that $f_{k+m}(x) = 0$ for all $m \in \mathbb{N}$, which would contradict lemma 3.6.

Next, let q be a positive integer such that $\frac{bq}{k}$, $\frac{kq}{x^2}$, and $\frac{q}{x^2}$ are all integers. Now, we will define a sequence G_n. First, $G_0(x) = f_k(x)$ and

$$
G_n(x) = \frac{q^n}{k(k+1)\cdots(k+n-1)} f_{k+n}(x) = \frac{q^n}{\prod\limits_{i=0}^{n-1}(k+i)} f_{k+n}(x)
$$

for $n \in \mathbb{N}$. Then, $G_0 = f_k(x) = ay$, $G_1 = \frac{q}{k}by = \frac{bq}{k}y$. Using lemma 3.5 we can see that

$$
G_{n+2} = \left(\frac{kq}{x^2} + \frac{q}{x^2}n\right) G_{n+1} - \left(\frac{q^2}{x^2}\right) G_n.
$$

This means G_n is an integer multiple of y for every value of n. We know from lemma 3.6 that $f_{k+n}(x)$ converges to 1. At the same time, $q^n / \big(k(k+1)\cdots(k+n-1)\big)$ is is non-zero and converges to 0. This means that G_n converges to 0. However, lemma 3.6 implies that G_n is non-zero for sufficiently large values of n. If we define $H_n = G_n/y$, then H_n is an integer for every n and converges to 0. However, a sequence of positive integers cannot converge to zero. This leads to a contradiction which means our assumption was wrong and $f_k(x) \neq 0$ and $f_{k+1}(x)/f_k(x)$ is irrational. □

Remark 3.8. In the original paper by Laczkovich, he argues that G_n is *positive* for all n instead of just non-zero. This is however not the case. To see this, consider the case where $k = -\frac{1}{2}$. The fraction

$q^n / \big(k(k+1)\cdots(k+n-1)\big)$ will be negative for each value of n and as we have seen in lemma 3.6, f_{n+k} tends to 1, making it positive for sufficiently large n. Here, we corrected this by arguing G_n is non-zero for large enough values of n. Alternatively, we could have restricted k to be strictly positive. This would have allowed for G_n to be positive and gets rid of the aforementioned counter-example. We also only ever use positive values of k in our examples and to show the irrationality of π^2.

Having seen this. We can prove the irrationality of π^2 rather straightforwardly. Namely consider the case where $k = \frac{1}{2}$ and $x = \frac{\pi}{4}$. As we have seen, this means that

$$f_{1/2}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0.$$

Since $f_k(x) = 0$, the condition for the theorem has not been met and since $x \neq 0$, this means that x^2 must be irrational. This then means that $\frac{\pi^2}{16}$ is irrational, and thus that π^2 is irrational.

Here, we have not used the full scope of the theorem. Since the theorem states that if $x \neq 0$ and x^2 is rational then $f_k(x) \neq$ and the fraction $f_{k+1}(x)/f_k(x)$ is irrational and we have only used the implication that $f_k(x) \neq 0$ to show that pi^2 is irrational. We can use the full result of the theorem to show the irrationality of a different set of numbers.

Theorem 3.9. *If $x \in \mathbb{Q} \setminus \{0\}$ then $\tan(x)$ is irrational.*

Proof. If x is non-zero and rational then so is $\left(\frac{x}{2}\right)^2$. It then follows from theorem 3.7 that

$$\frac{f_{3/2}\left(\frac{x}{2}\right)}{f_{1/2}\left(\frac{x}{2}\right)} = \frac{\sin(x)}{x \cdot \cos(x)} = \frac{\tan(x)}{x}$$

is irrational. We know that x is non-zero and rational so this means that $\tan(x)$ is irrational. □

3.6 Australian HSC exam problem

The Higher School Certificate (HSC) is the highest educational award that students can achieve in their high school career in New South Wales, Australia. In 2003 their Mathematics Extension 2 exams featured a question in which the students had to prove the irrationality of π by proving a number of steps (commmittee of the Education Standards Authority, 2003). We will adapt this proof and provide a bit more commentary. Generally, the structure of the proof as it was shown in the question, will be maintained.

The proof revolves around a family of definite integrals called I_n which involve π. We will show that I_n is a natural number for each $n \in \mathbb{N}$ and moreover that the sequence I_n tends to 0 for sufficiently large values of n, which will be the contradiction that leads to the conclusion that π cannot be rational.

