Generalisability of the LRL-vector: what the Kepler-Coulomb system can teach us about central potentials

Author: Savio Ming Hou Chen

Supervisor: Prof. Dr. Diederik Roest
Second examiner: Prof. Dr. Anupam Mazumdar

July 11, 2022

Abstract

Upon closer inspection of the Kepler-Coulomb problem a constant vector quantity is determined, revealing that the system admits a hidden symmetry. After investigating both the utility and interpretation of this invariant LRL-vector, other hidden symmetries are sought out. Through a mapping between the Kepler-Coulomb and two-dimensional harmonic oscillator, the latter is concluded to contain a conserved tensor. As only systems with specific potential energies could host a hidden symmetry, a modernised version of Bertrand's theorem is studied to learn about the qualified set of potentials. Classically, only the Kepler-Coulomb and harmonic oscillator potentials satisfy both the orbit equation and the closed orbit condition. Extension to include special relativistic effects provide a general solution of rather unusual form. Revealing that the classical solutions do not have closed orbits for all bound states in relativistic mechanics.

Contents

1 Introduction

The importance of classical mechanics in physics is hard to overemphasise. For example, it not only describes the phenomena of everyday life reliably, but also serves as the basis for most more advanced ideas. As a result, classical mechanics is introduced as one of the very first courses to students. Despite the fact that the subject is comprised of the lifework of countless great minds, course units on the subject are merely allowed to teach for a couple of weeks. Due to this, unfortunately, the omission of topics is inevitable.

One such excluded concept is the Laplace-Runge-Lenz (LRL) vector. This vector is generally associated with inverse square force laws and possesses several special characteristics. Consequently, the study of these features provides valuable insight on the nature of the system itself. For instance, in addition to complementing the information obtained from energy and angular momentum, the LRL-vector allows the description of the system in momentum space as well. Furthermore, its mere existence is of crucial importance for the understanding of the time evolution of states in phase space. Therefore, examining the LRL-vector can be conducted to determine properties of the system that are invisible to other approaches.

Aside from its diverse functionality, the history of the LRL-vector is also noteworthy. Over the course of time it seems that it has been independently discovered by numerous researchers. Thus, the proper name of the vector is sometimes still a subject of controversy [1]. The story begins when Lenz used the LRL-vector to describe the perturbed energy levels of the Kepler problem [7], in this paper he cited a text by Runge as his source for the LRL-vector. Even though Runge applied the vector to derive the orbit equation, similarly to Lenz, Runge did not claim ownership over the vector [7]. After Lenz's paper, Pauli used the vector to obtain the energy spectrum of hydrogen. Since Pauli cited Lenz's paper as his source, the vector temporarily became known as the "Runge-Lenz" vector. Although this name was used in books on quantum mechanics written during that time period, such as [14], people discovered the vector in older works by Laplace. Laplace demonstrated the vector's relation with the energy and angular momentum of the system in his *Traté de mécanique celeste* [6]. From this point on, people widely referred to the vector as the Laplace-Runge-Lenz vector. While this convention will also be used throughout this paper, the history of the LRL-vector did not start here. Some time after it came to light that the vector had been derived by Hamilton, in his *Applications of Quaternions to Some Dynamical Questions* the vector is referred to as the "eccentricity vector" [6]. Furthermore, by utilising the connection between the LRL-vector and hodographs shown by Hamilton, it became possible to trace this vector back to even older works such as Gibb's *Vector analysis* [6]. Despite rewinding the clock this far back, there are still earlier works where the vector can be found. The oldest mention of the LRL-vector seems to be in a letter exchange between Jakob Hermann and Johann I. Bernoulli. Here Hermann found the conserved magnitude and Bernoulli generalised it to obtain the LRL-vector [1]. As one might imagine after learning about its past, the LRL-vector has played a rather significant role throughout history. In spite of the fact that before Lenz's paper the vector was essentially unknown to the vast majority of scientists, just from the observation that the LRL-vector has been independently derived by different scientists for different goals speaks volumes of about its physical significance.

