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Extending the Double Copy for Scalar Effective Field Theories

MSc. Research Project in Theoretical Physics

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June 15, 2022

Abstract

General relativity and gauge theory provide the theoretical foundations of our current understanding of the universe, even though their field-theoretic formulations seem to share no common features. Recently, intriguing new insights from Bern, Carrasco and Johansson (BCJ) have shown that the group-theoretic and kinematic building-blocks of their scattering amplitudes share systematic relations. These relations have been proven to hold at tree-level and are conjectured to hold at loop-level, allowing gravitational amplitudes to be written as the "square", or as it is famously called, the *double copy*, of two gauge theories. Manifestations of the double copy have also been found for specific classes of exact classical solutions, and for a large web of theories involving different types of kinematic and group-theoretic information.

In this thesis, we first review the basics of the double copy and go through some relevant applications. Afterwards, we present a new double copy covariant formulation for the field equations and non-linear symmetries of a triplet of scalar effective field theories describing the physics of Goldstone modes. These involve the *non-linear sigma-model* (possibly coupled to gravity), (multi-field) *Dirac-Born-Infeld theory* and the *special Galileon*. Specifically, we show that their non-linear symmetries can be tuned to have the same type of terms, facilitating a mapping between the symmetries and field equations by systematically interchanging group-theoretic and kinematic information. In addition to this, we point out intriguing relations between their classical non-linear solutions, which all take the form of a (generalized) hypergeometric series. Finally, we investigate the double copy structure of their tree-level amplitudes and we outline mapping relations between their building-blocks. The results highlight that these theories share the same underlying structure, expressed in different flavour and kinematic spaces.

Contents

1 Introduction and Research Questions

During the last decade, two of the most successful physical theories have been tested by the largest experiments that have ever been performed. The first of these two events took place in 2015, when the ATLAS and CM collaborations discovered the Higgs particle in the Large Hadron Collider at CERN, located in Geneva [1, 2]. This discovery, among many others, has enforced the credence in the standard model, which describes three of the four fundamental forces (the weak, strong, and electromagnetic force), as a well-tested theory. Five years later, the Laser Interferometer Gravitational Observatory (LIGO) collaboration [3] verified Einstein's theory of general relativity with the first detection of gravitational waves originating from merging black holes. These great successes could have never found place without the huge effort of theoretical physicists who devote their careers developing theories that describe nature at every possible scale.

Although both theories are very successful, certain aspects are not fully understood yet. For example, astrophysicists have found experimental evidence for dark matter and dark energy, although both of these have no place in the current formulation of the SM. On the other hand, GR breaks down when quantum effects become important, such as at the centre of a black hole, where the curvature becomes singular, making no sense from a physical point of view. One of the biggest open questions in theoretical physics is to find a unified theory which seamlessly connects the SM and GR, resolving these issues. By now it is clear that the conventional methods will probably not lead us there, and that we will have to come up with radically new ideas such as new principles and symmetries.

The standard model is a non-abelian gauge theory, which is successfully formulated in the formalism of quantum field theory (QFT). In the conventional QFT approach, a theory is defined by thy the action which characterizes every possible state of the theory. The connection between theory and experiment is made by scattering amplitudes, which are the probability amplitudes associated with a particular particle scattering process. Since scattering amplitudes are physical observables, they have the property that they are independent of the (often) large redundancies of the theory, such as the gauge (or diffeomorphism) choice and field basis.

Following the textbook approach, one usually calculates scattering amplitudes from Feynman diagrams which serve as a diagrammatic representation of the process. Feynman diagrams schematically work as follows: given a Lagrangian, one constructs a set of so-called Feynman rules, depending on the specific interactions of the theory. Subsequently, one draws diagrams for every possible way that the scattering process can take place, and then one assigns, according to the Feynman rules, a certain value to each diagram. By summing up all individual contributions, one finally obtains the scattering amplitude. However, the problem with this method is that it is only manageable for the simplest theories and processes. For higher order processes (i.e., with more external particles), or more computationally intensive theories such as GR, the number of Feynman diagrams and the number of terms per diagram rapidly increase. As an example, we consider the n-

particle amplitude for *maximal helicity violating (MHV)* gluons[1]. Conservation of helicity refers to the conservation of total helicity, i.e., $\sum_{i=1}^{n} h_i = 0$, where h_i is the helicity of external particle i. In fact, for scattering processes, total helicity is not necessarily a conserved quantity. MHV refers to the helicity configurations for which the sum takes the maximum value, while the amplitudes are non-vanishing. For gauge theory, MHV configurations are configurations for which all but two gluons have the same helicity [4]. The rapid increase in the number of Feynman diagrams for MHV gluons can be seen in the table below.

n	4	5	6	7	8	9	10	...
Number of diagrams	4	25	220	2485	34300	559405	10525900	...

Table 1: The total number of Feynman diagrams that needed to compute the n-point (i.e., involving n external particles) scattering amplitude of MHV gluons [5].

In the early sixties, the *S-Matrix program* was initiated to calculate scattering amplitudes without relying on non-observable quantities; instead of this, people constructed scattering amplitudes by requiring simple fundamental assumptions such as unitarity, Lorentz invariance, analyticity and causality. The main goal of this program was to avoid the divergences that typically arise in perturbative QFT calculations. However, this program quickly came to an end when quantum chromodynamics (QCD) was able to solve the problems that plagued the field-theoretic description of the strong interaction. A major role in this breakthrough was played by Dutch physicist G. ‘t Hooft, who has shown that non-abelian gauge theory is renormalizable. For his contributions to the field, he received the 1999 Nobel prize, shared with his PhD supervisor M.J.G. Veltman.

However, a modern reincarnation of the S-matrix program was recently initiated with the arrival of new methods that were developed with the aim to remove redundancies of the QFT approach. The powerful computational methods that were developed have shown that certain scattering amplitudes take a much simpler form than the Feynman-diagrammatic approach would suggest. A very important insight came from Parke and Taylor, who found that the n-point MHV gluon tree-level amplitude can be expressed in a particularly compact form.[2] Taking all external legs except legs 1 and 2 to have positive helicity (as indicated by the superscripts), the n-point MHV tree-level amplitude can be written as [4]

$$A\left(1^-2^-3^+\dots n^+\right) = ig^{n-2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n-1n\rangle\langle n1\rangle}\,. \tag{1}$$

In the above expression, we used the notation $\langle ij\rangle = \lambda_{i\alpha}\lambda_{j\beta}\epsilon^{\alpha\beta}$, where $\lambda_{i\alpha}\lambda_{j\beta}$ is the outer

[1]Recall that helicity is the projection of spin along the direction of the particle momentum, that is, $h_i \equiv \frac{\mathbf{s_i}\cdot\mathbf{p_i}}{|\mathbf{s_i}\cdot\mathbf{p_i}|}$, where $\mathbf{s}_i$ and $\mathbf{p}_i$ respectively denote the spin and momentum of particle i.

[2]Tree-level refers to Feynman diagrams without loops: this means that quantum effects (i.e., the creation and annihilation of virtual particles) are not taken into account.

product of two spinors and $\epsilon^{\alpha\beta}$ is a polarization tensor (see [4] for details on this notation). The resulting amplitude is much simpler than one would anticipate from the Feynman diagrammatic approach, as can be seen in table 1. This difference shows that there are many cancellations among the individual terms of the Feynman diagrams; it therefore implies that the Feynman diagrammatic approach is ineffective for certain calculations. The main reason for this difference is related to the fact that individual Feynman diagrams are gauge dependent and contain off-shell propagators.

When one passes from Feynman diagrams to scattering amplitudes, all this nonphysical information is lost. This does, however, not explain why such a compact final result is obtained. Motivated by the above, and the increasing demand for precise calculations to verify experimental results, the research in so-called *on-shell analytical techniques*[3] has developed enormously. These techniques have revealed many hidden properties of scattering amplitudes that are not manifest in the field-theoretic formulation. One of the major discoveries in this field took place in the year 2004, when Britto, Cachazo, Feng and Witten (BCFW) discovered recursion relations [6, 7]. These relations state that higher-point tree-level amplitudes of certain theories, including GR and YM, can be written in terms of sums over products of lower-point tree-level amplitudes. These relations imply that the higher order (that is, higher than the leading three-point) interaction vertices of these theories only serve the purpose of making the gauge invariance manifest, and that they are in principle unnecessary to build the physical observables.

In the conventional QFT formulation, GR and YM theory seem to be two completely orthogonal theories, due to their different types of symmetries, fields and (non)-renormalization properties. However, their scattering amplitudes turn out to exhibit unexpected relations. This idea was first discovered in string theory, when Kawai, Lewellen, and Tye (KLT) found out that the tree-level amplitudes of gravity can be written as a product of two (possibly distinct) Yang-Mills (YM) tree-level amplitudes [8]. Years later, Bern, Carrasco, and Johannson (BCJ) came up with a more general and direct generalization of the KLT relations, when they realised that these follow from a duality between the colour and kinematic structures of these amplitudes [9, 10, 11]. BCJ found that all YM and gravity tree-level amplitudes can be organised in such a way that the numerators consist of two building blocks, or as it is often referred to, BCJ numerators. For YM theory there are two distinct building blocks, consisting of a so-called *colour factor*, containing (products of) structure constants, and a so-called *kinematic numerator*, depending on momenta and polarization. On the other hand, they found the remarkable result that the corresponding gravity amplitude can be built out of two of these kinematic numerators. Hence, given the YM n-point amplitude, the replacement of the (n-point) YM colour factor by a copy of the (n-point) kinematic numerator results in the gravity n-point amplitude and BCJ therefore referred to it as the *"double copy"*.

For this double copy procedure to work, it is crucial that the amplitudes can be expressed in terms of purely cubic diagrams (i.e., diagrams with only three-point vertices), with a colour factor and kinematic numerator associated with each diagram. This implies

[3] On-shell analytical techniques refer to analytical methods that are used to compute physical observables (see e.g. [4] for a comprehensive review)

that this diagrammatic description is different from the Feynman diagrammatic description where higher-point vertices are allowed. To be more specific, the n-point tree-level amplitude of YM can then be written as the following sum over cubic graphs i:

$$\mathcal{A}_n^{\text{tree}} = \sum_{i \in \text{ cubic}} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}\,, \tag{2}$$

where c_i and n_i are the colour and kinematic factors respectively. Furthermore, the denominator is the product of all propagators of the corresponding diagram. The colour and kinematic numerators must both be made to satisfy similar Jacobi identities and anti-symmetry properties; this is often referred to as *colour-kinematics (CK) duality*. Whenever this duality is manifest at e.g. the gauge theory side, then the double copy can be used to obtain the corresponding gravity amplitude. Naturally, this construction also works in the opposite direction.

By application of the BCFW recursion relations, the double copy is proven to hold at tree-level [10], and for certain nontrivial examples it has also been shown to hold at loop level [11, 12, 13]. The double copy does not only allow us to write gravity as "Yang-Mills squared", but it in fact relates a whole web of theories that are seemingly unrelated in the conventional field-theoretic formulation [4, 14]. The scattering amplitudes of these different theories are built out of different combinations of BCJ-numerators, all containing different forms and combinations of kinematic and group-theoretic information. An overview of the theories with a double copy formulation can be found in e.g. [14, 15].

The incredible effectiveness of the double copy can be understood by realising the fact that YM only contains three- and four-point interaction vertices, whereas GR has infinitely many higher-point interaction vertices (see figure 1). The latter, in combination with the complicated non-linear tensor structure of GR, have as a consequence that conventional calculations of GR amplitudes are much more computationally involved. These long calculations on the gravity side can, fortunately, be omitted by calculating the corresponding (much simpler) YM counterpart and using the double copy.

Naturally, the question arises whether an analogue of this double copy prescription also exists at the off-shell level, involving e.g. (classical) equations of motion and solutions. Recent studies (see for example [16]) have shown that double copy relations exist for certain perturbative classical solutions of GR and YM, where the polarizations of the external particles are identical. Perhaps this is not so surprising, since it turns out that tree-level scattering amplitudes can be extracted from the individual terms of a perturbative expansion of the classical solution. Double copies between exact classical solutions have also been found to exist for a highly restricted set of so-called *Kerr-Schild* solutions [17, 18, 19, 20]. For these solutions, the gravitational space-time metric must admit a specific form that linearizes the otherwise highly non-linear Einstein field equations. Despite of the success of finding exact solutions, these solutions are very special in the sense that they do not easily describe the systems that we naturally consider for the amplitudes double copy (which we will often refer to as the BCJ double copy). An example of

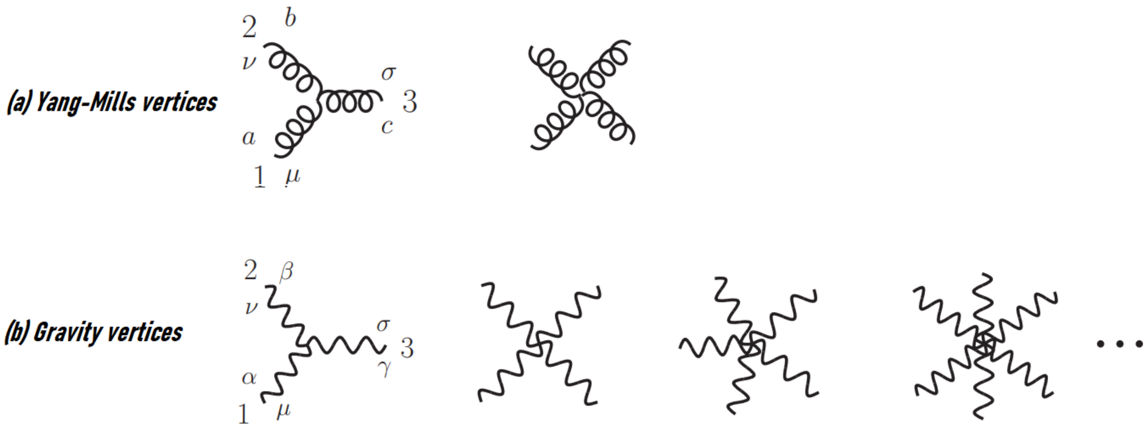


Figure 1: Interaction vertices contributing to (a) YM Feynman diagrams and (b) gravity Feynman diagrams. Note that each external gluon (indicated by the number) has one colour index and one Lorentz index, whereas the gravitons have two Lorentz indices. This figure is taken from [15].

this unnaturalness is that the non-linearities of Yang-Mills theory, which are related to gluon-gluon interactions, are completely absent. The Kerr-Schild double copy therefore only maps linear solutions from theory to theory.

The fact that there currently are limited examples of classical double copies, in combination with the fact that all of these examples omit the mapping non-linearities, brings us to the first research question of this thesis:

> *"Does there exists a double copy formulation that maps complete equations of motion and their non-linear solutions from theory to theory?"*

Instead of immediately trying to double copy non-linear solutions between YM and GR, we will investigate this in a simpler setting. Specifically, we investigate the (classical) double copy relations for three so-called *exceptional effective scalar field theories*. Effective field theories are simplifications of physical theories that describe their behaviour at certain energy scales. For the theories discussed in this thesis, this only involves the infrared regime, or in other words, the low energy regime.

The theories that will be central in this thesis are the (gravitationally-coupled) *non-linear sigma model (NLSM)*, *(multi-field) Dirac-Born-Infeld theory (DBI)*, and the *special Galileon (SG)*; these theories all describe the physics of massless Goldstone modes associated with the breaking of (different) internal symmetries. There are many motivations to investigate these theories, and the (perhaps) most important motivations are the following: (1) scalar theories facilitate mathematical calculations, (2) it is well known that their amplitudes obey double copy relations [4], and (3) these theories are invariant under specific non-linear symmetry transformations that are associated with spontaneous symmetry breaking patterns. In chapter 6, we will show that these non-linear symmetries, which are in fact global symmetries with consequences for the S-matrix [21, 22, 23, 24], can be cast in a very similar form, revealing off-shell double copy relations between their equations of motion and non-linear symmetry transformations.

Previous research on the double copy formulation of the triplet of theories has shown that their amplitudes can be constructed out of various types of BCJ numerators. Besides the YM colour factors of equation (2), this involves a scalar version of kinematic numerators (purely depending on momentum contractions), which we refer to as *scalar-kinematic numerators*, and another type of numerator that depends on both scalar-kinematics and group-theoretic information related to the fundamental representation of the e.g. special orthogonal group. Although research has been devoted to the structure of these numerators, little is known about their systematic structure and its generalization to higher order. Analogous to the off-shell mapping, it would be interesting to see whether there also exist systematic relations between the individual numerators; this leads to our second and last research question,

"Can we relate the different types of BCJ numerators in a systematical way?"

Previous research has already partially answered these questions. For instance, it is widely known that the YM colour factors can be written in terms of products of structure constants, with one additional structure constant for each additional external particle (see e.g. [15]). Furthermore, (rather complicated) mapping relations between the YM colour and kinematic numerators were already pointed out in [25]. Nevertheless, for the other types of numerators of the scalar EFTs, there is much left to discover.

The above questions will be addressed in chapter 7, where we impose algebraic constraints on BCJ numerators to explicitly construct the flavour factors and scalar-kinematic numerators, up to and including six-point. Furthermore, we identify the scalar EFTs that can be constructed using these numerators; interestingly, we find that the BCJ factorization of two flavour factors leads to the inclusion of graviton exchange for the NLSM. Analogous to the off-shell mapping as described before, we find that the flavour factors can be mapped onto the scalar-kinematic numerators by simple substitutions.

The existence of these relations at the on- and off-shell level implies that the three theories have the same underlying structure, expressed in different flavour and kinematic spaces.

Before we go into detail, let's concisely outline the content of this thesis. In **chapter 2**, we will start with a review of the duality between colour and kinematics. This will be done by studying the examples that originally led to the discovery of the double copy, relating the amplitudes of YM and GR. Thereafter, we review the most important relations between *partial amplitudes* (i.e., the part of an amplitude corresponding to a fixed colour configuration) and illustrate the motivation behind the double copy by comparison of the computationally intensive perturbative GR with its much simpler YM counterpart. Finally, we state the formal formulation of the amplitude double copy at tree-level.

In **chapter 3**, we show how the double copy manifests itself for a specific set of Kerr-Schild double copies and we review this construction for the Schwarzschild and Kerr black holes.

In **chapter 4**, we illustrate a perturbative method that allows classical solutions to be expressed in terms of connected correlation functions (or correlators), and using the LSZ method we show that tree-level amplitudes can easily be extracted from these correlators. This method not only emphasizes the close connection between tree-level amplitudes, classical solutions and the role of redundancy in field-theoretic formulations, but it also serves as a powerful computational tool that will be often used in chapters five and six.

In **chapter 5**, we derive the triplet of scalar EFTs by considering a general Lagrangian with a certain amount of derivatives and a so-called *enhanced soft limit*, serving as additional physical input. Furthermore, using the perturbation theory formalism of section 4, we explicitly show the amplitude double-copy relations between these theories as they are already known in the literature.

In **chapter 6**, we turn to our first research question: we construct the aforementioned off-shell double copy formulation for the triplet of scalar EFTs. In addition to this, we derive their classical non-linear solutions and point out intriguing relations between them.

Finally, in **chapter 7**, we turn to our second research question: we investigate the structure of the relevant BCJ numerators and amplitudes, and we outline the mapping relations between them.

The results of the latter two chapters follow from a collaborative effort between Diederik Roest (the first supervisor of this research project), Dijs de Neeling (PhD student at the University of Groningen) and the author of this thesis. Currently, a preprint of the resulting article is available via [26].

2 Scattering Amplitudes and the BCJ Double Copy

In this chapter, we will review the amplitude double copy which explicitly manifests the squaring relation between YM and gravity amplitudes.[4] We will do this by reviewing the main concepts that underlie the double copy of scattering amplitudes, loosely following [15, 19, 20, 27]. We will mostly illustrate these ideas on the hand of the few examples that initially lead to its discovery. This involves YM, governing the interactions between gluons, and on the other hand, GR, governing the interactions of the (hypothetical) gravitons.

2.1 Yang-Mills theory and colour-kinematics duality

YM theory is a gauge theory with symmetry group $SU(N)$, describing the interaction of massless spin-one fields A^a_μ (the gluons) that live in the adjoint representation of $SU(N)$. The importance of YM theory follows from the fact that the standard model is essentially described by a gauge theory with product group $U(1)\otimes SU(2)\otimes SU(3)$, respectively referring to the symmetry groups of the electromagnetic, weak, and strong interactions. The YM Lagrangian is given by [4]

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}\,\text{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right), \tag{3}$$

[4] With "amplitude" we always refer to tree-level amplitude, unless stated otherwise.

where the field strength tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right], \tag{4}$$

with g being the coupling constant. The gauge field $A_\mu(x)$ transforms in the adjoint representation of the Lie group, which implies that the Lagrangian (3) is invariant under the *gauge transformation*

$$\delta A_\mu^a = \partial_\mu \varepsilon^a + g f_{abc} \varepsilon^b A_\mu^c, \tag{5}$$

where the colour indices of the gauge field can be contracted with the generators via

$$A_\mu(x) = A_\mu^a(x) T^a. \tag{6}$$

The Lie-algebra of the gauge group $SU(N)$ has a total of $N^2 - 1$ generators, implying e.g. that the $SU(3)$ YM theory involves eight different "coloured" gluons. The generators obey the commutation relation

$$\left[T^a, T^b\right] = i f^{abc} T^c, \tag{7}$$

where f^{abc} are the structure constants of the Lie algebra. Naturally, these structure constants are anti-symmetric under the exchange of any two indices; additionally, they satisfy the *Jacobi identity* [15]

$$f^{abe} f^{ecd} + f^{bce} f^{ead} + f^{cae} f^{ebd} = 0. \tag{8}$$

The above relations allow us to rewrite the field strength (4) as

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \tag{9}$$

which implies that the Lagrangian (3) can be rewritten as

$$\mathcal{L}_{YM} = \frac{1}{2} A_\mu^a \Box A^{a\mu} + \frac{1}{2} \partial^\mu A^{\nu a} \partial_\nu A_\mu^a - g f^{abc} \left(\partial_\mu A_\nu^a\right) A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{abe} f^{cde} A_\mu^a A_\nu^b A^{c\mu} A^{d\nu}. \tag{10}$$

The attentive reader might have realized that the field strength (4) without the commutator term is equivalent to the field strength of Maxwell's equations of electrodynamics. In fact, if one considers YM with the abelian gauge group $U(1)$, then the generators of the Lie algebra commute and the last term of (4) vanishes, leading to Maxwell's equations of electrodynamics.[5] An important difference between abelian and non-abelian gauge theories is that the excitations of the former do not self-interact, whereas the excitations of the latter do. This can be seen as follows: due to the absence of cubic or higher order terms (i.e., terms containing three or more fields) in the electrodynamics Lagrangian, it follows that the equations of motion are linear in the field, which implies that superpositions of field configurations are also solutions of the equations of motion, and we therefore have that photons can pass through each other without interacting (see figure 2*a*). On the other

[5] The unitary group of one-dimensional matrices, denoted $U(1)$, is referred to as the abelian Lie group due to the fact that it is the only commutative Lie group.

hand, the equations of motion for gluons are non-linear, owing to the cubic and quartic terms in the Lagrangian (10). Therefore, gluons (of different colour) can interact with each other; this is visualized in figure 2*b* and 2*c* for contact and exchange interactions, respectively.

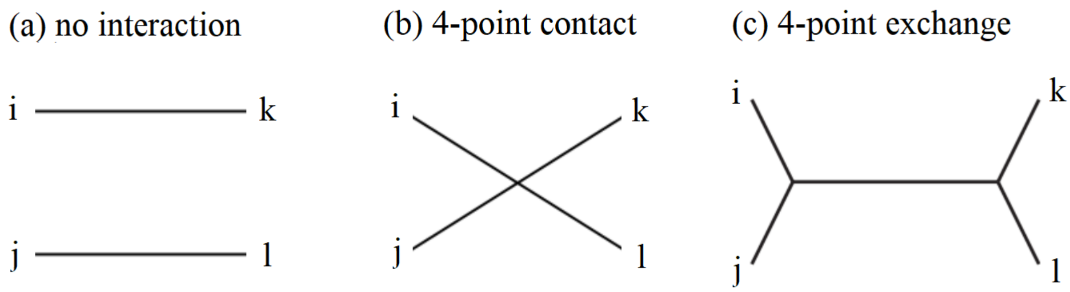


Figure 2: Maxwell's electrodynamics is quadratic in the gauge field, meaning that photons propagate without interacting, as visualized for a $2 \to 2$ *scattering process in* (*a*)*. On the other hand, non-abelian gauge theories involve up to quartic terms in the gauge field, giving rise to four-point Feynman diagrams of the following types: contact diagrams with one quartic vertex* (*b*) *and exchange diagrams with two cubic vertices* (*c*)*.*

In order to compute physically observable quantities, one should first remove the redundancy of the theory by performing so-called gauge fixing. Gauge fixing amounts to the addition of a term of the form $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\xi^{-1}G^a G^a$ to the Lagrangian. One of the most popular choices is the *Lorentz gauge*, corresponding to $\xi = 1$ and $G^a = \partial^\mu A^a_\mu$. Using this gauge choice, the Lagrangian (10) takes the particularly simple form

$$\mathcal{L} = \frac{1}{2} A^a_\mu \Box A^{a\mu} - g f^{abc} \left(\partial_\mu A^a_\nu\right) A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{abe} f^{cde} A^a_\mu A^b_\nu A^{c\mu} A^{d\nu} \,, \tag{11}$$

leading to the field equation

$$\Box A^a_\nu + g f^{abc} A^{b\mu} \left(2\partial_\mu A^c_\nu - \partial_\nu A^c_\mu\right) + g^2 f^{abc} f^{cde} A^{b\mu} A^d_\mu A^e_\nu = 0 \,. \tag{12}$$

In this form, we can easily derive the Feynman rules, e.g., the gluon propagator is given by the compact expression

$$\Delta^{ab}_{\mu\nu}(p) = -i \frac{\delta^{ab} \eta_{\mu\nu}}{p^2} \,, \tag{13}$$

where p^μ is the gluon four-momentum. We consider all external particles to be incoming; this implies that momentum conservation for a n-particle scattering event reads

$$\sum_{i=1}^{n} p^\mu_i = 0 \,. \tag{14}$$

Upon using this convention, we find that the three-point vertex function is given by

$$V^{\mu_1\mu_2\mu_3}_{3,a_1a_2a_3} = g f^{a_1a_2a_3} \left\{ \eta^{\mu_1\mu_2} \left(p_1 - p_2\right)^{\mu_3} + \eta^{\mu_2\mu_3} \left(p_2 - p_3\right)^{\mu_1} + \eta^{\mu_3\mu_1} \left(p_3 - p_1\right)^{\mu_2} \right\} , \tag{15}$$

where g is the coupling constant. Furthermore, the four-point vertex factor is given by

$$\begin{aligned} V_{4,a_1a_2a_3a_4}^{\mu_1\mu_2\mu_3\mu_4} = -ig^2 \{ & f^{a_1a_2e} f^{a_3a_4e} (\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}) \\ & + f^{a_1a_4e} f^{a_2a_3e} (\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}) \\ & + f^{a_1a_3e} f^{a_4a_2e} (\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} - \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}) \} \,. \end{aligned} \tag{16}$$

Note that the higher-point (≥ 5) vertices vanish, which can be easily seen from the fact that the Lagrangian contains at most four powers in the gauge field.

Since we are working with massless spin-one particles in four space-time dimensions, the polarizations are states living in the fundamental representation of the Lie group $SO(2)$, which means that we have two polarization states (one with positive and one with negative helicity) that can be embedded into the four-dimensional polarization vectors $\varepsilon_\mu(p)$. To calculate physical observables, the external particles must satisfy so-called *on-shell conditions*, meaning that the physical system obeys the classical equations of motion. For massless particles the momenta are required to satisfy

$$p_i \cdot p_i = 0 \,, \quad \sum_{i,j=1}^{n} p_i \cdot p_j = 0 \,, \tag{17}$$

where the latter follows from momentum conservation $p_1 + p_2 + \ldots + p_n = 0$. Furthermore, the polarization vectors satisfy the constraints $\varepsilon_i^\pm \cdot p_i = 0$ and $\varepsilon_i^\pm \cdot \varepsilon_j^\pm = 0$. Using the three-point vertex function and these on-shell conditions, we find that the three-point tree-level YM amplitude is given by

$$\begin{aligned} \mathcal{A}_3 &= igf^{a_1a_2a_3} \{ (\varepsilon_1 \cdot \varepsilon_2) [\varepsilon_3 \cdot (p_1 - p_2)] + (\varepsilon_2 \cdot \varepsilon_3) [\varepsilon_1 \cdot (p_2 - p_3)] + (\varepsilon_3 \cdot \varepsilon_1) [\varepsilon_2 \cdot (p_3 - p_1)] \} \\ &= -2igf^{a_1a_2a_3} \varepsilon_{\mu_1} \varepsilon_{\mu_2} \varepsilon_{\mu_3} \left(\eta^{\mu_1\mu_2} p_2^{\mu_3} + \eta^{\mu_2\mu_3} p_3^{\mu_1} + \eta^{\mu_3\mu_1} p_1^{\mu_2} \right) . \end{aligned} \tag{18}$$

The four-point tree-level amplitude follows from two types of Feynman diagrams, namely, contact diagrams and exchange diagrams. These two types are visualized in figure 3.

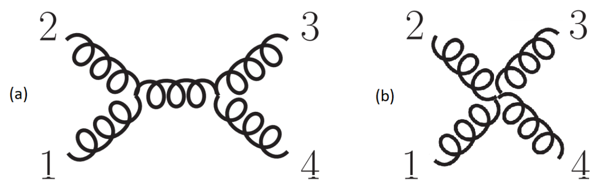


Figure 3: Two types of Feynman diagrams contributing to the YM four-point amplitude. On the left we have diagrams with intermediate gluon exchange, and on the right we have contact diagrams.

Upon cyclically permuting three external legs of the exchange diagram, while keeping one external leg fixed, we obtain a total of three (inequivalent) diagrams. These diagrams are referred to as the s, t and u-channel contributions respectively. The names of these channels are related to the momenta squared of the intermediate particle, in this case corresponding to the Mandelstam variables s, t and u, which are defined as

$$s=(p_1+p_2)^2\,,\quad t=(p_1+p_3)^2\,,\quad u=(p_1+p_3)^2\,. \tag{19}$$

To be a bit more specific, replacing $1\to 2\to 3\to 1$, while leg 4 is left untouched, corresponds to the replacement $s\to t\to u\to s$ of the Mandelstam variables. Using Mandelstam variables, we can summarize the four-point kinematics for massless on-shell particles as follows,

$$s+t+u=0\,,\quad s_{12}=s_{34}\,,\quad s_{13}=s_{24}\,,\quad s_{14}=s_{23}\,, \tag{20}$$

where $s_{ij}=(p_i+p_j)^2$ with $i,j=1,\ldots,4$.[6] The tree-level amplitude contribution of the exchange diagrams can then be written as

$$\begin{aligned}-i\frac{g^2}{s}f^{a_1a_2e}f^{a_3a_4e}&\Big((\varepsilon_1\varepsilon_2)(p_1-p_2)^\alpha+2\varepsilon_2^\alpha(\varepsilon_1p_2)-2\varepsilon_1^\alpha(\varepsilon_2p_1)\Big)\\&\times((\varepsilon_3\varepsilon_4)(p_3-p_4)_\alpha+2\varepsilon_{4\alpha}(\varepsilon_3p_4)-2\varepsilon_{3\alpha}(\varepsilon_4p_3))+\ldots\,,\end{aligned} \tag{21}$$

where the dots denote the t and u-channel contributions. In addition, we have the contact diagram (see figure 3), whose contribution to the amplitude reads

$$ig^2\left\{\frac{s}{s}c_s\left[(\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)-(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4)\right]\right\}+\ldots\,, \tag{22}$$

where we manually inserted a factor s/s. The latter amounts to the so-called *splitting procedure* of the contact terms. This procedure allows us to write the total amplitude as a sum over three terms of the form $\frac{nc}{D}$, where D are the propagators (here corresponding to the Mandelstam variables). Furthermore, n and c are numerators containing purely colour and kinematic information respectively. The polarization stripped four-point amplitude can finally be written as a sum over purely cubic diagrams,

$$\mathcal{A}_4=g^2\left\{\frac{n_sc_s}{s}+\frac{n_tc_t}{t}+\frac{n_uc_u}{u}\right\}\,, \tag{23}$$

where a colour factor and a kinematic numerator is assigned to each Mandelstam variable, i.e., for the colour numerators we have[7]

$$c_s=f^{a_1a_2e}f^{a_3a_4e},\quad c_t=f^{a_1a_4e}f^{a_2a_3e},\quad c_u=f^{a_1a_3e}f^{a_4a_2e}\,, \tag{24}$$

and the polarization-stripped s-channel kinematic numerator finally reads

$$\begin{aligned}n_s=&\Big[\Big(-\frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\lambda}p_1\cdot p_3-\eta^{\gamma\lambda}{p_2}^\alpha{p_3}^\beta-\eta^{\alpha\beta}{p_1}^\lambda{p_4}^\gamma-2\eta^{\beta\lambda}{p_2}^\alpha{p_4}^\gamma\Big)\\&-(1\leftrightarrow 2)-(3\leftrightarrow 4)+(1\leftrightarrow 2,3\leftrightarrow 4)+\Big(\eta^{\alpha\lambda}\eta^{\beta\gamma}-\eta^{\alpha\gamma}\eta^{\beta\lambda}\Big)p_1\cdot p_2\Big]\,.\end{aligned} \tag{25}$$

[6]Note that $s_{ii}\propto p_i^2$ vanishes on-shell, and that $s_{ij}\equiv s_{ji}$, so we really have $3!=6$ *dependent* kinematic invariants.

