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A Study of Lebesgue Integration via Category Theory

Abstract: This thesis aims to understand the Lebesgue integration operator using category theory. Particularly, it looks at whether this operator is uniquely characterized. We also discuss the traditional non-categorical way to do this and then give the similarities it has with the categorical approach. Finally, the thesis serves as a pedagogical exercise for the author to learn some basic notions of category theory such as universal properties, initial objects and unique characterization.

Key words: Category, Commutative Diagram, Initiality, Functor, Characterization, Banach Space, Lebesgue Space, Completion.

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Chapter 1

Introduction

Calculus has been an important part of Science and Mathematics since the 17-th century. Leibniz and Newton defined the integral operator as the inverse of taking the derivative. This method of anti-derivatives was later understood using the Fundamental Theorem of Calculus. However, it is already ineffective for functions with finite number of discontinuities [1]. In the 18-th century, with the advent of real analysis, Cauchy and Riemann defined the integral operator as the 'taking the area under the curve'. More precisely, this was done by partitioning the domain into sub-intervals in order to approximate the area between the curve and the x-axis by that of a series of rectangles, called the Riemann sum. Increasing the number of rectangles used to cover the region under the curve increases the accuracy of Riemann sum. So, in the limit, the sum and the integral values coincide. This method too, however, was not general enough. For instance, it fails[1] for functions like the Dirichlet function, which is nowhere continuous [1]. Also, since it depends on real analysis, the method fails to work when we want to integrate on spaces other than $\mathbb{R}^n$.

Later, in the 20-th century, Lebesgue introduced 'measurable' functions and defined the integral operator using the notions of measure theory. This extends integrability to a set of functions that is a super-set of the set of functions that are Riemann integrable. In fact, one can show that a function is Riemann integrable iff the set of its discontinuities has the Lebesgue measure 0, i.e. it is a zero set. Further, this method allows for integration over domains other than $\mathbb{R}^n$, provided that they are measure spaces [8]. This has huge consequences in Analysis, in particular the study of Partial Differential Equations and Fourier Analysis.

Despite being not as general as Lebesgue integration, Riemann integration is still widely used. This is because, for computation of most functions in calculus, science and engineering, approximating the integral using Riemann sums is a straightforward and useful method. But, then why do we do we need the theory of Lebesgue integration? Apart from the advantages listed before, it is important to note that integrability also plays a vital role in the completion of $\mathcal{C}[X]$, the space of continuous functions from X to $\mathbb{F}$, with respect to the L^p-norm (see Section 2.3). In fact, $L^p(X)$, the space of Lebesgue integrable functions on X, is indeed the completion of $\mathcal{C}[X]$. Here, by complete, we mean the property that every Cauchy sequence in some space converges to a point in that space. This is indeed a crucial property for spaces that are studied in Analysis, since without it we can't define such things as limits and continuity [4]. This completion, in fact, can be used to define Lebesgue integrability and integration. But, before introducing this, let us see what characterization means and why it is important.

[1] It only works for functions which are continuous almost everywhere.

1.1 Unique Characterization and Universal Properties

In Mathematics, to say that property P characterizes a space (or object) X is to mean that not only does X have the property P, but X is the unique (up to isomorphisms) space (or object) with property P. This plays an important role in the development and understanding of mathematics as often we have situations where a property P holds for a subset (usually satisfying a special set of conditions that other elements in X don't) of X and we are interested in finding ways to extend P (or a slightly modified version, say P) to all elements in X. While the first thing to check is whether this extension is possible, once it is found, one naturally wonders if it is the only possible extension. This has a wide range of consequences in all fields of mathematics and often shapes the research in these fields [5, 7].

While there are existence and uniqueness theorems in all fields of mathematics, the most natural way to check for unique characterization is by using the language of Category Theory. In Category Theory (see Section 3.1), one studies objects (roughly speaking, spaces) with some structure and morphisms (roughly speaking, maps) that preserves this structure. One way of showing that an object of a category is uniquely characterized is to show that it has a universal property. Roughly speaking, a universal property gives the unique (up to isomorphisms) way an object in some category is related to all other objects in a particular universe [5] containing the object. A universe, roughly akin to the universal set from set theory, is a category of categories[2] whose morphisms are the so called functors (see Definition 3.1.5).

Further, an initial object (roughly speaking) is an object of a category that is related to every other object in that category by a unique morphism of the category [7]. Using functors one can show that (up to isomorphisms of the universal category) the relationship between this initial object and any other object in a universe containing it is uniquely characterized. Therefore, initiality of objects is a universal property [5]. Moreover, one can show that every universal property can be expressed in terms of initiality of objects [6].

1.2 Traditional Methods

Now, we continue with the problems of extending (or generalizing) integrability and integration, beyond the Riemann integral. Like we discussed earlier, extension is Lebesgue integrability and integration. We also saw that this is connected to the problem of completion of $\mathcal{C}[X]$ in that the space $L^p(X)$ of Lebesgue integrable functions on X is indeed the completion of $\mathcal{C}[X]$ [4]. That is, by uniquely characterizing $L^p(X)$, one can uniquely characterize Lebesgue integrability and integration on X. This can be done in two ways [4, 6]:

(a) Provided integration is defined for continuous functions from $[a, b]$ to $\mathbb{F}$, theorem 2.2.1 can be used to find a unique isometry ι and a unique (up to isomorphisms) complete space $\overline{X}$. Further, using Corollary 2.2.3, it can be shown that this $\overline{X}$ is the unique completion of $\mathcal{C}[X]$. Therefore, the process of completion of $\mathcal{C}[X]$ not only leads to a unique complete extension of this space but also characterizes the completion uniquely. Now, using the isometry ι we can extend the (Riemann) integrability and integration defined for continuous functions on from X to $\mathbb{F}$ to those in a more general case on X. Since the pair $(\iota, \overline{X})$ is unique up to isomorphisms, this extension of integrability and integration must be unique as well. Therefore, the unique characterization of the completion of $\mathcal{C}[X]$ gives an unique characterization of integrability and integration.

(b) We can start with a measure space X, define the concepts of convergence almost everywhere

[2]Note that unlike in set theory this does not lead to paradoxes [5].

(a.e. for short), null sets and step functions. Once this is done, one defines integration for step functions and extends this integration operator to the set of functions that are a.e. limits of increasing sequences of step functions. Once this done, the integration operator is extended to $\mathcal{L}^1(X)$ which is the set precisely containing differences of elements of the set of functions that are a.e. limits of increasing sequences of step functions. Finally, we extend the integration operator once again to the Lebesgue space $L^1(X)$, which is the quotient space of $\mathcal{L}^1(X)$. In this thesis, we discuss this method very briefly and in conjunction with the previous method.

1.3 Via Category Theory

Following the methods of [6], we discuss a way to uniquely characterize the Lebesgue space $L^p(X)$, using Category Theory. As seen in Theorem Theorem 4.2.1, we show that $L^p(X)$ is an initial object of the category **Ban** of Banach spaces (with some additional categorical structure, see Sections 3.3 and 4.2). Once we have done this, we show that this unique characterization of $L^p(X)$ (for $p = 1$ particularly) uniquely characterizes Lebesgue integrability and the Lebesgue integration operator on any finite measure space X (see Theorems 4.2.2, 4.2.3). Note, we do the above (i.e. characterize $L^p(X)$ and use it characterize Lebesgue integrability and integration operator) first for the simple case of $X = [0, 1]$ in Section 3.3 (see Theorem 3.3.5 and Proposition 3.4.2). This method has several advantages not the least of which is that it provides new perspective on an old problem. See Section 3.2 for a full list of advantages to this method over the traditional methods.

1.4 Content Outline

Firstly, in Chapter 2, we list some basic notions required from Functional Analysis such as completion and use them to state and prove the completion theorem 2.2.1. Next we define the Lebesgue space $L^p[a, b]$ using some Measure theory, and use the completion theorem to uniquely characterize it as the completion of the space of continuous functions equipped with the L^p norm (see Section 2.4). We also discuss how we can uniquely define the Lebesgue integration operator using the completion theorem. Further, in Chapter 3, we first give some basic category theory definitions and notions. Then, after giving the motivation for a category theoretic approach to unique characterization of Lebesgue integration, we show that $L^p[0, 1]$ is an initial object of the category **Ban** (with some additional categorical structure) in Theorem 3.3.5 and use this to uniquely characterize and determine the integration operator $\int_0^1$ on $L^1[0, 1]$ (see Proposition 3.4.2 and Theorem 3.4.3). Lastly, in Chapter 4, we do the same for any arbitrary finite measure space X (see Theorems 4.2.1, 4.2.2 and 4.2.3).

Chapter 2

Integration via Functional Analysis

In Functional Analysis, we study normed linear spaces where every Cauchy sequence converges. Further, when dealing with spaces without such a property, we use the process of 'completion' to fix this. In this chapter, we first give the general theory of how to complete any normed linear space X (in Theorem 2.2.1, the so called completion theorem) and argue that such a completion of X is unique up to isometric isomorphism. Then, we study the space of continuous functions from an arbitrary interval $[a, b]$ to the field $\mathbb{R}$ or $\mathbb{C}$. We show that this space is a normed linear space that is not complete with respect to the L^p norm. Then, using the above mentioned completion theorem, we give an abstract completion of this space of continuous functions. Finally, we discuss how this leads to an unique characterization of the Lebesgue Space and Lebesgue Integration.

2.1 Basic Notions

Given a linear space V, we like to define a norm $\|.\|$ on V, as this allows us to do the necessary analysis.

Definition 2.1.1. Let V be a linear space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then, $\|.\| : V \times V \to \mathbb{R}$ is called a **norm** if,

1. $\|u\| \geq 0, \forall\, u \in V$ and $\|u\| = 0 \iff u = 0$,
2. $\|\lambda u\| = |\lambda|\|u\|, \forall\, \lambda \in \mathbb{F}, \forall\, u \in V$,
3. $\|u + v\| \leq \|u\| + \|v\|, \forall\, u, v \in V$.

With this, the linear space V equipped with norm $\|.\|$ is called the **Normed Linear Space**.

Definition 2.1.2. Let $(V, \|.\|)$ be a normed linear space:

(a) A sequence (x_n) in V is said to **converge** to a point x w.r.t $\|.\|$, if

$$\forall\, \epsilon > 0\,,\ \exists\, N \in \mathbb{N} \,:\, n \geq N \implies \|x_n - x\| < \epsilon,$$

(b) A sequence (x_n) in V is said to be **Cauchy**, if

$$\forall\, \epsilon > 0\,,\ \exists\, N \in \mathbb{N} \,:\, m,\, n \geq N \implies \|x_n - x_m\| < \epsilon,$$

(c) $(V, \|.\|)$ is said to be **complete** if every Cauchy sequence in $(V, \|.\|)$ converges to a point in V w.r.t $\|.\|$,

(d) Normed linear spaces that are complete are called **Banach Spaces**.

Now, we define some terms that relate to the algebraic structure of normed linear spaces.

Definition 2.1.3. Let X, Y be normed linear spaces,

(a) A **linear map** $T : X \to Y$ is such that for all $v, w \in X$ and $\lambda \in \mathbb{F}$, $T(v+w) = T(v)+T(w)$ and $T(\lambda v) = \lambda T(v)$.

(b) A linear map $T : X \to Y$ is called an **isometry** if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$[1].

(c) A linear map $T : X \to Y$ is called an **isomorphism** if it is a bijective map i.e. $\ker(T) = \{0\}$ and $\operatorname{im}(T) = Y$.

(d) A linear map $T : X \to Y$ is called an **isometric isomorphism** if it maps X isometrically to Y, i.e. if this linear map is an isomteric bijection.

(e) A set $A \subset X$ is said to be **dense** in X if the closure of A is X.

