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Minkowski's Question Mark Function

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Abstract

Minkowski's question mark function $?(x)$ establishes a relation between quadratic irrationals, non-dyadic rationals and dyadic rational numbers. This paper will work out the preliminary notions of modified Farey sequences, dyadic rational sequences and continued fraction expansions. We will prove certain properties of these notions, and use those to establish an equivalent but more workable definition of Minkwoski's question mark function. We will use both definitions to approximate some as of yet unknown fixed points of $?(x)$. We will conclude with a proof that this function is singular. This amounts to showing that it is continuous and non-constant, yet has derivative 0 almost everywhere.

Contents

1 Introduction

Hermann Minkowski first defined the question mark function $?(x)$ in 1904 [11, pp 50-51] with the aim to shed a new light on Lagrange's theorem on the relation between continued fractions and quadratic irrationals. This theorem states that any quadratic irrational can be written as a periodic continued fraction. Euler had already proven the converse, that any periodic continued fraction must be a quadratic irrational [5]. Minkowski's writing on the question mark function is quite short, only setting out its construction and concluding with the property he set out to achieve: $?(x)$ maps any quadratic irrational to a non-dyadic rational. Likewise, any non-dyadic rational gets mapped to a dyadic rational. Dyadic rationals are those rationals with a power of 2 as denominator.

Three decades after Minkowski's original definition, A. Denjoy [2] proved that Minkowski's question mark function is a singular function, meaning it is continuous and non-constant, yet has derivative 0 almost everywhere. Shortly after, R. Salem [12] provided an alternative proof for the singularity of $?(x)$. He also noted that, at the time, $?(x)$ is the only known singular function that is strictly increasing and relatively easy to construct. This paper will elaborate on Salem's proof of singularity by filling in many details and working out preliminaries. We will also approximate some of the fixed points of Minkowski's question mark function.

To be able to define Minkowski's question mark function, we will need some preliminary notions. These notions of modified Farey sequences and dyadic rational sequences will be introduced in section 2. In that section we will also introduce continued fraction expansions, which will be crucial for further proofs. We will also prove some useful properties of these concepts. Once those are established, section 3 will begin by providing Minkowski's original definition of $?(x)$. We will then follow Denjoy in establishing an equivalent definition of this function and use both definitions to calculate some example values. In section 4, we will make a slight detour to examine the fixed points of the question mark function. C. Bower has according to [3] made an unpublished note in 1999 on the fixed values of $?(x)$, conjecturing that the function has five fixed points of which two as of yet unknown. D. Gayfulin and N. Shulga are currently writing [4], where they prove that the unknown points are irrational and that there are exactly two of them. In this paper, we will use both definitions of $?(x)$ to make rudimentary approximations of the unknown fixed points through MATLAB. In section 5, we will elaborate on Salem's proof of the singularity of $?(x)$. We will find a subset of measure 1 on the interval $[0, 1]$ and prove that on this subset the derivative $?'(x)$ equals 0. This will let us conclude that Minkowski's question mark function is indeed singular.

2 Preliminary notions

In this section we will set the stage for defining and analysing $?(x)$ by defining some preliminary notions and proving some necessary properties.

2.1 Quadratic irrationals

Definition 2.1. *A quadratic irrational is an irrational number which is the root of some quadratic equation with integer coefficients.*

Examples of quadratic irrationals are therefore $\sqrt{2}$ (solution of $x^2-2=0$) and $\frac{1}{6}-\frac{\sqrt{13}}{6}$ (solution of $3x^2-x-1=0$).

2.2 Continued fraction expansion

This subsection follows [6]. A *continued fraction expansion* is a way to uniquely express any real number as a sequence of integers. This is done through repeated *mod one decomposition*: For a given x, the floor of x is $a=\lfloor x\rfloor$, the largest integer a such that $a\leq x$. The remainder $u=x-a$ then falls in the unit interval $[0,1)$. This lets us write any x uniquely as

$$x=a+u.$$

The first step towards a continued fraction expansion is simply applying this decomposition. Let

$$x=a_0+u_0.$$

If $u_0=0$, we are done and our continued fraction expansion can be denoted as $x=[a_0]$. If not, then since u_0 is on the unit interval and nonzero, $1/u_0$ is larger than 1 and therefore again an appropriate target for mod one decomposition: if

$$\frac{1}{u_0}=a_1+u_1$$

then

$$x=a_0+\frac{1}{a_1+u_1}$$

and again $u_1\in[0,1)$. This process repeats until $u_n=0$ for some n, or infinitely if this does not happen. The continued fraction expansion of x is denoted as

$$x=[a_0;a_1,a_2,\dots]=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots}}}. \tag{1}$$

Definition 2.2. *If $x=[a_0;a_1,\dots,a_n,\dots]$, then the nth convergent of x is*

$$\frac{p_n}{q_n}=[a_0;a_1,\dots,a_n]. \tag{2}$$

These convergents have the following property:

Proposition 2.3. *For any integer $n\geq 2$, if $x=[a_0;a_1,\dots,a_n,\dots]$, and $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_{n-2}}{q_{n-2}}$ are the $(n-1)$th and $(n-2)$th convergent of x, then the nth convergent of x is given by*

$$\frac{p_n}{q_n}=\frac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}}. \tag{3}$$

Proof. We will prove this by induction. Since

$$\frac{p_0}{q_0} = a_0, \qquad \frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1},$$

and

$$\frac{p_2}{q_2} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = a_0 + \frac{a_2}{a_2 a_1 + 1} = \frac{a_2 a_1 a_0 + a_0 + a_2}{a_2 a_1 + 1} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0},$$

the proposition holds for $n = 2$. Now assume it holds for some $n \in \mathbb{N}$. By construction of the convergent,

$$\frac{p_{n+1}}{q_{n+1}} = [a_0; a_1, \dots, a_{n-1}, a_n, a_{n+1}] = [a_0; a_1, \dots, a_{n-1}, a_n + \frac{1}{a_{n+1}}],$$

so

$$\frac{p_{n+1}}{q_{n+1}} = \frac{(a_n + \frac{1}{a_{n+1}})p_{n-1} + p_{n-2}}{(a_n + \frac{1}{a_{n+1}})q_{n-1} + q_{n-2}}.$$

Multiplying by $1 = \frac{a_{n+1}}{a_{n+1}}$ gives

$$\frac{p_{n+1}}{q_{n+1}} = \frac{(a_{n+1}a_n + 1)p_{n-1} + a_{n+1}p_{n-2}}{(a_{n+1}a_n + 1)q_{n-1} + a_{n+1}q_{n-2}} = \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}},$$

which, since equation (3) holds for n, gives

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}},$$

so equation (3) holds for $n + 1$ and by induction for all integer $n \geq 2$. ☐

Since the rationals are closed under addition and division we also get the following theorem:

Theorem 2.4. *The continued fraction expansion of a real number is finite if and only if that real number is rational.*

This begs the question of whether we can say anything useful about infinite continued fraction expansions. This is exactly what Minkowski set out to illustrate with his question mark function. Where Euler already proved that any infinite periodic continued fraction expansion must be a quadratic irrational, Lagrange proved the converse [5]. Together this gives the following theorem.

Theorem 2.5. *A real number is a quadratic irrational if and only if its continued fraction expansion is infinite and eventually periodic.*

2.3 Farey sequences

Minkowski defined his question mark function on the basis of modified Farey sequences and dyadic rational sequences. As regular Farey sequences are more common, we will introduce these first. Farey sequences are a way of ordering the rational numbers on the interval $[0, 1]$, defined as follows:

Definition 2.6. *The Farey sequence* F_n *of order* n *consists of all completely reduced rationals on the interval* $[0, 1]$ *with denominator no larger than* n*, arranged in ascending order.*

The first four Farey sequences are therefore:

$F_1 = \{\frac{0}{1}, \frac{1}{1}\}$
$F_2 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$
$F_3 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$
$F_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$.

By definition, any rational number $\frac{p}{q} \in [0, 1]$ is contained (in reduced form) in F_n for large enough n. An observation about the newly added fractions to each F_{n+1} follows from the definition of the mediant:

Definition 2.7. *The mediant of two fractions* $\frac{p}{q}$ *and* $\frac{p'}{q'}$ *is* $\frac{p+p'}{q+q'}$.

Note that the mediant of two fractions depends on the form the fractions are written in; a pair of non-reduced fractions may give a different resulting mediant than their reduced forms. We will generally only use reduced forms here.

Lemma 2.8 (from [1])**.** *If* $\frac{p}{q} < \frac{p'}{q'}$*, then their mediant has the property* $\frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}$.

Proof. For the first inequality, we have

$$\frac{p+p'}{q+q'} - \frac{p}{q} = \frac{p'q - pq'}{(q+q')q} = \frac{\frac{p'}{q'} - \frac{p}{q}}{(q+q')\frac{1}{q'}} > 0,$$

and in a similar way one can show the second inequality. □

In the first four Farey sequences it already becomes apparent that every new member of F_n is the mediant of its two neighbours. We still need to prove this however. The following theorems and proofs about Farey sequences follow [7, Ch. 3].

