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A Mathematical Study of Crochet

Bachelor's Project Mathematics

July 2022

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Contents

Abstract

In this paper, some of the mathematical properties of crocheting will be explored. The paper contains a proof that all topological surfaces can be crocheted, up to homeomorphism. Moreover, the connection between discrete and continuous differential geometry with respect to crochet, in particular the Gauss-Bonnet Theorem, will be examined. Finally, an introduction to utilizing a metric to crochet a surface will be given.

1. Introduction

Even though differential geometry is often studied in the continuous sense, when we look at the world around us, it becomes clear that many objects are not perfectly continuous, but discrete instead. In this thesis, we wish to examine discrete differential geometry. There is a lot to be said about this field, but we will restrict ourselves to one particular goal: to understand crocheted fabrics.

We will view crocheted fabric as a triangulation of a surface in order to be able to work with them mathematically in a discrete setting. A natural first question is whether crocheted fabric can imitate all surfaces we can think of, or whether there are some restrictions. As it turns out, we can crochet all topological surfaces, orientable and non-orientable, if we accept that we have to add a scar somewhere on the surface. A significant part of this thesis consists of proving the above. This will lead to a discussion about continuous and discrete differential geometry in order to justify some of the theorems used in proving all topological surfaces can be crocheted. In particular, we will draw parallels between the continuous and discrete Gauss-Bonnet theorem, as well as provide a proof for the discrete version, and a sketch of the proof for its continuous counterpart.

Finally, we explore a little about how to create crocheted fabrics from a metric, and how we can mathematically justify this procedure.

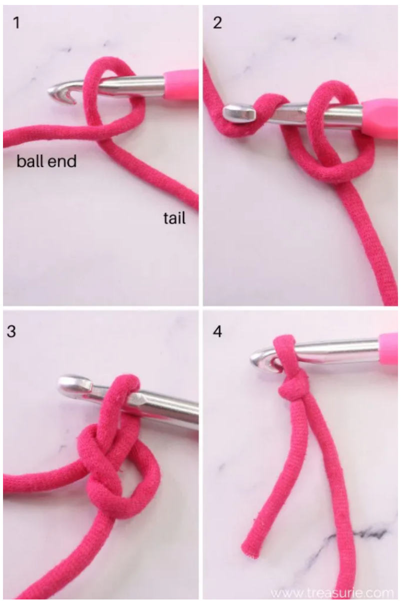
While there is some research to be found on the mathematical aspects of crocheting, a lot has been left unexamined. While we shed some light on some properties of crocheting, there is a lot more to explore and discover, as well as questions that came up during the writing of this paper, but which could unfortunately not be explored further at this time.

2. Background on Crocheting

Before we start with the mathematics, an introduction to what crocheting is will be given, as well as some differences between knitting and crocheting. Readers familiar with crochet might want to skip the first part of this chapter, but readers that are less confident in their crocheting skills are highly encouraged to read this chapter attentively. We highly recommend trying to crochet some stitches and surfaces throughout this paper!

To begin, crochet is a way to create a fabric from strands of some material, by using a crochet hook. What exactly is crocheted can widely depend, from blankets and clothes to stuffed animals. However, crochet has also proven to be a nice tool when discussing mathematics. For example, in [1] Osinga and Krauskopf create the Lorenz manifold by crochet, and in [2] crochet is used to make a model of the hyperbolic plane.

Crocheting creates a surface generating parts of the surface stitch by stitch. For every crochet project, first one makes a *slip knot* 1a to create a little loop in the yarn to put on the hook. Then, we create our first *row* out of *chain stitches*. Chain stitches 1b are the most basic type of stitch, and are created by looping the strand over the crochet hook, and pulling it through the loop that was already on the hook.



(A)

(B)

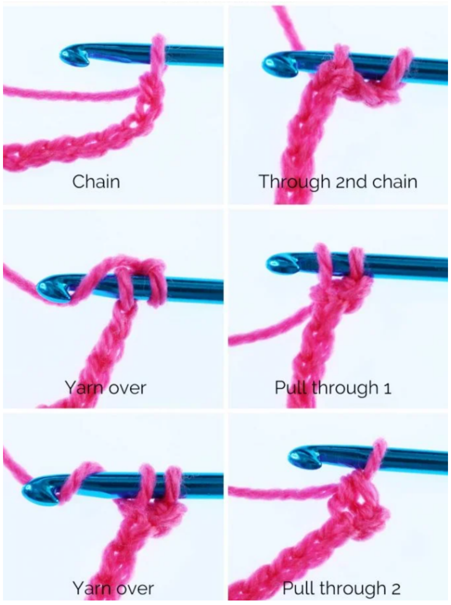
Figure 1. How to create a slip knot and chain stitch [3].

There are a few more basic stitches we will explain. We will be using the US naming conventions of the stitches. If the reader is more familiar with the UK terms, table 1 can be used to work out which stitches we mean.

UK Terms	US Terms	NL Terms
ss (slip stitch)	ss (slip stitch)	hv (halve vaste)
dc (double crochet)	sc (single crochet)	v (vaste)
htr (half treble crochet)	hdc (half double crochet)	hst (half stokje)
tr (treble crochet)	dc (double crochet)	st (stokje)
dtr (double treble crochet)	tr (treble crochet	ds (dubbel stokje)

TABLE 1. Crochet Abbreviations.

In this paper, we mostly use single crochets and half double crochets, and very sparingly treble crochets. These are also the only stitches we will explain how to make. For the single crochet stitch 2a, we insert the hook through a stitch in the previous row, yarn over, and pull through one loop. Then, we yarn over again, and pull through two loops, so only one loop remains on the crochet hook.



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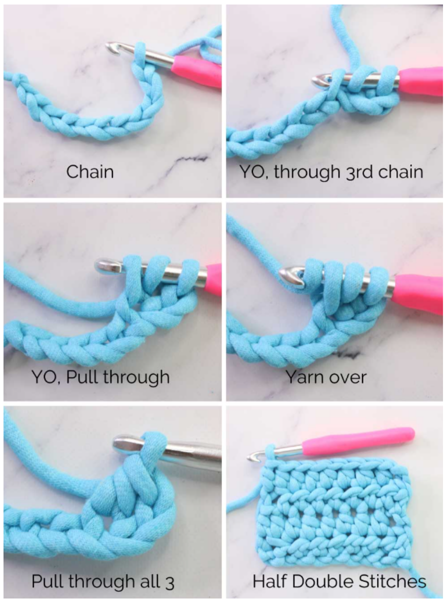


(B)

FIGURE 2. How to create a single crochet stitch [4] and a half double crochet stitch [5].

For the half double crochet stitch 2b, we yarn over already *before* we insert the hook through a stitch in the previous row. Then, we yarn over again, and pull through one loop. Then, we yarn over and pull through three loops, so only one loop remains on the crochet hook.

Lastly, a treble crochet stitch 3 is created by yarning over the hook twice *before* we insert the hook into a stitch from the previous row. Then, yarn over and pull through one loop. Then, yarn over and pull through two loops. Again, yarn over and pull through two loops, so only one loop remains on the crochet hook.

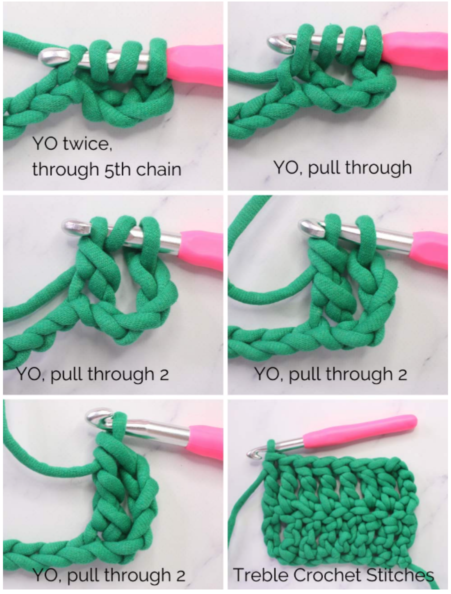


FIGURE 3. How to create a treble crochet stitch [6].

The most noticeable difference between these stitches in a crocheted fabric, is their height. A single crochet stitch is about as wide as it is tall, while the half double crochet is about $\frac{3}{2}$ times as tall as it is high. This continues for the other stitches mentioned in the table, as can be seen in figure 4.

Generally, we will be using single crochet stitches throughout the paper. When we need to add extra length in a stitch, we either use a half-double crochet stitch (to add only a little extra height) and the treble crochet stitch (to add more extra height). We will mention explicitly when the stitches we use are *not* the single crochet.

We can vary the length of these rows via *increases* and *decreases* to make the rows longer or shorter, respectively. As we said before, we create a fabric in rows. We do this in the following way:

- to increase, we crochet two stitches into the stitch of the previous row,
- to decrease, we crochet one stitch into two stitches of the previous row.

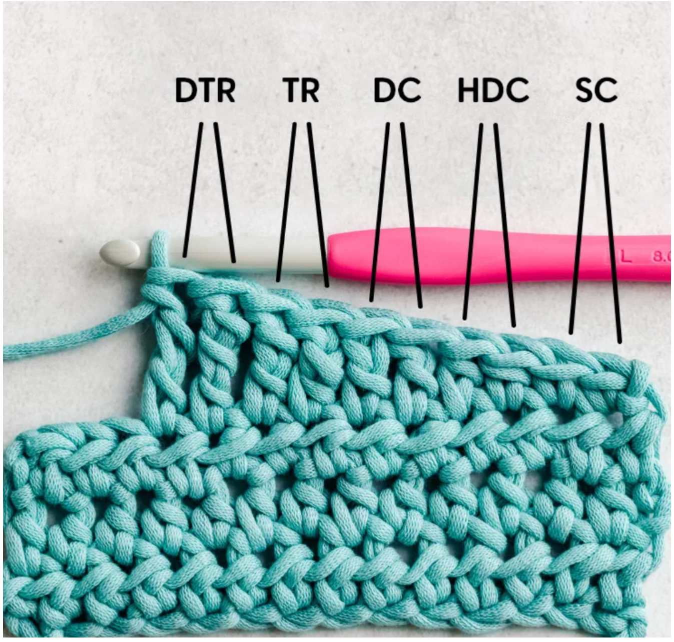


FIGURE 4. Difference in height in the basic stitches of crochet [7].

For the increase, stitching two stitches into one stitch of the previous row is as straightforward as it sounds. For the decrease, the procedure is as follows. In the step where we connect the new stitch to the previous row, instead of doing this *once*, we do this *twice* in a row so the new stitch will be connected to the two next stitches of the previous row, instead of only one stitch. Then, finish the stitch off as normal. Note that by doing this, we get an extra loop on our crochet hook. We solve this by, at the last step of finishing the stitch, pulling the hook through one extra loop than when we make the stitch normally. For example, for the single crochet, the last step is pulling the hook through two loops. When we decrease in single crochet, in the last step we instead pull the hook through three loops.

Lastly, we need to explain how we connected edges of crocheted fabric together. While there are many ways to do this, in this project we will use *single crochet seam* 5. This is quite straightforward. We line up the two edges we wish to crochet together, and, instead of only connecting the single crochet to one of the rows in the usual way, we connect the single crochet to both rows. This is done by inserting the hook through both stitches, yarning over and pulling the hook back through both stitches, and finish the stitch off as usual. A more detailed explanation can be found in [8].

2.1. **Difference between Crochet and Knitting.**

A significant part of this paper is proving that topological surfaces can be crocheted

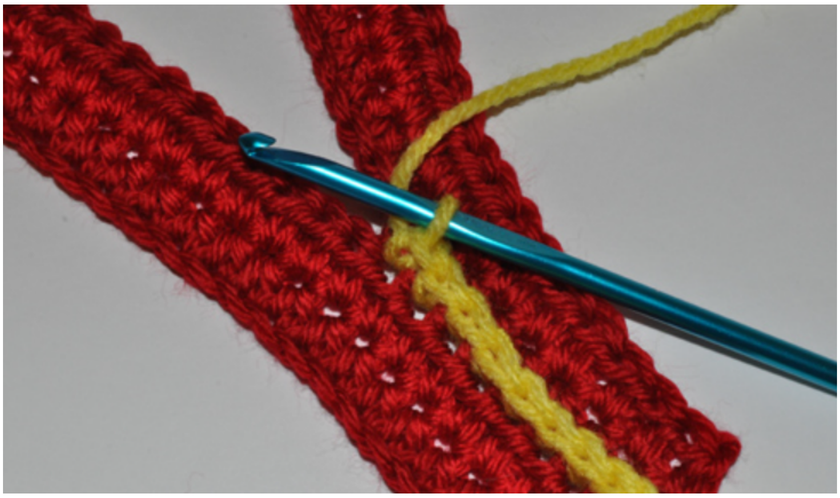


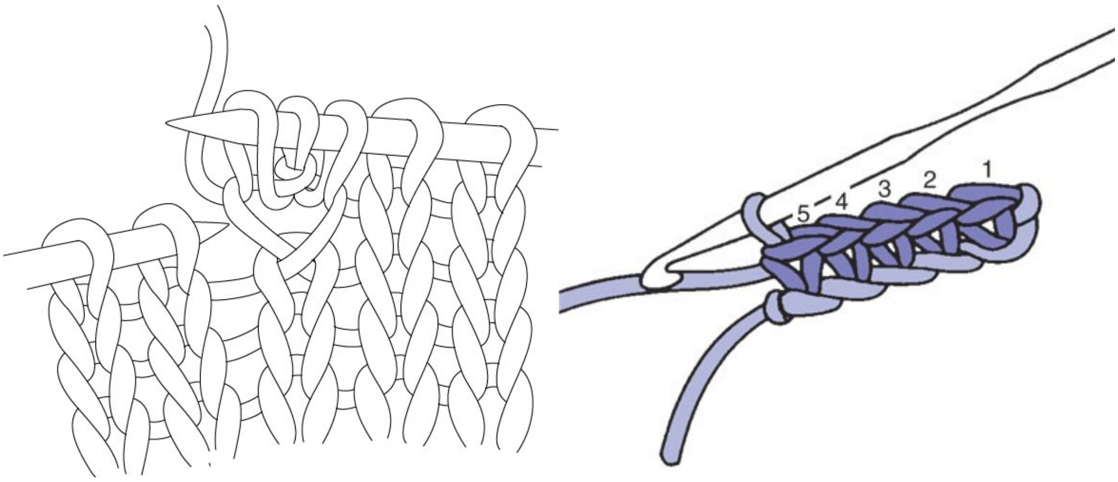
Figure 5. Fabric crocheted together using single crochet stitches [9].

using a single strand of yarn. We do this by reconstructing the proof of Sarah-Marie Belcastro in [10] for an equivalent statement for knitting. While both crocheting and knitting are ways to create textiles from strands of yarn, there are some inherent differences between the two crafts. This also leads to some modifications of the construction of the proof. We will explain the differences, which are crucial to the proof, below.