Assume that $\pi = \frac{p}{q}$ for two natural numbers p and q. We define the family of integrals I_n as

$$I_n = \frac{q^{2n}}{n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^n \cos(x)\,\mathrm{d}x$$

for all $n \in \mathbb{Z}^+$. We can compute that $I_0 = 2$ and $I_1 = 4q^2$. We want to show that each value of I_n is a natural number and we will do this by induction. First, we will do this by applying integration by parts twice.

Lemma 3.10. $I_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.

Proof. Assume $n \geq 2$. After one integration by parts we obtain

$$\begin{aligned}
I_n =& \frac{q^{2n}}{n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2 \right)^n \cos(x) \, \mathrm{d}x \\
=& \frac{q^{2n}}{n!} \left[\left(\frac{\pi^2}{4} - x^2 \right)^n \sin(x) \right]_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} + \frac{q^{2n}}{n!} \cdot 2n \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2 \right)^{n-1} \cdot x \cdot \sin(x) \, \mathrm{d}x \\
=& \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \left(\frac{\pi^2}{4} - x^2 \right)^{n-1} \sin(x) \, \mathrm{d}x
\end{aligned}$$

Integrating by parts a second time gives us

$$\begin{aligned}
I_n =& \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \left(\frac{\pi^2}{4} - x^2 \right)^{n-1} \sin(x) \, \mathrm{d}x \\
=& \frac{2q^{2n}}{(n-1)!} \left[-x \left(\frac{\pi^2}{n!} - x^2 \right)^{n-1} \cos(x) \right]_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \\
+& \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(\frac{\pi^2}{4} - x^2 \right)^{n-1} - 2(n-1)x^2 \left(\frac{\pi^2}{4} - x^2 \right)^{n-2} \right] \cos(x) \, \mathrm{d}x.
\end{aligned}$$

Rearanging the fraction $\frac{2q^{2n}}{(n-1)!}$ and the $\cos(x)$ term, we arrive at

$$I_n = \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2 \right)^{n-1} \cos(x) \, \mathrm{d}x - \frac{4q^{2n}}{(n-2)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \left(\frac{\pi^2}{4} - x^2 \right)^{n-2} \cos(x) \, \mathrm{d}x.$$

While this is quite a cumbersome expression for I_n, we recognise the expressions for I_{n-1} and I_{n-2} in this expression and with a simple substitution of $x^2 = \frac{\pi^2}{4} - \left(\frac{\pi^2}{4} - x^2 \right)$ we can rewrite this expression as

$$I_n = (4n-2)q^2 I_{n-1} - p^2 q^2 I_{n-2}, \text{ for } n \geq 2.$$

We know that $I_0 = 2$, $I_1 = 4q^2$, p, q, and n are all integers, hence I_n is an integer for all n. □

Now that we know I_n is an integer, we will show that $0 < I_n < 1$ for n sufficiently large. This will then contradict lemma 3.10 and prove that π cannot be written as a fraction of natural numbers. We will first bound I_n by approximating the integral it is defined by and then argue that the upper bound approaches 1 as n increases.

Lemma 3.11. $0 < I_n < \frac{p}{q} \left(\frac{p}{2} \right)^2 \frac{1}{n!}$.

Proof. Recall that I_n is defined as

$$I_n = \frac{q^{2n}}{n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2 \right)^n \cos(x) \, \mathrm{d}x,$$

with q assumed to be a positive whole number and $n \in \mathbb{Z}^+$. For any value of $n \in \mathbb{Z}^+$ we can see that $\frac{q^2 n}{n!} > 0$ and for any value of $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we can see that $\left(\frac{\pi^2}{4} - x^2\right) \geq 0$ and $\cos(x) \geq 0$. Hence $I_n > 0$. This proves the lower bound for I_n. For the upper bound, we will use the fact that a definite integral is the area under the curve of the function over which we integrate. Which can be bound by a rectangle that has a height equal the maximum value of the function in the domain over which we integrate and a length equal to the length of the domain. Hence

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^n \cos(x)\,\mathrm{d}x &< \pi \cdot \max_{x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \left\{ \left(\frac{\pi^2}{4} - x^2\right)^n \cos(x) \right\} \\
&\leq \pi \cdot \max_{x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \left\{ \left(\frac{\pi^2}{4} - x^2\right)^n \right\} \cdot \max_{x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \{\cos(x)\} \\
&= \pi \cdot \left(\frac{\pi^2}{4}\right)^n \cdot 1 \\
&= \frac{p}{q} \left(\frac{p}{2q}\right)^{2n}
\end{aligned}$$