Theorem 2.9. *If* $\frac{p}{q}$, $\frac{p''}{q''}$ *and* $\frac{p'}{q'}$ *are three consecutive fractions in* F_n *in that order, then*

$$\frac{p''}{q''} = \frac{p+p'}{q+q'}.$$

Or, as we will show, equivalently:

Theorem 2.10. *If* $\frac{p}{q}$ *and* $\frac{p'}{q'}$ *are two consecutive fractions in* F_n *with* $\frac{p}{q} < \frac{p'}{q'}$*, then* $qp' - q'p = 1$.

We will first prove that these two theorems are equivalent, and then prove that they both hold for all F_n.

Proof that theorem 2.10 implies theorem 2.9. Assume theorem 2.10 holds, then for three consecutive fractions in F_n with $\frac{p}{q} < \frac{p''}{q''} < \frac{p'}{q'}$ we have:

$$qp'' - q''p = 1, \quad q''p' - q'p'' = 1, \tag{4}$$

or, equivalently:

$$p'' = \frac{1 + q''p}{q}, \quad q'' = \frac{1 + q'p''}{p'}.$$

Multiplying by the denominator and substituting for respectively q'' and p'':

$$p''q = 1 + p\frac{1 + q'p''}{p'}, \quad q''p' = 1 + q'\frac{1 + q''p}{q}.$$

Again multiplying by the denominator and rearranging we get

$$p''(qp' - q'p) = p' + p, \quad q''(qp' - q'p) = q + q',$$

dividing the left equality by the right one (which is nonzero, as $q, q' > 0$),

$$\frac{p''}{q''} = \frac{p + p'}{q + q'}$$

which is theorem 2.9. □

For the other direction we need an extra lemma:

Lemma 2.11. *If $n > 1$, then no two consecutive terms of F_n have the same denominator.*

Proof. Let $\frac{p}{q} < \frac{p'}{q}$ be two consecutive terms of F_n with $q > 1$. Then $p + 1 \leq p' < q$, so

$$\frac{p}{q} < \frac{p}{q-1} < \frac{p+1}{q} \leq \frac{p'}{q},$$

so $\frac{p}{q-1}$ would appear between $\frac{p}{q}$ and $\frac{p'}{q}$, which is a contradiction. The middle inequality above follows from:

$$\frac{p}{q-1} - \frac{p+1}{q} = \frac{pq - (q-1)(p-1)}{(q-1)q} = \frac{p+1-q}{q^2 - q} < 0.$$

□

Proof that theorem 2.9 implies theorem 2.10. Assume theorem 2.9 holds in general, observe that theorem 2.10 holds for F_1, and assume that theorem 2.10 holds for F_n for some $n \in \mathbb{N}$. Let $\frac{p''}{q''}$ be an element of F_{n+1} but not of F_n. Then, since theorem 2.9 holds,

$$\frac{p''}{q''} = \frac{p + p'}{q + q'},$$

for $\frac{p}{q} < \frac{p''}{q''} < \frac{p'}{q'}$ consecutive. So

$$\lambda p'' = p + p', \qquad \lambda q'' = q + q'$$

for some integer λ. Since $\frac{p''}{q''}$ is irreducible (as it is an element of F_n) we have $\lambda \geq 1$, and since q and q' must both be less than q'' (due to lemma 2.11) we get $\lambda < 2$. Thus, $\lambda = 1$ and

$$p'' = p + p', \qquad q'' = q + q'. \tag{5}$$

Cross-multiplying these, we get

$$p''(q + q') = q''(p + p''),$$

or, equivalently,

$$p''q - q''p = q''p' - p''q'.$$

By substituting equations (5) we obtain:

$$\begin{aligned} p''q - q''p &= (q + q')p' - (p + p')q' \\ &= qp' - pq' = 1, \end{aligned}$$

with the final equality due to theorem 2.10 holding in F_n, where $\frac{p}{q}$ and $\frac{p'}{q'}$ are consecutive. If either were not in F_n, it must be new in F_{n+1} and therefore have the same denominator as $\frac{p''}{q''}$, which contradicts lemma 2.11. This shows that theorem 2.10 also holds for F_{n+1}, and thus by induction for all Farey sequences. □

Now we are ready to prove theorems 2.9 and 2.10 for all F_n.

Proof of theorems 2.9 and 2.10. Both theorems hold for F_1. We assume they hold for F_n for some $n \in \mathbb{N}$ and show that they hold for F_{n+1}, so by induction they hold for all Farey sequences. Suppose that $\frac{p}{q} < \frac{p'}{q'}$ are consecutive fractions in F_n, with $\frac{p''}{q''}$ between them in F_{n+1}. Then

$$\frac{p}{q} < \frac{p''}{q''}, \qquad \frac{p''}{q''} < \frac{p'}{q'},$$

so

$$\frac{p}{q} + \frac{r}{q''q} = \frac{p''}{q''}, \qquad \frac{p''}{q''} + \frac{s}{q''q} = \frac{p'}{q'}$$

for some integer $r, s > 0$, so

$$qp'' - pq'' = r, \qquad q''p' - p''q' = s. \tag{6}$$

Rearranging we get

$$p'' = \frac{r + q''p}{q}, \qquad q'' = \frac{s + q'p''}{p'}.$$

Multiplying by the denominator and cross-substituting gives

$$p''q = r + p\frac{s + q'p''}{p'}, \qquad q''p' = s + q'\frac{r + q''p}{q},$$

and multiplying by the denominator again and rearranging:

$$p''(qp'-q'p) = sp + rp', \quad q''(qp'-q'p) = sq + rq'.$$

Since theorem 2.9 holds for F_n we have $qp'-q'p = 1$, so

$$p'' = sp + rp', \quad q'' = sq + rq'.$$

Now all we need to do is show that $r = s = 1$. Let $gcd(r,s) = a$. Then $r = ab$, $s = ac$ for some $b, c \in \mathbb{N}$. This would give

$$p'' = a(cp + bp'), \quad q'' = a(cq + bq')$$

making a also a common factor of p'' and q''. However, $\frac{p''}{q''}$ was already in reduced form since it appears in a Farey sequence, so $gcd(r,s) = a = 1$.

Now consider the set of fractions

$$S = \left\{ \frac{P}{Q} : P = \mu p + \lambda p',\ Q = \mu q + \lambda q' \right\}$$

with μ and λ positive integers and $gcd(\mu, \lambda) = 1$. This set must contain $\frac{p''}{q''}$. Due to lemma 2.8, all fractions in S are between $\frac{p}{q}$ and $\frac{p'}{q'}$. We can also show that all fractions in S are in reduced form; assume a divides both P and Q. Then a also divides

$$\begin{aligned} q'P - p'Q &= q'(\mu p + \lambda p') - p'(\mu q + \lambda q') \\ &= (q'p - p'q)\mu + (q'p' - p'q')\lambda \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} qP - pQ &= q(\mu p + \lambda p') - p(\mu q + \lambda q') \\ &= (qp - pq)\mu + (qp' - pq')\lambda \\ &= \lambda, \end{aligned}$$

hence a must be 1. Since all fractions in S are in reduced form and between $\frac{p}{q}$ and $\frac{p'}{q'}$, all of them will appear in a Farey sequence at some point. The first to do so will clearly be the one with the lowest value for Q, so with $\lambda = \mu = 1$. This must be $\frac{p''}{q''}$, so we have

$$p'' = p + p', \quad q'' = q + q'.$$

Substitute this into equation (6), and we get:

$$\begin{aligned} q(p+p') - p(q+q') = r, &\quad (q+q')p' - (p+p')q' = s \\ qp - pq + qp' - pq' = r, &\quad qp' - pq' + q'p' - p'q' = s \\ 1 = r, &\quad 1 = s \end{aligned}$$

which proves theorem 2.10 for F_{n+1}, equivalent to theorem 2.9 for F_{n+1} and by induction both theorems hold for all Farey sequences. □

2.4 Modified Farey sequences

Now that we have established that all newly added members to the $(n+1)$th Farey sequence are mediants of fractions in the nth Farey sequence, we can construct the modified Farey sequences S_n [8].

Definition 2.12. *The* 0*th modified Farey sequence is* $S_0 = \{\frac{0}{1}, \frac{1}{1}\}$*. Any subsequent modified Farey sequence* S_n *is the union of* S_{n-1} *with all the reduced mediants of consecutive fractions in* S_{n-1}*, arranged in ascending order.*

The first 4 modified Farey sequences therefore are:

$S_0 = \{\frac{0}{1}, \frac{1}{1}\}$,
$S_1 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$,
$S_2 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$, and
$S_3 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$.

According to theorem 2.9 and lemma 2.11 every new Farey fraction in F_n is the mediant of two consecutive fractions in F_{n-1}, so we have

$$F_n \subset S_{n-1}.$$

Therefore, by definition 2.6, any reduced rational $\frac{p}{q} \in [0,1]$ will show up in S_n for large enough n.

2.5 Dyadic rationals

In order to be able to define Minkowski's question mark function, we need to introduce one more sequence of sequences.