The first and foremost difference between crochet and knitting is the *way* the rows get created. For crocheting, in every row, we crochet stitch by stitch. However, for knitting, the beginning of the stitches along the entire row is created (using 2 smooth knitting needles, rather than one hooked needle like the crochet hook), and then the stitches are finished one by one. The way the stitches are created is also of importance. In figure 6, we see knitted fabric and crocheted fabric (with single crochet stitches) side-by-side. It is clear that in the knitted fabric, the stitches are more plainly connected to each other than for the crocheted fabric.

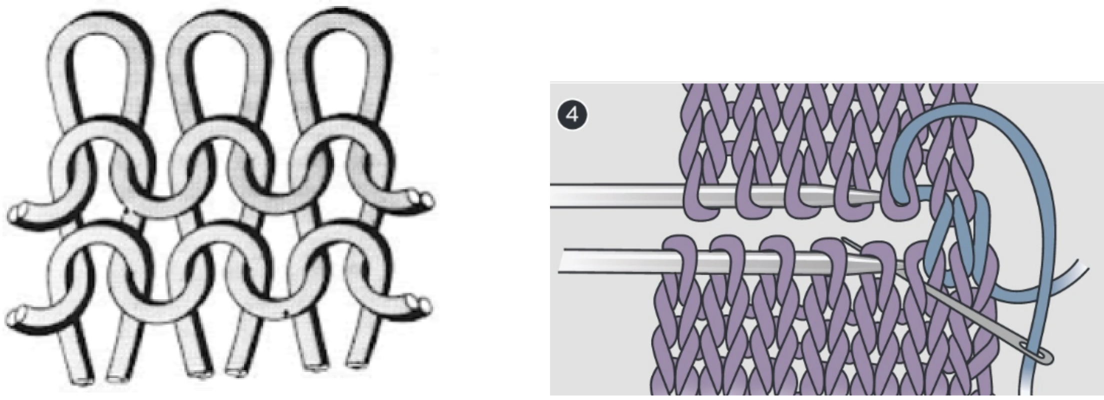
This fact, that knitted stitches are more plainly connected to each other, gives rise to a very important property for knitting, *grafting*. Grafting is a sewing technique where knitted fabric gets joined by sewing *via the same stitch* as the stitch that the knitted fabric consists of. In figure 7b we see how this technique works. Via a sewing needle, the structure in figure 7a can be fully imitated along the edges of the fabrics that are joined.

This grafting technique implies that knitted fabrics can be joined seamlessly. Unfortunately, crochet does not have this property. While sewing crocheted fabrics



(A) (B)

Figure 6. Knitted fabric [11] and crocheted fabric [12].



(A) (B)

Figure 7. Structure of knitted fabric [13] and grafting [14].

together can be done (almost) invisibly, the stitches are (in most cases[1]) significantly different. Because of this, we cannot fully imitate Belcastro's proof. Instead of the 'smooth' topological surfaces knitting produces, the crocheted surfaces will have a scar where the grafting procedure gets used. While on one hand we would rather have a smooth structure for crochet as well, it interestingly highlights some other properties of manifolds. We know that a manifold is a topological object

[1]There are some specific cases where connecting the edges of a crocheted fabric *can* be done seamlessly, like the *invisible join*[15] when crocheting in circles. However, especially for connecting longer edges together, this is not possible.

that consists of charts that locally look Euclidean, and agree with each other in the places where the charts overlap. In many cases we do not have a global chart to describe the entire manifold, but rather multiple local charts. The fact that we will have to add a scar in the crocheted surfaces highlights that being Euclidean is generally a local property.

3. Can Every Topological Surface Be Crocheted?

When discussing crocheted surfaces, the first thing that comes to mind is the question of existence. Do the manifolds we wish to discuss actually exist when it comes to crocheting? As it turns out, these surfaces can indeed be crocheted. In this section we will prove this rigorously. We will also provide crocheted examples of the surfaces discussed, and give directions on how to create them. We will start by stating a modified version of the theorem in [10] and proceed by following the general ideas in the proof, while making the appropriate modifications for crochet. For the proofs, we will make use of *crochet decompositions*. These are 2 dimensional polygon representations of the surfaces we create. The decompositions show the path the yarn takes, and how the 'sides' of the 2 dimensional representation are connected to each other. Let us now finally state the theorem.

Theorem 3.1. *Every topological surface can be crocheted with a single strand of yarn, up to adding a scar where we connect the surface to itself.*

We can split this theorem in two parts; for orientable surfaces and for non-orientable surfaces.

Theorem 3.2. *Every orientable surface can be crocheted with a single strand of yarn, up to adding a scar where we connect the surface to itself.*

We will prove the theorem by induction on the genus of surfaces. We first need to clarify what we mean by the statement above. We will prove that, up to homeomorphism, we can crochet all orientable topological surfaces by characterizing them by their genus. That orientable surfaces can be characterized by their genus relies on the *Euler-Poincaré characteristic.*

Definition 3.1. *Given a triangulation $\mathcal{J}$ of a compact connected surface S, the* Euler-Poincaré characteristic $\chi(S)$ *is defined as the number*

$$\chi(S) = V - E + F$$

where V, E and F are the number of vertices of the triangulation, the number of edges of the triangulation and the number of faces of the triangulation respectively.

The Euler-Poincaré characteristic and genus of an orientable surface S are related via the following formula,

$$\chi(S) = 2 - 2g, \tag{1}$$

where g is the genus of the surface [16].

With these ideas in mind, we state the following theorem.

Theorem 3.3. *Let $S \subset \mathbb{R}^3$ be a compact connected orientable surface; then the Euler-Poincaré characteristic $\chi(S)$ assumes one of the values $2,\ 0,\ -2, ...,\ -2n, ...$. Furthermore if $S' \subset \mathbb{R}^3$ is another compact surface and $\chi(S) = \chi(S')$, then S is homeomorphic to S'.*

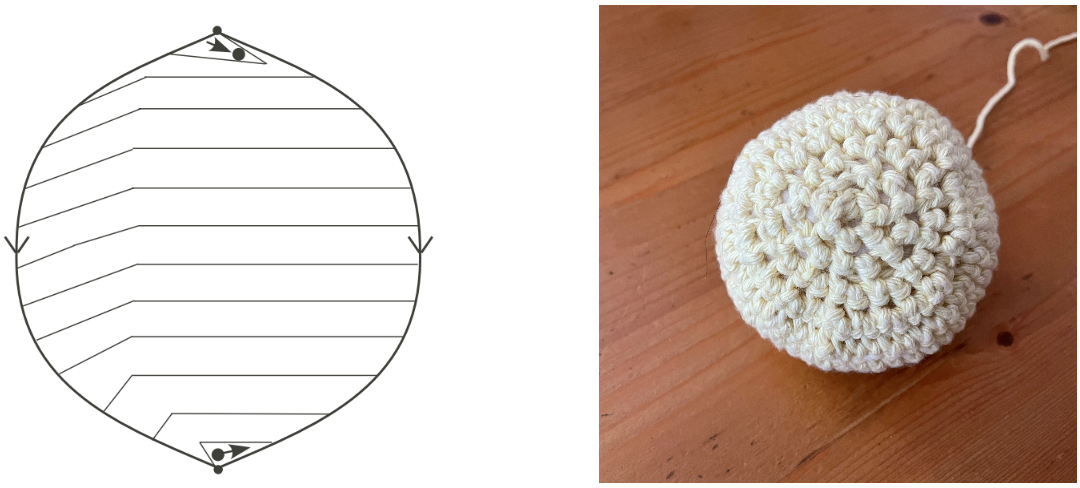
The proof of this theorem is outside of the scope of this paper, however, if the reader wishes to gain more insight, we recommend taking a look at [17], [16] or [18]. As the Euler-Poincaré characteristic and genus are linked by equation (1), this theorem in particular implies that an orientable compact connected surface is homeomorphic to any other orientable compact connected surface of the same genus. As the torus has genus 1, this means that we can see an orientable surface of genus n as the connected sum of n tori. Now, we have all the necessary tools to prove 3.2.

Proof of Theorem 3.2. We will prove the theorem via induction on the genus. The base cases for the orientable surfaces are for genus $n = 0$ and $n = 1$, the sphere and torus.

We will start with the sphere. It is enough to show that we can decompose the sphere into a long thin polygon that identifies with crocheted fabric. The decomposition is shown in figure 8a. This is the same decomposition as Belcastro uses in [10] for knitting, as the general techniques for creating a sphere for the two crafts agree.

We read the crochet decomposition in the following way. We start at the arrow with the dot at its tail. Then, we follow it in the direction as specified. When we arrive at the boundary on the right side, we move to the boundary on the left side *in the band where one winds up when we draw lines horizontally from the endpoints from the previous band.* We continue in this fashion until we reach the end of the decomposition diagram, marked by the arrow with the dot at its head. Practically, when we wish to crochet this, we create a sphere crocheting in circles, and each round increasing the amount of stitches. Then, we crochet the same amount of rounds without increasing *nor decreasing*, and finish up by decreasing the same amount of rows as we increased. A full pattern can be found in [19]. We start the sphere by crocheting a *magic ring*, i.e., we crochet six stitches in 1 loop, and we work on from there. The pattern ends the sphere by, via sewing, connecting the last 6 stitches in one loop again. As this is impossible to do with crochet stitches, we instead crochet the last stitches together, which forms a scar at the north pole of the sphere[2].

[2]In this particular case, we could use the invisible join, however, this would leave a gap at the north pole, and we would still have the singularity at the north pole, but in a different form.



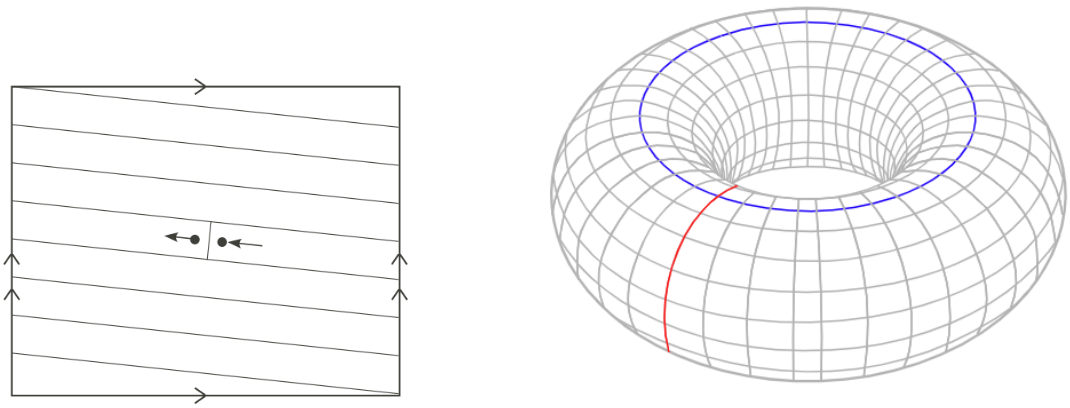
(A) (B)

FIGURE 8. Crochet decomposition of a sphere [10] and a crocheted sphere. Here, the scar is situated at the South Pole of the sphere, and cannot be seen.

For the torus, the general idea is similar. We decompose the torus into a polygon representation, as can be seen in figure 9a. This, again, is the same decomposition as Belcastro uses. As a torus is generated from two circles, see figure 9b, when we crochet a torus, we have to make a choice along which circle the scar has to appear. We can get rid of either scar, but not of both. In figure 10a, we see a crocheted torus with the scar along a *longitude* (circles parallel to the blue circle in figure 9b) and in figure 10b a crocheted torus with the scar along a *meridian* (circles parallel to the red circle).

