university of groningen

faculty of science and engineering

mathematics and applied mathematics

Reidemeister torsion and the classification of lens spaces

Bachelor's Project Mathematics

July 2022

Student: L. Taams

First supervisor: dr. R.I. van der Veen

Second assessor: dr. M. Seri

Abstract

Given coprime integers p and q we define the lens space $L(p,q)$ as a quotient of S^3. A lens space $L(p,q)$ is a 3-manifold with fundamental group isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We show that two lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic if and only if $q' = \pm q^{\pm 1} \mod p$. The sufficient condition is shown by constructing homeomorphisms between the lens spaces. The necessary condition is shown using a technique called Reidemeister torsion, which is a topological invariant. We give concrete calculations of Reidemeister torsion for the circle, the torus and the lens spaces. This quantity is then used to show the necessary condition of the classification. Lastly we show that only the fundamental group and the first homology group depend on the parameter p and that the higher homotopy are equal to those of S^3 and the higher homology groups do not depend on p and q.

Contents

Introduction

Lens spaces are a class of three manifolds that played a particularly important role in the history of algebraic topology. The classification lens spaces gave the first examples of 3-manifolds that are homotopy equivalent but not homeomorphic, thus showing shortcomings of algebraic topology.

Lens spaces were first described in detail by H. Tietze [7]. He described them in a different way than in this thesis. He defined them by identifying points on the boundary of a three dimensional ball. Later he also described them by gluing two solid tori along their boundaries in different ways.

Given coprime integers p and q. We define lens space $L(p, q)$ as a quotient by an action of the pth roots of unity on the 3-sphere embedded in $\mathbb{C}^2$. The question this thesis will answer is classifying when a different set of coprime integers p' and q' yields a lens space homeomorphic to the lens space $L(p, q)$. Meaning we will find necessary and sufficient conditions on the integers p, q, p' and q' so that the corresponding lens spaces are homeomorphic. Assuming correct conditions on these integers, we will construct homeomorphisms between the corresponding lens spaces, which proves half of the classification. The other half of the classification requires more work. Unfortunately the techniques of algebraic topology like the fundamental group, the homology groups or the higher homotopy groups are of little help. For this we have the notion of Reidemeister torsion of a topological space. With the help of Reidemeister torsion, we will show the necessary conditions on the lens spaces being homeomorphic.

A CW-complex is a type of topological space constructed inductively by gluing higher dimensional balls to lower dimensional CW-complexes, starting with a set of discrete points. The Reidemeister torsion of such a space is a quantity that is says something about the way the higher dimensional parts of the space are twisted when attached to the lowers. It is in some way a generalization of a determinant applied to the maps that attach higher dimensional parts to the lower dimensional parts.

Reidemeister was the first one to completely classify the lens spaces up to so called piecewise linear homeomorphisms. Franz later generalized his method using the now called Reidemeister torsion or Reidemeister-Franz torsion. Even later J.H.C Whitehead solved the homotopy classification of the lens spaces. He proved necessary and sufficient conditions on lens spaces for them to be homotopy equivalent.

This thesis uses many results from algebraic topology. Mainly results regarding covering spaces, specifically the universal covering of space and actions of the fundamental group of a space on its covering space.

1 Definition and basic properties of lens spaces

A lens space is defined by taking the quotient of a group action defined on S^3. Recall the definition of a group action.

Definition 1.1 (Group action)**.** Let X be a topological space and G a group. An action of G on X is a map Such that

- $1x = x$ for all $x \in X$,
- $g(hx) = (gh)x$ for all $x \in X$ and $g, h \in G$,
- the map $x \longmapsto gx$ is a homeomorphism for all $g \in G$.

The quotient space X/G of X with respect to the action of G is defined as the set of orbits under the action of G together with the quotient topology defined by the quotient map $x \longmapsto [x]$, where $[x]$ denotes the orbit of $x \in X$.

We embed the 3-sphere S^3 in the standard way in $\mathbb{C}^2$, so $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$. Let p and q be coprime integers. Let ζ be a generator of the pth roots of unity. Then the map $(z, w) \mapsto (\zeta z, \zeta^q w)$ is an automorphism of S^3, because $|\zeta| = 1$. The set of pth roots of unity is a group isomorphic to $\mathbb{Z}/p\mathbb{Z}$. It is easily seen that this map has order p. Thus the action generated by the map $\zeta(z, w) = (\zeta z, \zeta^q w)$ is a $\mathbb{Z}/p\mathbb{Z}$-action on S^3.

Definition 1.2 (Lens space)**.** Let $p, q \in \mathbb{Z}$ be coprime integers. We let $\mathbb{Z}/p\mathbb{Z}$ act on S^3 by the action defined above. The lens space $L(p, q)$ is then defined as the quotient space with respect to this action.

We pick the integers p and q to be coprime in order to have that the corresponding action is free. This yields nice properties namely that the lens space $L(p, q)$ is a manifold and that the quotient map $S^3 \to L(p, q)$ is a (universal) covering.

The main question here is that given coprime p, q and coprime p', q', whether or not the lens spaces $L(p, q)$ and $L(p', q')$ are homeomorphic or whether or not they are homotopy equivalent. The first observation we will make is that the fundamental group of a lens space $L(p, q)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. This means that for lens spaces $L(p, q)$ and $L(p', q')$ that they are homeomorphic or homotopy equivalent only if $p = p'$. Hence from now on we always assume $p = p'$. Thus our main focus will be on how lens spaces are different or the same depending on q and q'. This classification is given in the following theorem, which will be partly proven in this thesis.

Theorem 1.3. Let $L(p, q)$ and $L(p, q')$ be lens spaces. Then the following holds

1. $L(p, q)$ and $L(p, q')$ are homeomorphic if and only if $q' = \pm q^{\pm 1} \bmod p$.
2. $L(p, q')$ and $L(p, q')$ are homotopy equivalent if and only if $qq' = \pm n^2 \bmod p$ for some $n \in \mathbb{Z}$.

In this thesis we prove part 1 of this theorem. For the second part of the theorem we give a construction of a homotopy equivalence in the case $qq' = n^2 \bmod p$. However we do not fully prove that this is a homotopy equivalence.

The following figure illustrates where the name lens space comes from This picture arose

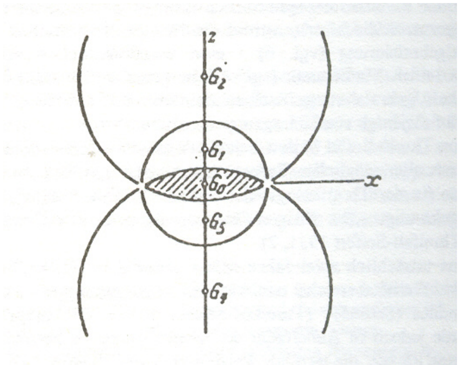


Figure 1: Lens shape in S^3

when Threlfall and Seifert [6] were studying a specific type of lens spaces that we would now call $L(p, 1)$. This image shows fundamental regions of the action on S^3 that defines the lens space. A fundamental region being a subset containing exactly one point of each orbit of the action, these will be useful when computing torsion later on. The lens shape can be seen in the middle of the image. Note how the lens shape is not actually inside the lens space, but in S^3 instead.

1.1 The fundamental group of lens spaces

As already mentioned, the first part of the classification follows from computing the fundamental group of the lens spaces. The fundamental group is computed using basic results from algebraic topology. Recall the following definition.

Definition 1.4 (Properly discontinuous action)**.** Let X be a topological space and G a group acting on X. Then the action of G is called **properly discontinuous** or **even** if for every $x \in X$ there exists an open neighbourhood $U \subset X$ of x. Such that $gU \cap U = \varnothing$ for every $g \in G$ with $g \neq 1$.

Lemma 1.5. A free action of a finite group G on a Hausdorff space is properly discontinuous.

Proof. Let $x \in X$. For a given $1 \neq g \in G$ we get that $gx \neq x$, since the action is free. Since X is Hausdorff we get that there exists an open neighbourhoods U_g and V_g of x and gx respectively such that $U_g \cap V_g = \varnothing$. Then we define $U = \bigcap_{g \in G} U_g$. This is open since G is finite and $gU \subset gU_g \subset V_g$. Thus $U \cap gU = \varnothing$, hence the action is properly discontinuous. ∎

Lemma 1.6. Let p, q be coprime integers and let ζ generate the pth roots of unity. Then the action on $S^3 \subset C^2$ generated by the homeomorphism

$$(z, w) \longmapsto (\zeta z, \zeta^q w)$$

is a properly discontinuous $\mathbb{Z}/p\mathbb{Z}$-action on S^3.

Proof. If $\zeta^k(z, w) = (\zeta^k z, \zeta^{kq} w) = (z, w)$, then $\zeta^k z = z$, and $\zeta^{kq} w = w$. If $z \neq 0$, then this implies that $\zeta^k = 1$. Otherwise if $z = 0$, then $w \neq 0$ this means that $\zeta^{kq} w = w$ implies

$\zeta^{qk} = 1$. Since p and q are coprime it follows that $\zeta^k = 1$, thus the action is free. Since $\mathbb{Z}/p\mathbb{Z}$ is finite and S^3 is a Hausdorff space it follows from Lemma 1.5 that the action is properly discontinuous. ■

We have the following general result from algebraic topology.

Theorem 1.7. [3] Let X be a simply connected topological space and G a group with an properly discontinuous action of G on X. Let X/G be the quotient space of X with respect to the action of G. Then the quotient map $X \longrightarrow X/G$ is a universal covering and the fundamental group $\pi_1(X/G)$ is isomorphic to G.

Corollary 1.8. The quotient map $S^3 \to L(p, q)$ is a universal covering and the fundamental group $\pi_1(L(p, q))$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Proof. It follows immediately from Lemma 1.6 and Theorem 1.7 ■

This implies the claim that lens spaces $L(p, q)$ and $L(p', q')$ are homeomorphic only if $p = p'$.

A covering projection is locally a homeomorphism. So if we have a covering projection $p : X \to Y$ then if either X or Y are n-manifolds, thus locally homeomorphic to $\mathbb{R}^n$, then we can 'pull' the local homeomorphisms of one space through p to get local homeomorphisms for the other space. We conclude that X is a manifold if and only if Y is a manifold. From this we conclude that since S^3 is a 3-manifold that our lens spaces $L(p, q)$ are 3-manifolds as well.

1.2 Homeomorphisms between lens spaces

A result that does not need any new techniques are that of the constructions of homeomorphisms between lens spaces. In this section we will prove the 'if' part of the second statement of Theorem 1.3. To do this we construct three types of homeomorphisms between lens spaces in for the different cases. First we need the following lemma.

Lemma 1.9. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a homeomorphism. Let $\sim$ and $\approx$ be equivalence relations on X and Y respectively. Assume that $[x]_\sim = [x']_\sim$ if and only if $[f(x)]_\approx = [f(x')]_\approx$. Then f induces a homeomorphism on the quotient spaces $\widetilde{f} : X/\sim \longrightarrow Y/\approx$.

Proof. We prove this as a consequence of the universal property of quotient spaces. Denote $\psi_x : X \longrightarrow X/\sim$ and $\psi_y : Y \longrightarrow Y/\approx$ for the quotient maps. Define the map $g = \psi_Y \circ f$. Then we show that $\psi_X(x) = \psi_X(x')$ if and only if $g(x) = g(x')$. Indeed $\psi_X(x) = \psi_X(x')$ is equivalent to $x \sim x'$ which by assumption is equivalent to $f(x) \approx f(x')$ thus equivalently $g(x) = \psi_Y(f(x)) = \psi_Y(f(x')) = g(x')$. Since f is a homeomorphism it follows that $g = f \circ \psi_Y$ is a quotient map. Thus by the universal property we get two unique maps $\widetilde{f} : X/\sim \longrightarrow Y/\approx$ and $\widetilde{h} : Y/\approx \longrightarrow X/\sim$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[d, "\psi_X"'] \arrow[dr, "g"] & \\
X/\sim \arrow[r, shift left, "\widetilde{f}"] & Y/\approx \arrow[l, shift left, "\widetilde{h}"]
\end{tikzcd}$$

By composing the maps $\widetilde{h}$ and $\widetilde{f}$ we get the following commutative diagram.

$$\begin{tikzcd}
X \arrow[d, "\psi_X"'] \arrow[dr, "\psi_X"] & \\
X/\sim \arrow[r, shift left, "\widetilde{h}\circ\widetilde{f}"] \arrow[r, shift right, "\mathrm{id}_{X/\sim}"'] & X/\sim
\end{tikzcd}$$

The uniqueness of the universal property gives that $\widetilde{h}\circ\widetilde{f} = \mathrm{id}_{X/\sim}$. Reversing the composition gives that $\widetilde{f} \circ \widetilde{h} = \mathrm{id}_{Y/\approx}$ in the same way. Hence $\widetilde{f} : X/\sim \longrightarrow Y/\approx$ is a homeomorphism. ∎

Proposition 1.10. If $q' = \pm q^{\pm 1} \bmod p$, then $L(p, q)$ and $L(p, q')$ are homeomorphic.