Definition 2.13. *A dyadic rational is a rational with a power of* 2 *as denominator.*

Dyadic rationals are thus all of the form $i \cdot 2^{-n}$, with $i, n \in \mathbb{Z}$. Like we did with the modified Farey sequences, we can construct sequences of reduced dyadic rationals on the interval [0,1].

Definition 2.14. *The* 0*th dyadic rational sequence is* $D_0 = \{\frac{0}{1}, \frac{1}{1}\}$*. Any subsequent dyadic rational sequence* D_n *is the union of* D_{n-1} *with all the averages of consecutive fractions in* D_{n-1}*, arranged in ascending order.*

The first 4 dyadic rational sequences therefore are:

$D_0 = \{\frac{0}{1}, \frac{1}{1}\}$,
$D_1 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$,
$D_2 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{1}\}$, and
$D_3 = \{\frac{0}{1}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \frac{1}{1}\}$.

Clearly we have $\#D_n = \#S_n$, as we have $\#D_0 = \#S_0 = 2$ and for both sequences $\#D_n = 2 \cdot \#D_{n-1} - 1$ and $\#S_n = 2 \cdot \#S_{n-1} - 1$. This leads to

Proposition 2.15. *For any* $n \in \mathbb{N}$*, the number of elements in* D_n *equals the number of elements in* S_n *and:*

$$\#D_n = \#S_n = 2^n + 1. \tag{7}$$

Proof. Equation (7) holds for $n = 0$. Now assume it holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\#S_{n+1} &= 2 \cdot \#S_n - 1 \\ &= 2(2^n + 1) - 1 \\ &= 2^{n+1} + 1,\end{aligned}$$

so it holds for $n + 1$ and by induction for all $n \in \mathbb{N}$. □

Definition 2.14 need not be recursive. An equal definition would be

Proposition 2.16. *The n-th dyadic rational sequence* D_n *is the set of all fractions with denominator* 2^n *in the interval* $[0, 1]$*, reduced where possible, arranged in ascending order.*

Proof. The proposition is true for $n = 0$. Assume it holds for some $n \in \mathbb{N}$. Let $d \in D_{n+1}$. Then either it was in D_n, in which case d was of the form $d = i \cdot 2^{-n} = 2i \cdot 2^{-(n+1)}$, or d was not in D_n. In that case it must be the average of two fractions in D_n, say $d' = i' \cdot 2^{-n}$ and $d'' = i'' \cdot 2^{-n}$. Then

$$d = \frac{d' + d''}{2} = \frac{i' + i''}{2^{n+1}},$$

so in either case d has denominator 2^{n+1}. Conversely, we can write any fraction with denominator 2^{n+1} as the average of two fractions with denominator 2^n. the proposition therefore holds for $n + 1$ and by induction for all D_n. □

Corollary 2.17. *If* $d(n, i)$ *is the ith element of* D_n*, then*

$$d(n, i) = \frac{i - 1}{2^n}.$$

3 Definition of $?(x)$

3.1 Minkowski's definition

There are multiple ways to define $?(x)$, but we will start with the method used by Minkowski himself. His method was to order the rationals according to modified Farey sequences as in subsection 2.4, and assign them through this ordering to the dyadic rationals. We have already seen that both sequences have an equal number of elements for equal orders, so this means we can simply map the nth modified Farey sequence to the nth dyadic rational sequence:

Definition 3.1. *If* $r(n, i)$ *is the i-th member of modified Farey sequence* S_n *and* $d(n, i)$ *is the i-th member of dyadic rational sequence* D_n*, then Minkowski's question mark function is:*

$$?(r(n, i)) = d(n, i). \tag{8}$$

Or, due to corollary 2.17, $?(r(n, i)) = \frac{i-1}{2^n}$.

Since the modified Farey sequence and the dyadic rational sequence can be defined recursively, we can also write this definition recursively:

Proposition 3.2. *For consecutive fractions* $\frac{a}{b}, \frac{c}{d} \in S_n$*, we have:*

$$?\left(\frac{a+c}{b+d}\right) = \frac{?\left(\frac{a}{b}\right) + ?\left(\frac{c}{d}\right)}{2}.$$

Proof. Since $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in S_n, we have

$$?\left(\frac{a}{b}\right) = \frac{i-1}{2^n}, \quad ?\left(\frac{c}{d}\right) = \frac{i}{2^n}$$

when $\frac{a}{b}$ is the ith element of S_n. By construction, $\frac{a+c}{b+d}$ comes between them in S_{n+1}, where it will be the $(2i)$th element. Therefore,

$$?\left(\frac{a+c}{b+d}\right) = \frac{2i-1}{2^{n+1}} = \frac{\frac{i-1}{2^n} + \frac{i}{2^n}}{2} = \frac{?\left(\frac{a}{b}\right) + ?\left(\frac{c}{d}\right)}{2}.$$

$\square$

Following proposition 2.15, S_{15} already has 32769 elements. Constructing these and plotting them gives us figure 1. The code for this follows [9] and can be found in appendix A.

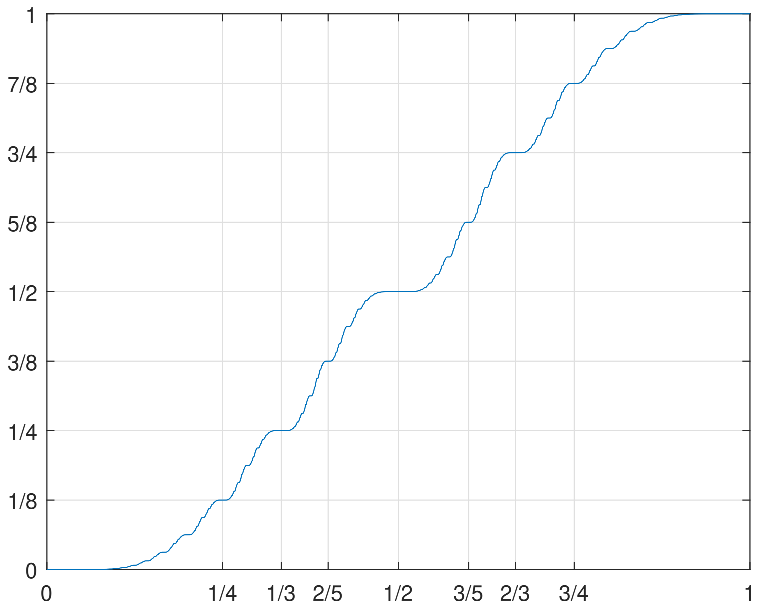

Figure 1: Minkowski's question mark function, $?(x)$. Along the x-axis is the 3rd modified Farey sequence, along the y-axis the 3rd dyadic rational sequence.

3.2 Continued fraction definition of $?(x)$

In this way we have only defined values of $?(x)$ for rational x. Since the rationals are dense in the reals, Minkowski simply defines the values of $?(x)$ for irrational x by continuity. How can this tell us anything about the quadratic irrationals though? This is done through their property of being uniquely written as periodic continued fractions. First, we must translate our current definition to continued fractions. This will allow for a more explicit definition by continuity for irrational x. We can then conclude several properties of $?(x)$ based on the properties of continued fractions. The following is based on Denjoy's work [2] as translated by Salem [12].

Theorem 3.3. *For any* $x \in [0,1]$ *with continued fraction expansion* $x = [0; a_1, a_2, \dots]$ *we have*

$$?(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{(a_1+\cdots+a_m)-1}}. \tag{9}$$

Proof. Let $\frac{p_n}{q_n} = [0; a_1, \dots, a_n]$ be the n-th convergent of x. While constructing the modified Farey sequences, at some point two successive convergents $\frac{p_{k-2}}{q_{k-2}}$ and $\frac{p_{k-1}}{q_{k-1}}$ will appear as consecutive in a modified Farey sequence, say S_m. This is sure to happen: $\frac{p_0}{q_0} = 0$ and $\frac{p_1}{q_1} = \frac{1}{a_1}$ will be consecutive in the $(a_1 - 1)$-th modified Farey sequence since $r(n, 2) = \frac{1}{n+1}$. Let $y_n = ?\left(\frac{p_n}{q_n}\right)$. Then

$$?\left(\frac{p_{k-1} + p_{k-2}}{q_{k-1} + q_{k-2}}\right) = \frac{y_{k-1} + y_{k-2}}{2}$$

due to proposition 3.2.