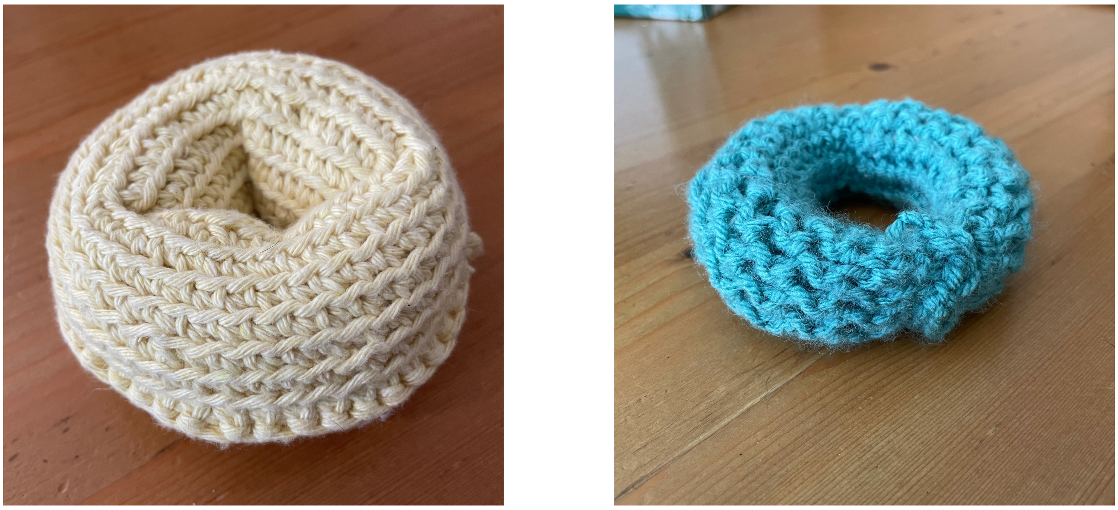
A torus is not too complicated to crochet. For the torus with the scar along the longitude, we start by crocheting a circle, and continue crocheting along circles, increasing and decreasing in the appropriate places to create the torus. We now have crocheted a band consisting of circles of varying radius. Then, in the final step, we crochet the first and last row that we crocheted together. For the torus with the scar along the meridian, we start by creating a circle as well, and then create a tube *without increasing or decreasing*[3]. Then, when our tube has the appropriate length, we crochet the first and last row together to finish the torus.

[3]As the 'outer part' of the tube represents a larger radius than the 'inner part', it can be nice to crochet the inner half of the tube by single crochet stitches, and the outer part by half double crochet stitches to illustrate this. When we cut open the crocheted fabric, this will also be more appropriate with respect to the stretching the stitches along the outsides would need to do if we only use the single crochet stitch.



(A) (B)

FIGURE 9. Crochet decomposition of a torus [10] and illustration that a torus is the product of 2 circles.



(A) (B)

FIGURE 10. Tori with the scar along a longitude and a meridian.

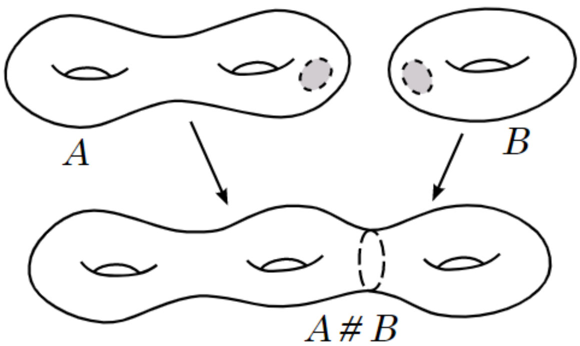
Now that we have the base cases, we state the induction hypothesis on the genus of the surface:

Any orientable surface of genus less than $n-1$ *can be crocheted with a single strand of yarn, up to adding a scar where we connect the surface to itself.*

We need one more definition before we can start proving the statement for surfaces of genus n.

Definition 3.2. *A* connected sum *of two m-dimensional manifolds is a manifold formed by deleting a ball inside each manifold and gluing the resulting boundary spheres together.*

We observe that an orientable surface of genus n can be seen as a connected sum of an orientable surface of genus $n-1$ and a torus. This is illustrated in figure 11a.



(A)

(B)

FIGURE 11. Connected sum of a two-holed torus and a torus [20] and a crocheted orientable surface of genus 2.

It was shown before that we can decide where the scar on the torus appears. The construction in this part of the proof is easier when we crochet along the meridians. We start by crocheting an orientable surface of genus $n-1$, but, once this surface has been crocheted, we do not crochet the edges together. By the induction hypothesis, we can crochet an orientable surface of genus $n-1$, and by the base case we can crochet a torus. Then we consider both the surface of genus $n-1$ and the torus, finished except for the final connecting of the edges. Then, we essentially have 4 boundaries. Let us call the boundary along the first row of the surface of genus $n-1$ b_2, and the row where we end b_1. Similarly, we call the boundary of the torus where we started crocheting b_4, and the boundary of the torus where we end b_3. We wish to connect these boundaries to each other to create a surface of genus n. To do this, we identify one third of b_2 with b_1, the remaining two thirds of b_1 with b_3, one third of b_3 with b_4 and the remaining two thirds of b_4 with b_2. These boundaries can be seen in figure 12. The diagram for the surface of genus $n-1$ is not complete. The full decomposition diagram can be seen in figure 13.

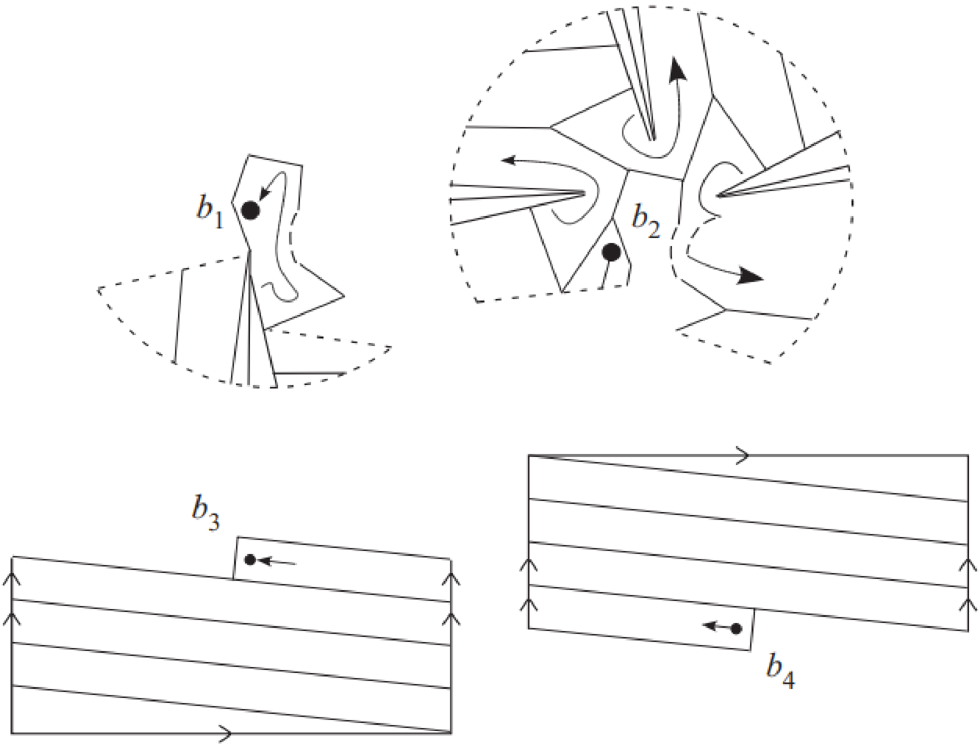


FIGURE 12. Decomposition diagram of an orientable surface of genus $(n-1)$ and a torus [10].

In figure 13, we see a decomposition diagram for an orientable surface of genus n. If we were to consider a surface of genus 4, then in between the 2 torus decomposition diagrams on the right side, the lines would be connected. Now, space is left to indicate this construction can be repeated an arbitrary amount of times to get the desired genus.

We can crochet the construction above in the following way. We start with crocheting the surface of genus $n-1$. When the only step remaining is for us to connect b_2 and b_1, we only connect one third of the stitches of b_2 and b_1. To have the surface appear more smooth, we crochet along the inner third to connect part of the boundaries to connect one third of the boundaries. Next, crochet along the remaining two thirds of b_1 and chain the same amount of stitches as that were crocheted together when we connected b_2 and b_1. We then connect the last chain stitch with the first stitch of b_1 that was *not* crocheted together with b_2. Note that we now have a circle with the same amount of stitches as we originally had along the boundaries. The chain that we crocheted corresponds to the one third of the boundary of b_4 that will be connected with one third of the boundary of b_3, while the other two thirds of b_4 (and b_1) are the stitches of b_1 that we did not crochet

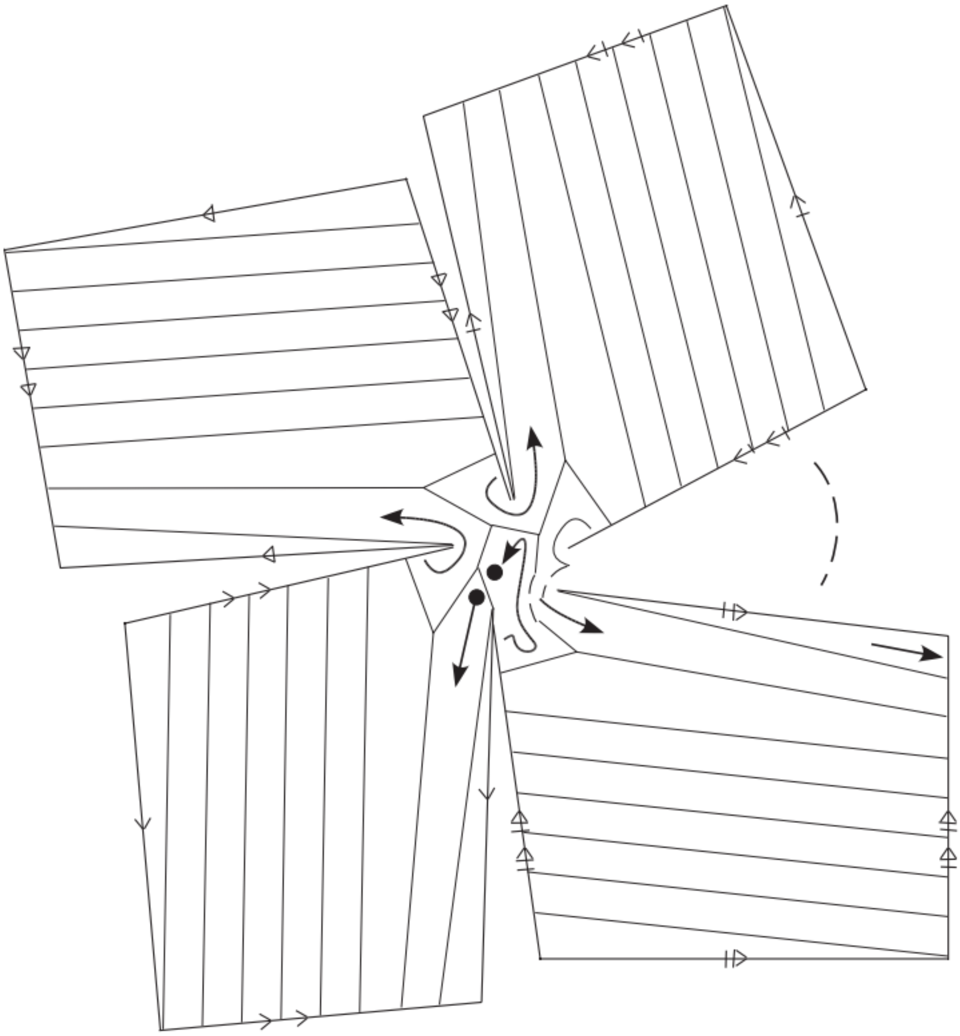


FIGURE 13. Decomposition diagram of an orientable surface of genus n [10].

together with b_2. For the next part, simply crochet a tube again, and form a torus with this tube. We now connect the inner part of b_3 and the inner part of b_4, and the remainder of b_3 with b_2. Note that where we crochet the boundaries together, we get the scar on the surface that we crochet.

This concludes the proof. □

Remark 1. With this construction of the proof, the connection of all the tori will be in the middle. While this looks aesthetically pleasing for lower genera, when we consider, for example, a surface of genus 6, the construction will start looking clunky. Another possibility would be to first crochet the surface of genus $n-1$

as a long chain of tori glued together like in figure 11a, by first doing the left (or right) halves of the tubes, and then work back to the first torus again. Once the only thing that remains is crocheting the beginning and end together, we add in another torus to make a surface of genus n. More specifically, we could do this by crocheting half of a tube, then chaining two thirds the amount of stitches that the circles consist of and then crochet along half of the tube. Crocheting along this new circle, we create the second half torus. We continue this until we have n half tori connected to each other, and then just continue onward to finish the last torus in the row. Then, we connect one half with the half chained circle where we started the last torus. After this, we crochet along the remaining half. We do this until we have completed the chain.

Remark 2. If the thread is long enough, one could sew the boundaries together instead of crocheting them to minimize the appearance of the scars, while still using 1 string of yarn (though the process is not necessarily very pleasant). However, as we will *necessarily* have scars when crocheting, and we cannot graft stitches together as in knitting, we have chosen to crochet them to explicitly indicate where the scars appear.

We now have the desired result for orientable surfaces. What remains to be shown is that we can also crochet non-orientable surfaces with one strand of yarn up to adding a scar. For good measure, let us state the theorem.

Theorem 3.4. *Every non-orientable surface can be crocheted with a single strand of yarn, up to adding a scar where we connect the surface to itself.*

Again, what we mean with the theorem, is that we can crochet all non-orientable surfaces up to homeomorphism. Before we prove this, we again need a theorem relating the genus to the Euler-Poincaré characteristic.

Theorem 3.5. *Let S_1 and S_2 be non-orientable compact connected surfaces without boundary. Then, S_1 and S_2 are topologically equivalent if and only if $\chi(S_1) = \chi(S_2)$.*

The proof is, once again, beyond the scope of this paper and quite involved, but the interested reader can find it in [16].

We can find a relation between the Euler-Poincaré characteristic and the genus for non-orientable surfaces as well. Here, we have

$$\chi(S) = 2 - g, \tag{2}$$

where g is the genus of the surface S [16].

