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A one-cocycle evaluated on the rotation loop of a knot

Bachelor's Project Mathematics

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Abstract

Thomas Fiedler formulated a framework for constructing knot invariants based on one-cocyles evaluated on loops in the space of knots[4][5]. We explain the necessary theory and clearly define and illustrate the definitions required for one of these cocycles, $R^{(2)}_{n,d^+}(a, r, K, \gamma)$, which was conjectured to be able to distinguish between mirrors. We give new direct but partial proofs of its invariance. It turns out that while the cocycle can not distinguish between mirrors its evaluation on the first half of the rotation loop gives rise to an equation that is able to distinguish between mirrors of knots, but is no longer an invariant.

Keywords: Knots, knot invariant, one-cocycle, Gramains loop, $R^{(2)}_{n,d^+}(a, r, K, \gamma)$.

Contents

1 Introduction

Knots are fascinating mathematical objects. The study of knot theory is interested in developing methods to distinguish different knots. This thesis will be about one such method that involves rotating a knot.

These methods to distinguish knots are called knot invariants and are usually computed on a knot diagram. Fiedler [4][5] has developed a theory of knot invariants that are instead computed on a sequence of knot diagrams. An example of such a sequence is to take a long knot and rotate it 360 degrees around its axis. Then we can discretize this continuous movement of a knot to a sequence of knot diagrams where going from one to the next is exactly one Reidemeister move. A one-cocycle is a function that associates a value to such a sequence. The main idea of Fiedler is to count the Reidemeister III moves while taking into account the rest of the structure of the knot. To do this we do not look at the knot itself but instead on a cabled version of the knot. Intuitively we get a cabled knot by tying the knot in multiple strands of rope and then gluing those together. We assign to each crossing their homological marking, which is the amount of times you need to travel through the knot between the under and over crossing. We use this additional information to weigh the Reidemeister moves depending on the other crossings in the knot. See figure 1 for a taster of these ideas.

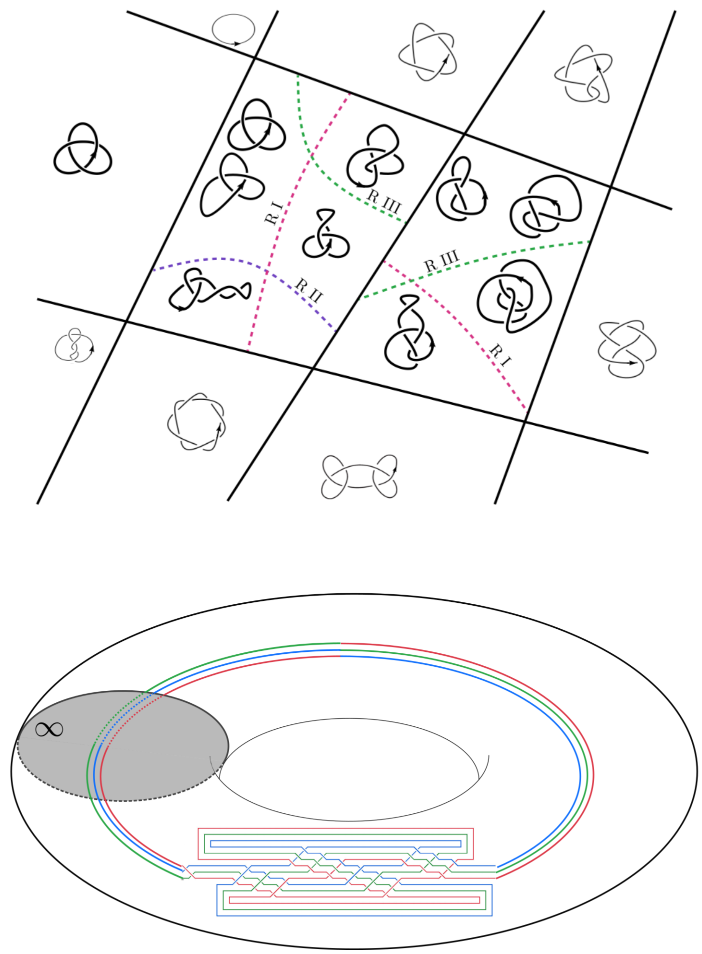


Figure 1: The space of all knots and a cabled knot in a torus.

Fiedler constructed many cocycles based on this framework and proved that they are invariant.

In this thesis we chose one of the cocycles, called $R^{(2)}_{n,d^+}(a, r, K, \gamma)$, that was deemed especially promising to distinguish mirrors. As Fiedlers definitions are rather hard to read we have wrote them out very explicitly and proved that the definition can be simplified a bit.

Our goal was to prove the invariance of this cocycle under RII and RIII in a more direct way. Although we were not able to proof this for all cases, the techniques we used could likely be used for the remaining case as well as for other cocycles. From the cases that we did proof it follows that, under certain conditions, dropping some of the terms is gives an equation that is still invariant under RII and likely also under RIII and modified stabilisation. However numerical results show this is no longer a knot invariant as it is not invariant under conjugation.

In the numerical results we also show that although the cocycle evaluated on the rotation loop is not able to distinguish between mirrors the contributions to the cocycle of a knot and its mirror do not correspond in a trivial way. This, together with the highly asymmetric behaviour between the first and second half of the rotation loop suggest that the cocycle could be modified to detect mirrors. Our equation is exactly the contributions of the first half of the cocycle and is in fact able to distinguish mirrors.

In chapter 2 we will give some basic knot theory background, then in chapter 3 we will give the setup for Fiedlers theory of cocycles and briefly outline his approach for constructing cocycles. Our main result is the invariance of R under RII which we will prove in chapter 4. In chapter 5 we will give the definitions of the cocycle on which R is based and give partial proofs of its invariance. We end with chapter 6 where we give and discuss some numerical results using the program written by Roland van der Veen. Finally we state some open questions and suggestions for further research.

2 Basics of knot theory

In this chapter we give the necessary definitions and terminology of knot theory that we will need in the subsequent chapters. We also state some results on when two knots are equal and when something is a knot invariants.

2.1 Knots

Definition 2.1. *A **knot diagram** is a planar inbedded oriented four valent graph where each vertex has a sign, which we depict as a crossing, see definition 2.2. The orientation is such that at each vertex you follow the path directly across, see figure 2.*

Figure 2: Mirrors and rotations of the first two vertices are allowed, the third is not allowed.

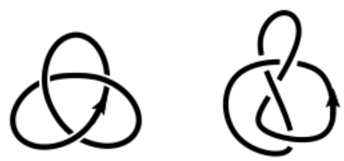
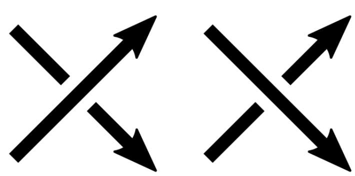


Figure 3: Some examples of knots.

For a source on graph theory one could look at [15].

Definition 2.2. *If c is a crossing in an oriented knot then its **sign** can be either positive or negative and is denoted by $(-1)^c$, see figure 4.*



$(-1)^c = -1$ $\quad$ $(-1)^c = +1$

Figure 4: The sign of a crossing

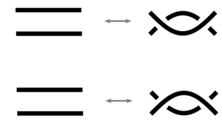
Thinking back to the more everyday notion of a knot we would like our knot to stay the same knot if we move around the rope a bit. Starting at a different definition of knots the following theorem gives necessary and sufficient conditions on when two knots are the same.

Theorem 2.3 (Reidemeister[12])**.** *Two knot diagrams represent the same knot(type) if and only if they are connected by a sequence of Reidemeister moves.*

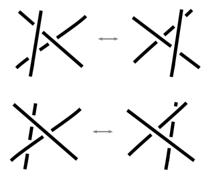
Definition 2.4. *There are three Reidemeister moves and planar isotopy also known as Reidemeister 0:*

(a) R0: Planar isotopy.

(b) RI: Create or remove a twist.



(c) RII: Move two parallel strands over or under each other.



(d) RIII: Move a strand over or under a crossing.

Figure 5: The three Reidemeister moves

Taking into account orientations the following generate all oriented Reidemeister moves [11].

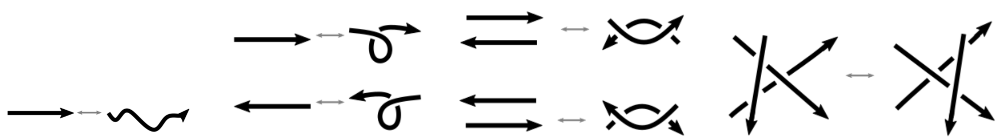


Figure 6: These 6 oriented Reidemeister moves generate all oriented Reidemeister moves.

A short remark on terminology. We will very often shorten the Reidemeister moves to RI, RII and RIII. Moreover we will use the term knot type to refer to an equivalence class of knots and knot diagram to refer to a specific representative of an equivalence class of knots. The term knot will be used for both instances where it is clear from the context which is implied.

There are many different ways to define knots, for some other definitions see [8]. A different way of looking at knots that agrees more with the real world knots that we encounter is that of a long knot:

Definition 2.5. *A* ***long knot*** *is defined in the same way as a knot diagram except that there are exactly two 1 valent vertices. In figure 7 the trefoil knot and the figure eight knot are shown as long knots.*

It can be shown that these definitions describe the same knots.

In the rest of this section we will introduce some properties and classes of kots.

Definition 2.6. *The* ***writhe*** *of a knot diagram is the sum of the signs of its crossings, see figure 7.*

Definition 2.7. *The* ***winding number*** *of a knot diagram is number of times the knot goes counterclockwise minus the number of times it goes clockwise, see figure 7.*

Definition 2.8. *The* ***mirror*** *of a knot K, denoted by K' is the knot one gets when changing the sign of all the crossings. Knots for which K and K' are the same knot type are called* ***amphichiral*** *knots and knots that are not equivalent to their mirror are called* ***chiral*** *knots.*

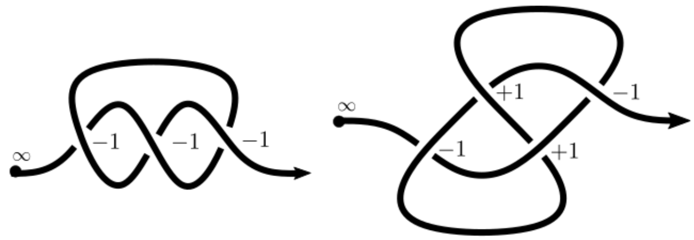


Figure 7: The writhe of this trefoil is -3, its winding number is 2. Both the writhe and the winding number of the figure eight knot is 0.

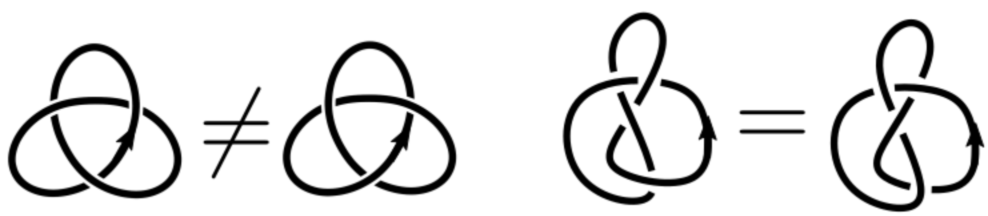


Figure 8: The trefoil knot and the figure eight knot with their mirrored version.

The trefoil knot is an example of the a chiral knot and the figure eight knot is an example of an amphichiral knot, see figure 8.

The (p, q)-torus knot is the knot constructed by going p times around the longitudinal direction and going q times through the hole in the torus, see figure 9.

Figure 9: The (2,3) torus knot (trefoil knot), the (2,5) torus knot (cinqefoil knot) and the (3,-8) torus knot.[3][7]

Definition 2.9. *A* ***cabled knot*** *is a knot where instead of knotting one strand you knot multiple parallel strands that twist around each other such that the resulting knot consists of one component, see figure 10.*

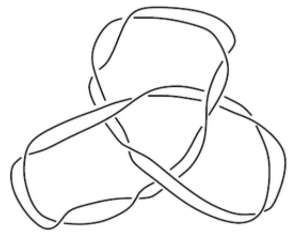


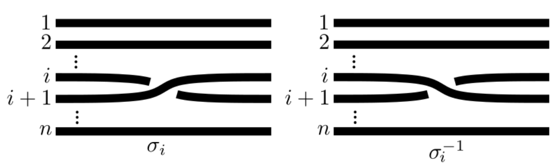
Figure 10: A cabled trefoil knot.[13]

2.2 Braids and Gauss diagrams

There are a couple other ways to represent knots that might be more convenient. One of these is the long knot as defined above. Another point of view is that of braids, which consists of multiple pieces of string that are braided and then joined up at the endpoints. Finally Gauss diagrams are more abstract way to encode where the crossings lies in relation to each other that we will use extensively through the rest of this thesis.