Since $\frac{p_{k-2}}{q_{k-2}}$ and $\frac{p_{k-1}}{q_{k-1}}$ were consecutive in S_m, $\frac{p_{k-1}+p_{k-2}}{q_{k-1}+q_{k-2}}$ will appear in between them in S_{m+1}. In S_{m+2} we will then have $\frac{2p_{k-1}+p_{k-2}}{2q_{k-1}+q_{k-2}}$, with

$$?\left(\frac{2p_{k-1} + p_{k-2}}{2q_{k-1} + q_{k-2}}\right) = \frac{y_{k-1} + (y_{k-1} + y_{k-2})/2}{2}.$$

From proposition 2.3 we have that for any convergent $\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$. By continuing in the way above, since $\frac{p_{k-1}}{q_{k-1}}$ is again consecutive to every mediant we construct, we get

$$?\left(\frac{p_k}{q_k}\right) = ?\left(\frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}\right) = \frac{y_{k-1}}{2} + \frac{y_{k-1}}{2^2} + \cdots + \frac{y_{k-1}}{2^{a_k}} + \frac{y_{k-2}}{2^{a_k}}.$$

Therefore,

$$y_k = \left(1 - \frac{1}{2^{a_k}}\right) y_{k-1} + \frac{y_{k-2}}{2^{a_k}}$$

and

$$y_k - y_{k-1} = \frac{-1}{2^{a_k}}(y_{k-1} - y_{k-2}).$$

Since $\frac{p_k}{q_k}$ is again consecutive to $\frac{p_{k-1}}{q_{k-1}}$, and as shown above $\frac{p_0}{q_0}$ and $\frac{p_1}{q_1}$ are also consecutive, we can extend this to y_n for any n, obtaining

$$y_n - y_{n-1} = \frac{-1}{2^{a_n}} \cdot \frac{-1}{2^{a_{n-1}}} \cdots \frac{-1}{2^{a_2}}(y_1 - y_0).$$

From equation (8) we get $y_0 = ?(0) = 0$ and $y_1 = ?\left(\frac{1}{a_1}\right) = ?(r(a_1-1,2)) = \frac{1}{2^{(a_1-1)}}$, so

$$y_n - y_{n-1} = \frac{(-1)^{n-1}}{2^{(a_1+\cdots+a_n)-1}}$$

and

$$?\left(\frac{p_n}{q_n}\right) = y_n = \sum_{m=1}^{n} \frac{(-1)^{m-1}}{2^{(a_1+\cdots+a_m)-1}}. \tag{10}$$

The demand for continuity then gives us

$$?(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{(a_1+\cdots+a_m)-1}},$$

concluding the proof. □

This indirectly proves a property of dyadic rationals:

Corollary 3.4. *Any dyadic rational* $d \in (0,1)$ *can be written in the form of the right hand side of equation (10).*

Proof. This follows from the established equality of equations (10) and (8), combined with the fact that any dyadic rational $d \in (0,1)$ will appear in D_n for large enough n. □

From this new definition we can also deduce several properties of $?(x)$, based on the properties of continued fraction expansions shown in subsection 2.2.

Corollary 3.5. *The sum (9) is infinite iff* x *is irrational.*

Proof. This also follows from theorem 2.4; if the sum is infinite, the continued fraction expansion of x must be infinite so the number cannot be rational. Conversely, if x is irrational, it cannot have a finite continued fraction expansion, so the sum cannot be finite. □

Corollary 3.6. $?(x)$ *is a non-dyadic rational iff* x *is a quadratic irrational.*

Proof. This is a consequence of the fact that quadratic irrationals can be written as infinite periodic continued fractions. Under $?(x)$, such an infinite periodic continued fraction gets mapped to an infinite dyadic periodic series. This series can be split up in a positive and a negative series, both of which converge to a rational number. What is left is the difference between two rational numbers, and therefore itself a rational number. Conversely, if $?(x)$ is a non-dyadic rational, we know that the continued fraction expansion of x cannot be finite. We also know that the sum must converge to a rational, so there must be periodicity in it. Hence the continued fraction representation of x must be infinite and after some point periodic, so x is a quadratic irrational according to theorem 2.5. □

3.3 Some examples

To illustrate the workings of $?(x)$ we will calculate it for some example values of x; A dyadic rational, a non-dyadic rational, a quadratic irrational and a non-quadratic irrational.

1. $x = \frac{5}{8}$: This is the mediant of the 6th and 7th members of S_3, so it will be the 12th member of S_4. It gets mapped to the 12th member of D_4 (the average of the 6th and 7th members of D_3), so $?(\frac{5}{8}) = \frac{11}{16}$, another dyadic rational.
2. $x = \frac{2}{7}$: Fourth member of S_4, so $?(\frac{2}{7}) = \frac{3}{16}$. Alternatively, using the algorithm in subsection 2.2:

$$\frac{2}{7} = 0 + \frac{1}{\frac{7}{2}} = 0 + \cfrac{1}{3 + \cfrac{1}{2}} = [0; 3, 2],$$

   for which equation (9) gives us

$$?\left(\frac{2}{7}\right) = ?\left([0; 3, 2]\right) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16},$$

   again a dyadic rational.
3. $x = \sqrt{6} - 2$: This is a quadratic irrational, as it is a root of $x^2 + 4x - 2$. It will not occur in any modified Farey sequence S_n, so we will have to do the continued fraction decomposition:

$$\sqrt{6} - 2 = 0 + \cfrac{1}{\cfrac{1}{\sqrt{6} - 2}} = \cfrac{1}{\cfrac{\sqrt{6} + 2}{2}} = \cfrac{1}{2 + \cfrac{\sqrt{6} - 2}{2}}$$

   Repeating this process once more:

$$\sqrt{6} - 2 = \cfrac{1}{2 + \cfrac{1}{\cfrac{2}{\sqrt{6} - 2}}} = \cfrac{1}{2 + \cfrac{1}{4 + (\sqrt{6} - 2)}}.$$

   Note that the left hand $\sqrt{6} - 2$ appears again in the right hand side of the decomposition so far, so the continued fraction expansion will be periodic:

$$\sqrt{6} - 2 = \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}} = [0; \overline{2, 4}].$$

Then equation (9) gives us:

$$?([0;2,4,2,4,\dots]) = \frac{1}{2} - \frac{1}{2^5} + \frac{1}{2^7} - \frac{1}{2^{11}} + \dots = \sum_{i=0}^{\infty} \frac{1}{2^{1+6i}} - \sum_{i=0}^{\infty} \frac{1}{2^{5+6i}}.$$

These series can be evaluated. Taking the left series first, multiplying by 2^6:

$$\begin{aligned} 2^6 \sum_{i=0}^{\infty} \frac{1}{2^{1+6i}} &= 2^6 \left(\frac{1}{2} + \frac{1}{2^7} + \frac{1}{2^{13}} + \cdots \right) \\ &= 2^5 + \frac{1}{2} + \frac{1}{2^7} + \frac{1}{2^{13}} + \cdots \\ &= 2^5 + \sum_{i=0}^{\infty} \frac{1}{2^{1+6i}}, \end{aligned}$$

so

$$2^5 = (2^6 - 1) \sum_{i=0}^{\infty} \frac{1}{2^{1+6i}},$$

and then

$$\sum_{i=0}^{\infty} \frac{1}{2^{1+6i}} = \frac{2^5}{2^6 - 1} = \frac{32}{63}.$$

A similar computation for the right hand series gives:

$$\sum_{i=0}^{\infty} \frac{1}{2^{5+6i}} = \frac{2}{2^6 - 1} = \frac{2}{63}.$$

Putting these results together we get:

$$?(\sqrt{6} - 2) = ?([0;2,4,2,4,\dots]) = \sum_{i=0}^{\infty} \frac{1}{2^{1+6i}} - \sum_{i=0}^{\infty} \frac{1}{2^{5+6i}} = \frac{32-2}{63} = \frac{10}{21}.$$

Which is, as expected, a rational. It is not a dyadic rational though, since $\sqrt{6} - 2$ is not a rational number.

4. $x = \pi - 3$: This will neither appear in any modified Farey sequence nor have a periodic continued fraction expansion. A quick search gives us the first part of its infinite continued fraction expansion:

   $$\pi - 3 = [0;7,15,1,292,\dots].$$

   and computing the sum (9) with this first part gives:

   $$?([0;7,15,1,292]) = \frac{1}{2^6} - \frac{1}{2^{21}} + \frac{1}{2^{22}} - \frac{1}{2^{314}} = 0.015624761581421\dots$$

   Which already gives such an immense amount of decimals (about 10^{94}) that we can no longer do anything meaningful with it. Since there is no periodicity in the continued fraction expansion either, we cannot evaluate the sum as for quadratic irrationals. It does however illustrate nicely that the value of $?(x)$ for any given x can be approximated quite closely with only the first few coefficients of its continued fraction expansion. As subsequent terms of the sum (9) will have even larger denominators, they will have progressively less impact on our estimate.

4 Fixed points of $?(x)$

While it is easily verified that 0, $\frac{1}{2}$, and 1 are fixed points of $?(x)$, there appear to be at least two more (see figure 2). Appendix B contains two approaches at finding these two apparent fixed points by iterative methods. Let us assume for now that these unknown fixed points are exactly two. We will denote the fixed point in $(0, \frac{1}{2})$ by x_1 and the one in $(\frac{1}{2}, 1)$ by x_2. Due to the symmetry of $?(x)$, we have $x_2 = 1 - x_1$.