2.2 Completion of Normed Linear Spaces

We can now consider completion the of normed linear spaces. The following theorem will be referred to as the completion theorem.

Theorem 2.2.1 (Completion)**.** Let X be a normed linear space, then there exists a complete normed linear space $\tilde{X}$ and a linear mapping $\iota : X \to \tilde{X}$ s.t. X is isometrically isomorphic to $\iota(X)$ and $\iota(X)$ is dense subset of $\tilde{X}$.

Proof. The idea is to find a suitable $\tilde{X}$ and $\iota : X \to \tilde{X}$; we do this by following [4]. Consider $\mathcal{X}$ the set of all Cauchy sequences in X. $\mathcal{X}$ is a linear space with pointwise addition and scalar multiplication of sequences. Further, consider $\mathcal{V}$ the set of all sequences that converge to zero. Note that $\mathcal{V}$ is a linear subspace of $\mathcal{X}$: (i) every convergent sequence is Cauchy, so $\mathcal{V}$ is a subset of $\mathcal{X}$; (ii) sum of any two sequences that converges to 0 is another sequence that converges to 0; and (iii) scalar multiple of any sequence that converges to 0 is another sequence that converges to 0. Therefore, we can consider the quotient space $\mathcal{X}/\mathcal{V}$ of equivalence classes of the form,

$$\mathbf{x} + \mathcal{V} := \{\mathbf{y} \in \mathcal{X} | \, \mathbf{x} - \mathbf{y} \in \mathcal{V}\},$$

for $\mathbf{x}$, $\mathbf{x}' \in \mathcal{X}$, with vector addition and scalar multiplication defined as follows:

$$\lambda_1(\mathbf{x} + \mathcal{V}) + \lambda_2(\mathbf{x}' + \mathcal{V}) = \lambda_1\mathbf{x} + \lambda_2\mathbf{x}' + \mathcal{V}, \;\; \lambda_1, \lambda_2 \in \mathbb{F}.$$

For any Cauchy sequence $\mathbf{x} = (x_i)$ in $\mathcal{X}$, we define the norm of $x + \mathcal{V}$ to be,

$$\|\mathbf{x} + \mathcal{V}\| = \lim_{i\to\infty} \|x_i\|.$$

Note, by the reverse triangle inequality we have,

$$|\|x_i\| - \|x_j\|| \leq \|x_i - x_j\|.$$

So, we get that the sequence $(\|x_i\|)$ in $\mathbb{F}$ is Cauchy and hence convergent in $\mathbb{F}$. So, $\|\mathbf{x}+\mathcal{V}\|$ is defined for all such Cauchy sequences $\mathbf{x} \in \mathcal{X}$. Moreover, $\|\mathbf{x}+\mathcal{V}\|$ is independent of the choice of representative since for any $\mathbf{x}' + \mathcal{V} = \mathbf{x} + \mathcal{V}$, we have,

$$\begin{aligned} 0 \leq | \, \|\mathbf{x} + \mathcal{V}\| - \|\mathbf{x}' + \mathcal{V}\| \, | &\leq \|(\mathbf{x} + \mathcal{V}) - (\mathbf{x}' + \mathcal{V})\| = \|(\mathbf{x} - \mathbf{x}') + \mathcal{V}\| \\ &= \lim_{i\to\infty} \|x_i - x_i'\| = 0, \end{aligned}$$

[1]i.e. T does not change the norms (and therefore distances) of (between) elements of X.

since $\mathbf{x} - \mathbf{x}' \in \mathcal{V}$, the set of all Cauchy sequences that converge to 0. Therefore, $\|\mathbf{x} + \mathcal{V}\|$ is well-defined. Further, for all $\|\mathbf{x} + \mathcal{V}\|$, we have,

$$\|\mathbf{x} + \mathcal{V}\| = \lim_{i\to\infty} \|x_i\| \geq 0,$$

$$\|\mathbf{x} + \mathcal{V}\| = \lim_{i\to\infty} \|x_i\| = 0 \quad \Longleftrightarrow \quad \mathbf{x} \in \mathcal{V} \quad \Longleftrightarrow \quad \mathbf{x} + \mathcal{V} = \mathbf{0} + \mathcal{V},$$

where $\mathbf{0}$ is the sequence with all zeros. Also,

$$\|\lambda\,(\mathbf{x} + \mathcal{V})\| = \lim_{i\to\infty} \|\lambda\, x_i\| = |\lambda| \lim_{i\to\infty} \|x_i\| = |\lambda|\,\|\mathbf{x} + \mathcal{V}\|,$$

$$\|(\mathbf{x} + \mathcal{V}) + (\mathbf{y} + \mathcal{V})\| = \lim_{i\to\infty} \|x_i - y_i\| \leq \lim_{i\to\infty} \|x_i\| - \lim_{i\to\infty} \|y_i\| = \|\mathbf{x} + \mathcal{V}\| + \|\mathbf{y} + \mathcal{V}\|,$$

for any Cauchy sequences $\mathbf{x}$, $\mathbf{y} \in \mathcal{X}$ and for any $\lambda \in \mathbb{F}$. So, the $\|\mathbf{x} + \mathcal{V}\|$ satisfies Definition 2.1.1 and is therefore a valid norm on $\mathcal{X}/\mathcal{V}$.

Now consider the map,

$$\iota : X \to \mathcal{X}/\mathcal{V}, \qquad \iota(x) = (x,\, x, \ldots) + \mathcal{V}.$$

Since, for each $x \in X$, the constant sequence $(x,\, x, \ldots)$ converges to x in X, it is a Cauchy sequence in X and hence belongs to $\mathcal{X}$, the set of all Cauchy sequences in X. Therefore, $\iota(x) = (x,\, x, \ldots) + \mathcal{V}$ is in $\mathcal{X}/\mathcal{V}$, i.e. the map ι is well-defined. Next we show that the above map ι is a linear isometry by showing that it satisfies Definition 2.1.3. For any $x, y \in X$ and for any $\lambda \in \mathbb{F}$,

$$\iota(x + y) = (x + y,\, x + y, \ldots) + \mathcal{V} = (x,\, x, \ldots) + \mathcal{V} + (y,\, y, \ldots) + \mathcal{V} = \iota(x) + \iota(y),$$

$$\iota(\lambda\, x) = (\lambda\, x,\, \lambda\, x, \ldots) + \mathcal{V} = \lambda\,(x,\, x, \ldots) + \mathcal{V} = \lambda((x,\, x, \ldots) + \mathcal{V}) = \lambda\, \iota(x),$$

by using addition and scalar multiplication as defined above for the linear space $\mathcal{X}/\mathcal{V}$. Also, for any $x \in X$, we have,

$$\|\iota(x)\| = \|(x,\, x, \ldots) + \mathcal{V}\| = \lim_{i\to\infty} \|x_i\| = \lim_{i\to\infty} \|x\| = \|x\|.$$

Further, ι is an injective map from X to $\mathcal{X}/\mathcal{V}$ since $x \in \ker(\iota)$ iff,

$$(0,\, 0, \ldots) + \mathcal{V} = \iota(x) = (x,\, x, \ldots) + \mathcal{V} \quad \Longleftrightarrow \quad (x,\, x, ..) - (0,\, 0, \ldots) = (x,\, x, \ldots) \in \mathcal{V}.$$

Since, $\mathcal{V}$ is the subspace of $\mathcal{X}$ containing only sequences that converge to 0 and since the constant sequence $(x,\, x, \ldots)$ converges to x, we have that $x = 0$. That is, $\ker(\iota) = \{0\}$, i.e. ι is injective. Moreover, this implies that ι is a bijection, i.e. a linear isomorphism, between the normed linear spaces[2] X and $\mathrm{im}(\iota) = \iota(X) \subset \mathcal{X}/\mathcal{V}$, by Definition 2.1.3. Together, we conclude that the linear mapping ι is an isometric isomorphism between X and $\iota(X)$. Therefore, we have shown that X and $\iota(X)$ are isometrically isomorphic. Further, for every $\epsilon > 0$, since $\mathbf{x}$ is Cauchy, there exists a natural number N such that for $y := x_N \in X$ we have, $\|x_i - y\| < \epsilon$ for $i \geq N$. Considering, $\iota(y) = (y,\, y, \ldots) \in \iota(X)$, we get,

$$\|(\mathbf{x} + \mathcal{V}) - \iota(y)\| = \lim_{i\to\infty} \|x_i - y\| = \lim_{i\to\infty} \|x_i - x_N\| < \epsilon.$$

So, the closure of $\iota(X) \subset$ is all of $\mathcal{X}/\mathcal{V}$ and therefore $\iota(X)$ is a dense linear subspace of $\mathcal{X}/\mathcal{V}$.

At this point, it is clear that for the map ι as we have defined, a good choice for $\overline{X}$ is the normed linear space $\mathcal{X}/\mathcal{V}$. We would be done if we could show that $\mathcal{X}/\mathcal{V}$ is complete. In other words, we want to show that every Cauchy sequence in $\mathcal{X}/\mathcal{V}$ converges to a point in $\mathcal{X}/\mathcal{V}$. This is a rather straightforward ϵ argument. For the full detail see Theorem 3.30, from [4]. □

[2]Note that $\iota(X)$ is a linear subspace of $\mathcal{X}/\mathcal{V}$, since it closed under addition and scalar multiplication as defined for sequences in $\mathcal{X}/\mathcal{V}$.

Remark 2.2.2. Therefore, for each normed linear space X, the Banach space $\overline{X} := \mathcal{X}/\mathcal{V}$ and the isometry,

$$\iota : X \to \overline{X}, \qquad \iota(x) = (x, x, \ldots) + \mathcal{V},$$

are such that X is isometrically isomorphic to $\iota(X)$ and $\iota(X)$ is dense in $\overline{X}$.

Further, we can show that for each normed linear space X, there is a unique completion of X, up to isometric isomorphism. This is done using the following theorem which allows us to uniquely extend any isometric isomorphism between dense linear subspaces of two Banach spaces to an isometric isomorphism between these two Banach spaces.

Theorem 2.2.3. Suppose X_1, X_2 are two complete normed linear spaces with Y_1, Y_2 as dense linear subspaces, respectively. If Y_1, Y_2 are isometrically isomorphic by a map $\phi : Y_1 \to Y_2$ then X_1, X_2 are isometrically isomorphic by the unique map $\varphi : X_1 \to X_2$, which is an extension of the map ϕ.

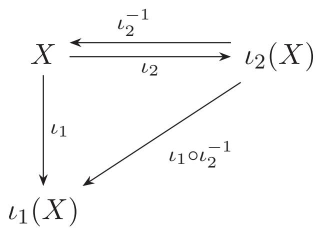
Proof. See Theorem 3.29 from [4]. □

Corollary 2.2.3.1 (Uniqueness of the Completion)**.** Let X be a normed linear space. Suppose that there is a pair of isometric isomorphism,

$$\iota_1 : X \to \iota_1(X) \subset \overline{X_1}, \qquad \iota_2 : X \to \iota_2(X) \subset \overline{X_2}$$

and a pair of completions $\overline{X_1}$, $\overline{X_2}$ of X, then $\overline{X_1}$, $\overline{X_2}$ are isometrically isomorphic.

Proof. If we suppose the hypothesis, we have a pair of isometric isomorphism ι_1 and ι_2. By definition, every isometric isomorphism is a bijection, so ι_2^{-1} exists and we can consider the map $\iota_1 \circ \iota_2^{-1}$. From the following diagram we see that $\iota_1 \circ \iota_2^{-1} : \iota_2(X) \to \iota_1(X)$.