Proof. We do this case by case.

Case $q' = q + kp$ for some integer k .

Since $\zeta^p = 1$ it follows that $\zeta^{q'} = \zeta^{q+kp} = \zeta^q$. Therefore the actions defined by $\zeta(z, w) = (\zeta z, \zeta^q w)$ and $\zeta(z, w) = (\zeta z, \zeta^{q'} w)$ are the same. Thus the corresponding quotient spaces are the same. So $L(p, q) \cong L(p, q + kp)$.

Case $q' = -q$ mod p ,

Let $f : S^3 \longrightarrow S^3$ be the homeomorphism defined by $(z, w) \longmapsto (z, \overline{w})$. Where $\overline{w}$ denotes the complex conjugate. Let $[(z, w)]_q$ denote the classes in $L(p, -q)$ and $[(z, w)]_{-q}$ for the classes in $L(p, q)$. We have to show that $[(z, w)]_q = [(z', w')]_q$ if and only if $[(z, \overline{w})]_{-q} = [f(z', \overline{w}')]_{-q}$. Note how $[(z, w)]_p$ has elements in the form $(\zeta^k z, \zeta^{kq} w)$. Direct computation shows the following

$$\begin{aligned}
[(z, w)]_q = [(z', w')]_q &\iff (z, w) = (\zeta^k z', \zeta^{kq} w') \\
&\iff (z, \overline{w}) = (\zeta^k z', \overline{\zeta^{kq} w'}) \\
&\iff (z, \overline{w}) = (\zeta^k z', \zeta^{-kq} \overline{w}') \\
&\iff [(z, \overline{w})]_{-q} = [(z', \overline{w}')]_{-q} \\
&\iff [f(z, w)]_{-q} = [f(z', w')]_{-q}
\end{aligned}$$

It follows from Lemma 1.9 that f induces a homeomorphism $\widetilde{f} : L(p, q) \longrightarrow L(p, -q)$.

Case $q' = q^{-1}$ mod p , This means $qq' = 1 \bmod p$ Let $f : S^3 \to S^3$ the homeomorphism defined by $(z, w) \mapsto (w, z)$. We show that this induces a homeomorphism $L(p, q) \to L(p, q')$. We can rewrite the ζ action in terms of a $\zeta^{q'}$ action because $\zeta^{q'}$ is also a generator of the pth roots of unity since p and q' are coprime. Thus we can see the following equivalences.

$$\begin{aligned}
[(z, w)]_q = [(z', w')]_q &\iff (z, w) = (\zeta^{q'k} z', \zeta^{kq'q} w') = (\zeta^{q'k} z', \zeta^k w') \\
&\iff (w, z) = (\zeta^k w', \zeta^{q'k} z') \\
&\iff [(w, z)]_{q'} = [(w', z')]_{q'} \\
&\iff [f(z, w)]_{q'} = [f(z', w')]_{q'}
\end{aligned}$$

From Lemma 1.9 if follows that f induces a homeomorphism $\widetilde{h} : L(p, q) \longrightarrow L(p, q')$.

These results together prove the statement. ■

It is not at all obvious that these are the only cases in which homeomorphisms exists between lens spaces. To show that this is actually the case we need the technique of Reidemeister torsion to distinguish the different lens spaces.

2 Reidemeister torsion

Torsion is a concept that can be seen as a generalised determinant of so called chain complexes. This concept of torsion is purely algebraically defined in terms of chain complexes. To make the torsion a topological notion, we need to add some structure on our topological spaces in order to create a chain complex using which we can compute the torsion. The topological notion of torsion is then called Reidemeister torsion.

2.1 Torsion of chain complexes

Definition 2.1 (Chain complex over a commutative ring)**.** A chain complex C is a set $\{C_i\}_{i\in\mathbb{Z}}$ of modules over a ring R together with a set of R-linear maps $\partial_i : C_{i+1} \to C_i$, called boundary maps. That have the property that $\partial_i \circ \partial_{i+1} = 0$ for each $i \in \mathbb{Z}$. The modules C_k are called the chain groups of the chain complex. This gives a sequence of modules and homomorphisms

$$\cdots \longrightarrow C_i \xrightarrow{\partial} C_{i-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

A chain complex of free modules is called based if we pick distinguished bases c_i for the chain groups C_i for each i.

From the fact that $\partial_i \circ \partial_{i+1} = 0$ it follows that $\operatorname{Im}(\partial_{i+1} : C_{i+1} \to C_i) \subset \ker(\partial_i : C_i \to C_{i-1}) \subset C_i$. Therefore we can define the homology groups of the chain complex as the quotient $H_i(C) = \ker(\partial_i : C_i \to C_{i-1})/\operatorname{Im}(\partial_{i+1} : C_{i+1} \to C_i)$

Definition 2.2 (Acyclic)**.** A chain complex C is called acyclic if $H_i(C) = 0$ for each i.

This is equivalent to the sequence of modules being exact. The torsion is a quantity that we will compute of these acyclic chain complexes. The definition of torsion requires the following observation.

Assume we have a based acyclic chain complex over a field, with distinguished bases $\{c_i\}$. Denote $Z_i = \ker(\partial_{i-1}) \subset C_i$ and $B_i = \operatorname{Im}(\partial_i) \subset C_i$. Then we can also write $H_i(C) = Z_i/B_i = 0$. In an acyclic chain complex we thus have $Z_i = B_i$. In other words we have that $\ker(\partial_{i-1}) = Z_i = B_i$. Thus we conclude that the following sequence is exact.

$$0 \longrightarrow B_i \hookrightarrow C_i \xrightarrow{\partial_{i-1}} B_{i-1} \longrightarrow 0$$

Since vector spaces are free modules it follows that the sequence splits. This means that $C_i = B_i \oplus B_{i-1}$. Thus choosing bases b_i and b_{i-1} for B_i and B_{i-1} gives a basis for C_i by choosing representative preimages of the basis vectors in b_{i-1} in C_i. The resulting basis of C_i is written as $b_i b_{i-1}$. Using this basis and the distinguished bases c_i.

Before defining the torsion we introduce the following notation. Let V be a vector space over a field $\mathbb{F}$. And let $v = (v_1, v_2, \cdots, v_n)$ and $w = (w_1, w_2, \cdots, w_n)$ be ordered bases for V. Then there exists a transition matrix T associated to the change of the basis from v to w. Then we write

$$[v/w] := \det(T) \in \mathbb{F}^*.$$

The following properties are apparent.

- $[v/v] = 1$ for any basis v.
- For any three bases v, w, z we have $[v/w] = [v/z][z/w]$.

Definition 2.3 (Torsion of an acyclic chain complex)**.** The torsion of an acyclic chain complex C with bases c_i and $b_i b_{i-1}$ as defined above is

$$\tau(C) = \prod_{i=1}^{m} [b_i b_{i-1}/c_i]^{(-1)^i} \in \mathbb{F}^*.$$

Remark 2.4. This torsion is the inverse of the torsion defined by Turaev in [9].

Lemma 2.5. The torsion $\tau(C)$ is independent of the choice of b_i.

Proof. We will show that the following quantity

$$[b_i b_{i-1}/c_i]^{(-1)^i} [b_{i+1} b_i/c_{i+1}]^{(-1)^{i+1}}$$

is independent of the choice of b_i. Let b_i' be another basis of B_i. Then we see the following.

$$\begin{aligned} [b_i' b_{i-1}/c_i] &= [b_i' b_{i-1}/b_i b_{i-1}][b_i b_{i-1}/c_i] \\ &= [b_i'/b_i][b_i b_{i-1}/c_i]. \end{aligned}$$

The first equality follows from the second property of the symbol and the second equality follows from the fact that the matrix corresponding to $[b_i' b_{i-1}/b_i b_{i-1}]$ is a block diagonal matrix with the base change matrix corresponding to the base change b_i'/b_i in the top left and the identity matrix in the bottom right. Thus $[b_i' b_{i-1}/b_i b_{i-1}] = [b_i'/b_i]$. In the same way we get the following

$$[b_{i+1} b_i'/c_{i+1}] = [b_i'/b_i][b_{i+1} b_i/c_i].$$

Thus

$$\begin{aligned} [b_{i+1} b_i'/c_{i+1}]^{(-1)^i} [b_i' b_{i-1}/c_i]^{(-1)^{i+1}} &= ([b_i'/b_i][b_{i+1} b_i/c_i])^{(-1)^i} ([b_i'/b_i][b_i b_{i-1}/c_i])^{(-1)^{i+1}} \\ &= [b_{i+1} b_i/c_{i+1}]^{(-1)^{i+1}} [b_i b_{i-1}/c_i]^{(-1)^i}. \end{aligned}$$

It follows that thus the torsion is independent of the choice of bases b_i. ∎

It is important to note that the torsion does depend on the choice of bases $\{c_i\}$. We have the following relation on how changing the distinguished bases impacts the torsion.

Lemma 2.6. Let C be a based acyclic chain complex with bases $\{c_i\}$ and C' the same chain complex with bases $\{c_i'\}$. Then

$$\tau(C') = \tau(C) \prod_{i=1}^{m} [c_i/c_i']^{(-1)^i}.$$

Proof. This follows from the fact that $[b_i b_{i-1}/c_i'] = [b_i b_{i-1}/c_i][c_i/c_i']$. ∎

In particular we have the following relations.

1. If $c_i = (c_i^1, c_i^2, c_i^3, \cdots)$ and $c_i' = (c_i^2, c_i^1, c_i^3, \cdots)$ for some i and $c_j = c_j'$ for $j \neq i$ then $\tau(C) = -\tau(C)$.
2. If $c_i = (c_i^1, c_i^2, c_i^3, \cdots)$ and $c_i' = (\alpha c_i^1, c_i^2, c_i^3, \cdots)$ for some i and $c_j = c_j'$ for $j \neq i$, then $\tau(C) = \alpha^{(-1)^i}\tau(C)$.

The following result will prove useful in later torsion computations.

Lemma 2.7. Let C be a chain complex of the form

$$0 \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

Then the chain complex is acyclic if and only if ∂ is an isomorphism. Let A be a matrix representing ∂. Then $\tau(C) = \det(A)$.

Proof. Acyclicity is equivalent to the sequence being exact, and the sequence is exact if and only if ∂ is an isomorphism. The map ∂ is surjective so $B_0 = C_0$. We choose $b_0 = c_0$. The change of the basis b_1 to c_1 corresponds to the map ∂^{-1}. It follows that $[b_1/c_1] = \det(A)^{-1}$. We have that b_1 and b_{-1} are empty. We then compute the torsion as follows

$$\tau(C) = [b_0 b_{-1}/c_0][b_1 b_0/c_1]^{-1} = [b_0/c_0][b_1/c_1]^{-1} = \det(A).$$

■

Given a topological space, our aim is to construct an acyclic chain complex of which we can compute the torsion, such that we get a value that is topologically invariant. This then gives us a way to distinguish different spaces by torsion. The way we construct this chain complex involves some steps, which are roughly as follows. First we construct a so-called CW-decomposition of our topological space. This decomposition is then lifted to the universal covering of the space, which yields a CW-decomposition of the universal covering. Using this decomposition we can construct a chain complex called the cellular chain complex. The last step is to apply a trick to make the resulting chain complex acyclic. Then we can compute the torsion. This torsion will depend on some choices we make along the way, which we will thus have to quotient out. In the end we end up with a well defined quantity, called the Reidemeister torsion, which turns out to be topologically invariant. The topological invariance of the Reidemeister torsion is a result from infinite dimensional topology and is not something we will prove here.