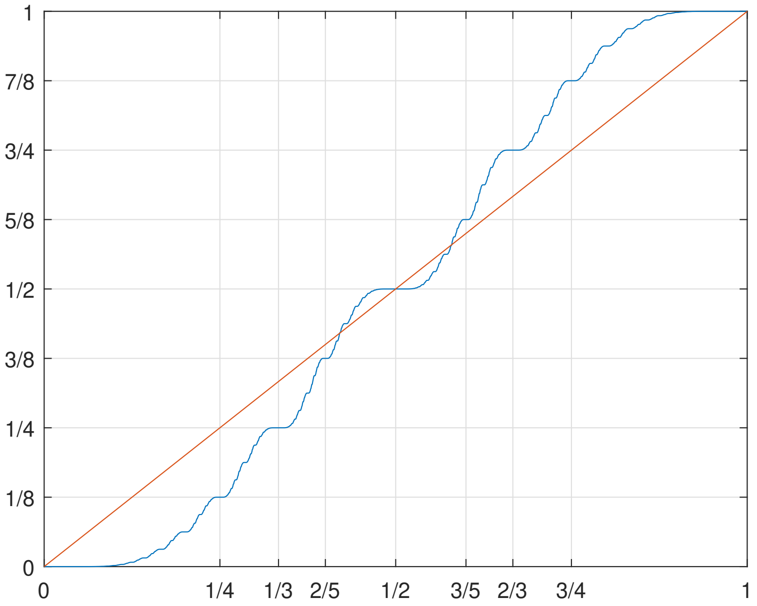

Figure 2: Minkowski's question mark function $?(x)$ together with $y(x) = x$. Note that there appear to be three intersections in $(0, 1)$.

Note that at both of these apparent fixed points $?'(x_i) > 1$ (assuming the derivative exists), so we can iteratively approach the fixed points by decreasing x when $x < ?(x)$ and increasing x when $x > ?(x)$.

4.1 Modified Farey sequences

The first approach is based on the iterative construction of modified Farey sequences. A starting value of x is taken, the ith member of modified Farey sequence S_n, and two members $r_{n,1}$ and $r_{n,2}$ of S_n are located for which $r_{n,1} > ?(r_{n,1})$ but $r_{n,2} < ?(r_{n,2})$. Our fixed point x_1 should then be between these two points, so we construct the mediant and repeat the process to find on which side of the mediant it is. By repeatedly applying this procedure, we obtain better approximations of x_1. By applying this until S_{50}, we obtained $x = 0.420372339423223\ldots$, with an error of $|x - ?(x)| \approx 2 \cdot 10^{-13}$. Further iterations were not pursued due to calculation time. The code could be made much more efficient to decrease calculation time, for example by not re-calculating every member of S_{n+1} when it was already a member of S_n).

4.2 Continued fraction expansions

The second approach is based on the continued fraction expansion of x and the calculation of $?(x)$ from its continued fraction as in equation (9). The following properties were used:

Proposition 4.1. *If $x = [a_0; a_1, a_2, \ldots, a_i, \ldots]$, then an increase in a_i will result in an increase in x for even i, and a decrease in x for odd i.*

Proof. We prove this by induction. Recall that the continued fraction expansion of x is

$$x = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}. \tag{11}$$

Increasing a_0 simply adds to the sum, so clearly it increases x. Likewise, increasing a_1 makes the denominator larger in the right hand side of the sum, thereby decreasing x. Now suppose that for a given $i \in \mathbb{N}_0$, increasing a_i will lead to an increase in x. We will look at the tail end t_i of the continued fraction expansion, starting from a_i. We can represent this by

$$t_i = [a_i, a_{i+1}, a_{i+2}, \ldots] = a_i + \cfrac{1}{a_{i+1} + \cfrac{1}{a_{i+2} + \cfrac{1}{\ddots}}}.$$

Note that

$$x = [a_0; a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots] = [a_0; a_1, a_2, \ldots, a_{i-1}, t_i].$$

Since increasing a_i would increase x, increasing t_i would also increase x. Observe that increasing a_{i+1} leads to a decrease in t_i, and as therefore a decrease in x. By the same reasoning, increasing a_{i+2} will again lead to an increase in t_i and therefore x. Since the effect of increasing a_i alternates on i, and increasing a_0 has positive effect on x, we conclude that increasing a_i leads to an increase in x for even i and a decrease in x for odd i. ☐

Proposition 4.2. *If $x_n = [a_0; a_1, a_2, \ldots, a_n]$ is the nth convergent of x, then $x_n < x$ for even n and $x_n > x$ for odd n.*

Proof. Note that for x_n, the continued fraction expansion terminates after n. This equates to setting $t_n = a_n$ in the notation of proposition 4.1. Since $t_i > a_i$ for all i as long as the continued fraction expansion does not terminate after a_i, this means a decrease in t_n and by the reasoning of proposition 4.1 a decrease in x for even n and an increase in x for odd n. Hence we conclude that $x_n < x$ for even n and $x_n > x$ for odd n. ☐

The fixed points x_1 and x_2 were then approximated by going through the successive convergents, where for each odd i the highest a_i was found for which $x_n < ?(x_n)$, and for each even i the highest a_i was found for which

$x_n > ?(x_n)$. This method proved much more efficient than the earlier one. The code for the first method was very ill-optimized, having to do recursive calculations through increasing layers of Farey sequences for every single evaluation. This second method also has the advantage of using the continued fraction expansion, which allows for direct computation of values of $?(x)$. As the example calculation of $?(\pi-3)$ in subsection 3.3 illustrated, the continued fraction expansions method also has rapidly increasing accuracy. It gave the following approximations:

$$\begin{aligned}x_1 &= [0;2,2,1,1,1,3,2,3,1,2,4,1,1,2,3,1,2,5,1,3,2,3,1,3,1,1,1]\\ &= \frac{802890961}{1909951930}\\ &\approx 0.420372339423223\ldots.\end{aligned}$$

Changing the initial value to [1,1,2,1,1] gave an approximation for the upper fixed point:

$$\begin{aligned}x_2 &= [0;1,1,2,1,1,1,3,2,3,1,2,4,1,1,2,3,1,2,5,1,3,2,3,1,3,1,1,1]\\ &= \frac{1107060969}{1909951930}\\ &= 1 - x_1\\ &\approx 0.579627660576777\ldots.\end{aligned}$$

The fact that $x_1 = 1 - x_2$ was to be expected due to the symmetry of $?(x)$. These approximations exceed the standard tolerance of MATLAB in accuracy, which is why the code terminated. It should be noted that the denominator 1909951930 is not a power of 2, so the approximations are non-dyadic rationals, which means they get mapped to dyadic rationals by the original definition of $?(x)$ and thus cannot be the exact fixed points.

5 The singularity of $?(x)$

This section will follow Salem's proof [12] that $?(x)$ is a singular function, with added details.

Definition 5.1. *A function f is singular on the interval $[a,b]$ if it has the following properties:*

1. *f is continuous on $[a,b]$,*
2. *the derivative of f vanishes almost everywhere, and*
3. *f is non-constant on $[a,b]$.*

The first and third conditions clearly hold for $?(x)$ on $[0,1]$. The second condition requires some additional specification before we can show that it also holds for $?(x)$. We will need to show that there is a subset N of $[0,1]$, where N has measure 1. We will then need to show that the derivative of $?(x)$ exists and equals zero on this subset.

5.1 The set N of measure 1

In this subsection we will show that the set of $x = [a_0; a_1, a_2, \dots] \in [0,1]$ such that $\sup_{i\in\mathbb{N}} a_i = \infty$ is of measure 1. We will follow [13, Ch. 19] with additional details from [10]. Measure is a generalization of ideas such as length, area and volume. The measure of a subset of the real numbers therefore coincides with the length of such a subset, or, if the subset is disjoint, the sum of the lengths of its parts. This allows us to make our first useful observation.

Lemma 5.2. *The measure of the set of* $x = [0; a_1, a_2, \dots] \in [0,1]$ *such that* $a_1 = k$ *is* $\frac{1}{k(k+1)}$.

Proof. Note that any number $x = [0; a_1, a_2, \dots]$ for which $a_1 = k$ is of the form

$$x = \cfrac{1}{k + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

This means that $x \le \frac{1}{k}$. Since $x \le \frac{1}{k+1}$ would imply $a_1 \ge k+1$ and vice versa, we get

$$x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \iff a_1 = k.$$

The measure of the set of $x = [0; a_1, a_2, \dots] \in [0,1]$ such that $a_1 = k$ is then

$$\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$$

□

We would like to generalize this to all a_n. Due to the influence of the preceding a_n this is however much more difficult, but we can obtain a general bound that does not depend on n.

Proposition 5.3. *The measure of the set* $I_{a_n=k} = \{x = [0; a_1, \dots, a_n, \dots] : a_n = k\}$ *is between* $\frac{1}{3k^2}$ *and* $\frac{2}{k^2}$.