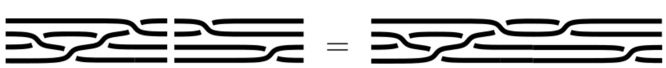
Definition 2.10. *A* ***braid*** *is an element from the braid group B_n, where n is the number of strings in the braid. The group B_n is generated by the set $\{\sigma_i | i = 1, \ldots, n-1\}$, with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ and $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1$.*

Geometrically the σ_i are twists between the ith and $i+1$ th strand, the operation is composition of braids. The inverse of σ_i is a twist in the opposite direction. Note that the second relation looks like the Reidemeister III move and the Reidemeister II move is recognizable in that $\sigma_i\sigma_i^{-1} = 1$, see figure 11.

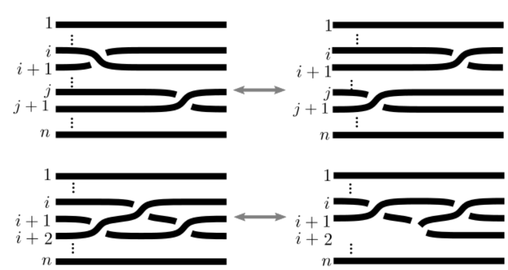


(a) A generator σ_i and its inverse σ_i^{-1}.

(b) The braid $\sigma_2^{-1}\sigma_3\sigma_2\sigma^{-1}$.



(c) Composition of braids.



(d) The braid relations.

Figure 11: Braids.

Definition 2.11. *A* ***closed braid*** *is constructed by joining up the opposite strings of a braid, see figure 12.*

Like for knot we want conditions on when two braids are the same. We also want to be able to connect the theory of braids to that of knots.

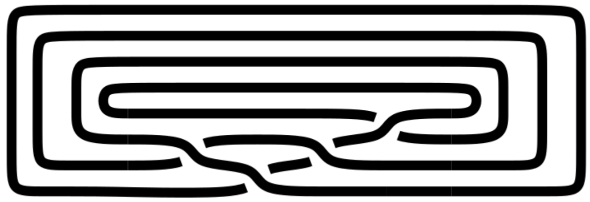


Figure 12: The closure of a braid.

Theorem 2.12 (Alexander[1])**.** *Every knot can be represented by a closed braid.*

Theorem 2.13 (Markov[10])**.** *Two braids represent the same knot when closed if and only if they are related by a sequence of Markov moves.*

Definition 2.14. *The conjugation move says that $ABA^{-1} = B$ for $A, B \in B_n$. The stabilisation moves says that $B = B\sigma_n$ and $B = B\sigma_n^{-1}$ for $B \in B_n$.*

Figure 13: The stabilisation move

Finally we will define the Gauss diagram of a knot.

Definition 2.15. *Let K be an oriented knot with a starting point marked. Alternatively let K be a long knot with a marked point. The* ***Gauss diagram*** *of K is a circle on which points are connected by arrows corresponding to crossings in the following way: starting at the starting point or the point of infinity, go through the knot following its orientation and mark each under crossing with a point and each over crossing with an arrow head and connect the point and arrowhead corresponding to the same crossing with a line. Finally we mark the signs, see figure 14 for two examples.*

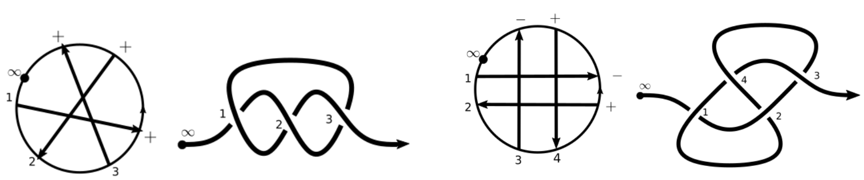


Figure 14: Gauss diagram of a trefoil knot and figure eight knot.

Note that the numbering of the crossings is usually omitted and the signs will be omitted or indicated by colour in the remainder of this thesis.

2.3 Invariants

One of the main goals in knot theory is to distinguish different knots. To this aim many knot invariants have been constructed. Knot invariants are things that one can compute for a knot, often on a single knot diagram. Their defining property is that they will give the same result for different representations of the same knot type. This means that if a knot invariant has a different value for two knot diagrams then the two knot diagram must represent different knot types.

Definition 2.16. *Let K and K' be two knots of the same knot type, then an **invariant** is a function on a knot such that it has the same value on K and K'.*

One example of a knot invariant is the minimal crossing number.

Example 2.17. *The **crossing number** is the minimal number of crossings needed in a knot diagram. For example the trefoil knot has minimal crossing number 3, the figure eight knot 4 and the cinquefoil knot 5.*

Some non-examples would be the number of crossings is a knot diagram, the writhe or the winding number.

We can use Theorem 2.3 and Theorem 2.12 to show that something is an invariant. We can also generalise the notion of invariants and look at functions that are invariant under a subset of these moves. For example the cocyle which we will discuss in chapter 5 is only invariant under RII and RIII. This is still a useful notion as we can replace the invariance under RI by a condition on the writhe and winding number.

Theorem 2.18 (Trace[14])**.** *Two knots are equivalent if they have the same writhe and winding number and are connected by a sequence of RII and RIII moves.*

One could likely prove an analogous result for Markov's theorem replacing the stabilisation move by adding two twists in opposites direction preserving the writhe and winding number.

Theorem 2.19 (Modified Markov)**.** *Two braids are equivalent if they have the same writhe and winding number and are connected by a sequence of conjugation and modified stabilisation moves. The modified stabilisation move says that $B = B'\sigma_1\sigma_n^{-1}$ or $B = B'\sigma_1^{-1}\sigma_n$ for $B \in B_n$, where B' is the braid B shifted one string down.*

That is if $f : B_n \longrightarrow B_{n+1}$
$\sigma_i \longmapsto \sigma_{i+1}$ then $B' = f(B)$, see figure 15.

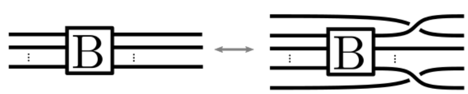


Figure 15: The modified stabilisation move.

In chapter 4 and 5.1 we will proof the invariance of R_n and $R^{(2)}_{n,d^+}(a, r, K, \gamma)$ under RII. We know from numerical results that R_n is not invariant under conjugation, but we suspect that it is invariant under RIII and the modified stabilisation move, see 6.4.

3 Loops in M_n

In this chapter we will explain some of the theory that Fiedler used to construct cocycles. We have used the introductory chapter of his latest work [5] as well as the master thesis of Aarnout Los [9].

3.1 M_n

The space of all knots, called $\mathcal{K}$, is the infinite dimensional space containing all knot diagrams. Each knot diagram is point in this space and each knot type lives in its own path connected component. In figure 16 these components are bounded by the black walls. A curve through this space is a sequence of knot diagrams. Performing a Reidemeister move is crossing one of the dashed coloured walls. These dashed coloured walls are called strata. Be aware that this picture should really be infinite dimensional which means that the strata which appear to be lines are instead of codimension one. Where the strata intersect we perform multiple Reidemeister moves at the same time and these strata have codimension two or higher.

A sequence of Reidemeister moves likewise corresponds to a path through this space. As we only care about the Reidemeister moves we can represent a path through this space as a sequence of knot diagrams with exactly one Reidemeister move between them.

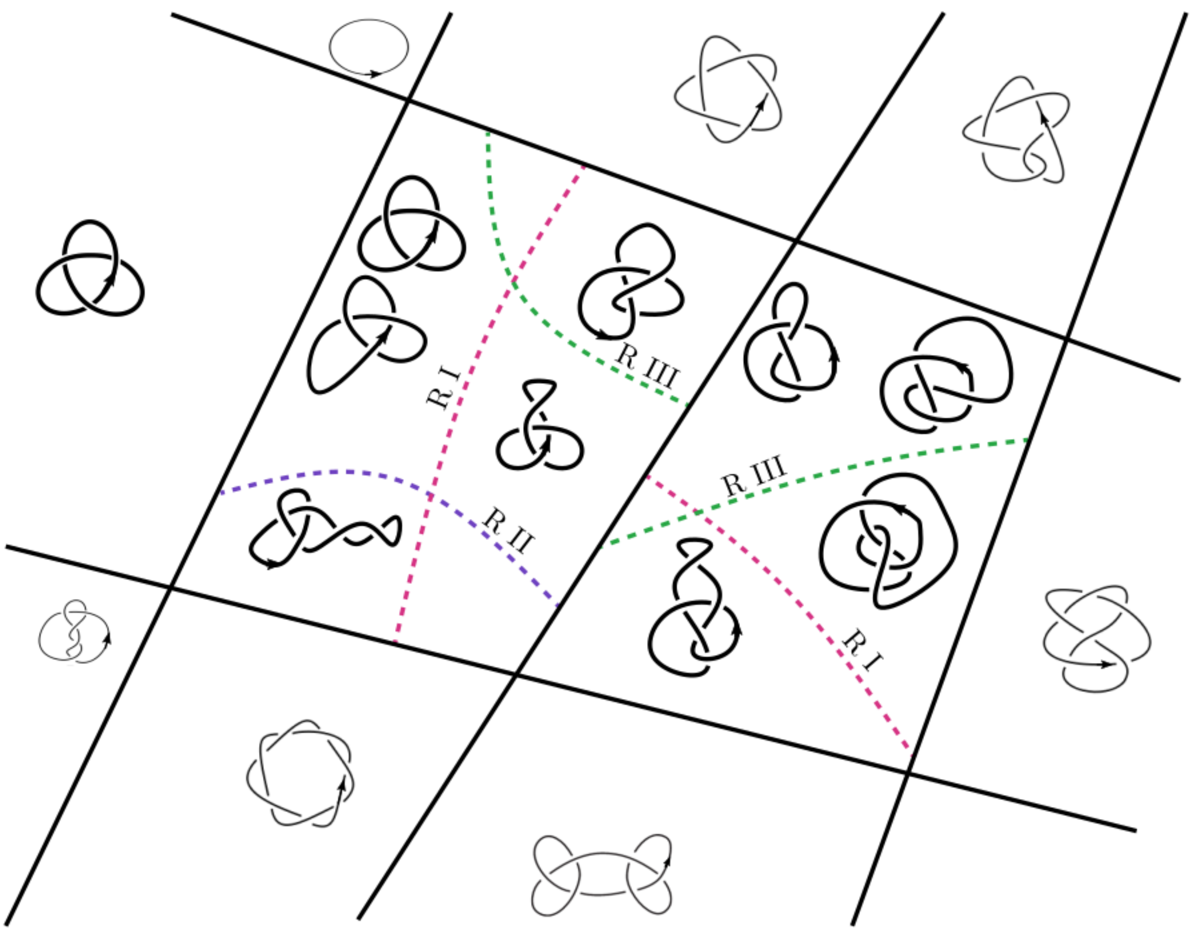


Figure 16: The space of all knots $\mathcal{K}$.

Analogous to the description above we can talk about the space of all long cabled knots M_n. Each component now contains a n-cabled long knot. This will be the space on which we walk loops and define our cocycles. For a more rigorous exposition of these spaces see [5].

3.2 Loops

There are two interesting loops that one could walk in M_n. These loops are nontrivial in the sense that they are non contractible, in the case of torus knots the Fox Hatcher loop is a multiple of the rotation loop but for all other loops they are distinct [6]. They are the rotation loop and the Fox Hatcher loop.

Definition 3.1. *The rotation loop is the loop obtained by rotating a long knot around its axis by 2π. It is also known as Gramains loop.*

There are of course two directions in which one can rotate a knot, in this thesis we will always assume it is the one depicted in figure 18.

Definition 3.2. *The Fox Hatcher loop, also known as the rolling loop, is the loop obtained by from right to left sequentially pulling each over strand over the whole knot and pulling each under strand under the strand, see figure 17 for the this loop applied to the figure eight knot.*

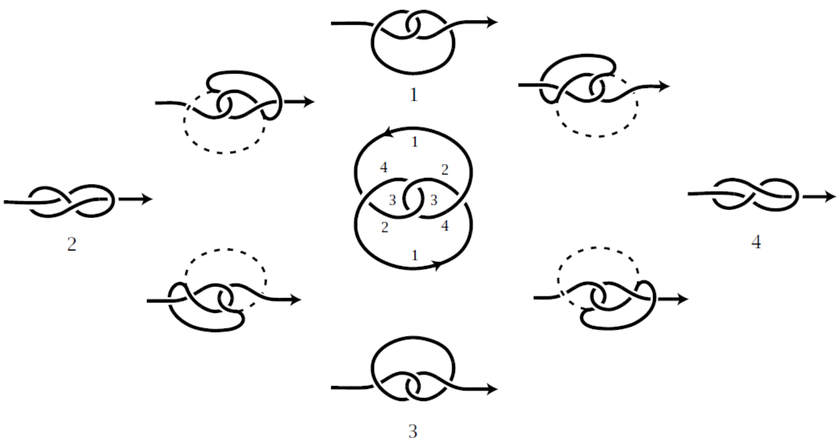


Figure 17: The Fox Hatcher loop applied to the figure eight knot.[6] (Start at 2 and go counterclockwise)

These loops are continuous but we only care about the Reidemeister walls that we cross. We can therefore represent these loops as a sequence of knot diagrams where each differs from the next by exactly one Reidemeister move. Moreover for our cocycles we only compute contributions from the Reidemeister III moves so those are the only ones we need.