A last note we have to make before we start the proof, is that non-orientable surfaces cannot be embedded into $\mathbb{R}^3$. This is also the case for our crocheted non-orientable surfaces. In all cases we need to remove a disk (we *can* for example embed the Klein bottle into $\mathbb{R}^3$ when we remove a disk), or let the surface intersect itself. We make the choice here to remove a disk from our surfaces, rather then letting the surfaces do a self-intersection. What this means practically will be explained in the constructions of the surfaces in the proof.

Proof of theorem 3.4. We proceed similarly to the proof of the orientable surfaces. Here, the base case would be the real projective plane of genus 1. While we could do induction on the genus identically to the orientable case, and consider connected sums of projective planes, this is more difficult to crochet. Instead, we will consider connected sums of projective planes and Klein bottles. Therefore, our base cases will be the real projective plane and the Klein bottle. Then, we will work by induction again:

- for a non-orientable surface of genus $2n$, we will prove that it can be crocheted as a connected sum of a non-orientable surface of $n-2$ and a Klein bottle,
- for non-orientable surfaces of genus $2n+1$, we will prove that it can be crocheted as a connected sum of a non-orientable surface of $2n$ and a real projective plane.

Let us now start on the base cases. As mentioned above, the real projective plane cannot be embedded into three dimensional Euclidean space. Instead, we will have to remove a disk so to represent it. Crocheting (a representation of) the projective plane is actually less difficult than one might expect. We start by crocheting a Möbius strip (there are many patterns available for this, for example [21]). Then, once the Möbius strip is sufficiently broad, we start to decrease along the one boundary of the Möbius strip until we cannot physically decrease any further. Here, we see that we cannot embed the projective plane into three dimensional space, and that we need to leave out a disk in the middle. While this crocheted construction might be a bit confusing, the following identification of the projective plane in figure 14 should make the idea more clear.

We take a square, and at each side of the boundary, we indicate an orientation. As we can see, opposite sides have opposite orientation. The next step is to connect the opposite sides in such a way that their orientation agrees. Note that if we only connect one colour, we obtain a Möbius strip.

The crochet decomposition is once again the same as the knitting decomposition Belcastro uses. Below the decomposition is displayed along with a crocheted version of the real projective plane in figure 15b.

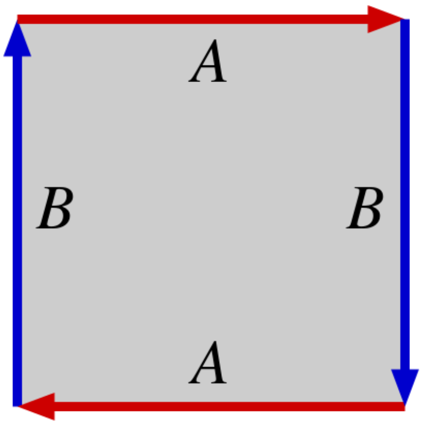
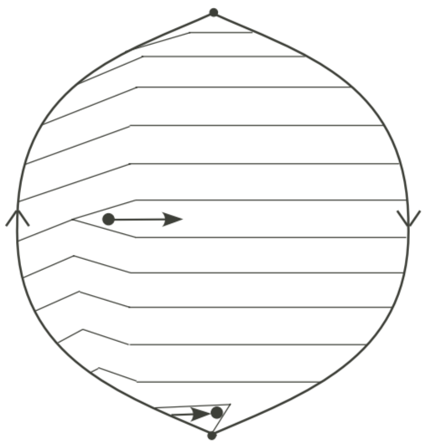


FIGURE 14. Projective plane [22].

The next base case is the Klein bottle. For the other cases discussed here, we used the same decompositions and general techniques to crochet the surfaces as the way Belcastro knitted the surfaces in [10]. However, for the Klein bottle, it is easier for us to consider another construction than the one she describes. Note that we can construct a Klein bottle by starting to construct a tube like we did for a torus. However, instead of connecting the edges of the tube according to the orientation, we instead reverse the orientation. This is illustrated visually in figure 16.

We again have to remove a disk so we can actually crochet the Klein bottle in three dimensional space. Here, we choose to crochet a Klein bottle (a pattern can be



(A)

(B)

FIGURE 15. Crocheting decomposition of the projective plane [10] and the crocheted result.

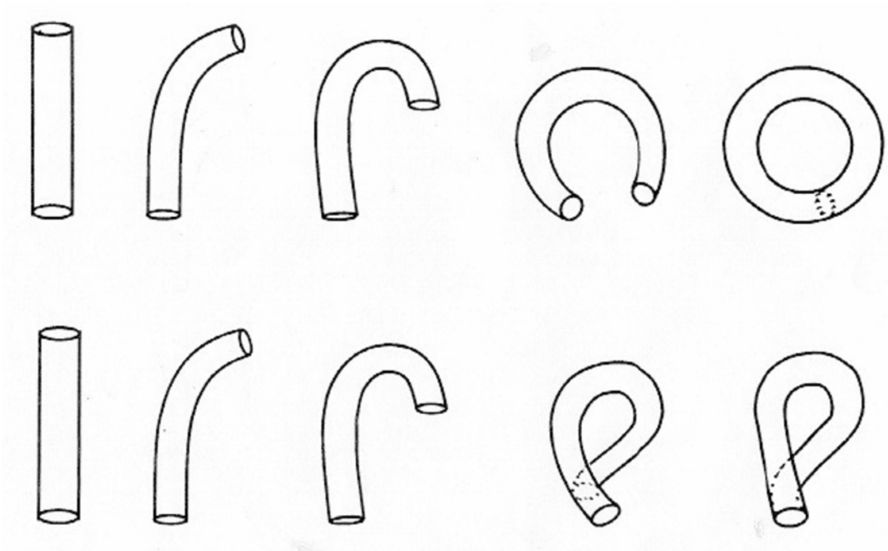


FIGURE 16. Construction of torus and Klein bottle [23].

found in [24]) where we simply start crocheting a tube, turn it inside out, and then increase. At some point, we will leave a disk in the body of the Klein bottle to allow the tube to pass through the rest of the surface, and then we start decreasing again to form the rest of the tube. We finally connect this to where we started crocheting the tube. This can be seen in figure 17.

For crocheting the Klein bottle, we make one more adjustment; we start with a (much) longer tube, so that the final connecting of the edges happens when the tube is about as long as in step 7 of figure 17. This is done purely for practical reasons, and is especially useful when we wish to connect the Klein bottle to another surface.

The crochet decomposition as well as a crocheted version of the Klein bottle can be found below in figure 18.

Now that we have established our two base cases, we can go on to the inductive step. Let us start by proving the statement for **a non-orientable surface of odd genus** $2n + 1$. We reconstruct the surface into a projective plane and a non-orientable surface of genus $2n$. As before, we will split up the boundaries of the two surfaces we want to connect. Note that the final step of the non-orientable surface of genus $2n$, connecting the edges, is the same as what we do for the Klein bottle.

We proceed in a fashion similar to the orientable case. We subdivide the boundaries into different segments, and connect these in a clever way. In figure 19 we see how we split up the boundaries of the projective plane. For the non-orientable surface

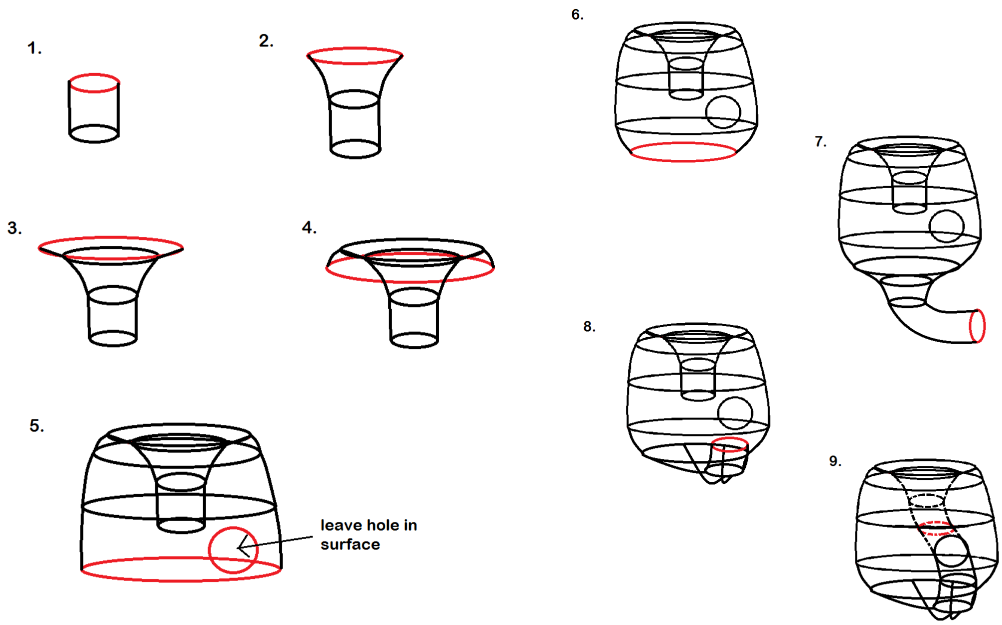
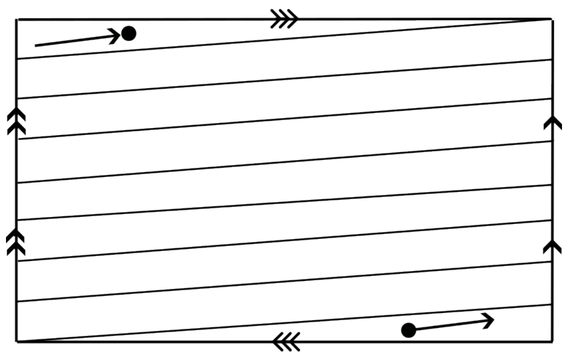


FIGURE 17. Construction of a crocheted Klein bottle [24].



(A)

(B)

FIGURE 18. Crochet decomposition of a Klein bottle and a crocheted Klein bottle.

of genus $2n$, the boundary simply appears where the tubes are crocheted together with opposing orientation. Let us call the boundaries of the the non-orientable surface of genus $2n$ $b_{1,\alpha}$ and $b_{1,\beta}$.

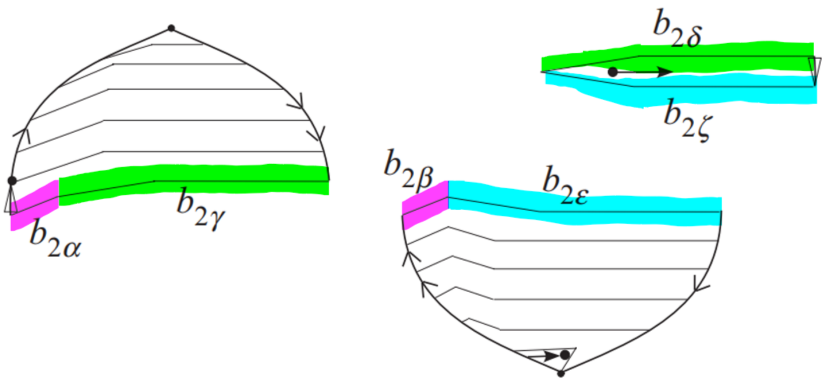


FIGURE 19. Splitting up the boundary of the projective plane [10]. The colours indicate how the boundaries were connected previously.

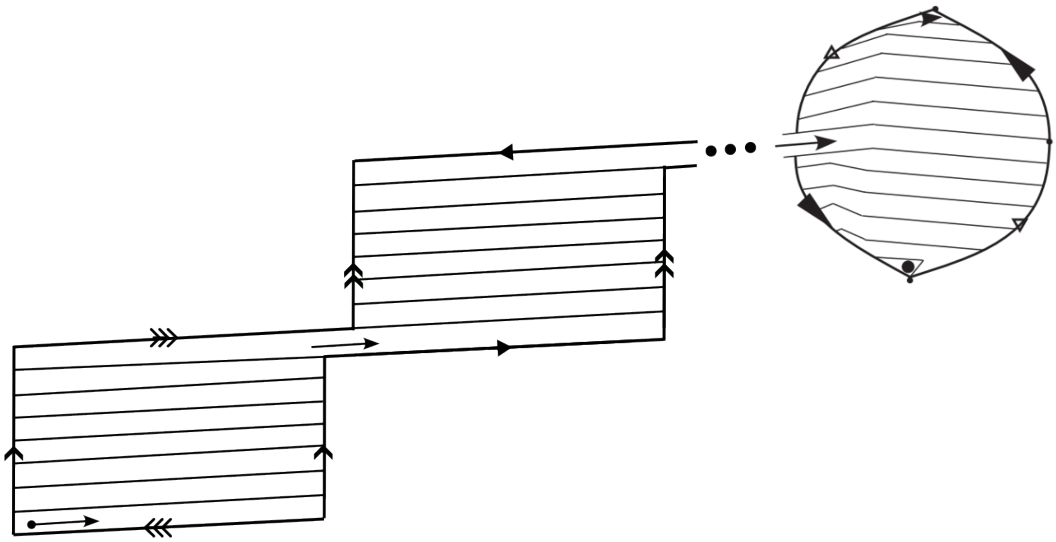


FIGURE 20. Crochet diagram of a non-orientable surface of genus $2n+1$.

We now connect the endpoints of the boundaries of $b_{1,\alpha}$ and $b_{1,\beta}$ via the the part of the boundary of the projective plane made up by $b_{2,\delta}$ and $b_{2,\zeta}$. Then, we connect $b_{2,\alpha}$ and $b_{a,\alpha}$, $b_{2,\beta}$ and $b_{1,\beta}$, $b_{2,\gamma}$ and $b_{2,\delta}$ and finally $b_{2,\epsilon}$ and $b_{2,\zeta}$.