This paper is written with the intent to introduce the reader to the LRL-vector and study its relevance both within the Kepler-Coulomb problem and to physics in general. This is done by first revisiting the classical Kepler-Coulomb system following the steps of [8] and [11]. By utilising the properties of the system alongside the Newtonian definition of force, the LRL-vector can be derived. Then the constant vector is studied by looking at the orbital motion in position space, phase space and momentum space. Secondly, largely following the footsteps of [12], the connection between the inverse square force law and Hooke's law is researched by studying a special complex mapping.

In this section it will become clear, that quantities in the Kepler-Coulomb system can be mapped to their counterparts in the harmonic oscillator system, when the orbital plane is reimagined as a complex plane. From this can be observed that the components of the LRL-vector are mapped towards a special tensor, which is described in more detail in [4]. This observation reveals that the additional constants of motion of the two systems are related to each other. Finally, to determine whether the LRL-vector could be applied to find other hidden symmetries, both a modernised version and a special relativistic extension of Bertrand's theorem are examined. More details on the aforementioned versions of the theorem can be found in [3] [5] and [9] respectively. After reading this paper the reader will have obtained a more complete portrayal of the Kepler-Coulomb problem and understand its implications to hidden symmetries.

2 The Kepler-Coulomb problem

2.1 The problem description

In order to learn about the LRL-vector it is beneficial to start with the Kepler-Coulomb system. This is a special case of the two-body problem where the interaction potential is given by the general form

$$V(r) = -\frac{k}{r}, \tag{1}$$

where r is the inter-particle distance and the constant k is dependent on whether the interaction is gravitational or electromagnetic. For the former, $k = Gm_1m_2$ with G being the Newtonian constant of gravitation. For the latter $k = \frac{q_1q_2}{4\pi\epsilon_0}$ with ϵ_0 being the vacuum permittivity. In the aforementioned expressions a subscript is given to the mass and the charge in order to be able to distinguish what point particle is being referred to. From Equation 1 can be seen that $V(r)$ is only dependent on the distance between the point particles. These types of potentials are referred to as "central potentials" and contain characteristics that can simplify the description of the system.

In principle, the motion of the two particles could be researched using the most general reference frame, which is illustrated in Figure 1a. As the origin in this frame is merely used as a point of reference, its position is chosen arbitrarily and does not contain any physical significance. The Lagrangian describing such a reference frame is given by

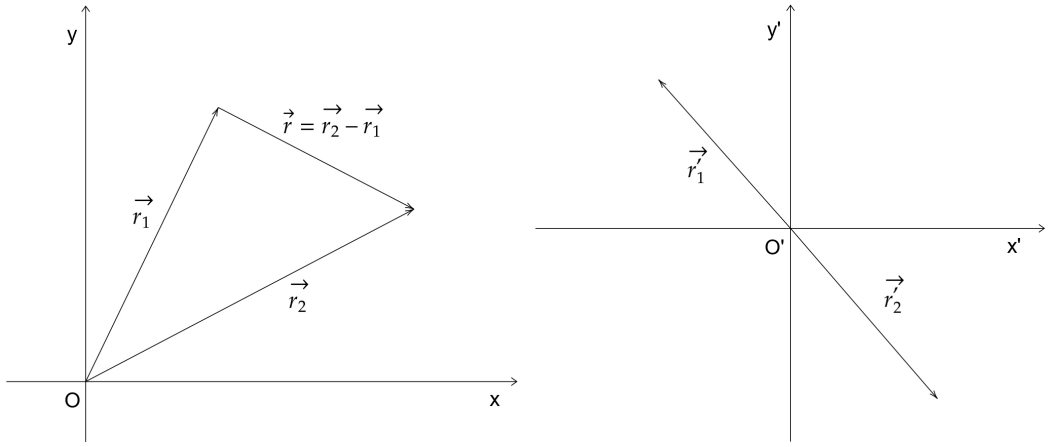
$$\mathcal{L} = T - V = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{Gm_1m_2}{|\vec{r_2} - \vec{r_1}|}.$$