2.2 CW-complexes

A CW-complex is a topological space that is constructed in the following inductive manner. Start with a space of discrete points called X^0. Then we take a set of 1 dimensional discs (closed intervals) and glue the boundary (the endpoints) of the intervals to the set of discrete points. The resulting space is then called X^1. Then we take a set of 2 dimensional discs and do the same; we take boundary of the discs and glue them to the space X^1. Resulting in the space called X^2. Continuing this procedure any number of times gives a CW-complex. Following this procedure a finite amount of times yields a finite CW-complex. It is possible to do this an infinite amount of time yielding an infinite CW-complex. This thesis will only consider finite CW-complexes. As an example, constructing the circle can be done by taking

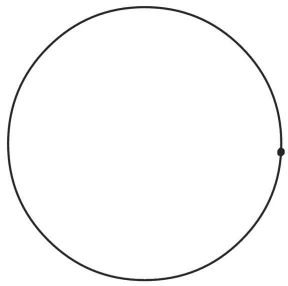


Figure 2: CW-decomposition of the circle

a point and an interval and gluing the end points of the interval to the point. Most 'nice' topological spaces can be constructed as a CW-complex.

Formally we define a CW-complex as follows. First we need the preliminary notion of adjoining k-cells. Which is means gluing the boundary of a set of k-disks to a topological space.

Definition 2.8 (adjoining k-cells)**.** Let $X \subset X'$ be two topological spaces. We say that X' is obtained by adjoining k-cells to X if there exist maps $f_i^k : D^k \to X'$. Such that the map $\bigsqcup_i f_i^k : \bigsqcup_i \operatorname{Int}(D^k) \to X' \backslash X$ is a homeomorphism.

In particular this definition gives that the space $X' \backslash X$ is a space consisting homeomorphic to a disjoint union of open k-balls. These k-balls are what we call the k-cells of the CW-complex. Specifically for each map f_i^k we define its open k-cell $e_i^k = f_i^k(\operatorname{Int}(D^k))$. The map f_i^k is then called the attaching map of the k-cell e_i^k.

Note how the restriction to the boundary does not have to be a homeomorphism, so when constructing a CW-complex when given a disc, we can tangle the boundary of the disc as much we want as long as the interior of the disk stays nice. Now a CW-complex is defined as follows.

Definition 2.9 (Finite CW-complex)**.** A CW-complex is a Hausdorff space X together with a sequence of closed subsets $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n = X$ such that

- X^0 is discrete,
- $X = \bigcup_i X^i$,
- for each k we have that X^k is obtained from X^{k-1} by adjoining k cells.

With this definition we see that the we can write a CW-complex as the union of all the cell, where the individual cells are all disjoint.

Example 2.10 (CW-decomposition of the torus). We construct a CW-decomposition of the torus. We start with X^0 equal to the one point space. Then we take two closed intervals and attach their boundaries to the point. This gives the space X^1, which is homeomorphic to the figure eight. Then we take disk and form it into a filled in square. Then we attach two opposite sides of the square to one of the circles and the other two opposite sides to the other circle.

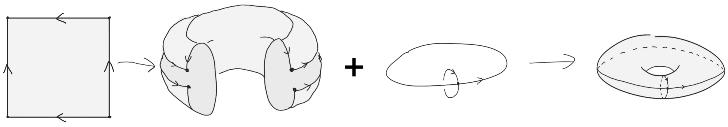


Figure 3: CW-decomposition of the torus

2.3 The cellular chain complex

Given a CW-complex X we can construct its *cellular chain complex* as follows, which we will use to compute the torsion. For a given k denote $\{e_i^k\}$ for the set of k-cells of the complex. We then define the cellular chain groups $C_k(X) = \bigoplus_i \mathbb{Z}e_i^k$ for $k \in \mathbb{Z}_{\geq 0}$. So the kth chain group is the free abelian group generated by the oriented k-cells of X. The construction of the boundary homomorphisms $\partial : C_k(X) \to C_{k-1}(X)$ first requires modifying the attaching maps f_i^k for each cell e_i^k, which is done as follows.

Take a k-cell e_i^k and its corresponding attaching map f_j^k. We restrict the map $f_i^k : D^k \to X^k$ to the boundary of the disc to get a map $\partial D^k = S^{k-1} \to X^{k-1}$. Note how we actually land in X^{k-1} here since we are only mapping the boundary of the disc. Then the space X^{k-1} gets collapsed by taking all the points in X^{k-2} and identifying them with each other. This results in the quotient space X^{k-1}/X^{k-2}. Since the boundary of each $(k-1)$-cell lies in X^{k-2} if follows that what happens here is that the boundary of each $(k-1)$-cell gets identified with a single point. These $(k-1)$-cells are open $k-1$-balls and when the boundary of the ball gets identified with a single point it results in a $(k-1)$-sphere. Hence the space X^{k-1}/X^{k-2} will be a bunch of spheres all glued together at a single point x. This space is thus a wedge sum of $(k-1)$ spheres $\bigvee_i S_j^{k-1}$. See how each S_j^{k-1} corresponds to a $(k-1)$-cell e_j^{k-1}. The last step is to define a map for each j that collapses each sphere except the sphere S_j^{k-1} to a single point. This leaves exactly S_j^k. Composing these maps we get for each j a map

$$f_{i,j}^k : S^{k-1} = \partial D^k \to X^{k-1} \to X^{k-1}/X^{k-2} = \bigvee_j S_j^{k-1} \to S_j^{k-1}. \tag{1}$$

Notice how this is a map from S^{k-1} to S^{k-1}, thus we can compute its degree. For a fixed e_i^k only finitely many of the $f_{i,j}^k$ will have non-zero degree. The proof is as follows. Take the open cover of $\bigvee_j S_j^{k-1}$ consisting $U_j = S_j^{k-1} \backslash \{x\}$ for each j and a small neighbourhood V of x. The image of S^{k-1} in $\bigvee_j S_j^{k-1}$ is compact, so it is contained in the union finitely many U_j and V. So for all but finitely many the map $f_{i,j}^k$ is not surjective thus homotopic to the constant map sending everything to the point x and thus has degree 0.

The degree of such map can be seen as number of times the boundary of a certain k-cell gets twisted in order to be attached to one of the boundary $(k-1)$-cells. We want to take this number into account when computing the boundary of a cell.

We define the boundary homomorphisms $\partial : C^k \to C^{k-1}$ for $k \geq 2$ by

$$\partial e_j^k = \sum_j \deg(f_{i,j}^k) e_j^{k-1}.$$

For $k = 1$ this definition does not work, as the degree of a map $S^0 \to S^0$ is not defined, since S^0 does not have 0th homology group equal to $\mathbb{Z}$. Instead for the case $k = 1$ we just use the standard simplicial boundary map i.e. the boundary of an oriented 1-cell is equal to the endpoint minus the starting point. Lastly the boundary of a 0-cell is defined to be 0.

Now we have defined a sequence of groups and homomorphisms

$$\cdots \longrightarrow C_i(X) \xrightarrow{\partial} C_{i-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow 0$$

Proposition 2.11. [3] The above is a chain complex, i.e. $\partial\partial = 0$ and the homology groups $H_i(C(X))$ are isomorphic to the singular homology groups $H_i(X)$.

Example 2.12. We consider the CW-decomposition of the torus defined in Example 2.10 and construct its cellular chain complex. Call the 0-cell e^0, the 1-cells e_1^2 and e_2^2 and the 2-cell e^2. The chain groups are the free abelian groups generated by the cells in each dimension. So C_0 and C_2 have rank 1 and C_1 has rank 2. The boundary maps are constructed as follows. The boundary map $\partial : C_1 \to C_0$ is the simplicial boundary map. Thus it is defined as $\partial e_1^1 = e^0 - e^0 = 0$ and $\partial e_2^1 = e^0 - e^0 = 0$. Thus the map $\partial : C_1 \to C_0$ is the zero map. For the boundary map $\partial : C_2 \to C_1$ we use the maps f_i^2. Where the attaching map of the 2-cell is called f^2. These maps are depicted in Figure 4. Notice how the space X^1/X^0 is a wedge sum of two circles so it is equal to X^1. We have to compute the degrees of these

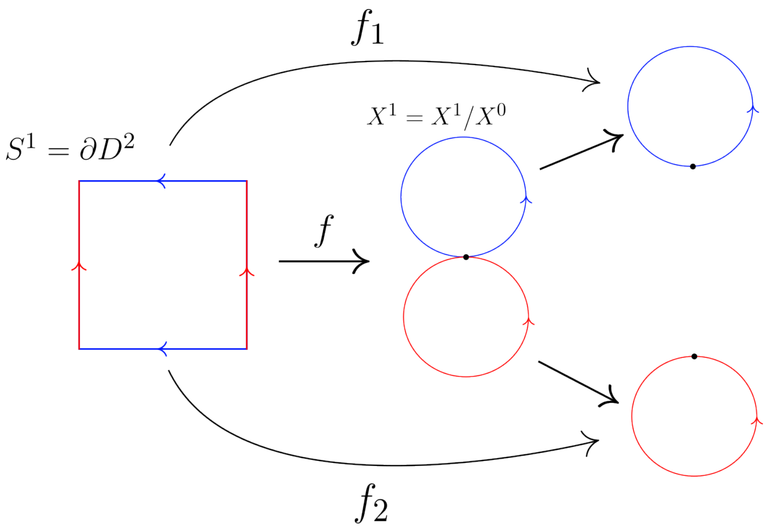


Figure 4

maps. The map f_1 the same as the map that takes the square and ignores the red sides by collapsing both red sides to two distinct points. Afterwards it takes what is left, which is a circle and maps it to the circle by first going around anticlockwise and then clockwise. This is illustrated in Figure 5. This map of going anticlockwise and then clockwise is homotopic to the constant map, by continuously moving turning point along the circle towards a single point. This means that the map has degree 0. Which in turn implies that the map f_1 also has degree 0. Quite similarly it also follows that the map f_2 has degree 0. Thus the

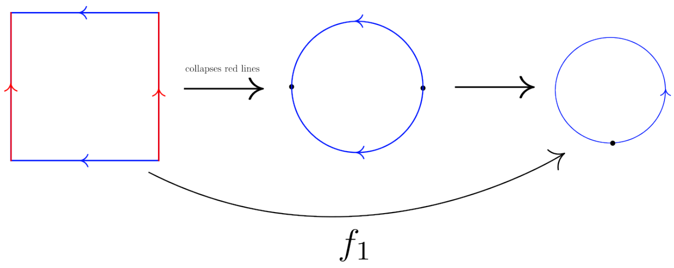


Figure 5

boundary map $\partial : C_2 \to C_1$ is the zero map. Thus we get the following cellular chain complex.

$$0 \longrightarrow \mathbb{Z}e^2 \xrightarrow{\;0\;} \mathbb{Z}e_1^1 \oplus \mathbb{Z}e_2^1 \xrightarrow{\;0\;} \mathbb{Z}e^0 \longrightarrow 0$$

From this chain complex we can read off the cellular homology groups as.

$$H_i(C(X)) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^2 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases}$$

These groups correspond to the known singular homology groups of the torus.

The next step in defining Reidemeister torsion requires the following lemma. It is known that every CW-complex is locally path-connected and semi-locally simply connected. It follows that every connected CW-complex X allows a universal covering $\widetilde{X}$.

Lemma 2.13. Let X be a CW-complex. Then CW-structure on X induces a CW-structure on the universal covering $\widetilde{X}$. Such that the fibre of a cell e^k consists of disjoint homeomorphic copies of e^k and the automorphism group $\mathrm{Aut}(\widetilde{X}/X) = \pi_1(X)$ acts freely and transitively on the set of cells in the fibre of each cell of X.

Proof. We set $\widetilde{X}^k = p^{-1}(X^k) \subset \widetilde{X}$. Then $\widetilde{X}^0 \subset \widetilde{X}^1 \subset \widetilde{X}^2 \subset \cdots$ is a chain of closed subsets and $\widetilde{X}^0$ is a discrete space. The cells and attaching maps of $\widetilde{X}$ are constructed as follows. Take a k-cell e^k of X with attaching map $f : D^k \to X^k$ and take a point $x \in D^k$. Then we consider the fibre of $f(x) \in X^k$. For every point $\widetilde{x} \in p^{-1}(f(x))$ there exists a unique lift $\widetilde{f}_{\widetilde{x}} : D^k \to \widetilde{X}^k$ of f such that $f(x) = \widetilde{x}$. Then for every $\widetilde{x}$ in the fibre of $f(x)$ we get the open cell $e^k_{\widetilde{x}} = \mathrm{Int}(\widetilde{f}(D^k))$. Since we have that $f = \widetilde{f}_{\widetilde{x}} \circ p$ it follows that the cell $e^k_{\widetilde{x}}$ gets mapped homeomorphically onto e^k. Thus $p^{-1}(e^k)$ consists of a number homeomorphic copies of e^k corresponding to the number of points in the fibre of $f(x)$.