For the rotation loop there is a nice way to generate these knot diagrams, see figure 18. For our com-

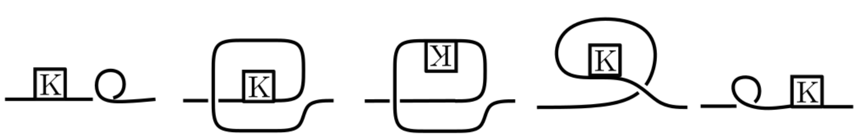


Figure 18: Rotation loop

putations we assume that our knot is in a standardised form called a balanced knot. This is necessary as the cocycle is not invariant under RI. Recall from theorem 2.19 that we can replace the invariance of RI by the condition that the writhe and winding number are equal for two knot diagrams. Also recall theorem 2.12 which states that every knot van be represent by a closed braid.

Definition 3.3. *A **balanced knot diagram** is a knot diagram of a partially closed braid, forming a long knot, such that their writhe and winding number is zero, we can always achieve such a form by adding loops using R I, see figure 19.*

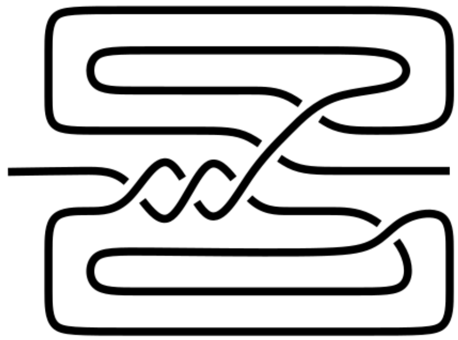


Figure 19: A balanced trefoil knot

In figure 20 you can see an example of the rotation loop applied to the figure eight knot.

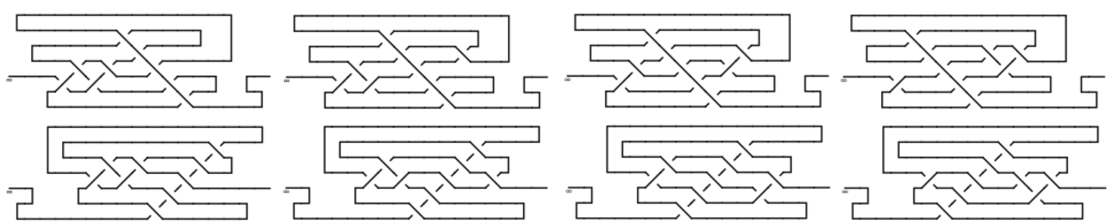


Figure 20: Knot diagrams for the rotation loop of the figure eight knot.

Definition 3.4. *We define γ to be the set of knot diagrams obtained by applying any loop in M_n to any knot. We define Rot to be the set of knot diagrams obtained by applying the rotation loop to a balanced knot in the method depicted in 18. We define Scan to be the first half of the set defined above, this is the part of the rotation loop corresponding to pulling the over strand over the knot. In each of these cases we will denote the knot diagram by the corresponding Reidemeister move.*

3.3 Homological markings

In order to define cocycles that are good at distinguishing knots we look at the cabling of a knot. We then use that additional information by taking into account the homological markings of the crossing. The homological marking of a crossing counts the number of times one needs to travel through the knot starting from the under strand of the crossing to the over strand. If we cable the knot in a specific way each crossing in an uncabled knot becomes a large crossing in the cabled knot. Within each large crossing the homological markings follows a predictable pattern and this allows us to give a direct proof of the RII invariance.

Definition 3.5. *The n **cabling** of a knot K is the cable knot constructed by taking n parallel strands, knotting them according to the long knot K and connecting them by the braid $\sigma_1\sigma_2\dots\sigma_{n-1}$. We call the* *parallel strands* ***segments*** *and the points at which the segments change* ***markings****. If the under crossing of c is in segment a we denote this by $c_1 = a$, similarly we denote that the over crossing is in segment b by $c_2 = b$.*

We often give each segment in the n cabled knot a different colour. By placing the knot in a torus we can choose some disc between the segment changes but before the braid to be the disc at infinity, see figure 23.

Definition 3.6. *The* ***homological marking*** *of a crossing c, denoted by $[c]$ is the number of time one needs to the knot intersects with the disc at infinity when traveling from the under strand to the over strand of a crossing. We will denote the large crossing of c by C and use the same notation for the matrix of homological markings of a large crossing as for the large crossing itself.*

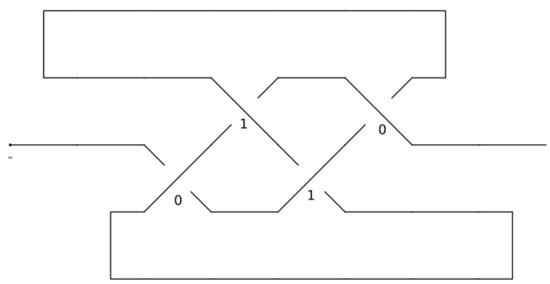


Figure 21: Homological markings of the uncabled figure eight knot.

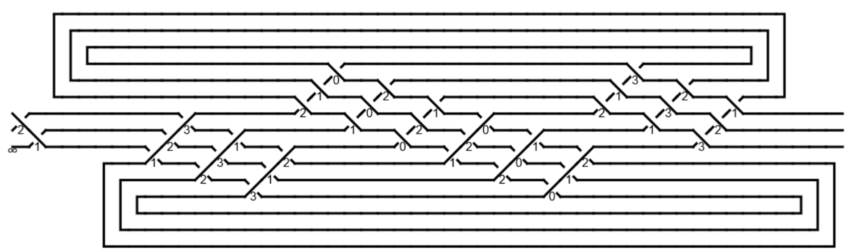


Figure 22: Homological markings of the 3-cabled figure eight knot.

Lemma 3.7. *1. We can compute the homological marking by $[c] = c_2 - c_1 \mod n$.*

2. The homological marking of a crossing is invariant under RIII.

Proof. The homological marking is the number of times one goes through the disc at infinity when traveling through from the foot to the head of a crossing. This is equivalent to counting the number of times we change colour, or if we number the strands instead of colouring then the homological marking is equal to the difference of the under strand and the over strand. That is $[c] = c_2 - c_1 \mod n$. By colouring the strands it also immediately becomes clear that an RIII move will not change the homological markings of the participating crossings. The crossings end up on a different spot but moving them does not impact the colours of their heads or feet and therefore also not their homological marking. ☐

As eluded to in the introduction of this section and as can be seen in figure 21 the homological markings follow a nice pattern.

Definition 3.8. *A* ***Latin square*** *is an n by n matrix such that each element of a set occurs exactly once in each column and each row.*

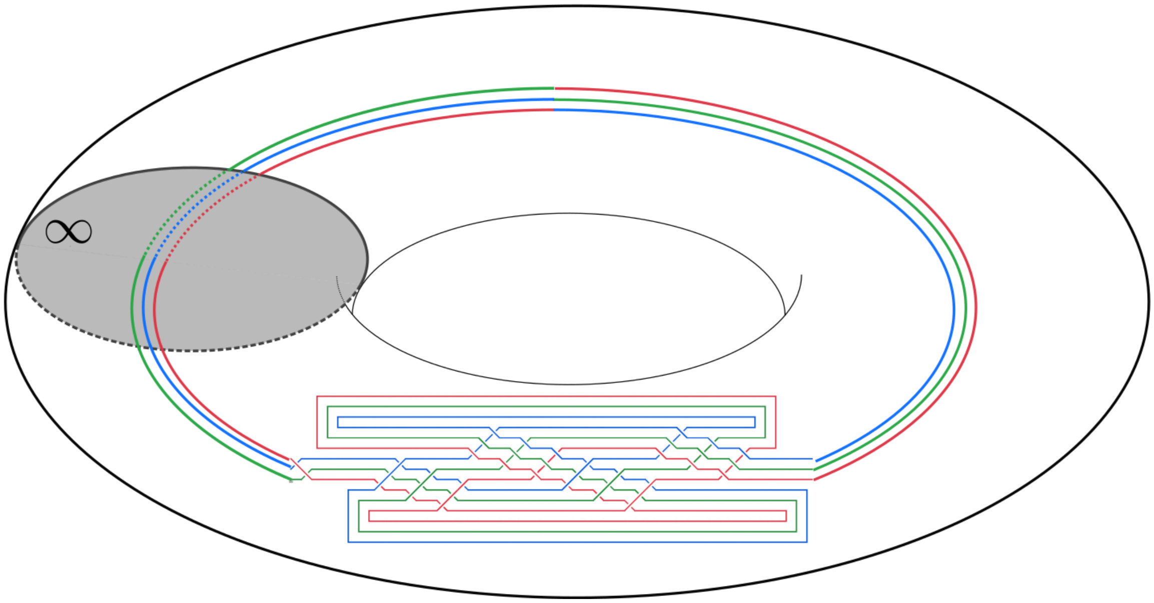


Figure 23: Cabling of a knot

Theorem 3.9. *The homological markings of a large crossings form a Latin square.*

Proof. First suppose that we close the cabled knot without a braid. This means that if we follow one segment we will always stay on that segment and that segment will always be on the same lane. Now we consider the cabled knot closed with a braid. Crucially this braid comes after the disc at infinity but before the rest of the knot. This means that when the segment we go through the disc at infinity and then change lane immediately afterwards.

Suppose we have any two crossings within a large crossing on the same row or column. Since they are on the same row or column either their feet or heads are in the same lane and same segment. Without loss of generality we can assume their heads are in the same lane which means their feet must be in different lanes. Since we have chosen a braid that turns our cabled knot into a one component link we can travel from one under strand to the other since these are on different lanes we need to change lane and the only place where we can do that is in the braid. In order to enter the braid we need to go through the disc at infinity and hence we will be on different segments. It then follows from $[c] = c_2 - c_1$ that they must have different homological markings.

This shows that in each row and column the homological markings are all equal. Depending on the homological marking of the uncabled crossing the possible homological markings in a large crossing are $\{0, \ldots, n-1\}$ or $\{1, \ldots, n\}$ and therefore the homological markings of a large crossing form a Latin square.

If we have a large crossings where the uncabled crossing has homological marking 1, then for any crossing in the large crossing we must encounter the disc at infinity at least once. Since going through the whole knot means we pass through infinity n times we conclude that in this case the homological matrix contains markings $\{1, \ldots, n\}$. The large crossings where the uncabled crossing has homological marking 0 have $\{0, \ldots, n-1\}$ as their homological markings. □

We know proof some results on how the homological markings of the crossings involved in a RII move behave.

Lemma 3.10. *Let a and b be the two crossings created by an RII move and let A and B be the corresponding large crossings. Then the following are true:*

1. *The homological markings of a and b are equal.*

2. *The crossings in A have the opposite sign of the crossings in B.*

3. *The large crossings A and B have mirrored homological matrices.*

4. *The crossings of A and B are in bijection in the following way: for every crossing c_A in A there exists a unique crossing c_B in B such that $c_{A1} = c_{B1}, c_{A2} = c_{B2}$ and $[c_A] = [c_B]$. We call these crossings* ***corresponding****.*

5. *The only crossings between c_A and c_B are in A or B. To be more precise, there are no crossings between the tails of A and the tails of B nor between the heads of A and the heads of B.*

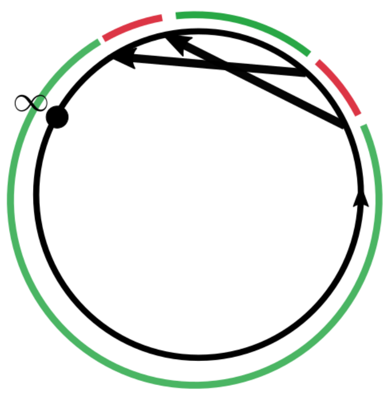


Figure 24: There can be no markings in the red section.

Proof.

1. This is easy to see if we draw the Gauss diagram containing only a and b. In the sections between the heads of a and b or the feet of a and b there can be no other crossings, see figure 24. In a RII move the arrows will point in the same direction and construction of n cabled knots excludes the possibility that ∞, or any other marking, is in the red sections, therefore the homological markings are equal.