Practically, what this means for crocheting the surface, is that we crochet one half of the boundary of the surface of genus $2n$ together. Then, we start in the next stitch at boundary $b_{1,\alpha}$, chain our desired amount of stitches for the Möbius strip, and connect this to *the last stitch* at boundary $b_{1,\beta}$, so that we get a 'diagonal' along the open part of the surface of genus $2n$. We continue by crocheting into the previous stitch at boundary $b_{1,\beta}$, and follow this by crocheting along the chain of stitches. In this step, we have to make sure that we get the twist of the Möbius strip[4]. For the remaining stitches along the boundary, we repeat this procedure, until there are no more stitches left. Then finally, we connect the last stitch we crocheted to the first stitch we crocheted to start closing the Möbius strip. We once again crochet along the strip, and connect the second to last stitch we crocheted with the second stitch we crocheted to fully close the strip. The Möbius strip is now formed, and we can continue creating the non-orientable surface of genus $2n+1$ by decreasing until we can no longer physically decrease. A crocheted non-orientable surface of genus 3 can be found below in figure 21.

FIGURE 21. Crocheted non-orientable surface of genus 3.

To complete the proof, let us consider the case of **a non-orientable surface of genus $2n$**. We decompose the surface into a non-orientable surface of genus $2n-2$ and a Klein bottle. We now have 2 boundaries of the 2 components, with 4 sub-boundaries. We will denote the 2 boundaries of the non-orientable surface of genus $2n-2$ $b_{1,\alpha}$ and $b_{1,\beta}$. Similarly, we denote the 2 boundaries of the Klein bottle $b_{2,\alpha}$ and $b_{2,\beta}$. Now, we proceed very similarly to how we connected the tori in the orientable case. We connect half of $b_{1,\alpha}$ to $b_{1,\beta}$, half of $b_{2,\alpha}$ to $b_{2,\beta}$, the remaining halves of $b_{1,\alpha}$ and $b_{2,\alpha}$ and the remaining halves of $b_{1,\beta}$ and $b_{2,\beta}$. The crocheting decomposition is given in figure 22.

[4]This is simply done by crocheting along one side of the chain, and then at the middle switching to the other side of the chain.

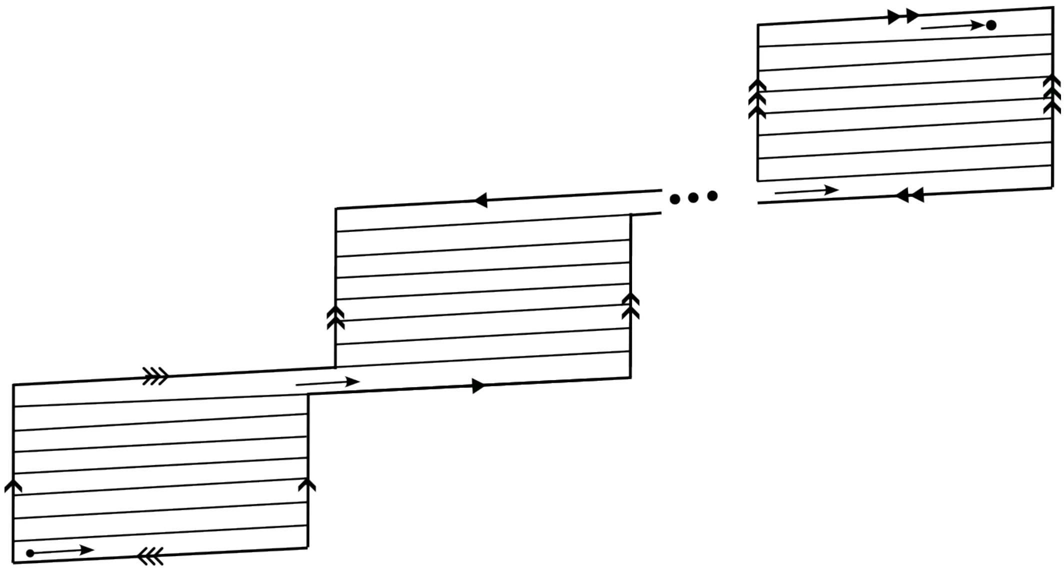


FIGURE 22. Crochet decomposition of a non-orientable surface of genus $2n$.

To create this, we crochet the surface of genus $2n-2$, and crochet half the boundary together (with opposite orientation). Then, we chain the same amount of stitches, and crochet along the boundary $b_{2,\beta}$. We now start to crochet a Klein bottle. Once we start to increase, when we create the hole[5] for the self-intersection, be sure to crochet *around* the tube as pulling the entirety of the non-orientable surface of genus $2n-2$ is quite difficult. After the hole, we can start to decrease again to form the last part of the tube. We then connect half of the boundary of the Klein bottle together, and the remaining two halves to the remaining halves of the non-orientable surface of genus $2n-2$. A crocheted non-orientable surface of genus 4 can be found in figure 23.

This concludes the proof. We can crochet any non-orientable surface with a single strand of yarn, but we have to include a scar somewhere when we connect edges of the surfaces. As with the orientable case, the scar appears where we crochet the boundaries of the different components of a surface together. □

Remark 3. Something the observant reader might have noticed is that the crochet decompositions of the base cases of the orientable and non-orientable surfaces are oddly similar. Indeed, in the orientable case, when we crochet a sphere, we first increase, then crochet a few rounds without either increasing or decreasing, and then decrease (do note that in the middle we have the greatest amount of

[5]Here, we create the hole by crochet a 2 half-double crochets, then a treble crochet, chain 3 stitches. Then, we move back to the single crochets again after another crocheting a treble crochet and 2 half-double crochets.

Figure 23. Crocheted non-orientable surface of genus 4.

stitches, and then when we move to the north/south pole of the sphere, the amount of stitches decreases symmetrically). When we crochet the projective plane, we crochet a Möbius strip, which respresents the 'middle' with the greatest amount of stitches, and then we decrease evenly along the one boundary. Where we could contract to 'a single point' for the sphere, this is impossible in 3 dimensional space for the projective plane, which is why we end up with our representation minus a disk when we decrease. We can make a similar observation for the similarities between the decomposition of a torus and a Klein bottle.

Now, with the theorems above, we have proven that it is possible to crochet any surface, orientable and non-orientable. Moreover, we have provided explicit explanations of how the abstract mathematical proofs can be used to practically crochet the surfaces.

However, due to the nature of crocheting, we will always have a scar where we connect certain edges. If, for some reason, one would want to crochet any of the surfaces, with as few singularities as possible, instead of crocheting the edges together (like we did to clearly indicate *where* the scars occur), the edges could be sewn together. It still will not make the surfaces completely scarless but will significantly reduce the noticeability.

4. Discrete vs Continuous Differential Geometry

After the previous discussion about being able to crochet any type of surface, an observant reader might rightfully complain that we cheated a little bit. Indeed, while we assumed that our crocheted fabrics and surfaces are continuous, and have the properties of continuous objects, in reality we are dealing with discrete ones. Does this mean that the previous result is less useful than expected? Fortunately not. While indeed the surfaces discussed are not smooth and nice objects, we can instead regard the stitches as a triangulation of the surface. In this chapter we will illustrate some of the parallels between discrete and continuous differential geometry, and why the result we have shown in the previous chapter is not weakened by the surfaces being discrete.

The key ingredient of our justification is the Gauss-Bonnet Theorem. This theorem connects the curvature on a surface with the Euler characteristic.

The continuous Gauss-Bonnet theorem is as follows:

Theorem 4.1 (Gauss-Bonnet (Continuous)). *Suppose M is a compact two-dimensional Riemannian manifold with boundary ∂M, Let K be the Gaussian curvature of M, and let k_g be the geodesic curvature of ∂M. Then,*

$$\int_M K dA + \int_{\partial M} k_g \, ds = 2\pi\chi(M)$$

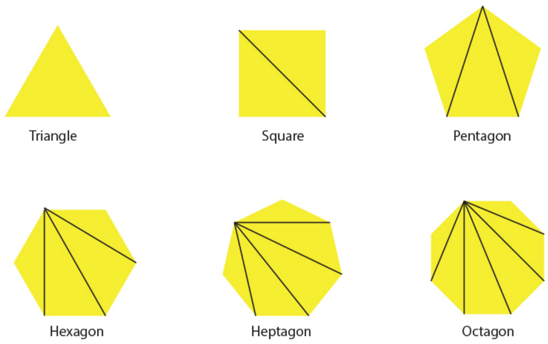
where dA is volume element which we integrate over and $\chi(M)$ denotes the Euler characteristic.

We will first look at the discrete counterpart of this theorem, and then return to continuous case to give a sketch of the proof and discuss the parallels between the discrete and continuous case. For our argumentation, it is not necessary to rigorously define the Gaussian and geodesic curvature. Fully diving into the continuous differential geometry needed for this will leave us at a significantly long discussion which will not add much to our understanding of the parallels between the continuous and discrete theorem. Therefore, we will omit this.

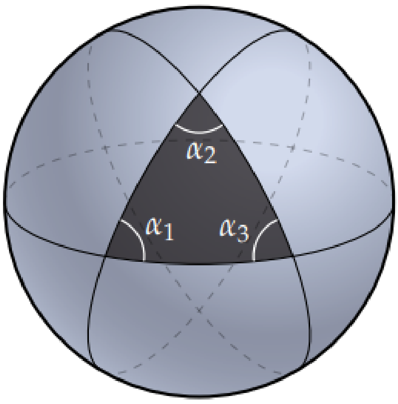
For the discrete case, we will need to introduce some definitions, notations, and intermediate results. We will follow the construction of the proof as given in [25]. While Crane gives the reader the intermediate steps, he leaves the actual proofs as exercises to the reader. We will fill in the gaps of the proofs to form a complete justification of the discrete Gauss-Bonnet theorem.

We start out with a discrete surface without boundary. Note that regardless of how the surface is discretized, we can always consider a triangulation of a surface.

Indeed, imagine that we have some polygon. Pick any vertex v of the polygon, and then draw lines between v and all the vertices that are not already connected to v. This way, for an $n-$gon, we will end up with triangulation of the $n-$gon of $n-2$ triangles. This procedure is shown for some polygons in figure 24a.



(A)



(B)

FIGURE 24. Polygons triangulated [26] and a spherical triangle [25].

For our crocheted surfaces, we can consider many different ways of triangulating it. Here, we will consider the triangulation as shown in figure 25.

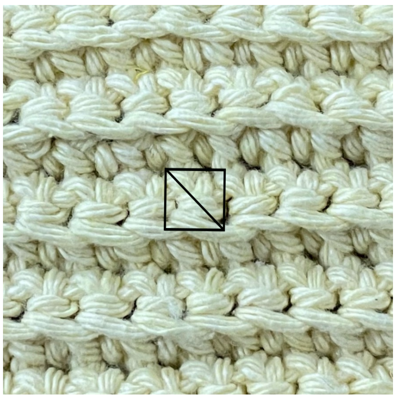


FIGURE 25. Triangulation of crocheted fabric.

We can actually describe this triangulation somewhat more formally. In fact, this triangulation of crocheted surfaces leads us to a concept called *simplicial surfaces*.

An abstract simplicial complex $\mathcal{K}$ is a rather intuitive way of describing discrete surfaces. The construction is easy enough, we start with labeling the vertices of our surface by $0, ..., n$, which we encompass in the set $V = \{0, 1, ..., n\}$. Then, from this set we form smaller subsets of $k+1$ elements, which declare which vertices form a k-simplex. We call k the degree of the simplex. This means that a triangle can be written as the 2-simplex $\{0, 1, 2\}$, a 1-simplex is a line $\{0, 1\}$ and vertices on their own are 0-simplices. Let us introduce some key concepts we will need.

Definition 4.1. *A* subcomplex *is a subset of the simplicial complex K which itself is also a simplicial complex.*

For example, we can consider 2 vertices and the corresponding edge within a triangle as a subcomplex of the triangle.

Definition 4.2. *A* pure k-simplicial complex *is a complex such that every simplex of the complex is contained in some simplex of degree k.*

Note that this means that the complex is made up out of a collection of k-simplices with fixed degree k. In particular, the triangulation of our crocheted surface is a pure $2-$simplicial complex.

Definition 4.3. *The star of a vertex v of a simplicial complex, denoted* $\mathrm{St(v)}$ *is the collection of all simplices $\sigma \in \mathcal{K}$ such that $v \in \sigma$.*

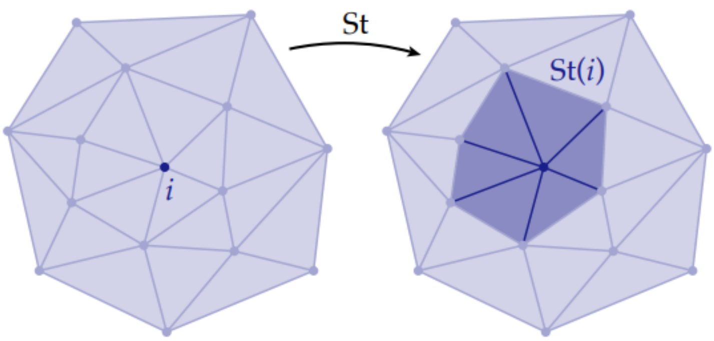


FIGURE 26. Star of a vertex i [25].

Definition 4.4. *An* abstract simplicial surface *is a pure simplicial $2-$complex where the star of every vertex is a combinatorial disk made of triangles.*

As we can see, our crocheted surface can be described as an abstract simplicial surface. Now, we can finally start with working our way to the discrete Gauss-Bonnet theorem.

The first result we need for the proof of the discrete Gauss-Bonnet Theorem is actually in a continuous setting.