Now, note that since the inverse of any isometric isomorphism is also an isometric isomorphism, ι_2^{-1} is an isometric isomorphism. Also, since composition of isometric isomorphism is an isometric isomorphism, $\iota_1 \circ \iota_2^{-1}$ is an isometric isomorphism from $\iota_2(X)$ to $\iota_1(X)$.

So, by the completion theorem (Theorem 2.2.1), we have that $\overline{X_1}$, $\overline{X_2}$ are Banach spaces, $\iota_1(X)$ is a dense linear subspace of $\overline{X_1}$, and $\iota_2(X)$ is a dense linear subspace of $\overline{X_2}$. Since we have already shown that $\iota_1(X)$ and $\iota_2(X)$ are isometrically isomorphic by the map $\iota_1 \circ \iota_2^{-1}$, by Theorem 2.2.3, we have that $\overline{X_1}$, $\overline{X_2}$ are isometrically isomorphic. □

2.3 Completion of Spaces of Continuous Functions

Now we will discuss the space of continuous functions from an arbitrary interval $[a, b]$ to $\mathbb{F}$. After giving the necessary definitions and notations, we first show that this space is a normed linear space that is not complete, and then use the completion theorem to complete this space uniquely up to isometric isomorphism. Note that in the introduction we considered the case of continuous functions from X to $\mathbb{F}$, but here we are restricting to the case where $X = [a, b]$. As long as we are concerned about spaces X that are either $\mathbb{R}^n$ or subsets of $\mathbb{R}^n$, for any $n \in \mathbb{N}$, what follows can be extended using the traditional methods of Multivariable Analysis.

However, if X is a (topological) space that is not a subset of $\mathbb{R}^n$, then we would have to define integration for continuous functions from X to $\mathbb{F}$, before attempting completion and unique characterization of integration of functions on X in the more general case. Once that is done, the rest is the same as for the $X = [a, b]$ case.

Definition 2.3.1. For a field $\mathbb{F}$ and a closed interval $[a, b] \subset \mathbb{R}$, we denote the space of all continuous maps $f : [a, b] \to \mathbb{F}$ by $\mathcal{C}[a, b]$.

Note that $\mathcal{C}[a, b]$ is a linear space with vector addition and scalar multiplication defined by

$$(f + g)(v) = f(v) + g(v), \qquad (\lambda f)(v) = \lambda f(v), \qquad v \in [a, b].$$

Further, as seen in [4], there are many choices for a norm on the linear space $\mathcal{C}[a, b]$. For example one can take the sup-norm or the L^p norm. Since this thesis focuses on (characterizing) Lebesgue integrability and integration, we focus on the L^p-norm.

Now, note that the integral in the definition of the L^p-norm, given below, is the Riemann integral. Provided that we have already defined (Riemann) integration of continuous functions from intervals to $\mathbb{F}$, we can continue with the completion of $\mathcal{C}[a, b]$ with respect to the L^p-norm and hope to extend the notion of integration to all functions in the resulting Banach space.

Lemma 2.3.2. For any $1 \leq p < \infty$, the L^p-norm,

$$\|.\|_p : \mathcal{C}[a, b] \to \mathbb{R},$$

$$\|f\|_p := \Big(\int_a^b |f(x)|^p dx \Big)^{1/p}, \qquad f \in \mathcal{C}[a, b], \tag{2.1}$$

is a norm on the linear space $\mathcal{C}[a, b]$.

Proof. For any $f \in \mathcal{C}[a, b]$, since f is continuous and the integrand is non-negative, we have,

$$\|f\|_p = \Big(\int_a^b |f(x)|^p dx \Big)^{1/p} \geq 0.$$

Further, for the same reason, $\|f\|_p = 0$ if and only if f is the zero map. Additionally, for any $\lambda \in \mathbb{F}$, we have,

$$\|\lambda f\| = \Big(\int_a^b |\lambda f(x)|^p dx \Big)^{1/p} = \lambda \Big(\int_a^b |f(x)|^p dx \Big)^{1/p} = \lambda \|f\|$$

Lastly, the triangle inequality follows from Minkowski's inequality- see Example 2.12 and Lemmas 2.3, 2.4 from [4]. □

Hence, $\mathcal{C}[a, b]$ is a normed linear space with respect to the L^p-norm. However, as seen in the following Lemma, this space is not complete.

Lemma 2.3.3. $\Big(\mathcal{C}[a, b], \|.\|_p\Big)$ is not complete.

Proof. Consider the following sequence of functions for some $c \in (a, b)$:

$$\varphi_n(x) = \begin{cases} 0 & \text{if } a \leq x \leq c - \frac{1}{n}, \\ nx + 1 - nc & \text{if } c - \frac{1}{n} \leq x \leq c, \\ 1 & \text{if } c \leq x \leq b. \end{cases}$$

For any $n \in \mathbb{N}$, $\varphi_n(x)$ is the zero function in $[a, c-1/n]$, a linear function on $[c-1/n, c]$ and a constant map on $[c, b]$. As such it is continuous on all three intervals individually. Further, the appropriate left and right hand limits (which are 0 and 1, respectively) agree at $c - 1/n$ and c. Therefore, $\varphi_n \in \mathcal{C}[a, b]$ for each $n \in \mathbb{N}$. Note that for $n \geq m$,

$$\|\varphi_n - \varphi_m\|_p^p = \int_{c-1/m}^{c-1/n} |mx + 1 - mc|^p dx + \int_{c-1/n}^{c} |nx - mx + mc - nc|^p dx,$$

$$\|\varphi_n - \varphi_m\|_p^p \leq m^p \int_{c-1/m}^{c-1/n} |x - c|^p dx + \int_{c-1/m}^{c-1/n} dx + c^p(n-m)^p \int_{c-1/n}^{c} |x - c|^p dx,$$

by Hölder's identity- see Example 2.12 from [4]. Further note that for $x \in [c - 1/m, c - 1/n]$, we have,

$$x - c \in [-1/m, -1/n], \qquad \text{i.e. } |x - c| \leq 1/m.$$

Similarly, for $x \in [c - 1/n, c]$, we have

$$x - c \in [-1/n, 0], \qquad \text{i.e. } |x - c| \leq 1/n.$$

Also note that $n - m \leq n$, so $(n-m)^p \leq n^p$. Together, we have,

$$\begin{aligned} \|\varphi_n - \varphi_m\| &\leq \frac{m^p}{m^p}\Big(c - \frac{1}{n} - c + \frac{1}{m}\Big) + \Big(c - \frac{1}{n} - c + \frac{1}{m}\Big) + c^p \frac{n^p}{n^p}\Big(c - c + \frac{1}{n}\Big), \\ &\leq \frac{2}{m} - \frac{2}{n} + \frac{c^p}{n} \to 0, \end{aligned}$$

for fixed c and p, and for $n \geq m \to \infty$. Therefore the sequence (φ_n) is Cauchy. Now, consider the map

$$\varphi(x) = \begin{cases} 0 & \text{if } a \leq x < c, \\ 1 & \text{if } c \leq x \leq b. \end{cases}$$

Note that for $x \in [c - 1/n, c]$, we have $nx + 1 - nc \in [0, 1]$ i.e. $|nx + 1 - nc| \leq 1$. So, it follows that,

$$\|\varphi_n - \varphi\|_p^p = \int_{c-1/n}^{c} |nx + 1 - nc|^p dx \leq \int_{c-1/n}^{c} dx = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Therefore, (φ_n) converges to φ pointwise. However, φ is not continuous on $[a, b]$ since it has a jump discontinuity at $x = c \in [a, b]$. Further, every normed linear space $(V, \|.\|)$ is a metric space [4] with the metric,

$$d(x, y) := \|x - y\|.$$

Since every metric space is Hausdorff [9], the limits in $(V, \|.\|)$ are unique, if they exist. So we have a Cauchy sequence of functions in $\mathcal{C}[a, b]$ that does not converge (with respect to the L^p-norm) to any function in $\mathcal{C}[a, b]$. Since limits in function spaces are unique when they exist and $\varphi \notin \mathcal{C}[a, b]$ is the limit of (φ_n), we have that $\Big(\mathcal{C}[a, b], \|.\|_p\Big)$ is not complete $\square$

We can now complete the normed linear space $\mathcal{C}[a, b]$ using Theorem 2.2.1 and Remark 2.2.2, that is, there exists a Banach space $\overline{X}$ and a (linear) isometry ι with,

$$\begin{aligned} \overline{X} &:= \mathrm{C}[a, b]/\mathrm{C}_*[a, b], \\ \iota : \mathcal{C}[a, b] \to \overline{X}, &\qquad \iota(x) = (x, x, \ldots) + \mathrm{C}_*[a, b], \end{aligned}$$

where $\mathrm{C}[a, b]$ is the set of all Cauchy sequences of continuous functions from $[a, b]$ to $\mathbb{F}$ and $\mathrm{C}_*[a, b]$ is the set of all sequences of continuous functions from $[a, b]$ to $\mathbb{F}$ that converge to 0, with respect to L^p-norm. Also, $\mathcal{C}[a, b]$ is isometrically isomorphic to $\iota(\mathcal{C}[a, b])$- which is a dense subset of $\overline{X}$. Furthermore, from Corollary 2.2.3, we can conclude that this completion $\overline{X}$ of $\mathcal{C}[a, b]$ is unique up to isometric isomorphism.

2.4 Unique Characterization of the Lebesgue Space

Thus far, we have shown how to complete the non-Banach space $\mathcal{C}[a,b]$ uniquely (up to isometric isomorphism) to get the Banach space $\overline{X} = \mathrm{C}[a,b]/\mathrm{C}_*[a,b]$. Theoretically, one could use the well-defined (Riemann) integration in the continuous case on $[a,b]$ and extend it to the Banach space $\overline{X}$ using the isometric isomorphism ι. Since ι and $\overline{X}$ are unique up to isometric isomorphism, i.e. uniquely characterized, this would lead to a unique characterization of integration in the more general case on $[a,b]$. However, since $\overline{X}$ is a quotient space of equivalence classes of function spaces, it is an abstract completion of $\mathcal{C}[a,b]$. Therefore in practice one usually uses Measure Theory to determine a general integration theory on $[a,b]$, called Lebesgue integration. Before we give a brief summary of how this done, consider the following definitions from Measure Theory [2, 4].

Definition 2.4.1. For any set X, the set $\mathcal{S}$ of subsets of X is called a **σ-algebra** if $\mathcal{S}$ contains X and is closed under taking complements and countable unions. Further, a **measure** $\mu : \mathcal{S} \to [0,\infty]$ on the σ-algebra $\mathcal{S}$ of X is any positive-definite map such that $\mu(\emptyset) = 0$ and for countably (possibly infinite) many sets S_n $(n \in \mathbb{N})$ in $\mathcal{S}$,

$$\mu\left(\bigcup_{n\in\mathbb{N}} S_n\right) = \sum_{n\in\mathbb{N}} \mu(S_n).$$

We then say $(X, \mathcal{S}, \mu)$ is a **measure space**. Finally, a **measurable function** is any map $f : (X,\mathcal{S},\mu) \to \overline{F}$ such that, for all $k \in \mathbb{F}$, $\{x \in X : f(x) < k\}$ belongs to $\mathcal{S}$.

In the above definition $\overline{F}$ is the field $\mathbb{F}$ with the corresponding points at infinity. We define **simple functions** to be measurable functions whose image is a finite set. Say, the image set of a simple function f is the finite set $a_1, \ldots, a_l$, then,

$$\int_X f \,\mathrm{d}\mu := \sum_{m=1}^{l} a_m \mu(f^{-1}(a_m)).$$

With this, we define the Lebesgue integral operator as follows.