2. The crossings of a and b have opposite sign and each crossing in a large crossing has the same sign as the uncabled crossing.

3. Let $x_A \in A$. Looking at the colours of the segments we can easily see that if $x_A \in A$ has a foot in segment p and a head in segment q then p and q cross in B as well, this crossing x_B will be the corresponding crossing. Since the foot and the head of this crossing have the same colours as x_A this crossing must have the same homological marking. In the case that the two colours are equal the two crossings will still have the same homological marking as the homological markings of a and b are equal. See figure 25.

4. Given a crossing in A the existence of the corresponding crossing in B with the desired properties follows from part 3. Since B is a Latin square the uniqueness follows.

5. This follows from there being no crossings between a and b.

□

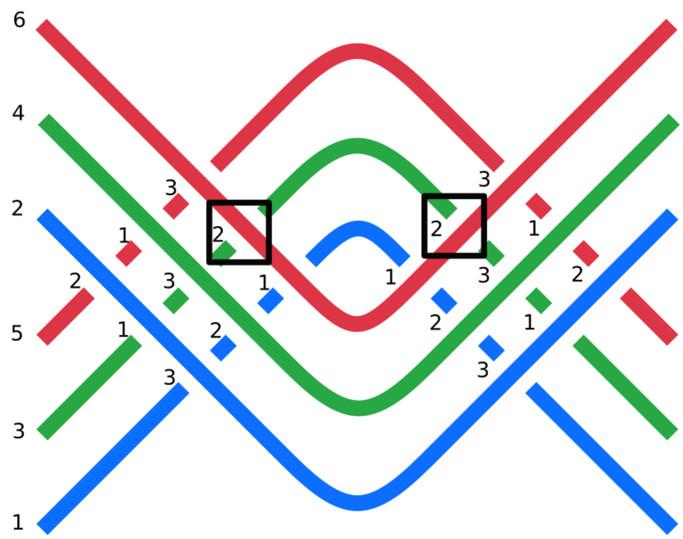


Figure 25: An RII crossing with an example of two corresponding crossings.

3.4 Cocycles

In this section we will outline the general method that Fiedler used to construct his cocycles. The loops in the space of knots are called one-cycles, they are formal $\mathbb{Z}$ linear combination of equivalence classes of loops. A one-cocycle is then a map from the one-cycles to $\mathbb{Z}$. As described in section 3.1 the space of knot contains strata which corresponds to the knot diagram at the moment a Reidemeister move is performed.

The strata of codimension one are the knot diagrams which contain only one point where a Reidemeister moves is performed. We can distinguish three types which correspond to the Reidemeister moves. The strata

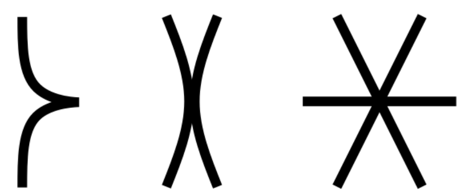


Figure 26: Possible singularities for strata of codimension one.

of codimension two are the intersection of two strata of codimension one. They correspond to knot diagrams where two Reidemeister moves are performed, either at different places in the knot or at the same place.

The types of cocycles that we want to construct will be based on counting the Reidemeister moves that the one-cycle crosses with a specific weight. In order to obtain an invariant we want this cocycle to be invariant under RII and RIII. This means that the cocycle evaluated on a small loop around a strata of codimension one, the red paths in figure 27, should be zero. Moreover we also want that the two paths in blue yield the same value, that is the order of performing Reidemeister moves should not change the value of the cocycle. To this end we also want the cocycle to be zero on loops around strata of higher dimensions, green in the picture. For the strata of codimension 2 there are 6 different kinds which each have multiple variants [5]. Fortunately one does not need to consider all of them as Fiedler was able to use the strata of codimension three to reduce the number of strata of codimension two that the cocycle should be invariant under. These requirements taken together then give rise to a series of equations called the triangle and global

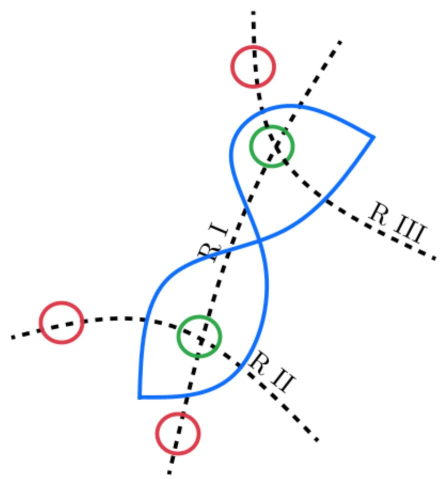


Figure 27: Paths through strata.

tetrahedron equation that need to be satisfied.

The cocycles are then constructed by guessing, or for cases with extra requirements systematically constructing (see the thesis of Los [9]), candidates for cocycles and proving that they satisfy the equations mentioned above. For the rest of this thesis we will focus on only one cocycle: $R^{(2)}_{n,d^+}(a, r, K, \gamma)$.

4 The function $R_n(a, r, K)$

In this chapter we will discuss the function R_n. This function is based on the cocycle discussed in the next chapter. If we evaluate that cocycle on the rotation loop but only look at the contributions of the first half we get R_n, see lemma 5.9. We will first give the definition and then give a proof of its invariance under RII. We end with the conjecture that it is invariant under RIII. Note that R_n is neither a cocycle, we can not evaluate it on any loop, nor is it an invariant, see 6.4.

4.1 Definition

The function R_n will be a sum over certain RIII moves in the first half of the rotation loop. For each of these valid RIII moves we consider the Gauss diagram of the knot at that point in the rotation loop. In this Gauss diagram we then count a certain configuration of crossing, which means that we take into account information on the other crossings in the knot. Since our aim is to acquire something that is (at least partially) invariant we take into account the orientations.

Definition 4.1. *The function R_n is defined for each natural number $n > 1$, $0 < a < n$,$0 \leq r < n$ and any knot K by*

$$R_n(a, r, K) = \sum_S (-1)^p W_2(p)$$

.

Before we can define S and W_2 we need some more terminology. We will also define the sign of a RIII move, for the sign of a crossing recall def 2.2.

Definition 4.2. *Each crossing divides the Gauss diagram into arcs which we will denote by the name of that crossing. These arcs are open so the crossing itself is not included. We also indicate on which side of the crossing the arc lies. If going counterclockwise we first encounter the head of the crossing the arc is positive and if we first encounter its foot it is negative. See figure 28.*

Definition 4.3. *The sign of a Reidemeister III move p denoted by $(-1)^p$ is defined to be $+1$ if in the Gauss diagram the arrows intersect an even number of times before the RIII move and an odd number of times after and -1 if vice versa. See figure 29.*

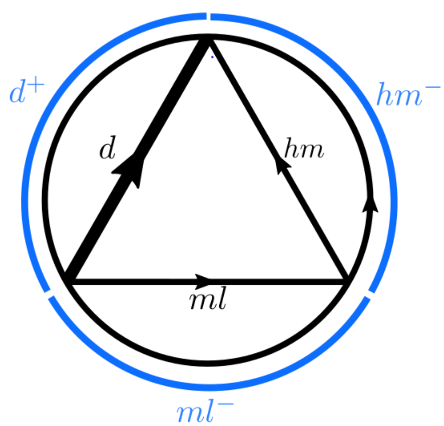


Figure 28: In this diagram the arcs are d^+, hm^- and ml^-.

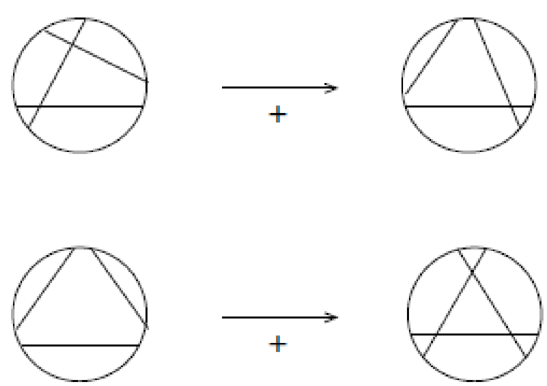


Figure 29: The sign of an RIII move.

We can now define S and W_2, we will give examples at the end of this section.

Definition 4.4. *A Reidemeister III is of type $r(a,n,a)$ if $[d]=a,[hm]=n,[ml]=a$ and the ml is a right crossing, that is the foot comes first. The set S is defined as the set of $r(a,n,a) \in Scan$ such that $\infty \in d^+$ and , $d_1 = ml_1 = r$.*

Recall that $Scan$ is the first half of the rotation loop, see figure 18.

Definition 4.5. *The weight of a RIII move, denoted by $W_2(p)$ is defined as the sum over all diagrams A containing two crossings x and y such that when going counterclockwise from ∞ we first encounter the foot of x then the head of y then the head of x and finally the foot of y. Furthermore we require $[x]=n,[y]=0$ and the foot of x to be in the open arc hm^+. We then sum over the signs of x and y. That is*

$$W_2(p) = \sum_{A_{W_2}(x,y)} (-1)^x(-1)^y.$$

See figure 30.

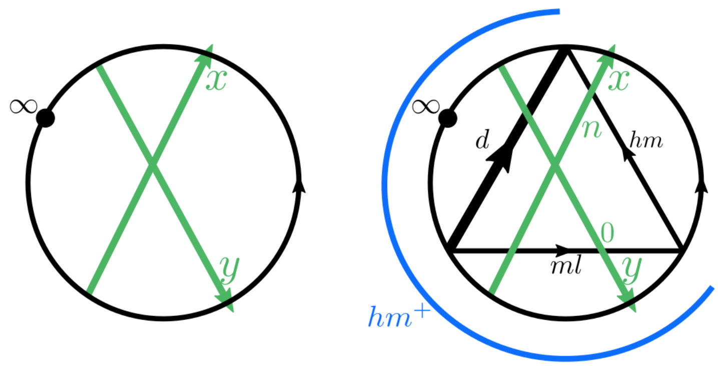


Figure 30: A diagram $A(x,y)$ and an example of $A_{W_2}(x,y)$

In figure 31 we have given a part of an 3-cabled knot diagram in which we have indicated an example of a valid move p in S, we have chosen $n=3$ and $a=2$. When computing this formula it is more convenient to check the validity using a Gauss diagram. As an example for W_2 we have computed its value for the first

Figure 31: A valid move p, instead of drawing the rest of the knot we have indicated the order in which one enters this section of the knot.

valid move of the two cabled figure eight knot. In the Gauss diagram in figure the signs of the crossings are indicated by colour and the arrow heads are replaced by dots, see figure 32. In green the two instances of a diagram $A(x,y)$ are indicated. In both cases the signs of x and y were opposite which gives a value of -2 for the weight term. In table 1 we give the weight terms and signs for all the valid p moves. We can then compute that $R_2(1,0,4_1)=0$. For more numerical results see chapter 6.

sign	W_2
1	-2
-1	-2
1	-2
-1	-2

Table 1: Example W_2 on 4_1.

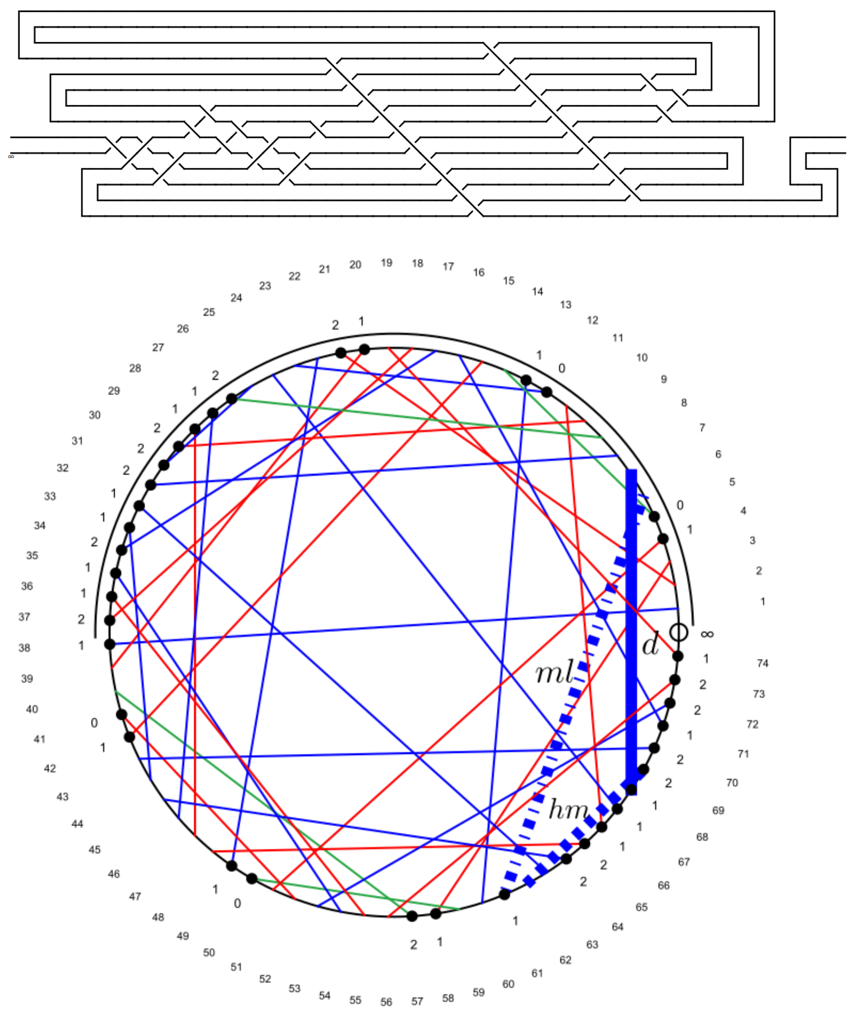


Figure 32: Weight term for the first valid p move.