[7]Note that the correct notation for the colour factors and kinematic numerators, $(c_s)^{a_1a_2a_3a_4}$ and $(n_s)^{\alpha\beta\gamma\lambda}$ respectively, is suppressed for notational simplicity.

Like before, the t and u-channel kinematic numerators can be obtained by cyclic permutation of three of the four external legs. The key point of this splitting procedure is that we have rearranged the amplitude in terms of a set of purely cubic diagrams, see figure 4.

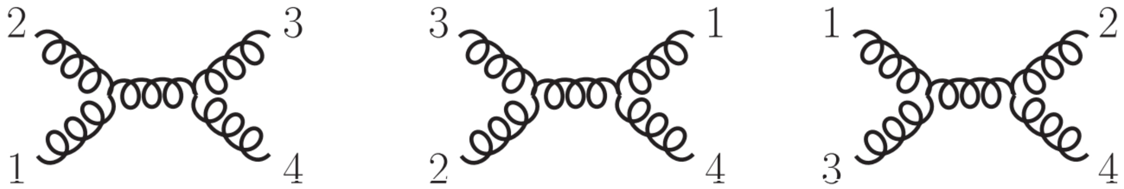


Figure 4: Cubic s,t and u-channel diagrams (from left to right), contributing to the YM four-point amplitude. The curly lines represent gluons. As explained in the text, it is important to note that these "BCJ diagrams" are different from Feynman diagrams. This figure was taken from [15].

It turns out that many field theories allow their higher-point ($n \geq 4$) tree-level amplitudes to be expressed in terms of purely cubic diagrams [15]. As will soon become clear, this is property is satisfied by all theories that have a double copy formulation.

The above discussion naturally extends to higher order. For instance, at five-point, where we have a total number of 15 distinct BCJ diagrams, following from $5! = 25$ distinct Feynman diagrams of two different types. Firstly, we have 15 cubic diagrams (see figure 5a, and secondly we have 10 diagrams with one cubic and one quartic vertex (see figure 5b. The latter number of diagrams can be understood as follows: given a diagram with four external legs and two three-point vertices, we have two possible vertices to attach the fifth leg, resulting in the diagram of figure 5b, or its mirrored counterpart. Either of these choices gives rise to five inequivalent permutations, adding up to a total number of 10 distinct diagrams.

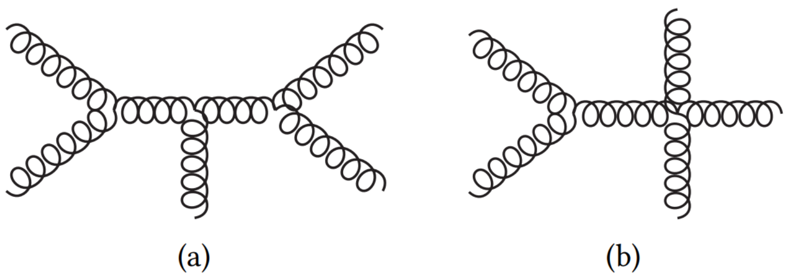


Figure 5: Two types of Feynman diagrams that contribute to the YM five-point amplitude: (a) cubic diagrams and (b) diagrams with one cubic and one quartic vertex [27]. Inequivalent permutations must be taken into account when considering the full amplitude.

In this case, the splitting procedure works analogously to the four-point case. For instance, we have one contribution for each of the three colour structures that one can write down for the diagram in figure 5b. Including its mirrored counterpart, this adds up to six. Going through all five inequivalent permutations, we eventually find a total number of 30 contributions, which implies that each of the 15 colour factors (one per cubic diagram) obtain two contributions from the diagrams with a quartic vertex. At five-point, the kinematic factors are more complicated higher-point generalizations of four-point kinematic factors (25), while the 15 colour factors naturally take the simple form

$$c_i = f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5}\,, \quad c_j = f^{a_1 a_2 b} f^{b a_4 c} f^{c a_5 a_3}\,, \quad c_k = f^{a_1 a_2 b} f^{b a_5 c} f^{c a_3 a_4}\,, \tag{26}$$

where each of the above factors contributes five inequivalent permutations.
By now, it should be obvious that this new BCJ diagrammatic description is different from the Feynman diagrammatic description since we write the complete amplitudes as a sum over solely cubic graphs. For general, say n-point scattering events, the amplitude sums over a total number of $(2n-5)!!$ cubic graphs.[8] This number can be understood as follows: cubic diagrams with $n = j+1$ external legs can be built from a j-point diagram by attaching an external leg to any of the $2j-3$ edges of the j-point diagram. By induction, we find the result

$$\#\text{diagrams} = 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2j-3) = (2n-5)!!\,. \tag{27}$$

Hence, for $n=5$, we indeed find a total number of $5!! = 15$.

Returning to the previously discussed four-point example, it is easy to see that the colour factors (24) satisfy the Jacobi identity

$$c_s + c_t + c_u = 0\,. \tag{28}$$

In addition to this, it was first realised by [9] that the kinematic numerators (25) satisfy a similar algebraic relation, which is often referred to as the *"kinematic Jacobi identity"*,

$$n_s + n_t + n_u = 0\,, \tag{29}$$

where the on-shell conditions (17) were imposed. The Jacobi identity (28) and its kinematic analogue (29) are schematically visualized in figure 6. These algebraic conditions naturally extend to higher order and a detailed discussion on this will follow in chapter 7.

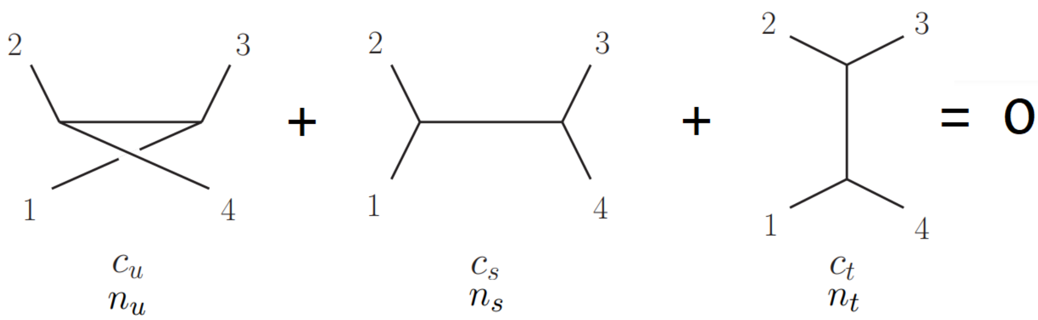


Figure 6: Schematic visualization of the colour and kinematic Jacobi identities. The sum of s,t and u-channel BCJ numerators, here denoted by c_i and n_i below the corresponding diagram, vanishes on-shell.

[8] Note that the double factorial is different from the factorial iterated twice. The double factorial can be written as $n!! = \prod_{k=0}^{\left[\frac{n}{2}\right]-1}(n-2k) = n(n-2)(n-4)\cdots$, where the product is taken over all integers ranging from 1 up to n, including only integers of the same parity as n.

Similar relations hold for the higher-point factors, such as (26) and their kinematic counterparts. The fact that both types of numerators satisfy similar relations indicates some sort of algebraic similarity between colour and kinematics, despite their radically different group-theoretic and kinematic natures. This provides one of the simplest examples of colour-kinematics duality. Furthermore, the kinematic numerators are not unique in the sense that they mix via so-called (possibly non-local) *generalized gauge transformations*

$$n_s' = n_s + s\alpha\,(p_i\,,\varepsilon_i)\,,\quad n_t' = n_t + \mathrm{t}\,\alpha\,(p_i,\varepsilon_i)\quad n_u' = n_u + u\alpha\,(p_i,\varepsilon_i)\,, \tag{30}$$

which leaves the amplitude invariant as the additional part is always proportional to an on-shell vanishing quantity. This invariance can be seen by substituting the above in (23). We obtain

$$\mathcal{A}_4' = \mathcal{A}_4 + (c_s + c_t + c_u)\,\alpha = \mathcal{A}_4\,, \tag{31}$$

where the latter equality follows from the Jacobi identity (138).

Both colour-kinematics duality and generalized gauge invariance are crucial aspects of the double copy between the amplitudes of different theories. As mentioned before, CK-duality was first proposed by Bern, Carrasco and Johansson [9] and it is therefore also referred to as BCJ-duality. BCJ found out that this is a general property of YM amplitudes at tree-level, in the sense that one can always find a form of the numerators such that the n-point tree-level amplitude can be written as

$$\mathcal{A}_n^{\text{tree}} = \sum_{i\in\text{ cubic}} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}\,, \tag{32}$$

where the sum is taken over all $(2n-5)!!$ cubic diagrams. The generalized gauge transformations (30) represents a gauge transformation, possibly combined with a field-redefinition or the addition of higher derivative terms that leave the theory invariant by force of the group-theoretic Jacobi identity (138). In addition to the colour and kinematic Jacobi identities, BCJ later proposed that the colour and kinematic numerators share equivalent anti-symmetry properties, that is,

$$c_i = -c_j \implies n_i = -n_j\,. \tag{33}$$

This can be understood as follows: if we consider two diagrams α and β, where β is related to α by the interchange of two external legs that are attached to the same vertex, then the anti-symmetry of the structure constants $f^{abc} = -f^{acb}$ implies that the colour factors are related by $c_\alpha = -c_\beta$. The anti-symmetry of the kinematic numerators then naturally follows from the Feynman rules.

Although BCJ proposed that it is always possible to find CK-dual numerators, it often is a nontrivial task to manifest the duality in the usual field-theoretic formalism. One way to obtain manifest BCJ duality is to change field basis or to add (or subtract) terms that vanish on-shell, or a combination of both. When done properly, the Feynman rules will automatically generate BCJ dual diagrams, having the same effect as specific generalized gauge transformations. However, the problem with this approach lies in the fact each subsequent order requires additional on-shell vanishing terms and field redefinitions, making it hard, if not impossible, to find a closed form Lagrangian that manifests CK-duality at any given order. An example of such a construction is given in [28], where infinitely many non-local gauge fixing terms were added to the YM Lagrangian, yielding manifest BCJ duality for all Feynman diagrams.

2.2 Relations between partial amplitudes

Before we move on to discuss the double copy, we will first briefly review the most relevant properties of colour ordered (or partial) amplitudes as these will play an important role throughout this thesis. For a YM theory with gauge group $SU(N)$, we already saw that the amplitude numerator consists of kinematic numerators multiplied by (products of) structure constants. We can express the structure constants (and thus the colour factors) in terms of traces of generators by using the relation

$$if^{abc} = \mathrm{Tr}([T^a, T^b], T^c) = \mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c)\,. \tag{34}$$

Next, we can use the *Fierz-identity*, given by [15]

$$(T^a)_i{}^j\,(T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \frac{1}{N}\delta_i{}^j \delta_k{}^l\,, \tag{35}$$

to write the higher-point colour factors in terms of traces of generators. For instance, the four-point s-channel colour factor can be written as

$$\begin{aligned} f^{a_1 a_2 e} f^{e a_3 a_4} =& \,\mathrm{tr}\,(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) - \mathrm{tr}\,(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) \\ & - \mathrm{tr}\,(T^{a_2} T^{a_1} T^{a_3} T^{a_4}) + \mathrm{tr}\,(T^{a_2} T^{a_1} T^{a_4} T^{a_3})\,, \end{aligned} \tag{36}$$

and similar considerations apply to higher orders. In the single-trace basis, the n-point amplitude can be written as [27]

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{\sigma \in S_{n-1}} A_n[1\sigma(2)\sigma(3)\ldots\sigma(n)]\,\mathrm{tr}\left(T^{a_1} T^{\sigma(a_2)} T^{\sigma(a_3)} \ldots T^{\sigma(a_n)}\right), \tag{37}$$

where the sum is taken over all $(n-1)!$ inequivalent permutations of single-traces and the $A_n[1\sigma(2)\sigma(3)\ldots\sigma(n)]$ are so-called *partial (or colour-ordered) amplitudes*; these correspond to the part of the amplitude given a fixed colour configuration of the external legs. Partial amplitudes are naturally easier to compute than the full amplitude since we only have to take one colour configuration into account. This basis of $(n-1)!$ traces is, however, over-complete as a consequence of the following symmetry relations between the partial amplitudes [20]:

1. **Cyclicity**: due to the cyclicity of the trace, meaning that only the relative ordering is relevant, we have that the partial amplitudes obey the relation

$$A_n[1,2,\ldots,n] = A_n[2,\ldots,n,1]\,.$$

2. **Reflection-(anti) symmetry**: by looking at the definition of colour-ordered amplitudes (37), we naturally observe that

$$A_n[1,2,\ldots,n] = (-1)^n A_n[n,\ldots,2,1].$$

This property can be traced back to the (anti)-symmetry of the structure constants, for instance, at three-point this can be easily seen to be the consequence of the anti-symmetry relation $f^{abc} = -f^{acb}$.

3. **Photon decoupling**: by taking all generators to be the unit matrix we obtain the relation

$$A_n[1,2,3,\ldots,n] + A_n[2,1,3,\ldots,n] + A_n[2,3,1,\ldots,n] + \cdots + A_n[2,3,\ldots,1,n] = 0.$$

If we then replace one gluon with a photon (which has unit matrix generators), the photon does not interact with the gluons, yielding a vanishing amplitude and hence leading to the above relation.

4. **Kleiss-Kluijf (KK) relations**: The KK relations were first conjectured in [11] and eventually proven in [29]. The proof is quite complicated and beyond the scope of this thesis, we therefore only state the result. First, we introduce ordered permutations $\mathrm{OP}(\alpha,\beta)$ of two sets, say α and β. The ordered permutations are all permutations of the merging of $\{\alpha\}$ and $\{\beta\}$ that maintain the original ordering of their individual elements. For example, the ordered permutations of $\{\alpha\}$ and $\{\beta\}$ could be

$$\begin{aligned}
&\{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n\}\,,\\
&\{\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots,\alpha_n,\beta_n\}\,,\\
&\{\beta_1,\ldots,\beta_j,\alpha_1,\ldots,\alpha_n,\beta_{j+1},\ldots,\beta_n\}\,,\\
&\{\beta_1,\ldots,\beta_j,\alpha_1,\ldots,\alpha_k,\beta_{j+1},\ldots,\beta_n,\alpha_{k+1}\ldots,\alpha_n\}\,.
\end{aligned} \tag{38}$$

The KK relations then state that partial amplitudes are related via

$$A_n[1,\{\alpha\},n,\{\beta\}] = (-1)^{|\beta|} \sum_{\sigma\in\mathrm{OP}\left((\alpha),(\beta)^T|\right)} A_n[1,\sigma,n]\,, \tag{39}$$

where the subsets $\{\alpha\}$ and $\{\beta\}$ contain external particles, $|\beta|$ denotes the number of particles within $\{\beta\}$, and the superscript T denotes reverse ordering of the set. The KK relations imply that partial amplitudes can be expressed as a linear combination of the $n-2!$ partial amplitudes that remain upon fixing two external legs. Hence, so far we have further reduced the number of independent partial amplitudes to $(n-2)!$.

5. **BCJ relations:** This number can be even further reduced to $(n-3)!$, by virtue of the BCJ relations [9], which are closely related to CK duality. The BCJ relations relate different partial amplitudes with three fixed external legs, and with kinematic invariants as their coefficients. The BCJ relations explicitly read [9]

$$\sum_{i=3}^{n}\sum_{j=3}^{i} s_{2j} A_n[1,3,\ldots,i,2,i+1,\ldots,n] = 0\,, \tag{40}$$

where s_{2j} are kinematic invariants given by $p_2 \cdot p_j$.

The KK and BCJ relations together give rise to the group-theoretic and kinematic Jacobi Identities that form the basis of CK duality. Initially, the BCJ relations were only conjectured in [9], however, later they have been proved many times in string-theoretic context [30], as well as in field-theoretic settings [31].

2.3 Perturbative gravity and the KLT relations

In this section, we will briefly review the theory of gravity in its traditional setting, which is the well known general GR. In the introduction, we have already mentioned that GR plays a central role in this thesis since it was initially found to be the double copy of YM. Here we will illustrate that the calculations of gravitational amplitudes are much more computationally involved compared to YM, emphasizing the peculiarities of the double copy.

In GR, the space-time metric tensor $g_{\mu\nu}$ plays the role of a spin-two field whose classical equations of motion are the so-called Einstein field equations (EFE). Famous examples of exact solutions to the EFE are the Schwarzschild and Kerr metric (for the original papers see [32] and [33] respectively), which have the physical interpretation of stationary and rotating black holes respectively. Although these exact solutions obey double copy relations with their YM counterparts, we will here focus on amplitudes instead. GR is field-theoretically described by the Einstein-Hilbert (EH) action

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R + S_{\mathrm{matter}} \ , \tag{41}$$

where g is the metric determinant, R the Ricci scalar, S_{matter} the contribution from the matter fields (if present), and $\kappa^2 = 8\pi G$ the gravitational constant.[9] Functional differentiation of the Einstein Hilbert action (41) with respect to the space-time metric $g_{\mu\nu}$ yields the EFE

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu} \,, \tag{42}$$

where the stress-energy tensor $T_{\mu\nu}$ follows from the matter term. In order to apply flat-space field-theoretical methods, we will write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \,, \tag{43}$$

where $\eta_{\mu\nu}$ is the flat background metric and $h_{\mu\nu}$ is the fluctuation over the background, which is often referred to as the graviton field. Substituting this Ansatz into the Einstein-Hilbert action allows us to expand in powers of coupling constant

$$S_{\mathrm{EH}} = S^{(0)}_{\mathrm{EH}} + \kappa S^{(1)}_{\mathrm{EH}} + \kappa^2 S^{(2)}_{\mathrm{EH}} + \dots \,, \tag{44}$$

where the ellipses denote the infinitely many higher order contributions, giving rise to interaction vertices of arbitrary order. The explicit expression can be derived by using

[9]Note that we work with the units $c = 1$.

basic mathematical manipulations such as expanding $\sqrt{-g}$ in terms of the graviton, however, for the sake of this thesis, we skip the derivation and immediately state the original result from [27]. In this paper, the authors used the so-called *de-Donder gauge*, which is defined by $\partial_\mu h^\mu_\lambda - \partial_\lambda h^\mu_\mu/2 = 0$, where the Lorentz indices of the graviton can be raised and lowered by the Minkowski metric $\eta^{\mu\nu}$. The advantage of the de-Donder gauge is that it yields a particularly simple Ricci tensor. The three-point vertex is obtained by functionally differentiating the cubic part of the action with respect to the three gravitons. This leads to the result

$$
\begin{aligned}
V_3^{\mu\nu\sigma\tau\rho\lambda}(p_1,p_2,p_3) &= \frac{\delta S^{(1)}_{\text{EH}}}{\delta h_{\mu\nu}\delta h_{\sigma\tau}\delta h_{\rho\lambda}} \\
&= \text{sym}\Big[-\frac{1}{4}P\left(p\cdot q\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\right) - \frac{1}{4}P\left(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\right) + \frac{1}{4}P\left(p\cdot q\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\right) \\
&\quad + \frac{1}{2}P\left(p\cdot q\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\right) + P\left(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho}\right) - \frac{1}{2}P\left(p^\tau q^\mu\eta^{\nu\sigma}\eta^{\rho\lambda}\right) - \frac{1}{2}P\left(p^\rho q^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\right) \\
&\quad + \frac{1}{2}P\left(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\right) + P\left(p^\sigma q^\lambda\eta^{\tau\mu}\eta^{\nu\rho}\right) + P\left(p^\sigma q^\mu\eta^{\tau\rho}\eta^{\lambda\nu}\right) - P\left(p\cdot q\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\right)\Big] ,
\end{aligned}
\tag{45}
$$

where sym implies symmetrization over the pairs of indices $\mu\nu$, $\sigma\tau$ and $\rho\lambda$. The symbol P_n indicates that summation should be performed over the n distinct permutations of the momentum-index triplets (μ,ν,p_1), $(\sigma\tau p_2)$ and $(\rho\lambda p_3)$. Hence, when fully expanding the above, we already have more than 100 terms. Furthermore, the corresponding graviton can easily be found to be given by

$$
G^{\mu\nu\sigma\tau\rho\lambda}(p^2) = \frac{(\eta_{\mu\sigma}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\sigma\tau})}{p^2} . \tag{46}
$$

Given the above, one can easily calculate the three-graviton amplitude. Similar considerations apply to higher order, where the number of terms drastically increases, making it extremely tedious to calculate amplitudes.

By comparison of the three-point GR and YM vertices, it should be obvious that perturbative gravity is much more involved in the current approach, and that there seems to be no obvious relation between their vertices. The reason for the complexity is that the diffeomorphism invariance of GR causes the vertices to have immense gauge freedom. Field redefinitions and gauge choice can drastically simplify the vertices, however, perturbative gravity remains computationally intense, even for computers. To remove the gauge redundancy we should instead require all external particles to be on-shell, requiring the conditions

$$
\varepsilon^{\mu\rho} = \varepsilon^{\rho\mu}, \quad p_\mu\varepsilon^{\mu\rho} = 0, \quad p_\rho\varepsilon^{\mu\rho} = 0, \quad \varepsilon^\mu_\mu \equiv \eta^{\mu\nu}\varepsilon_{\mu\nu} = 0 , \tag{47}
$$

where $\varepsilon^{\mu\nu}$ is the graviton polarization tensor. Imposing these on-shell conditions on the three-graviton vertex (48) reduces it to the much simpler form

$$
G_{3\mu\rho,\nu\lambda,\sigma\tau}(p_1,p_2,p_3) = -i\left[(p_1-p_2)_\sigma\eta_{\mu\nu} + \text{ cyclic }\right]\left[(p_1-p_2)_\tau\eta_{\rho\lambda} + \text{ cyclic }\right], \tag{48}
$$

and we therefore conclude that the canonical formulation of GR is not very efficient for calculations in perturbation theory.

Intriguingly, we recognise the three-graviton vertex (48) to be equal to the product of two YM three-point vertices (15). The existence of similar relations between YM and GR amplitudes was also discovered in string theory, where Kawai, Lewellen and Tye (KLT) showed that tree-level gravity amplitudes can be written as the product of two YM partial amplitudes [8].

The KLT relations state that the n-point tree-level amplitudes of closed can be written as the sum over products of open-string partial amplitudes, with coefficients depending on kinematic invariants and so-called string tension. These relations are valid in all dimensions. In the field-theory limit, which is the limit of infinite string tension (or low energies), the tree-level amplitude of a massless open-string vector becomes the YM colour-ordered partial amplitude, whereas the massless closed-string amplitude becomes the graviton amplitude. At four and five-point, the KLT relations explicitly read

$$\begin{aligned} M_4^{\text{tree}}(1,2,3,4) &= -is_{12}A_4^{\text{tree}}(1,2,3,4)\tilde{A}_4^{\text{tree}}(1,2,4,3)\,, \\ M_5^{\text{tree}}(1,2,3,4,5) &= -is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)\tilde{A}_5^{\text{tree}}(2,1,4,3,5) \\ &\quad + is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\tilde{A}_5^{\text{tree}}(3,1,4,2,5)\,, \end{aligned} \tag{49}$$

where M_n are n-point graviton amplitudes and A_n are colour ordered YM amplitudes. At four-point, we have a manifest relation between gravity and YM since the gravity amplitude consists of two YM amplitudes with different colour ordering. Nevertheless, for higher-point amplitudes ($n >$)5, these relations quickly become much more complicated in the sense that the YM partial amplitudes have increasingly complicated kinematic coefficients, see e.g. [19].

2.4 The BCJ double copy

Although the KLT relations clearly manifest relations between tree-level amplitudes of gravity and colour ordered tree-level gauge theory amplitudes, they fail to hold at loop-level and the relations become increasingly unclear at higher-point. Here we will review the BCJ double copy, which provides an explicit squaring relation (without kinematic coefficients) that is proven to hold at tree-level, and has been verified to hold at loop level [15]. The general statement of the BCJ double copy can be seen as a consequence of CK-duality that we discussed in section 2.1.

To get started, we recall that the n-point YM amplitude can be written as a sum over products of CK-duality satisfying colour and kinematic numerators, that is,

$$\mathcal{A}_n^{\text{tree}} = \sum_{i\in\,\text{cubic}} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}\,. \tag{50}$$

BCJ found out that replacing the n-point colour factors c_i with n-point kinematic numerators n_i yields exactly the n-point graviton amplitude

$$\mathcal{M}_n^{\text{tree}} = \sum_{i \in \text{ cubic}} \frac{n_i^2}{\prod_{\alpha_i} p_{\alpha_i}^2} \, . \tag{51}$$

The interchange of colour factors by kinematic numerators is what we refer to as the *(BCJ) double copy*; similarly, we refer to the inverse as the *single copy*. This double copy statement goes even further: for instance, we can have two distinct sets of kinematic numerators, say n_i and $\tilde{n}_i$, where only one set satisfies (say n_i) CK-duality, while the other set may be in an arbitrary form. The YM amplitudes are corresponding to these distinct sets are given by

$$\mathcal{A}_n^{\text{tree}} = \sum_{i \in \text{ cubic}} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \quad \mathcal{A}_n^{\text{tree}} = \sum_{i \in \text{ cubic}} \frac{c_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \, . \tag{52}$$

If we now take the Ck-duality satisfying kinematic numerators, say n_i, and replace the corresponding colour factor c_i with the arbitrary form kinematic numerators $\tilde{n}_i$ of the other (possibly different) YM theory, we still obtain the n-point gravity amplitude

$$\mathcal{M}_n^{\text{tree}} = \sum_{i \in \text{ cubic}} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \, . \tag{53}$$

To prove this, we consider the difference between the two distinct numerators,

$$\Delta_i = n_i - \tilde{n}_i \, , \tag{54}$$

which by force of (52) must satisfy

$$\sum_{i \in \text{ cubic}} \frac{c_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0 \, . \tag{55}$$

The only way the above can be satisfied is by force of the Jacobi identity on the colour factors. Since the kinematic numerators n_i satisfy the kinematic analogue of the Jacobi identity, we can replace c_i by n_i. We obtain the expression

$$\sum_{i \in \text{ cubic}} \frac{n_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0 \, , \tag{56}$$

which obviously establishes the equivalence relation

$$\sum_{i \in \text{ cubic}} \frac{n_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = \sum_{i \in \text{ cubic}} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \, . \tag{57}$$

This relation allows for the mixing of symmetries in the gravitational amplitudes since it is possible to use two sets of different kinematic numerators corresponding to two YM theories with different (super) symmetry content.

In addition to the above, we can also take the *"zeroth copy"* by replacing the kinematic numerators of YM with colour factors, possibly from a different gauge theory. The tree-level amplitude of the zeroth copy can then be written as

$$\mathcal{A}_n^{\text{tree}} = \sum_{i \in \text{ cubic}} \frac{c_i \tilde{c}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{58}$$

and coincides with the Bi-adjoint scalar theory (BAS) amplitude. The BAS is a massless cubic scalar theory, with the fields $\phi^{a\bar{a}}$ living in the bi-adjoint representation of to (possibly different) Lie groups G and $\bar{G}$. The corresponding field equations read [34]

$$\Box\phi^{aa'} - y f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{b\bar{b}} \phi^{c\bar{c}} = 0, \tag{59}$$

where y is a coupling constant, and f^{abc} and $f^{\bar{a}\bar{b}\bar{c}}$ are the structure constants of the unbarred and barred Lie groups respectively. The BAS enjoys the special property that its BCJ and Feynman-diagrammatic descriptions are equivalent because the theory solely involves three-point interactions [25].

3 Classical Kerr-Schild Double Copies

At this point, it should be obvious that the double copy is intrinsically defined in a perturbative sense, since the double copy provides a mapping of scattering amplitudes between different theories. However, in the introduction, we already mentioned that scattering amplitudes can be extracted from the individual terms of the perturbative expansion of the classical field. This already hints at a close connection between the on-shell amplitudes (observables), and on the other hand, the off-shell equations of motion. A natural question that arises is whether this symmetry underlies the whole dynamics of theory, or if it is some trick that merely connects the scattering amplitudes. Hints at the former have recently been coming from the so-called *Kerr-Schild double copies* (for examples, see [17, 18, 19, 37]), which prescribes a mapping between complete classical solutions of BAS, YM and GR.

As we discussed before, gluons only interact if they carry different colour charges, which can be seen as a consequence of the fact that the equation of motion for a single colour charge becomes linear in the field. The key strength of the Kerr-Schild double copy is that a specific Ansatz on the space-time metric causes the otherwise highly non-linear Einstein Equations to become linear, with the consequence that it potentially satisfies a relatively simple mapping relation with its YM counterpart, which can also be cast into a linear form. Classical solutions, contrary to scattering amplitudes, do depend on gauge fixing, field redefinitions and coordinate changes. For this classical double copy to be manifest, it will therefore be essential to pick the right field basis and gauge choice.

Recently, the Kerr-Schild double copy successfully managed to map many different solutions from YM to GR and vice-versa. A few of the most popular examples are the Schwarzschild black hole [17], the Kerr black hole [17], Taub-NUT space-times [37] and accelerated point-particles (18). In this chapter, we will review the basics of the Kerr-Schild double copy on the hand of a few examples, loosely following [17, 19].

3.1 General relativity in Kerr-Schild coordinates

To get started, we recall from chapter 2 that the Einstein-Hilbert Lagrangian is given by

$$\mathcal{L}_{\mathrm{EH}} = \frac{1}{2\kappa^2}\sqrt{-g}R + \mathcal{L}_{\mathrm{matter}}\,, \tag{60}$$

with the corresponding field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu}\,. \tag{61}$$

We wish to solve this equation for the metric tensor $g_{\mu\nu}$, in the presence of a particular (classical) source J, contributing the term $\mathcal{L}_{\mathrm{m}}$ to the Lagrangian. The choice of coordinates that will eventually facilitate the classical double copy are the so-called *Kerr-Schild (KS) coordinates*; in KS coordinates, the metric takes the special form

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu} \\ &\equiv \eta_{\mu\nu} + k_\mu k_\nu \phi, \end{aligned} \tag{62}$$

where ϕ is a scalar function and k_μ a four-vector with the additional property that it is null with respect to the two metrics $\eta_{\mu\nu}$ and $g_{\mu\nu}$. We will follow our previously introduced language and refer to $h_{\mu\nu}$ as the graviton field. Mathematically, the nullity properties of k_μ implies

$$k_\mu \eta^{\mu\nu} k_\nu = 0 = k_\mu g^{\mu\nu} k_\nu\,. \tag{63}$$

As a consequence of the above, the inverse space-time metric takes the simple form

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa k^\mu k^\nu \phi\,. \tag{64}$$

Using the metric and its inverse, we can compute the Christoffel symbols,

$$\Gamma^\rho_{\mu\nu} = \frac{\kappa}{2}\left(\partial_\mu \hat{k}^\rho \hat{k}_\nu \phi + \partial_\nu \hat{k}^\rho \hat{k}_\mu \phi - \partial_\rho \hat{k}_\mu \hat{k}_\nu \phi + \kappa\left(\hat{k}^\rho \hat{k}^\sigma \phi\right)\left(\partial_\sigma \hat{k}_\mu \hat{k}_\nu \phi\right)\right), \tag{65}$$

and the Ricci and scalar curvature,

$$\begin{aligned} R^\mu{}_\nu &= \frac{1}{2}\left(\partial^\mu \partial_\alpha \left(\phi k^\alpha k_\nu\right) + \partial_\nu \partial^\alpha \left(\phi k_\alpha k^\mu\right) - \partial^2 \left(\phi k^\mu k_\nu\right)\right), \\ R &= \partial_\mu \partial_\nu \left(\phi k^\mu k^\nu\right), \end{aligned} \tag{66}$$

where we used the convention $\partial^\mu \eta^{\mu\nu} \equiv \partial^\mu$. This convention is essential since the Ricci tensor with mixed indices $R^\mu{}_\nu$ obtains the highly remarkable property that it is linear in the graviton.