Definition 2.4.2. For a measure space $(X,\mathcal{S},\mu)$ and measurable non-negative function f, we define $\int_X f \,\mathrm{d}\mu := \sup\{\int_X g \,\mathrm{d}\mu \mid\ g \text{ is simple}, 0 \leq g \leq f\}$. We then extend this to any measurable function f as, $\int_X f \,\mathrm{d}\mu := \int_X f^+ \,\mathrm{d}\mu + \int_X f^- \,\mathrm{d}\mu$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

Once the Lebesgue integral operator is defined we can redefine the L^p-norm of any $\mathcal{S}$-measurable function $f : X \to \mathbb{F}$ as,

$$\|f\|_p := \left(\int |f(x)|^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}}.$$

Note that, we need $X = [a,b]$ as we are considering the completion of $\mathcal{C}[a,b]$. With this, once again for any $1 \leq p < \infty$, we can define the Lebesgue space $L^p[a,b]$ to be the set of $\mathcal{S}$-measurable functions (which is a super-set of the set of continuous functions from X to $\mathbb{F}$, [2]) $f : X \to \mathbb{F}$ such that $\|f\|_p < \infty$, [2]. Once one shows that the Lebesgue space $L^p[a,b]$ is indeed a Banach space and a completion of $\mathcal{C}[a,b]$ (see Theorem 7.24 from [2]), it is immediate from the unique characterization of completion of $\mathcal{C}[a,b]$ that indeed $\overline{X}$ and $L^p[a,b]$ are the same (up to isometric isomorphism). That is,

$$L^p[a,b] \cong \mathrm{C}[a,b]/\mathrm{C}_*[a,b].$$

Similarly, the unique extension of Riemann integration and integrability in the continuous case on $[a,b]$ is identifiable with the more general Lebesgue integrability and integration. Thus we have shown that $L^p[a,b]$ is the unique completion of $\mathcal{C}[a,b]$ and that this uniqueness leads to a unique characterization of (Lebesgue) integrability and integration.

Chapter 3

Integration via Category Theory

In Chapter 2 we saw how the completion $L^p(a, b)$ of the normed linear space $\mathcal{C}[a, b]$ is uniquely characterized by the completion theorem (Theorem 2.2.1) and how this unique characterization, starting from (Riemann) integrability and integration for continuous functio- -ns from $[a, b]$ to $\mathbb{F}$, leads to a unique characterization of the more general (Lebesgue) integrability and integration. We also briefly discussed how one could do the same using Measure Theory. Notably, both these methods are grounded in Analysis. In the current chapter, however, we aim to uniquely characterize the Lebesgue space $L^p(a, b)$ and the (Lebesgue) integration operator using Category Theory- which naturally bridges various fields of mathematics in a structured way using 'universal properties'. Roughly speaking, universal properties are broad general mathematical statements that can be used to describe the similarities and connections between two (often times, seemingly distant) mathematical theories.

Before giving the motivation for using a Category Theoretic approach to characterize Lebesgue integration and integrability, here are some required basic notions from category theory.

3.1 Basic Notions

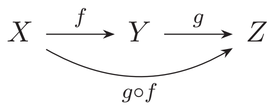
In Category Theory, we usually work with 'objects' and 'morphisms' between these objects. Objects are mathematical structures like sets, groups and topological spaces, while morphisms are roughly structure preserving maps between them. For example, the category of topological spaces has topological spaces as its objects and continuous maps as morphisms [7, 9]. Following is a formal definition of a category [5, 7]:

Definition 3.1.1. A category C consists of collections Ob(C) and Mor(C), where,

$$\text{Ob(C)} := \{X : X \text{ is an object of } C\}$$
$$\text{Mor(C)} := \{f : f \text{ is a morphism of } C\}$$

such that:

1. For each morphism f, there exist objects X, Y such that X is a source (or domain) of f and Y is the target (or range) of f.
2. For each object X there exists a morphism $\mathbf{id}_X : X \to X$, called the identity morphism.
3. For each pair of morphisms $f : X \to Y$ and $g : Y \to Z$ there exists a morphism $g \circ f : X \to Z$, called the composite morphism, given by

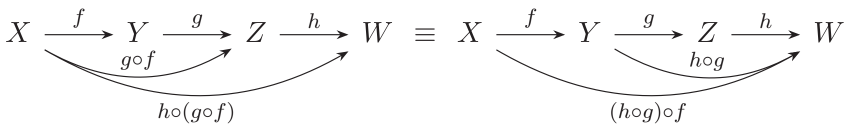


and satisfying the following axioms:

A.1 (Unitality): $\forall f : X \to Y$, the compositions $f \circ \mathbf{id}_X = \mathbf{id}_Y \circ f = f$.

A.2 (Associativity): $\forall f : X \to Y,\ g : Y \to Z,\ h : Z \to W$, we have,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

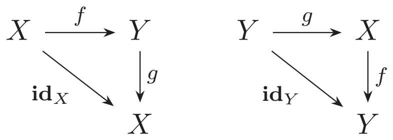


Note that as long as the above definition is satisfied, we have a category. This means there are categories whose objects are not sets and/or whose morphisms are nothing like maps or functions [5]. For example the category of a directed graph has vertices as objects and edges or arrows between adjacent vertices as morphisms [7]. Next, we would like to define the notion of inverse morphisms.

Definition 3.1.2. For objects X and Y in a category C, an isomorphism is a pair of morphisms $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f = \mathbf{id}_X \text{ and } f \circ g = \mathbf{id}_Y.$$

That is, if the following two diagrams commute

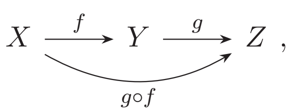


If such an isomorphism exists between X and Y, then we say that X and Y are isomorphic. We also call f the inverse of g and vice-versa.

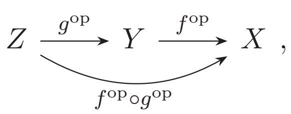
For example, the isomorphism of the category of topological spaces are homeomorphisms, which are precisely pairs of bijective continuous maps whose inverse is one another. In the category of vector spaces, the isomorphism are pairs of bijective linear maps whose inverse is one another.

Definition 3.1.3. A category is said to be small if the collection of all its morphisms is a set. The term small here refers to the fact that some 'collections' or 'categories' such as those containing all sets are too 'large' to be a set.

Definition 3.1.4. Let $\mathbf{C}$ be a category, then the **Opposite Category** $\mathbf{C}^{\text{op}}$ is the category whose objects are objects of $\mathbf{C}$ and each morphism $f : X \to Y$ of $\mathbf{C}$ gives a morphism $f^{\text{op}} : Y \to X$ of $\mathbf{C}^{\text{op}}$. For each object X, identity morphism and the opposite identity morphism coincide; and if f, g and $g \circ f$ are such that,



then, f^{op}, g^{op} and $f^{\text{op}} \circ g^{\text{op}}$ are such that



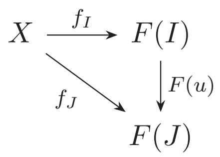
with $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$.

Definition 3.1.5. Let **C** and **D** be categories. $F : \mathbf{C} \to \mathbf{D}$ is said to be a **functor** if for each object X in **C**, there is an object $F(X)$ in **D** and for each morphism f in **C**, there is a morphism $F(f)$ in **D**, such that the following axioms hold:

(a) For every object X of **C**, $F(\mathbf{id}_X) = \mathbf{id}_{F(X)}$ (i.e. the identity of X in **C** is mapped by the functor to the identity of $F(X)$ in **D**).

(b) For all $f : X \to Y$ and $g : Y \to Z$ in **C**, we have $F(g \circ f) = F(f) \circ F(g)$ (i.e. the composition of f, g in **C** is mapped by the functor to the composition of $F(f)$ and $F(g)$ in **D**).

Definition 3.1.6. Let C be a category, S be a small category and $F : S \to C$ be a diagram in C, then we define a cone on F to be an object $X \in C$ and an indexed family of morphisms $f_I : X \to F(I),\ I \in S$ on C, such that for all morphisms $u : I \to J$ of S, the following triangle commutes:

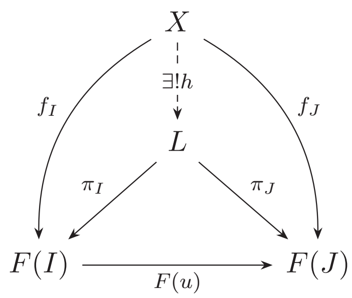


We represent this cone on F as $\left(X \xrightarrow{f_I} F(I)\right)_{I \in S}$.

Definition 3.1.7. For the same diagram $F : S \to C$ as in the above definition, we define the **limit** of F to be a cone $(L \xrightarrow{\pi_I} F(I))_{I \in S}$ such that for any cone on F there exists a unique map

$$h : X \to L,\ \text{s.t. } \pi_I \circ h = f_I,\ \forall I \in I.$$

The diagram for limit of F is as follows:



Definition 3.1.8. Let C be a category and S be a small category with the functor $F : S \to C$ as a diagram in C. Further, let the corresponding opposite functor be $F^{\text{op}} : S^{\text{op}} \to C^{\text{op}}$. Then we define a co-limit of F as a limit of F^{op}.

Definition 3.1.9. Let C be a category and let S be a small category. Then we call a functor $F : S \to C$ a diagram in C of shape S.

Definition 3.1.10. An object Y in a category C is said to be an initial object if for all objects X in C there exists a unique morphism from Y to X.

For a trivial example of an initial object consider the category **Set** of sets as objects and functions as morphisms. Then, the empty set is an initial object of **Set** because for any set X in Obj(**Set**), there is exactly one function $f : \emptyset \to X$ called the empty function whose graph is empty [5]. Another example of initial objects is the ring $\mathbb{Z}$ of integers in the category of rings with objects as rings and morphisms as ring homomorphisms [5]. This is because for any ring R with identity 1_R, we can define $\varphi_R : \mathbb{Z} \to R$, $\varphi_R(n) := n.1_R$, where

$n.1_R = 1_R + \ldots + 1_R$ (n times) if $n \geq 0$, and $n.1_R = -1_R - \ldots - 1_R$ ($|n|$ times) otherwise. Note, φ_R is a well-defined map for each ring R and one can easily show that it is a morphism of the category of rings by checking that it is a ring homomorphism. Moreover, for each R, φ_R is the only morphism from $\mathbb{Z}$ to R in the category of rings. This is because if η is another ring homomorphism from $\mathbb{Z}$ to R, then for any $n \geq 0$ we have,

$$\eta(n) = \eta(1 + \ldots + 1) = \eta(1) + \ldots + \eta(1) = 1_R + \ldots + 1_R = n.1_R = \varphi_R(n).$$

Similarly, if $n < 0$ then,

$$\eta(n) = \eta(-1 - \ldots - 1) = \eta(-1) + \ldots + \eta(-1) = -1_R - \ldots - 1_R = n.1_R = \varphi_R(n).$$

We now show that if a category has an initial object then it is unique up to isomorphism.

Lemma 3.1.11. Let $\mathbf{C}$ be a category. If I_1 and I_2 are two initial objects of $\mathbf{C}$, then $I_1 \cong I_2$.