4.2 Proof of partial invariance

In this section we prove that R_n is invariant under RII. We also conjecture that it is invariant under RIII. As we discussed in section 2.3 this is not sufficient to show that R_n is a knot invariant even if we disregard R I. Since we are working with braids we need to use theorem 2.19 which means we also have to show invariance under conjugation and stabilisation. We only investigated this numerically for a small number of knots see 6.4.

Theorem 4.6. *The function R_n is invariant under RII.*

Proof. If we perform a RII move that creates two large crossings these new crossings can contribute to R_n in two ways. The new crossings can feature in the $W_2(p)$ term from a p that was already there before the RII move. The new crossings also create new valid p moves which can have nonzero contributions.

(Contributions to other crossings) Let A and B be the large crossings created by the RII move. Let p be any RIII move in the loop where the new crossings do not participate. We will show that any contributions from crossings in A to $W_2(p)$ cancel out with contributions from B.

Suppose x and y are crossings in A and B that satisfy the conditions to contribute to W_2. Lemma 3.10.1 implies that the homological matrices of A and B either both contain $0, \ldots, n-1$ or $1, \ldots, n$. This means that crossings in A or B can only contribute together with a crossing outside A or B. Therefore we only need to consider the case where either x or y is in A or B.

Let x_A be a crossing in A and let y be outside A and B. By lemma 3.10.4 there exists a unique crossing x_B such that $[x_A] = [x_B]$. By lemma 3.10.5 the crossing y can not be between x_A and x_B. This implies that if $x_{A1} < y_2 < x_{A2} < y_1$ then $x_{B1} < y_2 < x_{B2} < y_1$. Since hm^+ can not be between A and B we have that if $x_{A1} \in hm^+$ then also $x_{B1} \in hm^+$.

Suppose (x_A, y) are a pair of crossings which contribute to W_2, then there exists a x_B that contributes with the same y to W_2. Since the signs of x_A and x_B are opposite the contribution of x_B to W_2 is minus the contribution of x_A. Hence their total contribution is zero.

Now suppose that y is in either A or B. By the same argument as above any pair (x, A_y) that satisfies to conditions has a pair (x, B_Y) which also satisfies the condition. As A_y and B_y will have opposite sign the total contribution to W_2 is zero.

(Contributions from the new RIII moves) Now we consider the additional p moves in which the new crossings participate.

We will first show that the new valid RIII moves are symmetric. That is for each valid RIII move in which a crossing in B participates, the corresponding crossing in A participates in the corresponding RIII move (by which we mean that the same segment of the track participates in it). This will turn out to be a valid RIII move with the opposite sign.

As the crossings of the 'track' are now also involved we will denote the corresponding matrices by T_1 and T_2. We will denote with R the part of the knot consisting of the large crossings A, B, T_1 and T_2.

Suppose p is a valid move. Without loss of generality we can assume that $x_B \in B$ participates in it. The other crossings t_1 and t_2 are in T_1 and T_2 respectively. If we follow the lines of the crossing of x_B back to A they intersect in the corresponding crossing. We then claim that this crossing x_A together with the same crossings t_1 and t_2 forms a valid move p, See figure 42 for an example.

Since x_A and x_B are corresponding crossings they will have the same homological markings and lie on the same segments.

We need to check that ml stays a right crossing. If a crossing is right it means that we first encounter their feet and then their head. If the track is under then ml is in the track for both p_A an p_B which means that $ml_A = ml_B$. If the track is over then ml is the crossing coming from the RII move and since the two heads lie on the same segment and the two feet also lie on the same segment we have that ml_1 comes before ml_2 for crossing ml_A if and only if that is the case for crossing ml_B.

Finally we need that ∞ stays in d^+, but this will follows from Theorem 5.8.

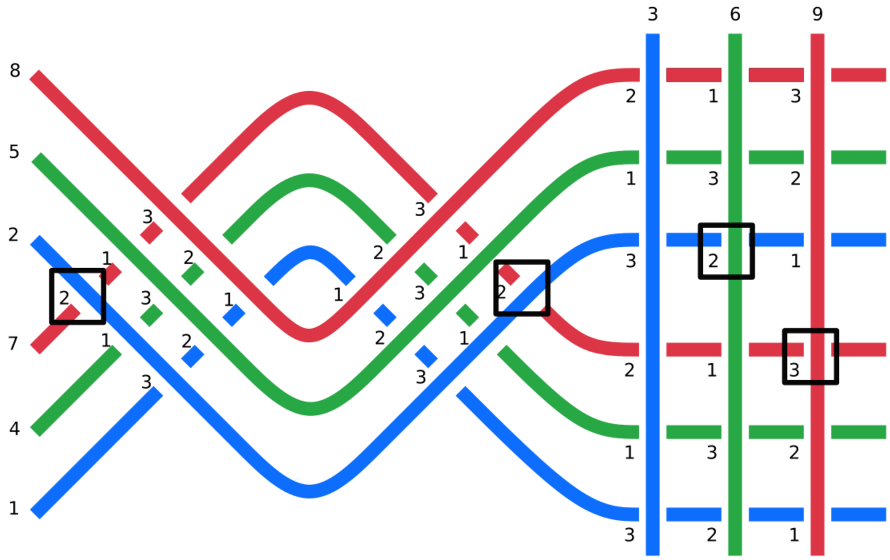


Figure 33: An RII crossing with an example of two corresponding moves.

To summarise, the valid RIII moves that arise from the new RII crossings come in pairs. Furthermore the two moves, which we will henceforth call p_A and p_B, have opposites signs. To see this notice that the sign of p is determined by the order in which one encounters the various under and over crossings. Comparing p_A and p_B the amount of arrows that cross always changes at exactly one spot. If we then look at the definition of the sign of a Reidemeister III moves we deduce that p_A and p_B must have opposite signs.

Let p_A and p_B be two corresponding $r(a, n, a)$ moves and consider how W_2 depends on p. Since a RIII move does not change where the crossings lie in relation to ∞ this only depends on p in that we need the foot of x to be in hm^+.

Therefore we investigate how hm^+ changes in R. We distinguish three relevant sections, see figure 34.

The red ends indicates where hm^+ lies outside of R and is equal for p_A and p_B. The green section is from hm_2 to the next marking, this does change depending on hm but it contains only over crossings, so this does not change the possible feet of x. The blue section has some crossings that are there for p_B but not for p_A, indicated in purple. If we look carefully these are always on the middle strand of the RIII move and are part of A or B. So the only differences in blue are over crossings which means they do not impact the possible feet for x. So comparing p_A and p_B the only difference in hm^+ are over crossings which means that W_2 is equal. Since $(-1)^{p_A} = -(-1)^{p_B}$ we have that $(-1)^{p_A} W_2(p_A) + (-1)^{p_B} W_2(p_B) = 0$ and hence R_n is invariant under RII. □

Corollary 4.7. *The function R_n is also defined for $a = n$ and is in that case still invariant under RII.*

Proof. At no point in the proof of theorem 4.6 did we use that $a < n$. □

Based on numerical results we believe the following conjecture to be true.

Conjecture 4.8. *The function R_n is invariant under RIII.*

Unlike an RII move an RIII move does not change the number of crossings. We were able to show that an RIII move will change the position of the valid p moves but will not create or destroy any. Showing that the repositioning of p is not enough to change the value of $W_2(p)$, perhaps in a similar way as in the proof of 4.6, is then enough to show invariance.

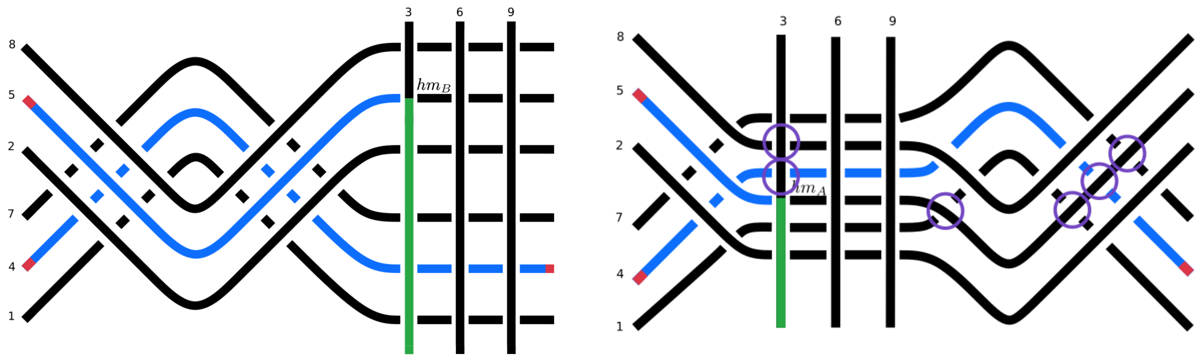


Figure 34: hm^+ in R_A and R_B

Although there are many possible RIII moves every choice one makes, signs, order of entry and homological marking of uncabled knot, is reflected in the homological matrices. To proof the conjecture without resorting to an unreasonable amount of case distinction would therefore be required to not make any assumptions on the homological matrices.

Lemma 4.9. *The valid p moves are permuted by a RIII move.*

Proof. We will first need some additional notation: The RIII move in the uncabled knot involves 6 crossings. We have the three crossings of the RIII move which we will denote by a, b, c where a is the distinguished (high low), b is the high middle and c is the middle low crossing. We also have 3 crossings in the track which we will denote by t_1, t_2, t_3. As usual the matrices of the corresponding large crossings are denoted by A, B, C and T_1, T_2, T_3, we will call all these crossings together R. Any p move in the cocycle will now involve one crossings from A, B or C and two crossings in from two distinct T_i.

Let p be a valid move. It involves one crossing in $\{A, B, C\}$ and two crossings in the track. If we perform the RIII move this only influence A, B and C and can be seen as a sequence of RIII moves of individual crossings. We know from lemma 3.7.2 that the homological marking is not affected by an RIII move. The crossings changes place but its homological marking and the segments it is in remain the same. It then follows that the RIII move of the large crossing simply changes the position of the crossings but does not affect the homological matrices.

If we now look at the definition of a valid p move the crossings need to satisfy the following types of conditions: homological markings, segments and ml needs to be right. Note that whether ml is right depends only on the homological matrix and is not affected by the rest of the knot. All of these properties are determined locally by the homological matrices.

So assume p is a valid move. The crossings in the track are not affected by the RIII move. The third crossing is now in a different place but this does not affect it validity. Hence an RIII move does not change the amount of valid p moves but it does change their relative placement.

□

5 The cocycle $R^{(2)}_{n,d^+}(a,r,K,\gamma)$

In this chapter we will give the definitions of the refined cocycle $R^{(2)}_{n,d^+}(a,r,K,\gamma)$ as described in [5]. We will proof that the refinement of this cocycle implies that these definitions can be slightly simplified. We will show a partial proof of their invariance under RII where we use that the first half of the rotation loop reduce to R_n.

5.1 Definition

The cocycle is defined on any loop γ however after stating the definitions we will assume that we are evaluating on the rotation loop, that is $\gamma = Rot$. The cocycle consists of two large sums that each sum over a specified type of Reidemeister III move. Each large sum then consists of a combination of signs, weights and linking numbers which we will define after the main definition. The whole cocycle depends on the number of strands in the cabling n, a variable a and the refinement variable r. For completeness we repeat the definition of W_2 which was already stated in the previous chapter.