Proof. We want to find the measure of the subset of $(0,1)$ for which $a_n = k$ for given n and k, independent of the a_i for $i < n$. We will start by trying to find this measure for given preceding a_i. Let

$$I_k = \{x = [0; b_1, b_2, \dots] : b_i = a_i \text{ for } i < n, b_i = k \text{ for } i = n\},$$

and likewise

$$I_{a_{n-1}} = \{x = [0; b_1, b_2, \dots] : b_i = a_i \text{ for } i < n\}.$$

Recall that for given $x = [0; a_1, a_2, \dots, a_{n-1}, k]$, proposition 2.3 gives us

$$[0; a_1, a_2, \dots, a_{n-2}, a_{n-1}, k] = \frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}},$$

where $\frac{p_n}{q_n} = x_n$ again denotes the nth convergent of x, i.e. $x_n = [0; a_1, a_2, \dots, a_n]$. The proof of theorem 3.3 also shows that any consecutive convergents will be neighbours in some Farey sequence, and therefore theorem 2.10 holds here as well:

$$|p_{n-1}q_{n-2} - p_{n-2}q_{n-1}| = 1,$$

with the absolute bars due to the fact that we do not know which convergent is larger. By the same reasoning as for $a_1 = k$ above, I_k is equal to the interval between $[0; a_1, \dots, a_{n-1}, k]$ and $[0; a_1, \dots, a_{n-1}, k+1]$. Depending on whether n is even or odd (see prop 4.1), this gives us either

$$I_k = \left[\frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}}, \frac{(k+1)p_{n-1} + p_{n-2}}{(k+1)q_{n-1} + q_{n-2}}\right) \tag{12}$$

or

$$I_k = \left(\frac{(k+1)p_{n-1} + p_{n-2}}{(k+1)q_{n-1} + q_{n-2}}, \frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}}\right].$$

Without loss of generality, let us assume the interval is of the form (12). The measure of this interval is

$$|I_k| = \left|\frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}} - \frac{(k+1)p_{n-1} + p_{n-2}}{(k+1)q_{n-1} + q_{n-2}}\right|. \tag{13}$$

The interval $I_{a_n - 1}$ is then equal to the union of this interval over all possible k:

$$I_{a_{n-1}} = \bigcup_{k \in \mathbb{N}} I_k,$$

which allows us to compute its measure. Note that the intervals I_k are pairwise disjoint, so we get

$$|I_{a_{n-1}}| = \sum_{k \in \mathbb{N}} |I_k| = \sum_{k \in \mathbb{N}} \left|\frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}} - \frac{(k+1)p_{n-1} + p_{n-2}}{(k+1)q_{n-1} + q_{n-2}}\right|.$$

Observe that this is a telescoping sum. This allows us to compute the sum by taking $k = 1$ on one side and $\lim_{k\to\infty}$ on the other:

$$|I_{a_{n-1}}| = \lim_{k\to\infty} \left|\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} - \frac{kp_{n-1} + p_{n-2}}{kq_{n-1} + q_{n-2}}\right| = \left|\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} - \frac{p_{n-1}}{q_{n-1}}\right|$$

Cross-multiplying this gives

$$|I_{a_{n-1}}| = \left|\frac{p_{n-2}q_{n-1} - p_{n-1}q_{n-2}}{q_{n-1}^2 + q_{n-2}q_{n-1}}\right|,$$

and applying theorem 2.10 we get

$$|I_{a_{n-1}}| = \left|\frac{1}{q_{n-1}^2(1 + \frac{q_{n-2}}{q_{n-1}})}\right|.$$

This measure still depends on the denominators of the convergents and therefore on the a_i, but it will have to do for now. Let us shift our focus to the measure of I_k itself.

For the measure of I_k, we go back to equation (13). Cross-multiplying and working out some brackets gives

$$|I_k| = \left| \frac{k(p_{n-1}q_{n-2} - q_{n-1}p_{n-2}) + (k+1)(p_{n-2}q_{n-1} - q_{n-2}p_{n-1})}{(kq_{n-1} + q_{n-2})((k+1)q_{n-1} + q_{n-2})} \right|.$$

Using theorem 2.10 we can simplify the numerator to:

$$|k(p_{n-1}q_{n-2} - q_{n-1}p_{n-2}) + (k+1)(p_{n-2}q_{n-1} - q_{n-2}p_{n-1})| = |\pm k \mp (k+1)| = 1,$$

which gives us

$$|I_k| = \left| \frac{1}{q_{n-1}^2 k^2 + q_{n-1}^2 k + 2q_{n-1}q_{n-2}k + q_{n-1}q_{n-2} + q_{n-2}^2} \right|.$$

Factoring out the term $q_{n-2}^2 k^2$ for reasons that will become apparent soon:

$$|I_k| = \left| \frac{1}{q_{n-1}^2 k^2 (1 + \frac{1}{k} + \frac{2q_{n-2}}{kq_{n-1}} + \frac{q_{n-2}}{k^2 q_{n-1}} + \frac{q_{n-2}^2}{q_{n-1}^2 k^2})} \right|,$$

and once again factoring:

$$|I_k| = \left| \frac{1}{q_{n-1}^2 k^2 (1 + \frac{1}{k} + \frac{q_{n-2}}{kq_{n-1}})(1 + \frac{q_{n-2}}{kq_{n-1}})} \right|.$$

Now that we have an expression for both $|I_k|$ and $|I_{a_{n-1}}|$, let us compare them:

$$\frac{|I_k|}{|I_{a_{n-1}}|} = \frac{q_{n-1}^2 (1 + \frac{q_{n-2}}{q_{n-1}})}{q_{n-1}^2 k^2 (1 + \frac{1}{k} + \frac{q_{n-2}}{kq_{n-1}})(1 + \frac{q_{n-2}}{kq_{n-1}})}.$$

This simplifies to

$$\frac{|I_k|}{|I_{a_{n-1}}|} = \frac{1}{k^2} \cdot \frac{1 + \frac{q_{n-2}}{q_{n-1}}}{(1 + \frac{1}{k} + \frac{q_{n-2}}{kq_{n-1}})(1 + \frac{q_{n-2}}{kq_{n-1}})}. \tag{14}$$

We can bound the rightmost fraction in this equation. Taking $\lim_{k\to\infty}$ minimizes the denominator, giving us

$$\lim_{k\to\infty} \frac{1 + \frac{q_{n-2}}{q_{n-1}}}{(1 + \frac{1}{k} + \frac{q_{n-2}}{kq_{n-1}})(1 + \frac{q_{n-2}}{kq_{n-1}})} = \frac{1 + \frac{q_{n-2}}{q_{n-1}}}{1}.$$

As shown in the proof of theorem 3.3, higher order convergents only appear in higher order modified Farey sequences, and therefore have larger denominator. This gives us $\frac{q_{n-2}}{q_{n-1}} < 1$, so

$$\frac{1 + \frac{q_{n-2}}{q_{n-1}}}{(1 + \frac{1}{k} + \frac{q_{n-2}}{kq_{n-1}})(1 + \frac{q_{n-2}}{kq_{n-1}})} < \frac{1 + \frac{q_{n-2}}{q_{n-1}}}{1} < 2.$$

The lower bound for this same fraction we obtain by maximizing the denominator, by setting $k = 1$:

$$\frac{1 + \frac{q_{n-2}}{q_{n-1}}}{(1 + \frac{1}{k} + \frac{q_{n-2}}{kq_{n-1}})(1 + \frac{q_{n-2}}{kq_{n-1}})} \geq \frac{1 + \frac{q_{n-2}}{q_{n-1}}}{(2 + \frac{q_{n-2}}{q_{n-1}})(1 + \frac{q_{n-2}}{q_{n-1}})} = \frac{1}{2 + \frac{q_{n-2}}{q_{n-1}}} > \frac{1}{3}.$$

Which gives us

$$\frac{1}{3k^2} < \frac{|I_k|}{|I_{a_{n-1}}|} < \frac{2}{k^2},$$

or

$$\frac{1}{3k^2}|I_{a_{n-1}}| < |I_k| < \frac{2}{k^2}|I_{a_{n-1}}|.$$

Summing over all possible a_i for $i < n$ gives

$$\sum_{1\leq a_1,\ldots,a_{n-1}<\infty} |I_{a_{n-1}}| = 1 \quad \text{and} \quad \sum_{1\leq a_1,\ldots,a_{n-1}<\infty} |I_k| = |I_{a_n=k}|,$$

so we finally get

$$\frac{1}{3k^2} < |I_{a_n=k}| < \frac{2}{k^2}.$$

□

Corollary 5.4. *For all $k, n \in \mathbb{N}_+$,*

$$\frac{1}{3k} < |\{x = [0; a_1, a_2, \ldots] : a_n \geq k\}| < \frac{4}{k}.$$

Proof. For any given $k, n \in \mathbb{N}_+$, we have

$$|\{x = [0; a_1, a_2, \ldots] : a_n \geq k\}| = \sum_{i=k}^{\infty} |I_{a_n=i}|$$

and

$$\sum_{i=k}^{\infty} \frac{1}{3i^2} < \sum_{i=k}^{\infty} |I_{a_n=i}| < \sum_{i=k}^{\infty} \frac{2}{i^2}.$$

Now we need to find bounds for the left- and rightmost sums in this inequality. Note that both are some multiple of

$$\sum_{i=k}^{\infty} \frac{1}{i^2} = \frac{1}{k^2} + \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \cdots.$$