Proof. We want to show that there exists an isomorphism between I_1 and I_2. Note that by the initiality of I_1, there is the unique morphism between it and I_2, say f_1. Similarly, there is the unique morphism, say f_2 from I_2 to I_1. Note that the map,

$$f_2 \circ f_1 : I_1 \to I_2 \to I_1,$$

is a morphism from I_1 to I_1. Once again by the initiality of I_1, $f_2 \circ f_1$ is the unique morphism between I_1 and itself. But, the identity map id_{I_1} on I_1 is also a map from I_1 to itself. So, by the uniqueness of the morphism from I_1 to itself, we have that $f_2 \circ f_1 = id_{I_1}$. Similarly, we have, $f_1 \circ f_2 = id_{I_2}$, the identity map from I_2 to itself. So, the pair of morphisms f_1, f_2 is an isomorphism between the two initial objects, by Definition 3.1.2. In other words, I_1 and I_2 are isomorphic and therefore initial objects are unique up to isomorphism. $\square$

Remark 3.1.12. We will use this later to prove the main theorems of this thesis, namely Theorems 3.3.5, 4.2.1, that show that the spaces $L^p(0,1)$ and $L^p(X)$ (for any measure space X) are unique (up to isomorphism) initial objects of certain categories (roughly speaking the category **Ban** of Banach spaces with some extra structure). This is because, since an initial object is unique (up to isomorphism) in a category, it is the only object (up to isomorphism) with any properties it has. So any theorem showing that some object is initial in a particular category, characterizes this object uniquely up to isomorphism. Moreover, at least in principle, all the properties of this initial object can be derived solely from this unique characterization [6].

Note how the statement and proof of Lemma 3.1.11 above is similar to the statement and proof of Corollary 2.2.3, which we used to show that the completion $L^p(a,b)$ of $C[a,b]$ is uniquely characterized up to isometric isomorphism. Further, mirroring what we did in Chapter 2 using completion (Theorem 2.2.1), we will show in this chapter that the Lebesgue space $L^p[0,1]$ (as defined in Section 2.4, for $X = [0,1]$) is an initial object of the category **Ban** of Banach spaces with some extra structure (Theorem 3.3.5). We also show that this initiality leads to unique characterization of $L^p[0,1]$ in general. Finally, we show that the unique characterization of $L^1(a,b)$ leads to the unique characterization of the integration operator $\int_0^1$ on $[0,1]$. In the following chapter, Chapter 4, we do the same but for any measure space X. But, before that, let us discuss the motivation and advantages for taking the category-theoretic approach to this problem.

3.2 Motivation for the Category Theoretic Approach

Firstly, new methods of showing statements in one field of mathematics, especially when they include methods from an entirely different field of mathematics, gives valuable new perspective into both fields and helps us understand mathematical structures in a broader setting. In this vein, having a category theoretic approach to integration and integrability is beneficial. Following are the specific reasons that help justify this claim:

Characterizing objects using category theory has the advantage of establishing uniqueness of this object at two levels. This is because it not only shows that an initial object in a specific category is unique up to isomorphisms of the category, but also that any morphism from this object to any other object is unique (see Definition 3.1.10). So, this way once the unique initial object of a category is found, we can know which other objects of this category are isomorphic to the initial object and hence share its properties, and which aren't. Hence in any category, once we find an initial object and any of the properties it has, then it automatically is uniquely characterized to have these properties. While characterization is possible using traditional methods as seen in Chapter 2, because of the above reasons a category theoretic approach gives characterization immediately once initiality is established (which can be done a whole slew of techniques and theorems from category theory) [6, 5, 7].

Secondly, following [6], we note that initiality of objects (as defined in Definition 3.1.10) is an universal property and every universal property can be expressed in terms of initial objects. This allows us to discuss our category in relation to other categories in the category of categories using functors (see Definition 3.1.5). So, the properties the initial object has in our category will be shared (albeit in a slightly different but equivalent formulation) by the initial object of every other category we consider in our category of categories. So this allows us to uniquely characterize objects with properties of interest across a vast array of mathematical fields, no matter how distant they may seem to us in a more traditional viewpoint [5, 7].

Further, the traditional methods of completion and measure theory used to characterize Lebesgue integration and integrability depend on the concepts of integration for continuous functions, and/or step functions. On the other hand, the category theoretic approach does not depend of any of these concepts [6]. As we shall see in Section 3.3 and Chapter 4, the unique characterization of the Lebesgue space and the resulting concepts of Lebesgue integration and integrability follow immediately from the notion of Banach spaces alone. That is, all we need is to find the initial object in the category **Ban** of Banach spaces.

So, in this sense it is a completely different way to not only formalize integration but also serves as a model to extend integration as we know it to an even more general case. Historically, as discussed in the Introduction 1, integration has its roots in geometry (as area under a curve) and analysis. But, the category theoretic approach does not depend on these notions and is an algebraic theory of integration. This opens doors for the discovery of new theories of integration [6]. For instance, if one wants to integrate in some arbitrary space of interest where traditional methods of integration makes no sense, one could try to follow the methods used in Theorems 3.3.5, 4.2.1, 3.4.3 and Proposition 3.4.2 to find a way to integrate functions in this space.

3.3 Integration on [0,1]

Working on the special case of $X = [0, 1]$ gives us a way to better understand and generalize characterization of Lebesgue integrability and integration over arbitrary measure spaces (see Definition 2.4.1) X. So, in this section we will work on characterizing $L^p[0, 1]$ using category theory. First, consider the following definition:

Definition 3.3.1. Let $(V_1, \|.\|_1)$ and $(V_2, \|.\|_2)$ be two normed linear spaces. A map $f : V_1 \to V_2$ is said to be a linear contraction if f is linear and for any u, $v \in V_1$, there exists a constant $0 \le L \le 1$, such that,

$$\|f(u) - f(v)\|_2 \le L\|u - v\|_1.$$

Let us consider the category **Ban** with Banach spaces as objects and linear contractions as morphisms. That is,

$$\text{Ob}(\textbf{Ban}) := \{X : \ X \text{ is a Banach space } \},$$

$$\text{Mor}(\textbf{Ban}) := \{f : X \to Y \ : \ f \text{ is a linear contraction, and } X, Y \text{ are Banach spaces}\}.$$

Lemma 3.3.2. **Ban** is a category.

Proof. For each Banach space V with norm $\|.\|_V$, the identity morphism is the identity map from V to V, which is a linear contraction since for all $u, v \in V$,

$$\|\textbf{id}_V(u) - \textbf{id}_V(v)\|_V = \|u - v\|_V.$$

Let V_i be a Banach space with norm $\|.\|_i$ for $i \in \{1, 2, 3\}$. Let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be two linear contractions. Then, the composite morphism $g \circ f : V_1 \to V_3$ is also a linear contraction since

$$\|g(f(u)) - g(f(v))\|_3 \le L\|f(u) - f(v)\|_2 \le LL'\|u - v\|_1,$$

where $0 \le L$, $L' \le 1$ are constants, so $0 \le LL' \le 1$ is a constant as well. Further, we have unitality since the identity morphism is the identity map for all Banach spaces. Lastly, we have associativity since linear contractions are functions and composition of functions is always associative [11]. So, we are done. ☐

The isomorphism of **Ban** are pairs of invertible linear contractions whose inverse is one another. Note that for any such isomorphism pair $f : V_1 \to V_2$ and $g : V_2 \to V_1$, f and g are linear contractions and inverses of each other. So, for any $u, v \in V_1$ there exists a constant $0 \le L \le 1$ and for any $w, z \in V_2$ there exists a constant $0 \le L' \le 1$, such that,

$$\|u - v\|_1 = \|g(f(u)) - g(f(v))\|_1 \le L'\|f(u) - f(v)\|_2 \le LL'\|u - v\|_1,$$

So, we have $LL' \ge 1$. Further, since $0 \le L, L' \le 1$, we have that $L = L' = 1$. Now, since by definition every vector space has a zero vector (say, denoted by 0) and the image of 0 under any linear map is 0, we have that,

$$\|f(u)\|_2 = \|f(u) - f(0)\|_2 \le \|u\|_1 = \|g(f(u))\|_1 \le \|f(u)\|_2.$$

So, for any $u \in V_1$, $\|f(u)\|_2 = \|u\|_1$, i.e. by definition 2.1.3, we have that the linear invertible contraction is an isometry. Similarly, g, the inverse of f is also a linear invertible isometric contraction. So, the isomorphism of the category **Ban** are precisely isometric isomorphisms, a fact that will be used in proving Lemmas 3.3.3, 3.3.2.

Now, our aim is to show that $L^p[0, 1]$ (equipped with some extra categorical structure) is an initial object of the category **Ban** (also equipped with some extra categorical structure which we shall call A_p, see below). Using this we shall define the integration operator as the unique morphism from $L^1[0,1]$ to $\mathbb{F}$, thus showing that a unique characterization (via initiality) of $L^p[0,1]$ leads to a unique characterization of integration operator on $L^1[0,1]$. Define $\mathbb{N}_0$ as the set that contains all the natural numbers along with the number 0. Let $1 \leq p < \infty$ and consider the closed interval $[0,1]$. We know from our discussion in Chapter 2 that $L^p[0,1]$ is a Banach space. We define the direct sum of two Banach spaces $(V_i, \|.\|_i)$ for $i \in \{1,2\}$, as the linear space [6]

$$V_1 \oplus V_2 = \{(v_1, v_2) : \; v_i \in V_i\},$$

with the norm

$$\|(v_1, v_2)\|_p = \left(\frac{\|v_1\|_1^p + \|v_2\|_2^p}{2} \right)^{\frac{1}{p}}.$$

The factor 2 in the above norm is chosen for convenience and will be used later in the characterization of the integration operator, see Proposition 3.4.2. Further, consider the function Γ in $L^p[0,1]$ such that

$$\Gamma(x) = 1, \quad \forall x \in [0,1],$$

and the map γ, called the juxtaposition map, such that,

$$\gamma : L^p[0,1] \oplus L^p[0,1] \to L^p[0,1],$$

$$(\gamma(f,g))(x) := \begin{cases} f(2x) & \text{if } x \leq \frac{1}{2}, \\ g(2x-1) & \text{if } x > \frac{1}{2}, \end{cases}$$

for $f, g \in L^p[0,1]$. Note that Γ is indeed the indicator function of $[0,1]$ used in building Lebesgue integration in Measure theory. In Measure Theory, for a measurable set X with measure μ_X, the indicator Γ_X takes value 1 on X and 0 elsewhere. To make its integral on X consistent with the measure μ_X (which, roughly, indicates the size of the set X), $\int_X \Gamma_X d\mu_X = \int_X d\mu_X$ is set to $\mu_X(X)$. Similarly, we set $\|\Gamma\|_p = 1 - 0 = 1$ as Γ is the indicator function on $[0,1]$ which has length $1 - 0 = 1$. With this assumption, we will later show (see Proposition 3.4.2 and the discussion that follows it) that $\int_0^1$, i.e. the integration operator on $L^1[0,1]$, is the only bounded linear operator that maps Γ to 1. Furthermore, let A_p be a category [6] whose objects are triples (V, v, δ), where v is an element of some Banach space V with norm less than or equal to 1 and $\delta : V \oplus V \to V$ such that $\delta(v, v) := v$. Note that we do not require δ to be bi-linear. Let the morphisms of A_p be maps $\eta : (V_1, v_1, \delta_1) \to (V_2, v_2, \delta_2)$ such that $\eta : V_1 \to V_2$ is a morphism of the category **Ban** with

$$\eta(v_1) = v_2, \tag{3.1}$$

$$\eta(\, \delta_1(v_1^1, \, v_1^2\,)\,) = \delta_2(\, \eta\,(v_1^1\,), \, \eta\,(v_2^2\,)\,), \tag{3.2}$$

for any v_1^1, v_2^2 in V_1.

Lemma 3.3.3. A_p is indeed a category.