Furthermore if $h \in \mathrm{Aut}(\widetilde{X}/X)$ then $h \circ \widetilde{f}_{\widetilde{x}}$ is another lift of f. And since $h \circ \widetilde{f}_{\widetilde{x}}(x) = h(\widetilde{x})$ by uniqueness of lifts it follows that $h \circ \widetilde{f}_{\widetilde{x}} = \widetilde{f}_{h(\widetilde{x})}$. So we see that $\mathrm{Aut}(\widetilde{X}/X)$ acts on the set of lifts and since $\mathrm{Aut}(\widetilde{X}/X)$ acts freely and transitively on the set of points in the fibre of $f(x)$ it follows that it also acts freely and transitively on the set of lifts and thus the cells. ∎

Using this induced CW-structure we obtain the cellular chain complex $C(\widetilde{X})$ of $\widetilde{X}$

$$\cdots \longrightarrow C_k(\widetilde{X}) \xrightarrow{\partial} C_{k-1}(\widetilde{X}) \longrightarrow \cdots \longrightarrow C_2(\widetilde{X}) \xrightarrow{\partial} C_1(\widetilde{X}) \longrightarrow 0$$

Let $\pi = \pi(X)$ be the fundamental group of X. The group ring $\mathbb{Z}[\pi]$ is the ring consisting of formal $\mathbb{Z}$-linear combinations of elements of π, where the multiplication is induced by the multiplication of π. Let the group π act in the natural way on the cells of $\widetilde{X}$. Then the group $\mathbb{Z}[\pi]$ acts in a natural way on the chain groups $C_k(\widetilde{X})$ by extending the action of π on the k-cells linearly. In this way the chain groups $C_k(\widetilde{X})$ become $\mathbb{Z}[\pi]$-modules. It is easily verified that the boundary homomorphisms ∂ are also $\mathbb{Z}[\pi]$-linear, since an element $h \in \pi$ acts in the same way on the boundary of a k-cell e^k as on the k-cell itself, thus giving that $h(\partial e^k) = \partial h(e^k)$, hence ∂ is also $\mathbb{Z}[\pi]$-linear. For each k-cell e_i^k of X we pick a representative lift $\tilde{e}_i^k$ in $\widetilde{X}$. Then there is the following result

Lemma 2.14. The $\mathbb{Z}[\pi]$-modules $C_k(X)$ are free with basis $\widetilde{e} = \{\tilde{e}_i^k\}$. Thus we can write

$$C_k(\widetilde{X}) = \bigoplus_i \mathbb{Z}[\pi]\tilde{e}_i^k.$$

Proof. This is an immediate consequence of the fact that π acts freely and transitively on the set of lifts of each e_i^k. ■

In order to make this chain complex into something of which we can compute the torsion we need the following definition, which is basically a trick to make certain elements invertable to make the chain complex acyclic.

Definition 2.15 (Twisted tensor product)**.** Let R and R' be nonzero commutative rings and $\varphi : R \to R'$ a ring homomorphism. Let M be a R-module. Then we define the R'-module $R' \otimes_\varphi M$, called the twisted tensor product, which is generated by elements of the form $r \otimes_\varphi m$, with $r' \in R'$ and $m \in M$. With the relation $\varphi(\lambda)r' \otimes m = r' \otimes \lambda m$ for $\lambda \in R$. Together with the usual tensor product relations.

What this does is change a free R module into a free R' module with the same basis, which is illustrated in the following lemma

Lemma 2.16. [9] If $M = \bigoplus_i Re_i$ is a free R-module, then $R' \otimes_\varphi M$ is a free R'-module isomorphic to $\bigoplus_i R'e_i$.

Proof. Define the ring homomorphism

$$\begin{aligned} f : R' &\longrightarrow R' \otimes_\varphi R \\ r' &\longmapsto r' \otimes 1. \end{aligned}$$

The standard bilinear relations on the tensor product tell us that this is a ring homomorphism. Furthermore the map is injective since if $r' \otimes 1 = 0$, then $r' = 0$.

We can write an element $\sum_i r_i' \otimes r_i \in R' \otimes R$ as $\sum_i (\varphi(r_i) r_i' \otimes 1) = \sum_i (\varphi(r_i) r_i') \otimes 1$. Where $\sum_i (\varphi(r_i) r_i') \in R'$, hence the map f is surjective. It follows that $R' \otimes_\varphi R \cong R'$. Using the distributivity of the tensor product i.e. $R' \otimes_\varphi (\bigoplus_i R) = \bigoplus_i (R' \otimes_\varphi R)$ the statement follows. Distributivity of the twisted tensor product follows in a similar way as the distributivity of the normal tensor product. ■

Let $\varphi : \mathbb{Z}[\pi] \to \mathbb{F}$ be a ring homomorphism then we define $C_k^{\varphi}(X) = \mathbb{F} \otimes_{\varphi} C_k(\widetilde{X})$. The matrix of the boundary homomorphism $\partial : C_k^{\varphi}(X) \to C_{k-1}^{\varphi}(X)$ is given by taking the matrix of the boundary homomorphism $\partial : C_k(\widetilde{X}) \to C_{k-1}(\widetilde{X})$ and applying φ to each entry [9]. Then we get the chain complex

$$\cdots \longrightarrow C_k^{\varphi}(X) \xrightarrow{\partial} C_{k-1}^{\varphi}(X) \longrightarrow \cdots \longrightarrow C_2^{\varphi}(X) \xrightarrow{\partial} C_1^{\varphi}(X) \longrightarrow 0$$

with the given boundary homomorphisms. Since the chain groups $C_k(\widetilde{X})$ are free $\mathbb{Z}[\pi]$-modules it follows from Lemma 2.16 that the chain groups here become free modules over the field $\mathbb{F}$. This means that under the assumption that this chain complex is acyclic (whether or not this is the case depends on φ), we can define the torsion of X with respect to the basis of representative lifts $\widetilde{e}$ as

$$\tau_{\varphi}(X, \widetilde{e}) = \tau(C^{\varphi}(X)) \in \mathbb{F}^*.$$

We want a quantity that is independent of the choice $\widetilde{e}$. For this we want to know in what way this torsion depends on the choices we made when picking $\widetilde{e}$. Specifically the choices of representative lifts of the CW-structure of X and the ordering of the bases.

Theorem 2.17. The quantity $\tau_{\varphi}(X) := \tau_{\varphi}(X, \widetilde{e})$ is well defined up to multiplication by elements in $\pm\varphi(\pi) \subset \mathbb{F}^*$. Hence we have a well defined torsion $\tau_{\varphi}(X) = \tau_{\varphi}(X, \widetilde{e}) \in \mathbb{F}^*/ \pm \varphi(\pi)$.

Proof. When defining the torsion we make two choices. Namely the choice of representative lifts and the choice of ordering of these lifts to form a basis. By Lemma 2.6 we have that changing the order of the lifts introduces a ± 1 factor in the torsion depending on the sign of the permutation. If $\widetilde{e}_j^k$ is a lift of the k-cell e_j^k and $\overline{e}_j^k$ is another lift of e_j^k, then by the transitivity of the fundamental group action we have that $\overline{e}_j^k = g\widetilde{e}_j^k$ for some $g \in \pi$. Thus in the twisted chain complex we have $\overline{e}_j^k = \varphi(g)\widetilde{e}_j^k$. This gives that the torsion $\tau_{\varphi}(X, \widetilde{e})$ gets multiplied by $\varphi(g)$. Thus these different choices have no effect modulo $\pm\varphi(\pi)$. ■

The quantity $\tau_{\varphi}(X)$ is called the Reidemeister torsion of the CW-complex X. One thing to note from the previous theorem is that the Reidemeister torsion fails to see orientation of the CW-complex. Since picking opposite orientations of cells introduces minus signs in the final computations. However the different choices for ordering the bases of the the chain complex also introduces minus signs, which we have to quotient out. Therefore information of orientations is lost.

Furthermore it is good to remember that the torsion is only defined under the assumption that we are given a ring homomorphism φ such that the resulting chain complex is acyclic. Thus for computing torsion we find right conditions on what φ does in order to make the chain complex acyclic. Then there is the task of actually finding such a ring homomorphism satisfying the conditions. Luckily for our applications finding this ring homomorphism is not a problem as we will see later.

We want to use the torsion as a topological invariant in order to determine when two spaces are different. It turns out that whether or not the Reidemeister torsion is a topological invariant is nontrivial. It is not clear that two CW-decompositions of a topological space yield the same Reidemester torsion, as it is defined using the CW-decomposition. There is the following result from infinite dimensional topology that says that it is invariant under homeomorphisms of CW-complexes.

Given a homeomorphism $f : X \to Y$ between CW-complex then the induced group isomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ naturally induces a ring isomorphism $f_* : \mathbb{Z}[\pi(X)] \to \mathbb{Z}[\pi(Y)]$.

Theorem 2.18 (Topological invariance of torsion)**.** [1] Let X and Y be CW-complexes. And let $f : X \to Y$ be a homeomorphism, then given a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(Y)] \to \mathbb{F}$ define $\psi = \varphi \circ f_* : \mathbb{Z}[\pi_1(X)] \to \mathbb{F}$ then we have $\tau_\psi(X) = \tau_\varphi(Y)$.

When looking at the lens spaces, we show that the Reidemeister torsion completely classifies them up to homeomorphisms. Meaning that two lens spaces are homeomorphic if and only if they have the same torsion.

3 Calculation of torsions

As described in the previous section we have the following recipe for calculating the torsion given a topological space X.

1. Find a CW-decomposition for X.
2. Lift the CW-decomposition of X to its universal cover $\widetilde{X}$.
3. Pick representative lifts of the cells of X.
4. Describe the boundary maps in terms of the representative lifts.
5. Find conditions on a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(X)] \to \mathbb{F}$ such that the chain complex $C^\varphi(X)$ is acyclic.
6. Assuming $C^\varphi(X)$ is acyclic, find bases b_i for the spaces B_i.
7. Compute the terms $[b_i b_{i-1}/c_i]$.
8. Compute the torsion with formula

$$\tau_\varphi(X) = \prod_{i=1}^{m} [b_i b_{i-1}/c_i]^{(-1)^i}.$$

Step 6 and 7 are in general not as easy as it seems and are not applicable to every calculation, specifically if the chain complex is infinitely long, which happens in the case of infinite CW-complexes. However in our case this method is sufficient. Examples of more general ways of these calculations are done using τ-chains or chain contractions as described in [9].