We can transform this into a telescoping sum by adding to the denominator in every summand, thereby decreasing the total sum:

$$\sum_{i=k}^{\infty} \frac{1}{i^2} \geq \sum_{i=k}^{\infty} \frac{1}{i(i+1)} = \sum_{i=k}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{k} - \lim_{i\to\infty} \frac{1}{i} = \frac{1}{k}.$$

Decreasing the summand in the denominator instead gives us the other bound:

$$\sum_{i=k}^{\infty} \frac{1}{i^2} \leq \sum_{i=k}^{\infty} \frac{1}{(i-1)i} = \sum_{i=k}^{\infty} \left(\frac{1}{i-1} - \frac{1}{i}\right) = \frac{1}{k-1} - \lim_{i\to\infty} \frac{1}{i} = \frac{1}{k-1} \leq \frac{2}{k}.$$

Note that this last bound only works for $k > 1$. For $k = 1$ however, we can easily see that

$$|\{x = [0; a_1, a_2, \ldots] : a_n \geq 1\}| = 1.$$

Therefore, we obtain

$$\frac{1}{3k} < |\{x = [0; a_1, a_2, \ldots] : a_n \geq k\}| < \frac{4}{k}.$$

□

This last result allows us to get to the point of this subsection: the measure of the set of $x = [a_0; a_1, a_2, \dots] \in [0, 1]$ such that $\sup_{i\in\mathbb{N}} a_i = \infty$.

Theorem 5.5. *The set* $N = \{x = [a_0; a_1, a_2, \dots] \in [0, 1] : \sup_{i\in\mathbb{N}} a_i = \infty\}$ *has measure 1.*

Proof. Consider the complement of N in $[0, 1]$:

$$B = \{x = [a_0; a_1, a_2, \dots] \in [0, 1] : \sup_{i\in\mathbb{N}} a_i = K \text{ for some } K < \infty\}.$$

If we then define, for given K,

$$B_K = \{x = [a_0; a_1, a_2, \dots] \in [0, 1] : \sup_{i\in\mathbb{N}} a_i = K\}$$

and

$$B_{K,n} = \{x = [a_0; a_1, a_2, \dots] \in [0, 1] : \sup_{1\leq i\leq n} a_i = K\},$$

we get

$$\bigcup_{K=1}^{\infty} B_k = B,$$

and

$$\bigcap_{n=1}^{\infty} B_{K,n} = B_K.$$

Since every next $B_{K,n}$ is contained in all previous $B_{K,i}$ for $i < n$,

$$|B_K| = \left|\bigcap_{n=1}^{\infty} B_{K,n}\right| = \lim_{n\to\infty} |B_{K,n}|.$$

Now let us determine the measure of $B_{K,n}$ inductively. For $B_{K,1}$, we have

$$|B_{K,1}| = |\{x = [a_0; a_1, a_2, \dots] \in [0, 1] : a_1 < K\}|,$$

so by corollary 5.4 we have

$$|B_{K,1}| = 1 - |\{x = [0; a_1, a_2, \dots] : a_1 \geq K\}| < 1 - \frac{1}{3K}.$$

Assume we have $|B_{K,n}| < \left(1 - \frac{1}{3K}\right)^n$ for some $n \in \mathbb{N}$. Then

$$|B_{K,n+1}| = |B_{K,n}| - \frac{|\{x = [0; a_1, a_2, \dots] : a_{n+1} \geq K\}|}{|B_{K,n}|},$$

since $B_{K,n+1} \subset B_{K,n}$ and we only lose that part of $B_{K,n}$ for which $a_{n+1} \geq K$. Since a_{n+1} is independent of the earlier a_i, this can be represented by the fraction shown above. Therefore,

$$|B_{K,n+1}| = (1 - |\{x = [0; a_1, a_2, \dots] : a_{n+1} \geq K\}|) \cdot |B_{K,n}| < (1 - \frac{1}{3K})^{n+1}.$$

This proves by induction that $|B_{K,n}| < (1 - \frac{1}{3K})^n$. Since

$$0 < \frac{1}{3K} < 1,$$

we have $0 < (1 - \frac{1}{3K}) < 1$, so we get

$$|B_K| = \lim_{n\to\infty} |B_{K,n}| \leq \lim_{n\to\infty} (1 - \frac{1}{3K})^n = 0.$$

Since B is the countable union of B_K, this gives us

$$|B| \leq \sum_{K=1}^{\infty} |B_K| = \sum_{K=1}^{\infty} 0 = 0.$$

Since the complement of $N \subset [0, 1]$ has measure 0 while $|[0, 1]| = 1$, we get

$$|N| = |[0, 1]| - |B| = 1.$$

□

5.2 The derivative of $?(x)$ on N

Now that we have a subset $N \subset (0, 1)$ with measure $|N| = 1$, we can go back to Salem's proof and look at the derivative of $?(x)$ on that subset. We will do so by looking at the convergents.

Proposition 5.6. *For any $x = [0; a_1, a_2, \dots] \in N$ with $\frac{p_n}{q_n} = x_n = [0; a_1, a_2, \dots, a_n]$ the nth convergent of x, we have*

$$\frac{1}{(a_{n+1} + 2)q_n^2} < |x - x_n| < \frac{1}{a_{n+1}q_n^2}. \tag{15}$$

Proof. We will denote the 'tail end' t_n of the continued fraction expansion as in the proof of proposition 4.1:

$$t_n = [a_n, a_{n+1}, a_{n+2}, \dots] = a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{\ddots}}}.$$

Note again that

$$x = [0; a_1, a_2, \dots] = [0; a_1, a_2, \dots, a_n, t_{n+1}].$$

This gives us, by proposition 2.3,

$$x = \frac{t_{n+1}p_n + p_{n-1}}{t_{n+1}q_n + q_{n+1}}$$

Subtracting $x_n = \frac{p_n}{q_n}$ from both sides gives

$$x - x_n = \frac{t_{n+1}p_n + p_{n-1}}{t_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{t_{n+1}p_nq_n + p_{n-1}q_n - t_{n+1}q_np_n - q_{n-1}p_n}{(t_{n+1}q_n + q_{n-1})q_n}.$$

Applying theorem 2.10 gives

$$x - x_n = \frac{t_{n+1}p_nq_n - t_{n+1}q_np_n \pm 1}{(t_{n+1}q_n + q_{n-1})q_n},$$

so

$$|x - x_n| = \frac{1}{(t_{n+1}q_n + q_{n-1})q_n}.$$

Note that since

$$t_{n+1} = a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} + \cfrac{1}{\ddots}}},$$

we have $a_{n+1} < t_{n+1} < a_{n+1} + 1$, so

$$\frac{1}{((a_{n+1}+1)q_n + q_{n-1})q_n} < |x - x_n| < \frac{1}{(a_{n+1}q_n + q_{n-1})q_n}.$$

As shown in the proof of theorem 3.3, higher order convergents only appear in higher order modified Farey sequences, and therefore have larger denominator. Therefore $q_n > q_{n-1}$, so we can simplify the above inequalities to

$$\frac{1}{(a_{n+1}+2)q_n^2} < |x - x_n| < \frac{1}{a_{n+1}q_n^2}.$$

□

Proposition 5.7. *For any* $x = [0; a_1, a_2, \dots] \in N$ *with* $\frac{p_n}{q_n} = x_n = [0; a_1, a_2, \dots, a_n]$ *the nth convergent of* x*, and* $y = ?(x)$*,* $y_n = ?(x_n)$*, we have*

$$\frac{1}{2^{a_1+\cdots+a_{n+1}}} < |y - y_n| < \frac{1}{2^{(a_1+\cdots+a_{n+1})-1}}. \tag{16}$$

Proof. From theorem 3.3 we have

$$y = ?(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{(a_1+\cdots+a_m)-1}}$$

and

$$y_n = ?(x_n) = \sum_{m=1}^{n} \frac{(-1)^{m-1}}{2^{(a_1+\cdots+a_m)-1}},$$

so

$$y - y_n = \sum_{m=n+1}^{\infty} \frac{(-1)^{m-1}}{2^{(a_1+\cdots+a_m)-1}} = (-1)^n \left(\frac{1}{2^{(a_1+\cdots+a_{n+1})-1}} - \frac{1}{2^{(a_1+\cdots+a_{n+2})-1}} + \cdots \right).$$

Note that, since all the a_i are positive integers, each consecutive summand is of smaller magnitude than the previous one, so

$$|y - y_n| \leq \left| \frac{1}{2^{(a_1+\cdots+a_{n+1})-1}} - \frac{1}{2^{(a_1+\cdots+a_{n+2})-1}} + \cdots \right| < \frac{1}{2^{(a_1+\cdots+a_{n+1})-1}}.$$

Also,

$$|y - y_n| > \frac{1}{2^{(a_1+\cdots+a_{n+1})-1}} - \frac{1}{2^{(a_1+\cdots+a_{n+2})-1}} \geq \frac{1}{2^{(a_1+\cdots+a_{n+1})-1}} - \frac{1}{2^{(a_1+\cdots+a_{n+1})}},$$

which leads to

$$|y - y_n| > \frac{1}{2^{(a_1+\cdots+a_{n+1})-1}} - \frac{1}{2^{(a_1+\cdots+a_{n+1})}} = \frac{1}{2^{(a_1+\cdots+a_{n+1})}}.$$

Combining this gives

$$\frac{1}{2^{a_1+\cdots+a_{n+1}}} < |y - y_n| < \frac{1}{2^{(a_1+\cdots+a_{n+1})-1}}.$$

□

Now that we have bounds for both the error in the convergent and the difference in value between $?(x)$ and $?(x_n)$, we can combine them.