Theorem 4.2. *The area A of a spherical triangle with interior angels $\alpha_1,\ \alpha_2,\ \alpha_3$ on a unit sphere is*

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi.$$

Proof. We want to compute the area as shown in figure 24b. Let us consider the areas of the sphere enclosed by each of the interior angles, A_1, A_2 and A_3 as shown in figure 27. We see that the shaded regions overlap exactly at the spherical triangle we want to compute, and at the area opposite to this triangle. Note that this second triangle can be characterized by drawing a line through the origin and the vertices of the spherical triangle, and then looking at their second intersection with the sphere. Connecting these points gives us the second triangle, with the same angles and area as A. This means that when we add up the areas of A_1, A_2 and A_3 to get the area of the entire sphere, we add up A 4 times too many. In other words,

$$4\pi r^2 = 4\pi = A_1 + A_2 + A_3 - 4A.$$

Now, note that $A_1 = 2 * (2 * \alpha_1 * r^2) = 4\alpha_1$. Similarly, $A_2 = 4\alpha_2$ and $A_3 = 4\alpha_3$. Then, we end up with

$$4\pi = 4\alpha_1 + 4\alpha_2 + 4\alpha_3 - 4A.$$

Rearranging the terms and dividing by 4 we end up with

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi,$$

which is the desired result. □

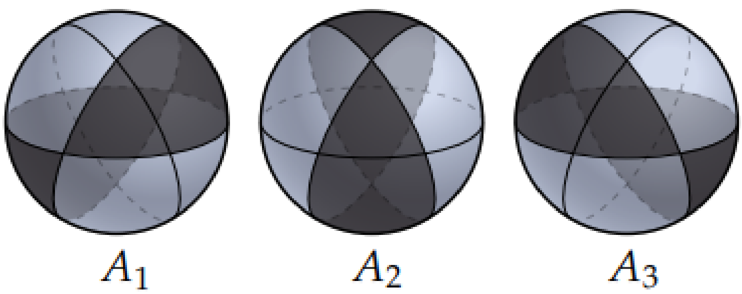


FIGURE 27. The areas of the sphere enclosed by each of the interior angles [25].

Theorem 4.3. *The area A of a spherical polygon with consecutive interior angles $\beta_1,\ \beta_2,\ \beta_3, \ldots,\ \beta_k$ is*

$$A = (2-k)\pi + \sum_{i=1}^{k} \beta_i.$$

Proof. Remember from figure 24a that any $n-$gon can be reconstructed into $n-2$ triangles. We can do the exact same thing for spherical triangles. Let us denote the area of these spherical triangles by $T_1,\ T_2, ...,\ T_{k-2}$. Then, we find that

$$A = \sum_{i=1}^{k-2} T_i.$$

Let $\alpha_{i_1},\ \alpha_{i_2},\ \alpha_{i_3}$ denote the angles of triangle T_i. Then, we see that

$$A = \sum_{i=1}^{k-2} (\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} - \pi).$$

Note that by the construction of the spherical triangles,

$$\sum_{i=1}^{k-2} (\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3}) = \beta_1 + \beta_2 + ... + \beta_k.$$

Combining the previous two equations we see that

$$A = \sum_{i=1}^{k} \beta_i - (k-2)\pi = (2-k)\pi + \sum_{i=1}^{k} \beta_i.$$

□

Now that we have these results, we need to introduce some definitions.

Definition 4.5. *The* discrete Gaussian curvature *at a vertex v is is the area on the unit sphere bounded by the spherical polygon whose vertices are the unit normals of the faces around v.*

While this definition may seem alien and technical, the image in figure 28 will clear up the concept.

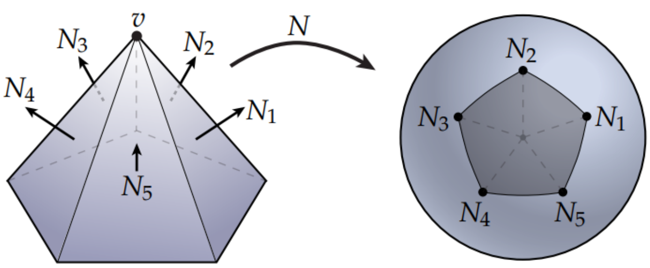


FIGURE 28. Gaussian curvature [25].

Definition 4.6. *The* angle defect $d(v)$ *is defined by*

$$d(v) = 2\pi - \sum_{f \in F_v} \angle_f(v),$$

where F_v *is the set of faces containing* v *and* $\angle_f(v)$ *is the interior angle of face* f *at vertex* v*.*

As it turns out, the above concepts are one and the same.

Theorem 4.4. *The discrete Gaussian curvature at a vertex* v *is equal to the angle defect* $d(v)$*.*

Proof. Let us consider the case where we take an arbitrary vertex v, with k distinct faces connected to it. Let us denote the unit normals on the k faces by $n_1, n_2, ..., n_k$ Then, it follows that for the Gaussian curvature, we have to compute the spherical n-gon created by the unit normals. Let us denote the angle between unit normal n_{i-1} and unit normal n_{i+1} by α_i for $1 < i < k$. Lastly, α_k and α_1 are defined as the angles between n_{k-1} and n_1 and n_2 and n_k respectively. Then, theorem 4.3 tells us that

$$A = (2-k)\pi + \sum_{i=1}^{k} \alpha_i = 2\pi + \sum_{i=1}^{k} \alpha_i - k\pi = 2\pi + \sum_{i=1}^{k} (\alpha_i - \pi). \tag{3}$$

What remains to be shown is that $\alpha_i - \pi$ equals the interior angle. Consider the situation we have as is depicted in figure 29, namely a triangular face f_i, with interior angle v_i at the vertex v. The edges AV and BV denote the shared edge between this face and faces f_{i+1} and f_{i-1} respectively. Note that we indicate the angle between AV' and BV' by α_i. Indeed, we can create a similar construction on face f_{i+1}. Then, let us call the parallel to AV', AW'. Then, AW' is also perpendicular to AV, i.e., the union of AV' and AW' is a straight line between n_i and n_{i+1}. The exact same holds for the face f_{i-1} and its intersection with f_i. When we elevate this construction to the unit sphere, we see that the angle of the spherical triangle at n_i is indeed the angle between AV' and BV'. Then, when we look at the quadrilateral, it is clear that

$$\alpha_i = 2\pi - \frac{1}{2}\pi - \frac{1}{2}\pi - v_i = \pi - v_i. \tag{4}$$

Reorganizing the terms we see that equation (4) becomes

$$A = 2\pi + \sum_{i=1}^{k} (\alpha_i - \pi) = 2\pi + \sum_{i=1}^{k} (-v_i) = 2\pi - \sum_{i=1}^{k} v_i = 2\pi - \sum_{f \in F(v)} \angle_f(v).$$

This is precisely what we wished to show. □

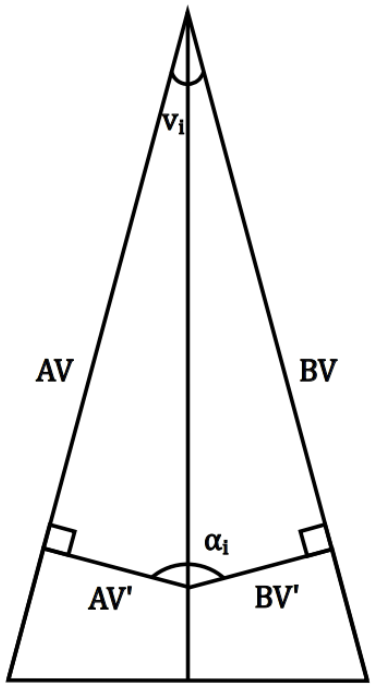


FIGURE 29

We can finally state, and prove, the discrete Gauss-Bonnet theorem for surfaces without a boundary.

Theorem 4.5 (Gauss-Bonnet without Boundary (Discrete))**.** *Consider a (connected, orientable) simplicial surface M with finitely many vertices V, edges E and faces F. We have that*

$$\sum_{v \in V} d(v) = 2\pi\chi(M)$$

where $\chi(M) = |V| - |E| + |F|$ is the Euler characteristic of the surface M.

Proof. Without loss of generality, let us assume that we have a simplicial surface. We see that

$$\sum_{v \in V} d(v) = \sum_{v \in V} \left((2 - k(v))\pi + \sum_{i=1}^{k(v)} \alpha_i \right) = 2\pi|V| - \sum_{v \in V} k(v)\pi + \sum_{v \in V} \sum_{i=1}^{k(v)} \alpha_i.$$

Note that $k(v)$ is determined by the amount of interior angles we have at a specific vertex v, hence the explicit dependence on v. Moreover, when we have $k(v)$ interior angles at v, we also have $k(v)$ edges that connect to v. As we sum over all $v \in V$, note that the total sum of $k(v)$ will be *twice* the total amount of edges $|E|$, as a single edge is connected to two vertices. Therefore

$$\sum_{v\in V} k(v)\pi = 2\pi|E|.$$

Now that we have expressed the first two sums in terms of $|V|$ and $|E|$, we now wish to express the last sum in terms of $|F|$. Using equation (4) from the proof of theorem 4.4, we write

$$\sum_{v\in V}\sum_{i=1}^{k(v)} \alpha_i = \sum_{v\in V}\sum_{i=1}^{k(v)} (\pi - v_i) = \sum_{v\in V}\sum_{f\in F_v} (\pi - v_f),$$

where for the last equation, we related the number of interior angles $k(v)$ with the faces F_v at the vertex v, and called the corresponding interior angles v_f. Since these are finite sums, we can interchange the them to obtain

$$\sum_{v\in V}\sum_{i=1}^{k(v)} \alpha_i = \sum_{f\in F}\sum_{v\in V_f} (\pi - v_f),$$

where V_f denotes the set of vertices of the face f. Note now that

$$\sum_{v\in V_f} (\pi - v_f) = \sum_{v\in V_f} \pi - \sum_{v\in V_f} v_f = 3\pi - \pi = 2\pi,$$

as each of our faces is a triangle, so we have 3 vertices connected to each face, and the 3 interior angles of a triangle sums to π as well. Then, we see that

$$\sum_{f\in F}\sum_{v\in V_f} (\pi - v_f) = \sum_{f\in F} 2\pi = 2\pi|F|.$$

Finally, putting all this together, we see that (see definition 3.1)

$$\sum_{v\in V} d(v) = 2\pi|V| - 2\pi|E| + 2\pi|F| = 2\pi\chi(M),$$

showing us the desired result. $\square$

The final step is stating and proving the discrete Gauss-Bonnet Theorem for surfaces with a boundary.

Theorem 4.6. *Consider a connected simplicial surface M with finitely many vertices V, edges E and faces F. We have that*

$$\sum_{v\in M_{int}} d(v) + \sum_{v\in\delta M} \alpha_{ext} = 2\pi\chi(M)$$

where $\chi(M) = |V| - |E| + |F|$ is the Euler characteristic of the surface M, and $\alpha_{ext} = \pi - \sum_{f\in F_v} \angle_f(v)$.

Proof. We start the proof by considering two copies of M, and gluing these together along the boundary to create the surface M'. This way, we have created a surface without boundary. The angle defect for the interior vertices remains the same. When we take a look at the vertices which were boundary vertices before we glued them together, we see that the angle defect at these vertices is twice the exterior angle. Then, theorem 4.5 tells us that

$$2\pi\chi(M') = \sum_{v\in M'} d(v) = 2\sum_{v\in M_{int}} d(v) + 2\sum_{v\in\partial M} \alpha_{ext}.$$

Dividing both sides by 2, we obtain

$$\pi\chi(M') = \sum_{v\in M_{int}} d(v) + \sum_{v\in\partial M} \alpha_{ext}. \tag{5}$$

Note that all that remains to be shown now, is that $\chi(M') = 2\chi(M)$. From [27], [28], we have that

$$\chi(M') = \chi(M) + \chi(M) - \chi(\partial M) = 2\chi(M) - \chi(\partial M). \tag{6}$$

Now, note that on the boundary of M, the number of edges must be equal to the number of vertices. Therefore, the boundary has Euler-Poincaré characteristic $\chi(\partial M) = 0$. Then, equation (6) is reduced to

$$\chi(M') = 2\chi(M).$$

It follows that equation (5) becomes

$$2\pi\chi(M) = \sum_{v\in M_{int}} d(v) + \sum_{v\in\partial M} \alpha_{ext},$$

proving the theorem. □

When we compare the continuous Gauss-Bonnet theorem 4.1 and the discrete theorem 4.5, we see that for each, there is a term for the interior of the surface, a term for the boundary, and the Euler-Poincaré characteristic. This is, of course, no coincidence. We will not be proving the entirety of the continuous Gauss-Bonnet theorem here, as we would need to introduce a lot of preliminaries from differential geometry, but we would rather focus on a sketch to convince the reader that we are justified in treating our discrete objects as 'continuous' in some sense.

Sketch of the proof of Theorem 4.1. The proof for the continuous Gauss-Bonnet Theorem actually consists of 2 main parts; the first is the *local* Gauss-Bonnet theorem, and the second part is the *global* Gauss-Bonnet theorem. As these names suggest, the first concerns itself with only a subset U of the entire surface S, whereas the second one gives the theorem for the entire surface S, and uses the local theorem.