Proof. Any morphism in A_p is of the form $\eta : (V_1, v_1, \delta_1) \to (V_2, v_2, \delta_2)$ for some objects (V_i, v_i, δ_i) for $i \in \{1,2\}$ and for each object (V, v, δ) the identity morphism is the identity map $\mathbf{id}_V$ on V, since the identity map is a linear contraction(specifically an isometric isomorphism), $\mathbf{id}_V(v) = v$ by definition and for any v^1, v^2 in V, we have,

$$\mathbf{id}_V(\, \delta\,(v^1, v^2)\,) = \delta(v^1, v^2) = \delta(\, \mathbf{id}_V(v^1), \, \mathbf{id}_V(v^2)).$$

Further, for any two morphisms $\eta_1 : (V_1, v_1, \delta_1) \to (V_2, v_2, \delta_2)$ and $\eta_2 : (V_2, v_2, \delta_2) \to (V_3, v_3, \delta_3)$ of A_p, the composition $\eta_2 \circ \eta_1 : (V_1, v_1, \delta_1) \to (V_3, v_3, \delta_3)$ is the composition morphism since composition of linear contractions is a linear contraction[1], and for all v_1^1, v_1^2 in V_1 we have,

$$\eta_2 \circ \eta_1\,(v_1) = \eta_2(\eta_1(v_1)) = \eta_2(v_2) = v_3,$$
$$\eta_2 \circ \eta_1(\delta_1(v_1^1, v_1^2)) = \eta_2(\delta_2(\eta_1(v_1^1), \eta_1(v_1^2))) = \delta_3(\eta_2 \circ \eta_1(v_1^1), \eta_2 \circ \eta_1(v_1^2)).$$

Unitality holds, since $\eta \circ \mathbf{id}_V(u) = \eta(u) = id_V \circ \eta(u)$ for all $u \in V$. Finally, associativity holds since the morphisms of A_p are linear contractions (specifically they are functions) and therefore their composition is associative [11]. So, indeed A_p is a category. □

Lemma 3.3.4. $(L^p[0,1],\, \Gamma,\, \gamma)$ is an object of A_p.

Proof. As discussed above $L^p[0,1]$ is a Banach space. Further, $\|\Gamma\|_p = 1$, and,

$$\begin{aligned}(\gamma(\,\Gamma,\,\Gamma\,))(x) &= \begin{cases}\Gamma(2x) = 1 & \text{if } x \le \frac{1}{2},\\ \Gamma(2x-1) = 1 & \text{if } x > \frac{1}{2},\end{cases}\\ &= 1 = \Gamma(x).\end{aligned}$$

So, $\gamma(\,\Gamma,\,\Gamma) = \Gamma$ and therefore, by definition, $(L^p[0,1],\, \Gamma,\, \gamma)$ is an object of A_p. □

Theorem 3.3.5. $(L^p[0,1],\, \Gamma,\, \gamma)$ is the initial object of A_p.

Proof. From Lemma 3.3.4, we have that $(L^p[0,1],\, \Gamma,\, \gamma)$ is an object of the category A_p. So, to show that $(L^p[0,1],\, \Gamma,\, \gamma)$ is an initial object in A_p it suffices to show that, for any arbitrary object (V, v, δ) of A_p, there exists a unique morphism

$$\tau :\, (L^p[0,1],\, \Gamma,\, \gamma) \to (V, v, \delta).$$

But before we do so, note that the Banach space $L^p[0,1]$ is the completion of the normed linear space $\mathbb{C}[0,1]$, the space of continuous functions from $[0,1]$ to $\mathbb{F}$. So, we can take a sequence of linear subspaces E_i, for each $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, of $L^p[0,1]$ with each E_i containing the equivalence classes of step functions that are constant on the open intervals

$$\left(\frac{j-1}{2^i}, \frac{j}{2^i}\right), \text{ for each } 1 \le j \le 2^i.$$

Let us define a set

$$E := \bigcup_{i \in \mathbb{N}_0} E_i. \tag{3.3}$$

Clearly, E is the space of step functions with the only discontinuities at rational numbers with denominators equal to a power of two. Further, since [6] we take $p < \infty$, E is dense in the set of all step functions from $[0,1]$ to $\mathbb{F}$. Also, the set of all step functions is dense [10] in $L^p[0,1]$. So, by the property that if A is dense in B and B is dense in C, then A is dense in the category C [9], we have that E is dense in $L^p[0,1]$. Then, from [3] we have that in the category **Ban**, $L^p[0,1]$ is the co-limit of the diagram

$$E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots\,,$$

where $\hookrightarrow$ denotes inclusion[2]. Further, the restriction of the juxtaposition map γ to $E_i \oplus E_i$ gives an isomorphism [6],

$$E_i \oplus E_i \to E_{i+1}, \text{ for all } i \in \mathbb{N}_0.$$

[1] See the proof of Lemma 3.3.2.

[2] Note, the inclusion arrows used here are valid by construction of the E_i's.

If such a morphism τ exists, then it is unique: We proceed to prove that τ is unique by arguing that τ is uniquely determined on E. This in turn implies that τ is uniquely determined on $L^p[0,1]$ since E is a dense subspace of $L^p[0,1]$ as shown above and τ is bounded [6]. To prove that τ is determined uniquely on E, it suffices (by definition of E, see Equation 3.3) to show that τ is determined uniquely on each E_i for all i in $\mathbb{N}_0$. We do this by induction, as follows:

Basis step: Note that, if it exists, the morphism

$$\tau : (L^p[0,1], \Gamma, \gamma) \to (V, v, \delta), \tag{3.4}$$

is a morphism in A_p. So, from Equation 3.1, we have that $\tau(\Gamma) = v$. By linearity, this implies that τ is determined uniquely on E_0 [6].

Inductive step: Let us assume that τ is uniquely determined on E_i, for some arbitrary i in $\mathbb{N}_0$. We want to show that $\tau|_{E_{i+1}} : E_{i+1} \to V$ is uniquely determined. Since, τ is a morphism of A_p, its restriction to E_i is such that,

$$\tau|_{E_i} \oplus \tau|_{E_i} : E_i \oplus E_i \to V \oplus V.$$

From previous discussion, we know that,

$$\gamma : E_i \oplus E_i \to E_{i+1},$$

is an isomorphism and is therefore invertible i.e. γ^{-1} exists. Further, by definition of objects in A_p, $\delta : V \oplus V \to V$. Now, consider the following square:

$$\begin{array}{ccc} E_i \oplus E_i & \xrightarrow{\gamma} & E_{i+1} \\ {\scriptstyle \tau|_{E_i} \oplus \tau|_{E_i}} \downarrow & & \downarrow {\scriptstyle \tau_{E_{i+1}}} \\ V \oplus V & \xrightarrow{\delta} & V \end{array}$$

As τ is a morphism of A_p, by Equations 3.2 and 3.4, the square above commutes and

$$\tau|_{E_{i+1}} \circ \gamma = \delta \circ (\tau|_{E_i} \oplus \tau|_{E_i}).$$

So, since γ^{-1} exists as discussed above, we have,

$$\tau|_{E_{i+1}} = \delta \circ (\tau|_{E_i} \oplus \tau|_{E_i}) \circ \gamma^{-1}.$$

Since, δ, $\tau|_{E_i}$ and γ^{-1} are uniquely determined, so is their composition. In other words, τ is determined uniquely on E_{i+1}.

Therefore, by the principle of Mathematical Induction, τ is determined uniquely on each E_i for all i in $\mathbb{N}_0$. With this, we have shown that τ is determined uniquely on $L^p[0,1]$, by the discussion given before the induction proof. Equivalently, τ, if it exists, is a unique map from the object $(L^p[0,1], \Gamma, \gamma)$ of A_p to the arbitrary object (V, v, δ) of A_p. So, if we show that such a τ exists, then we will have that $(L^p[0,1], \Gamma, \gamma)$ is an initial object of A_p and we would be done.

Existence of the morphism τ: Consider the morphisms $\tau_i : E_i \to V$ of the category **Ban**, for each i in $\mathbb{N}_0$, defined as:

$$\tau_0 : E_0 \to V, \quad \tau_0(\Gamma) = v,$$

$$\tau_{i+1} = \left(E_{i+1} \xrightarrow{\gamma^{-1}} E_i \oplus E_i \xrightarrow{\tau_i \oplus \tau_i} V \oplus V \xrightarrow{\delta} V \right), \tag{3.5}$$

defined inductively on i. Now, as discussed above, $L^p[0,1]$ is the co-limit of $E_0 \hookrightarrow E_1 \hookrightarrow \cdots$, and therefore, by the density of E in $L^p[0,1]$ and the definitions of limit(3.1.7) and co-limit(3.1.8), there exists a unique map

$$\tau : \; L^p[0,1] \to V \quad \text{with} \quad \tau|_{E_i} = \tau_i, \text{ for each } i \in \mathbb{N}_0,$$

where $\tau|_{E_i}$ is the restriction of τ to E_i as used in above in the uniqueness proof. Now, we want to show that the above map is a morphism of the category **Ban**. Note that, by definition, τ_i is a morphism of the category **Ban**. So, the restriction of τ to E_i is a morphism of **Ban**, for each i in $\mathbb{N}_0$. Moreover, by the definition of E

$$\tau : \; E \to V,$$

is a morphism of the category **Ban**. Finally, since E is dense in $L^p[0,1]$,

$$\tau : L^p[0,1] \to V,$$

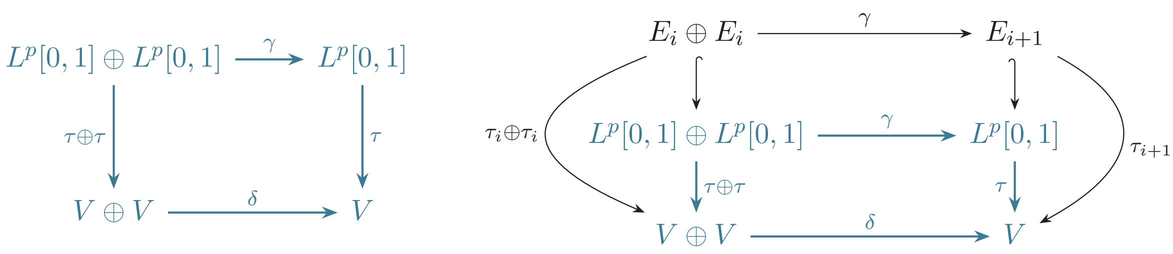
is a morphism of the category **Ban**. Now, note that,

$$\tau(\Gamma) = \tau_0(\Gamma) = \tau|_{E_0}(\Gamma) = v.$$

So, if we show that

$$\tau(\gamma(f,g)) = \delta(\tau(f), \tau(g)), \text{ for all } f, g \text{ in } L^p[0,1],$$

then we can conclude that τ, as defined in Equation 3.4, is indeed a morphism of the category A_p, by definition. This means, we want to show that $\tau \circ \gamma = \delta \circ (\tau \oplus \tau)$, or equivalently the teal square on the left side of the following figure commutes.



In order to do so, let us consider a second diagram (on the right side of the previous image) that contains the teal square. Let us call this diagram 'the inclusion diagram'. The upper square commutes since $\gamma \circ (\iota \oplus \iota)(f,g)(x) = \iota \circ \gamma\,(f,g)(x)$ for all f, g in E_i and x in $[0,1]$. The outer square commutes as well since $\delta \circ (\tau_i \oplus \tau_i) = \tau_{i+1} \circ \gamma$ by the inductive definition of τ_i in Equation 3.5. The left triangle commutes since $\tau_i \circ \tau_i = (\tau \oplus \tau) \circ \iota$, since τ_i is the restriction of τ to E_i, and the inclusion map ι restricts τ to E_i. Similarly, the right triangle commutes since $\tau_{i+1} = \tau \circ \iota$. Together, the lower square (indicated in teal) commutes on $E_i \oplus E_i$ for all i in $\mathbb{N}_0$ [6].