Definition 5.1. *Let γ be any oriented loop in M_n.*

The integer-valued 1-cochain $R^{(2)}_{n,d^+}(a,r,K,\gamma)$ is defined for each natural number $n > 1$, $0 < a < n$ and $0 \leq r < n$, by

$$R^{(2)}_{n,d^+}(a,r,K,\gamma) = \sum_{p\in F}(-1)^p\Big(W_2(p) + (-1)^{hm}W_1(hm)(l^{(ana)}(p) + (-1)^{hm} - 1)\Big)$$

$$- \sum_{p=r(n,n,n)\in\gamma}(-1)^p l^{(nnn)}(p)(-1)^{hm}W_1(hm).$$

We follow Fiedlers notation for this cocycle. The subscript d^+ refers to the choice that for a valid p move infinity has to be in d^+. The superscript $^{(2)}$ refers to the which diagrams to count in both the W_2 and W_1 terms, see Fiedlers work for more on these cocycles.

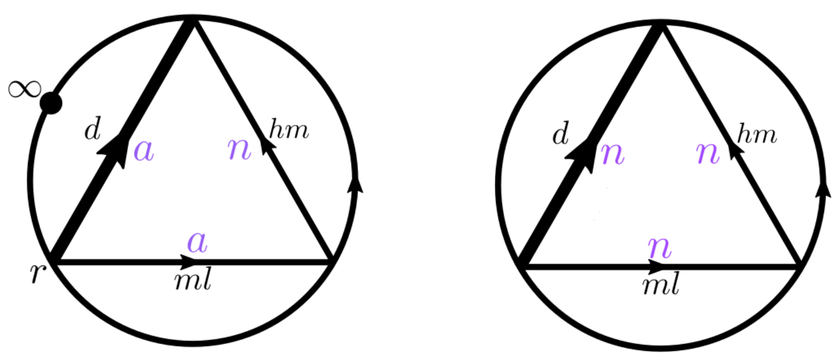


Figure 35: A RIII move of type $r(a,n,a)$ and of type $r(n,n,n)$

Definition 5.2. *A Reidemeister III is of type $r(a,n,a)$ if $[d] = a, [hm] = n, [ml] = a$ and ml is right, that is its foot comes first. It is of type $r(n,n,n)$ if $[d] = n, [hm] = n, [ml] = n$ and ml is right. See figure 35.*

The set F is defined as the set of $r(a,n,a) \in \gamma$ such that $\infty \in d^+$ and , $d_1 = ml_1 = r$.

Definition 5.3. *The weight of a RIII move, denoted by $W_2(p)$ is defined as the sum over all diagrams A containing two crossings x and y such that when going counterclockwise from ∞ we first encounter the foot of x then the head of y then the head of x and finally the foot of y. Furthermore we require $[x] = n, [y] = 0$ and the foot of x to be in the open arc hm^+. We then sum over the signs of x and y. That is*

$$W_2(p) = \sum_{A_{W_2}(x,y)} (-1)^x(-1)^y.$$

See figure 30.

For an example of a valid p move and a computation of $W_2(p)$ one can refer back the example in chapter 4.

Definition 5.4. *The weight of a hm crossing, denoted by $W_1(hm)$ is defined as the sum over all diagrams $A(x, y)$ as defined above where $x = hm$. That is*

$$W_1(hm) = \sum_{A_{W_1}(hm,y)} (-1)^{hm}(-1)^y.$$

See figure 36.

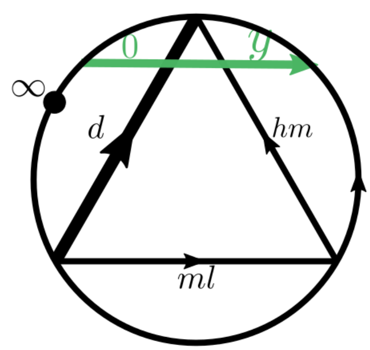


Figure 36: An example of $A_{W_1}(hm, y)$.

Definition 5.5. *The refined linking number for the type $r(a, n, a)$, denoted by $l^{(ana)}(p)$ is defined as the sum of the signs of all crossings x such that $[x] = n$, x intersects the crossing ml, the foot of x is in the open arc ml^- and $x \neq hm$. That is*

$$l^{(ana)}(p) = \sum_{x_{lana}} (-1)^x$$

where x_{lana} is as described above. See figure 37.

Definition 5.6. *The refined linking number for the type $r(n, n, n)$, denoted by $l^{(nnn)}(p)$ is defined as the sum of the signs of all crossings x such that $[x] = n - a$, $x_2 = r$, x intersects the crossing ml, the foot of x is in the open arc ml^- and $x \neq hm$. That is*

$$l^{(nnn)}(p) = \sum_{x_{nnn}} (-1)^x$$

where x is as described above. See figure 38.

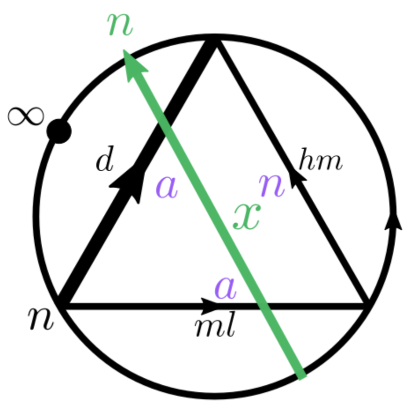


Figure 37: An example of a possible x_{lana}, the head of x could also be before the head of d.

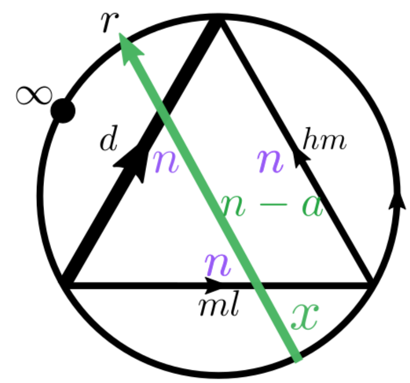


Figure 38: An example of a possible x_{lnnn}.

5.2 Refinement

Fiedler originally defined the cocycle $R^{(2)}_{n,d^+}(a,r,K,\gamma)$ without the refinement variable r. So in all definitions any conditions involving r were not there. We obtained the definition of the refinement through personal communication. Fiedler claims that summing over all possible r gives back the originally defined cocycle.

In this section we will show that using the refinement one can simplify the definition of the cocycle can be simplified and that the cocycle is zero for a certain choice of variables.

First we state as a conjecture the claim that the original cocycle is the sum of the refined cocycles.

Conjecture 5.7. *Let $R^{(2)}_{n,d^+}(a,K,\gamma)$ be the unrefined cocycle, then*

$$R^{(2)}_{n,d^+}(a,K,\gamma) = \sum_{r=0}^{n-1} R^{(2)}_{n,d^+}(a,r,K,\gamma).$$

Note that this equality is not at all obvious. Considering where r plays a part in the definitions the refinement essentially refines which $r(a,n,a)$ moves are valid and which crossings play a roll in the linking number for $r(n,n,n)$. If only the former were true the equality follows immediately but is nontrivial to show that the second refinement has no impact on this equality.

Theorem 5.8. *The cocycle with refinement is zero if $r \geq a$. Moreover the condition $\infty \in d^+$ is automatically true if p satisfies the other conditions of the set F.*

Proof. We will show that in the refinement the contribution of moves of type $r(n,n,n)$ can only be nonzero if $a > r$ and that the contribution of moves of type $r(a,n,a)$ can only be nonzero if $a > r$.

Let us indicate the place on the Gauss diagram where we change component by a marker and note that ∞ also has a marker. Then the homological marking of a crossing is exactly the amount of markers one encounters on the Gauss diagram when traveling from the head to the foot of the corresponding arrow. The component that a foot or head is in is equal to the number of markers from infinity to that foot or head, excluding infinity.

Assume that we have a move of type $r(a,n,a)$ satisfying the conditions of the refinement. Then the foot of d and the foot of ml should be in component r. Recall that we denote this by $d_1 = r$ and $ml_1 = r$. We also know that $[d] = a$ and that ∞ is in d^+.

Since the foot of d is in component r we know that there are exactly r markers between ∞ and d_1. Since we first encounter the foot of d and then its head this implies that between the foot and head of d there are at least $r+1$ markers, r between ∞ and d_1, one more for ∞ itself and an unknown number between d_2 and ∞. See figure 39.

As the homological marking of d is a we can now conclude that $a \geq r+1$.

Hence for any move of type $r(a,n,a)$ to contribute to the cocycle we must have $a > r$.

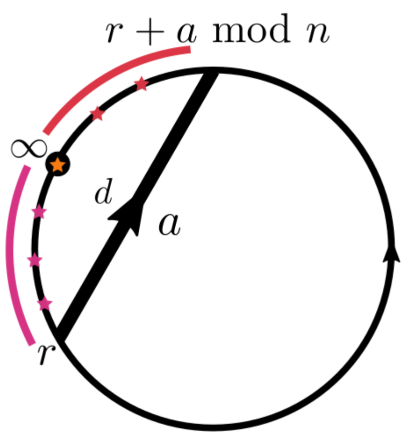


Figure 39: Refinement for $r(a,n,a)$. The markings are indicated by a star. Here $r=3$ and $a=6$.

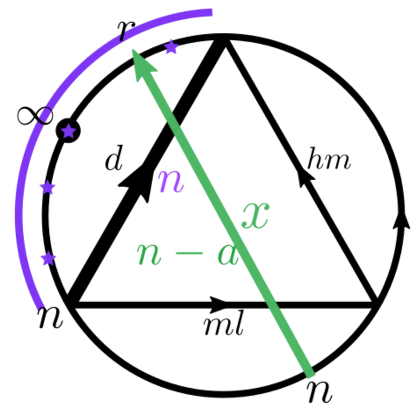


Figure 40: Refinement for $l^{(n,n,n)}$. Here $n=4$ and $r=3$.

Now assume we have a move of type $r(n,n,n)$ and a crossing x contributing to $l^{(ana)}(p)$ that satisfy the conditions for the refinement. This means that $x \neq hm$, $[x] = n-a$ and $x_2 = r$. Also x intersects ml and $x_1 \in ml^-$.

When we look at the location of the markings we see that they must all be in d^+ as $[d] = n$. This implies that $ml_1 = ml_2 < n$. Since $x_1 \in ml^-$ we also have that $x_1 < n$. See figure 40.

Finally we know that $[x] \bmod n = x_2 - x_1$. Combining this with the above we get: $x_1 = n - a + r < n$. Since $0 < a < n$ and $0 \leq r < n$ this means that $r < a$. So $l^{(n,n,n)}(p)$ is nonzero only if $r < a$

As the contribution of $r(n,n,n)$ can only be nonzero if the linking number is nonzero this means that for $r(n,n,n)$ moves to contribute r must be smaller than a.

To proof that $\infty \in d^+$ if p satisfies the other conditions we assume that p satisfies the other conditions but $\infty \notin d^+$. If $d_1 = a$ and $\infty < d_2$ then the markings between d_2 and d_1 can only be equal to d_1. However this means that $[d] = d_1$ so $a = r$ which is not possible. $\square$

5.3 Invariance

In this section we will give a partial proof of the invariance of the cocyle under Reidemeister II moves. We will then briefly discuss invariance under Reidemeister III. Before we give the proof of invariance we will proof a lemma that allows us to reuse the results in chapter 4.

Lemma 5.9. *Suppose γ is the rotation loop. If the track is over then $W_1 = 0$. Additionally this implies that $R^{(2)}_{n,d^+}(a,r,K,\gamma)$ evaluated on the first half of the rotation loop is equal to R_n.*

Proof. Let p be a RIII move of the rotation loop in the first half, that is $p \in Scan$. Suppose y is a crossing participating in W_1. Then we have $[y] = 0$ and $hm_1 < y_2 < hm_2 < y_1 < \infty$ (note that inequality signs denote the ordering, the homological markings may still be equal.) Suppose the head of hm is in segment k. Since the track is over there will be only over crossings between hm_2 and the next marking. If γ is not the rotation loop there could be other crossings between the "track" and the next marking. This means that the foot of y must lie on one of the next segments, that is $y_1 > k$(see figure41). However since $[hm] = n$, its foot and head will be on the same segment and since y_2 lies between them it will also be on the same segment. Therefore $y_2 = k$. But now $[y] = y_2 - y_1 \neq 0$. So we have found a contradiction which means no such y exists and hence $W_1(p) = 0$. □

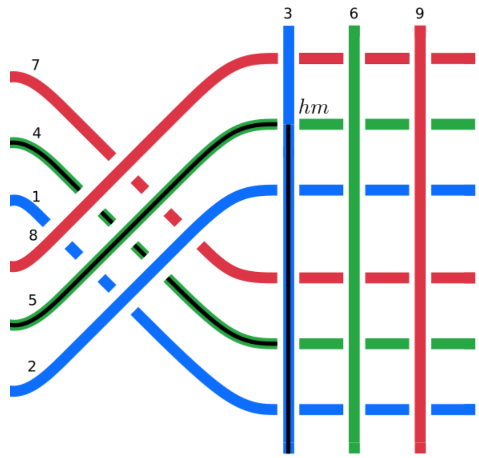


Figure 41: Black indicates hm^+, the colours indicate different segments.