However, by choosing the position of the origin more carefully the Kepler-Coulomb problem can be simplified. A new reference frame can be constructed by choosing the position of the origin to coincide with the center of mass of the point masses, this is the center of mass frame. Since the origin of this frame is always between the two point masses, the position vectors in this frame can also be expressed in terms of the displacement vector $\vec{r}$. As a result of that, the expression for the Lagrangian can be rewritten into

$$\mathcal{L} = \frac{1}{2}\mu v^2 + \frac{Gm_1m_2}{r} \tag{2}$$

where $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass and $\vec{v}$ is the time derivative of the displacement vector. Consequently, by choosing the center of mass reference frame the description of the Kepler-Coulomb problem reduces to the motion of single body described by the vector $\vec{r}$.

Despite the fact that a physical particle is in principle allowed to move in three spatial dimensions, the reference frames described presented in Figure 1a and Figure 1b are only two-dimensional.



(a) *This reference frame is considered the most general reference frame due to the fact that the position of the origin was chosen arbitrarily. Regardless of its position one can always construct position vectors towards the point masses without changing its physical interpretation.*

(b) *By repositioning the origin to coincide with the center of mass the new primed position vectors will always be (anti-)parallel to the displacement vector $\vec{r}$. Making use of this fact the two body problem can be reduced to the description of a single particle.*

Figure 1

This apparent contradiction can be justified by studying the Lagrangian given in Equation 2. As the Lagrangian does not contain any explicit angular dependence, the system can be concluded to be symmetric under rotations. As Noether's theorem relates symmetries to conservation laws, this implies that the system conserves angular momentum. An alternative way to obtain the invariance of the angular momentum is by noting that a central potential is not capable of creating any torque, resulting in the fact that the angular momentum is not allowed to be time dependent. The conservation of angular momentum forces the motion described by $\vec{r}$ to be confined within a plane, permitting the description of the orbital motion to be given in just two dimensions. However, it is important to keep in mind that while the orbits are two-dimensional, the system is described by all three spatial dimensions.

2.2 Conserved quantities and phase space

Due to the inherent symmetries of the Kepler-Coulomb problem the description of motion is more illuminating in polar coordinates than in Cartesian coordinates. Rewriting Equation 2 in terms of polar coordinates gives rise to the following

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{Gm_1m_2}{r}, \tag{3}$$

where ϕ is the anti-clockwise angle defined between the position vector and the x-axis. Note that this Lagrangian is independent of ϕ despite being dependent on $\dot{\phi}$. By solving the Euler-Lagrange equation for ϕ it becomes clear that there is a conserved quantity

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

$$\frac{\partial}{\partial t}(\mu r^2 \dot{\phi}) = 0.$$

The physical significance of this quantity is more readily understood by also looking at the Hamilton's equations of motion

$$p_\phi = \frac{\partial H}{\partial \phi} = \frac{\partial \mathcal{L}}{\partial \phi} \equiv L,$$

where p_ϕ is the momentum conjugate of the angle ϕ, also known as the angular momentum L, and H is the Hamiltonian of the system. Combining these equations together then results in the following angular momentum equation

$$L = \mu r^2 \dot{\phi}. \tag{4}$$

Aside from angular momentum there are other symmetries within this system. As shown in the previous section, the symmetries of the system become more visible by inspecting the Lagrangian. From Equation 3 it can be observed that the Lagrangian does not contain any explicit time dependence. Hence, Noether's theorem demands the total energy of the system to be invariant and given by

$$E = T + V = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{Gm_1m_2}{r}. \tag{5}$$

Within the energy and angular momentum conservation laws a lot of information about the orbital motion is contained. By combining Equation 4 with Equation 5 we obtain the orbit equation [8] [11]

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r},$$

which can be easily rearranged into

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - \left[\frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}\right]\right)}$$

giving rise to the two integral equations

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \left[\frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}\right]\right)}} \tag{6}$$

and

$$d\phi = \frac{L}{\mu r^2}\frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \left[\frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}\right]\right)}}.. \tag{7}$$