Which means that we can bound I_n as well after canceling out the q^{2n} terms.

$$I_n < \frac{q^{2n}}{n!} \cdot \frac{p}{q} \left(\frac{p}{2q}\right)^{2n} = \frac{p}{q} \left(\frac{p}{2}\right)^{2n} \frac{1}{n!}.$$

□

We know that we can make this arbitrarily small with a large enough value of n, so we can certainly make it smaller than 1, which means that we can conclude that $0 < I_n < 1$ for n sufficiently large, which clearly contradicts with lemma 3.10, meaning π must be irrational.

3.7 Discussion

In this section, we looked at proofs of irrationality of π. These are again all proofs by contradiction. We saw how we could apply theorem 2.9 to the continued fraction of $4/\pi$ to prove it's irrationality. Furthermore, we saw in section 3.3, section 3.4 and section 3.5 that both Proofs the THE BOOK, Niven, and Laczkovich use a family of functions f_n to show the irrationality of π in very similar ways, which again made use of the technique of a quantity depending on n being both an integer and between 0 and 1 for a large enough value of n. Lastly, in section 3.6 we again saw the use of restricting an integer to the interval $(0, 1)$.

4 Differences between methods

We have seen several methods of proving the irrationality of e and π. We would like to see if these methods are roughly interchangeable. Specifically, the methods for π almost all rely on families of functions (Proofs from THE BOOK, Niven, Laczkovich), while the proofs for e revolve more around some expression of e as a series are generally more straightforward. Therefore, we would like to see if we can use the methods for e, or variations thereof, to prove the irrationality of π.

4.1 Kifowit

In this proof, we relied on integrating by parts several times. There are several identities of π that involve integrals. In the original proof, we used the identity

$$\int_0^1 e^{-x}\,\mathrm{d}x = 1 - \frac{1}{e}$$

We could try using a similar interval

$$\int\limits_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

or an interval over a finite domain such as

$$\frac{\pi}{2} = \int\limits_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x \text{ or } \frac{\pi}{4} = \int\limits_{0}^{1} \sqrt{1-x^2}\,\mathrm{d}x.$$

When we try the first, we will find that applying integration by parts the same way we did in the original proof will leave us with some meaningless results, first, we cannot use the fundamental theorem of calculus, since it is only defined for finite intervals. Instead, we have to rewrite it as a limit and continue from there. We would have

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \int\limits_{-\infty}^{\infty} e^{-x^2}\cdot 1\,\mathrm{d}x \\
&= \lim_{a\to\infty}\int\limits_{0}^{a} e^{-x^2}\cdot 1\,\mathrm{d}x + \lim_{a\to-\infty}\int\limits_{a}^{0} e^{-x^2}\cdot 1\,\mathrm{d}x \\
&= \underbrace{\lim_{a\to\infty}\left[xe^{-x^2}\right]_0^a - \lim_{a\to-\infty}\left[xe^{-x^2}\right]_a^0}_{=0} + 2\int\limits_{-\infty}^{\infty} e^{-x^2}\cdot x^2\,\mathrm{d}x
\end{aligned}$$

To see that the difference of limits is equal to zero, we have to rewrite the limits and apply l'Hospital's rule for limits to get

$$\begin{aligned}
\lim_{a\to\infty}\left[xe^{-x^2}\right]_0^a - \lim_{a\to-\infty}\left[xe^{-x^2}\right]_a^0 &= \lim_{a\to\infty}\frac{a}{e^{a^2}} - \lim_{a\to-\infty}\frac{a}{e^{a^2}} \\
&= \lim_{a\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}a}a}{\frac{\mathrm{d}}{\mathrm{d}a}e^{a^2}} - \lim_{a\to-\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}a}a}{\frac{\mathrm{d}}{\mathrm{d}a}e^{a^2}} \\
&= \lim_{a\to\infty}\frac{1}{2a\cdot e^{a^2}} - \lim_{a\to-\infty}\frac{1}{2a\cdot e^{a^2}} \\
&= 0.
\end{aligned}$$

So we are left with

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = 2\int_{-\infty}^{\infty} e^{-x^2}\cdot x^2\,\mathrm{d}x.$$

If we apply integration by parts again, we will see that that the evaluated term vanished after applying l'Hospital, so we are left with