Proposition 5.8. *For any* $x = [0; a_1, a_2, \dots] \in N$ *with* $\frac{p_n}{q_n} = x_n = [0; a_1, a_2, \dots, a_n]$ *the nth convergent of* x*, and* $y = ?(x)$*,* $y_n = ?(x_n)$*, with*

$$\delta_n = \left| \frac{y - y_n}{x - x_n} \right|$$

we have

$$\liminf_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = 0.$$

Proof. From inequalities (15) and (16) we get

$$\delta_n = \left| \frac{y - y_n}{x - x_n} \right| < \frac{(a_{n+1}+2)q_n^2}{2^{(a_1+\cdots+a_{n+1})-1}}$$

and

$$\delta_{n-1} = \left| \frac{y - y_{n-1}}{x - x_{n-1}} \right| > \frac{a_n q_{n-1}^2}{2^{a_1+\cdots+a_n}}.$$

Therefore,

$$\frac{\delta_n}{\delta_{n-1}} < \frac{(a_{n+1}+2)q_n^2 \cdot 2^{a_1+\cdots+a_n}}{2^{(a_1+\cdots+a_{n+1})-1} \cdot a_n q_{n-1}^2} = \frac{2}{2^{a_{n+1}}} \cdot \frac{a_{n+1}+2}{a_n} \left(\frac{q_n}{q_{n-1}} \right)^2. \tag{17}$$

Since

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}},$$

we have

$$q_n \leq a_n q_{n-1} + q_{n-2}$$

and therefore

$$\frac{q_n}{q_{n-1}} \leq a_n + \frac{q_{n-2}}{q_{n-1}} < a_n + 1.$$

Combining this with inequality (17) gives us

$$\frac{\delta_n}{\delta_{n-1}} < \frac{2}{2^{a_{n+1}}} \cdot \frac{a_{n+1}+2}{a_n} (a_n+1)^2 = \frac{2a_n a_{n+1} + 4a_n + 4a_{n+1} + 8 + \frac{2a_{n+1}}{a_n} + \frac{4}{a_n}}{2^{a_{n+1}}}.$$

Since we always have $a_n, a_{n+1} \geq 1$, we get

$$\frac{\delta_n}{\delta_{n-1}} < 24 \frac{a_n a_{n+1}}{2^{a_{n+1}}}.$$

Recall that we took $x \in N = \{x = [a_0; a_1, a_2, \dots] \in [0,1] : \sup_{i\in\mathbb{N}} a_i = \infty\}$. This ensures that there is an infinite subsequence (a_{n_k}) of the a_n such that $a_{n_k} < a_{n_{k+1}}$ and $\lim_{k\to\infty} a_{n_k} = \infty$. Therefore,

$$\liminf_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} \le \lim_{k\to\infty} 24 \frac{a_{n_k} a_{n_{k+1}}}{2^{a_{n_{k+1}}}} \le \lim_{x\to\infty} 24 \frac{x^2}{2^x} = 0.$$

□

Theorem 5.9. *Minkowski's question mark function $?(x)$ is a singular function.*

Proof. Lebesgue's theorem for the differentiability of monotone functions tells us that the monotone function $?(x)$ is differentiable almost everywhere. The subset of N for which $?'(x)$ exists and is finite is therefore also of measure 1. Now assume that x is in this subset and $?'(x)$ is nonzero. Then we should have

$$?'(x) = \lim_{h\to 0} \frac{?(x+h) - ?(x)}{h} \neq 0$$

Since the convergents x_n converge to x, we should get

$$\liminf_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = \liminf_{n\to\infty} \frac{\left|\frac{y-y_n}{x-x_n}\right|}{\left|\frac{y-y_{n-1}}{x-x_{n-1}}\right|} = \lim_{h\to 0} \frac{\left(\frac{?(x+h)-?(x)}{h}\right)}{\left(\frac{?(x+h)-?(x)}{h}\right)} = 1.$$

Proposition 5.8 however gives us

$$\liminf_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = 0,$$

so by contradiction $?'(x)$ cannot be nonzero. Since this implies $?'(x) = 0$ on a subset measure 1 of the interval $[0,1]$, and since $?(x)$ is continuous and non-constant on $[0,1]$, we conclude that $?(x)$ is singular on $[0,1]$. □

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A Plotting $?(x)$

The following MATLAB code was used to construct the first 15 modified Farey sequences and corresponding dyadic fraction sequences, and plot $?(x)$ from this.

Listing 1: p.m

```
function [output] = p(x,y)
if x == 0
    if y == 0
        output = 0;
    elseif y == 1
        output = 1;
    end
elseif floor(y/2) == y/2
    output = p(x-1,y/2);
else
    output = p(x-1, (y+1)/2 - 1) + p(x-1, (y+1)/2);
end
```

Listing 2: q.m

```
function [ output ] = q(x,y)
if x == 0
    if y == 0
        output = 1;
    elseif y == 1
        output = 1;
    end
elseif floor(y/2) == y/2
    output = q(x-1,y/2);
else
    output = q(x-1, (y+1)/2 - 1) + q(x-1, (y+1)/2);
end
```

Listing 3: r.m

```
function [ output ] = r(x,y)
output = p(x,y)/q(x,y);
end
```

Listing 4: Kinney.m

```
for n = 1:15
    i = 0;
```

```
    check = 0;
    while check == 0
        S(n,i+1)=r(n,i);
        M(n,i+1)=i*2.^(-n);
        i = i+1;
        if S(n,i) == 1
            check = 1;
        end
    end
end
plot(S(15,:),M(15,:))
```

B Finding fixed points of $?(x)$

B.1 Modified Farey sequences

The following Matlab code was used to approximate the fixed point of $?(x)$ that lies between 0 and $\frac{1}{2}$, using the function r.m as shown in appendix A, and a starting point as read off from figure 2.

Listing 5: fixedpts.m

```
i = 13775;
rval = r(16,2*i);
success = 0;
for n = 16:50
    i = 2*i;
    check = 0;
    while check == 0
        rmin = rval;
        i = i+1;
        rval = r(n,i);
        if i*2.^(-n) == rval
            success = 1;
            break
        end
        if i*2.^(-n) > rval
            if (i-1)*2.^(-n) < rmin
                i = i-1;
                check = 1;
            else
                i = i-2;
            end
        end

    end
end
```

```
rval
```

with as best approximation $x = 0.420372339423223\ldots$. ($|x - ?(x)| \approx 2\cdot 10^{-13}$)

B.2 Continued fraction expansions

The following is another approach at finding the same fixed point, instead using the continued fraction representation and calculation.

Listing 6: infsum.m

```
function [que] = infsum(frac)
n = size(frac,2);
que = 0;
a = 0;
for i = 1:n
    a = a + frac(i);
    que = que + (-1)^(i-1) / 2^(a-1);
end
end
```

Listing 7: contfrac.m

```
function [value] = contfrac(frac)
n = size(frac,2);
value = 0;
for i = 1:n
    a = n+1-i;
    value = 1/(frac(a)+value);
end
end
```

Listing 8: fixedpts.m

```
frac = [2,2,1,1,1];
n = 5;
while infsum(frac)-contfrac(frac) ~= 0
    d = infsum(frac)-contfrac(frac);
    fracplus = [frac(1:(n-1)),frac(n)+1];
    dplus = infsum(fracplus)-contfrac(fracplus);
    if rem(n,2) == 0
        if dplus < 0
            frac = fracplus;
        else
            n = n+1;
            frac(n) = 1;
        end
```

```
    else
        if dplus > 0
            frac = fracplus;
        else
            n = n+1;
            frac(n) = 1;
        end
    end
end
```

with as best approximation

$$\begin{aligned} x_1 &= [0; 2, 2, 1, 1, 1, 3, 2, 3, 1, 2, 4, 1, 1, 2, 3, 1, 2, 5, 1, 3, 2, 3, 1, 3, 1, 1, 1] \\ &= \frac{802890961}{1909951930} \\ &\approx 0.420372339423223\ldots. \end{aligned}$$

Changing the initial value to [1,1,2,1,1] gave an approximation for the upper fixed point:

$$\begin{aligned} x_2 &= [0; 1, 1, 2, 1, 1, 1, 3, 2, 3, 1, 2, 4, 1, 1, 2, 3, 1, 2, 5, 1, 3, 2, 3, 1, 3, 1, 1, 1] \\ &= \frac{1107060969}{1909951930} \\ &= 1 - x_1 \\ &\approx 0.579627660576777\ldots. \end{aligned}$$