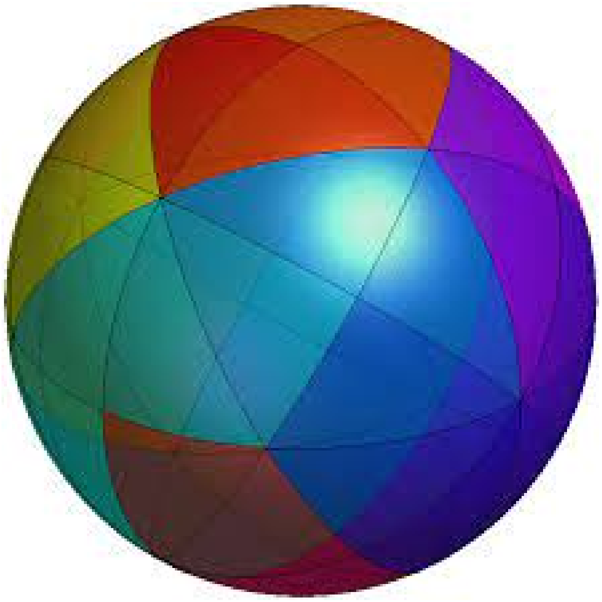


FIGURE 30. Some triangulation of a sphere [29].

Let us first discuss the *global theorem*. The main idea here is to triangulate the surface and use the local Gauss-Bonnet theorem on all these smaller subsets. As an example, consider a sphere with a triangulation, as seen in figure 30.

Then, at each of the subsets U created by the triangulation, we apply the local Gauss-Bonnet theorem. This tells us that, at each subset, we have

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(M)ds + \int\int_U K\ dA + \sum_{i=0}^{k} \theta_i = 2\pi.$$

Here, we let the boundary of the subset ∂U be parameterized by a curve α. We can see from the picture that the boundary is not smooth. The angles between the curves that make up α are denoted by θ_i. This explains the final term in the result of the local Gauss-Bonnet theorem.

The next step is adding up the results from applying the local theorem to all the subsets produced by the triangulation. This results in the following equality:

$$\int_{\partial M} k_g(M)ds + \int\int_M KdA + \sum_{j,\ k=1}^{F,\ 3} \theta_{jk} = 2\pi F,$$

where F denotes the number of triangles and $\theta_{j1},\ \theta_{j2}$ and θ_{j3} are the *external angles* of the triangle T_j. This third term on the left-hand side is proven to be equal to $2\pi E - 2\pi V$. This is done by counting the interior and exterior edges of the triangulation in a smart way, as well as the fact that the sum of angles around each internal vertex is 2π.

Moving the terms involving E and V from the left-hand side to the right-hand side then gives us the continuous Gauss-Bonnet theorem.

For the *local Gauss-Bonnet theorem*, the proof relies on the theorem of turning tangents, which states that a closed loop on a surface turns by 2π. We pick one of the subsets U given by the triangulation. Let $\mathbf{x}$ denote an appropriate parameterization of the subset, and choose α such that it is a parameterization of the boundary ∂U by its arc length s. Lastly, we denote by $\alpha(s_0), ..., \alpha(s_k)$ the vertices of α, and $\theta_0, ..., \theta_k$ the external angles at these vertices.

First, we have a sum of integrals of the geodesic curvature over small segments of the boundary,

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) ds.$$

Then, the geodesic curvature is rewritten in a specific way, so the Gauss-Green theorem[6] can be applied. This, as well as rewriting the integral, transforms the expression into

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(M) ds = -\int\int_U K\lambda \, dA + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\phi_i}{ds} ds$$

where the ϕ_i are differentiable functions which measure the positive angle from $\mathbf{x}_u$ to α' in $[s_i, s_{i+1}]$, and λ is a function describing the metric.

The theorem of turning tangents, combined with the fact that α is positively oriented, tells us that

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\phi_i}{ds} ds = 2\pi - \sum_{i=0}^{k} \theta_i.$$

[6]The Gauss-Green theorem states that if $P(u, v)$, $Q(u, v)$ are differential functions in a simple region $A \subset \mathbb{R}^2$, the boundary of which is given by $u = u(s)$, $v = v(s)$, then

$$\sum_{i=1}^{k} \int_{s_i}^{s_{i+1}} \left(P\frac{du}{ds} + Q\frac{dv}{ds}\right) ds = \int\int_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) [17].$$

Then, reshuffling the terms gives us the desired result, namely

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(M)ds + \int\int_U K \ dA + \sum_{i=0}^{k} \theta_i = 2\pi.$$

The full proofs of these theorems can be found in [17].

Note that in the sketch above, we split up the problem twice. We once split it on a global level in the triangulation of the surface, and then on a local level by splitting up the boundary in segments, and working over these little segments. Remember that a Riemannian integral is nothing but an infinite sum.

This construction of the proof very clearly shows the parallels between the discrete and continuous Gauss-Bonnet theorems. We see that we can relate the curvature over the surface, an analytic property of continuous manifolds, to the Euler-Poincaré characteristic, a topological property linked to the triangulation of the surface. Since taking the discrete surface to infinity does not change the Euler-Poincaré characteristic, this in particular shows that we were justified in using continuous results, as the discrete surfaces share the same topological invariant as their continuous counterparts.

5. Creating a Crocheted Surface from the Metric

In section 3, we have seen that we can crochet topological surfaces up to adding some boundary. This chapter will focus on creating crocheted fabric from a metric. Phrased differently, given a surface, and a metric describing said surface, we wish to investigate how we can crochet this. This section is loosely based on slides of a talk about *Geometric knitting* by Hugh Griffiths [30]. We have taken the main ideas, but modified them to apply for crochet. Moreover, we will give and work out several examples to explain the, somewhat abstract, procedures. We will first need to introduce some theory on metrics.

The natural inner product on a surface $S \subset \mathbb{R}^3$ induces on each tangent plane at a point p, $T_p(S)$, an inner product $\langle ., . \rangle_p$. Let g be the metric on the surface. The inner product is computed as follows. If $w_1,\ w_2 \in T_p(S)$, then $\langle w_1, w_2 \rangle = g(w_1, w_2)$ is computed as the inner product of w_1 and w_2 as vectors in $\mathbb{R}^3$ [7]. We know by the *polarization identity* that we can characterize the inner product by its diagonal,

$$||w|| = \sqrt{\langle w, w \rangle}.$$

This leads us to the definition *first fundamental form* of the surface S at the point p.

Definition 5.1. *The quadratic form $I_p : T_p(S) \to \mathbb{R}$ given by*

$$I_p(w) = \langle w, w \rangle = ||w||^2,$$

is called the first fundamental form *of the regular surface $S \subset \mathbb{R}^3$ at the point $p \in S$.*

This fundamental form tells us how the surface S inherits the natural inner product on Euclidean spaces. Now, note that when we have a parameterization $x(u, v)$ of the surface at p, $T_p(S)$ has a basis given by $\{x_u, x_v\}$. We know that we can see an element $w \in T_p(S)$ as the tangent vector to some parameterized curve α through $p \in S$. Let us write this parameterized curve as

$$\alpha(t) = x(u(t), v(t))$$

for $t \in (-\epsilon, \epsilon)$, with $\alpha(0) = (u(0), v(0)) = p$. Then, we can compute the first fundamental form.

$$\begin{aligned} I_p(w) = I_p(\alpha'(0)) = I_p(x_u(0)u'(0) + x_v(0)v'(0)) &= \langle x_u u' + x_v v', x_u u' + x_v v' \rangle_p \\ = \langle x_u u', x_u u' \rangle_p + 2\langle x_u u', x_v v' \rangle_p + \langle x_v v', x_v v' \rangle_p & \\ = (u'(0))^2 \langle x_u, x_u \rangle_p + u'(0)v'(0)\langle x_u, x_v \rangle_p + (v'(0))^2 \langle x_v, x_v \rangle_p. & \end{aligned}$$

[7] Remember that the standard inner product on Euclidean space is the dot product.

Then, by setting $E(u(0), v(0)) = \langle x_u, x_u \rangle_p$, $F(u(0), v(0)) = \langle x_u, x_v \rangle_p$ and $G(u(0), v(0)) = \langle x_v, x_v \rangle_p$, the first fundamental form reduces to

$$I_p(w) = E(u'(0))^2 + Fu'(0)v'(0) + G(v'(0))^2.$$

The first fundamental form gives us a convenient way to compute inner products, and as such, characterizes the metric on the surfaces we are interested in. This form will also help us greatly once we move to crocheting surfaces.

But how can we relate this mathematical concept to crochet? Let us explain this with an example.

Consider a cylinder of radius 1, centered at the origin. Let us call this surface U. Then, we can find a parameterization[8] $x : U \to \mathbb{R}^3$ given by

$$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 1,\ 0 \leq v \leq 10\}.$$

and

$$x(u, v) = (\cos\ 2\pi u,\ \sin\ 2\pi u,\ v).$$

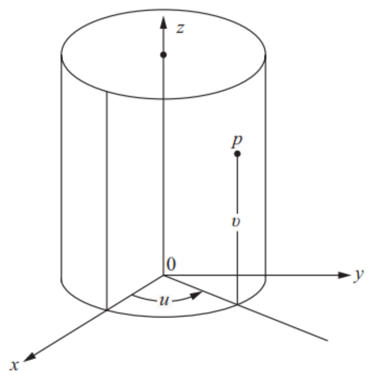


FIGURE 31. Illustration of a cylinder [17].

Computing the first fundamental form, we see that

$$x_u = (-2\pi \sin\ 2\pi u, 2\pi \cos\ 2\pi u, 0)$$
$$x_v = (0, 0, 1).$$

[8]For this construction to work, we want a parameterization where one coordinate ranges between 0 and 1, while we want the other coordinate to range from 0 to some integer. This is because, for the first coordinate, we wish to compute an integral, which is easier when the integral in computed over the interval 0 to 1, but not strictly necessary. For the second coordinate, we want the range to be 0 to some integer as we can only crochet integer rows.

Then, we see that

$$\begin{aligned}
E &= \langle x_u, x_u \rangle_p = |x_u|^2 = 4\pi^2 \sin^2 2\pi u + 4\pi^2 \cos^2 2\pi u = 4\pi^2,\\
F &= \langle x_u, x_v \rangle_p = 0,\\
G &= \langle x_v, x_v \rangle_p = |x_v|^2 = 1.
\end{aligned}$$

Thus, all in all, our first fundamental form looks like

$$I_p(w) = 4\pi^2 (u'(0))^2 + (v'(0))^2.$$

Now, we go back to crocheting. Imagine that we have a crocheted cylinder. Notice that u describes in which column we are, whereas v tells us in which row we are. In particular, we can compute the length L of our rows:

$$L(v) = \int_0^1 \sqrt{\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle} du = \int_0^1 \left| \frac{\partial}{\partial u} \right| du = \int_0^1 4\pi^2 \; dv = 4\pi^2 = E(u, v).$$

This now gives us the information necessary to crochet the surface. Note that as $L(v)$ is identically $4\pi^2$ for all v, our rows neither increase nor decrease. This is also exactly what we would expect from a surface describing a cylinder; when we cut open the cylinder, we end up with a rectangle.

The crocheting procedure is as follows. As we have that $0 \leq v \leq 10$, we will crochet 11 rows, indicated by $v = 0,\; v = 1, ..., v = 10$. We start by crocheting the first row, which consists of $L(0) = 4\pi^2 \approx 39.478$ stitches. As the number of stitches need to be integer numbers, we crochet 39 stitches. Then, for the second row, we see that, again, the number of stitches we should crochet is $L(1) = 39$. We continue in this fashion up until $v = 10$. We now have a rectangular piece of fabric which can be sewed together to form the surface of the cylinder.

Remark 4. In this construction, we have assumed that we are working with single crochet stitches. These have the property that they are as wide as they are tall. If we were to crochet with different stitches, we would need to add in an appropriate constant when determining how many stitches to crochet in a next row. If a stitch is $\frac{a}{b}$ wider than they are tall, then we multiply the numbers found in the procedure above by $\frac{b}{a}$ in each row to compensate for this. For example, say we wish to crochet a square of length 6, but our stitches are twice as wide as they are tall. Then, if we crochet 6 rows and 6 columns, we will end up with a rectangle of 12 by 6. Instead, we need to multiply the amount of stitches we crochet in the rows by $\frac{1}{2}$ to end up with a square of 6 rows as 3 columns.

An important part has not come into play yet. As the cylinder remains of the same radius, we have no increases or decreases. If we have a surface that *does* have increases or decreases, the crochet procedure becomes slightly more complicated.

Let us imagine that in row $n+1$, we change the amount of stitches we need to crochet. This can be seen via computing $E(u,n+1) - E(u,n)$, where we subtract the amount of stitches in row n from the amount of stitches in row $n+1$. If this quantity is *positive*, then we increase with $\frac{b}{a}[E(u,n+1) - E(u,n)]$ many stitches. However, if the quantity is *negative*, we decrease with $|\frac{b}{a}[E(u,n+1) - E(u,n)]|$ many stitches. If the quantity is 0, we neither increase nor decrease, as is to be expected.

With these ideas in mind, let us turn to another example, now with increases.

Let us consider a *helicoid* H with its parameterization $x : H \to \mathbb{R}^3$ given by

$$x(u,v) = (v\cos 2\pi u, v\sin 2\pi u, a2\pi u)$$

where a is the radius of the helicoid, and $0 \leq u \leq 1$, and $0 \leq v \leq 10$.

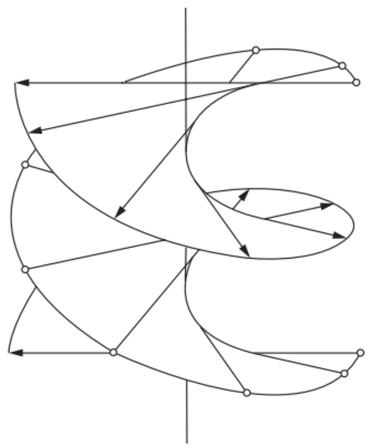


FIGURE 32. Illustration of a helicoid [17].