Let's now consider the time-independent case, where all time derivatives vanish. Without loss of generality, we can set $k^0 = 1$, such that the time-like component is completely contained in the scalar function ϕ. Under these conditions, the Ricci and scalar curvature

(66) respectively read

$$
\begin{aligned}
R^0_0 &= \frac{1}{2}\partial_i\partial^i\phi\,,\\
R^i_0 &= -\frac{1}{2}\partial_j\left[\partial^i\left(\phi k^j\right)-\partial^j\left(\phi k^i\right)\right],\\
R^i_j &= \frac{1}{2}\partial_l\left[\partial^i\left(\phi k^l k_j\right)-\partial_j\left(\phi k^l k^i\right)-\partial^l\left(\phi k^i k_j\right)\right],\\
R &= \partial_i\partial_j\left(\phi k^i k^j\right).
\end{aligned}
\tag{67}
$$

To make the connection with SDYM, we define the gauge field $A^\mu=\phi k^\mu$, and refer to it as the single copy of the graviton. This vector field has the associated (abelian) field strength

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,. \tag{68}$$

The EFE in vacuum $R_{\mu\nu}=0$ now implies that the single copy satisfies the Maxwell's equations, that is,

$$\partial_\mu F^{\mu\nu}=\partial_\mu\left[\partial^\mu\left(\phi k^\nu\right)-\partial^\nu\left(\phi k^\mu\right)\right]=0\,. \tag{69}$$

We could make the above more general by assigning colour to the single copy, i.e., we can write the gauge field as $A^a_\mu=k_\mu\phi^a$. As a consequence of this trivial colour dependence, the non-linear part of the Yang-Mills equation vanishes, effectively recovering Maxwell's equation

$$\Box A^a_\nu+f^{abc}A^{b\mu}\left(2\partial_\mu A^c_\nu-\partial_\nu A^c_\mu\right)=0\,. \tag{70}$$

Hence, we see that the non-abelian gauge field A^a_μ solves the Maxwell's equations.

We can go even further and take the zeroth copy by removing another vector k_v from the graviton $h_{\mu\nu}$. The remaining object is the scalar field ϕ, which satisfies the free field equation

$$\partial^i\partial_i\phi=0\,. \tag{71}$$

This equation turns out to be equivalent to the equation of motion of the BAS (59) with the fields living in the adjoint representation of abelian Lie group $U(1)$.

3.2 Kerr-Schild approach to the self-dual sectors

In the previous section, we discussed general vacuum Kerr-Schild solutions in general relativity. It turns out that the vacuum solutions to the self-dual equations of YM and GR are exactly of this form. Before we discuss the double copy, we will review the equations of motion of the self-dual sectors, following [17].

The SDYM equation of motion in four dimensions is given by

$$F_{\mu\nu}=\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \tag{72}$$

with all solutions A^μ being complex. Since the self-dual configurations physically represent waves of positive polarization, it will be useful to transform to the $(2+2)$-dimensional light-cone coordinates:

$$u=t-z\,,\quad v=t+z\,,\quad w=x+iy\,. \tag{73}$$

Using the above, the line-element takes the form

$$ds^2 = -dudv + dwd\bar{w}\,. \tag{74}$$

Furthermore, using the light cone gauge with $A_u = 0$, the SDYM equations (72) imply

$$A_w = 0\,, \quad A_v = -\frac{1}{4}\partial_w \Phi\,, \quad A_{\bar{w}} = -\frac{1}{4}\partial_u \Phi\,, \tag{75}$$

where the Lie-algebra valued scalar field, $\Phi \equiv \phi^a T^a$, satisfies the field equation

$$\partial^2 \Phi - ig\,[\partial_w \Phi, \partial_u \Phi] = 0\,. \tag{76}$$

The above turns out to be equivalent to the SDYM equations of motion (72). On the other hand, SDG has the equation of motion

$$R_{\mu\nu\lambda\delta} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma}_{\lambda\delta}\,. \tag{77}$$

Expanding the metric $g_{\mu\nu}$ as in (62), and using diffeomorphism freedom, we find that the nontrivial components of the graviton $h_{\mu\nu}$ read

$$h_{vv} = -\frac{1}{4}\partial_w^2 \phi\,, \quad h_{\bar{w}\bar{w}} = -\frac{1}{4}\partial_u^2 \phi\,, \quad h_{v\bar{w}} = h_{\bar{w}v} = -\frac{1}{4}\partial_w \partial_u \phi\,. \tag{78}$$

Furthermore, the scalar field ϕ satisfies the equation of motion

$$\partial^2 \phi - \kappa \left(\partial_w^2 \phi \partial_u^2 \phi - (\partial_w \partial_u \phi)^2\right) = 0\,, \tag{79}$$

which we identify to be equivalent to the SDG equation of motion (77).

In order to make the connection with scattering amplitudes, it would be natural to work in momentum-space, where the Kerr-Schild Ansatz (62) becomes

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa \hat{k}_\mu \hat{k}_\nu(\phi)\,, \tag{80}$$

where $\hat{k}_\mu$ is now a linear differential operator. Since the metric $g_{\mu\nu}$ must be symmetric, the commutator of $\hat{k}$ with itself is required to vanish, i.e., $[\hat{k}_\mu, \hat{k}_\nu] = 0$. Furthermore, we restrict to double copies without a dilaton field, which forces us to require the graviton field $h_{\mu\nu}$ to be trace free, corresponding to $\eta^{\mu\nu}\hat{k}_\mu \hat{k}_\nu(\phi) = 0$. In light cone coordinates, the linear momentum operators are given by

$$\hat{k}_u = 0\,, \quad \hat{k}_v = \frac{1}{4}\partial_w\,, \quad \hat{k}_w = 0\,, \quad \hat{k}_{\bar{w}} = \frac{1}{4}\partial_u\,. \tag{81}$$

Using the above, we straightforwardly find that the vacuum EFE becomes

$$R_{\mu\nu} = \frac{\kappa}{2}\left[-\hat{k}_\mu \hat{k}_\nu \partial^2 \phi + \kappa \left(\hat{k}_\mu \hat{k}_\nu \partial_\rho \partial_\sigma \phi\right)\left(\hat{k}^\rho \hat{k}^\sigma \phi\right) - \kappa \left(\hat{k}_\mu \hat{k}_\rho \partial^\sigma \phi\right)\left(\hat{k}_\nu \hat{k}_\sigma \partial^\rho \phi\right)\right] = 0\,, \tag{82}$$

which is different from the position space EFE, in the sense that the Ricci tensor is no longer linear in the graviton field. However, we recover linearity when we force the linear operator $\hat{k}_\mu$ to be a vector. Using (86), we find that the Einstein vacuum equation $R_{\mu\nu} = 0$ is equivalent to the scalar equation

$$\partial^2\phi - \frac{\kappa}{2}\left(\hat{k}^\mu\hat{k}^\nu\phi\right)(\partial_\mu\partial_\nu\phi) = 0\,, \tag{83}$$

which coincides with the so-called *Plebanski equation* for SDG [16]. This equation is well known to provide an example of a classical double copy with SDYM at the gauge-theory side [16].

Next, we move on to the gauge-theory counterpart by assuming that the gauge field takes the form

$$A^a_\mu = \hat{k}_\mu\phi^a\,, \tag{84}$$

which will, as we will soon see, turn out to be a solution to the SDYM equations of motion. The Yang-Mills equations (12) for this gauge field reads

$$\hat{k}_\nu\partial^2\phi^a + 2gf^{abc}\left(\hat{k}^\mu\phi^b\right)\left(\hat{k}_\nu\partial_\mu\phi^c\right) = 0\,. \tag{85}$$

By multiplication with the generator T^a of the Lie group, and using equations (81), we find that the above reduces to the Lie algebra valued scalar equation

$$\hat{k}^\nu\left(\partial^2\Phi - ig\left[\partial_w\Phi, \partial_u\Phi\right]\right) = 0, \tag{86}$$

which is equivalent to the SDYM equation (76).

The above clearly illustrates that the momentum space description of SDYM and SDG in Kerr-Schild coordinates is an example of a classical double copy. That is, given the fact that (62) with a specific scalar field ϕ is a solution to the EFE, then upon taking the single copy, or in other words, by removing one vector k_μ from the graviton field $h_{\mu\nu}$ and interchanging the charges and coupling constants, we obtain the non-abelian gauge field $A^a_\mu = \phi^a k_\mu$, which satisfies the SDYM equations of motion (76).

In the upcoming sections, we will illustrate specific examples of Kerr-Schild double copies by taking well known GR vacuum solutions as a starting point. Thereafter, we take the (Kerr-Schild) single copy in order to construct and interpret their Yang-Mills counterparts.

3.3 The Schwarzschild black hole

The simplest spherically symmetric and time-independent solution of the vacuum EFE is the Schwarzschild black hole. The line element of this solution in Schwarzschild coordinates is given by [38]

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2\,, \tag{87}$$

where G is the gravitational constant and M the mass of the black hole. To utilize the double copy, we must exploit the gauge freedom to transform the above to Kerr-Schild coordinates. It turns out that the metric takes the KS form after the basis transformation

$$t' = t - 2GM\log\left(\frac{r}{2GM} - 1\right)\,, \tag{88}$$

which brings the line-element (87) to the form

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + 4GM dt dr + (1 + \frac{2GM}{r})dr^2 + r^2 d\Omega^2 \,, \tag{89}$$

from which we identify that the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r} k_\mu k_\nu \,, \tag{90}$$

where $k^\mu = (1, \hat{r})$. Comparing this solution to the general Kerr-Schild solution (62), we easily identify

$$h_{\mu\nu} = \frac{\kappa}{2}\phi k_\mu k_\nu, \qquad \phi = \frac{M}{4\pi r} \,. \tag{91}$$

Following the Kerr-Schild double copy prescription, we obtain the single copy (i.e., the gauge field) by replacing the coupling constant, removing one momentum vector k_μ, and replacing the mass M by colour charge $Q \equiv c_a T^a$. This can be summarized by the following mappings

$$\frac{\kappa}{2} \to g, \quad M \to c_a T^a, \quad k_\mu k_\nu \to k_\mu \,. \tag{92}$$

The resulting single copy reads

$$A^\mu = \frac{g c_a T^a}{4\pi r}\left(1, \frac{x^i}{r}\right) \equiv \frac{g c_a T^a}{4\pi r} k_\mu \,, \tag{93}$$

which turns out to be a solution to the (abelian) Yang-Mills equation [17]. In its current form, it is not clear how we should physically interpret this single copy. To make it easier to physically interpret this solution, we exploit the gauge freedom, meaning that we can add the derivative of any scalar function, say χ, to the gauge field. Mathematically, this transformation can be written as

$$A^a_\mu \to A^a_\mu + \partial_\mu \chi^a(x) \,. \tag{94}$$

Our task is to carefully choose χ^a, such that the spatial parts of A^a_μ vanish. This can be achieved by the choice

$$\chi^a = -\frac{g c_a}{4\pi} \log\left(\frac{r}{r_0}\right) \,, \tag{95}$$

where the length scale r_0 does not affect the result and is only introduced to make the argument of the logarithm dimensionless. Using the above, the single copy takes the form

$$A_\mu = \left(\frac{g c_a T^a}{4\pi r}, 0, 0, 0\right) \,, \tag{96}$$

which we immediately recognise to be the Coulomb solution of a static (colour) charge located at the origin. This result is not surprising, since this Coulomb solution is the most general static spherically symmetric solution of gauge theory [17]. Hence, by fixing the gauge in a specific way, we have established the classical double copy between a point-like solution in general relativity and its gauge theory counterpart. It turns out that this result generalizes straightforwardly to higher dimensions [17].

In addition to the classical solutions, we will also investigate the double copy structure of the classical sources. To understand how this works, we will begin by analyzing the (relatively simple) sources of the above example. From literature it is well known that the Schwarzschild solution is sourced via the energy-momentum tensor [17],

$$T^{\mu\nu} = M v^\mu v^\nu \delta^{(3)}(\vec{x}) \,, \quad v^\mu = (1,0,0,0) \,, \tag{97}$$

which has the interpretation of a point mass M, located at the origin $\vec{x}=0$. Returning to gauge theory, we substitute the single copy (96) into the (abelian) Yang-Mills equations; we obtain the equation

$$\partial_\mu F^{\mu\nu} = j^\nu \,, \tag{98}$$

where the current j^v is given by

$$j^\nu = -g\,(c_a T^a)\,u^\nu \delta^3(\vec{x}) \,. \tag{99}$$

This current represents a static colour charge at the origin. Hence. a point charge $c_a T^a$ at the origin double copies to a point mass M, also located at the origin. In this example, we already anticipated that the Schwarzschild metric and its single copy correspond to point-like charges, so the above learns us nothing new.

In the next, slightly more complicated example, we will see that the double copy structure of the sources is less intuitive. Specifically, we will see that the different natures of the charges (i.e., mass and colour charge), and the way the corresponding forces act on them, leads to classical sources with a different physical interpretation.

3.4 The Kerr black hole

Next, we consider the *Kerr black hole*, which is another vacuum solution of the EFE with the physical interpretation of an uncharged rotating black hole (see [33] for the original paper). In Kerr-Schild coordinates, the graviton and scalar field respectively take the form

$$h_{\mu\nu} = \frac{\kappa}{2}\phi(r) k_\mu k_\nu \,, \qquad \phi(r) = \frac{M}{4\pi}\frac{r^3}{r^4 + a^2 z^2} \,, \tag{100}$$

where a is a constant, and k_μ is given by

$$k_\mu = \left(1, \frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right) \,. \tag{101}$$

The radial distance r, which was previously given by $r = \sqrt{x^2+y^2+z^2}$, is now implicitly defined via the equation

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1 \,, \tag{102}$$

which is valid everywhere, except for the region in the xy-plane where the massive disc of radius a is located. This disc can be parametrized as $\{x^2+y^2 \leq a^2, z=0\}$, and it contains a ring-like singularity at its boundary $x^2+y^2 = a^2$.

In the original paper from Kerr, the solution was given in Cartesian Kerr-Schild coordinates, but the calculations can be significantly simplified by using the spheroidal coordinate system, parametrized by

$$x = \sqrt{r^2 + a^2} \sin\theta \cos\phi, \quad y = \sqrt{r^2 + a^2} \sin\theta \sin\phi, \quad z = r\cos\theta, \tag{103}$$

where r, θ and ϕ are the radial, polar, and azimuthal coordinates respectively (see figure 7).

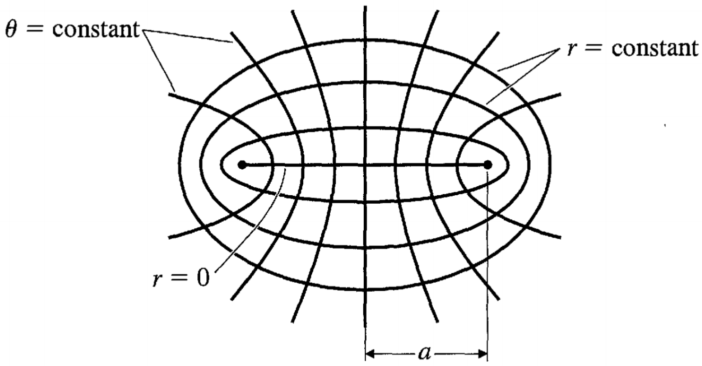


Figure 7: Cross section (with constant φ) of the spheroidal coordinate system. The singularity is located at the endpoints of the horizontal line, with coordinates $(r = 0, \theta = \pi/2)$. These endpoints form a ring when considering all values of φ. This image was taken from [39].

In terms of these coordinates, the Minkowski line-element reads

$$ds^2 = -dt^2 + \frac{\rho^2}{a^2 + r^2} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2\right) \sin^2\theta d\varphi^2, \tag{104}$$

where we defined $\rho^2 \equiv r^2 + a^2 \cos^2\theta$. Furthermore, in this basis, the vector k_μ and the scalar function $\phi(r)$ read

$$k_\mu dx^\mu = dt + \frac{\rho^2}{a^2 + r^2} dr - a\sin^2\theta d\varphi, \quad \phi(r) = \frac{M}{4\pi} \frac{r}{\rho^2}. \tag{105}$$

Analogous to the previous section, we now take the single copy of the graviton (100) by the replacements (92). The resulting single copy reads

$$A^a_\mu = \frac{g}{4\pi} \phi(r) c^a k_\mu, \tag{106}$$

and solves the (abelian) Yang-Mills equations in the exterior of the ring-like singularity. This single copy is equivalent to the Schwarzschild single copy (96), apart from the interchange of the variables r and ρ. Hence, we see that sending the angular momentum to zero, $a \to 0$, results in $\rho \to r$. In this limit, the Kerr black hole no longer rotates and therefore becomes a Schwarzschild black hole.

The classical source corresponding to the Kerr black hole and its single copy counterpart turn out to have a different physical interpretation [17]. Let's therefore proceed in a similar fashion as before by taking the Kerr black hole solution as a starting point. To obtain the classical source corresponding to the Kerr black hole, we use the fact that the Ricci curvature tensor can be written as a divergence, as can be seen from equation (66). Using this property, and the fact that the scalar curvature vanishes, we find that the EFE (61) equates the energy-momentum tensor with a divergence. We can then calculate the energy-momentum tensor by using the divergence theorem.[10] The usefulness of this approach lies in the fact that we can use the same method to identify the source on the gauge theory side. In spheroidal coordinates, parametrized by (t, r, θ, φ), we find that the energy-momentum tensor reads [41]

$$T^{\mu\nu} = \sigma \left(w^\mu w^\nu + \zeta^\mu \zeta^\nu \right), \quad \sigma = -\frac{M}{8\pi^2 a \cos\theta}, \tag{107}$$

where w^μ and ζ^μ are the space-like vectors

$$w^\mu = \tan\theta \left[\right] 1, 0, 1/(a\sin^2\theta), 0 \Big], \quad \zeta^\mu = [0, 1/(a\cos\theta), 0, 0]. \tag{108}$$

The energy-momentum tensor represents a negative proper surface density σ, rotating around the z-axis with a superluminal velocity, see w^μ. The centrifugal force that arises due to this rotation is balanced by a radial pressure following from the term involving ζ^μ.

We now use the method of the previous section to interpret the single copy source. Substitution of the single copy (106) into the abelian Yang-Mills equations yields the source current

$$j^\mu = -\delta(z)\Theta(\rho - a) \frac{g\,(c_a T^a)}{4\pi} \frac{1}{a^2\cos\theta} \left(\sec^2\theta, 0, \frac{\sec^2\theta}{a}, 0 \right). \tag{109}$$

By introducing the vector

$$\xi^\mu = \left(1, 0, a^{-1}, 0 \right), \tag{110}$$

we can rewrite the current (109) as

$$j^\mu = q\xi^\mu, \quad q = -\delta(z)\Theta(a - \rho) \frac{g c_a T^a}{4\pi a^2} \sec^3\theta, \tag{111}$$

which can be interpreted as a colour charge distribution rotating about the z-axis, analogous to the rotating massive disc. At this point, it is not entirely clear that the energy-momentum tensor (107) is a double copy of the source current (111), however, we can make the double copy relation more obvious by writing the energy-momentum tensor (107) in terms of the vector (110), which yields

$$T^{\mu\nu} = \delta(z)\Theta(\rho - a) \left(-\frac{M\sec^3\theta}{8\pi a^2} \right) \left[\xi^\mu \xi^\nu - \cos^2\theta\, \tilde{\eta}^{\mu\nu} \right], \quad \tilde{\eta}^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 0), \tag{112}$$

[10] The divergence theorem states that the volume integral of the divergence of a vector field is equal to the surface integral of the vector field over the boundary of the volume. Mathematically, the divergence theorem can be written as $\int_V (\nabla \cdot \mathbf{F}) dV = \int_{\partial V} \mathbf{F} \cdot d\mathbf{a}$, where V, $\mathbf{F}$, ∂V and $d\mathbf{a}$ respectively denote the volume, the vector field, the boundary of V and the infinitesimal surface element [40].

where $\tilde{\eta}^{\mu\nu}$ is defined in Cartesian coordinates (t, x, y, z). We can now interpret the first term of (152) as the double copy of the source current (111), where the colour charge distribution is replaced by a mass distribution. The second term corresponds to a stabilizing radial pressure, which is needed to prevent the massive disc from undergoing a gravitational collapse.

The above examples leave some questions unanswered. The first question is: why are all single copies in these examples (and the other examples in the references) abelian? Or in other words, why do the non-abelian single copies A^μ satisfy the abelian YM equations? This could logically be explained by the fact that the EFE for Kerr-Schild solutions are linear in the graviton field [17], which implies that we automatically obtain a linear single copy after extraction of a vector k^μ. Previous research [42] involving stationary sources has shown that it is always possible to gauge away the non-abelian part of the solutions. It might therefore be the case that the necessary condition for the existence of a Kerr-Schild double copy is that the YM part must be linear. Another explanation could be that there exist non-abelian single copies that double copy to the same space-times as abelian single copies. The latter could be related to the information loss during the process. For example, in [43] it was shown that infrared singularities of YM theory and Maxwell's electrodynamics both double copy to the same infrared singularity in gravity.

Currently, the double copy construction of any complete non-linear classical solutions of YM to its gravity counterpart remains elusive. However, the examples that we have illustrated in this chapter, in combination with the well understood double copy of their amplitudes, raise hope that a more general formulation of the classical double copy could be possible.

In this chapter, we have only seen a restricted selection of the many possible KS double copies. For extensive detail and many more examples such as KS double copy in curved space-times and higher dimensions, or the double copy of multiple Kerr Black holes, the interested reader is invited to have a look at the overview, provided by [19, ch. 3,4,5,6].

4 A Perturbative Approach: Amplitudes from Classical Solutions

In chapter 2, we have seen the double copy in its original and most well known formulation: that is, by simply interchanging BCJ-duality satisfying numerators, one can hop from the amplitudes of one theory to the tree-level amplitudes of another theory. This way, the double copy is naturally a perturbative statement.

On the other hand, in chapter 3 we have also seen that there exist double copy relations between exact (but linear) classical solutions of the BAS and the self-dual sectors of GR and YM. This exact double copy crucially relies on (1) the Kerr-Schild Ansatz, which linearizes the otherwise highly non-linear EFE, and (2) the trivial colour dependence on the gauge theory side, effectively transforming YM into Maxwell's electrodynamics. The above already indicates that the amplitude double copy is only one of the (perhaps) many more manifestations of the double copy.

In this chapter, we will, to a certain extent, bridge the gap between these two seemingly unrelated manifestations. As mentioned before, it is well known that perturbative classical solutions can be expressed in terms of connected correlation functions (which we shall often refer to as *correlators*). The classical solution, which is a one-point correlator, can be obtained by expanding the field in the coupling, followed by subsequently solving the perturbed equation of motion at each order in the coupling. The solution, say at order n in the coupling, corresponds to $(n+2)$-point correlators (here also referred to as *perturbative corrections*). In momentum space, these perturbative corrections take the form of integrals that implicitly encode scattering amplitudes. In addition, these integrals also explicitly depend on the classical source. Finally, we can use the so-called *LSZ (Lehmann-Symanzik-Zimmermann)* formalism [19] to extract n-point tree-level amplitudes from n-point perturbative corrections. The resulting amplitudes do, as they should, not depend on the source, as a consequence of the fact that these will be differentiated out by the LSZ formula. The methods illustrated here will also be used in the next chapter to calculate amplitudes and to illustrate double copy relations for a triplet of scalar effective field theories.

On the other hand, by substituting the explicit form of the source into the perturbative corrections and evaluating the integrals, we can iteratively construct the (possibly non-linear) classical solution. When considering amplitudes in the double copy context, the off-shell form of the integrand, which can be thought of as the off-shell extension of the amplitude (or *off-shell amplitude*), is only relevant for manifest BCJ-duality of the numerators, since the final amplitude solely depends on the on-shell information. On the other hand, for classical solutions, the off-shell terms (i.e., terms that vanish on-shell) do change the outcome of the integral and consequently also the form of the classical solution. This off-shell form of the integrand is a consequence of field basis and gauge choice, which means that these should be chosen carefully to (possibly) manifest the double copy of non-linear classical solutions and equations of motion.

Below, we will first introduce the perturbation theory formalism in momentum space and illustrate the LSZ formalism on the hand of a simple example, loosely following [16]. Thereafter, we illustrate the perturbative formalism for the BAS, SDYM and SDG. Specifically, we show how the BCJ numerators arise in the momentum-space integrands, and how the kinematic factors that arise in these integrands have similar algebraic properties as the colour factors. As a consequence of leaving the classical sources J implicit, higher order non-linear terms arise, contrary to the exact vacuum Kerr-Schild solutions of the previous chapter. When we explicitly substitute the source, the classical solution can be obtained by evaluating the momentum space integrands of the perturbative corrections.

4.1 The classical field as a generating functional

In the introduction, we already mentioned that most QFT textbooks study scattering processes by using the path integral formulation. In this formulation, the scattering amplitudes are calculated using Feynman diagrams and the corresponding Feynman rules. These Feynman diagrams arise as a pictorial representation of the perturbative expansion of the field equations in response to a particular source J. Following this approach, the starting point is the generating functional $Z[J]$, which is given by [44]

$$Z[J] \equiv \int \mathcal{D}\phi e^{\frac{i}{\hbar}\int d^4x[\mathcal{L}+J\phi]} = e^{\frac{i}{\hbar}W[J]}\,, \tag{113}$$

where $\mathcal{L}$ is the Lagrangian, $W[J]$ generates connected correlators, and $\int\mathcal{D}\phi$ denotes integration over all possible values of the field $\phi(x^\mu)$. One particularly useful feature of the path integral formalism is the convenience in which the classical limit can be taken. That is, one simply takes the limit $\hbar \to 0$, where after the equations of motion coincide with the classical field equations that follow from the well known Euler-Lagrange equations [35]

$$\frac{\delta S[\phi, J]}{\delta\phi} \equiv \quad \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\right) - \frac{\delta\mathcal{L}}{\delta\phi} = J\,, \tag{114}$$

where J denotes the classical source. In the classical limit, or in other words, in the absence of quantum effects, the virtual (off-shell) particles can no longer be created and annihilated. At the level of Feynman diagrams, this corresponds to the absence of loop-level diagrams. Then the diagrammatic representation of the perturbative classical solution entirely consists of tree-level diagrams. In appendix A.1 we briefly review the difference between perturbative classical and quantum field theory; we explicitly show that loop diagrams result from an additional $\mathcal{O}(\hbar)$ term contributing to the equation of motion. However, in this research, we will purely restrict to the tree-level double copy.

Since the aim of this is to study classical solutions, we will take this limit, $\hbar \to 0$, upon which the generating functional (113) is dominated by the classical solution $\phi_{cl}(J)$ of the equations of motion (114). Using the saddle point approximation, we can approximate the generating functional as (113)

$$Z[J] \simeq e^{i\left(S[\phi_{\rm cl}]+\int d^4x J\phi_{\rm cl}\right)}\,, \tag{115}$$

and by comparison with the path integral (113), we identify the relation

$$W[J,\phi_{\rm cl}] \equiv S[\phi_{\rm cl}] + \int d^4x J\phi_{\rm cl}\,. \tag{116}$$

Functionally differentiating $W[J,\phi_{cl}]$ with respect to the source J, and application of the equations of motion, we obtain

$$\begin{aligned}\frac{\delta W[J,\phi_{\rm cl}]}{\delta J} &= \frac{\delta S[\phi_{\rm cl}]}{\delta\phi_{\rm cl}}\frac{\delta\phi_{\rm cl}[J]}{\delta J} + \phi_{\rm cl}[J] + J\frac{\delta\phi_{\rm cl}[J]}{\delta J}\,,\\ &= \left(\frac{\delta S[\phi_{\rm cl}]}{\delta\phi_{\rm cl}} + J\right)\frac{\delta\phi_{\rm cl}[J]}{\delta J} + \phi_{\rm cl}[J] = \phi_{\rm cl}[J]\,.\end{aligned} \tag{117}$$

As previously mentioned, $W[J]$ is a generating functional of connected correlators, and therefore the above relation implies that the classical solution ϕ_{cl} can also be regarded as the generating functional of correlation functions. To make this statement more precise, we note that we can obtain the n-point correlators by functionally differentiating the classical solution $n-1$ times with respect to J. Thereafter, we can extract n-point tree-level amplitudes from the n-point correlators by applying the so-called LSZ method, which we now turn to.

4.2 Perturbative solutions and the LSZ reduction formula

To illustrate the extraction of tree-level amplitudes from classical solutions, we start by considering the simplest self-interacting theory: a massless cubic scalar theory with the Lagrangian [16]

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{3!}g\phi^3 + J\phi\,. \tag{118}$$

The equation of motion follows from the Euler-Lagrange equation (114) and explicitly reads

$$\partial^2\phi + \frac{1}{2}g\phi^2 = J\,. \tag{119}$$

In chapter 2, we have seen that kinematic numerators of the BCJ double copy are expressed in terms of momenta, and since the aim of this chapter is to establish the relation between amplitudes and classical solutions, it will be convenient to Fourier transform the equation of motion to momentum space. Upon Fourier transforming the above, we obtain

$$p_1^2\phi(p_1) - \frac{1}{2}g\int đp_2 đp_3 \delta\!\!\!{}^{-(4)}(p_2+p_3-p_1)\,\phi(p_2)\,\phi(p_3) = -J(p_1)\,, \tag{120}$$

where we introduced the short-hand notation

$$đp \equiv \frac{d^4p}{(2\pi)^4}\,, \quad \delta\!\!\!{}^{-}(p) \equiv (2\pi)^4\delta^4(p)\,. \tag{121}$$

The goal is to solve this equation order by order in the coupling g, therefore we expand the scalar field as

$$\phi = \phi^{(0)} + g\phi^{(1)} + g^2\phi^{(2)} + \dots\,, \tag{122}$$

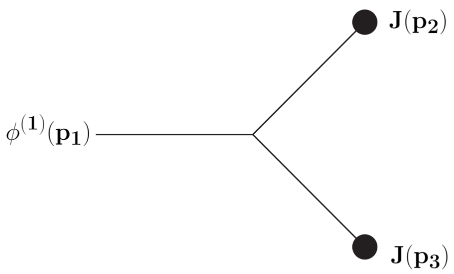


Figure 8: Diagrammatic representation of the first order perturbative correction $\phi^{(1)}(p_1)$. Two sources $J(p_2)$ and $J(p_3)$ propagate according to the equations of motion and terminate at the field $\phi^{(1)}$, which has the same argument as ϕ_{cl}.

where we assume that the $\phi^{(n)}$ are independent of the coupling constant g. Substituting this expansion into the equation of motion (120), and collecting all terms of the same order in g, we obtain a differential equation for each n'th order perturbative correction $\phi^{(n)}$. For instance, at zeroth order we have

$$p^2\phi^{(0)}(p_1) = -J(p_1) \qquad \Longrightarrow \qquad \phi^{(0)}(p_1) = -\frac{J(p_1)}{p_1^2}\,. \tag{123}$$

Furthermore, by collecting terms at order g we find that sub-leading correction $\phi^{(1)}$ satisfies the equation

$$p_1^2\phi^{(1)}(p_1) = \frac{1}{2}g\int \mathrm{đ}^4p_2\mathrm{đ}^4p_3\bar{\delta}^{(4)}\left(p_2+p_3-p_1\right)\phi^{(0)}\left(p_1\right)\phi^{(0)}\left(p_2\right). \tag{124}$$

By using equation (123), the above can be rewritten in terms of the source J

$$\phi^{(1)}(p_1) = \frac{1}{2}\int \mathrm{đ}^4p_2\mathrm{đ}^4p_3\bar{\delta}^{(4)}\left(p_2+p_3-p_1\right)\frac{1}{p_1^2}\frac{J\left(p_2\right)}{p_2^2}\frac{J\left(p_3\right)}{p_3^2}\,. \tag{125}$$

This first order correction can be diagrammatically visualized as is shown in figure 8. Throughout this thesis, we will by convention choose the external leg at which the sources terminate to have the momentum dependence p_1, and we will refer to it as the *"root leg"* (as in [25]) since this leg is special in the sense that the perturbative corrections $\phi^{(n)}$ are (generally) only permutation invariant in the other external legs; we shall refer to these as the *"leaf legs"*.