3.1 Torsion of the circle

Theorem 3.1. Let S^1 be the circle and $\pi = \pi(S^1) \cong \mathbb{Z}$ be its fundamental group. Let T be a generator of π. Let $\mathbb{F}$ be a field and let $\varphi : \mathbb{Z}[\pi] \to \mathbb{F}$ be a ring homomorphism. Set $t = \varphi(T) \in \mathbb{F}^*$. Then $H_*^\varphi(C) = 0$ if and only if $t \neq 1 \in \mathbb{F}^*$. In which case the torsion is

$$\tau_\varphi(S^1) = t - 1 \in \mathbb{F}^*/\{\pm t^n\}.$$

Proof. Let $p : \mathbb{R} \to S^1$ be the universal covering of $S^1 \subset \mathbb{C}$ defined by $p(x) = e^{2\pi i x}$. The generator T acts on $\mathbb{R}$ as $T(x) = x + 1$. We choose the CW-decomposition of S^1 consisting of one 0-cell e_0 and one 1-cell e_1. Where the endpoints of the one cell are both attached to the 0-cell. We orient them as depicted in Figure 6. Lifting this decomposition gives the decomposition of $\mathbb{R}$ as also depicted in the figure. We pick a representative 0-cell $\widetilde{e}_0$ and a representative 1-cell $\widetilde{e}_1$

We thus get the following cellular chain complex for $\mathbb{R}$

$$0 \longrightarrow \mathbb{Z}[\pi]\widetilde{e}_1 \xrightarrow{\partial} \mathbb{Z}[\pi]\widetilde{e}_0 \longrightarrow 0$$

The boundary map is given by the simplicial boundary map which is defined by

$$\partial \widetilde{e}_1 = T^{n+1}\widetilde{e}_0 - T^n \widetilde{e}_0 = T^n(T-1)\widetilde{e}_0.$$

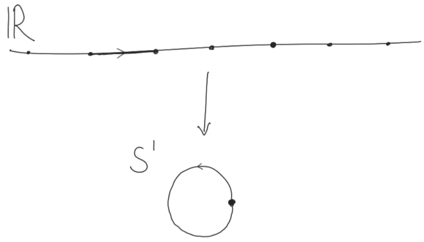


Figure 6: Universal covering of S^1

Then the chain complex $C^\varphi(S^1)$

$$0 \longrightarrow \mathbb{F}\widetilde{e}_1 \xrightarrow{\partial} \mathbb{F}\widetilde{e}_0 \longrightarrow 0$$

has the induced boundary map $\partial\widetilde{e}_1 = t^n(t-1)\widetilde{e}_0$. For the chain complex to be acyclic we need that ∂ is an isomorphism, this is the case if and only if $t-1$ is invertable. So the chain complex is acyclic if and only if $t-1 \in \mathbb{F}^*$. Then by Lemma 2.7 the torsion is the determinant of the matrix representing ∂, thus $\tau_\varphi(S^1) = t^n(t-1) = t-1 \in \mathbb{F}^*/\{\pm t^n\}_{n\in\mathbb{Z}}$ ■

3.2 Torsion of the torus

Theorem 3.2. Let $T = S^1 \times S^1$ be the torus. Define $\pi = \pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let $\varphi : \mathbb{Z}[\pi] \to \mathbb{F}$ be a ring homomorphism. Let A and B be the generators of π and denote $a = \varphi(A)$ and $b = \varphi(B)$. Then $C^\varphi(T)$ is acyclic if $a-1 \neq 0$ or $b-1 \neq 0$. In which case the torsion is

$$\tau_\varphi(T) = 1 \in \mathbb{F}^*/\{a^n b^m\}_{n,m\in\mathbb{Z}}.$$

Proof. We have the universal covering of the torus $S^1 \times S^1$:

$$\begin{aligned} p : \mathbb{R}^2 &\longrightarrow T = S^1 \times S^1 \\ (x,y) &\longmapsto (e^{2\pi i x}, e^{2\pi i y}). \end{aligned}$$

Lifting the CW-decomposition of the torus defined in Example 2.10 gives us the CW-decomposition of $\mathbb{R}^2$ with $X^0 = \mathbb{Z}^2 \subset \mathbb{R}^2$, X^1 equal to the grid as depicted in figure 7 and $X^3 = \mathbb{R}^2$. The fundamental group $\pi = \pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$ acts on the universal cover by horizontal and vertical translation. More precisely we can pick generators A and B of π such that for a point $(x,y) \in \mathbb{R}$ we have $A(x,y) = (x+1,y)$ and $B(x,y) = (x,y+1)$. The ring $\mathbb{Z}[\pi]$ is isomorphic to the ring of Laurent polynomials in two variables $\mathbb{Z}[A, A^{-1}, B, B^{-1}]$. The 0-cells of the CW-decomposition of $\mathbb{R}^2$ are generated by a single 0-cell, we pick a representative cell $\widetilde{e}^0$. The 1-cells consist of horizontal 1-cells and vertical 1-cells. The action of $\mathbb{Z}[\pi]$ gives that one horizontal and one vertical 1-cell generate the 1-cells. For this we pick generators $\widetilde{e}^1_-$ and $\widetilde{e}^1_|$ representing the horizontal and vertical 1-cells respectively.

We get the following cellular chain complex.

$$0 \longrightarrow \mathbb{Z}[\pi]\widetilde{e}^2 \longrightarrow \mathbb{Z}[\pi]\widetilde{e}^1_| \oplus \mathbb{Z}[\pi]\widetilde{e}^1_- \longrightarrow \mathbb{Z}[\pi]\widetilde{e}^0 \longrightarrow 0$$

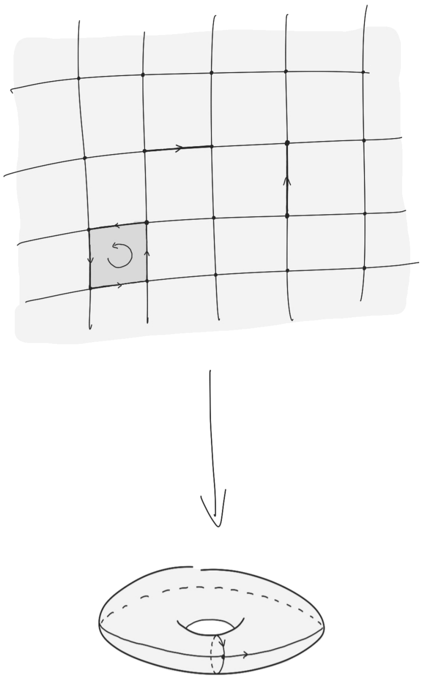


Figure 7: Universal covering of the torus

The boundary maps are as follows (remember that A denotes horizontal translation and B denotes vertical translation).

$$\begin{aligned}
\partial\widetilde{e}^0 &= 0 \\
\partial\widetilde{e}^1_- &= A^n B^m e^0 - A^{n-1} B^m \widetilde{e}^0 \\
&= A^{n-1} B^m (A-1)\widetilde{e}^0 \\
\partial\widetilde{e}^1_| &= A^k B^l \widetilde{e}^0 - A^k B^{l-1} \widetilde{e}^0 \\
&= A^k B^{l-1}(B-1)\widetilde{e}^0 \\
\partial\widetilde{e}^2 &= A^c B^d \widetilde{e}^1_- - A^c B^{d+1}\widetilde{e}^1_- + A^f B^g \widetilde{e}^1_| - A^{f-1} B^g \widetilde{e}^1_| \\
&= A^c B^d (1-B)\widetilde{e}^1_- + B^c A^{d-1}(A-1)\widetilde{e}^1_|
\end{aligned}$$

The exponents $n, m, k, l, c, d, f, g \in \mathbb{Z}$ depend on the chosen representatives of the lifts. For simplicity we can pick the representatives such that the values are all equal to 0. Then the boundary maps are simplified to

$$\begin{aligned}
\partial\widetilde{e}^0 &= 0 \\
\partial\widetilde{e}^1_- &= (A-1)\widetilde{e}^0 \\
\partial\widetilde{e}^1_| &= (B-1)\widetilde{e}^0 \\
\partial\widetilde{e}^2 &= (1-B)\widetilde{e}^1_- + (A-1)\widetilde{e}^1_|
\end{aligned}$$

Let $\varphi : \mathbb{Z}[\pi] \to \mathbb{F}$ be a ring homomorphism. Denote $a = \varphi(A)$ and $b = \varphi(B)$. The boundary maps become

$$\begin{aligned}
\partial \widetilde{e}^0 &= 0 \\
\partial \widetilde{e}^1_- &= (a-1)\widetilde{e}^0 \\
\partial \widetilde{e}^1_| &= (b-1)\widetilde{e}^0 \\
\partial \widetilde{e}^2 &= (1-b)\widetilde{e}^1_- + (a-1)e^1_|
\end{aligned}$$

We have to find conditions on φ under which we have that the corresponding chain complex is acyclic.

$$0 \longrightarrow \mathbb{F}\widetilde{e}^2 \xrightarrow{\partial_1} \mathbb{F}\widetilde{e}^1_| \oplus \mathbb{F}\widetilde{e}^1_- \xrightarrow{\partial_0} \mathbb{F}\widetilde{e}^0 \longrightarrow 0$$

For the chain complex to be acyclic it is required that the last boundary map ∂_0 is surjective. The map is surjective if either $a-1 \neq 0$ or $b-1 \neq 0$. We assume that $a-1 \neq 0$. Then the map ∂_0 has a one dimensional kernel. The map ∂_1 is defined as $\partial_1 e^2 = (1-b)\widetilde{e}^1_- + (a-1)e^1_|$. Since we assumed that $(a-1)$ is nonzero, we get that the image of ∂_1 is nonzero. And hence one dimensional. This implies that $H_1(C^\varphi) = \ker(\partial_0)/\mathrm{Im}(\partial_1) = 0$. Lastly we have that the map ∂_1 is non-zero and thus injective thus $H_2(C^\varphi) = 0$. So under the assumption that $a-1 \neq 0$ we get that the chain complex C^φ is acyclic. This means we can compute the torsion. We pick the following bases for the b_i

$$\begin{aligned}
b_{-1} &= \varnothing \\
b_0 &= (\widetilde{e}^0) = c_1 \\
b_1 &= ((1-b)\widetilde{e}^1_- + (a-1)\widetilde{e}^1_|) \\
b_2 &= \varnothing
\end{aligned}$$

This means that the term $[b_0 b_{-1}/c_0] = [b_0/c_0] = 1$ and pulling back the basis b_1 to $\mathbb{F}\widetilde{e}^2$ gives the basis $(\widetilde{e}^2) = c_2$. Hence $[b_2 b_1/c_2] = [b_1/c_1] = 1$. The term $[b_1 b_0/c_1]$ is computed as follows. We pull the basis b_0 back to $\mathbb{F}\widetilde{e}^1_| \oplus \mathbb{F}\widetilde{e}^1_-$ giving the basis $b_0 = ((a-1)^{-1}\widetilde{e}^1_-)$. Therefore we get that the matrix A representing the base change from c_1 to $b_1 b_0$ is

$$A = \begin{bmatrix} (a-1)^{-1} & b-1 \\ 0 & a-1 \end{bmatrix}$$

Thus the term $[b_1 b_0/c_1] = \det(A^{-1}) = (a-1)(a-1)^{-1} = 1$. Thus we get the torsion

$$\tau_\varphi(T) = 1 \in \mathbb{F}^*/\{\pm a^n b^m\}_{n,m\in\mathbb{Z}}.$$

By symmetry we also get the same quantity if assuming $b-1 \neq 0$. ■

This result of the torsion being 1 generalises as follows.

Theorem 3.3. [4] Let M be a compact orientable manifold without boundary. Then the Reidemeister torsion $\tau_\varphi(M)$ is equal to 1.

This implies that Reidemeister torsion is generally of no use when trying to distinguish odd dimensional manifolds.

3.3 Torsion of the lens spaces

In order to compute the torsion of a lens space we need a CW-decomposition for it. This is done by constructing a CW-decomposition of S^3, which is invariant under the $\mathbb{Z}/p\mathbb{Z}$-action defined on S^3. This way we get that the projection of the CW-decomposition of S^3 gives a CW-decomposition of our lens space. Lifting this CW-decomposition to the universal cover S^3 gives us the original CW-decomposition of S^3. This way we do not have to explicitly describe a CW-decomposition of the lens spaces since for the torsion calculation we only need the lift of a CW-decomposition to the universal cover.

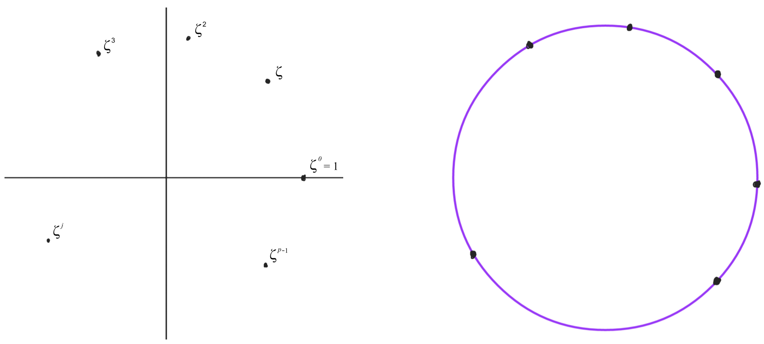
3.3.1 CW-decomposition of S^3

Let $\zeta \in \mathbb{C}$ generate the pth roots of unity. And denote for $j = 0, 1, 2, \cdots, p-1$ the arc on the unit circle between the points ζ^j and ζ^{j+1} as $I_j \subset \mathbb{C}$.