Let us compute the partial derivatives.

$$\begin{aligned} x(u,v)_u &= (-2\pi v\sin 2\pi u, 2\pi v\cos 2\pi u, a2\pi) \\ x(u,v)_v &= (\cos 2\pi u, \sin 2\pi u, 0). \end{aligned}$$

We can now compute E, F and G. We see that

$$\begin{aligned} E &= |x_u|^2 = 4\pi^2 v^2 \sin^2 2\pi u + 4\pi^2 v^2 \cos^2 2\pi u + a^2 4\pi^2 = 4\pi^2(v^2 + a^2) \\ F &= \langle x_u, x_v \rangle = -2\pi v \sin 2\pi u \cos 2\pi u + 2\pi v \cos 2\pi u \sin 2\pi u + 0 = 0 \\ G &= |x_v|^2 = \cos^2 2\pi u + \sin^2 2\pi u = 1. \end{aligned}$$

Then, we compute once again the length of our rows:

$$L(v) = \int_0^1 \sqrt{\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle} du = \int_0^1 \left| \frac{\partial}{\partial u} \right| du = \int_0^1 4\pi^2(v^2 + a^2) du = 4\pi^2(v^2 + a^2).$$

We see that $L(v) = 4\pi^2(v^2 + a^2)$, so as we crochet our rows, we increase in each row by $4\pi^2(v+1)^2 - 4\pi^2 v^2 = 4\pi^2(2v+1)$ stitches. For example, in our first row, $v = 0$, we crochet $L(0) = 4\pi^2 a^2$ stitches. Then, in the second row, we crochet $L(1) = 4\pi^2(1 + a^2)$ stitches, so we increase by an amount of $4\pi^2 = 4\pi^2(2 \cdot 0 + 1)$ stitches. We continue onward in this fashion until we have reached and finished our eleventh row at $v = 10$.

One thing these two examples above have in common is that $G = 1$. When this is the case, this implies that $v'(0)$ does not change, i.e., *the stitches do not change in height, only the rows change in length.* The construction explained above will work for all such surfaces, provided we find the correct parameterization. However, we can expand on these ideas to also include cases where both G and E can be function of both u and v (we still want F to be 0 so that the parameterization is orthogonal). This is slightly more involved, but the main ideas do not change. Instead of looking at the surface row by row, and seeing how much we should increase/decrease, we instead look at the surface on a bit more detailed level. We look, still row by row, how much we should increase or decrease, but also stitch by stitch if the stitches should shrink or grow in height.

We start, as before with the first row. We crochet

$$L(0) = \int_0^1 E(u, 0) du$$

stitches. Note that we do nothing different then before, the only change is that E now more explicitly depends on u. Once we have computed this and crocheted the stitches, we let the stitches we just made represent the points $(u_k, 0) = (\frac{k}{n}, 0)$, and we organize them in a list. Then, for each stitch we compute

$$\int_0^{u_k} E(u, v) du - \int_0^{u_{k-1}} E(u, v) du - 1. \tag{7}$$

This tells us where to increase or decrease along the row. We can see that this is truly the case in the following way. If at the point u_k and u_{k+1} integrals are the same, then this implies that in between u_k and u_{k-1} there is no extra space, so we crochet these two stitches together. However, if the difference of the integrals is 1 (so the entire expression is 0), this means that for the stitch at u_k, we find that it has width 1. This clearly implies we neither increase nor decrease. Then, lastly, if the difference is 2, then we have a width of 2 stitches at the stitch u_k. Thus, we increase by 1 in the stitch u_k. Once we have increased/decreased, we directly need to alter the procedure to make sure that, in the case of a decrease, the point is removed from the list where we store our points, and in the case of an increase, an extra point is added between in the list between u_k and u_{k-1}.

Once we have done this, we can start with computing the change in length. For this, we compute the following expression at each stitch:

$$\int_0^1 G(u,v)dv - 1. \tag{8}$$

Note that if we are in row 3, then the integral ranges from 3 to 4 instead of 0 to 1. The procedure is the same as for the change in width. If the expression is 0, then the length of the stitch is the 'expected' length of the stitch so we do not do anything. If the expression is -1, then the stitch adds no length at that position. If the expression is 1, then the stitch adds the length of 2 stitches. Now that we computed both the change in width and length for all the stitches, we can crochet the row. Once we have done this, we continue on the second row by again computing the change in width and length as we did for the first row. We continue this until we have crocheted all our rows.

Remark 5. Something to note here is that, generally, when we compute the integral expressions, equation (7) and equation (8), we will not end up with a 1 or -1, perfectly corresponding to an increase/decrease. We can have 2 options here. The first being that we can approximate for the change of width and length of the stitches to which integer the expressions are the closest, and work like this. Then, increasing and decreasing within a row are as we did before, while increasing and decreasing within the columns is a bit more involved. The most straightforward way to change the length of the stitches, is to change the stitch. If we need to increase the length, we can make a *double crochet* instead of a single crochet, as the double crochet stitch is approximately twice as tall as the single crochet stitch. If we need to decrease in the length of a stitch, the most straightforward thing to do is *to add a slip stitch on the side of the previous stitch.*This makes sure that we do not add any extra length to the stitch, but do end up in the correct position to crochet into the next stitch again. Note that if we just skip the stitch, instead of a stitch without length, we get a hole. Another possibility would be the following approach. We do not approximate the changes of width and length to the nearest integer, but instead let them remain the way they are. Then, as crochet lets us change the size of the crochet hook very easily, we pick the crochet hook such that the length of the stitch is the closest to the computed length. (Of course, to do this, one would first need to crochet some samples with the hooks of different sizes to gauge when to use which hook).

Remark 6. While these constructions above explain mathematically how we could crochet the surfaces, in practice this might be much more complicated. For some surfaces, the procedures might not be suitable for any number of reasons. For example, if we have a surface where a stitch needs to decrease in length by *more than* 1 we run into some issues, as we cannot decrease into a previous stitch. While we could to some degree get rid of this issue by doubling the total amount of rows, the general shortcoming persists (what if we need to decrease 17 stitches in length? do we multiply the amount of rows by 17? This is generally not very useful or even possible). However, for surfaces that do not change in ways that are 'too strange' what we described above is a good way to roughly crochet a surface.

Let us consider a final example, the sphere V. We parameterize V in the following way $x : V \to \mathbb{R}^3$ defined by

$$V = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \theta < 1,\ 0 < \phi < 10\}$$

and

$$x(\theta, \phi) = (\sin \pi\theta \cos \frac{\pi}{5}\phi, \sin \pi\theta \sin \frac{\pi}{5}\phi, \cos \pi\theta).$$

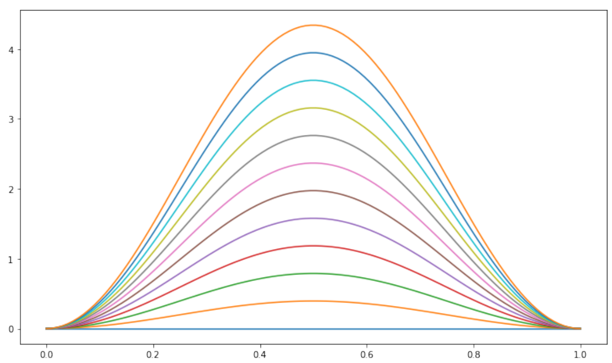
We compute that

$$\begin{aligned} x_\theta(\theta, \phi) &= (\pi \cos \pi\theta \cos \frac{\pi}{5}\phi, \pi \cos \pi\theta \sin \frac{\pi}{5}\phi, -\pi \sin \pi\theta) \\ x_\phi(\theta, \phi) &= (\theta, \phi) = (-\frac{\pi}{5} \sin \pi\theta \sin \frac{\pi}{5}\phi, \frac{\pi}{5} \sin \pi\theta \cos \frac{\pi}{5}\phi, 0). \end{aligned}$$

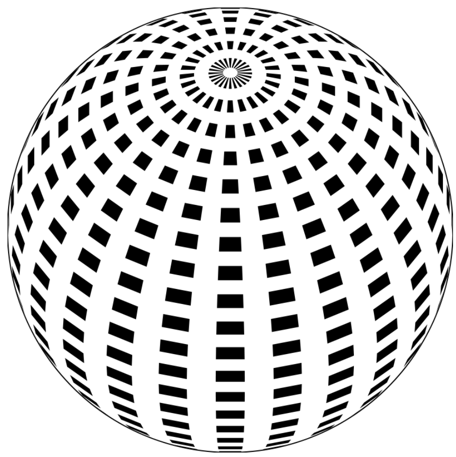
Now, we compute $E,\ F$ and G:

$$\begin{aligned} E &= \langle x_\theta, x_\theta \rangle = \pi^2 \cos^2 \pi\theta \cos^2 \frac{\pi}{5}\phi + \pi^2 \cos^2 \pi\theta \sin^2 \frac{\pi}{5}\phi + \pi^2 \sin^2 \pi\theta \\ &= \pi^2 \cos^2 \pi\theta (\cos^2 \frac{\pi}{5}\phi + \sin^2 \frac{\pi}{5}\phi) + \pi^2 \sin^2 \pi\theta = \pi^2 \cos^2 \pi\theta + \pi^2 \sin^2 \pi\theta = \pi^2, \\ F &= \langle x_\theta, x_\phi \rangle = -\frac{\pi^2}{5} \cos \pi\theta \cos \frac{\pi}{5}\phi \sin \pi\theta \sin \frac{\pi}{5}\phi + \frac{\pi^2}{5} \cos \pi\theta \cos \frac{\pi}{5}\phi \sin \pi\theta \sin \frac{\pi}{5}\phi + 0 \\ &= 0, \\ G &= \langle x_\phi, x_\phi \rangle = \frac{\pi^2}{25} \sin^2 \pi\theta \sin^2 \frac{\pi}{5}\phi + \frac{\pi^2}{25} \sin^2 \pi\theta \cos^2 \frac{\pi}{5}\phi + 0 \\ &= \frac{\pi^2}{25} \sin^2 \pi\theta (\sin^2 \frac{\pi}{5}\phi + \cos^2 \frac{\pi}{5}\phi) = \frac{\pi^2}{25} \sin^2 \pi\theta. \end{aligned}$$

Now, how do we interpret this result? For E, the value remains constant at $\pi^2 \approx 9.7$. We know that this implies that we do not increase or decrease along the rows. Then, when we look stitch by stitch, we see that in the first stitch (where $\theta_0 = 0$) we have a stitch of no height. As we are dealing with a sin^2 function, we also see that at the fifth stitch where $\theta_5 \approx \frac{1}{2}$, we reach the maximum of sin^2, i.e., we have the maximum height of the stitch, which is $\frac{\pi^2}{25}$. Then, we start decreasing again, and at the tenth stitch, the height of the stitch will be 0 once more. This procedure gets repeated for 10 rows. Then, at this point, we have 11 rows of 10 stitches each, where in every row, the stitches at the end points have no length, and in the middle the highest length. A sketch of this procedure is shown in 33a.



(A)



(B)

FIGURE 33

Now, as the rows have the same amount of stitches, we can easily connect the 2 edges. While this might seem counter-intuitive from figure 33a as the seem curves become longer, the reader should convince themselves that the rows are all the same length as each consists of 10 stitches, and that the figure is a limitation of representing three-dimensional objects onto a two-dimensional page. The extra space we have created by increasing the stitches in the middle will be pushed outward to form the sphere.

We can also explain why this work in another, maybe more familiar, way. When we consider figure 33b [31], we can notice some things. First off, the closer we get to the North (and South) pole, the thinner the rectangles are. However, their length does not change. Now, we can pick any one vertical 'strand' of rectangles on the sphere. Note that all these strands are identical. Then, when we take a closer look at one such strand, we see that at the end points, the rectangles are incredibly thin, while in the middle the rectangles are wider. This is exactly what we mimicked with our crocheting before! As such, when we have an inner product on our surface, and as long as we can find an orthogonal parameterization of it, we can create a crocheted version of the surface.

6. Conclusion

It is now clear that any surface can be crocheted as per the discussion in section 3. We have proved that, up to homeomorphism, any orientable and non-orientable surface can be crocheted. While for the orientable case, the induction hypothesis is quite straightforward, the non-orientable case requires a bit more thought about how to properly connect the boundaries. The proof was largely reconstructed and modified from the proof for knitting by Sarah-Marie Belcastro. While knitting and crocheting are fundamentally different ways to create textiles, we were able to modify the theorem and proofs for crochet. The price we have to pay for this modification of the proof for crochet is the scar that is created when we connect the edges of the crocheted fabrics to create the surfaces. Interestingly, this scar illustrates the local nature of charts on a manifold. Section 3 led us into a discussion about the Gauss-Bonnet theorem in section 4, where we justified talking about crocheted surfaces as continuous objects, even though they are discrete. Lastly, we showed in section 5 that, given a particular class of *conformal metrics*[9], we can crochet a surface that somehow preserves the properties of the metric. However, there are some questions remaining. A question that came up, but that we could not deliberate extensively due to time constraints, is if we can find an algorithm when $F \neq 0$, i.e., when the surface also changes in the $u'v'$ direction, and what this would practically mean for crocheting. Moreover, the parameterization for the algorithm to work needs to be of a very particular form as we have seen. Another extension could be considered for more relaxed restrictions on the parameterization of the surface.

Another topic would be the automation of crochet. As of now, no crocheting machines exist (in contrast, sewing machines as well as knitting machines have existed for quite some time now). In particular, no machinery exists that is able to imitate the transverse chains that crochet uses. As the absence of the existence of such a machine does not mean it cannot be done, interesting further research would be to prove that such a machine indeed cannot exist, or instead to prove or make a machine that can crochet.

[9] Conformal metrics are precisely the metrics with diagonal quadratic form as we discussed. In the quadratic form, $F = 0$ for these metrics. In particular, this means that the metrics preserve angles.

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