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = 4\int_{-\infty}^{\infty} e^{-x^2}\cdot \frac{1}{3}x^4\,\mathrm{d}x.$$

Every time we integrate, the exponent adds a 2 to the term in front of the integral and increases the power of x by one. The integration of the monomial increases the power of x by one and divides everything by the new power, which is now 2 higher than before we applied integration by parts. The term we evaluate at a and vanishes each time before we take the limit so the limit is also equal to zero. In general we then have the expression

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = 2^n\int_{-\infty}^{\infty} e^{-x^2}\cdot \prod_{i=1}^{n}\frac{1}{2i-1}\cdot x^{2n}\,\mathrm{d}x.$$

We can rewrite this expression slightly by recognizing the product can be written in a closed form. Namely

$$\prod_{i=1}^{n}\frac{1}{2i-1} = \frac{1}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} = \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{1\cdot 2\cdot 3\cdot \ldots \cdot (2n-1)\cdot (2n)} = \frac{2^n n!}{(2n)!}.$$

Plugging this into our equality and taking out the 2^n term to the front, we get that

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = 4^n\int_{-\infty}^{\infty} e^{-x^2}\cdot \frac{n!}{(2n)!}\cdot x^{2n}\,\mathrm{d}x.$$

If we want to roughly follow the structure of the proof for the irrationality of e. We would assume that $\pi = a/b$ or maybe that $\sqrt{\pi} = a/b$ for two integers a and b and then use some cleaver bound to show part of the integral is strictly between 0 and 1 while also being an integer. But we can't take any integer terms out of the expression easily. Additionally, because we cannot separate the original integral into a sum and an integral, the value of resulting integral doesn't change if n changes so we cannot steer the result by changing n either. Moreover, we had to work with limits to infinity to obtain our result, which wasn't the case when we treated e.

The second identity also doesn't give us much to work with. It doesn't allow us to apply integration by parts nicely, since there are no two functions being multiplied and the square root also complicates things. We could attempt to apply a u substitution or and Euler substitution but both of those methods are considerably less straightforward than the original prove and do not guarantee giving us an expression that allows us to prove our result, as they are more often used to evaluate the value of the integral directly.

4.2 Sondow

The core of this proof relies on showing that e cannot be written as a fraction with denominator $n!$ for any n in the natural numbers. We could also try to use this concept for π by, for example, taking the identity

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and define s_n as the sum of the first n terms, which can be written in the form

$$s_n = \sum_{i=0}^{n} \frac{1}{i^2} = \frac{a_n}{(n!)^2}$$

for some natural number a_n depending on n. We would like to repeat our earlier reasoning and conclude that π^2 is irrational. But as we will see, this isn't quite as straightforward as we would like. Suppose that

$$\frac{a_n}{(n!)^2} < \frac{\pi^2}{6} < \frac{a_n + 1}{(n!)^2}.$$

On the left had side we have that $\pi^2/6$ is less than s_n. On the left hand side, it must be true that

$$\begin{aligned}
\frac{\pi^2}{6} &< \frac{a_n + 1}{(n!)^2} \\
&= \frac{a_n}{(n!)^2} + \frac{1}{(n!)^2} \\
&\leq \frac{a_n}{(n!)^2} + \frac{1}{(n+1)^2} \\
&= s_{n+1}
\end{aligned}$$

This contradicts the fact that $\pi^2/6$ is greater than s_n for all natural values of n. As such, the argument that $\pi^2/6$ cannot be written as a fraction with a denominator of the form $(n!)^2$ cannot be made.

4.3 Proofs from THE BOOK & Niven

In Proofs from THE BOOK, specifically in sections 2.6 and section 3.3 we used a family of functions f_n and a function F_n where we summed multiples of the different derivatives of f_n which satisfied desirable properties which allowed us to define a quantity we called $\mathcal{N}$. In both cases we use an identity in which the first or second derivative of F_n depended on F_n and f_n. We then showed that $\mathcal{N}$ depended solely on (integer multiples of) $F(1)$ and $F(1)$ which we could show were also integers. We then showed that $\mathcal{N}$ would also have to be contained in the interval $(0, 1)$, leading to a contradiction.