At order g^2, we find that the second order correction $\phi^{(2)}$ satisfies the equation

$$p_1^2\phi^{(2)}(p_1) = \int \mathrm{đ}^4p_2\mathrm{đ}^4p_3\bar{\delta}^{(4)}\left(p_2+p_3-p_1\right)\phi^{(0)}\left(p_2\right)\phi^{(1)}\left(p_3\right). \tag{126}$$

Upon substituting the first-order correction (125), the above can be rewritten as

$$\phi^{(2)}(p_1) = \int \mathrm{đ}^4p_2\mathrm{đ}^4p_3\mathrm{đ}^4p_4\bar{\delta}^4\left(p_2+p_3+p_4-p_1\right)\phi^{(0)}\left(p_2\right)\phi^{(0)}\left(p_3\right)\phi^{(0)}\left(p_4\right)\left[\frac{1}{p_1^2}\frac{1}{\left(p_3+p_4\right)^2}\right], \tag{127}$$

which is diagrammatically visualized in figure 9. The perturbative corrections, $\phi^{(n)}$ in this example, are the $(n+2)$-point correlators from which we can finally extract the $(n+2)$-point scattering amplitudes.

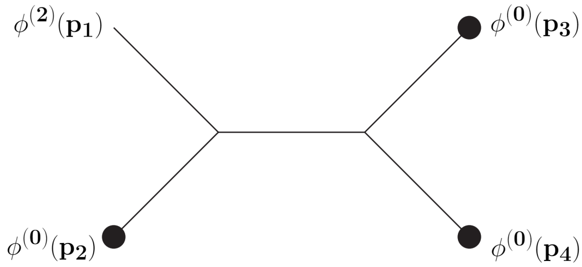


Figure 9: Diagrammatic visualization of the second order correction $\phi^{(2)}(p_1)$, which consists of two particles merging into an intermediate particle that eventually splits into two other particles.

Summing up all perturbative corrections and multiplying by the corresponding power of the coupling, we obtain the perturbative classical solution (122) in terms of momentum space integrals containing delta functions.

We now turn to the extraction of scattering amplitudes. For example, to extract the 3-point scattering amplitude from the three-point correlator $\phi^{(1)}$, we functionally differentiate $\phi^{(1)}$ twice with respect to the sources J, multiply by the appropriate power of the coupling constant, and amputate the external legs. Mathematically, this corresponds to

$$\mathcal{A}_3(p_1,p_2,p_3) = g \lim_{p_1^2\to 0}\lim_{p_2^2\to 0}\lim_{p_3^2\to 0} p_1^2 p_2^2 p_3^2 \frac{\delta}{\delta J(p_1)}\frac{\delta}{\delta J(p_2)}\phi^{(1)}(-p_3) = g\,, \tag{128}$$

where the delta function serves the purpose of imposing momentum conservation $(p_1+p_2+p_3=0)$. It should by now be obvious that the scattering amplitudes, contrary to the classical solutions, are independent of the form of the source J, owing to the fact that the $n-1$ sources are eventually removed by functional differentiation. Since amplitudes are physical observables, on-shell conditions have been applied to the result of the above, corresponding to $p_i^2=0$ for massless particles. However, from the three-point amplitude (128) one can see that the root leg (corresponding to external momenta p_1) can always be taken to be off-shell without changing the result. In addition, appropriate contractions with polarization tensors (as we have seen in section 2.1) should be made in case we are dealing with theories involving spin.

The LSZ formula straightforwardly generalizes to higher order scattering processes, i.e., the general LSZ formula for a n-point scattering amplitude can be written as

$$\mathcal{A}_n(p_1,\dots,p_n) = g^{n-2}\lim_{p_1^2\to 0}\dots\lim_{p_n^2\to 0} p_1^2\dots p_n^2 \frac{\delta^{n-1}\phi^{(n-2)}(-p_n)}{\delta J(p_1)\,\delta J(p_2)\dots\delta J(p_{n-1})}\,. \tag{129}$$

From the above, it is easy to see that we can rewrite this formula in terms of the leading-order corrections $\phi^{(0)}$, by noting that functional differentiation with respect to $\phi^{(0)}(p)$ is equivalent to multiplication by p^2, in combination with functional differentiating with respect to $J(p)$. Hence we can also rewrite (129) as

$$\mathcal{A}_n\left(p_1, \ldots, p_n\right)=g^{n+2} \lim_{p_n^2 \rightarrow 0} p_n^2 \frac{\delta^{n-1} \phi^{(n-2)}\left(-p_n\right)}{\delta j\left(p_1\right) \delta \phi^{(0)}\left(p_2\right) \ldots \delta \phi^{(0)}\left(p_{n-1}\right)} . \tag{130}$$

Application of this formula to the four-point correlator (127) yields

$$g^2 \frac{1}{\left(p_3+p_4\right)^2}=\frac{g^2}{t} . \tag{131}$$

As we discussed in section 2.2 for YM, we should here also include all inequivalent permutations, since assigning the labels $2,3,4$ to the momenta in (127) is arbitrary. Keeping external leg 1 fixed, while replacing external legs $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$, we can finally write the amplitude as a sum over the s, t and u-channel contributions:

$$\mathcal{A}_4=g^2\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right) . \tag{132}$$

In practice the LSZ formula is much less used than the Feynman diagrammatic approach. However, these two different methods are closely related since the LSZ formalism is implicitly encoded in the Feynman rules [19].

By now, it is hopefully clear that tree-level amplitudes and perturbative classical solutions are very directly related. In the next section, we will, on the hand of the previously studied SDYM and SDG, see how the integrands of the perturbative coefficients can be made to satisfy CK duality and eventually be double copied.

4.3 A perturbative approach to the double copy of SDG and SDYM

In the previous section, we have seen that scattering amplitudes can be easily extracted once we have constructed a perturbative classical solution in momentum space. To see how CK numerators arise, and to see how the double copy can be used to map perturbative classical solutions between different theories, we will here establish the perturbative double copy formulation between BAS and the self-dual sectors of YM and GR.

In the previous chapter, we already saw these theories can be related via the (exact) Kerr-Schild double copy. It should, however, be stressed that the double copy is by no means restricted to the self-dual sectors of GR and YM and applies to the full theories, which is perhaps the biggest triumph of the whole story. The reason that we consider the self-dual sectors is, as we already saw in the previous chapter, because of the striking simplicity and similarity of the equations of motion. This way, the self-dual sectors can be viewed as the dynamical analogue of the MHV sector at the amplitude level, since the MHV amplitudes and self-dual equations of motion both take strikingly simple forms. Another motivation to study the self-dual sectors is to show how the exact classical double copy that we discussed in the previous chapter extends to perturbative classical solutions and scattering amplitudes.

To get started, it is useful to realise that the equations of motion of the BAS (59) and the cubic scalar theory (118) have the same structure, apart from the colour structure and a constant factor. Fourier transforming the BAS equation of motion (59) to momentum space gives

$$\phi^{a\bar{a}}(-p_1) = -y \int \text{đ}^4 p_2 \text{đ}^4 p_3 \bar{\delta}^{(4)}(p_1+p_2+p_3) \frac{1}{p_1^2} \left[f^{abc} f^{\bar{a}\bar{b}\bar{c}} \right] \phi^{b\bar{b}}(p_2) \phi^{c\bar{c}}(p_3) \,. \tag{133}$$

Next, we perform a perturbative expansion of ϕ in the coupling y,

$$\phi = \phi^{(0)} + y\phi^{(1)} + y^2\phi^{(2)} + \dots \,, \tag{134}$$

and we repeat the perturbative method of the previous section. At $\}'$, we obtain the equation

$$\Box \phi^{(0)a\bar{a}} = 0 \,, \tag{135}$$

while the sub-leading correction is given by

$$\phi^{(1)a\bar{a}}(-p_1) = -\frac{1}{p_1^2} \int \text{đ} p_2 \text{đ} p_3 \bar{\delta}(p_1+p_2+p_3) \left[f^{abc} f^{\bar{a}\bar{b}\bar{c}} \right] \phi^{(0)b\bar{b}}(p_2) \phi^{(0)c\bar{c}}(p_3) \,, \tag{136}$$

From this expression, we identify the integrand to be the product of the following structure constants:

$$c = f^{abc} \,, \quad \bar{c} = f^{\bar{a}\bar{b}\bar{c}} \,. \tag{137}$$

Like we have seen for YM theory in chapter 2, it can be easily checked that the three-point colour factors are already anti-symmetric under the interchange of any of the colour indices, and that both satisfy the Jacobi identity. For example, for the unbarred colour factors (and similar for the barred), the Jacobi identity reads

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \,. \tag{138}$$

At second order in the coupling g, we find the equation

$$\phi^{(2)a\bar{a}}(p_1) = \frac{2}{p_1^2} \int \text{đ} p_2 \text{đ} p_3 \text{đ} p_4 \left[\frac{f^{abc} f^{\bar{a}\bar{b}\bar{c}} f^{bde} f^{\bar{b}\bar{d}\bar{e}}}{(p_3+p_4)^2} \right] \phi^{(0)c\bar{c}}(p_2) \phi^{(0)d\bar{d}}(p_3) \phi^{(0)e\bar{e}}(p_4) \,. \tag{139}$$

Including all inequivalent permutations, we find that the integrand between square brackets is proportional to

$$\frac{f^{abd} f^{bec} f^{\bar{a}\bar{b}\bar{d}} f^{\bar{b}\bar{e}\bar{c}}}{s} + \frac{f^{abc} f^{bde} f^{\bar{a}\bar{b}\bar{c}} f^{\bar{b}\bar{d}\bar{e}}}{t} + \frac{f^{abe} f^{bcd} f^{\bar{a}\bar{b}\bar{e}} f^{\bar{b}\bar{c}\bar{d}}}{u} \,, \tag{140}$$

from which we identify new BCJ colour factors, consisting of products of two three-point colour factors (137). For instance, the four-point s-channel numerators are given by

$$C_s = f^{abd} f^{bec}, \quad \tilde{C}_s = f^{\bar{a}\bar{b}\bar{d}} f^{\bar{b}\bar{e}\bar{c}}. \tag{141}$$

The four-point colour factors can also be easily checked to satisfy the kinematic Jacobi identity,

$$C_s + C_t + C_u = 0, \quad \tilde{C}_s + \tilde{C}_t + \tilde{C}_u = 0. \tag{142}$$

Hence, we now have the desired result that the three- and four-point numerators satisfy BCJ-duality.

Before, we already mentioned that the BCJ and Feynman diagrammatic descriptions are equivalent for the BAS. In addition to this, the BAS has manifest permutation invariance in all external legs, as a consequence of Bose symmetry [25]. The other theories that we will encounter in this thesis turn out to be only permutation invariant in the leaf-legs.

Next, we repeat the above for SDYM and SDG. Recall from the previous chapter that the SDYM equation of motion in four-dimensional space-time can be phrased in terms of light-cone coordinates, in which it reduces to the lie-algebra valued scalar equation

$$\Box\Phi - ig\left[\partial_w\Phi, \partial_u\Phi\right] = 0. \tag{143}$$

In momentum space the above becomes

$$\Phi^a(p_1) = -\frac{1}{2}\int \bar{d}^4 p_2 \bar{d}^4 p_3 \left[\frac{F_{p_2p_3}{}^{p_1}(p_2,p_3) f^{abc}}{p_1^2}\right] \Phi^b(p_2)\,\Phi^c(p_3)\,, \tag{144}$$

where we introduced the short-hand notation

$$F_{p_2p_3}{}^{p_1}(p_2,p_3) = \bar{\delta}^{(4)}(p_2+p_3-p_1) X(p_2,p_3)\,, \quad X(p_2,p_3) = \left[p_{1w}p_{2u} - p_{1u}p_{2w}\right]. \tag{145}$$

Using the above, the indices of the kinematic factors F are contracted using an integral analogue of the Einstein summation convention,

$$\begin{aligned} F_{p_1q}{}^{k} F_{p_2p_3}{}^{q} &\equiv \int dq\delta\left(p_1+q-k\right) X\left(p_1,q\right)\delta\left(p_2+p_3-q\right) X\left(p_2,p_3\right) \\ &= \delta\left(p_1+p_2+p_3-k\right) X\left(p_1,p_2+p_3\right) X\left(p_2,p_3\right). \end{aligned} \tag{146}$$

In addition to this, we raise and lower indices using

$$\delta^{pq} \equiv \delta(p+q) = \delta_{pq} \quad \Longrightarrow \quad \delta_{pq}\delta^{qk} = \delta_p^k = \delta(p-k). \tag{147}$$

The above notation is used to emphasise the algebraic similarity between the kinematic factors F^{p_1,p_2,p_3} and the colour factors. It turns out that these kinematic numerators are the structure constants of the Lie algebra of an infinite-dimensional area-preserving diffeomorphism group, see [16] for details. The next step is to perturbatively expand the field in the coupling,

$$\Phi^a = \Phi^{(0)a} + g\Phi^{(1)a} + g^2\Phi^{(2)a} + \ldots, \tag{148}$$

and substitute this expansion into the equation of motion (144). Repeating the previously discussed method, we find that the momentum-space perturbative corrections (up to four-point) are given by the following integrals

$$\begin{aligned}
\Phi^{(0)a}(p_1) &= -\frac{J(p_1)}{p_1^2}\,, \\
\Phi^{(1)a}(p_1) &= \frac{1}{2}\int dp_2 dp_3 \frac{F_{p_2p_3}{}^{p_1} f^{b_2b_3a}}{p_1^2}\Phi^{(0)b_2}(p_2)\,\Phi^{(0)b_3}(p_3)\,, \\
\Phi^{(2)a}(p_1) &= \frac{1}{2}\int dp_2 dp_3 dp_4 \frac{F_{p_2q}{}^{p_1} F_{p_3p_4}{}^{q} f^{b_2ca} f^{b_3b_4c}}{p_1^2\,(p_3+p_4)^2}\Phi^{(0)b_2}(p_2)\,\Phi^{(0)b_3}(p_3)\,\Phi^{(0)b_4}(p_4)\,.
\end{aligned} \tag{149}$$

The integrand of the sub-leading correction is dressed with the colour factor $f^{b_2b_3a}$ and the kinematic structure constant $F_{p_2p_3}{}^{p_1}$, which is anti-symmetric under the interchange of any two external legs (all assigned to a momentum label and a corresponding flavour index). For instance, the s-channel numerators of the second order correction $\Phi^{(2)}$ are given by

$$G_s = F_{p_2q}{}^{p_1} F_{p_3p_4}{}^{q} = \delta(p_1+p_2+p_3-k)\,X(p_1,p_2+p_3)\,X(p_2,p_3)\,, \quad C_s = f^{b_2ca} f^{b_3b_4c}\,. \tag{150}$$

It is easy to see that, in addition to the colour factors C, the kinematic structure constants F also satisfy the (kinematic) Jacobi Identity (29). This is a consequence of the identity

$$X(p_2,p_3)\,X(p_4,p_2+p_3)+X(p_3,p_4)\,X(p_2,p_3+p_4)+X(p_4,p_2)\,X(p_3,p_4+p_2)=0\,. \tag{151}$$

Hence, we conclude that the four-point SDYM numerators are in BCJ-dual form.

Similar consideration apply to the SDG equation of motion (83), which can be written as

$$\phi(p_1) = -\frac{1}{2}\kappa\int đ^4p_2 đ^4p_3 \frac{F^{p_1}_{p_2p_3} X(p_2,p_3)}{p_1^2}\phi(p_2)\,\phi(p_3)\,. \tag{152}$$

Once again, we expand the scalar field in the coupling constant κ and substitute it into the equation of motion (152). Up to $\mathcal{O}(\kappa^2)$, we find the implicit solutions

$$\begin{aligned}
\phi^{(0)}(p_1) &= -\frac{J}{p_1^2}\,, \\
\phi^{(1)}(p_1) &= -\frac{1}{2}\int đp_2 đp_3 \frac{F^{p_1}_{p_2p_3} X(p_2\,,p_3)}{p_1^2}\phi^{(0)}(p_2)\,\phi^{(0)}(p_3)\,, \\
\phi^{(2)}(p_1) &= -\frac{1}{2}\int đp_2 đp_3 đp_4 \frac{X(p_2,q)\,F_{p_2q}{}^{p_1} X(p_3,p_4)\,F^{q}_{p_3p_4}}{p_1^2\,(p_3+p_4)^2}\phi^{(0)}(p_2)\,\phi^{(0)}(p_3)\,\phi^{(0)}(p_4)\,.
\end{aligned} \tag{153}$$

Each of the above integrands contains combinations of kinematic factors X and F, with the same algebraic properties as we discussed earlier. At this point, we could make the connection with the BCJ double copy, however, in this specific case, it turns out to be slightly more subtle than naively interchanging the colour and kinematic structure constants. The relation with the amplitude factorization in terms of BCJ numerators will be the topic of the next section.

4.4 Link with BCJ double copy

By observing the perturbative corrections that we found in the previous section, it should be obvious that we can obtain SDG from SDYM by replacing the colour factors with appropriate kinematic structures, and of course, the YM coupling constant by its gravitational counterpart. We have also seen that the kinematic factors X are closely related to the structure constants F of the kinematic algebra. However, in the current notation, taking the double copy is a bit more subtle than simply replacing colour factors by kinematic structure constants, due to the fact that this replacement would involve taking the square of a delta function (which is included in the kinematic structure constant). One way to identify the true BCJ kinematic numerators, following [16], is to contract the kinematic structure constants according to (146), followed by integrating out the auxiliary momentum (previously denoted by q). From the resulting integral, we can then simply extract the overall momentum conserving delta function $\delta^{(4)}(p_2+\dots p_n-p_1)$ and read off the true BCJ numerators.

To illustrate the above, we consider the second order perturbative correction of SDYM (149), as in [16]. We perform the contraction of the kinematic structure constants according to (146) and integrate over the auxiliary momentum q. We obtain the expression

$$\Phi^{(2)a}(p_1) = -\frac{1}{2}\int đqđp_2đp_3dp_4\delta\left(p_2+q-p_1\right)\delta\left(p_3+p_4-q\right)\frac{X\left(p_2,q\right)X\left(p_3,p_4\right)f^{b_2ca}f^{b_3b_4c}}{p_1^2\left(p_3+p_4\right)^2} \times\phi^{(0)b_2}\left(p_2\right)\phi^{(0)b_3}\left(p_3\right)\phi^{(0)b_4}\left(p_4\right)+\dots, \tag{154}$$

where the dots indicate the s and u-channel contributions. By functional differentiation with respect to the $\phi^{(0)}$'s, amputation of the off-shell leg, and multiplication by g^2, we find that the (polarization-stripped) 4-point amplitude is given by

$$\mathcal{A}\left(p_1,p_2,p_3,-k\right) = -\frac{g^2}{2}\frac{X\left(p_1,p_2+p_3\right)X\left(p_2,p_3\right)f^{b_1ca}f^{b_2b_3c}}{\left(p_2+p_3\right)^2}+\dots, \tag{155}$$

where the ellipses denote the u and s-channel contributions. From the above expression, we identify the true BCJ kinematic numerators as

$$n_t = X\left(p_1,p_2+p_3\right)X\left(p_2,p_3\right). \tag{156}$$

Now we can hop between BAS, SDYM and SDG perturbative solutions and amplitudes by simply interchanging the appropriate BCJ numerators. For instance, starting from the SDYM second order correction (154), one can obtain its BAS counterpart by simply replacing n_i (where i runs over all n-point numerators) with the corresponding colour factors C_i respectively. Similarly, one can obtain the second order SDG correction (153) by replacing the SDYM colour factors C_i by n_i.

Fortunately, this subtlety involving the identification of the true BCJ numerators is solely a consequence of introducing the kinematic structure constants F, which by definition include delta functions. This was done in [16], to show that colour and kinematic algebras have similar properties. This difficulty does not arise in the remaining of this thesis, where we can identify the BCJ numerators directly from the perturbative corrections.

In this chapter, we have seen how the Kerr-Schild and BCJ double copies are closer related than one would naively expect. Furthermore, we have seen that the BCJ numerators in the integrands of the perturbative corrections were manifestly in BCJ dual form, without requiring field redefinitions or the addition of off-shell terms. This had the consequence that the double copy prescription could immediately be used to hop between the different theories. However, we will soon see that this is an ideal case as it might be challenging to obtain manifest BCJ duality in some cases; with increasing difficulty at higher order. For the latter, the main challenge lies in choosing a specific gauge and field basis. Ideally, one picks the field basis and the gauge choice that yields the closest-to BCJ duality of the off-shell integrands, meaning that little as possible additional off-shell terms and symmetrizations are required to massage the numerators into BCJ dual form.

5 A Triplet of Scalar EFTs and the Double Copy

Most research on the double copy was initially devoted to the duality between gauge theory and gravity. However, in the last decades, many more theories sharing double copy relations have been found. Among the large web of double-copy related theories that are currently known [4, 15, 45], there are three effective scalar theories involving so-called *Goldstone modes* which are the massless scalar modes associated with spontaneous symmetry breaking. The triplet of effective scalar theories involves

1. The ***non-linear sigma model*** .
2. The (multi-field) ***Dirac-Born-Infeld theory*** .
3. The ***special Galileon*** .

These theories are often referred to as being *"exceptional"* [24], not only because of their double copy relations, but also in the sense that their amplitudes have so-called *enhanced soft limits*. The notions of soft limits and enhanced soft-limits characterize the specific behaviour of amplitudes when taking one of the external momenta to be negligible. As will be discussed in chapter 5, the enhanced soft limit is related to nontrivial cancellations among Feynman diagrams of different topology; this is closely related to the invariance under non-linear symmetry transformations.

In addition to the double copy relations, the three theories theories have recently been shown to admit a *Cachazo-He-Yuan (CHY) representation* [46]. The existence of a CHY-representation implies that the tree-level amplitudes can be represented as integrals over the moduli space of punctured Riemann spheres. These integrands carry all information about the theory and the integrands of different theories are related by specific operations on their constitutes. The details of the CHY-representation go beyond the scope of this thesis, however, we should stress that the results of this section are closely related to the corresponding operations performed in [46].

In this chapter, we will first derive the aforementioned scalar EFTs by forcing a specific soft limit on the most general amplitudes of a scalar theory with a maximum of either one or two derivatives per field. As will become clear soon, requiring an enhanced soft limit forces nontrivial cancellations among Feynman diagrams of different topology; consequently completely restricting the form of the Lagrangian and the corresponding non-linear symmetry transformation. For a more formal (and less detailed) treatment of the content in this chapter, the reader is invited to have a look at the corresponding article [26].

5.1 Soft limits and soft theorems

The soft limit of a tree-level amplitude is defined by sending one of the external momentum to zero. In this limit, the amplitude scales as

$$\lim_{p\to 0} A(p) \propto \mathcal{O}(p^{\sigma})\,, \tag{157}$$

where $\sigma \in \mathbb{Z}$ is known as the *soft degree*. If this limit approaches zero or a known factor multiplied by a lower-point tree-level amplitude, then the theory under consideration is said to satisfy a so-called *soft-theorem* [4]. The notion of a soft theorem was initially discovered by Steven Weinberg [47, 48], who found out that gauge theory and gravity amplitudes are singular in the soft limit, with soft degree $\sigma = -1$. For gravity and gauge theory amplitudes, the *soft particle* with negligible momentum interacts with another external particle (let's call it the *hard particle*) through a cubic vertex, implying that the propagator of the hard particle becomes singular as a consequence of the negligible momentum transferred by the soft particle. Specifically, Weinberg found that photon amplitudes have the soft limit [4]

$$\lim_{p\to 0} A_{n+1} = e_\mu S^\mu A_n\,, \tag{158}$$

where e_μ is the polarization vector. Furthermore, the *Weinberg soft factor* S^μ reads

$$S^\mu = \sum_i^n q_i \frac{p_i^\mu}{p_i p}\,, \tag{159}$$

where n labels the hard particles and q_i is the coupling constant associated with each cubic vertex involving a hard particle. The Ward identity then [4] states that the n-point amplitude A_n should be invariant under the transformation $e_\mu \to e_\mu + p_\mu$, leading to the identity

$$\left(\sum_i^n q_i\right) A_n = 0\,, \tag{160}$$

where q_i can be interpreted as the electrical charge of the hard particles. Since A_n is non-vanishing, this identity implies conservation of charge $\sum_i^n q_i = 0$.

Analogous considerations apply to the soft limit of graviton amplitudes. In this case, the vector structure generalizes to the tensor structure [4]

$$\lim_{p\to 0} A_{n+1} = e_{\mu\nu} S^{\mu\nu} A_n \,, \tag{161}$$

where the Weinberg soft factor $S^{\mu\nu}$ is similar to (159), but with an additional momentum factor in the numerator:

$$S^{\mu\nu} = \sum_i^n \kappa_i \frac{p_i^\mu p_i^\nu}{p_i p} \,. \tag{162}$$

In this case, diffeomorphism invariance implies that A_n should be invariant under the transformation $e_{\mu\nu} \to e_{\mu\nu} + \alpha_\mu p_\nu + \alpha_\nu p_\mu$, leading to the identity

$$\left(\sum_i^n \kappa_i p_i^\mu \right) A_n = 0 \,, \tag{163}$$

where κ_i is the coupling constant associated with each cubic vertex involving a hard graviton. Naturally, the total momentum is conserved ($\sum_i^n p_i^\mu = 0$); this implies that (163) can only be satisfied when $\kappa_i \equiv \kappa$. Hence, we have deduced the equivalence principle, which states that *the coupling strength of gravity is universal* [4].

The scalar theories that we consider in this chapter turn out to have amplitudes that vanish in the soft limit, or in other words, they have Adler zero [4, 15]. Contrary to GR and YM, scalar EFTs are not constructible by solely considering physical input such as Lorentz invariance, dimensional analysis and locality. The additional input that is required is non-linearly realised symmetry, which is a global symmetry with consequences for physical observables, contrary to the (local) gauge and diffeomorphism invariance of YM and GR. The most general Lagrangian for a scalar theory satisfying the Adler zero condition reads

$$\mathcal{L}_{(\rho)} = (\partial\phi)^2 F\left(\partial^m \phi^n\right) \tag{164}$$

where $\rho = m/n$ specifies the number of derivatives per field, F is a general function, and m is even to ensure Lorentz invariance.

To understand the enhanced soft limit, we consider the case of one derivative per field, corresponding to $\rho = 1$. Naively, one would expect an amplitude with a linear dependence in each of the momenta, corresponding to soft degree one. However, it could be possible to have highly non-trivial cancellations among diagrams of different topologies as a consequence of relations between the coefficients $\lambda_{m,n}$ of the Lagrangian. For theories with enhanced soft limits, for which the latter is the case, these relations among the coefficients can be seen as a consequence of underlying (possibly non-linearly realised) symmetries that leave the action invariant. The nontrivial cancellations between diagrams of different topology make the part of the amplitude that scales with, for example, $\sigma = 1$, completely vanish. This implies that the leading behaviour becomes $\mathcal{O}(p^2)$, corresponding to soft degree $\sigma = 2$. Hence, we can speak of an enhanced soft limit when the soft degree is greater than one would naively expect based on the number of derivatives per field, or equivalently, when $\sigma > \rho$.

5.2 Scalar EFTs from soft limits

In the conventional field-theoretic approach, one considers symmetries and actions to be primary, while soft theorems are seen as a consequence of the specific properties of the theories such as non-linearly realised symmetries. This philosophy can be inverted by building amplitudes purely based on physical assumptions. The idea of building theories from scattering amplitudes has been a well known strategy. For example, Weinberg derived GR and YM solely based on physical assumptions such as Lorentz invariance, factorization on propagator poles[11], and locality. A similar strategy can be applied to derive scalar EFTs, albeit under the additional assumption of a specific (enhanced) soft degree [21, 22, 23, 24]. The latter then serves as additional physical input.

In this section, we will consider a general scalar EFT Lagrangian of the form (164), where specific choices of derivative counting ρ and (enhanced) soft degree σ completely restrict the form of the Lagrangian $\mathcal{L}_{(\rho,\sigma)}$. Throughout this section, we will be loosely following [23, 24].

5.2.1 Amplitude Ansatze

First we have to constrain the space of all allowed EFTs by constructing an amplitude Ansatz, which is fixed by the following physical requirements [24]: (1) Lorentz invariance, (2) factorization on propagator poles, and (3) a given set of parameters (ρ,σ). If this Ansatz can be satisfied, then the corresponding theory possibly exists.

As we have seen many times in this thesis, every n-point amplitude can be written in terms of kinematic invariants

$$s_{ij} = (p_i + p_j)^2 = 2\,(p_i \cdot p_j)\,, \quad i,j \in \{1,\ldots,n\}\,. \tag{165}$$

This space of kinematic invariants is redundant since the different s_{ij} are related. For example, by conservation of total momentum, $p_1 + \ldots + p_n = 0$, we can already single out one of the momenta. By taking the dot product between the latter and any of the momenta, and by realizing that ${p_i}^2$ vanishes on-shell, we find the additional constraint $\sum_{i,j\neq n} s_{ij} = 0$. Furthermore, five or more particles living in a four-dimensional space-time must be necessarily linearly dependent, leading to even more relations (see [24] for details on these relations). These relations make it difficult to determine a set of independent kinematic invariants, especially at higher-order. Instead, it is more convenient to calculate amplitudes using the complete set of kinematic invariants, and mod out the dependent ones afterwards, reducing the amplitude to the simplest possible form.

For a general $(n+2)$-point amplitude with derivative counting $\rho = m/n$, the schematic

[11] Factorization can be stated as $\lim_{P^2\to 0} A_{n,m} = \sum \frac{A_L A_R}{P^2}$, where $P = \left(p_{i_1} + p_{i_2} + \cdots + p_{i_k}\right)$ and the sum is taken over all internal legs [24]. This implies that the numerators of amplitudes factorize into two lower-point amplitudes (denoted by A_L and A_R) when the propagator P is taken to be on-shell.

amplitude Ansatz is of the form [23]

$$A_{n+2} = \sum_{\alpha} c_{\alpha}^{(0)} \left(s_{\alpha_1} \dots s_{\alpha_{m/2+1}}\right) + \sum_{\alpha,\beta} \frac{c_{\alpha}^{(1)} \left(s_{\alpha_1} \dots s_{\alpha_{m/2+2}}\right)}{s_{\beta}} + \sum_{\alpha,\beta} \frac{c_{\alpha}^{(2)} \left(s_{\alpha_1} \dots s_{\alpha_{m/2+3}}\right)}{s_{\beta_1} s_{\beta_2}} + \dots , \tag{166}$$

where α labels external legs and β labels factorization channels with propagator poles s_{β}. The first sum corresponds to contact diagrams, while the other sums correspond to exchange diagrams of different topologies. In the case of single-scalar theories, this Ansatz should respect invariance under the exchange of any external legs and the individual diagrams of the same topology must be related by permutations of external legs.

For simplicity, we define the soft limit by sending $p \mapsto zp$ for one of the external particles. Then the amplitude can be expanded as

$$A_n = \sum_{s=0}^{\infty} A_{n,s} z^s \,. \tag{167}$$

If we now choose to enforce a soft degree of, say $\sigma = s$, then we require $A_{n,s}$ to vanish for all $s < \sigma$. The latter leads to a set of equations that must be solved for the coefficients c_{α}^{k}. Plugging these coefficients back into the Ansatz (166) results in the n-point amplitude of a specific theory with soft degree s and derivative counting ρ.