The strategy of the proof is as follows. As in the proof of theorem 4.6 we need to show that when we add two large crossings to the cabled long knot using a RII move the total contribution of those crossings to the cocycle is zero. This then implies that the cocyle is invariant under RII.

The new crossings can again contribute to the cocycle in two ways. The first is the contribution the RII crossings might have as crossings in the weights or linking numbers of other RIII moves. The second is the contributions of the RIII moves in which the new crossings participate. In the proof we will show that in both cases the contributions will cancel out.

We consider the first half and second half of the rotation loop separately as lemma 5.9 reduces the first half to theorem 4.6.

Theorem 5.10. *The cocycle as defined in definition 5.1 applied to the rotation loop is invariant under Reidemeister II moves.*

Proof. (Contributions to other crossings) The first part of this proof is identical to the proof of 4.6. Let A and B be the large crossings created by the RII move. Let p be any RIII move in the loop where the new crossings do not participate. We will show for each of the components of the cocycle that any contributions from crossings in A cancel out with contributions from B.

Suppose x and y are crossings in A and B that satisfy the conditions to contribute to W_2. Lemma 3.10.1 implies that the homological matrices of A and B either both contain $0, \ldots, n-1$ or $1, \ldots, n$. This means that crossings of A or B can only contribute together with a crossing outside A or B. So we only need to consider the case where either x or y is in A or B.

Let x_A be a crossing in A and let y be outside A and B. By lemma 3.10.4 there exists a unique crossing x_B such that $[x_A] = [x_B]$. Also by lemma 3.10.5 the crossing y can not be between x_A and x_B. This implies that if $x_{A1} < y_2 < x_{A2} < y_1$ then $x_{B1} < y_2 < x_{B2} < y_1$. Since hm^+ can not be between A and B if $x_{A1} \in hm^+$ then also $x_{B1} \in hm^+$.

Let x_A and y be a pair of crossings which contribute to W_2, then there exists a x_B that contributes with the same y to W_2. Since the signs of x_A and x_B are opposite the contribution of x_B to W_2 is minus the contribution of x_A. Hence their total contribution is zero.

Now suppose that y is in either A or B. By the same argument of above the crossings that satisfy the conditions always come in pairs with opposite sign so the total contribution to W_2 is zero.

The cocycle has more terms then R_n so we also need to show that the contributions in those terms cancel out. If we consider W_1, $l^{(ana)}$ and $l^{(nnn)}$ they all involve only one additional crossing x (or y) that has to satisfy conditions for the homological marking and the placement of its foot or head relative to the crossings of p. (In the case of $l^{(nnn)}(p)$ there is also a condition on the segment of x_2).

Suppose this crossing x (or y) is in A or B. By Lemma 3.10.4 and 3.10.5 we are guaranteed that any crossing in A that satisfies the conditions to contribute has a corresponding crossing in B and vice versa. Since all these crossings enter with their sign and each crossing is paired with a crossing of opposite sign the total contribution of each pair is 0.

It follows that the contribution of the RII move to the already existing RIII moves p is zero.

(Contributions from the new RIII moves) Now we consider the additional p moves in which the new crossings participate. We can again reuse most of the reasoning as for theorem 4.6, however we also need to take into account the p moves of type $r(n, n, n)$.

We will first show that the valid RIII moves are symmetric. That is for each valid RIII move in which a crossing in B participates, the corresponding crossing in A participates in the corresponding RIII move (by which we mean that the same segment of the track participates in it). And this will turn out to be a valid RIII move with the opposite sign.

As the crossings of the 'track' are now also involved we will denote the corresponding matrices by T_1 and T_2. We will denote with R the part of the knot consisting of the large crossings A, B, T_1 and T_2.

Suppose p is a valid move (so either $p \in F$ or $p = r(n, n, n)$). Without loss of generality we can assume that $x_B \in B$ participates in it. The other crossings t_1 and t_2 are in T_1 and T_2 respectively. If we follow the lines of the crossing of x_B back to A they intersect in the corresponding crossing. We then claim that this crossing x_A together with the same crossings t_1 and t_2 forms a valid move p. See figure 42 for an example. Since x_A and x_B are corresponding crossings they will have the same homological markings and lie on the same segments.

Next we need to check that ml stays a right crossing. If a crossing is right it means that we first encounter their feet and then their head. If the track is under then ml is in the track for both p_A an p_B so then ml_A is right if and only if ml_B is right. If the track is over then ml is the crossing coming from the RII move and since the two heads lie on the same segment and the two feet also lie on the same segment we have that ml_1 comes before ml_2 for crossing ml_A if and only if that is the case for crossing ml_B.

Finally in the case of an $r(a, n, a)$ move we need that ∞ stays in d^+, but this follows from Theorem 5.8.

To summarise, the valid RIII moves that arise from the new RII crossings come in pairs. Each pair is of the same type ($r(a, n, a)$ or $r(n, n, n)$) and the participating RII crossings x_A and x_B correspond. Furthermore the two moves, which we will henceforth call p_A and p_B, have opposites signs.

For the rest of the proof we need the assumption that we are evaluating at the rotation loop and make the case distinction between the first half of the loop where the track is over and the second half where the track is under.

If the track is over hm is in T_1 or T_2 so the sign stays the same. If the track is under however hm moves between A and B so $(-1)^{x_A} = -(-1)^{x_B}$.

(Track is over) This follows immediately from theorem 4.6.

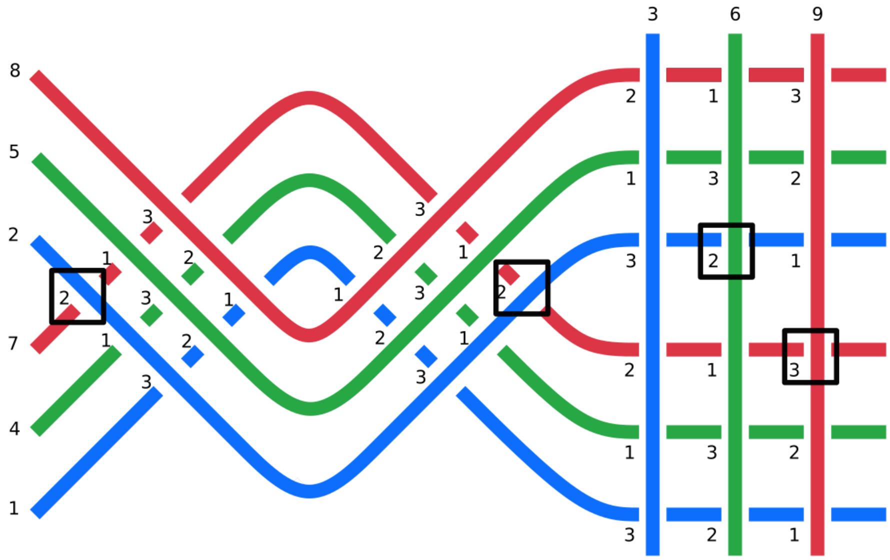


Figure 42: An RII crossing with an example of two corresponding moves.

	p_A	p_B
$p^{(-1)}$	-1	1
$hm(-1)$	1	-1
W_2	-2	-4
W_1	-2	2
$l^{(ana)}$	0	1
$Sum_{(ana)}$	2	-2

Table 2: The contributions cancel out in a non trivial manner.

(Track is under)

Note that for both $r(a,n,a)$ and for $r(n,n,n)$ we need a hm crossing with homological marking n. Since hm is now no longer in the track but in A or B such a move only happens if A and B contain n crossings which means that the uncabled crossings of the RII move must have homological marking 1.

We will show that in this case $W_1(p_A) = -W_1(p_B)$. Let p_A and p_B be two corresponding moves of either type $r(a,n,a)$ or $r(n,n,n)$. The value of W_1 depends on how many crossings are in hm^+. Like we did for W_2 we can colour in the different sections of R to indicate how hm_A^+ compares to hm_B^+, see figure 34 again. We can see that the difference of hm_A^+ and hm_B^+ is located entirely within R. Since we must have $[y] = 0$ there are no y in this difference which means that hm_A^+ and hm_B^+ contain the same set of y crossings. However when computing the sum of W_1 we do take into account the sign of hm and as the sign of hm_A and hm_B are opposites it follows that $W_1(p_A) = -W_1(p_B)$.

Let us compare what we know about the elements of the sum of $r(n,n,n)$: the sign of p is opposite, the sign of hm is opposite, $W_1(p_A) = -W_1(p_B)$. So to show that it is invariant we need that $l^{(nnn)}(p_A) = l^{(nnn)}(p_B)$. In order to show this one could use a similar method of colouring as we did for the W_2 and W_1 term.

Finally we are left with the last case: the invariance for an $r(a,n,a)$ move in the second half of the loop where the track is over and the homological marking of the uncabled crossings is 1. It is clear how the sign of p and hm as well as how W_1 changes but it turns out that W_2 and $l^{(ana)}$ behave in a more complex way. Using the computer program we have found a set of crossings in the 5_2 knot satisfying the conditions of the last case, see figures 43 and 44. The program then gives the values in table 2. We can see that the

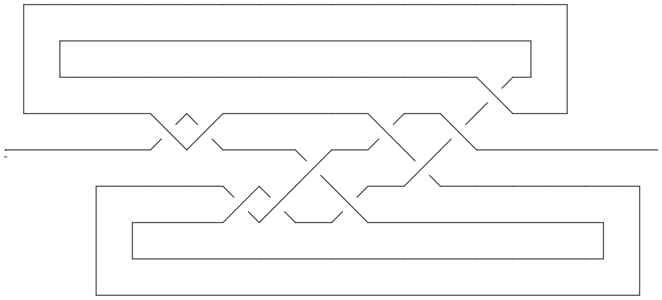


Figure 43: The 5_2 knot with an RII move.

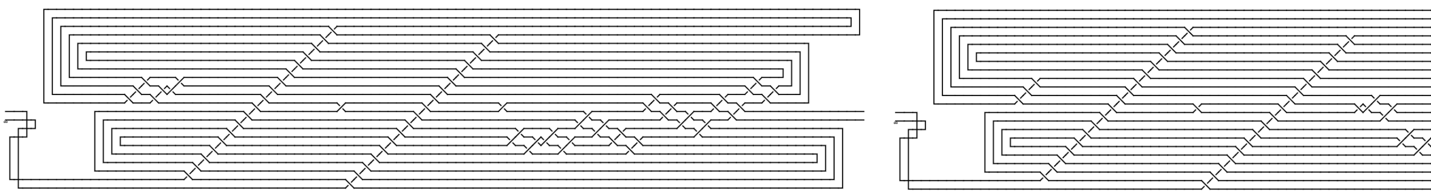


Figure 44: The two $r(a,n,a)$ moves.

final contributions will indeed cancel out but there is no simple relation between $W_2(p_A)$ and $W_2(p_B)$ not between $l^{(ana)}(p_A)$ and $l^{(ana)}(p_B)$ as there was for all the other cases so far. This suggests that the amount of crossings x_{ana} and the crossing x, y of W_2 depend on each other in a more complex way. □

Theorem 5.11. *The cocycle as defined in definition 5.1 applied to the rotation loop is invariant under Reidemeister III moves.*

As Fiedler proved that this cocycle is an invariant we knot that the above is true. It might be possible to give a proof for this using a similar approach as we have had so for. However already in the proof of the invariance of R_n under RIII we got stuck due to the large number of crossings to keep track of and the more complex difference in segments that the RIII move causes.

6 Numerical results

Using the program written by Roland van der Veen we have computed the cocycle as well as R on some small knots.

6.1 Prime knots

First we evaluated the cocycle and R on the first 20 prime knots, see table 3. The cocycle on its own already does a pretty good job of distinguishing these knots.

We also see that R gives very different results and is for example able to distinguish between 3_1 and 6_3 while the cocyle is not. In fact the two taken together can distinguish all but 2 knots from the first 20. We have also ran the program for R with $a = n$ for both $r = 1$ and $r = 1$, while $a = n$ is not allowed for the cocycle and indeed only gives 0, it is permitted for the function R_n, see corollary 4.7, and yields different values.