Now note that for each i, by construction, E_i is a subspace of $L^p[0,1]$. By definition of direct sum, $E_i \oplus E_i$ is a linear subspace of $L^p[0,1] \oplus L^p[0,1]$. Further, as discussed earlier, E is dense in $L^p[0,1]$, so $E \oplus E$ is dense in $L^p[0,1] \oplus L^p[0,1]$. Finally, by construction, we have

$$\bigcup_{i \in \mathbb{N}_0} E_i \oplus E_i = E \oplus E,$$

and therefore the lower square (indicated in teal) commutes. Hence, τ, as defined in Equation 3.4, is indeed a morphism of the category A_p. This completes the proof. $\square$

Remark 3.3.6. Theorem 3.3.5 shows that $L^p[0,1]$ with some additional structure is the initial object of A_p (which is **Ban** with some extra structure by construction). Note that, as we have shown in Lemma 3.1.11, $(L^p[0,1], \Gamma, \gamma)$ is 'the' unique initial object of A_p up to isomorphisms of A_p. Hence, up to isomorphisms $L^p[0,1]$ is the only object that has the properties that it has, i.e. $L^p[0,1]$ is uniquely characterized. It follows, using information from Measure Theory, that $L^p[0,1]$ is the only Banach space that contains all Lebesgue integrable functions and is the completion of the space of continuous functions $\mathcal{C}[0,1]$.

3.4 The Integration Operator $\int_0^1$

Lastly, we use the unique characterization of $L^p[0,1]$ to define the integration operator as the unique morphism from the initial object $L^1[0,1]$ to $\mathbb{F}$. Let us take the mean $m(x,y)$ of $x, y \in [0,1]$, given by $m(x,y) = \frac{x+y}{2}$.

Lemma 3.4.1. $(\mathbb{F}, 1, m)$ is an object of A_1.

Proof. The vector space $\mathbb{F}$, with the Euclidean norm, is a Banach space. 1 is an element of $\mathbb{F}$ with $\|1\| = 1$ and $m : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ satisfies $m(z,z) = z$ for an z in $\mathbb{F}$, since we have $m(z,z) = \frac{z+z}{2} = z$. □

Proposition 3.4.2. The integration operator $\int_0^1$ is the unique morphism from $(L^1[0,1], \Gamma, \gamma)$ to $(\mathbb{F}, 1, m)$ in the category A_1.

Proof. From Theorem 3.3.5, since $(L^1[0,1], \Gamma, \gamma)$ is an initial object of A_1. From Lemma 3.4.1, we have that $(\mathbb{F}, 1, m)$ is an object of A_1. So, by Definition 3.1.10, we have that there exists a unique morphism

$$\phi : (L^1[0,1], \Gamma, \gamma) \to (\mathbb{F}, 1, m).$$

So, if we show that the integration operator is in fact a morphism from $(L^1[0,1], \Gamma, \gamma)$ to $(\mathbb{F}, 1, m)$, then we would be done. Note that by the linearity of integration, we have that the integration operator $\int_0^1 : L^1[0,1] \to \mathbb{F}$ is linear. Further, it is a contraction since for any two functions $f, g \in L^1[0,1]$, we have,

$$\left| \int_0^1 f(t)\,\mathrm{d}t - \int_0^1 g(t)\,\mathrm{d}t \right| = \left| \int_0^1 f(t) - g(t)\,\mathrm{d}t \right| \le \int_0^1 |f(t) - g(t)|\,\mathrm{d}t = \|f - g\|_1,$$

where $|.|$ is the standard Euclidean norm on $\mathbb{F}$, and $\|.\|_1$ is the L^p norm for $p = 1$ as defined in Equation 2.1 and extended to the Banach space $L^p[0,1]$ via completion of $\mathbb{C}[a,b]$. So, $\int_0^1 : L^1[0,1] \to \mathbb{R}$ is a morphism of the category **Ban**. Further, since $\Gamma(t) = 1$ for all $t \in [0,1]$,

$$\int_0^1 \Gamma(t)\,\mathrm{d}t = \int_0^1 1\,\mathrm{d}t = 1.$$

Finally, for any f, g in $L^1[0,1]$, we have,

$$\int_0^1 \gamma(f,g)(t)\,\mathrm{d}t = \int_0^{\frac{1}{2}} f(2t)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 g(2t-1)\,\mathrm{d}t = \frac{1}{2}\int_0^1 f(x)\,\mathrm{d}x + \frac{1}{2}\int_0^1 g(y)\,\mathrm{d}y,$$
$$= m\left(\int_0^1 f(x)\,\mathrm{d}x, \int_0^1 g(y)\,\mathrm{d}y\right).$$

Hence, for the objects $(L^1[0,1], \Gamma, \gamma)$ and $(\mathbb{F}, 1, m)$ in A_1, we have for any f, g in $L^1[0,1]$,

$$\int_0^1 \gamma(f,g) = m\left(\int_0^1 f, \int_0^1 g\right).$$

Therefore, by definition of a morphism of the category A_1, the integration operator $\int_0^1$ is the unique morphism from $(L^1[0,1], \Gamma, \gamma)$ to $(\mathbb{F}, 1, m)$ in A_1. □

So, this shows that the integration operator $\int_0^1$, abstractly, is the unique morphism from $L^1[0,1]$ to $\mathbb{F}$ that maps Γ to 1 and γ to m. Since, the morphisms of the category A_p are indeed those of **Ban** (by construction) with some additional structure, $\int_0^1$ is a linear contraction and hence is a bounded linear operator. That is, we have shown that the integration operator $\int_0^1$ is the unique bounded linear functional on $L^1[0,1]$ such that $\int_0^1 1 = 1$. While this gives an abstract characterization of the integration operator, the following Theorem gives a concrete definition of the same. That is, we will concretely give the definite integration operator $\int_0^x$ for any $x \in [0,1]$.

Let $Z[0,1]$ be the space of continuous functions from $[0,1]$ to $\mathbb{F}$ that map 0 to 0. Then, $Z[0,1]$ provided with the sup-norm is a Banach space [6]. Suppose we like to integrate some function $F \in L^1[0,1]$ from 0 to some $x \in [0,1]$. Then, define I_x to be a continuous function on $[0,1]$ that preserves x, i.e. $I_x(x) = x$ and $I_x \in Z[0,1]$. Consider, $\psi : Z[0,1] \oplus Z[0,1] \to Z[0,1]$, given by,

$$(\psi(f,g))(x) := \begin{cases} \frac{1}{2} f(2x) & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{2}\left(f(1) + g(2x-1)\right) & \text{if } x > \frac{1}{2}. \end{cases}$$

This is very similar to how γ is defined. One can show, similar to how we showed $(L^p[0,1], \Gamma, \gamma)$ is an object in A_p (see Lemma 3.3.4), that $(Z[0,1], I, \psi)$ is an object of A_1. With all this, we can finally give a concrete definition for the integration operator.

Theorem 3.4.3. Let $x \in [0,1]$. Then, the definite integration operator $\int_0^- : L^1[0,1] \to Z[0,1]$, such that $\int_0^-(f) := \int_0^x f$ is the unique morphism from $(L^1[0,1], \Gamma, \gamma)$ to $(Z[0,1], I_x, \psi)$ [6].

Proof. To show that $\int_0^-$ is the unique morphism from $(L^1[0,1], \Gamma, \gamma)$ to $(Z[0,1], I_x, \psi)$, we note that $(L^1[0,1], \Gamma, \gamma)$ is initial in the category A_1 by Theorem 3.3.5 and show that $\int_0^-$ is indeed a morphism from $(L^1[0,1], \Gamma, \gamma)$ to $(Z[0,1], I_x, \psi)$. Then the uniqueness of the morphism follows directly from the initiality of $(L^1[0,1], \Gamma, \gamma)$ in A_1. We check that $\int_0^-$ is a morphism of A_1 in the same way as we checked ϕ is a morphism of A_1 in the proof of Proposition 3.4.2. In short, this follows from the linearity of integration and the following: Let $x \in [0,1]$; for any $f, g \in L^1[0,1]$, we have,

$$\left| \int_0^x f \right| \leq \int_0^1 |f|, \qquad \int_0^x 1 = x, \text{ and,}$$
$$\int_0^x \gamma(f,g) = \left(\psi\Big(\int_0^- f, \int_0^- g\Big) \right)(x).$$

□

Chapter 4

Integration on Arbitrary Measure Spaces

Now that we have seen how to uniquely characterize $L^p[0,1]$ and use this to uniquely define an integration operator in Chapter 3, we will give an outline of how to do the same for $L^p(X)$, where X is any measure space (see Definition 2.4.1). In what follows, we will first characterize the L^p functor from the category **Meas** of measure spaces to the category **Ban** of Banach spaces. To that end let us consider the following basic notions.

4.1 Basic Notions

Firstly we would like to define the category **Meas** with finite measure spaces as objects and measure-preserving partial maps as morphisms. So, we give the necessary definitions.

Definition 4.1.1. A measure space $(X, \mathcal{S}_X, \mu_X)$ (see Definition 2.4.1) is said to be a **finite measure space** if $\mu_X(X) < \infty$.

From here on we write any measure space simply $(X, \mathcal{S}_X, \mu_X)$ simply as X. The measure on any measure space X will be simply labelled μ_X.

Definition 4.1.2. Let X, Y be measure spaces and let $\xi : X \to Y$ be a map that preserves the measure structure of X and Y. For any $A \in \mathcal{S}$ (i.e. a measurable subset of X), we define $\xi_*\mu_X$ as the map that takes A to $\mu_X(\xi^{-1}(A))$.

Definition 4.1.3. With the same setup as in the previous defintion, $\xi : X \to Y$ is said to be a **measure preserving map** if $\xi_*\mu_X = \mu_Y$ [7].

Definition 4.1.4. For a pair of measure spaces X, Y, an **embedding** $\eta : Y \to X$ is an injective map such that $\eta(B)$ is measurable subset of X iff B is a measurable subset of Y. Moreover, $\mu_X(\eta(B)) = \mu_Y(B)$.

Definition 4.1.5. A **measure preserving partial map** is $(A, \sigma) : X \to Y$, where $\sigma : A \to Y$ is a measure preserving map and A is a measurable subset of X where the inclusion $\iota : A \to X$ is an embedding [6, 2].

With this in mind, one defines **Meas** to be the category containing finite measure spaces as objects and measure preserving partial maps as morphisms [6, 7]. Having defined **Meas**, we now give an useful functor from the category **Meas** of finite measure spaces to the category **Ban** of Banach spaces following [6]. For the definitions of opposite morphisms and functors used in the following definition, see Definitions 3.1.4 and 3.1.5, respectively.

Definition 4.1.6. Let $1 \leq p < \infty$. The L^p **functor**,

$$L^p : \mathbf{Meas}^{\text{op}} \to \mathbf{Ban},$$

maps $X \in \mathbf{Meas}^{\text{op}}$ to $L^p(X)$, which is the usual Lebesgue space on the measure space X (see Section 2.4). Further, it maps any measure preserving partial map $(A, \sigma) : X \to Y$ to the induced map $L^p(Y) \to L^p(X)$ given by $g \mapsto (g \circ \sigma)^X$. Here $(g \circ \sigma)^X$ is the composite map $g \circ \sigma : A \to \mathbb{F}$ that is extended to all of X by $(g \circ \sigma)(x) = 0$, $\forall x \in X - A$.

Lemma 4.1.7. L^p, as defined above, is indeed a functor.

Proof. We check this by showing that L^p satisfies the defintion of functor given in Definition 3.1.5. Firstly, note that L^p maps any measure space X to the Lebesgue space $L^p(X)$ which is a Banach space and hence in **Ban**. Secondly, for each $(A, \sigma) : X \to Y$, the induced map $L^p(Y) \to L^p(X)$ is a morphism of the category **Ban** as it is a linear contraction. Further note that $\mathbf{id}_X = (X, I) : X \to X$ where $I : X \to X$ is the identity map on X. So, $L^p(\mathbf{id}_X) = L^p(X, I)$ is the induced map on $L^p(X)$ such that $g \mapsto (g \circ I)^X = g^X$. That is, the induced map on $L^p(X)$ is a function that maps $g \in L^p(X)$ to g, i.e. it is the identity morphism on $L^p(X)$. So, we have, $L^p(\mathbf{id}_X) = \mathbf{id}_{L^p(X)}$.