At three-point scattering, all scattering amplitudes of scalar field theories involving only a single field vanish since the required permutation invariance always causes the amplitude to be proportional to $s+t+u$ (which vanishes on-shell). However, this does generally not hold in the presence of a flavour (or colour) structure. In this case, the sum $s+t+u$ possibly becomes tangled with group-theoretic structures such as traces over products of generators. So far we can state that, in the case of single-scalar theories, the first nontrivial amplitudes can be found at four-point.

For arbitrary ρ, the amplitude Ansatz is fixed by the $2\rho+2 = 2m+2$ derivatives of the theory. In combination with the fact that the amplitude is required to respect permutation invariance, this leads to a four-point amplitude Ansatz of the form

$$A_4 = \sum_{a_1 a_2 a_3} c_{a_1 a_2 a_3} \left(s_{12}\right)^{a_1} \left(s_{23}\right)^{a_2} \left(s_{31}\right)^{a_3} , \tag{168}$$

where we deduce $a_1 + a_2 + a_2 = \rho + 1$, since every kinematic invariant results from two derivatives. Each kinematic invariant is of soft degree $\sigma = 1$, meaning that we can conclude that each four-point amplitude A_4 is of soft degree $\sigma = \rho + 1$.

5.2.2 $\rho = 0$: Non-linear sigma model

We proceed by considering the simplest case, that is, a theory with no derivatives, corresponding to $\rho = 0$. In this case, the theory is of the schematic form

$$\mathcal{L}_{(0)} = (\partial\phi)^2 F(\phi) \,, \tag{169}$$

and up to a simple field redefinition, the above corresponds to a free theory with trivial amplitudes.

A more interesting situation arises when we allow the scalars to carry colour charge. One of these theories is the non-linear sigma model with Goldstone bosons arising from internal symmetry breaking. The Lagrangian of the NLSM associated with the symmetry breaking pattern $U(N)\times U(N)\mapsto U(N)$, in the commonly studied exponential parametrization, is given by [20]

$$\mathcal{L}_{(0,1)}=\frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\right),\quad U=e^{\frac{i\phi}{F}}, \tag{170}$$

where the Goldstone bosons $\phi\equiv\phi^aT^a$ live in the adjoint representation of $U(N)$ and F is the NLSM coupling constant. This theory is invariant under the non-linear symmetry transformation

$$\phi\to\phi+\epsilon+\mathcal{O}(\phi^2), \tag{171}$$

where ϵ is a constant and $\mathcal{O}(\phi^2)$ accounts for terms that are of higher order in ϕ; this term implies the breaking of axial symmetry. The four-point colour-ordered amplitude, or more specifically, the part of the amplitude that is proportional to $\mathrm{Tr}[T^aT^bT^cT^d]$, is given by

$$A_4[1,2,3,4]=-\frac{F^2}{2}(t+s). \tag{172}$$

Alternatively, one could rewrite the amplitude in terms of $U(N)$ structure constants f^{abc} by using relations of the form (36). The full four-point amplitude in terms of structure constants will be derived in section 5.3, where we make the connection with the double copy.

5.2.3 $0<\rho<1$: A non-vanishing soft limit

In the case $0<\rho<1$, not every field is equipped with a derivative. This corresponds to so-called *non derivatively coupled theories* and it turns out that these theories do not have a vanishing soft limit [23]. This can easily be shown by contradiction. For instance, a vanishing soft limit requires that $A_n\to 0$ when $p\to 0$, for each external leg. The latter, together with permutation invariance, leads to the (schematic) amplitude Ansatz

$$A_n=p_1^{\mu_1}p_2^{\mu_2}\dots p_n^{\mu_n}L_{\mu_1\mu_2\dots\mu_n}, \tag{173}$$

where $L_{\mu_1\mu_2\dots\mu_n}$ is a symmetric tensor which contains momenta and Minkowski metrics. The n momenta in the above follow from n derivatives, while the tensor $L_{\mu_1\mu_2\dots\mu_n}$ possibly contains even more momenta, implying that the total number of derivatives is equal to or greater than the number of fields, or $\rho\geq 1$. This obviously contradicts our initial assumption $0<\rho<1$ and hence we conclude that derivative coupling is a necessary requirement to have a vanishing soft limit.

5.2.4 $\rho=1$: Dirac-Born-Infeld theory

The next case that we consider is one derivative per field, corresponding to $\rho=1$, such that the theory is derivatively coupled and has a vanishing soft limit with $\sigma=1$. In this case, the Lagrangian is schematically given by

$$\mathcal{L}_{(1),(1)} = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda_4}{4!}(\partial\phi)^4 + \frac{\lambda_6}{6!}(\partial\phi)^6 + \dots, \tag{174}$$

and corresponds to the well known *Nambu-Goldstone Boson* [4]. Since every field carries a derivative, all particle labels must appear in the amplitude; this respectively leads to the four- and six-point amplitude Ansatzes

$$\begin{aligned} A_4 &= c_4 \left(s_{12}s_{34} + s_{13}s_{24} + s_{14}s_{23}\right), \\ A_6 &= 2c_4^2 \left[\frac{\widetilde{s_{123}}\widetilde{s_{456}}}{s_{123}} + \dots\right] + c_6 \left(s_{12}s_{34}s_{56} + \dots\right), \end{aligned} \tag{175}$$

where $s_{123} = s_{12} + s_{23} + s_{31}$, $\widetilde{s_{123}} = s_{12}s_{23} + s_{23}s_{31} + s_{31}s_{12}$, and the ellipses denote permutations. In addition to this, we enforce an enhanced soft limit $\sigma = 2$, meaning that we force the linear part $A_{n,1}$ to vanish. For A_4, this is obviously satisfied for any c_4, since each kinematic invariant has soft degree one and the product of two invariants vanishes as $\mathcal{O}(z^2)$, corresponding to soft degree $\sigma = 2$.

At six-point, this forces cancellations between the contributions of $\mathcal{O}(z)$ of the different topologies, leading to the constraint $c_6 = 2c_4^2$ on the coefficients. This straightforwardly extends to arbitrary higher order, eventually fixing all higher order coefficients in terms of c_4. It turns out that the corresponding theory is given by a Lagrangian of the form [23]

$$\mathcal{L}_{(1,2)} = -\frac{1}{g}\sqrt{1 + g(\partial\phi)^2} = -\frac{1}{2}(\partial\phi)^2 + \frac{g}{8}(\partial\phi)^4 - \frac{g^2}{16}(\partial\phi)^6 + \dots, \tag{176}$$

where we defined the coupling constant $g = 2c_4$. This theory corresponds to the well known *Dirac-Born-Infeld theory*, describing massless Goldstone modes (here in four dimensions) corresponding to the symmetry breaking pattern $ISO(1,4)/ISO(1,3)$. The theory is invariant under the non-linear symmetry transformation $\delta\phi = c + b_\mu x^\mu + b_\mu \phi \partial^\mu \phi$, where b_μ is an arbitrary four-vector. This non-linear symmetry manifests invariance under higher dimensional Lorentz boosts and rotations from the higher dimensional Lorentz group. Physically, the DBI can be interpreted as a four-dimension Dirichlet-brane (i.e., a brane with fixed boundary conditions) fluctuating in an additional fifth dimension. The scalar field ϕ then describes the deviation into the fifth dimension, whereas its derivative $\partial\phi$ has the interpretation of velocity.

5.2.5 $\rho = 2:$ The special Galileon

We now increase the amount of derivatives by allowing more than one derivative per field. Specifically, we distribute $2n-2$ derivatives over n fields, with each field equipped with at least one derivative. The schematic Lagrangian then reads

$$\mathcal{L}_{(2,1)} = (\partial\phi)^2 \sum_{n=2}^{\infty} F_n\left(\partial^{2n-2}\phi^n\right). \tag{177}$$

Additionally, we assume a soft degree $\sigma = 2$, which means that we force $A_{n,1}$ to vanish. At four-point, we have two structures that are compatible with the Ansatz,

$$A_4 = c_1\left(s_{12}^3 + s_{23}^3 + s_{31}^3\right) + c_2\left(s_{12}s_{23}s_{31}\right). \tag{178}$$

Both terms are of order three in kinematic invariants, implying that the four-point amplitude naturally has soft degree $\sigma = 3$, which is one more than we initially required. Up to seven-point, the requirement $A_{2,n} = 0$ uniquely fixes the coefficients, with soft behaviour $A_5 = \mathcal{O}\left(z^2\right)$, $A_6 = \mathcal{O}\left(z^3\right)$, and $A_7 = \mathcal{O}(z^2)$, respectively [23]. At eight-point, $\sigma = 2$ yields two solutions, whereas $\sigma = 3$ yields one unique solution. The solution with only even-point interactions therefore has an enhanced soft limit $\sigma = 3$. This theory can be obtained by setting $c_1 = 0$, such that each subsequent term with an even number of fields is uniquely fixed in terms of c_2; this works analogously to what we have seen for DBI. The theory corresponds to the *special Galileon*. The Lagrangian in the quartic formulation (and four space-time dimensions) contains only two terms and explicitly reads [4]

$$\mathcal{L}_{\mathrm{SG}} = -\frac{1}{2}(\partial\phi)^2\left[1 - \frac{1}{6L^6}\left((\Box\phi)^2 - (\partial^\mu\partial^\nu\phi)(\partial_\mu\partial_\nu\phi)\right)\right], \tag{179}$$

where we will later use the redefined coupling constant $\Lambda = L^{-6}/6$ for notational simplicity. The Goldstone mode has the non-linear symmetry

$$\delta\phi = c + c_\mu x^\mu + c_{\mu\nu}L^{-6}\left(x^\mu x^\nu + \partial^\mu\phi\partial^\nu\phi\right), \tag{180}$$

where $s_{\mu\nu}$ is a symmetric tensor. The non-linear part of this symmetry corresponds to the coset $ISU(4)/SO(4)$. The special Galileon is sometimes referred to as the scalar analogue of GR, owing to the fact that its non-linear symmetry has a correspondence with the diffeomorphism invariance of GR [49].

The soft-behaviour of the triplet of scalar EFTs that we discussed in this section is visualized in the *"periodic table"* of figure 10; it is graphically indicated that these theories cannot be constrained any further. Note that this figure contains two theories that have not yet been discussed. The first of these is the $P(X)$ theory (as it is referred to in the context of inflationary cosmology), which is a general class of theories of the form

$$\mathcal{L} = g^d P\left(\frac{\partial\phi\cdot\partial\phi}{g^d}\right), \tag{181}$$

where d is the number of space-time dimensions. It is easy to see that the DBI theory as discussed above is a special case of the $P(x)$ theory. The $P(X)$ theory is invariant under the shift symmetry $\phi \to \phi + c$, and has a trivial soft limit since the number of derivatives per field is equal to the soft degree ($\rho = \sigma = 1$).

The second theory (denoted by WZW) is given by the Lagrangian

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu\phi\partial^\mu\phi\right) + \lambda\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\phi\partial^\mu\phi\partial^\nu\phi\partial^\alpha\phi\partial^\beta\phi\right), \tag{182}$$

and it follows from the $\phi \to 0$ limit of the so-called Wess-Zumino-Witten (WZW) term

$$\mathcal{L}_{WZW} = i\lambda\epsilon^{\mu\nu\rho\alpha\beta}\,\mathrm{Tr}\left(U^\dagger\partial_\mu U U^\dagger\partial_\nu U U^\dagger\partial_\rho U U^\dagger\partial_\alpha U U^\dagger\partial_\beta U\right) \tag{183}$$

where U is defined as in (170), $\epsilon^{\mu\nu\rho\alpha\beta}$ is the totally anti-symmetric Levi-Civita tensor, and λ is a constant. This theory is shift symmetric; in the four-dimensional formulation as it is given here, this theory has derivative counting $\rho = 3/2$ and soft degree $\sigma = 1$, and hence an enhanced soft limit.

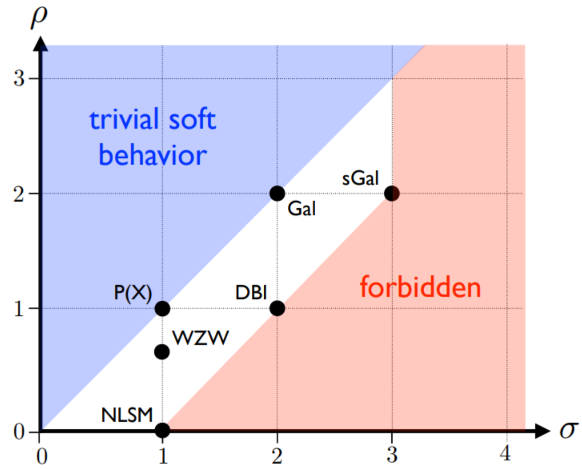


Figure 10: Visualization of the allowed EFTs for given derivative counting ρ and soft degree σ. The blue region contains EFTs with trivial soft-behaviour which can be anticipated by naive derivative counting. The white region contains EFTs with enhanced soft limits and the red region is forbidden by consistency constraints. The triplet of exceptional scalar EFTs (NLSM, DBI and SG) lie on the boundary of the white and red region, indicating that they cannot be constrained any further. This figure was taken from [24].

5.3 The canonical double copy of the NLSM, DBI and SG

Before we discuss our main results, we first review the well known double copy relations between the three theories, since it served as the initial motivation to study possible off-shell aspects. To explicitly show the amplitude double copy relations, we will use the perturbative formalism of the previous chapter. We show how (new types) of BCJ numerators arise and that these can be made to satisfy colour-kinematics duality upon addition of off-shell terms. Together with the earlier encountered YM colour factors and tensor-kinematic numerators, the new types of numerators discussed in this thesis constitute the full set of BCJ numerators that are currently known. Here we restrict to the already well known amplitude double copy relations between the triplet of theories, and in chapter 7 we will turn to an in-depth investigation of the relevant BCJ numerators and amplitude structures.

5.3.1 NLSM and the BCJ formulation

Let's first turn to the $U(N)$ non-linear sigma model (as discussed in the previous section (170)), which is one of the most studied non-linear sigma models together with the very similar $SU(N)$ NLSM. Expanding the Lagrangian (170) in terms of the Goldstone bosons yields [20]

$$\mathcal{L}_{\text{NLSM}} = -\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi^a + \frac{\lambda^2}{6} f_{abe} f^{cde} \phi^a \partial_\mu \phi^b \phi^c \partial^\mu \phi^d + \cdots \,, \tag{184}$$

which gives rise to the field equation

$$\Box \phi^a - \frac{\lambda^2}{3} f^{abe} f^{ecd} \left(\partial^\mu \phi^b \phi^c \partial_\mu \phi^d + \phi^b \phi^c \Box \phi^d\right) + \ldots = J^a \,, \tag{185}$$

where J^a is a classical source. By expanding the field in the coupling λ and subsequent substitution into the equation of motion (up to quartic interactions), we obtain the differential equations

$$\begin{aligned} \Box \phi^{(0)a} &= J^a \,, \\ \Box \phi^{(1)a} &= \frac{1}{3} f^{abe} f^{ecd} \left(\partial^\mu \phi^{(0)b} \phi^{(0)c} \partial_\mu \phi^{(0)d} + \phi^{(0)b} \phi^{(0)c} \Box \phi^{(0)d}\right) . \end{aligned} \tag{186}$$

Fourier transforming to momentum space leads to

$$\phi^{(0)a}(p_1) = -\frac{J^a}{p_1^2} \,, \tag{187}$$

$$\phi^{(1)a}(-p_1) = \frac{1}{3p_1^2} \int đ^4 p_2 đ^4 p_3 đ^4 p_4 \delta\!\!\!{}^-{}^{(4)}(\sum_{i=1}^{4} p_i) f^{abe} f^{ecd} \left[(p_2 \cdot p_4) + p_4^2\right] \phi^b(p_2) \phi^c(p_3) \phi^d(p_4) \,, \tag{188}$$

where all ϕ's on the right-hand side are actually $\phi^{(0)}$'s. Interchanging external legs $3 \leftrightarrow 4$ (corresponding to the replacements $c \leftrightarrow d$ and $p_3 \leftrightarrow p_4$) on the latter yields

$$\phi^{(1)a}(-p_1) = -\frac{1}{3p_1^2} \int đ^4 p_2 đ^4 p_3 đ^4 p_4 \delta\!\!\!{}^-{}^{(4)}(\sum_{i=1}^{4} p_i) f^{abe} f^{ecd} \left[(p_2 \cdot p_3) + p_3^2\right] \phi^b(p_2) \phi^c(p_3) \phi^d(p_4) \,, \tag{189}$$

where we used the anti-symmetry property of the structure constants $f^{edc} = -f^{ecd}$. Next, we add (187) and (189) such that we obtain

$$\begin{aligned} \phi^{(1)a}(-p_1) = -\frac{1}{6p_1^2} \int & đ^4 p_2 đ^4 p_3 đ^4 p_4 \delta\!\!\!{}^-{}^{(4)}(p_1 + p_2 + p_3 + p_4) f^{abe} f^{ecd} \\ & \times \left[(p_2 \cdot p_4) + p_4^2 - (p_2 \cdot p_3) - p_3^2\right] \phi^b(p_2) \phi^c(p_3) \phi^d(p_4) \,, \end{aligned} \tag{190}$$

where the second and last term of the integrand in square brackets vanish on-shell. Application of the LSZ formula (129) and including all inequivalent permutations yields the NLSM four-point amplitude

$$A_4^{\text{NLSM}} = \frac{\lambda^2}{12} \left[f^{abe} f^{ecd}(t-u) + f^{bce} f^{ead}(u-s) + f^{cae} f^{ebd}(s-t)\right] . \tag{191}$$

Upon performing the splitting procedure that we discussed in 2 (i.e., multiplying each term by the appropriate s_{ij}/s_{ij}), we can finally write the NLSM four-point amplitude as the BCJ factorization[12]

$$A_4^{\text{NLSM}} \sim \left(\frac{c_s R_s}{s} + \frac{c_t R_t}{t} + \frac{c_u R_u}{u} \right) , \tag{192}$$

where the $c_s = f^{abe} f^{ecd}$ are the previously encountered colour factors and the $R_s = s(t-u)$ will be referred to as *scalar-kinematic numerators*, owing to their dependence on (solely) kinematic invariants. The scalar-kinematic numerators can be easily checked to satisfy CK duality, that is,

$$R_s + R_t + R_u \sim s(t-u) + t(u-s) + u(s-t) = 0 \,. \tag{193}$$

These numerators are well known in the literature (see e.g. [36]) and we will soon see that these, along with other types of numerators, constitute the amplitude building blocks of the three theories.

When discussing double copy relations (like above), we only consider the structure of the amplitudes, indicated by the symbol $\sim$, while we neglect the constant prefactors that should additionally be mapped onto each other; this can always be achieved by allowing numerical coefficients in the ratios of their coupling constants.

Although the BCJ formulation of the amplitude (192) contains a pole structure, the poles are absent in the final amplitude (191); this reflects the fact that the NLSM does not possess three-point vertices, and therefore that the corresponding Feynman diagrams are contact diagrams.

5.3.2 DBI and the BCJ formulation

Next, we turn to the DBI. The DBI Lagrangian (176) gives rise to the equation of motion

$$\Box\phi - \frac{1}{2} l^2 \left(\Box\phi (\partial\phi)^2 + \partial^\mu \phi \partial_\nu \phi \partial^\nu \partial_\mu \phi + \partial^\mu \phi \partial^\nu \phi \partial_\mu \partial_\nu \phi \right) + \ldots = J \,. \tag{194}$$

Like before, we expand the field in the coupling constant l and upon substituting this expansion into the equation of motion, we obtain one equation for each perturbative correction. For the leading and sub-leading perturbative corrections, these equations respectively read

$$\begin{aligned} &\Box\phi^{(0)} = J \,, \\ &\Box\phi^{(1)} = \frac{1}{2} l^2 \left(\Box\phi^{(0)} (\partial\phi^{(0)})^2 + \partial^\mu \phi^{(0)} \partial_\nu \phi^{(0)} \partial^\nu \partial_\mu \phi^{(0)} + \partial^\mu \phi^{(0)} \partial^\nu \phi^{(0)} \partial_\mu \partial_\nu \phi^{(0)} \right) . \end{aligned} \tag{195}$$

By Fourier transforming to momentum space, the latter can be expressed as

$$\phi^{(1)}(-p_1) = -\frac{1}{2p_1^2} \int đ^4 p_2 đ^4 p_3 đ^4 p_4 \delta\!\!\!^{-}(p_1 + p_2 + p_3 + p_4) \left[p_2^2 + (p_3 \cdot p_4)(p_2 \cdot p_3) + (p_2 \cdot p_4)(p_3 \cdot p_4) \right] . \tag{196}$$

[12] The BCJ factorization of the NLSM was first discovered in [50]

Finally, we include all inequivalent permutations and use the LSZ method to extract the four-point amplitude. We obtain

$$A_4^{\rm DBI} = \frac{l^2}{4}\,(us+st+tu)\,. \tag{197}$$

Addition of an off-shell term proportional to $(s+t+u)^2$ finally allows us to write the four-point amplitude as the BCJ factorization

$$A_4^{\rm DBI} \sim (\frac{\tilde{f}_s R_s}{s} + \frac{\tilde{f}_t R_t}{t} + \frac{\tilde{f}_u R_u}{u})\,, \tag{198}$$

where $\tilde{f}_s = t-u$ are a new type of BCJ numerators, which we will refer to as *flavour factors*. The "flavour" part of the name refers to the fact that the above form actually represents the single-flavour version (as indicated by the tilde) of the flavour factors $f_s = \delta^{ab}\delta^{cd}(t-u) - s(\delta^{ca}\delta^{bd} - \delta^{cb}\delta^{ad})$.[13] These flavour factors without the tilde are BCJ numerators belonging to the *multi-field extension of DBI* (or multi-DBI), with the Lagrangian in arbitrary space-time dimensions given by [51]

$$\mathcal{L}_{\rm mDBI} = -\frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi + \frac{1}{4}\left(\partial_\mu\phi\cdot\partial_\nu\phi\right)\left(\partial^\mu\phi\cdot\partial^\nu\phi\right) - \frac{1}{8}\left(\partial_\mu\phi\cdot\partial^\mu\phi\right)^2 + \ldots\,, \tag{199}$$

where the scalar fields live in the fundamental representation of $SO(N)$. Furthermore, we used dot-products to denote flavour contractions, i.e., $\phi^a\phi^a \equiv \phi\cdot\phi$. The corresponding non-linear symmetry, associated with the coset $ISO(4+D)/SO(D)\times SO(N)$, is given by

$$\delta\phi^a = c^a + c^a{}_\mu x^\mu + c^b{}_\mu \phi^b \partial^\mu \phi^a\,. \tag{200}$$

From this multi-field extension, the single-DBI flavour factors, non-linear symmetry and Lagrangian can easily be recovered by (1) setting Kronecker deltas equal to unity $\delta^{ij}=1$, and (2) removing the flavour indices from the fields $\phi^i = \phi^j \equiv \phi$.

5.3.3 SG and the BCJ formulation

Finally, we turn to the special Galileon (179), with the equation of motion

$$\Box\phi + \Lambda^6\left[(\Box\phi)^3 - 3(\Box\phi)\left(\partial^\mu\partial^\nu\phi\right)\left(\partial_\mu\partial_\nu\phi\right) + 2\left(\partial^\mu\partial^\alpha\phi\right)\left(\partial_\alpha\partial_\beta\phi\right)\left(\partial^\beta\partial_\mu\phi\right)\right] = J\,. \tag{201}$$

Expanding the field in the coupling Λ, and substitution into the equation of motion yields the following equations,

$$\begin{aligned} &\Box\phi^{(0)} = J\,,\\ &\Box\phi^{(2)} = \Lambda^6\left[-(\Box\phi^{(0)})^3 + 3\Box\phi^{(0)}\partial^\mu\partial^\nu\phi^{(0)}\partial_\mu\partial_\nu\phi^{(0)} - 2\partial^\mu\partial^\alpha\phi^{(0)}\partial_\alpha\partial_\beta\phi^{(0)}\partial^\beta\partial_\mu\phi^{(0)}\right]. \end{aligned} \tag{202}$$

[13]Similar to the scalar-kinematic numerators, these can easily be checked to satisfy BCJ duality.

In momentum space, the latter can be written as

$$\phi^{(2)}(-p_1) = \frac{1}{p_1^2}\int d\!\!{}^-{}^4 p_2 d\!\!{}^-{}^4 p_3 d\!\!{}^-{}^4 p_4 \delta\!\!{}^-{}^4(p_1+p_2+p_3+p_4)$$
$$\times\left[-p_2^2p_3^2p_4^2 + 3p_2^2(p_3\cdot p_4)^2 - 2(p_2\cdot p_3)(p_3\cdot p_4)(p_4\cdot p_2)\right]\phi^{(0)}(p_2)\phi^{(0)}(p_3)\phi^{(0)}(p_4)\,, \tag{203}$$

where the first two terms inside the integrand vanish on-shell. The remaining term is already symmetric under the exchange of any leaf legs. By application of the LSZ formula (as given in (129)), and by including all inequivalent permutations, we obtain the four-point amplitude

$$A_4^{\mathrm{SG}} = -\frac{1}{24L^6}stu\,, \tag{204}$$

where we recovered the original cut-off scale L. The above coincides (only on-shell) with the BCJ factorization

$$A_4^{\mathrm{SG}} \sim \left(\frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}\right). \tag{205}$$

By comparison of each four-point amplitude (see equations (192, 198, 205)) in terms of BCJ factorization, we note that they all contain a scalar-kinematic numerator and therefore obey a double copy relation.

Interestingly, in chapter 7, we will point out a different type of NLSM, coupled to gravity, which follows from the BCJ factorization of two flavour factors. This excludes flavour factors and hence establishes a double copy involving only two types of numerators.

6 Off-Shell Flavour-Kinematics Duality for Goldstone Modes

In the previous chapter, we have seen that the amplitudes of the NLSM, DBI and SG share double copy relations. However, in the canonical formulation (which was used in the previous section) there seems to be no obvious relation between their equations of motion. Looking back to chapter 3, we have seen that it was essential to work in the right field basis and gauge choice for the Kerr-Schild double copies to be manifest. In this case, similar considerations apply but instead of gauge and diffeomorphism freedom we are now dealing with non-linear symmetries; these directly depend on field basis.

In this section, we first present new considerations on the non-linear symmetries that result in manifest flavour-kinematics duality at the level of (off-shell) equations of motion and non-linear symmetries. To achieve this, we will see that we only have to adapt a specific field basis for the NLSM, whereas the (multi-field) DBI and the SG remain in their canonical (but D-dimensional) formulation.[14] Thereafter, we calculate the classical non-linear solutions in response to a point-like charge and investigate possible mapping relations between them.

[14] As will become clear soon, the D-dimensional DBI and NLSM are equivalent to the 4-dimensional formulation, whereas the SG contains higher order terms.

6.1 Non-linear symmetry transformations of the same type

Perhaps the intuitive strategy to proceed would be to adapt field bases in such a way that each Lagrangian generates Feynman rules that manifest BCJ duality, or in other words, such that the integrands of the perturbative corrections (or *off-shell amplitudes*) exactly match the off-shell BCJ factorization.

As an example, we consider DBI theory. In section 5.3, we found that the off-shell amplitude of single-DBI (in the canonical field basis) is given by[15]

$$-\frac{1}{4}(us+ut+ts)\,, \tag{206}$$

whereas the BCJ factorization reads

$$\frac{R_s f_s}{s}+\frac{R_t f_t}{t}+\frac{R_u f_u}{u}=8(s^2+t^2+u^2-ut-ts-us)\,. \tag{207}$$

Hence, we should perform a field redefinition that transforms the off-shell amplitude such that the result is proportional to the BCJ factorization. In this particular case, this can be achieved by performing a field redefinition of the form $\phi \longmapsto \alpha\phi+\beta\phi(\partial\phi)^2$, where α and β are constants. This Ansatz gives rise to additional terms of the form Mandelstam squared, which allows the new integrand to potentially match (207). Demanding that the resulting off-shell amplitude after the field redefinition is proportional to the BCJ sum yields a system of two equations in α and β, with the unique solution given by $(\alpha,\beta)=(1,1/6)$. Similar considerations apply to higher orders, where the field redefinition must be extended by the addition of higher order terms, implying that it eventually becomes an arbitrarily long series. As one could imagine, it would be a tedious job to do the above for each theory and order by order. Furthermore, the non-linear symmetries also become arbitrary long expressions due to the field dependence.

In this section, we will show that the field basis for which an off-shell double copy relations manifest itself is one where all non-linear symmetries contain the same type of terms, or as we will sometimes refer to it, a "comparable field basis".

By comparison of the non-linear symmetries of DBI and SG, we note that each non-linear symmetry consists of two parts. The first part is a generalized shift term, consisting of parameters (possibly with space-time contractions) with no field dependence, while the second term is quadratic in the field and consists of parameters and one or two space-time derivatives (i.e., one for DBI and two for SG). On the other hand, the non-linear symmetry of the NLSM consists of a generalized shift term and field dependent terms of at least quadratic order, without any derivatives. Based on these observations, the most natural way to have non-linear symmetries with the same type of terms would be to perform a field redefinition such that the NLSM also obtains a non-linear symmetry of the schematic form $\delta\phi=\mathcal{O}(\phi^0)+\mathcal{O}(\phi^2)$.

[15]Note that we have included the appropriate prefactors, while we have set the DBI coupling constant equal to unity.

In order to make the connection with the multi-DBI involving flavour structure, we will here work with a NLSM corresponding to the following coset structure:

$$\frac{SO(M+N)}{SO(M)\times SO(N)}\,. \tag{208}$$

Note that, in case of a relative minus sign between the two terms of the non-linear symmetry, the above would have isometry group $SO(M,N)$ instead. The scalars (denoted by $\phi^{a\bar{a}}$) live in the bi-fundamental representation of $SO(N)$ and $SO(M)$. These flavour structures (as indicated by the indices a and $\bar{a}$) are independent and could have different dimensions (i.e., when $N\neq M$). The non-linear symmetry that realises this coset is given by

$$\delta\phi = c+\phi c^T\phi\,, \tag{209}$$

where we adopted the matrix notation $\phi^{a\bar{a}}\equiv\phi$, such that the above corresponds to $\delta\phi^{a\bar{b}} = c^{a\bar{b}}+\phi^{a\bar{c}}c^{d\bar{c}}\phi^{d\bar{b}}$, in terms of flavour indices. Note that we have suppressed the NLSM coupling constant in the field dependent term. The motivation to study this specific NLSM with two flavour structures, instead of the NLSM containing colour, will be further motivated when we turn to the on-shell amplitude in section 7. There we will see that the amplitude of this NLSM coupled to gravity follows from the BCJ factorization involving two flavour factors (instead of the colour factor and scalar-kinematic numerator of the $SU(N)$ NLSM).

The above NLSM obviously differs from the $SU(N)$ model discussed earlier. However, it is not totally unrelated: when identifying the barred and unbarred indices $a=\bar{a}$ (which requires $M=N$), we can specialize to a symmetric or anti-symmetric field $\phi\equiv\pm\phi^T$; the symmetric case corresponds to the coset structure $SL(N)/SO(N)$, whereas the anti-symmetric case corresponds to coset $(SO(N)\times SO(N))/SO(N)_{\rm diag}$. Since it is possible to embed $SU(N/2)$ in $SO(N)$, one could recover the $SU(N)$ NLSM.