Equation 6 and Equation 7 completely specify the orbit. If one manages to integrate these expressions, then the Kepler-Coulomb problem is solved. In the past the analytical techniques to perform these integrations were not as sophisticated, therefore the solutions were expressed in terms of integrals and the Kepler-Coulomb problem was said to be"solved to quadrature" [8] [15]. Fortunately, analytical solutions to these equations have been found [8] [11] and the orbit of a particle obeying Equation 7 is given by

$$r(\phi) = \frac{a(1-e^2)}{1 + e\cos(\phi - \phi_0)}, \tag{8}$$

where a and e are defined by

$$a = -\frac{Gm_1m_2}{2E}$$

$$e = \sqrt{1 + \frac{2EL^2}{\mu k^2}}.$$

Closer inspection of Equation 8 reveals that the orbiting particle traverses an ellipse with semi-major axis a and eccentricity e.

What follows from the elliptical orbits found in Equation 8 is that the bound states of the Kepler-Coulomb system have closed orbits, this means that the same trajectory is repeated every time. At first glance this phenomenon might be overlooked, but having closed orbits has large implications that are more visible when the problem description is altered. For example, the orbital motion in position space is studied by keeping track of the combinations of spatial coordinates that are available to the bound states. Likewise, the orbits can be studied in phase space by keeping track of both the allowed spatial coordinates and their associated momenta. These specific combinations are encoded as points in phase space and are interpreted as specific states of the system. The advantage of the phase space description is that every point already contains all the information required to compute its time evolution by utilising Hamilton's equations. Since every point is an unique combination of spatial coordinates and momenta, every state has a distinct time evolution. Therefore, from a single point in phase space the specific path to all future states can be determined and the crossing of paths in phase space is forbidden [15]. From the specific shape of the Kepler-Coulomb orbits, one can conclude that both the radial and angular time evolution are related to each other, as this shape is only possible when their periods are exactly identical. In addition to that, since this shape is preserved indefinitely, this ratio must stay the same. Upon recalling the conservation of angular momentum in Equation 4, it is known that the angular velocity $\dot{\phi}$ does not have an explicit time dependence, therefore both the angular and radial time evolution are not allowed to change with time. Using this knowledge combined with the fact that both the radial and angular components are periodic, one is able to conclude that orbits in phase space are required to be periodic as well. As a consequence of this and the fact that path is not allowed to intersect with itself, the orbit must also be a closed curve in phase space.

The phase space curve of an orbit is directly related to the symmetries that the system admits. For instance, for energy to be conserved Equation 5 must hold, restricting the points in phase space that are valid for the system. In principle, the maximum number of conserved quantities coincide with the total number of spatial coordinates and momenta, however this would only correspond to a single state in phase space. In order to allow for dynamical behaviour, at most five independent quantities can be conserved. For the Kepler-Coulomb system in particular, closed curves in phase space are associated with the conservation of five independent quantities.

The previous description of the orbits in spatial coordinates only divulged the conservation of energy and angular momentum. The former is a scalar, consisting of a single quantity, and the latter is a vector, corresponding to three quantities. Although four of the independent constant quantities could be discovered by scrutinizing the Lagrangian, Equation 2 does not provide any indication of another invariant quantity. Yet, the fact that orbits are also closed in phase space would require its existence. Since this unexpected symmetry could not be inferred from examining the Lagrangian, it is referred by many as the "hidden" or "accidental" symmetry of the system [2].

2.3 The LRL-vector and its implications

Perhaps the most natural response would be to search for a conserved scalar quantity, but that turns out to be incorrect. This will be demonstrated clearly by investigating the problem using the Newtonian definition of force. In this paper the derivations are simplified to highlight the most important details, the full derivation can be found within [8]. Since the interaction potential is central, the force must be radial since

$$\vec{F} \equiv -\vec{\nabla}V(r) = f(r)\frac{\vec{r}}{r},$$

where $\vec{\nabla}$ is the del operator $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ and $f(r)$ is a scalar valued function. The Newtonian definition of force is then given by

$$\vec{F} = \frac{d\vec{p}}{dt} = f(r)\frac{\vec{r}}{r}. \tag{9}$$

By taking the outer product of Equation 9 and the angular momentum vector, the relation can be rewritten into