We define the following subsets of S^3. Which will serve as our decomposition.

$$
\begin{aligned}
E_j^0 &= \{(\zeta^j, 0)\} \subset S^3 \\
E_j^1 &= I_j \times 0 \subset S^3 \\
E_j^2 &= \{(z_1, t\zeta^j) \in \mathbb{C}^2 \mid 0 \leq t \leq 1, |z_1| = 1 - t^2\} \subset S^3 \\
E_j^3 &= \{(z_1, t\xi) \in \mathbb{C}^2 \mid \xi \in I_j,\ 0 \leq t \leq 1, |z_1| = 1 - t^2\} \subset S^3
\end{aligned}
$$

Here j runs from 0 to $p-1$. Here we define $X^0 = \bigcup_j E_j^0$ and $X^k = X^{k-1} \cup \left(\bigcup_j E_j^k\right)$. The set X^0 is the discrete space consisting of all pth roots of unity, illustrated in Figure 8a.



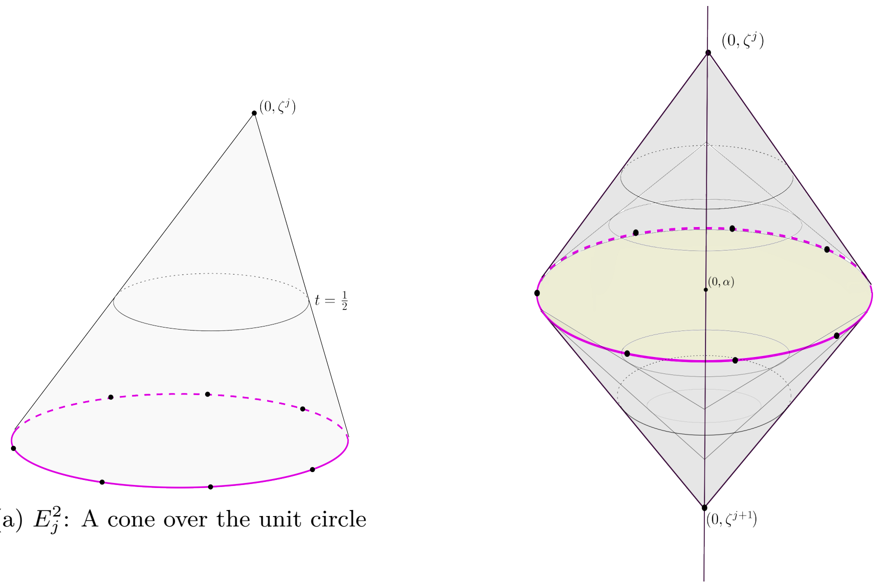
(a) X^0: The pth roots of unity

(b) $X^1 = S^1 \times 0 \subset \mathbb{C}^2$

Figure 8: X^0 and X^1

Then X^1 is the unit circle $S^1 \times 0 \subset \mathbb{C}^2$, where the sets E_j^1 are the arcs connecting the roots of unity, as illustrated in purple in Figure 8b.

The equation for E_j^2 satisfies the equation for a cone over the circle X^1 with as tip the point $(0, \zeta^j)$. This gives rise to the picture in Figure 9a. Consider the equation defining the set E_j^3. Fixing $\xi \in I_j$ gives us the subset of E_j^3, which is a cone over the circle with as tip the point $(0, \xi)$. Therefore we see that the set E_j^3 is the union of cones over the circle with tip $(0, \xi)$, where ξ runs along the arc I_j. At the endpoints of I_j we get that ξ is ζ^j or



(a) E_j^2: A cone over the unit circle

(b) E_j^3 as a union of cones over a circle

Figure 9: E_j^2 and E_j^3

ζ^{j+1}. These values of ξ give rise to the sets E_j^3 and E_{j+1}^3 respectively. Thus we see that the boundary of E_j^3 is equal to $E_j^2 \cup E_{j+1}^2$. Figure 9b illustrates this subset.

We define the open cells $e_j^k = \text{Int}(E_j^k)$. It is easily verified that these cells are disjoint and cover S^3.

We want to show that the sets E_j^k cells are homeomorphic to a closed disk and that the zero cells form a discrete space, which would then imply that the k-cells are open balls. For E_j^0 and E_j^1 this is clear. The sets E_j^2 are cones of a circle along a point, the projection $(z_1, t\zeta^j) \mapsto (z_1, 0)$ is a homeomorphism from E_j^2 to $D^2 \subset \mathbb{C}$. Now we show that E_j^3 is a closed 3-ball. As before fixing a $\xi \in I_j$ gives a subset of E_j^3 homeomorphic to a disk. All these discs have the same boundary, namely the unit circle $S^1 \times 0 \subset \mathbb{C} \times 0$ and are disjoint besides that. The discs corresponding to $\xi = \zeta^j$ and $\xi = \zeta^{j+1}$ can be shaped into the upper and lower hemisphere of a sphere respectively. Then the discs corresponding to the other values of ξ are layered on top of each other only connecting at their boundaries to fill the between these hemispheres thus giving a closed 3-ball.

We conclude that the sets E_j^k are all closed discs. We have the following topological boundaries:

$$
\begin{aligned}
\partial E_j^0 &= \varnothing \\
\partial E_j^1 &= E_j^0 \cup E_{j+1}^0 = \{(\zeta^j, 0), (\zeta^{j+1}, 0)\} \\
\partial E_j^2 &= E_0^1 \cup E_1^1 \cup \cdots \cup E_{p-1}^1 = S^1 \times 0 \subset \mathbb{C}^2 \\
\partial E_j^3 &= E_j^2 \cup E_{j+1}^2
\end{aligned}
$$

Since all these sets are subsets of S^3 and homeomorphic to k-discs, we can use the inclusion map to get the attaching map for each k-cell. Thus for a given k-cell e_j^k we define the attaching map $f_j^k : D^k \xrightarrow{\sim} E_j^k \hookrightarrow S^3$. This yields a CW-composition with $X^k = \left(\bigcup_j E_j^k\right) \cup X^{k-1}$.

Lemma 3.4. The $\mathbb{Z}/p\mathbb{Z}$-action of ζ permutes the cells in an orientation preserving way by $\zeta(e_j^k) = e_{j+1}^k$ for $k = 0, 1$ and $\zeta(e_j^k) = e_{j+q}^k$ for $k = 2, 3$.

Proof. For $k = 1, 2$ this is clear as the action is just multiplication by ζ in the first argument and multiplication by ζ is orientation preserving. For the set E_j^2 we get that applying ζ to it gives

$$\begin{aligned}
\zeta(E_j^2) &= \{(\zeta z_1, t\zeta^{j+q}) \in \mathbb{C} \mid 0 \leq t \leq 1, |z_1| = 1 - t^2\} \\
&= \{(z_1, t\zeta^{j+q}) \in \mathbb{C} \mid 0 \leq t \leq 1, |\zeta^{-1} z_1| = 1 - t^2\} \\
&= \{(z_1, t\zeta^{j+q}) \in \mathbb{C} \mid 0 \leq t \leq 1, |z_1| = 1 - t^2\} \\
&= E_{j+q}^2
\end{aligned}$$

This implies that $\zeta(e_j^2) = e_{j+q}^2$. Similar argument shows that $\zeta(e_j^3) = e_{j+q}^3$. ■

3.3.2 Cellular chain complex of S^3

The next step is to consider the cellular chain complex of the defined CW-decomposition and compute its boundary homomorphisms. For a fixed k, the chain group C_k is the free abelian group generated by the k-cells $\{e_j^k\}$. The boundary map $\partial : C_1 \to C_0$ is the simplicial boundary map, thus by picking the right orientations this boundary map is defined by $\partial e_j^1 = e_{j+1}^0 - e_j^0$. The second boundary map $\partial : C_2 \mapsto C_1$ is computed as follows. The space $X^1 = X^0 \cup \bigcup_j e_j^1 = S^1 \times 0$ inherits an orientation from the orientations of the 1-cells e_j^1, depicted by an arrow pointing anticlockwise. Given a 2-cell e_j^2 we orient its boundary such that the boundary gets attached to X^1 in an orientation preserving way. Thus for each 2-cell we get that the maps $f_{i,j}^2$ have degree 1. This way the boundary map is defined as $\partial e_j^2 = e_0^1 + e_1^1 + \cdots + e_{p-1}^1$. To find the boundary of the cells e_j^3, we observe that it will be a linear combination of the cells e_j^2 and e_{j+1}^2, since the topological boundary other 2-cells only intersect in the common circle with the topological boundary of e_j^3. To know what coefficients these cells have we have to look at the maps f_j^1 and f_j^2 as defined in equation 1 corresponding to the cells e_j^2 and e_{j+1}^2 respectively. Figure 10 illustrates these maps.

We want to compute the degrees of these maps. The degree of the maps depend on the orientations we choose of the depicted spheres. The map f_j^1 collapses the upper hemisphere to a point and the map f_j^2 collapses the lower hemisphere to a point. Then we see that if we take a reflection swapping the upper and the lower hemisphere called r, then the map $f_j^2 \circ r$ will collapse the upper hemisphere to a point. Thus we see that $r \circ f_j^2 = f_j^1$. Since degree of maps is multiplicative and the degree of a reflection is -1, we conclude that $\deg(f_j^1) = -\deg(f_j^2)$. We can then pick orientations such that $\deg(f_j^2) = 1$. This means that the boundary map $\partial : C_3 \to C_2$ is defined by $\partial e_j^3 = e_{j+1}^2 - e_j^2$.

3.3.3 Torsion of lens spaces

Theorem 3.5. Let p and q be coprime positive integers. Denote $T \in \pi(L(p,q)) =: \pi$ a distinguished generator and let $r \in \mathbb{Z}$ such that $rq = 1 \bmod p$. Let $\mathbb{F}$ be a field and

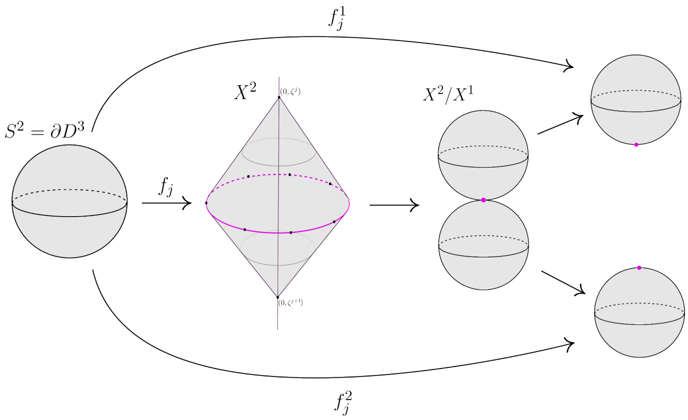


Figure 10: The maps f_j^1 and f_j^2.

$\varphi : \mathbb{Z}[\pi] \to \mathbb{F}$ be a ring homomorphism. Set $t = \varphi(T) \in \mathbb{F}^*$. If $t \neq 1$ then $H_*^\varphi(L(p,q)) = 0$ and the Reidemeister torsion of $L(p,q)$ is

$$\tau_\varphi(L(p,q)) = (t-1)(t^r-1) \in \mathbb{F}^*/\{\pm t^j\}_{j\in\mathbb{Z}/p\mathbb{Z}}.$$

Proof. We use the CW-composition described above. From Lemma 3.4 it follows that under the $\mathbb{Z}/p\mathbb{Z}$-action that for a the k-cells of a given dimension all lie in the same orbit. This means that under the projection this gives a CW-decomposition of $L(p,q)$, where there is one cell in each dimension. Lifting this decomposition to S^3 gives back the original decomposition of S^3. As before we pick representative lifts of the decomposition and let $\pi = \pi_1(L(p,q))$ act on it to get the chain complex of $\mathbb{Z}[\pi]$-modules, where each chain group has one generator since de CW-decomposition of $L(p,q)$ has one cell in each dimension. In each dimension we pick the cell e_0^k as the representative lift. This yields the following chain complex.

$$0 \longrightarrow \mathbb{Z}[\pi]e_0^3 \xrightarrow{\partial} \mathbb{Z}[\pi]e_0^2 \xrightarrow{\partial} \mathbb{Z}[\pi]e_0^1 \xrightarrow{\partial} \mathbb{Z}[\pi]e_0^0 \longrightarrow 0$$

With the boundary maps

$$\begin{aligned}
\partial e_0^0 &= 0 \\
\partial e_0^1 &= e_1^0 - e_0^0 \\
\partial e_0^2 &= e_0^1 + e_1^1 + \cdots + e_{p-1}^1 \\
\partial e_0^3 &= e_1^2 - e_0^2
\end{aligned}$$