Knot	3_1	4_1	5_1	5_2	6_1	6_2	6_3	7_1	7_2	7_3
$R^{(2)}_{n,d^+}(1,0,K)$	3	-3	9	6	-6	-3	3	18	9	15
$R_2(1,0,K)$	0	0	0	-1	1	0	2	0	-7	5
$R_2(2,0,K)$	-1	1	-4	-3	3	2	0	-10	-6	7
$R_2(2,1,K)$	-1	1	-4	-3	3	2	0	-10	-6	7
Knot	7_4	7_5	7_6	7_7	8_1	8_2	8_3	8_4	8_5	8_6
$R^{(2)}_{n,d^+}(1,0,K)$	0	12	4	-2	-9	0	-12	-9	-3	-6
$R_2(1,0,K)$	3	1	-3	0	1	0	1	3	-1	2
$R_2(2,0,K)$	5	-7	-3	1	4	2	5	4	-3	4
$R_2(2,1,K)$	5	-7	-3	1	4	2	5	4	-3	4

Table 3: The cocycle and R evaluated on the first 20 prime knots for $n = 2$

In table 4 we have ran the cocycle for more combinations of n, a and r. In the second and third row we see that if $r \geq a$ then the cocycle is zero as the theory predicts. We also notice that the rows $(n, a, r) = (3, 1, 0)$ and $(n, a, r) = (3, 2, 1)$ are equal. In Table 5 this pattern appears to continue. Table 5 shows that this pattern does not hold for R. From the last two rows in table 4 we conclude that only changing the variable r can give different results. Based on all the numerical results we have computed for R this does not seem to be the case for R.

Notice that almost all evaluation of the cocycle are divisible by 3. This could be a coincidence or something interesting happens for those knots. Furthermore we notice that all the rows of table 4 are multiples of each other, for example $R^{(2)}_{2,d^+}(1,0,3_1) = \frac{3}{4}R^{(2)}_{3,d^+}(1,0,K)$. However as $R^{(2)}_{2,d^+}(1,0,7_6)$ is not divisible by 3 and all evaluations are integers this pattern must break for that knot. Indeed computing $R^{(2)}_{3,d^+}(1,0,7_6)$ we get 5. So if there is a relation between $R^{(2)}_{n,d^+}(a,r,K)$ and $R^{(2)}_{n+1,d^+}(a,r,K)$ it is more subtle than multiplying by a factor.

From table 5 we do not recognize any clear relation between R for $n = 2$ and $n = 3$.

We observe that for all (n, a, r) that we tested and for both the cocycle and R the values of 3_1 and 4_1 as well as 5_2 and 6_1 are opposite. For the first pair we have computed some more values for $n = 4$ and the pattern seems to continues, see table 6. Looking at table 3 there are no other such pairs up to 8_6.

6.2 Torus knots

We have also run the program on the first few $(p, 2)$ torus knots. The table is incomplete as the computation time increases rapidly for knots with more crossings.

In the first row we recognize the triangle numbers (multiplied by 3) which suggests a connection with the finite type invariant of order 2, a connection between these invariants and cocycles is also proven in [9].

Knot	3_1	4_1	5_1	5_2	6_1
$R^{(2)}_{2,d^+}(1,0,K)$	3	-3	9	6	-6
$R^{(2)}_{2,d^+}(1,1,K)$	0	0	0	0	0
$R^{(2)}_{2,d^+}(2,0,K)$	0	0	0	0	0
$R^{(2)}_{3,d^+}(1,0,K)$	4	-4	12	8	-8
$R^{(2)}_{3,d^+}(2,0,K)$	1	-1	3	2	-1
$R^{(2)}_{3,d^+}(2,1,K)$	4	-4	12	8	-8

Table 4: The cocycle evaluated on the first 5 prime knots for $n = 3$.

Knot	3_1	4_1	5_1	5_2	6_1
$R_3(1,0,K)$	1	-1	3	1	-1
$R_3(2,0,K)$	1	-1	3	-3	3
$R_3(2,1,K)$	1	-1	3	-3	3
$R_3(3,0,K)$	-1	1	4	-3	3

Table 5: R evaluated on the first 5 prime knots for $n = 3$

Knot	3_1	4_1
$R_4(1,0,K)$	2	-2
$R^{(2)}_{4,d^+}(1,0,K)$	5	-5

Table 6: 3_1 and 4_1 for $n = 4$.

Torus knot	(3,2)	(5,2)	(7,2)	(9,2)	(11,2)
$R^{(2)}_{2,d^+}(1,0,K)$	3	9	18	30	45
$R^{(2)}_{3,d^+}(1,0,K)$	4	12	-	-	-
$R^{(2)}_{3,d^+}(2,0,K)$	1	3	-	-	-
$R^{(2)}_{3,d^+}(2,1,K)$	4	12	-	-	-

Table 7: The cocycle evaluated on the first 5 $(p, 2)$ torus knots.

For more on finite type invariants one could consult [2]. One could probably proof this pattern by explicitly computing the contributions added when increasing p by 2. In the other rows the first two entries fit the sequence of a multiple of the triangle numbers. However due to the computation time we could not check if the pattern continues.

We have also evaluated R on the first few torus knots. Interestingly the triangle number patterns do not continue for $R_2(2,0,k)$, instead the first few terms of the tetrahedral numbers appear.

Torus knots	(3,2)	(5,2)	(7,2)	(9,2)
$R_2(1,0,K)$	0	0	0	0
$R_2(2,0,K)$	-1	-4	-10	-20
$R_3(1,0,K)$	1	3	-	-
$R_3(2,0,K)$	-1	-3	-	-
$R_3(2,1,K)$	1	3	-	-

Table 8: R evaluated on the torus knots.

6.3 Mirrors

Recall that we hoped that the cocycle $R^{(2)}_{n,d^+}(a,r,K,\gamma)$ would be able to distinguish mirror, unfortunately it does not. However the contributions of the valid moves do not cancel in a nice symmetric way. For example the nonzero contributions to $R^{(2)}_{2,d^+}(1,0,3_1)$ are $2,2,2,-2,-2,-2|1,-2,-2,2,-1,3$ while the nonzero contributions to $R^{(2)}_{2,d^+}(1,0,3_1')$ are $-2,-2,-2,2,2,1,1|2,-2,-1$. The numbers before $|$ are the contributions of the first half of the rotation loop. Notice that although they both sum to 3 there is no clear correspondence between the contributions as there are not even the same amount of valid p moves.

We can explain these differences by considering what effect mirroring has on the cocycle. Mirroring the knot changes the sign of all the crossings in the knot. The arcs, such as hm^+ will change to their complement. Furthermore each homological marking will change to the additive inverse mod n. All of this means that the valid p moves of a knot are not preserved by mirroring.

As R_n effectively sums these contributions up to $|$ it was our hope that R_n could distinguish between knots and their mirror. It does, see table 9, however evaluating R_n on amphichiral knots like 4_1 and 6_3 also shows that R at least sometimes gives different values for the same knot.

Another thing to notice it that the mirror knots of 5_2 and 6_1 no longer give the opposite value.

Knot	3_1	4_1	5_1	5_2	6_1	6_2	6_3
$K = K'$?	No	Yes	No	No	No	No	Yes
$R_2(1,0,K)$	0	0	0	-1	1	0	2
$R_2(1,0,K')$	1	-1	3	2	0	-1	1

Table 9: R_2 evaluated at mirrors

6.4 Conjugation and modified stabilisation.

As we could already conclude from Table 7 R_n is not a knot invariant.

We can moreover say that it is not invariant under conjugation. As an example let $conj4_1$ be the figure eight knot conjugated by the braid $\sigma_1\sigma_2$, see figure 45. Evaluating the equation we get $R_2(1,0,conj4_1) = -2 \neq R_2(1,0,4_1)$.

We have also evaluated $R_2(1,0,K)$, $R_2(2,0,K)$ and $R_3(1,0,K)$ on the modified stabilization of the trefoil knot, the figure eight knot and the 6_1 knot. The values are all consisted with the results on the normal balanced knot diagrams of these knots. This suggests that R_2 is invariant under modified stabilisation.

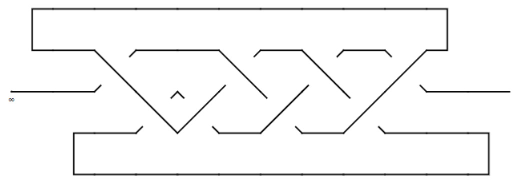


Figure 45: 4_1 conjugated with $\sigma_1\sigma_2$.

The crossings created by modified stabilisation have a similar property as the crossings created by an RII move which can perhaps be used to proof this conjecture. However the new crossings are much further away from each other and it is not known how this affects the manner in which their contributions cancel.

6.5 Complexity of the cocycle

A very nice property of the cocycle is that it can be computed in polynomial time.

Theorem 6.1. *Let K be a knot with k crossing. The cocycle evaluated on the rotation loop has order of complexity $\mathcal{O}(m^3)$ in the number of ring operators in $\mathbb{Z}$. Here m is the number of crossings in the n-cabled balanced knot. We can bound m by $m \leq 6kn^2$.*

Proof. Given a knot diagram with k crossings we first need to construct the balanced knot diagram. This involves computing the writhe and the winding number which both depend linearly on the number of crossings.

The balanced knot diagram might have more crossings then the original knot diagram but the additional crossings to make the writhe 0 can not be more than k and the same is true for the winding number. Hence the number of crossings for the balanced knot diagram is at most $3k$.

Then we need to construct the sequence of knot diagrams corresponding to the rotation loop. Each crossing is involved in an RIII move twice which means we have at most $6k$ knot diagrams.

We need the n-cabled version of the knot so we will have at most $6kn^2$ crossings. We will define m to be the number of crossings and we know that $m \leq 6kn^2$.

Next we need to compute the homological markings. Using that $[c] = c_2 - c_1$ we only need to find in which segment each of the crossings is. Therefore computing the homological markings is linear in m. From lemma 3.7 we know that the homological markings are invariant under RIII which means that we have to compute them only once.

So far the amount of operations we need depend linearly on k.

Next we need to check which are the valid p moves. The conditions on homological markings, order and segments are all things that can be checked as this information is already computed. Doing this takes a number of operations that do not depend on k or n. We need to do this for at most m moves.

For each valid move p we might need to compute W_2. Computing this term involves looking for diagrams with to crossings involving certain conditions. This means we need an amount of operations that is quadratic in the number of crossings. We need to potentially do this for all p moves, which means the total amount of operations of the order $\mathcal{O}(m^3)$.

For each of the other terms in the equation we need to find diagrams involving one extra crossings and we need to compute these terms for up to all crossings, which means they are of order $\mathcal{O}(m^2)$.

The highest order term is the cubic term so we can conclude that evaluating the cocycle on the rotation loop has complexity $\mathcal{O}(m^3)$, where m can be bounded by $6kn^2$. □

Corollary 6.2. *Let K be a knot with k crossing. The formula R_n has order of complexity $\mathcal{O}(m^3)$ in the number of ring operators in $\mathbb{Z}$. Here m is the number of crossings in the n-cabled balanced knot. We can bound m by $m \leq 3kn^2$.*

Proof. The computations necessary for R_n on any knot are a strict subset of the computation necessary for evaluating the cocycle on that knot. We can bound the number of crossing by $3kn^2$ as *Scan* has exactly half the number of knot diagrams as the full rotation loop. $\square$

7 Conclusion

In this thesis we have looked at the cocycle $R^{(2)}_{n,d^+}(a, r, K, \gamma)$. In order to do investigate it we first had to explain the necessary background in knot theory and Fiedlers theory of cocycles. In particular we have explicitly written out and illustrated the definitions of each term in this cocycle in a manner that is more accessible then it is Fiedlers work. We set out to proof explicitly the invariance of this cocycle and in doing so proved some results regarding the homological markings of a cabled knot. We also slightly simplified the definition of the refinement of the cocycle.

In proving the invariance under RII we discovered that the W_1 term is zero for the first half of the rotation loop which lead us to the equation R_n. We were not able to proof the invariance under RIII. As the cocycle is a knot invariant we know that it is true, but for R_n we can only conjecture it based on numerical results. It might be possible to apply the methods that we used in our proof for RII but as one needs to considers much more crossings in the case of RIII it could be that this approach is unworkable due to the many case distinctions. Finally we have used the program by written by Roland van der Veen to investigate some numerical results. This led to some more questions, which we have stated below, but also answered questions regarding invariance and ability to distinguish mirrors.

We end with some open questions and suggestions for further research.

- Proof that R_n is invariant under R III and modified stabilisation.
- Can R_n be modified to be an invariant without losing its property of distinguishing mirror knots?
- Fiedler described many more invariants in [5], what happens if these are evaluated on only the first half of the rotation loop? Do they distinguish mirrors, are they invariants or could they be modified to be invariants?
- What is the connection between the cocycle and finite invariants? Can all evaluations on $(p, 2)$ torus knots be expressed as multiples of triangle numbers, tetrahedral numbers or higher dimensional generalisations of these.? What happens if we instead look at (p, q) torus knots.
- The values of $R_2(a, r, K)$ seem to be independent of r. Can this be proven and does this also hold for higher n? Similarly R does seem to be independent of r, is this always true.
- Is it a coincidence that the values for 3_1 and 4_1 as well as 5_2 and 6_1 are opposites? If not, are there other such pairs and can we explain their behaviour?

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