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = -mf(r)r^2\frac{d}{dt}\left(\frac{\vec{r}}{r}\right), \tag{10}$$

where from now on the reduced mass is written as $\mu = m$. For the Kepler-Coulomb problem Equation 10 simplifies by using that

$$f(r) = -\frac{k}{r^2}$$

resulting in

$$\frac{d}{dt}\left(\vec{p}\times\vec{L} - \frac{mk\vec{r}}{r}\right) = 0, \tag{11}$$

which implies the existence of a conserved vector

$$\vec{A} = \vec{p}\times\vec{L} - \frac{mk\vec{r}}{r}. \tag{12}$$

Equation 11 discloses that the final invariant object is the vector defined in Equation 12 rather than a scalar.

The vector presented in Equation 12 is the LRL-vector. Since both the outer product $\vec{p}\times\vec{L}$ and the other term points in the direction perpendicular to both the linear and angular momentum, the LRL-vector must be contained in the plane of the orbit. The LRL-vector is best visualised in position space, where it is represented graphically by a constant vector extending from the center of attraction, located at the leftmost focus of the ellipse, to the periapsis, the point of the orbit closest to the center of attraction. These relations have been illustrated in Figure 2.

At first glance the discovery of the LRL-vector could seem contradictory to what was concluded in the previous section. A vector consists of three components while the Kepler-Coulomb system can only allow one more independently conserved quantity, therefore finding a conserved vector might sound suspicious. Fortunately, this apparent paradox vanishes when the LRL-vector is examined in more detail. By taking the inner product of the LRL-vector with the position vector the following will be obtained

$$\vec{A}\cdot\vec{r} = Ar\cos\phi = (\vec{p}\times\vec{L})\cdot\vec{r} - mk.$$

Cyclic permutation of the triple product allows the first term to be rewritten into

$$(\vec{p}\times\vec{L})\cdot\vec{r} = (\vec{r}\times\vec{p})\cdot\vec{L} = L^2.$$

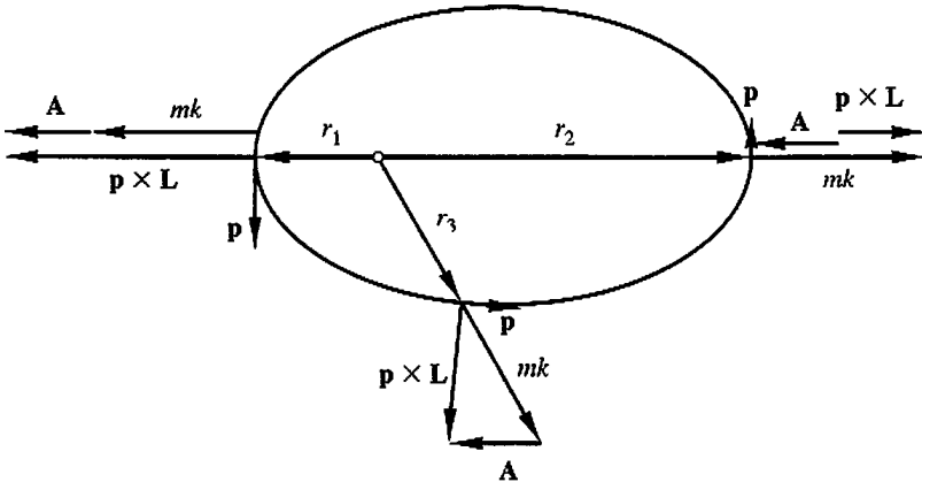


Figure 2: *This is a graphical representation of the position space orbits associated with the Kepler-Coulomb problem. The center of attraction is located at the leftmost focus of the ellipse and represented by a white circle. The LRL-vector, defined in Equation 12, is the constant vector extending from the center of attraction towards the periapsis, the point on the orbit that is closest to the center of attraction. To exhibit the invariance of the LRL-vector, three points on the orbit have been chosen where the components of the LRL-vector are explicitly drawn. This figure has been obtained from [8].*