Using Lemma 3.4 we can rewrite everything in terms of the generator $\zeta \in \pi$ and the representative lifts.

$$\begin{aligned}
e_1^0 &= e_{rq}^0 = \zeta^r e_0^0 \\
e_j^1 &= \zeta^j e_0^1 \\
e_1^2 &= e_{rq}^2 = \zeta^r e_0^2
\end{aligned}$$

Here we use that $rq = 1 \bmod p$. So we write the boundary maps as follows

$$\begin{aligned}
\partial e_0^0 &= 0 \\
\partial e_0^1 &= \zeta e_0^0 - e_0^0 = (\zeta - 1)e_0^0 \\
\partial e_0^2 &= e_0^1 + \zeta e_0^1 + \cdots + \zeta^{p-1} e_0^1 \\
&= (1 + \zeta + \cdots + \zeta^{p-1})e_0^1 \\
\partial e_0^3 &= \zeta^r e_0^2 - e_0^2 = (\zeta^r - 1)e_0^2
\end{aligned}$$

Given the ring homomorphism $\varphi : \mathbb{Z}[\pi] \to \mathbb{F}$, we get the following chain complex with induced boundary homomorphisms.

$$0 \longrightarrow \mathbb{F}e_0^3 \xrightarrow{\partial} \mathbb{F}e_0^2 \xrightarrow{\partial} \mathbb{F}e_0^1 \xrightarrow{\partial} \mathbb{F}e_0^0 \longrightarrow 0$$

$$\begin{aligned}
\partial e_0^0 &= 0 \\
\partial e_0^1 &= (t - 1)e_0^0 \\
\partial e_0^2 &= (1 + t + \cdots + t^{p-1})e_0^1 \\
\partial e_0^3 &= (t^r - 1)e_0^2
\end{aligned}$$

We assumed that $t \neq 1$, and thus $t - 1 \neq 0$ and since $t^p = \varphi(\zeta)^p = \varphi(\zeta^p) = \varphi(1) = 1$ we get that $1 + t + \cdots + t^{p-1} = \frac{t^p - 1}{p-1} = 0$. It follows that the boundary map $\partial : \mathbb{F}e_0^2 \to \mathbb{F}e_0^1$ is the zero map. Thus the chain complex reduces to

$$0 \longrightarrow \mathbb{F}e_0^3 \xrightarrow{\partial} \mathbb{F}e_0^2 \xrightarrow{0} \mathbb{F}e_0^1 \xrightarrow{\partial} \mathbb{F}e_0^0 \longrightarrow 0$$

For the chain complex to be acyclic we now need that the two non-zero boundary maps are isomorphisms, which is in this case equivalent to them not being the zero maps. We have that $t \neq 1$ and since r is coprime with p it follows that $t^r - 1 \neq 0$. Thus the maps are isomorphisms and so the chain complex is acyclic.

Extending the argument in Lemma 2.7 we see that the torsion is then the product of the determinants of the matrices representing the remaining non-zero boundary maps. Thus we get the torsion

$$\tau_\varphi(L(p,q)) = (t - 1)(t^r - 1) \in \mathbb{F}^*/\{\pm t^j\}_{j \in \mathbb{Z}/p\mathbb{Z}}.$$

■

3.4 Limitations of torsion calculations

Unfortunately with the way we have defined Reidemeister torsion it is not possible to calculate the torsion of every CW-complex. The problem that can arise is that for a given CW-complex X, it is not possible to make the chain complex $C^\varphi(X)$ acyclic, regardless of what ring homomorphism φ is chosen. To illustrate we give the example of the figure eight space. The CW-decomposition consists of one 0-cell e^0 and two 1-cells e_1^1 and e_2^1, where the endpoints of the 1-cells are attached to the 0-cell e^0. Picking representative lifts of these cells to the universal cover and letting $\mathbb{Z}[\pi] = \mathbb{Z}[\pi_1(S^1 \vee S^1)]$ act on them gives the chain complex.

$$0 \longrightarrow \mathbb{Z}[\pi]e_1^1 \oplus \mathbb{Z}[\pi]e_2^1 \longrightarrow \mathbb{Z}[\pi]e^0 \longrightarrow 0$$

Let $\varphi : \mathbb{Z}[\pi] \to \mathbb{F}$ be a ring homomorphism. Then we get the chain complex $C^\varphi(X)$ equal to

$$0 \longrightarrow \mathbb{F}e_1^1 \oplus \mathbb{F}e_2^1 \longrightarrow \mathbb{F}e^0 \longrightarrow 0$$

For this chain complex to be acyclic we need that the middle map is an isomorphism. This is not possible since the two vector spaces are not of the same dimension. So regardless of what the boundary map is, we can never find a ring homomorphism φ such that the chain complex is acyclic. Thus the torsion of the figure eight is not yet defined. To remedy this there are are more general notions of torsion. Examples of these are described in [4] and [9].

4 Classification of lens spaces

In this section we will complete the homeomorphism classification partially proved in Proposition 1.10. We will show the necessary conditions on q and q' in order for $L(p,q)$ and $L(p,q')$ to be homeomorphic.

Theorem 4.1. If $L(p,q) \cong L(p,q')$ are homeomorphic lens spaces, then $q' = \pm q^{\pm 1} \mod p$.

Partial proof. Let $\pi = \pi_1(L(p,q))$ and $\pi' = \pi_1(L(p,q'))$. We choose generators σ and σ' of π and π' respectively. Then for a given pth root of unity $\zeta \neq 1$ we define the map $\varphi_\zeta : \mathbb{Z}[\pi] \to \mathbb{C}$ defined by sending σ to ζ and we define the map $\varphi'_\zeta : \mathbb{Z}[\pi'] \to \mathbb{C}$ sending σ' to ζ. Let $f : L(p,q) \to L(p,q')$ be a homeomorphism. The induced map $f_* : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi']$ sends σ to a power of σ'. Thus the map $\psi_\zeta = \varphi_\zeta \circ f_*$ sends σ to ζ^d for some $d \in \mathbb{Z}$. By Theorem 2.18 it follows that $\tau_{\varphi_\zeta}(L(p,q)) = \tau_{\psi_\zeta}(L(p,q'))$. Let r and r' be integers such that $rq \equiv 1 \mod p$ and $r'q' = 1 \mod p$. Theorem 3.5 then gives us

$$(\zeta - 1)(\zeta^r - 1) = (\zeta^d - 1)(\zeta^{r'd} - 1) \in \mathbb{C}^*/\{\pm\zeta^n\}_{n\in\mathbb{Z}}.$$

This gives us

$$(\zeta - 1)(\zeta^r - 1) = \pm\zeta^l(\zeta^d - 1)(\zeta^{r'd} - 1) \in \mathbb{C}$$

for some $l \in \mathbb{Z}$. Then we take the square of the norm on both sides, which is the same as multiplying each side with its complex conjugate. This yields the following

$$(\zeta - 1)(\zeta^{-1} - 1)(\zeta^r - 1)(\zeta^{-r} - 1) = (\zeta^d - 1)(\zeta^{-d} - 1)(\zeta^{r'd} - 1)(\zeta^{-r'd} - 1) \in \mathbb{C}.$$

Showing that up to some permutation that the exponents of ζ in this equation are equal modulo p will prove the statement. Indeed if $d \equiv \pm 1 \mod p$ then automatically $r' \equiv \pm r \mod p$, which implies that $q' \equiv \pm q \mod p$. Otherwise if $r'd \equiv \pm 1 \mod p$, then $r' \equiv \pm d^{-1} \mod p$ and $d \equiv \pm r \mod p$. Hence $r' \equiv \pm r^{-1} \mod p$, which implies $q' \equiv \pm q^{-1} \mod p$. Other cases of exponents being equal are not possible. Showing that these exponents are equal in this way is not easy in general. To show it in general we need a number theoretic result. We will first prove the theorem for the special case of $d = 1$, which does not require any number theory.

Assume $d = 1$. This means that the induced map of the homeomorphism f sends the distinguished generator of π to the distinguished generator of π'. Assuming $d = 1$ gives the equality.

$$(\zeta - 1)(\zeta^{-1} - 1)(\zeta^r - 1)(\zeta^{-r} - 1) = (\zeta^1 - 1)(\zeta^{-1} - 1)(\zeta^{r'} - 1)(\zeta^{-r'} - 1)$$

Since $\zeta \neq 1$ this reduces to

$$(\zeta^r - 1)(\zeta^{-r} - 1) = (\zeta^{r'} - 1)(\zeta^{-r'} - 1)$$

Working out the brackets gives. $\zeta^r + \zeta^{-r} = \zeta^{r'} + \zeta^{-r'}$. We pick $\zeta = e^{\frac{2\pi i}{p}}$. This gives us the formula for twice the cosine on both sides namely.

$$e^{\frac{2\pi r i}{p}} + e^{-\frac{2\pi r i}{p}} = e^{\frac{2\pi r' i}{p}} + e^{-\frac{2\pi r' i}{p}}$$

This means
$$\cos\left(\frac{2\pi r}{p}\right) = \cos\left(\frac{2\pi r'}{p}\right).$$
Solving this gives
$$2\pi r' = 2\pi r + 2\pi pm$$
$$\text{or}$$
$$2\pi r' = -2\pi r + 2\pi pm$$
This implies $r' \equiv \pm r \bmod p$, thus $q' = \pm q \bmod p$. ■

Here we see that if we have a homeomorphism that preserves the distinguished generator of the fundamental group, then we get a stricter condition. These type of homeomorphisms are called 'enriched'. For the case of non-enriched homeomorphism we use the following number theoretic result

Lemma 4.2 (Franz Independence Lemma)**.** [5] Let $S = (\mathbb{Z}/p\mathbb{Z})^\times$. And let $\{a_j\}_{j\in S}$ be a set of integers satisfying.

1. $\sum_{j\in S} a_j = 0$
2. $a_j = a_{-j}$
3. $\prod_{j\in S}(\zeta^j - 1)^{a_j} = 1$ for each pth root of unity $\zeta \neq 1$.

Then $a_j = 0$ for each $j \in S$.

Full proof of Theorem 4.1. For each root of unity ζ, we have the equality
$$(\zeta - 1)(\zeta^{-1} - 1)(\zeta^r - 1)(\zeta^{-r} - 1) = (\zeta^d - 1)(\zeta^{-d} - 1)(\zeta^{r'd} - 1)(\zeta^{-r'd} - 1) \tag{2}$$
It remains to show that the sequences $(1, -1, r, -r)$ and $(d, -d, r'd, -r'd)$ are equal modulo p up to some permutation. For each $j \in S$ define m_j as the number of elements in the sequence $(1, -1, r, -r)$ equal to j modulo p. Similarly define m'_j as the number of elements in $(d, -d, r'd, -r'd)$ equal to j. Since the map $f_* : \zeta \mapsto \zeta^d$ is an isomorphism we must have that ζ^d generates the pth roots of unity, thus d is coprime with p. We conclude that the elements of both sequences are all units modulo p. Therefore we get that $\sum_{j\in S} m_j = \sum_{j\in S} m'_j = 4$. Furthermore we see that $m_j = m_{-j}$. Now define $a_j = m_j - m'_j$. It is clear that $\sum_{j\in S} a_j = 0$ and that $a_j = a_{-j}$. The way we defined m_j implies that the product $\prod_{j\in S}(\zeta^j - 1)^{m_j}$ is equal to the left hand side of the equation 2. Similarly $\prod_{j\in S}(\zeta^j - 1)^{m'_j}$ is equal to the right side of equation 2. This gives the following
$$\begin{aligned}\prod_{j\in S}(\zeta^j - 1)^{a_j} &= \prod_{j\in S}(\zeta^j - 1)^{m_j}(\zeta^j - 1)^{-m'_j} \\ &= \prod_{j\in S}(\zeta^j - 1)^{m_j}\left(\prod_{j\in S}(\zeta^j - 1)^{m'_j}\right)^{-1} = 1\end{aligned}$$
By Lemma 4.2 it follows that $a_j = 0$ for each j. Thus $m_j = m'_j$ for each j. Thus each exponent of ζ occurs equally often on the left as on the right side of the equation. Thus the sequences $(1, -1, r, -r)$ and $(d, -d, r, -r)$ are equal modulo p up to some permutation, which proves the claim. ■

5 Homotopy and homology of lens spaces

In this section we will briefly introduce the concepts of homotopy groups of a topological space. Then we compute the homotopy and homology groups of the lens spaces. It turns out for lens spaces $L(p, q)$ that between these groups only the fundamental group and the first homology group depends on p and none of them depend on q. This is an interesting observation that further exemplifies that standard techniques of distinguishing topological spaces from algebraic topology do not work on lens spaces. We conclude with a description of how one could construct a homotopy equivalence between lens spaces. Leaving the full proof of it being a homotopy equivalence as an exercise for the reader.