Returning to the above $SO(M+N)$ NLSM (on which we will focus here), the lowest order invariant Lagrangian contains two derivatives, and it can be phrased in terms of the group element g as follows:

$$\mathcal{L} = \frac{F^2}{4}\left[\partial g\partial g^{-1}\right], \tag{210}$$

where [...] denotes a trace over flavour indices and F is the NLSM cut-off scale, which we set to unity for readability. A possible representation of the $SO(M+N)$ group element is given by

$$g=\begin{pmatrix} A & B\\ -B^T & C\end{pmatrix}, \tag{211}$$

where the matrices $\{A,B,C,D\}$ can be written terms of the $M\times N$ Goldstone modes ϕ as follows:

$$A=\frac{1-\phi\phi^T}{1+\phi\phi^T}\,,\qquad B=\frac{2}{1+\phi\phi^T}\phi\,,\qquad C=\frac{1-\phi^T\phi}{1+\phi^T\phi}\,. \tag{212}$$

These indeed lead to the correct dimensions and properties for the $SO(M+N)$ group element. Using the above, the Lagrangian (210) can be written as

$$\mathcal{L} = -\frac{1}{2}\left[\frac{1}{1+\phi\phi^T}\partial\phi\frac{1}{1+\phi^T\phi}\partial\phi^T\right], \tag{213}$$

where we used $[\ldots]$ to denote the trace over flavour indices. Upon isolating the d'Alembertian, the corresponding equations of motion can be written as

$$\Box\phi = \sum_{n=1}^{\infty}(-1)^{n-1}2\left(\partial_\mu\phi\right)\phi^T\left(\phi\phi^T\right)^{n-1}\left(\partial^\mu\phi\right), \tag{214}$$

where we used the $M\times N$ matrix notation.

Next, we turn to multi-DBI theory, defined by the Lagrangian (199) and non-linear symmetry (200). Since this the non-linear symmetry is already of the form $\delta\phi = \mathcal{O}(\phi^0)+\mathcal{O}(\phi^2)$, it is already in the desired form. In D-dimensions, the symmetry breaking pattern reads

$$\frac{ISO(D+N)}{SO(D)\times SO(N)}, \tag{215}$$

where one of the dimensions D is time-like. In the case of a relative minus sign between the terms of the non-linear symmetry, the isometry group is given by $ISO(D,N)$ instead. The corresponding equation of motion (after simple manipulations) can be written as

$$\Box\phi^a = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\left(\partial\partial\phi^a\right)\left(\partial\phi\cdot\partial\phi\right)^n\right], \tag{216}$$

where the trace involves space-time indices.

Finally, we have the SG with non-linear symmetry (180); it is also already in the desired form with $\delta\phi = \mathcal{O}(\phi^0)+\mathcal{O}(\phi^2)$. However, in D dimensions the symmetry breaking pattern generalizes to

$$\frac{ISU(D)}{SO(D)}, \tag{217}$$

or with $ISL(D)$ instead of $ISU(D)$, when there is a relative sign difference in the non-linear symmetry (180). The D-dimensional Lagrangian can be written as an infinite series over all Galileon terms with an even number of fields [52], and it explicitly reads

$$\mathcal{L}_{\mathrm{SG}} = -\frac{1}{2}\sum_{n=1}^{\lfloor\frac{D+1}{2}\rfloor}\frac{(-1)^{n-1}}{(2n-1)!}(\partial\phi)^2\mathcal{L}^{TD}_{2n-2}, \tag{218}$$

where the total-derivative terms (denoted by $\mathcal{L}^{TD}$) are given by

$$\mathcal{L}^{\mathrm{TD}}_n = \sum_p(-1)^p\eta^{\mu_1p(\nu_1)}\eta^{\mu_2p(\nu_2)}\cdots\eta^{\mu_np(\nu_n)}\left(\Phi_{\mu_1\nu_1}\Phi_{\mu_2\nu_2}\cdots\Phi_{\mu_n\nu_n}\right), \tag{219}$$

with the sum running over all permutations p of the indices ν. Furthermore, $(-1)^p$ is the sign of the permutation. In the above, we adopted the matrix notation $\Phi^\mu{}_\nu \equiv \partial^\mu \partial_\nu \phi$, such that $[\Phi]^2 = (\Box\phi)^2$, $[\Phi^2] = \partial^\mu \partial^\nu \phi_\mu \partial_\nu \phi$ and so forth. Adopting this notation, the three leading terms are given by

$$\mathcal{L}_0^{\mathrm{TD}} = 1, \quad \mathcal{L}_2^{\mathrm{TD}} = [\Phi]^2 - \left[\Phi^2\right], \quad \mathcal{L}_4^{\mathrm{TD}} = [\Phi]^4 - 6\left[\Phi^2\right][\Phi]^2 + 8\left[\Phi^3\right][\Phi] + 3\left[\Phi^2\right]^2 - 6\left[\Phi^4\right], \tag{220}$$

and the resulting field equation can be written in the compact form

$$\Box\phi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}\left[\Phi^{2n+1}\right]. \tag{221}$$

Finally, it is important to note that the NLSM is special in the sense that it allows for a specific type of freedom in its construction. This can be seen from the following two different perspectives. First, we note that the NLSM non-linear symmetry (209) is the only one with constant (that is, space-time independent) parameters; this implies that this NLSM is unaffected after coupling it to gravity, and it is left unaffected by the choice of gravitational background. The coupling to gravity introduces an additional coupling constant, which in this case corresponds to the (reduced) Planck mass (or equivalently, Newtons gravitational constant).

Secondly, this can be seen from the NLSM coset construction (208). Since the parameters of the DBI and SG non-linear symmetries are space-time dependent, they necessarily respect Lorentz symmetry. On the other hand, the space-time independence of the NLSM non-linear symmetry implies that the gravitationally coupled NLSM corresponds to a product of two cosets, with one corresponding to the NLSM scalar sector, and the other one forming the gravitational background. Restricting to a flat gravitational background, the latter takes the form of the Poincaré group over the Lorentz group, and the coset product reads

$$\frac{ISO(D)}{SO(D)} \times \frac{SO(M+N)}{SO(M)\times SO(N)}, \tag{222}$$

where the space-time coordinates x^μ and the NLSM scalars $\phi_{a\bar{b}}$ are the broken generators. The existence of this coset product leads to the possibility to introduce a coupling constant (or cut-off scale) for each coset.

With the triplet of theories in the above formulation, the non-linear symmetries have the same type of terms consisting of a generalized shift-term and a non-linear term that is quadratic in the field, possibly including space-time derivatives. Furthermore, after isolating the d'Alembertian, each equation of motion can be written as a simple infinite series, with the different theories being distinguished by (1) their constant coefficients, (2) the number of space-time derivatives per field in the $\mathcal{O}(\phi^2)$ term of the symmetry, and (3) the number of flavour structures. The non-linear symmetries and equations of motion are summarized in table 2, which emphasizes their similarities.

	Non-linear symmetry	Equation of motion
NLSM	$\delta\phi = c + \phi c^T \phi$	$\Box\phi = \sum_{n=1}^{\infty} (-1)^{n-1} 2 (\partial_\mu \phi)\, \phi^T \left(\phi\phi^T\right)^{n-1} (\partial^\mu \phi)$
DBI	$\delta\phi^a = c^a + c^a{}_\mu x^\mu + c^b{}_\mu \phi^b \partial^\mu \phi^a$	$\Box\phi^a = \sum_{n=1}^{\infty} (-1)^{n-1} [(\partial\partial\phi^a)(\partial\phi\cdot\partial\phi)^n]$
SG	$\delta\phi = c + c_\mu x^\mu + c_{\mu\nu}(x^\mu x^\nu + \partial^\mu\phi\partial^\nu\phi)$	$\Box\phi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} \left[\Phi^{2n+1}\right]$

Table 2: The three scalar EFTs with their non-linear symmetries and field equations. The off-shell mapping, as will be discussed in section 6, relates the non-linear symmetries and equations of motion.

6.2 Off-shell flavour-kinematics duality

As mentioned before, it turns out that the above formulation facilitates mapping relations between the three theories. In this section, we will illustrate how a systematic interchange of flavour and kinematic information maps the non-linear symmetries and equations of motion onto each other.

We will start with the SG non-linear symmetry and map this subsequently onto the DBI and NLSM non-linear symmetries. First, we note that the SG non-linear symmetry can be written as

$$\delta\phi = p + \frac{1}{2}\partial^\mu\phi\partial_{\mu\nu}p\partial^\nu\phi\,, \tag{223}$$

where p is a quadratic polynomial in space-time coordinates, which can be written as $p = c + c_\mu x^\mu + c_{\mu\nu}x^\mu x^\nu$. In order to transform kinematic to flavour information and thereby unify the three theories, we expand p linearly in the *auxiliary flavour coordinate* θ^a, i.e., we write $p = \theta^a p_a$, where p_a is at most linear in space-time coordinates. By substituting this expansion into the SG non-linear symmetry (223), and summing over both types of indices, we obtain[16]

$$\delta\phi = p + \partial^\mu\phi\partial_{\mu a}p\partial^a\phi = p + \partial^\mu\phi\partial_\mu p_a\phi^a\,, \tag{224}$$

where we also expanded the field via $\phi = \phi^a\theta_a$ in the latter expression. Note that ϕ^a only depends on space-time coordinates. The latter expression exactly coincides with the multi-DBI non-linear symmetry (200) after expanding along the flavour coordinates.[17]

It is easy to check that this mapping is invertible by interchanging flavour coordinates by derivatives via $\phi^a = \partial^a\phi$ where the index on the derivative is finally identified with a space-time index.[18] Substituting this into the multi-DBI non-linear symmetry, one recovers the SG non-linear symmetry (180).

[16]To be more specific: we let the indices on the derivatives run over both types of indices (μ, a).

[17]A similar mapping from kinematic to flavour information at the level of currents was outlined in [25].

[18]Note that, if we let an index on a derivative, say α, run over both flavour and space-time indices (μ, a), one obtains $\partial^\alpha\phi = \partial^\mu\phi + \partial^a\phi = \partial^\mu\phi$, where the latter follows from the fact that ϕ only depends on space-time coordinates.

We can go further by expanding the field and parameter p bi-linearly instead. We do this by introducing another independent auxiliary flavour coordinate $\bar{\theta}^{\bar{a}}$; this means that the original Ansatz takes now the form $p = p_{a\bar{a}}\theta^a\theta^{\bar{a}}$ (or $p_a = p_{a\bar{a}}\theta^{\bar{a}}$ in order to go from the multi-DBI non-linear symmetry to its NLSM counterpart). Plugging this new Ansatz into the SG non-linear symmetry yields

$$\delta\phi = p + \partial^a\phi\partial_{a\bar{a}}p\partial^{\bar{a}}\phi = p + \bar{\theta}^{\bar{a}}\phi_{a\bar{a}}p_{a\bar{b}}\phi^{b\bar{b}}\theta_b\,, \tag{225}$$

which we identify to be the NLSM non-linear symmetry (209) after expansion along the flavour coordinates.

In a similar fashion, we can map the complete equations of motion onto each other. However, as will soon become clear, here we have to impose an additional constraint on the auxiliary flavour dimensions.

To illustrate this, we start with the mapping from the SG equation of motion (221) onto its multi-DBI counterpart (216), and for simplicity, we restrict to the cubic interactions of the right-hand sides. Plugging in our field Ansatz $\phi = \theta_a\phi^a$ into the left-hand side of (221) yields

$$\theta_a\Box\phi^a\,. \tag{226}$$

Summing over both space-time and flavour indices and using the fact that $\partial^\mu\partial^b\phi \rightarrow \theta_a\partial^\mu\partial^b\phi^a + \partial^\mu\phi^b$ and $\partial^\mu\partial^\nu\phi \rightarrow \theta_a\partial^\mu\partial^\nu\phi^a$, we find that we could write down three possible terms for the right-hand-side. The first type follows from three space-time contractions and no flavour contractions; it explicitly reads

$$[\partial^\mu\partial^\nu(\phi^a\theta_a)\partial_\mu\partial_\rho(\phi^b\theta_b)\partial_\nu\partial^\rho(\phi^c\theta_c)] = \theta^a\theta^b\theta^c\left[\Pi_a\Pi_b\Pi_c\right]\,, \tag{227}$$

where we used the notation $\Pi_a \equiv \partial\partial\phi_a$, and the trace is over the space-time indices. Note that the higher order contributions of the field equation will also generate terms that are cubic in θ. Secondly, we could write down a term involving three flavour contractions. However, this term vanishes as a consequence of the initially assumed linearity of ϕ in flavour coordinates.[19] Finally, we have a term that involves a single flavour contraction:

$$\frac{1}{3}[\partial^d\partial^\nu(\phi^a\theta_a)\partial_\mu\partial_\rho(\phi^b\theta_b)\partial_d\partial^\rho(\phi^c\theta_c)] = \theta^a\left[\Pi_a(\partial\phi\cdot\partial\phi)\right]\,, \tag{228}$$

where, once again, the trace is over space-time indices and the dot denotes flavour contraction. Note that the original coefficient $1/3$ of the SG equation of motion has become unity due to the three-fold choice to distribute the free flavour index within the trace.

Stripping the auxiliary coordinate from both the left-hand side (226) and the latter right-hand side term (228) exactly results in the cubic part of the multi-field DBI equation of motion (216).

Based on the above, we conclude that our Ansatz maps the SG equation of motion onto its multi-DBI counterpart if we constrain the auxiliary dimension in such a way that

[19] It is easy to see that two flavour derivatives acting on $\phi^a(x)\theta_a$ results in zero.

the higher order (in the flavour coordinate) terms of the form (227) vanish. There are two possibilities to achieve this. The simplest way is to simply truncate the expression at linear order in θ, while another possibility is to take the auxiliary dimensions to be Grassmannian. As a consequence of the latter, the flavour coordinates satisfy the anti-symmetry property $\theta_a\theta_b = -\theta_b\theta_a$, which leads to the vanishing of (227) after contraction with the trace.

To illustrate that this mapping is invertible, we now take the cubic part of the DBI equation of motion (216) to be the starting point:

$$\partial^\mu\partial^\nu\phi^a\partial_\mu\phi^b\partial_\nu\phi^b\,. \tag{229}$$

Replacing the colour indices with derivatives and interpreting them as space-time indices yields

$$\partial^\mu\partial^\nu\partial^a\phi\partial_\mu\partial^b\phi\partial_\nu\partial_b\phi\,, \tag{230}$$

which is equivalent to

$$\frac{1}{3}\partial^a[\Phi^3]\,. \tag{231}$$

We identify the above to be the derivative of the cubic part of the SG field equation. Turning to the complete equation of motion, the above strategy would eventually give

$$\Box\partial^a\phi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}\partial^a\left[\Phi^{2n+1}\right]\,, \tag{232}$$

which coincides with the SG equation of motion (221) after extraction of the derivative on both sides.

In a similar fashion, we can go from DBI to the NLSM. In this case, we expand in a second (independent) auxiliary flavour dimension, i.e,

$$\phi^a = \phi^{a\bar{a}}\bar{\theta}_{\bar{a}}\,, \tag{233}$$

where we introduced an additional (barred) flavour structure. Note that it is (like for the non-linear symmetries) also possible to map the NLSM directly to the SG by expanding ϕ bi-linearly in the two auxiliary dimensions:

$$\phi = \phi^{a\bar{a}}\theta_a\bar{\theta}_{\bar{a}}\,. \tag{234}$$

Substituting the Ansatz (233) into the left-hand side of the DBI equations of motion (216) results in $\theta_{\bar{a}}\Box\phi^{a\bar{a}}$. Turning to the right-hand side, we will again focus on the cubic contribution, which is in this case given by $[\partial\partial\phi(\partial\phi\cdot\partial\phi)]$. The terms containing two flavour contractions once again vanish due to the linearity in flavour coordinates. Therefore, we remain with two possible types of terms. Firstly, we have a term with zero flavour contractions, which reads

$$\partial^\mu\partial^\nu(\phi^{a\bar{a}}\bar{\theta}_{\bar{a}})\partial_\mu(\phi^{b\bar{b}}\bar{\theta}_{\bar{b}})\partial_\nu(\phi^{b\bar{c}}\bar{\theta}_{\bar{c}}) = \bar{\theta}_{\bar{a}}\bar{\theta}_{\bar{b}}\bar{\theta}_{\bar{c}}\partial^\mu\partial^\nu\phi^{a\bar{a}}\partial_\mu\phi^{b\bar{b}}\partial_\nu\phi^{b\bar{c}}\,. \tag{235}$$

Like before, this term vanishes under the assumption that the auxiliary dimension is Grassmannian. Secondly, we have a term involving a single (barred) flavour contraction,

$$\partial^{\bar{d}}\partial^{\nu}\left(\phi^{a\bar{c}}\bar{\theta}_{\bar{c}}\right)\partial_{\bar{d}}\left(\phi^{b\bar{b}}\bar{\theta}_{\bar{b}}\right)\partial_{\nu}\left(\phi^{b\bar{a}}\bar{\theta}_{\bar{a}}\right)=2\bar{\theta}_{\bar{a}}\partial^{\nu}\phi^{a\bar{b}}\phi^{b\bar{b}}\partial_{\nu}\phi^{b\bar{a}}\,, \tag{236}$$

where the factor 2 results from the two-fold possibility to distribute the flavour indices on the left-hand side. After stripping the auxiliary coordinate $\bar{\theta}_{\bar{a}}$ from both sides, we recover the cubic part of the NLSM equation of motion (214). Once again, similar considerations apply to the higher order terms.

Finally, we can go from NLSM to DBI (and vice-versa) by interpreting flavour indices as derivatives acting on the field. We will not go into detail here, since the considerations are fully analogous to going from DBI to SG (and vice-versa).

The systematic replacement of colour and kinematic information as outlined here results in invertible mappings between the three Goldstone modes with identical coupling constants (which are set equal to unity here). By additionally allowing for numerical coefficients in these mappings, such as e.g. $\phi=(M_{\rm SG}/M_{\rm DBI})\,\phi_a\theta^a$, it is always possible to introduce arbitrary numerical ratios; restricting to the same sign will always map compact onto compact cosets. Note that these mapping relations do not enforce any relations or identifications on the coupling constants of the NLSM. In section 7, we will see that this works differently at the level of amplitudes.

6.3 Classical non-linear solutions

In this section, we turn to relations between the classical solutions of the triplet of scalar theories. Specifically, we will solve the field equations in the presence of a point-like (classical) source at the origin, and we restrict to four space-time dimensions. Motivated by the off-shell flavour-kinematics duality as outlined in the previous section, it would be natural to expect some sort of double copy relation between their classical solutions. Finally, we will discuss the implications of flavour-kinematics duality, and we outline the similarities that initially inspired us to further investigate their off-shell double copy formulation.

6.3.1 Classical solutions in response to a point-like charge

Since we consider stationary spherically symmetric solutions in position space, the derivatives on the fields can be written purely in terms of the radial coordinate $r=\sqrt{x^i x_i}$, where $i=1,2,3$ runs over the space-like indices. Assuming spherical symmetry and time independence ($\sqrt{x^\mu x_\mu}\equiv r$), the following replacements can be made in the DBI and SG equations of motion (as written in (194) and (201) respectively):

$$\partial^\mu\phi\rightarrow\frac{x^i}{r}\phi'(r),\quad (\partial\phi)^2\rightarrow\left(\phi'(r)\right)^2,\quad \partial^\mu\partial^\nu\phi\rightarrow\frac{x^ix^j}{r^2}\phi''(r)-\frac{x^ix^j}{r^3}\phi'(r),\quad \Box\phi\rightarrow\frac{2\phi'(r)}{r}+\phi''(r), \tag{237}$$

where we denote $\phi'(r) \equiv \mathrm{d}\phi(r)/\mathrm{d}r$. After substituting the above into the (quartic) SG field equation (201), we obtain the closed-form ordinary differential equation (ODE)

$$\left[\Lambda^6(\phi')^2 + r^2\right]\phi'' + 2r\phi' = 0\,, \tag{238}$$

which is of order one in the first order derivative ϕ'. The only real solution in terms of ϕ' reads [55]

$$\phi'(r) = \frac{1}{\sqrt{6}L^3}\left[\left(\sqrt{\frac{27}{2}}L^3\mu + \sqrt{r^6 + \frac{27}{2}L^6\mu^2}\right)^{\frac{1}{3}} - r^2\left(\sqrt{\frac{27}{2}}L^3\mu + \sqrt{r^6 + \frac{27}{2}L^6\mu^2}\right)^{-\frac{1}{3}}\right], \tag{239}$$

where μ is a constant with the interpretation of SG scalar charge. This equation is finally solved by the generalized hypergeometric series

$$\begin{aligned}\phi_{SG} &= -\frac{\mu}{r}\,{}_4F_3\left(\frac{1}{6},\frac{2}{6},\frac{4}{6},\frac{8}{6};\frac{9}{6},\frac{8}{6},\frac{7}{6};-\frac{\sigma^2\mu^2}{r^6}\right)\\ &= -\frac{\mu}{r} + \frac{4\sigma^2\mu^3}{189r^7} - \frac{16\sigma^4\mu^5}{3159r^{13}} + \frac{256\sigma^6\mu^7}{124659r^{19}} + \dots\,,\end{aligned} \tag{240}$$

where we redefined the coupling constant as $\sigma^2 = (27/2)\Lambda^{-6}$.[20] A (generalized) hypergeometric series, which we will encounter many times in this section, can be written as

$${}_pF_q\left(a_1,\dots,a_p;b_1,\dots,b_q;z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}\,, \tag{241}$$

where the so-called *rising factorials* are given by

$$\begin{aligned}&(a)_0 = 1\,,\\ &(a)_n = a(a+1)(a+2)\cdots(a+n-1)\,, \quad n\geq 1\,.\end{aligned} \tag{242}$$

Next, we turn to the equation of motion of the DBI (194). This equation cannot be solved in the elegant way of the SG since it contains an infinite number of terms; in this case we use perturbation theory to solve order by order. The spherical-symmetry Ansatz $\phi \equiv \phi(r)$ leads to the ODE

$$0 = \phi'' r + 2\phi' - \frac{1}{2}l^2\left(\phi'\right)^2\left[2\left(\phi'\right) + 3\phi'' r\right] + \frac{3}{8}l^4\left(\phi'\right)^4\left[2\left(\phi'\right) + 5\phi'' r\right] + \dots \tag{243}$$

Expanding the field in the coupling,

$$\phi = \phi^{(0)} + l\phi^{(1)} + l^2\phi^{(2)} + \dots\,, \tag{244}$$

[20] Note that this solution can be simplified, but in order to show the similarities between all solutions, we will refrain from doing so.

and substituting the expansion into the equation of motion (243), leads to a set of differential equations for each perturbative correction $\phi^{(i)}$. By solving for the first few $\phi^{(i)}$ (see Appendix for details), we eventually find the unique closed-form solution

$$\begin{aligned}\phi_{\text{DBI}} &= -\frac{\rho}{r}{}_3F_2\left(\frac{1}{4},\frac{2}{4},\frac{4}{4};\frac{5}{4},\frac{4}{4};-\frac{\rho^2}{F^2r^4}\right)\\ &= -\frac{\rho}{r}+\frac{\rho^3}{10F^2r^5}-\frac{\rho^5}{24F^4r^9}+\frac{5\rho^7}{208F^6r^{13}}+\dots,\end{aligned} \tag{245}$$

where ρ is the DBI scalar charge.

Finally, we consider the $SO(M,N)$ NLSM minimally coupled to GR, which is stated in equation (281). There are two reasons to consider the classical to gravity. In the first place, one should note that contrary to the uncoupled NLSM, the NLSM$_g$ possesses non-linear interactions via graviton exchange, even when restricting to a single flavour. Instead of the $1/r$ fall-off of the normal NLSM, we therefore already anticipate that the result will be a non-linear classical solution. And secondly, as will outlined in the next chapter, it turns out that the amplitude corresponding to the BCJ factorization of two flavour factors gives rise to exactly this theory. The presence of two coupling constants will allows us to explore the possible solutions in special cases, such as e.g. in the absence of NLSM contact-interactions.
[21]

The derivation of the classical solution of NLSM$_g$ is somewhat more complicated than the previous solutions. However, in previous research [56, 57], similar calculations were performed for the case of a free scalar field coupled to gravity. Given our knowledge that the NLSM (170) reduces to a free theory for a single scalar, in combination with the fact that our single-field NLSM is related to the free theory by the field redefinition of the form $\phi_{\text{NLSM}} = \tan\phi_{\text{free}}$, allows us to take the results of the free theory coupled to gravity and simply take its tangent.[22]

Following [56], we briefly review the derivation of the scalar solution for a free scalar field minimally coupled to gravity. The Lagrangian of a free scalar field coupled to GR reads

$$\mathcal{L}_{\text{EH}+\phi} = \frac{1}{2}(R - g_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi)\,, \tag{246}$$

where R is the Ricci scalar curvature for the space-time metric $g_{\mu\nu}$. Furthermore, we work with units in which $8\pi G = c = \hbar = 1$. Variation of (246) with respect to the metric and scalar field respectively leads to the following two field equations:

$$G_{\mu\nu} = T_{\mu\nu}\,,\quad g_{\mu\nu}\nabla^\mu\nabla^\nu\phi = 0\,. \tag{247}$$

[21] Note that the single-colour (or flavour) NLSM reduces to a free theory.

[22] This field redefinition can be derived by comparison of the single-field non-linear symmetries of our NLSM (209) and the canonical NLSM (171). Division of these gives $\frac{\delta\phi}{\delta\phi_c} = \frac{1}{1+\phi^2}$. Hence, we deduce $\phi = \tan\phi_c$, where the subscript c refers to the canonical NLSM (which is equivalent to a free theory for a single scalar).

Since we wish to solve these equation in the presence of a point-like scalar charge, the only contribution to $T_{\mu\nu}$ is generated by the energy of the scalar field. Furthermore, we assume that the stationary spherically-symmetric space-time metric takes the form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + R(r)^2\, \mathrm{d}\Omega^2\,, \tag{248}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin\theta^2\, \mathrm{d}\phi^2$. With this metric choice, the non-vanishing elements of the Einstein tensor are given by

$$\begin{aligned} G_r^r &= f\left(\frac{R_{,r}}{R}\right)^2 + \frac{R_{,r}}{R} f_{,r} - \frac{1}{R^2}\,, \\ G_t^t &= G^r{}_r + 2f\frac{R_{,rr}}{R}\,, \\ G_\theta^\theta = G_\phi^\phi &= \frac{f_{,rr}}{2} + \frac{R_{,r}}{R} f_{,r} + \frac{R_{,rr}}{R} f\,, \end{aligned} \tag{249}$$

and the Ricci curvature reads

$$R = -f_{,rr} - \frac{4}{R}\left(fR_{,r}\right)_{,r} - 2f\left(\frac{R_{,r}}{R}\right)^2 + \frac{2}{R^2}\,. \tag{250}$$

The spherically symmetric scalar field $\phi(r)$ with respect to this metric Ansatz generates the energy-momentum tensor

$${}^{\phi}T^{\mu}{}_{\nu} = \frac{f\phi_{,r}^2}{2}\,\mathrm{diag}\{-1,1,-1,-1\}\,, \tag{251}$$

and the scalar field equation of motion (247) with respect to our metric leads to

$$f\phi_{,r}R^2 = C_1\,, \tag{252}$$

where we set the constant C_1 equal to zero without losing generality. Next, we integrate the above and obtain

$$\phi(r) = C_2\int \frac{\mathrm{d}r}{f(r)R(r)^2} + C_3\,, \tag{253}$$

where C_2 and C_3 are integration constants. Without losing generality we set $C_3 = 0$, corresponding to vanishing ϕ at spatial infinity. The EFE and scalar field equation (247) together fix $f(r)$ and $R(r)$; it turns out that the simplest solution reads [57]

$$f(r) = 1\,, \qquad R(r) = \sqrt{r^2 - \chi^2}\,, \tag{254}$$

where χ is the NLSM scalar charge. Finally, we substitute the above into (253) and integrate: we find that the solution of the scalar field reads

$$\phi(r) = -\sqrt{2}\,\mathrm{arctanh}\left(\frac{\chi}{r}\right)\,. \tag{255}$$

Finally, we take the tangent of the above and obtain the classical solution of the NLSM minimally coupled to gravity. The result is given by

$$\phi_{\mathrm{NLSM}_g} = M_{\mathrm{NLSM}} \tan\left(-\frac{\sqrt{2}M_{\mathrm{pl}}}{M_{\mathrm{NLSM}}} \operatorname{arctanh}\left(\frac{\chi}{M_{\mathrm{pl}} r}\right)\right), \tag{256}$$

where we included the right dimensionality.

Contrary to the classical solutions of DBI (245) and SG (240), in this case we have two coupling constants (due to the reasons discussed in 6.1). Since this solution is in response to a single scalar charge, the scalar interactions only take place via graviton exchange. One choice would therefore be to take the limit of $M_{\mathrm{NLSM}} \to \infty$, such that all information that can be traced back to the interactions of differently coloured NLSM scalars is removed. By taking this limit, the solution reduces to the form

$$\begin{aligned}\lim_{M_{\mathrm{NLSM}}\to\infty} \phi_{\mathrm{NLSM}_g} &= -M_{\mathrm{pl}} \operatorname{arctanh}\left(\frac{\chi}{M_{\mathrm{pl}} r}\right) \\ &= -\frac{\mu}{r} {}_2F_1\left(\frac{1}{2}, \frac{2}{2}; \frac{3}{2}; \frac{\chi^2}{M_{\mathrm{pl}}^2 r^2}\right) \\ &= -\frac{\chi}{r} + \frac{\chi^3}{3M_{\mathrm{pl}}^2 r^3} - \frac{\chi^5}{5M_{\mathrm{pl}}^5 r^5} + \frac{\chi^7}{7M_{\mathrm{pl}}^7 r^7} + \dots,\end{aligned} \tag{257}$$

which is a hypergeometric series that no longer depends on M_{NLSM}. Taking this limit is, however, slightly debatable. An alternative option to get rid of one cut-off scale is to compare the amplitudes of the NLSM and NLSM$_g$. As will be discussed later, a comparison of the amplitude resulting from BCJ factorization (involving two flavour factors, it turns out) and the amplitude resulting from the Lagrangian (i.e., the NLSM, as given in (213), minimally coupled to GR), leads to the identification of $M_{\mathrm{pl}} = M_{\mathrm{NLSM}}$. Using this identification, one can single out one of the coupling constants. However, it turns out that this strategy leads to a non-closed solution that does not fit in line with the DBI and SG classical solutions.

6.3.2 Relations between classical solutions

The three classical solutions that we have found clearly show striking similarities since each solution takes the form of a hypergeometric series ${}_pF_q(\{a_1, \dots, a_p\}; \{b_1, \dots, b_q\}; z)$, with the (as we shall refer to it) *expansion parameter* z being proportional to r^{-2}, r^{-4} or r^{-6}, for the NLSM$_g$, DBI and SG respectively. These similarities are visible from the summary of solutions in table 3.

Theory	Classical solution
NLSM$_g$	$-\frac{\chi}{r}\,{}_2F_1\left(\frac{1}{2},\frac{2}{2};\frac{3}{2};\frac{\chi^2}{M_{\text{pl}}^2 r^2}\right)$
DBI	$-\frac{\rho}{r}\,{}_3F_2\left(\frac{1}{4},\frac{2}{4},\frac{4}{4};\frac{5}{4},\frac{4}{4};-\frac{\rho^2}{F^2 r^4}\right)$
SG	$-\frac{\mu}{r}\,{}_4F_3\left(\frac{1}{6},\frac{2}{6},\frac{4}{6},\frac{8}{6};\frac{9}{6},\frac{8}{6},\frac{7}{6};-\frac{\sigma^2\mu^2}{r^6}\right)$

Table 3: The classical solutions of the NLSM$_g$, DBI and SG in response to a point-like charge located at the origin.

When writing the arguments $\{a_1,\ldots,a_p;b_1,\ldots b_n\}$ of the hypergeometric series as fractions with denominators $2,4$ and 6 (for NLSM$_g$, DBI, and SG respectively), we found that the numerators belonging to the elements of the set p respectively read $\{1,2\}$, $\{1,2,4\}$ and $\{1,2,4,8\}$. The numerators of the set q can then be systematically derived from the numerators of p. This works as follows: one of the numerators is given by the greatest numerator of p plus unity, while the remaining $q-1$ numerators follow from subtracting unity from the latter.

In section 6.1, we already saw that NLSM$_g$, DBI and SG have the schematic BCJ numerator content $f\times f$, $f\times R$ and $R\times R$, where f and R respectively refer to flavour factors and scalar-kinematic numerators. Starting from the classical solution of NLSM$_g$, we note that the replacement of a flavour factor by a scalar-kinematic numerator at the amplitude level corresponds to the mapping ${}_2F_1 \mapsto_3 F_2$ at the level of classical solutions. Furthermore, the "central number", in this case corresponding to 2, is mapped to the DBI central number 4; by this we mean that the r^{-2} dependence of the expansion parameter is mapped to a r^{-4} dependence, and the denominators of p and q are also mapped to 4. As mentioned above, the numerator of the additional element of p is two times the greatest element of the set p of NLSM$_g$, that is, $2\times 2=4$, while the two numerators of q are $4+1=5$ and $5-1=4$. Finally, the NLSM$_g$ scalar charges and coupling constants (or cut-off scales) are trivially mapped onto its DBI counterpart. Remarkably, there is a sign flip in the expansion parameter of the hypergeometric series.