Finally, for any $(A_1, \sigma_1) : X \to Y$, $(A_2, \sigma_2) : Y \to Z$ in $\mathbf{Meas}^{\text{op}}$, we have $L^p((A_2, \sigma_2) \circ (A_1, \sigma_1))$ is the composition of the induced map $L^p(X) \to L^p(Y)$ with $g \mapsto (g \circ \sigma_1)^X$, and the induced map $L^p(Y) \to L^p(Z)$ with $h \mapsto (h \circ \sigma_2)^Y$. Hence, $L^p((A_2, \sigma_2) \circ (A_1, \sigma_1)) = L^p((A_1, \sigma_1)) \circ L^p((A_2, \sigma_2))$. Therefore, by definition, L^p is a functor from **Meas** to **Ban**. □

Before characterizing L^p, let us first define a few more terms needed.

Let **NLS** be the category of normed linear spaces as objects and linear contractions as morphisms. For $1 \leq p < \infty$, define $\mathbf{N}^p$ to be the category with pairs (F, u) where,

$$F, u : \mathbf{Meas}^{\text{op}} \to \mathbf{NLS},$$

such that $u_X := u(X) \in F(X)$ for measure space $X \in \mathbf{Meas}^{\text{op}}$. Finally let $\mathbf{B}^p$ be the subcategory of $\mathbf{N}^p$ containing pairs (F, u) such that,

$$F, u : \mathbf{Meas}^{\text{op}} \to \mathbf{Ban} \subset \mathbf{NLS}.$$

Furthermore, let $\Gamma : \mathbf{Meas}^{\text{op}} \to \mathbf{Ban}$ be such that $\Gamma_X := \Gamma(X) \in L^p(X)$ with,

$$\Gamma_X(x) = 1, \quad \forall x \in X.$$

Similar to how we set $\int_0^1 \Gamma = \mu([0, 1]) = 1 - 0 = 1$ in Section 3.3, we set,

$$\int_X \Gamma_X = \mu_X(X). \tag{4.1}$$

Lastly, we note the following definition of left adjoints:

Definition 4.1.8. Let **M**, **N** be two categories and $F : \mathbf{M} \to \mathbf{N}$ and $G : \mathbf{N} \to \mathbf{M}$ be two functors. Then, F is said to be a left adjoint to G if $\mathbf{N}(F(M), N) \cong \mathbf{M}(M, G(N))$ by the canonical map for any $M \in \mathbf{M}, N \in \mathbf{N}$.

Here, $\mathbf{N}(F(M), N)$ means the set of all morphisms $F(M) \to N$ in the category **N** for the objects $F(M), N \in \mathbf{N}$ (note, F is a functor from **M** to **N**). Similarly, $\mathbf{M}(M, G(N))$ is understood to be the set of all morphisms $M \to G(N)$ in the category **M**. In other words, for any $M \in \mathbf{M}$, $N \in \mathbf{N}$, F the left adjoint to G is such that morphisms $F(M) \to N$ in **N** can be bijectively identified (by an isomorphism provided by the canonical map) to the morphisms $M \to G(N)$ in **M**.

4.2 Unique Characterization of L^p

Now, we give the generalization of Theorem 3.3.5 to any finite measure space X. That is, we shall show that $L^p(X)$ is uniquely characterized in **Ban**, provided some additional structure. We do this by showing that the L^p functor is an initial object in $\mathbf{B}^p$, as follows:

Theorem 4.2.1. Let $1 \leq p < \infty$. Then (L^p, Γ) is an initial object of $\mathbf{B}^p$.

Proof. Note, here we give only a proof outline; the full details can be seen in Theorem 3.7 of [6]. Firstly, note that the functor $U \mapsto \overline{U}$ from **NLS** to **Ban** is a left adjoint to the inclusion map (which is a functor) $\iota : \mathbf{Ban} \to \mathbf{NLS}$. For an arbitrary pair $(F, u) \in \mathbf{B}^p$ (see the last part of the previous section), let us define $\overline{F} : \mathbf{Meas} \to \mathbf{Ban}$ such that $\overline{F}(X) = \overline{F(X)}$ where $X \in \mathbf{Meas}$. Now, note that the pair $(\overline{F}, u) \in \mathbf{B}^p$, and the morphism $(F, u) \mapsto (\overline{F}, u)$ is the left adjoint of the functor $\mathbf{B}^p \to \mathbf{N}^p$. Since left adjoints preserve initial objects [5], using Proposition 3.6 of [6], we conclude that (L^p, Γ) is the initial object of $\mathbf{B}^p$. □

So, thus far we have shown that L^p is a functor from the (opposite) category of finite measure spaces $\mathbf{Meas}^{\mathrm{op}}$ to the category of Banach spaces **Ban** (see, Lemma 4.1.7); and that L^p (with some additional structure i.e. Γ) is an initial object of the category $\mathbf{B}^p$. Therefore, since the morphisms of the category **Ban** are linear contractions, we have that for any[1] finite measure space X, $L^p(X) \in \mathbf{Ban}$ are characterized uniquely up to isometric isomorphisms (which are precisely the isomorphisms of **Ban**).

In other words, for any finite measure space X and for any $1 \leq p < \infty$, $L^p(X)$ is the unique object (up to isometric isomorphisms) in the category **Ban** that has the properties it has, i.e. the Banach space which is the completion of the space of continuous functions on X, containing all the Lebesgue integrable functions on the measure space X, and being the set of $\mathcal{S}$-measurable functions $f : X \to \mathbb{F}$ such that $\|f\|_p < \infty$ (see, Section 2.4) [6].

Now, similar to how we used the unique characterization of $L^1[0,1]$ to give a unique characte- -rization of the integration operator $\int_0^1$ as the unique morphism from $L^1[0,1]$ to $\mathbb{F}$ (with some extra structure) in Proposition 3.4.2, we will now use the unique characterization of $L^1(X)$ for each finite measure space X to give a unique characterization of the integration operator $\int_X$. Consider the family of operators,

$$\int_{\_} := \left(\int_X : L^1(X) \to \mathbb{F} \right)_{\text{measure space } X}.$$

That is, for each finite measure space X, we get a uniquely characterized Lebesgue space $L^1(X)$ in **Ban** (from the functor L^1) and an integration operator $\int_X : L^1(X) \to \mathbb{F}$ (which is in the family of operators $\int_{\_}$). So, just like we were able to uniquely characterize $L^1(X)$ (for each finite measure space X) using the initiality of L^1 functor, we uniquely characterize the integration operator $\int_X$ (for each finite measure space X) by the uniqueness of the family of operators $\int_{\_}$ as defined above. Finally for the ground field $\mathbb{F}$, we define for each finite measure space X, a map $t : \mathbf{Meas}^{\mathrm{op}} \to \mathbb{F} \subset \mathbf{Ban}$ such that $t(X) := t_X = \mu_X(X) = \int_X \Gamma_X$ (see, Eqn. 4.1).

Theorem 4.2.2. The family of operators $\int_{\_}$ is the unique morphism $(L^1, \Gamma) \to (\mathbb{F}, t)$ in $\mathbf{B}^1$.

Proof. We use the same techniques as used in Proposition 3.4.2 i.e. since (L^1, Γ) is an initial in $\mathbf{B}^1$, if $\int_{\_}$ is a morphism from (L^1, Γ) to $(\mathbb{F}, t)$, then it is the (literal) unique morphism $(L^1, \Gamma) \to (\mathbb{F}, t)$ in $\mathbf{B}^1$. This follows from the linearity of integration, the triangle inequality and $\int_X \Gamma_X = \mu_X(X)$. For details, see Proposition 3.8 of [6]. □

[1]Technically we need $X \in \mathbf{Meas}^{\mathrm{op}}$, but since the objects of the opposite category of a category are by definition (see, Definition 3.1.4) precisely those of the category, $X \in \mathbf{Meas}$.

So, the above Theorem (and the discussion preceding it) uniquely characterizes and therefore gives an abstract formulation of the integration operator $\int_X$ for any finite measure space X. But in order to get a more practical definition of the integration operator and thus of Lebesgue integration we consider (similar to Theorem 3.4.3) the following:

Take, for each finite measure space X, $M(X) \in \mathrm{Ob}(\mathbf{Ban})$ to be the space of finite signed measures (if $\mathbb{F} = \mathbb{R}$) or complex measures (if $\mathbb{F} = \mathbb{C}$) on the underlying $\sigma-$algebra (see, Definition 2.4.1). Then, for any category Y, $M(Y) \to M(X)$ is an isometry that extends measure by taking zero [6]. So, $M : \mathbf{Meas}^{\mathrm{op}} \to \mathbf{Ban}$ is a functor and (M, μ) is an object of $\mathbf{B}^1$ with $\mu : \mathbf{Meas}^{\mathrm{op}} \to \mathbf{Ban}$ such that $\mu(X) := \mu_X \in M(X)$ [6]. So, since (L^1, Γ) is initial in $\mathbf{B}^1$, the morphism $(L^1, \Gamma) \to (M, \mu)$ exists and is unique up to isometric isomorphisms (see, Definition 3.1.10 and Lemma 3.1.11).

Theorem 4.2.3. For each finite measure space X, the unique morphism $(L^1, \Gamma) \to (M, \mu)$ in $\mathbf{B}^1$ has $X-$component $L^1(X) \to M(X)$ such that $f \mapsto f\mu_X$.

Proof. Here we give a brief outline of the proof; for the full details see the proof of Proposition 3.10 in [6]. For each finite measure space X, let $\pi_X : L^1(X) \to M(X)$ be such that $\pi_X(f) = f\mu_X$. If we show that π is the unique map $(L^1, \Gamma) \to (M, \mu)$ in $\mathbf{B}^1$, then we are done since π_X is its $X-$component by construction. Indeed this is true since $\pi_X(\Gamma_X) = \Gamma_X \circ \mu_X = \mu_X$ by definition of Γ_X, π_X is an isometry, and,

$$\int_A g^X \, \mathrm{d}\mu_X = \int_{A\cap Y} g \, \mathrm{d}\mu_Y,$$

for any finite measure space Y and for any measurable subset A of X. Here, g^X is the map $g : A \to \mathbb{F}$ that is extended to all of X by $g(x) = 0$, $\forall x \in X - A$. □

Remark 4.2.4. So, Theorem 4.2.2 states that $\int_{\_}$ is the unique morphism $(L^1, \Gamma) \to (M, \mu)$ in $\mathbf{B}^1$, and Theorem 4.2.3 states that the X component of $\int_{\_}$ i.e. $\int_X$ is the morphism from $L^1(X)$ to $M(X)$ such that $\int_X f \, \mathrm{d}\mu_X = f\mu_X(X)$. Restricting the morphism $\int_X$ to some measurable subset A of X, we get [6],

$$\int_A f \, \mathrm{d}\mu_X := (f\mu_X)(A).$$

Thus given any finite measure space X, starting from the notion of Banach spaces and some extra categorical structure arising from the nature of integration, one can uniquely characterize the space of Lebesgue integrable functions $L^p(X)$ on X (see, Theorem 4.2.1). Further, using this specifically for the case $p = 1$, one can both give a unique characterization of the integration operator abstractly (see, Theorem 4.2.2) and finally give a concrete formula--tion of the definite integration operator on any $A \in \mathcal{S}_X$.

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