5.1 Homotopy groups

The fundamental group is the group of homotopy classes of loops with fixed basepoint in a topological space. Loops are paths that begin and end at the same point, this means we can view loops as maps from S^1 to a given topological space X with fixed base points. More specifically the fundamental group $\pi_1(X, x)$ consists of homotopy classes of pointed maps $\gamma : (S^1, y) \to (X, x)$. Homotopy groups are a natural generalization of this. We define the ith homotopy group $\pi_i(X, x)$ as the set of homotopy classes of pointed maps $\gamma : (S^i, y) \to (X, x)$. How the specific group structure works will be left out here. However an interesting thing to note about the group structure of the higher homotopy groups is that they are all abelian in contrast with the fundamental group, which does not have to be abelian. For a more thorough introduction to homotopy groups refer to [3].

Example 5.1 (Homotopy groups of the spheres). While homology groups of the spheres are very simple ($H_i(S^n) = \mathbb{Z}$ if $i = n$ and 0 otherwise), the homotopy groups of the spheres exhibit much more seemingly irregular and chaotic patterns. A couple of the homotopy groups of different spheres are given in Figure 11 to illustrate this behaviour. What is

$\pi_i(S^n)$

$n \downarrow$, $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Figure 11: $\pi_i(S^n)$ for some values of i and n from [8]

known, and can been seen in the figure, is that the homotopy groups $\pi_n(S^k)$ are trivial in the case $n < k$ and equal to $\mathbb{Z}$ if $k = n$. A lot more patterns in the homotopy groups can be described using tools like fibre bundles.

Given a map $f : X \to Y$ between topological spaces there is the induced homomorphism $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$. Defined by $f_*[\gamma] = [f \circ \gamma]$. The following result gives that the homotopy groups are a homotopy invariant as one would expect.

Theorem 5.2. [3] Let $f : X \to Y$ be a homotopy equivalence. Then the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 2$.

Furthermore we have the following nice property with respect to covering spaces.

Theorem 5.3. [3] Let $p : (Y, y) \to (X, x)$ be a covering projection. Then the induced map $p_* : \pi_n(Y, y) \to \pi_n(X, x)$ is an isomorphism for each $n \geq 2$.

The following general is a partial converse to Theorem 5.2 applied to CW-complexes.

Theorem 5.4 (Whitehead's theorem)**.** [3] Let $f : (X, x) \to (Y, y)$ be a map between pointed CW-complexes such that the induced homomorphism $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 1$, then f is a homotopy equivalence

Note that it is not true in general that if X and Y have isomorphic homotopy groups that then X and Y are homotopy equivalent. The important thing is that all isomorphisms come from a single map.

5.2 Homotopy and singular homology groups of lens spaces

Computing the homotopy groups of a lens space $L(p, q)$ is easy. Since we have a covering projection $S^3 \to L(p, q)$ it follows that for all the lens spaces that their homotopy groups are isomorphic to the homotopy groups of S^3, except for the fundamental group, which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Let $L(p, q)$ be a lens space. To compute the homology groups we use Proposition 2.11. The CW-decomposition of S^3 described earlier induces a CW-decomposition of $L(p, q)$ with one cell in each dimension by projecting each cell. Denote $e^k = \pi(e^k_j)$, with $\pi : S^3 \to L(p, q)$ the covering projection map. Recall the boundary homomorphism of the CW-decomposition of S^3.

$$\begin{aligned}
\partial e^0_0 &= 0 \\
\partial e^1_0 &= \zeta e^0_0 - e^0_0 \\
\partial e^2_0 &= e^1_0 + \zeta e^1_0 + \cdots + \zeta^{p-1} e^1_0 \\
\partial e^3_0 &= \zeta^r e^2_0 - e^2_0
\end{aligned}$$

Then we can compute the boundary homomorphism for the CW-decomposition of $L(p, q)$ as follows.

$$\begin{aligned}
\partial e^0 &= 0 \\
\partial e^1 &= \partial \pi e^1_0 \\
&= \pi \partial e^1_0 = \pi(\zeta e^0_0 - e^0_0) = 0 \\
\partial e^2 &= \partial \pi e^2_0 = \pi \partial e^2 \\
&= \pi(e^1_0 + \zeta e^1_0 + \cdots + \zeta^{p-1} e^1_0) = p e^1 \\
\partial e^3 &= \partial \pi e^3_0 = \pi \partial e^3_0 \\
&= \pi(\zeta^r e^2_0 - e^2_0) = 0
\end{aligned}$$

Thus we get the cellular chain complex of $L(p, q)$

$$0 \longrightarrow C_3 \xrightarrow{0} C_2 \xrightarrow{p} C_1 \xrightarrow{0} C_0 \longrightarrow 0$$

From this we can read off the homology groups $H_i(L(p, q))$ as

$$H_i(C(L(p,q))) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/p\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \mathbb{Z} & \text{if } i = 3 \\ 0 & \text{if } i \geq 4 \end{cases}$$

Proposition 2.11 then gives us that these groups are isomorphic to the singular homology groups. We conclude that between the singular homology groups and the homotopy groups, only H_1 and π_1 actually depend on the parameter p and there is no dependence at all on the parameter q.

5.3 Homotopy classification of lens spaces

Recall the homotopy classification of lens spaces mentioned in chapter 1.

Theorem 5.5. [9] Lens spaces $L(p, q)$ and $L(p, q')$ are homotopy equivalent if and only if $qq' = r^2 \bmod p$ for some integer r.

So while we've shown that for a given p that the homotopy and singular homology groups of a lens space $L(p, q)$ are independent on q, we see that the lens spaces are not always homotopy equivalent, thus showing that homotopy groups and homology groups do not fully determine the homotopy of a topological space. Furthermore this result gives us examples of 3-manifolds that are homotopy equivalent, but not homeomorphic. These lens spaces were actually the first examples of 3-manifolds that exhibit this behaviour [2].

The next proposition shows the sufficient condition for the existence of a homotopy equivalence. We will construct a map that will turn out to be a homotopy equivalence, however we will not show that this is a homotopy equivalence and leave out some details.

Proposition 5.6. If $qq' = n^2 \bmod p$ for some $n \in \mathbb{Z}$ then there exists a homotopy equivalence $g : L(p, q) \to L(p, q')$.

Partial proof sketch. Denote $L = L(p, q)$ and $L' = L(p, q)$. Since q and q' are both coprime with p, there exists a $k \in \mathbb{Z}$ such that $q' = kq \bmod p$. Define the map $f : S^3 \to S^3$ by $f(e^{\theta_1 i}, e^{\theta_2 i}) = f(e^{\theta_1 i}, e^{k\theta_2 i})$. Given that $q' = kq \bmod p$ then this map induces a map $\bar{f} : L \to L'$. Then our homotopy equivalence is constructed as the composition of the following maps. First we take the map that takes our lens space L and a small 3-dimensional ball within it and collapses the boundary of that small ball to a point. This yields back our lens space with a 3-sphere wedged to it, so $L \vee S^3$. How this works is illustrated in Figure 12 with the line and the endpoints of an intervals collapsed to a point.

The second map takes S^3 in $L \vee S^3$ and applies a degree d map to it. Yielding a map $L \vee S^3 \to L \vee S^3$. The last map applies $\bar{f}$ to L in $L \vee S^3$ and applies the covering projection $S^3 \to L'$ to S^3, this yields a map $L \vee S^3 \to L'$. Then the map $g : L \to L'$ is the composition

$$g : L \to L \vee S^3 \to L \vee S^3 \to L'.$$

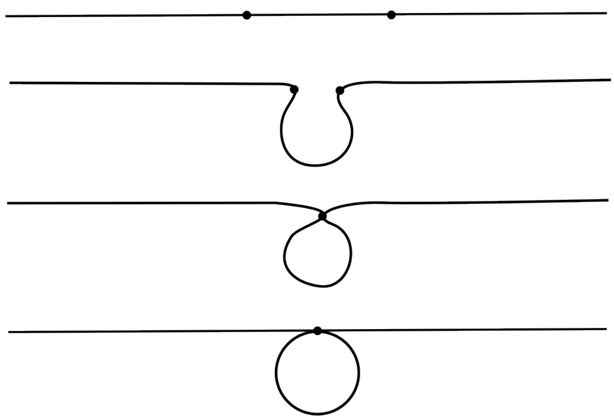


Figure 12: $\mathbb{R}$ becomes $\mathbb{R} \vee S^1$

It can be shown that given the assumption that $qq' = n^2 \bmod p$ for some $n \in \mathbb{Z}$ then we can pick d such that the degree of this map g is ± 1 i.e. the map $g_* : H_3(L) \to H_3(L')$ is multiplication by ± 1 on $H_3(L) = H_3(L') = \mathbb{Z}$. This can be shown to imply that the induced map g_* is an isomorphism on all homotopy groups, then by Whiteheads theorem it follows that g is a homotopy equivalence. Why this implies that the induced map is an isomorphism on all homotopy groups is a good point for further research, as it is not in general the case that degree ± 1 maps are homotopy equivalences. $\blacksquare$

6 Conclusion

We can say we have classified all the lens spaces up to homeomorphism. Furthermore we saw a construction of a homotopy equivalence. Looking at the conditions for the existence of this homotopy equivalence, they are strictly weaker than the conditions for the existence of a homeomorphism. Meaning that there are homotopy equivalent lens spaces that are not homeomorphic. Like said before these are the first examples of 3-manifolds that show this behaviour and it points out a weak point of algebraic topology applied to manifolds. Apparently homotopy is not enough to describe all manifolds, thus requiring the search for other techniques like Reidemeister torsion.

Good points for further research are how we can generalize the technique of Reidemeister torsion to apply it to more types of spaces. As we've seen Reidemeister torsion is not defined for the figure eight space, which would restrict its usage. Furthermore there is no way to distinguish orientation using Reidemeister torsion. More general ways of defining torsion using Euler structure remedy this problem.

Lens spaces as spaces themselves are also an interesting point of further research. For example how the different ways of constructing the lens spaces (gluing solid tori along their boundary or identifying points on the boundary of a ball) are related.

Lastly lens spaces can be generalised in a natural way to a class of $(2n-1)$-manifolds, by embedding S^{2n-1} in $\mathbb{C}^n$. These spaces are also classified using Reidemeister torsion, also following the same procedure as in this thesis.

In conclusion, as it turns out, while constructing lens spaces is quite easy, they have deep properties that requires us to reassess how we study 3-manifolds.

References

[1] T. A. Chapman. "Topological Invariance of Whitehead Torsion". In: *American Journal of Mathematics* 96.3 (1974), pp. 488–497. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2373556.

[2] C. McA. Gordon. "CHAPTER 15 - 3-Dimensional Topology up to 1960". In: *History of Topology*. Ed. by I.M. James. Amsterdam: North-Holland, 1999, pp. 449–489. ISBN: 978-0-444-82375-5. DOI: https://doi.org/10.1016/B978-044482375-5/50016-X. URL: https://www.sciencedirect.com/science/article/pii/B978044482375550016X.

[3] Allen Hatcher. *Algbraic Topology*. 2001.

[4] J. Milnor. "Whitehead torsion". In: *Bulletin of the American Mathematical Society* 72.3 (1966), pp. 358–426. DOI: bams/1183527946. URL: https://doi.org/.

[5] G. Rham, S. Maumary, and M. A. Kervaire. *Torsion et Type Simple d'Homotopie*. 1967.

[6] W. Threlfall and H. Seifert. "Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes". In: *Mathematische Annalen* 104 (1931), pp. 1–70. URL: http://eudml.org/doc/159448.

[7] Heinrich Tietze. "Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten". In: *Monatshefte für Mathematik und Physik* 19 (1908), pp. 1–118.

[8] Hirosi Toda. *Composition Methods in Homotopy Groups of Spheres. (AM-49), Volume 49*. Princeton University Press, 1962. URL: http://www.jstor.org/stable/j.ctt1bgzb5t (visited on 07/01/2022).

[9] Vladimir Turaev. *Introduction to Combinatorial Torsions*. 2001.