Mapping DBI to SG works fully analogous to going from NLSM$_g$ to DBI, with the DBI central number 4 being mapped to 6, and with the DBI scalar charge and coupling constant being mapped to their SG counterparts. In this case, however, there is no sign flip.

It is clearly visible that the classical solutions have a very similar structure, and that there is (up to a certain extent) a systematic pattern between the different solutions when taking the limit $M_{\text{NLSM}}\to\infty$. These similarities initially inspired us to further investigate the off-shell double copy relations, eventually leading to the results of this chapter. Despite the remarkable similarities, it is not entirely clear how one should interpret these results in the context of the double copy. Especially the limit $M_{\text{NLSM}}\to\infty$ (which forbids NLSM contact interactions) seems to be out of line with the previous results, since the off-shell mapping of chapter 6, on the other hand, involves the NLSM without graviton exchange.

In combination with the different sign of the expansion parameter of the $\phi_{\mathrm{NLSM_g}}$, this means that all we can conclude so far is that the similar forms of the classical solutions are (at least) highly remarkable.

7 Amplitudes and On-Shell Flavour-Kinematics Duality

Motivated by the already established double copy relations (as discussed in section 5.3), and the off-shell double copy formulation of the previous chapter, we now turn to an in-depth investigation of the relevant amplitudes and BCJ numerators. To get started, we will discuss general numerator and amplitude structures; we show that amplitudes involving two flavour factors naturally lead to the inclusion of graviton exchange for the $SO(M+N)$ NLSM. Finally, we point out mapping relations between the numerators, including the tensor-kinematic YM numerators.

7.1 The BCJ formulation and algebraic properties of numerators

To refresh our memory on the algebraic properties of BCJ numerators, we recall from section 2.4 that the four-point tree-level amplitude of theories admitting a BCJ formulation can be written as

$$A_4 = \sum_{\text{exchange}} \frac{N_{ijkl}\tilde{N}_{ijkl}}{s_{ij}}\,, \tag{258}$$

where the sum is taken over cubic diagrams (see figure 11). At four-point, this corresponds to the permutations $(ijkl) = (1234, 2314, 3124)$, or as we previously referred to these: the s, t and u exchange diagrams. In total, one could write down 4! permutations; the reduction down to three inequivalent permutations follows from the following three $\mathbb{Z}_2$ symmetries. Firstly, we have anti-symmetry in the first and last pairs of legs[23], and secondly, we have reflection-(anti) symmetry, as we saw in equation (2).

As discussed in section 2.1, we have a total of $(2n-5)!!$ inequivalent cubic diagrams, which means that we find $5!! = 5!/2^3 = 15$ inequivalent half-ladders at five-point (see figure 11). At six-point, two inequivalent topologies add up to a total of $7!! = 115$ diagrams; these topologies consist of half-ladders (similar to four- and five-point) and snow-flake diagrams (see figure 11). In total, there are $6!/2^3 = 90$ inequivalent half-ladders and $6!/\left(2^3 \cdot 3!\right)$ snow-flake diagrams. The reduction by 3! follows from the equivalence of the three centre (internal) legs, while the 2^3 reduction follows from the anti-symmetry in each pair of external legs. The kinematic numerators of the snow-flake diagrams are related

[23] This anti-symmetry property can be easily seen by considering the s-channel four-point scalar-kinematic numerator $N_{1234} = (s_{14} - s_{13})/s_{12}$. Upon interchange of the first pair of legs, the numerator becomes $N_{2134} = (s_{24} - s_{23})/s_{12}$; upon using four-point kinematic identities, the latter can be written as $-(s_{14} - s_{13})/s_{12} = -N_{1234}$.

via kinematic Jacobi identities and take the form $N_{ijklmn} - N_{ijlkmn}$. Hence, by including both topologies, the six-point amplitude takes the schematic form

$$A_6 = \sum_{\text{half}-\text{ladder}} \frac{N_{ijklmn}\tilde{N}_{ijklmn}}{s_{ij}s_{ijk}s_{mn}} + \sum_{\text{snow-flake}} \frac{\left(N_{ijklmn} - N_{ijlkmn}\right)\left(\tilde{N}_{ijklmn} - \tilde{N}_{ijlkmn}\right)}{s_{ij}s_{kl}s_{mn}} . \tag{259}$$

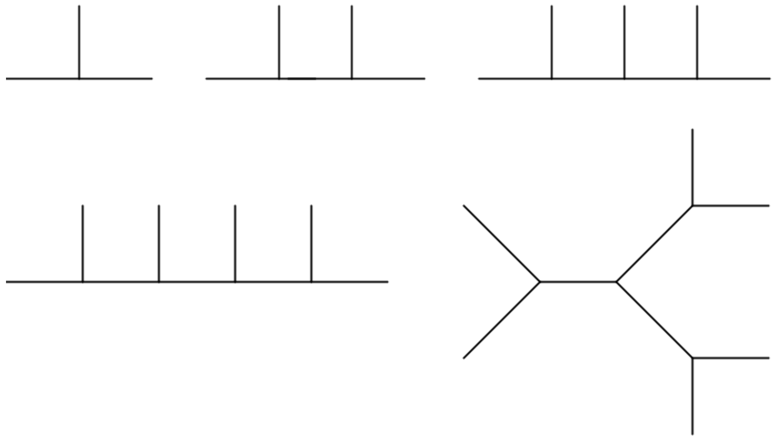


Figure 11: The trivalent diagrams relevant at lower-point scattering; all have half-ladder topology at three-, four- and five-point, while at six-point there are half-ladder and snow-flake topologies.

Next, we turn to the algebraic properties of BCJ numerators. As discussed before, the four-point kinematic BCJ numerators are required to satisfy the following symmetry conditions:

$$N_{ijkl} = -N_{jikl}, \quad N_{ijkl} = N_{lkji}, \quad N_{ijkl} + N_{jkil} + N_{kijl} = 0 . \tag{260}$$

The first two identities respectively correspond to anti-symmetry and reflection symmetry, whereas the latter is the kinematic analogue of the Jacobi identity for structure constants. These conditions naturally generalize to higher order, say n-point, where the kinematic Jacobi identities can be written as

$$-N_{ijkl\ldots} = N_{jikl\ldots} = N_{k[ij]l\ldots} = N_{l[[ij]k]\ldots} = \ldots , \tag{261}$$

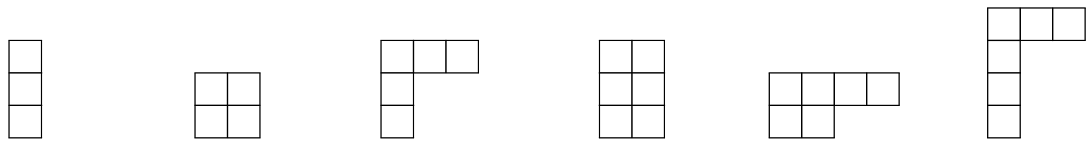
with multiple commutators on the first $n-1$ indices. These conditions lead to $n-2$ conditions on the indices. In addition to the above, we have the reflection (anti) symmetry

$$N_{ijklm\ldots} = (-)^n N_{\ldots mlkji} , \tag{262}$$

with the sign depending on the parity (i.e., the number of particles).

Imposing these algebraic relations constrain the representations that the BCJ numerators can take. The Young-Tableaux corresponding to these representations are shown in figure 12, and at each order, these are given by the following:

- At three-point, we can only have the anti-symmetric tensor. Its dimension (as element of the symmetric group) is 1.
- At four-point, the unique representation is the window tensor with dimension 2.
- At five-point, the unique representation is the equal-arms hook tensor with dimension 6.
- At six-point, we find three irreps with different Young tableaux, with dimensions 5, 9 and 10, respectively, adding up to a total dimension of 24.



(a) *(b)* *(c)* *(d)* *(e)* *(f)*

Figure 12: Young tableaux for the BCJ numerators at: (a)*, three-,* (b) *four-,* (c) *five- and* (d,e,f) *six-point.*

Let us now consider specific representations that satisfy these constraints. Starting with colour, we can represent the colour factors in terms of structure constants, and at arbitrary order, the colour factors are given by[24]

$$N_{1234\ldots} = f_{AB}{}^{P} f_{PC}{}^{Q} f_{QD}{}^{R} \ldots \,, \tag{263}$$

where the capitals refer to the adjoint representation. For example, at three- and four-point, the colour factors read

$$N_{123} = f_{ABC} \,, \quad N_{1234} = f_{AB}^{P} f_{PCD} \,. \tag{264}$$

These indeed correspond to the earlier discussed YM colour factors, and can easily be checked to satisfy the algebraic constraints. Note that we have assumed the existence of a metric to raise and lower the last index.

7.2 Flavour and kinematic BCJ numerators

In this section, we will investigate numerators structures that are different from the original colour factors and YM (or tensor-kinematic) numerators involved in YM and GR. As we have discussed before, these could, for instance, solely depend on kinematic invariants (without polarisation) and hence describe the scattering of massless scalars without any additional structure. First, we will restrict to purely scalar-kinematics, and thereafter we generalize with the inclusion of flavour.

[24]Note that the left-hand side should be read as $(N_{123\ldots})_{ABC\ldots}$, where adjoint index A is assigned to the adjoint representation of particle 1 and so forth. For notational simplicity, we will leave this implicit.

In the purely scalar-kinematic case, the absence of colour structure implies that no anti-symmetric three-point numerators can possibly exist, since all momentum contractions vanish as a consequence of momentum conservation. Hence, at three-point, there does not exist a scalar-kinematic numerator.

To find the four-point numerator, we impose the identities (260), and we will restrict to numerators that are quadratic in the Mandelstam variables[25], in order to obtain the same order in momenta as the SG. The most general solution is given by[26]

$$N_{ijkl} = \lambda_4 s_{ij}(s_{jk} - s_{ik})\,, \tag{265}$$

where λ_4 is a constant. The four-point amplitude then follows from the BCJ factorization (258) and explicitly reads

$$A_4 = -9{\lambda_4}^2 s_{12}s_{23}s_{13}\,. \tag{266}$$

This amplitude exactly coincides with the SG four-point amplitude (204) after fixing the coefficient λ_4 in terms of the coupling constant.

Next, we consider five-point amplitudes. As we discussed earlier, the triplet of scalar EFTs all have vanishing odd-point amplitudes. However, here we will provide an in depth explanation. In the five-point case, there are four constraints on the numerators. In addition to the anti-symmetry and reflection (anti) symmetry constraints, we have two Jacobi-like identities involving commutators. Upon imposing these constraints, we find that the first nontrivial solution is found at cubic order, involving a single parameter. However, the BCJ factorization of these numerators indeed leads to a vanishing total amplitude; this parameter therefore corresponds to a generalized gauge transformation, as discussed in section 2.1. This can be seen as follows: suppose that the five-point numerator is of the form

$$N_{ijklm} \sim s_{ij}G_{ijklm} - s_{lm}G_{mlkji}\,, \tag{267}$$

where G is the gauge parameter. When the BCJ factorization is taken with another numerator $\tilde{N}$, the amplitude will receive contributions of the form

$$\sum \frac{\tilde{N}_{ijklm}G_{ijklm}}{s_{lm}} - \frac{\tilde{N}_{ijklm}G_{mlkji}}{s_{ij}}\,. \tag{268}$$

Let us now recall from section 2.1 that there are 15 distinct cubic diagrams at five-point, and hence the above structure will contribute a total of 30 terms to the amplitude. Provided that the gauge parameters G_{ijklm} are fully anti-symmetric in the first three

[25] Note that there also exist solutions of the form $N_{ijkl} \sim (s_{jk} - s_{ik})$. These will, however, have a more natural interpretation as a special case of flavour factors, as we will discuss later.

[26] Note that we work with the notation $s_{ij} = (p_i + p_j)^2$ (already at four-point), instead of Mandelstam variables, since we will work with higher-point amplitudes, and it gives explicit insight into the algebraic structure of the numerators.

indices (ijk), the 30 terms will combine in a triplet of 10 terms that share a common denominator. For example, this involves terms of the form

$$\frac{(\tilde{N}_{ijklm}+\tilde{N}_{jkilm}+\tilde{N}_{kijlm})G_{ijklm}}{s_{ij}}, \tag{269}$$

and all of these terms combined will vanish by force of the Jacobi identity. Note that this only requires the $\tilde{N}_{ijklm}$ to satisfy the Jacobi identities, while it does not require any specific form. The most general numerator turns out to be of the form of equation (267), with the gauge parameter given by

$$G_{ijklm}=(s_{il}s_{jm}-(l\leftrightarrow m))+(cyclic)\,, \tag{270}$$

where $(cyclic)$ refers to the two terms resulting from cyclic permutations in the first three indices (ijk).

Next, we turn to six-point, and we will be interested in numerators that are of quartic order in Mandelstam, in order to coincide with the SG six-point interaction.[27] Imposing the algebraic constraints (261) and (262) results in an expression involving 23 free parameters. Furthermore, we require factorization into even-point amplitudes; this means that we have to impose the vanishing of terms involving single-Mandelstam poles, since these correspond to a splitting into a three- and five-point vertex (which is prohibited by the BCJ approach). The latter reduces the number of free parameters to a total of six; we checked that the final amplitude (which follows from the BCJ factorization) only contains a single linear combination of these. Therefore, we conclude that these parameters consist of one physical parameter and five gauge parameters.

Inspired by the kinematic dependence of the four- and five-point amplitudes, we naturally extend the amplitude Ansatz to six-point:

$$N_{ijklmn}=s_{ij}P_{ijklmn}+\text{order reversed}\,. \tag{271}$$

This Ansatz is checked to have one physical parameter and one gauge parameter; the former explicitly reads

$$\begin{aligned}P_{ijklmn}=&-s_{mn}\Big(s_{jk}\Big(-4s_{in}+4\Big(s_{jl}+s_{kl}+s_{km}\Big)+5s_{lm}\Big)+\\&+s_{ik}\Big(-4s_{in}+4\Big(s_{jl}+s_{kl}\Big)+s_{jk}+9s_{lm}\Big)+5s_{ij}s_{ik}+4s_{ij}^2+s_{ik}^2\Big)+\\&+4\Big(s_{ij}+s_{ik}+s_{jk}\Big)\Big(\Big(s_{ik}+s_{jk}\Big)\Big(-s_{in}+s_{jk}+s_{jl}+s_{kl}\Big)+s_{jk}s_{km}\Big)\\&-4s_{lm}\Big(s_{ik}\Big(s_{jl}+s_{kl}\Big)+s_{jk}s_{jl}\Big)+s_{mn}^2\Big(4s_{ij}+5s_{ik}\Big)\,,\end{aligned} \tag{272}$$

which is, of course, not unique due to the possibility of generalized gauge transformations. Again, it has been checked that the resulting six-point amplitude coincides with the SG six-point interaction.[28]

[27] Note that there also exist quadratic and cubic solutions. However, again it will be more natural to interpret these as special cases of the flavour factors.

[28] Six-point scalar-kinematic numerators were also constructed in [36]. However, in this paper a different basis was used; in which the reflection symmetry was not imposed on the numerators.

The expectation is that this structure extends to higher order, with the first non-trivial solution to the algebraic constraints coming in at order $n-2$ in kinematic invariants. As mentioned earlier, the odd-point numerators all consist of pure gauge parameters, leading to vanishing odd-point amplitudes.

As we have seen above, the first non-trivial solution involving purely scalar-kinematics occurs at order $n-2$ in the kinematic invariants. Below this order, there are no solutions to the algebraic conditions. However, solutions below order $n-2$ in kinematic invariants do exist when the scalar fields carry additional structure. One possibility is to consider scalar fields in the adjoint representation, leading to the inclusion of colour factors (i.e., structure constants). Another option, which we will consider here, is the scalar fields living in the fundamental representation of e.g. $SO(N)$, which we have seen in section 5.3, yielding amplitudes that can be factorized in terms of flavour factors. Since these factors will involve the invariant metric δ^{ab}, the odd-point flavour factors (and hence amplitudes) naturally vanish.

Turning to explicit flavour factors, the first solution at four-point is found at linear order in kinematic invariants:

$$F_{1234} = f_1(\delta_{ab}\delta_{cd}(s_{23}-s_{13})-(\delta_{ac}\delta_{bd}-\delta_{bc}\delta_{ad})s_{12})+f_2(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{db}+\delta_{ad}\delta_{bc})(s_{13}-s_{23})\,, \tag{273}$$

where we (like before) assign momentum p_1 and flavour index a to external particle 1 etc. Due to the mixing of flavour and kinematic, the Jacobi identities can be solved in multiple ways. For example, the above expression is anti-symmetric under the exchange of particles 1 and 2 as a consequence of anti-symmetry in flavour and symmetry in kinematics, or vice-versa. As a consequence of this freedom, we already find that the four-point flavour factor already contains a free parameter. When restricting to a single flavour (by setting $\delta^{ab}=1$ etc.), both parameters collapse onto the same factor, and we recover the numerators as given in footnote 25. The latter therefore has a natural interpretation as a special case of the above flavour factor.

The amplitude resulting from the BCJ factorization involving one scalar-kinematic numerator and flavour factor can schematically be written as

$$A_4 = \sum_{\text{exchange}} \frac{N_{ijkl}F_{ijkl}}{s_{ij}}\,, \tag{274}$$

and we have checked that both parameters (f_1 and f_2) are physical instead of generalized gauge. The resulting amplitude consists of two types of contributions with different flavour structures. The partial amplitude (which is e.g. the part proportional to $\delta_{ab}\delta_{cd}$), for instance, is given by

$$A_4^{ab,cd} = -6f_1\lambda_4(s_{23}s_{13})-6f_2\lambda_4(s_{23}s_{13}-s_{12}^2)\,. \tag{275}$$

By comparison with the multi-DBI field equation (216), we conclude that the part proportional to f_1 coincides with the multi-DBI amplitude; this follows from a specific relation between the terms $\mathrm{Tr}\left[(\partial\phi^a\partial\phi^n)^2\right]$ and $\mathrm{Tr}\left[(\partial\phi^a\partial\phi^n)\right]^2$ of the field equation. The second term, on the other hand, solely results from the latter, and for arbitrary f_2 the corresponding theory has no clear Goldstone interpretation associated with spontaneous symmetry breaking. Therefore, we set $f_2 = 0$ so that the above amplitude corresponds to the BCJ formulation of the multi-DBI theory.

Next, we consider the BCJ factorization involving two flavour factors. For notational convenience, and an improved understanding of these amplitudes, it will be useful to define the following notation for the flavour structures:

$$[AB] \equiv \delta_{ab}\delta_{\bar{a}\bar{b}}\,, \qquad [ABCD] \equiv \delta_{ab}\delta_{cd}\delta_{\bar{b}\bar{c}}\delta_{\bar{a}\bar{d}}\,, \tag{276}$$

where again δ and $\bar{\delta}$ belong to the distinct numerators F and $\bar{F}$, respectively.[29] The BCJ factorization,

$$A_4 = \sum_{\text{exchange}} \frac{F_{ijkl}\bar{F}_{ijkl}}{s_{ij}}\,, \tag{277}$$

now contains two different structures, which we will outline below.

- Firstly, we have contributions involving a single trace; one example of this part is explicitly given by

$$\sim f_2^2 \frac{(s_{12}^2 - s_{23}s_{13})^2}{s_{12}s_{23}s_{13}}([ABCD] + [ADCB]) + (\text{cyclic})\,, \tag{278}$$

  where (cyclic), like before, denotes cyclic permutation of three external legs while keeping one leg fixed. Note that single-trace contributions could correspond to the four-point amplitudes of the NLSM. However, in the above we find (single) poles and these contributions can therefore not originate from the NLSM (which only has contact interactions and hence no poles). Like for the multi-DBI, we have to set $f_2 = 0$ in order to connect the amplitude to a known theory.

  The second type of single-trace contribution reads

$$A_4^{\text{contact}} = -4f_1^2 s_{13}([ABCD] + [ADCB]) + (\text{cyclic})\,, \tag{279}$$

  and these indeed have the same structure as the $SO(M+N)$ NLSM amplitudes, which can be trivially seen from the field equation (214). These contact interactions are visualized in the left part of figure 13.

- Secondly, we have double-trace structures of the form

$$A_4^{\text{exchange}} = -4f_1^2 \frac{s_{23}s_{13}}{s_{12}}[AB][CD] + (\text{cyclic})\,. \tag{280}$$

  As a consequence of the poles, these structures represent exchange diagrams; it turns out that these correspond to graviton exchange between the four NLSM scalars.

[29]Note that this trace is not invariant under cyclic permutations.

Since the gravitons do not carry any flavour, the flavours of the scalars connected to the vertices should coincide, leading to the double-traces (see the right part of figure 13).[30]

The sum of the above amplitude structures follows from the non-compact $SO(M,N)$ NLSM minimally coupled to gravity, with the Lagrangian given by

$$\mathcal{L}_{\mathrm{GR+NLSM}} = \sqrt{-g}\left(\frac{1}{2}M_{\mathrm{Pl}}^2 R - \frac{1}{2}\left[\frac{1}{1-\phi\phi^T/F^2}\nabla^\mu\phi\frac{1}{1-\phi^T\phi/F^2}\nabla_\mu\phi^T\right]\right), \tag{281}$$

where M_{Pl} is the reduced Planck mass and F is the NLSM cut-off scale. A derivation of the amplitudes (including F and M_{Pl} and the appropriate coefficients) is reviewed Appendix A.3. By comparison of the relative strength of this amplitude (see (319)) and the two contributions resulting from the BCJ factorization (see equations (279) and (280)), we conclude that the latter corresponds to the gravitationally coupled NLSM with the identification $F = M_{\mathrm{Pl}}$.

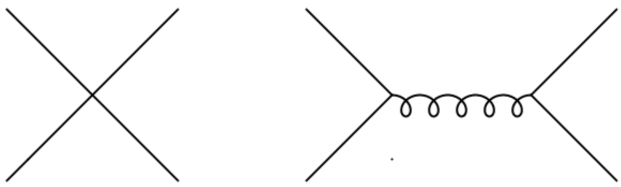


Figure 13: The two types of Feynman diagrams contributing to the NLSM$_g$ four-point amplitude. These consist of contact-interactions (left) and graviton exchange (right). The straight lines are scalars, whereas the curly line represents an intermediate graviton. For the full amplitude, permutations should be taken into account.

Moving on to six-point, we have 15 inequivalent BCJ diagrams, with inequivalent flavour structures consisting of products of three delta functions. These are multiplied by quadratic expressions in one of the nine independent kinematic invariants, leading to 45 different terms per single-trace contribution; hence adding up to a total of $15\cdot 45 = 675$ terms. The algebraic conditions constrain 642 of the parameters, leaving a total of 33 free parameters. These are contained in the three irreps of figure 12 as follows: (d) contains 9 parameters, (e) contains 15 parameters and (f) contains 9 parameters. Like before, we additionally impose the factorisation constraint; this leaves us with a total of 6 unfixed parameters (similar to the scalar-kinematic numerators), with components in all three irreps. The complete six-point flavour factor is given by a complicated expression. However, restricting to a specific parameter (not equal to pure gauge), the part that is proportional to s_{12}^2 explicitly reads

$$\begin{aligned}
&28\delta_{af}\delta_{be}\delta_{cd} - 28\delta_{ae}\delta_{bf}\delta_{cd} + 11\delta_{ab}\delta_{cd}\delta_{ef} + 13\delta_{af}\delta_{bd}\delta_{ce} - 13\delta_{ad}\delta_{bf}\delta_{ce} + \\
&- 34\delta_{ae}\delta_{bd}\delta_{cf} - 5\delta_{ad}\delta_{be}\delta_{cf} + 25\delta_{af}\delta_{bc}\delta_{de} - 26\delta_{ac}\delta_{bf}\delta_{de} + 8\delta_{ab}\delta_{cf}\delta_{de} + \\
&- 7\delta_{ae}\delta_{bc}\delta_{df} + 26\delta_{ac}\delta_{be}\delta_{df} + 13\delta_{ab}\delta_{ce}\delta_{df} - 9\delta_{ad}\delta_{bc}\delta_{ef} + 7\delta_{ac}\delta_{bd}\delta_{ef}
\end{aligned} \tag{282}$$

[30] For future research, it would be interesting to see if the pole structures proportional to f_2 have a similar form, corresponding to e.g. gluon exchange.

The other parts of the six-point factor that are proportional to the other (eight) independent kinematic invariants squared then follow from the algebraic and factorization constraints.

For the six-point amplitude, we find similar results as in the purely scalar-kinematic case, since this six-point flavour amplitude also depends on a linear combination of the six parameters. Hence, we again conclude that five of these are gauge parameters and one is a physical parameter. As a consequence of the long expressions for the six-point flavour factors, the resulting amplitude is also a long expression, and therefore we refrain from explicitly stating it.

7.3 On-shell flavour-kinematics duality

In this section, we naturally extend the off-shell flavour-kinematics duality of section 6 to the on-shell level: we propose (non-invertible) mapping relations between the different BCJ numerators, extending the off-shell mapping to the on-shell level.

To illustrate these relations, we take our starting point to be the four-point flavour factor

$$F_{1234} = \delta_{ab}\delta_{cd}\,(s_{23} - s_{13}) - (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad})\,s_{12}\,, \tag{283}$$

where we map flavour information to kinematic information via

$$\delta_{ab} \longmapsto 1 + \lambda s_{12}\,, \tag{284}$$

where λ is a constant. Next, we isolate the part of the resulting expression at order $\mathcal{O}(\lambda)$; upon using momentum conservation ($s_{12}+s_{13}+s_{14}=0$), the coefficient of λ turns out to be exactly the scalar-kinematic numerator

$$N_{1234} = s_{12}\,(s_{23} - s_{13})\;. \tag{285}$$

Note that this is a non-invertible mapping, since we throw away quadratic terms in λ. This on-shell mapping mimics its off-shell counterpart: here we map a four-derivative interaction to a six-derivatives interaction, analogous to the off-shell mapping where the interaction also increased by a total number of two derivatives.

In addition to the above, we have pointed out a similar mapping from tensor-kinematic numerators to flavour factors. To illustrate this, we recall that the four-point s-channel tensor-kinematic numerator is given by

$$\begin{aligned} n_s = -\Big\{\Big[(\varepsilon_1\cdot\varepsilon_2)\,p_1^\mu + 2\,(\varepsilon_1\cdot p_2)\,\varepsilon_2^\mu - (1\leftrightarrow 2)\Big]\,[(\varepsilon_3\cdot\varepsilon_4)\,p_{3\mu} + 2\,(\varepsilon_3\cdot p_4)\,\varepsilon_{4\mu} - (3\leftrightarrow 4)] \\ +2s_{1,2}\,[(\varepsilon_1\cdot\varepsilon_3)\,(\varepsilon_2\cdot\varepsilon_4) - (\varepsilon_1\cdot\varepsilon_4)\,(\varepsilon_2\cdot\varepsilon_3)]\Big\}\;. \end{aligned} \tag{286}$$

In order to transform tensor-kinematic information into flavour information, we map

$$\varepsilon_i\cdot p_j \longmapsto 0, \quad \varepsilon_i\cdot\varepsilon_j \longmapsto \delta_{ij}\,, \tag{287}$$

and the result exactly coincides with the flavour factor (283). Naturally, all odd-point numerators vanish due to the odd numbers of polarizations and momenta.

Alternatively, one could take the starting point to be the polarization-stripped version of (286), which is given by

$$\begin{aligned} n_s = \Big[&\Big(-\frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\lambda} p_1 \cdot p_3 - \eta^{\gamma\lambda} p_2{}^{\alpha} p_3^{\beta} - \eta^{\alpha\beta} p_1{}^{\lambda} p_4^{\gamma} - 2\eta^{\beta\lambda} p_2{}^{\alpha} p_4^{\gamma} \Big) \\ &- (1 \leftrightarrow 2) - (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) + \Big(\eta^{\alpha\lambda}\eta^{\beta\gamma} - \eta^{\alpha\gamma}\eta^{\beta\lambda} \Big) p_1 \cdot p_2 \Big] \,. \end{aligned} \tag{288}$$

In this case, the mapping is given by

$$\eta^{\alpha\beta} \mapsto \delta^{ab} \,, \tag{289}$$

whereas all other terms (i.e., terms with uncontracted momenta) vanish. Note that the space-time indices of the Minkowski metric are mapped to flavour indices on the right-hand side.

Having studied the off-shell field equation, non-linear symmetries and the on-shell amplitude structure, one might wonder which picture emerges from these findings. It turns out that the double copy relations can be graphically represented as the tetrahedron visualized in figure 14: at the nodes of the tetrahedron, we find theories involving two BCJ numerators of the same type, while theories with mixed numerators can be found along the edges. The vertical levels of the tetrahedron are related to the spin of the theory and therefore to the numerator content. At the top of the tetrahedron, we find general relativity with purely tensor-kinematic BCJ numerators. Moving one level down, we find spin-one theories involving one tensor-kinematic numerator, combined with another type. These involve Born-Infeld (BI) non-linear electrodynamics [53], with the complement numerator being scalar-kinematic, YM theory with a complement colour factor, and the third theory describes photons interacting via graviton exchange (indicated by the subscript g), with a complement flavour numerator.[31] At the bottom level we find scalar theories; at the front edge we find the theories that we discussed in this chapter, involving the $SO(M, N)$ NLSM (possibly) coupled to gravity (and with the identification $F = M_{\text{pl}}$), (multi-) DBI theory, and the SG. One of the bottom-level theories that we have not encountered so far is the scalar sector of the so-called *Yang-Mills scalar theory*, which corresponds to YM minimally coupled to a quartic scalar theory (see e.g. [54] for details).

All theories that lie on the *self-interaction face* possess self-interactions that are reflected by their non-linear equations of motion, even when restricted to a single species. Note that this tetrahedron does not contain all theories that have a double copy formulation.

[31] This theory is often referred to as Einstein-Maxwell theory since it corresponds to Maxwell's electrodynamics coupled to GR.

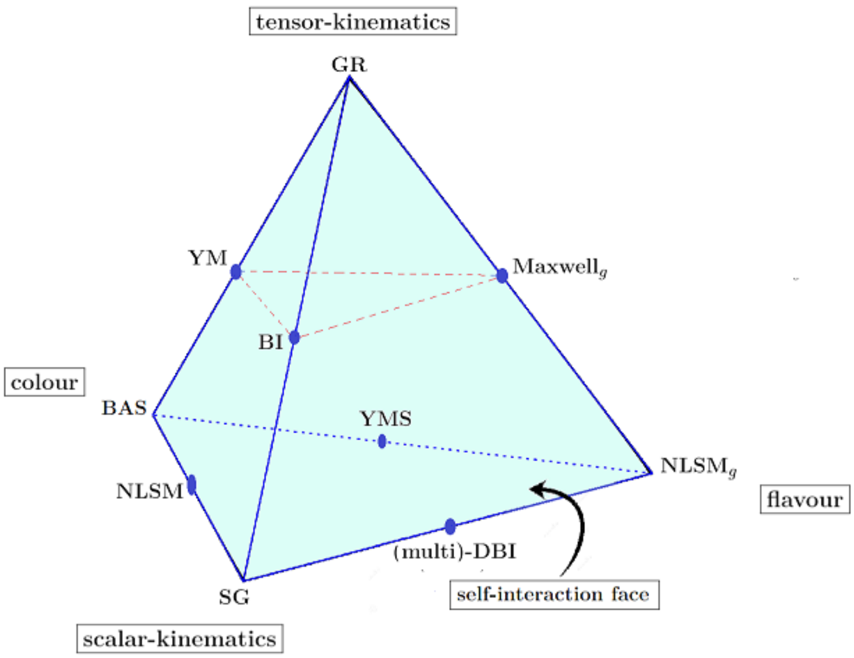


Figure 14: The "double copy tetrahedron", indicating the BCJ numerator content of a set of well known theories (as explained in the text). Moving vertically from top to bottom reduces the spin from two to one, and to zero, respectively.