In Niven's proof in section 3.4 we saw a similar approach as in section 3.3. Here, we notice a number of things. First is that we use the method of reaching a contradiction through showing that something which is necessarily an integer is also strictly between 0 and 1. This is the same as we have seen in sections 2.1 through 2.3. Second, when looking at the aforementioned identities involving the derivative of F_n and its dependence on F_n and f_n, if we look at the proof revolving around e, we had F_n' depending on F_n and f_n while in the proof revolving around π we had F_n'' depending on F_n and f_n. This is because when working with e, functions involving powers of e usually have derivatives involving the original functions and when working with π there are usually trigonometric functions like sin and cos involved, whose second derivatives involve themselves. While this didn't complicate the proof a lot, it is worth noticing. Lastly, for comparable effort to show that π is irrational, we get that e^r is irrational for all non-zero rational values of r. So even though the methods are very comparable and applicable for both constants, being roughly equally complicated, we get a much stronger result for e than we do for π.

4.4 Continued fractions

We have seen two ways of checking the irrationality of a number using continued fractions. One for simple continued fractions and one for general continued fractions. For e, we used the continued fraction

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cdots}}}},$$

which showed us very straightforwardly that e is irrational by using either the theorem or the lemma. On the other hand, when we try to do the same for π, we run into a few issues. We have already seen that the continued fraction given by

$$\frac{4}{\pi} = 1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cfrac{11^2}{2+\cdots}}}}}}.$$

Satisfies theorem 2.9 and can therefore be shown to be irrational. But showing this via theorem 2.8 remains elusive. Like we have seen, if we consider the simple continued fraction for π, we have to deal with

$$\pi = \cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}.$$

which does not have an established pattern (Weisstein, 2022). Since the continued fraction is simple, the theorems are equivalent and the only thing we need is for the continued fraction to be infinite, and because no pattern is apparent, we cannot assume this without external knowledge. Of course, we know π is irrational and so we know this continued fraction has to be infinite, but if we can only argue that based on the knowledge that π is irrational, then we cannot use this line of reasoning to prove its irrationality. If we instead consider the continued fraction for half π, we surprisingly get a very well behaved continued fraction, namely

$$\frac{\pi}{2} = 1-\cfrac{1}{3-\cfrac{2\cdot 3}{1-\cfrac{1\cdot 2}{3-\cfrac{4\cdot 5}{1-\cfrac{3\cdot 4}{3-\cfrac{6\cdot 7}{1-\cdots}}}}}}.$$

While this seems helpful at first, as we try to apply theorem 2.9, we find that b_n is increasing and a_n is always either 1 or 3 for all n. So the term $1 + b_n$ will quickly surpass 3 and so we cannot apply the theorem. Using theorem 2.8 is also not a possibility because the continued fraction is not simple.

4.5 Summary

Having compared the methods, we can comfortably conclude that showing the irrationality of π is substantially more difficult than showing the irrationality of e. When we considered methods using series or identities involving either constant, it has been consistently simpler to rewrite the expression in a way that isolates integer terms and work towards a conclusion that a certain quantity is both an integer and in the interval $(0, 1)$. When we looked at Sondow's method using intervals, it was not possible to bound π as easily in an interval while this was not at all difficult for e. This may be due to the structure of the series expansion of e that allows us to multiply by a factorial to reduce a specific range of terms to integers without multiplying by an absurdly large term and without creating more integer terms after that point, which happens when we try to do this for the series involving π. We also saw that identities involving π more often involve limits to infinity which makes, among other things, methods using integration a lot less simple. These more 'complicated' properties all seem to be integral to π. Additionally, the methods used to proof the irrationality of e more often used sums than they used integrals, which is exactly the other way around for methods that proved the irrationality of π. Of course, though it might be harder or more complicated, in many cases, to show the irrationality of π, this does not make the irrationality of π any less true.

Of course, we have not checked every single possible series or identity involving either constant, so it remains entirely possible that there is some series that, when paired with the right method, gives a much simpler result, but working through established methods, no such combination or proof has been found. We also have to mention that some methods worked about equally well for both constants. The polynomial method that we first saw in section 2.6 allowed us to show the irrationality of both π and e^r with relatively similar amounts of work / complication. It should be mentioned that showing the irrationality of e^r for all rational non-zero r is a much stronger result than showing the irrationality of any specific power of e. Not only because we can just fill in any rational value for r but also because if e^r is irrational then the irrationality of e^d is immediate for d a divisor of r. So with this, even when we can apply the method to both constants, we can get a stronger result for e than we get for π, fitting with the conclusion that the irrationality of e is much easier to show than the irrationality of π.

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