The off-shell mapping that we discussed earlier systematically relates the non-linear symmetries and field equations of the three theories at the bottom level of the self-interaction face and it works in both directions. On the other hand, the on-shell mapping between flavour and scalar-kinematic numerators only works in the direction flavour $\rightarrow$ scalar-kinematic. Including the mapping involving tensor-kinematic numerators allows us to construct the tree-level amplitudes of all theories that lie on the self-interaction face by simply using the mapping operations on the tensor-kinematic numerator.

8 Conclusions and Outlook

In the first part of this thesis, we reviewed the well known amplitude double copy in its original setting, which systematically relates the tree-level amplitudes of GR and YM. We discussed the implications of colour-kinematics duality, emphasizing the algebraic similarity between colour and kinematic BCJ numerators. By comparison of the three- and four-point amplitudes of GR and YM in perturbation theory, we have seen that the double copy serves as an incredibly powerful computational tool, as it allows one to omit the intensive gravity calculations by instead calculating the YM counterpart and using the double copy.

In chapter 3, we have seen examples of manifest double copy relations at the level of classical solutions. Although we have seen that classical double copy relations can be made manifest for a specific set of vacuum solutions of GR and YM, a general procedure that takes into account non-linear solutions remains elusive. This difficulty lies in the fact that classical solutions depend on off-shell information, such as gauge choice and field basis, while the amplitudes are physical observables and therefore unaffected by the off-shell formulation. The immense freedom in the field-theoretic formulation makes it highly non-trivial to manifest the classical double copy, and it is unclear whether there exist systematic considerations on the gauge (or diffeomorphism) choice and field basis.

In chapter 4, we have shown how classical solutions can be perturbatively expanded in terms of momentum-space correlators, and we have seen how these encode for tree-level amplitudes. In addition to this, the perturbative formalism revealed that it is often possible to choose a combination of field basis and gauge (or diffeomorphism) choice for which the off-shell amplitudes (i.e., the integrands of the perturbative corrections) exactly match the off-shell amplitude that results from BCJ factorization. In case the latter two match, it is obvious that the perturbative classical solutions double copy under the interchange of the appropriate structures. This double copy of perturbative classical solutions was performed in e.g. [61], where perturbative space-times such as the Schwarzschild metric were constructed by "squaring" the perturbative YM counterpart. However, these results did not yet lead to a well understood double copy structure for explicit equations of motion and classical solutions.

In chapter 6, we addressed our first (main) research question, which we formulated as follows:

> *"Does there exists a double copy formulation that maps complete equations of motion and their non-linear solutions from theory to theory?"*

We investigated this for a triplet of effective scalar theories, involving the $SO(M,N)$ *NLSM* (possibly) coupled to GR, (multi-field) *DBI* and the *SG*; we saw that each of these theories is invariant under a specific non-linear symmetry transformation associated with the breaking of internal symmetry.

The canonical formulations of these theories (including the $SU(N)$ or $U(N)$ NLSM) give rise to amplitudes that can be made to double copy onto one another under the interchange of colour factors, flavour factors, or scalar-kinematic numerators.

In order to naturally make the connection with the multi-DBI theory, it was natural to include the $SO(M+N)$ NLSM, involving two flavour structures instead of a colour structure.

As we pointed out in section 6.1, the equations of motion of the three theories take a similar form (see table 2) when adopting a field basis in which their non-linear symmetries are of the schematic form $\delta\phi = \mathcal{O}(\phi^0) + \mathcal{O}(\phi^2)$. The resulting similarity of the equation of motion facilitates a systematic and invertible mapping between the theories by expanding the field linearly (to go from SG to DBI or from DBI to NLSM) or bi-linearly (to go directly from SG to NLSM) in auxiliary flavour coordinates. Remarkably, for this to work, we had to assume that the auxiliary flavour coordinates are Grassmannian.

Motivated by this off-shell formulation, we investigated potential double copy manifestations at the level of classical non-linear solutions in section 6.3. It was shown that each spherically symmetric classical solution can be written in terms of a distinct (generalized) hypergeometric series, with intriguing relations between the coefficients within each series, as well as between the coefficients of the different theories (see table 3).

Although these relations initially appeared to be very promising, it was necessary to get rid of the NLSM coupling by taking the limit $M_{\text{NLSM}} \to \infty$, which essentially means that the NLSM scalars can only interact via graviton exchange. The latter, in combination with a sign flip in the NLSM expansion parameter, leads us to conclude that we, currently, can not convincingly speak of a true manifestation of the double copy.

At the level of on-shell amplitudes, as discussed in section 7, we have seen that tree-level amplitudes different theories can be constructed using different combinations of BCJ numerators, and we provided explicit expressions for these numerators up to and including six-point. In addition to the well known colour and kinematic numerators, we have included the flavour factors that are subject to the same algebraic relations, ensuring that they give rise to the appropriate amplitude factorisation. Furthermore, we pointed out that the amplitude corresponding to the BCJ factorization of two (possibly different) flavour factors corresponds to the $SO(M, N)$ NLSM minimally coupled to gravity.

Finally, we turned to our second (main) research question, which we formulated as

"Can we relate the different types of BCJ numerators in a systematical way?"

Analogous to the off-shell mapping of chapter 6, the tensor-kinematic, flavour, and scalar-kinematic numerators were shown to be related by the mappings of equations (284) and (287) respectively. However, in contrast to the off-shell mappings, the numerators can only be mapped along the fixed direction NLSM $\mapsto$DBI $\mapsto$ SG, as a consequence of throwing away specific terms.

The results of this research provide deeper insight into the duality between the group-theoretic and kinematic structures of different field theories; it shows that it is indeed possible (at least for scalar EFTs) to double copy off-shell information, when appropriately dealing with the field-theoretic redundancy. The mapping relations between the non-linear symmetries and equations of motion, in combination with the on-shell mapping relations, emphasize that the three theories are different manifestations of the same underlying structure.

For future research, it would be interesting (and challenging) to investigate whether similar off-shell constructions are feasible for higher-spin theories such as YM and GR, involving local symmetries which do not affect the S-matrix. However, due to the different spin and the different nature of the symmetries involved, the procedure of identifying appropriate field bases and symmetry considerations is expected to be much more challenging, or even impossible.

In addition to this, it would also be interesting to further investigate the mapping relations between the BCJ numerators. Recently, the authors of [25] managed to establish a mapping between tensor-kinematic numerators and colour factors. However, the mapping proposed in this article skips the flavour factors, and they also worked in a so-called

Duca-Dixon-Maltoni (DDM) basis (originally proposed in [62]), instead of the half-ladder that was used here. The identification of simple expressions for the BCJ numerators at arbitrary order, accompanied by simple mapping relations, would not only improve the understanding of their duality, but also serve as a powerful computational for amplitude calculations.

Related to the original results of chapters 6 and 7, it would also be interesting to investigate if (and how) this duality extends to other off-shell aspects such as wavefunctions and correlators. Recent progress on the latter has been made in [63], where it was found that the double copy relations in their original (or simplest) form do not hold. Given the off-shell mappings for the (gravitationally coupled) NLSM, multi-DBI and the SG as proposed in this thesis, it would be interesting to see if the correlators of these theories can be mapped onto each other.

Furthermore, the flavour factors that we discussed in chapter 3 are already known in the form of higher-derivative corrections to an $SO(M+1)/SO(M)$ coset. The addition of colour and flavour factors results in the tree-level amplitudes of the so-called *extended DBI theory* [59, 60]. This particular combination of numerators is possible due to the special coset structure in the $N=1$ case, and with both flavour and colour in the fundamental representation of $SO(M)$. It would also be interesting to further restrict to the single-flavour case, i.e., $(M=1)$, where the flavour and scalar-kinematic factors both live in the trivial representation (and hence only depend on kinematic invariants). At four-point these factors are given by a linear combination of the expression in footnote (25) and equation (265). The expectation is that the resulting theory corresponds to a free scalar field minimally coupled to gravity, with DBI and SG as higher-derivative corrections.

Acknowledgement

First and foremost, I would like to thank my supervisor Diederik Roest for his excellent guidance throughout the project, the thoughtful discussions and his enthusiasm for physics. His creative ideas made me stay curious and greatly helped me to improve as a researcher. Furthermore, I am thankful to the following people. To Dijs de Neeling for the nice collaboration and guidance during this research. To Anupam Mazumdar for his time spent on the assessment of this thesis and the presentation. To Shubham Maheshwari for explaining the basic concepts at the beginning of this research project. Last but not least, to everyone who was part of the journal clubs: these greatly helped me to stay motivated while also providing a broader overview on the current research in theoretical physics.

A Appendix

A.1 Invitation: classical versus quantum field theory

Before we move on to study the classical and amplitude double copies of effective scalar field theories, we will first briefly illustrate the difference between and classical field theory and QFT, motivated by the fact that the double copy has also been verified for some highly nontrivial loop-level examples, see for example [11, 12, 13]. We previously mentioned that at the classical level there are no loop-level Feynman diagrams, corresponding to the absence of the creation and annihilation of virtual particles. In this appendix, we will first specify the non-perturbative relation between the classical and quantum equations of motion that are obeyed by the connected correlation functions of non-linear field theories. Thereafter, we will provide insight in how the loop-level contributions arise as a consequence of a subtle difference between the classical and quantum equations satisfied by the correlators. Throughout this discussion, we will be loosely following [35]. We start

by considering the free scalar theory

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi\,, \tag{290}$$

where the scalar field ϕ satisfies the equations of motion

$$\Box\phi = 0\,, \tag{291}$$

simultaneously with the well known commutation relations of quantum mechanics

$$\begin{aligned} \left[\phi(\vec{x},t),\phi\left(\vec{x}',t\right)\right] &= 0\,, \\ \left[\phi(\vec{x},t),\partial_t\phi\left(\vec{x}',t\right)\right] &= i\hbar\delta^3\left(\vec{x}-\vec{x}'\right). \end{aligned} \tag{292}$$

The first commutation relation is a condition for causality, i.e., at the same instant of time, all operators should be observable and commute, which means that two points in space cannot exchange information faster than the speed of light. The latter commutation relation is equivalent to the Heisenberg uncertainty principle ($[\hat{x},\hat{p}] = i\hbar$ in quantum mechanics). Let's first recall from our introductory quantum field theory course that the time-ordered correlation function for the free theory (or Feynman propagator) is given by[32]

$$D_F(x_1,x_2) = \langle 0|T\{\phi_0(x_1)\phi_0(x_2)\}|0\rangle = \lim_{\varepsilon\to 0}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}e^{ik(x_1-x_2)}, \tag{293}$$

where the pole at $k^2=0$ will eventually be removed by the LSZ reduction formula. Our goal is to calculate the time-ordered n-point correlation function for the interacting theory, that is, when (290) contains an additional interaction term denoted by $\mathcal{L}_{\text{int}}$. It is instructive to first state the intermediate result,

$$\Box\left\langle\Omega\middle|T\left\{\phi(x)\phi\left(x'\right)\right\}\middle|\Omega\right\rangle = \left\langle\Omega\middle|T\left\{\Box\phi(x)\phi\left(x'\right)\right\}\middle|\Omega\right\rangle - i\hbar\delta^4\left(x-x'\right), \tag{294}$$

[32] See [35] for the derivation and detailed explanation.

where $|\Omega\rangle$ denotes the vacuum of the interacting theory. The latter term of this equation encodes the difference between a quantum and a classical theory, in a way that will become clear soon. To derive the above equation, we first calculate

$$\begin{aligned}\partial_t \left\langle \Omega \middle| T\left\{\phi(x)\phi\left(x'\right)\right\} \middle| \Omega \right\rangle &= \partial_t \left[\left\langle \Omega \middle| \phi(x)\phi\left(x'\right) \middle| \Omega \right\rangle \theta\left(t-t'\right) + \left\langle \Omega \middle| \phi\left(x'\right)\phi(x) \middle| \Omega \right\rangle \theta\left(t'-t\right)\right] \\ = \left\langle \Omega \middle| T\left\{\partial_t\phi(x)\phi\left(x'\right)\right\} \middle| \Omega \right\rangle &+ \left\langle \Omega \middle| \phi(x)\phi\left(x'\right) \middle| \Omega \right\rangle \partial_t\theta\left(t-t'\right) + \left\langle \Omega \middle| \phi\left(x'\right)\phi(x) \middle| \Omega \right\rangle \partial_t\theta\left(t'-t\right) \\ &= \left\langle \Omega \middle| T\left\{\partial_t\phi(x)\phi\left(x'\right)\right\} \middle| \Omega \right\rangle + \delta\left(t-t'\right)\left\langle \Omega \middle| \left[\phi(x),\phi\left(x'\right)\right] \middle| \Omega \right\rangle,\end{aligned} \tag{295}$$

where the last line follows from $\partial_t\theta(t) = \delta(t)$. We note that the second term of the last line vanishes by force of the commutation relations (292), i.e., the delta function $\delta(t-t')$ forces the commutator $[\phi(x),\phi(x')]$ to be evaluated at $t = t'$, which means that the second term of the last line reduces to $\langle\Omega|[\phi(x,t),\phi(x',t)]|\Omega\rangle = 0$. The second derivative of the above is given by

$$\begin{aligned}\partial_t^2\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle &= \left\langle \Omega \middle| T\left\{\partial_t^2\phi(x)\phi\left(x'\right)\right\} \middle| \Omega \right\rangle + \delta\left(t-t'\right)\left\langle \Omega \middle| \left[\partial_t\phi(x),\phi\left(x'\right)\right] \middle| \Omega \right\rangle \\ &= \left\langle \Omega \middle| T\left\{\partial_t^2\phi(x)\phi\left(x'\right)\right\} \middle| \Omega \right\rangle - i\hbar\delta^4\left(x-x'\right)\end{aligned} \tag{296}$$

where the last line follows from the commutation relation (292). It is then easy to see that the above generalizes to the result (294). We now introduce the short-hand notation for the time-ordered correlation function, $\langle\cdots\rangle = \langle\Omega|T\{\cdots\}|\Omega\rangle$, which we use to rewrite equation (294) as

$$\left(\Box + m^2\right)\left\langle\phi(x)\phi\left(x'\right)\right\rangle = \left\langle\left(\Box + m^2\right)\phi(x)\phi\left(x'\right)\right\rangle - i\hbar\delta^4\left(x-x'\right). \tag{297}$$

We can easily generalize the above for time-ordered correlation functions involving more fields, by noting that the time derivative acting on the time-ordering operator $T\{\ldots\}$ yields additional terms of the form $[\partial_t\phi(x),\phi(x_j)]$. The general result for n fields is given by

$$\begin{aligned}\Box_x\langle\phi(x)\phi(x_1)\cdots\ \phi(x_n)\rangle &= \langle\Box_x\phi(x)\phi(x_1)\cdots\phi(x_n)\rangle \\ &-i\hbar\sum_j\delta^4(x-x_j)\langle\phi(x_1)\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n)\rangle.\end{aligned} \tag{298}$$

Let's now compare this result with an interacting theory with Lagrangian $\mathcal{L} = -\frac{1}{2}\phi\Box\phi + \mathcal{L}_{\text{int}}$. In this case, the equation of motion of the (quantum) field read

$$\Box\phi + \mathcal{L}'_{\text{int}}(\phi) = 0, \tag{299}$$

where the prime denotes functional differentiation. Using the above and denoting $\phi_x = \phi(x)$, we can rewrite (298) for the interacting theory as

$$\Box_x\langle\phi_x\phi_1\cdots\phi_n\rangle = \left\langle\mathcal{L}'_{\text{int}}(\phi_x)\phi_1\cdots\phi_n\right\rangle - i\hbar\sum_j\delta^4(x-x_j)\langle\phi_1\cdots\phi_{j-1}\phi_{j+1}\cdots\phi_n\rangle, \tag{300}$$

which are the well known *Dyson-Schwinger equations*. It is important to note that we derived this equation without specifying the interaction term of the Lagrangian, and only by using the quantum mechanical commutation relations (292). The Dyson-Schwinger equations illustrate the difference between quantum and classical theories, which can be seen from the fact that classical fields satisfy $[\phi(\vec{x},t), \partial_t \phi(\vec{x}',t)] = 0$, leading to the absence of the second term of the Dyson-Schwinger equations. Another (equivalent) way to obtain the classical equations from the quantum equations is by simply taking the classical limit $\hbar \to 0$. These observations imply that for classical theories the fields within the correlation functions satisfy the same equations as the correlation functions themselves. For quantum theories, this only holds modulo so-called *contact terms*, which are the terms proportional to the delta functions. These contact terms allow virtual particles to be created and destroyed, giving rise to Feynman diagrams with closed loops.

In the above, we have seen that the Dyson-Schwinger equations give a non-perturbative relationship between classical and quantum theories. Here we will illustrate how loop-diagrams arise by solving the Dyson-Schwinger equations in perturbation theory. In order to keep the equations compact, we introduce the notations $\delta_{xi} = \delta^{(4)}(x - x_i)$, $D_{ij} = D_{ji} = D_F(x_i, x_j)$, and we will work in natural units ($\hbar = 1$). Using this notation, the free equation of the Feynman propagator reads

$$\Box_x D_{x1} = -i\delta_{x1}, \tag{301}$$

which implies that the D_{ij} are Green's functions for the free equation. Using this relation, we can write the two-point correlation function as

$$\langle \phi_1 \phi_2 \rangle = \int d^4x \delta_{x1} \langle \phi_x \phi_2 \rangle = i \int d^4x \left(\Box_x D_{x1} \right) \langle \phi_x \phi_2 \rangle = i \int d^4x D_{x1} \Box_x \langle \phi_x \phi_2 \rangle, \tag{302}$$

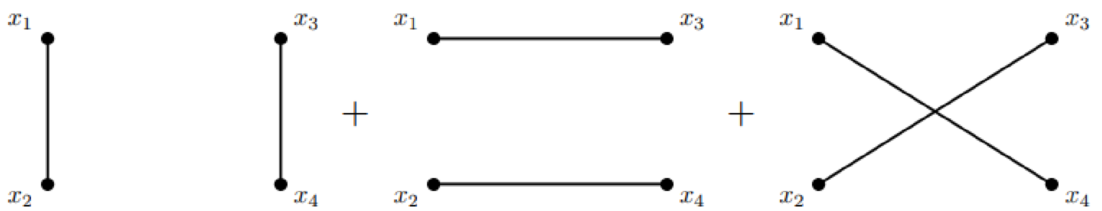
where the last step is obtained by partial integration. Suppose that we consider the free theory, then we can use the Dyson-Schwinger equation $\Box_x \langle \phi_x \phi_y \rangle = -i\delta_{xy}$ to rewrite the above as

$$\langle \phi_1 \phi_2 \rangle = \int d^4x D_{x1} \delta_{x2} = D_{12}. \tag{303}$$

In a similar fashion, we find that the four-point correlation function is given by

$$\begin{aligned} \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle =& i \int d^4x D_{x1} \Box_x \langle \phi_x \phi_2 \phi_3 \phi_4 \rangle = \int d^4x D_{x1} \left\{ \delta_{x2} \langle \phi_3 \phi_4 \rangle + \delta_{x3} \langle \phi_2 \phi_4 \rangle + \delta_{x4} \langle \phi_2 \phi_3 \rangle \right\} \\ =& D_{12} D_{34} + D_{13} D_{24} + D_{14} D_{23}, \end{aligned} \tag{304}$$

which can be diagrammatically visualized as



,

where the lines represent propagators and the x_j denote the points of evaluation. Next we "turn on" the interactions by considering the cubic scalar theory (118) that we previously discussed. The equations of motion read

$$\partial^2\phi + \frac{1}{2}g\phi^2 = 0, \tag{305}$$

from which we identify the interaction term $\mathcal{L}_{\text{int}} = \frac{g}{3!}\phi^3$. Application of the Dyson-Schwinger equation now yields the two-point correlation function

$$\langle\phi_1\phi_2\rangle = i\int d^4x D_{1x}\left(\frac{g}{2}\left\langle\phi_x^2\phi_2\right\rangle - i\delta_{x2}\right). \tag{306}$$

This two-point correlation function can be simplified by using $\delta_{2y} = i\Box_y D_{y2}$. We obtain

$$\begin{aligned}\langle\phi_1\phi_2\rangle =& D_{12} - \frac{g}{2}\int d^4x d^4y D_{x1}D_{y2}\Box_y\left\langle\phi_x^2\phi_y\right\rangle \\ =& D_{12} - \frac{g^2}{4}\int d^4x d^4y D_{x1}D_{2y}\left\langle\phi_x^2\phi_y^2\right\rangle + ig\int d^4x D_{1x}D_{2x}\left\langle\phi_x\right\rangle,\end{aligned} \tag{307}$$

where the last line follows from partial integration. To keep the discussion clear and concise, we are only interested in the one loop-level diagrams, which arise from the $\sim g^2$ term of the last line. Therefore, we must expand the second and third term up to $\mathcal{O}(g^0)$ and $\mathcal{O}(g)$, respectively. The second term of the last line can be simplified by recognising the similarity with the four-point correlation function of the free equation, (304). We obtain

$$\left\langle\phi_x^2\phi_y^2\right\rangle = 2D_{xy}^2 + D_{xx}D_{yy} + \mathcal{O}(g). \tag{308}$$

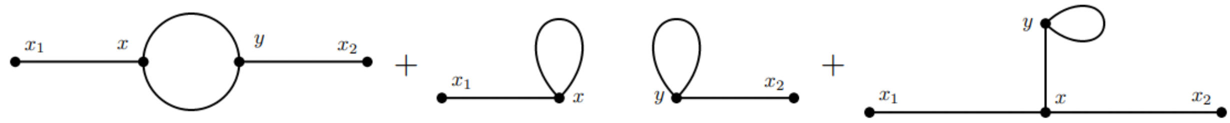
Furthermore, we can expand $\langle\phi_x\rangle$ using the Dyson-Schwinger equation (300),

$$\langle\phi_x\rangle = i\int d^4y D_{xy}\Box_y\langle\phi_y\rangle = i\frac{g}{2}\int d^4y D_{xy}\left\langle\phi_y^2\right\rangle = i\frac{g}{2}\int d^4y D_{xy}D_{yy} + \mathcal{O}\left(g^2\right), \tag{309}$$

which leads to the final result

$$\langle\phi_1\phi_2\rangle = D_{12} - g^2\int d^4x d^4y\left(\frac{1}{2}D_{1x}D_{xy}^2 D_{y2} + \frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2} + \frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}\right) + \mathcal{O}(g^3). \tag{310}$$

The second term of this result corresponds to the (one-loop) diagrams



where the loops represent the creation and annihilation of virtual particles. We should remark that this corresponds to the lowest loop-level and that the higher order terms of (310) give rise to diagrams containing more loops. In contrast, the classical counterpart of this two-point correlation function only contains the term D_{12}, which represents a particle propagating from x_1 to x_2, and therefore leads to a vanishing tree-level scattering amplitude.

A.2 DBI classical solution

The Mathematica code that was used to derive the classical solution to DBI (245) is shown below. Finally, we used WolframAlpha to identify the closed form hypergeometric series, given the calculated coefficients. Note that a similar approach works for the SG classical solution.

```
In[1]:= (*Below we have the DBI equation of motion for a stationary spherically symmetric field.*)
        ODE = p''[r] r + 2 p'[r] - (1 / 2) g (p'[r])^2 (2 p'[r] + 3 p''[r] r) + (3 / 8) g^2 (p'[r])^4 (2 p'[r] + 5 p''[r] r) -
           (5 / 16) g^3 (p'[r])^6 (2 p'[r] + 7 p''[r] r) + (35 / 128) g^4 (p'[r])^8 (2 p'[r] + 9 p''[r] r);
        (*expanding the field, denoted p[r_], in the coupling constant g.*)
        p[r_] := p0[r] + g * p1[r] + g^2 * p2[r] + g^3 * p3[r] + g^4 * p4[r];

        (* Below we solve the ODE iteratively for eacht perturbative correction P#[r_] *)

In[3]:= p0[r_] := -c / r (*The leading order correction is always proportional to 1/r*)
        firstcorrODE = Coefficient[ODE, g]
        sol1 = DSolve[Coefficient[ODE, g] == 0, p1[r], r]
        p1[r_] := -c^3 / (10 r^5)
        sol2 = DSolve[Coefficient[ODE, g^2] == 0, p2[r], r]
        p2[r_] := -c^5 / (24 r^9)
        sol3 = DSolve[Coefficient[ODE, g^3] == 0, p3[r], r]
        p3[r_] := -5 c^7 / (208 r^13)
        sol4 = DSolve[Coefficient[ODE, g^4] == 0, p4[r], r]
```

Out[4]= $\frac{2\,c^3}{r^6} + 2\,p1'[r] + r\,p1''[r]$

Out[5]= $\left\{\left\{p1[r] \to \frac{1}{2}\left(-\frac{c^3}{5\,r^5} - \frac{2\,c_1}{r}\right) + c_2\right\}\right\}$

Out[7]= $\left\{\left\{p2[r] \to -\frac{c^5}{24\,r^9} - \frac{c_1}{r} + c_2\right\}\right\}$

Out[9]= $\left\{\left\{p3[r] \to -\frac{5\,c^7}{208\,r^{13}} - \frac{c_1}{r} + c_2\right\}\right\}$

Out[11]= $\left\{\left\{p4[r] \to -\frac{35\,c^9}{2176\,r^{17}} - \frac{c_1}{r} + c_2\right\}\right\}$

A.3 GR minimally coupled to the NLSM

Here we derive the four-point amplitudes of the compact $SO(M+N)/(SO(M)\times SO(N))$ NLSM coupled to gravity.[33] First, we will use the perturbation theory method as outlined in e.g. [61, 16], after which we apply the *Lehmann, Symanzik and Zimmermann (LSZ) formula* to extract amplitudes.

Recall from section 7 that the Lagrangian of the $SO(M+N)$ NLSM minimally coupled to gravity, now including coupling constants, reads

$$\mathcal{L}_{\text{GR+NLSM}} = \sqrt{-g}\left(\frac{2}{\kappa^2}R - \frac{1}{2}[\frac{1}{1+\frac{\phi\phi^T}{F^2}}\nabla^\mu\phi\frac{1}{1+\frac{\phi^T\phi}{F^2}}\nabla_\mu\phi^T]\right), \tag{311}$$

[33] This appendix is taken from the preprint version of our article [26].

where $\kappa^2 = 4/M_{\rm Pl}^2$ in terms of the (reduced) Planck mass, and F is the NLSM cut-off scale. Following e.g. [61], we work with the so-called *gothic graviton* $\mathfrak{h}^{\mu\nu}$, such that

$$\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} - \kappa\mathfrak{h}^{\mu\nu}\,. \tag{312}$$

Furthermore, we adopt the *De Donder gauge*, with $\partial_\mu \mathfrak{h}^{\mu\nu} = 0$. These choices lead to the particularly useful properties that the Einstein tensor is given by $G_{\mu\nu} = -\frac{\kappa}{2}\Box\mathfrak{h}_{\mu\nu}$, and the curved space-time d'Alembertian, denoted by $\Box_c \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, reduces to $\Box_c = g^{\mu\nu}\partial_\mu\partial_\nu$ [64]. The field equation for the graviton and scalar field respectively read

$$\begin{aligned} G_{\mu\nu} =& \frac{\kappa^2}{4}(g_{\rho\mu}g_{\sigma\nu} - \tfrac{1}{2}g_{\mu\nu}g_{\rho\sigma})[\frac{1}{1+\frac{\phi\phi^T}{F^2}}\partial^\rho\phi\frac{1}{1+\frac{\phi^T\phi}{F^2}}\partial^\sigma\phi^T]\,, \\ \Box_c\phi =& 2\sum_{n=1}^{\infty}(-1)^{n-1}\partial_\mu\phi\frac{\phi^T}{F^2}\left(\frac{\phi\phi^T}{F^2}\right)^{n-1}\partial^\mu\phi\,. \end{aligned} \tag{313}$$

By expanding $\mathfrak{h}^{\mu\nu}$ and ϕ in their coupling constants,

$$\mathfrak{h}^{\mu\nu} = \mathfrak{h}^{(0)\mu\nu} + \kappa\mathfrak{h}^{(1)\mu\nu} + \kappa^2\mathfrak{h}^{(2)\mu\nu} + \dots\,, \qquad \phi = \phi^{(0)} + \frac{\phi^{(1)}}{F^2} + \frac{\phi^{(2)}}{F^4} + \dots\,, \tag{314}$$

and substituting these expansions into the field equations (313), we obtain a differential equation for each perturbative correction $\mathfrak{h}^{(k)\mu\nu}$ and $\phi^{(k)}$. The relevant equations for four-scalar scattering are given by

$$\begin{aligned} \Box\mathfrak{h}^{(0)\mu\nu} &= -\frac{\kappa}{2}\left(\partial^\mu\phi^{(0)}\partial^\nu\phi^{(0)T} - \frac{\kappa}{2}\eta^{\mu\nu}\partial^\rho\phi^{(0)}\partial_\rho\phi^{(0)T}\right)\,, \\ \Box\phi^{(1)} &= \kappa\partial_\mu\partial_\nu\phi^{(0)}\mathfrak{h}^{(0)\mu\nu} + \frac{2}{F^2}\partial^\mu\phi^{(0)}\phi^{(0)T}\partial_\mu\phi^{(0)}\,. \end{aligned} \tag{315}$$

Fourier transforming the above to momentum space leads to

$$\mathfrak{h}^{(0)\mu\nu}(-p_1) = -\frac{1}{p_1^2}\int đ^4p_2đ^4p_3\frac{\kappa}{2}\left\{(p_2^\mu p_3^\nu) - \frac{1}{2}\eta^{\mu\nu}(p_2\cdot p_3)\right\}[CD][\phi^{(0)c\bar{c}}(p_2)\phi^{(0)d\bar{d}}(p_3)]\,, \tag{316}$$

$$\begin{aligned} \phi^{(1)a\bar{a}}(-p_1) = -\frac{1}{p_1^2}\int đ^4p_2đ^4p_3đ^4p_4\Bigg\{&\frac{\kappa^2}{4}\left(\frac{s_{23}s_{24}}{2s_{12}} - \frac{1}{2}(p_2)^2\right)[AB][CD]\phi^{(0)b\bar{b}}(p_2)[\phi^{(0)c\bar{c}}(p_3)\phi^{(0)d\bar{d}}(p_4)] \\ &- \frac{s_{13}}{2F^2}([ABCD]+[ADCB])\phi^{(0)b\bar{b}}(p_2)[\phi^{(0)c\bar{c}}(p_3)\phi^{(0)d\bar{d}}(p_4)]\Bigg\}\,, \end{aligned} \tag{317}$$

where we have explicitly included the flavour indices and suppressed the momentum-conserving delta functions $\delta\!\!\!{}^{-}{}^{(4)}(p_1+\dots+p_n)$. Additionally, the common short-hand notation

$$đ^4p \equiv \frac{d^4p}{(2\pi)^4}\,, \quad \delta\!\!\!{}^{-}{}^{(4)}(p) \equiv (2\pi)^4\delta^{(4)}(p)\,, \tag{318}$$

was employed for legibility.

Next, we note that the term proportional to ${p_2}^2$ in (317) vanishes on-shell and use the LSZ formula (see e.g. [16] for similar calculations) in order to extract the four-scalar partial amplitude from $\phi^{(1)}$. The result in terms of $M_{\rm Pl}$ reads

$$\begin{aligned} A_4 &= \lim_{p_1^2 \to 0} p_1^2 \frac{\delta^3 \phi^{(1)}(-p_1)}{\delta\phi^{(0)}(p_2)\delta\phi^{(0)}(p_3)\delta\phi^{(0)}(p_4)} \\ &= -\frac{1}{2M_{\rm Pl}^2} \frac{s_{14}s_{13}}{s_{12}}[AB][CD] + \frac{s_{13}}{2F^2}([ABCD]+[DCBA]) + (\text{cyclic}) \, , \end{aligned} \tag{319}$$

where the first term corresponds to graviton exchange diagrams and the second to contact interactions (see figure 13). The structures of these amplitudes coincide with the graviton exchange amplitude in equation (11) of [65] and the NLSM amplitude in equation (1.4) of [66]. Note that we have opposite signs in the above amplitude, in contrast to what we found in section 7; the latter therefore corresponds to a non-compact scalar manifold with non-linear $SO(M,N)$ symmetry.

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