Then the inner product between the LRL-vector and the radial vector is then described by

$$Ar\cos\phi = L^2 - mk,$$

which can be rearranged into

$$\frac{1}{r} = \frac{mk}{L^2}(1 + \frac{A}{mk}\cos\phi). \tag{13}$$

Equation 13 might not seem illuminating at the moment, but it is actually equivalent to Equation 8, also capable of describing the orbits of the bound states. Their connection can be seen more clearly by rewriting Equation 8 by substituting the identity

$$a(1-e^2) = \frac{L^2}{mk},$$

which results into

$$r = \frac{a(1-e^2)}{1+e\cos\phi} \rightarrow \frac{1}{r} = \frac{mk}{L^2}(1 + e\cos\phi). \tag{14}$$

Clearly Equation 13 and Equation 14 are related. If both expressions are interpreted as representing the same orbits, then the relationship between the constants of motion can be determined to be

$$A^2 = (mke)^2 = m^2k^2 + 2mEL^2, \tag{15}$$

which constrain what values A is allowed to have. Equation 15 has explicitly shown that the magnitude of the LRL-vector is neither independent of energy or angular momentum. Moreover,

from the fact that $\vec{A}$ is necessarily within the plane of motion, one can also conclude that it is constraint by the following relation

$$\vec{A} \cdot \vec{L} = 0.$$

Now the invariance of the LRL-vector becomes more comprehensible. Despite the fact that the vector introduces three different components, these components are related to both the energy and angular momentum in two different ways. Thus, effectively the LRL-vector only contributes one independent conserved quantity to the system.

Although the preceding analysis of the LRL-vector has identified the final invariant quantity of the system, its physical interpretation has yet to become clear. As Noether's theorem relates conserved quantities to continuous symmetries of the system, the hidden symmetry of the system can be expected to provide clues about the physical significance of the LRL-vector. Possibly the most straightforward method to study this symmetry is by studying the Kepler-Coulomb problem in momentum space. It turns out that the LRL-vector enables the orbits to be described purely in terms of momenta. This is achieved by taking the outer product of the angular momentum with the LRL-vector. By recalling the vector triple product identity, the aforementioned outer product can be rewritten to obtain

$$\vec{L} \times \vec{A} = \vec{p}L^2 - mk\frac{\vec{L} \times \vec{r}}{r}.$$

By describing the motion in Cartesian coordinates and orienting the axes of our reference frame such that $\vec{L} = (0, 0, L)$ and $\vec{A} = (A, 0, 0)$, the vector product gives rise to the following two equations

$$p_x L^2 = mk\frac{yL}{r}$$

$$p_y L^2 = AL - mk\frac{yL}{r}.$$

These relations can be combined to describe the orbits purely in terms of system characteristics, such as the interaction strength and constants of motion, and momentum components

$$p_x^2 + \left(p_y - \frac{A}{L}\right)^2 = \left(\frac{mk}{L}\right)^2. \tag{16}$$

Equation 16 is recognised as the equation of a circle, therefore one can conclude that the Kepler-Coulomb orbits are always given by perfect circles in momentum space. These momentum space representations, shown graphically in Figure 3, are known as "hodographs" and are employed as an alternative way to specify the state of the system. By comparing Equation 15 and Equation 16 a connection between hodographs and the energy of the state can be determined. Whenever the momentum in the y-direction vanishes, $p_y = 0$, Equation 16 can be rewritten using Equation 15 to read

$$p_x^2 = \frac{m^2k^2 - A^2}{L^2} = -2mE, \tag{17}$$

where E here is the Kepler-Coulomb bound state energy which obeys $E < 0$. By defining $p_x = p_0$ for when p_y vanishes the relation becomes

$$p_0 = \sqrt{-2mE}.$$

By researching this momentum representation of the system the physical interpretation of the hidden symmetry becomes more apparent. Firstly, from Equation 17 it has become clear that the Kepler-Coulomb problem contains energy degenerate states, which have differing magnitudes for the LRL-vector A, the angular momentum L, and the two scalars quantifying the interaction strength m and

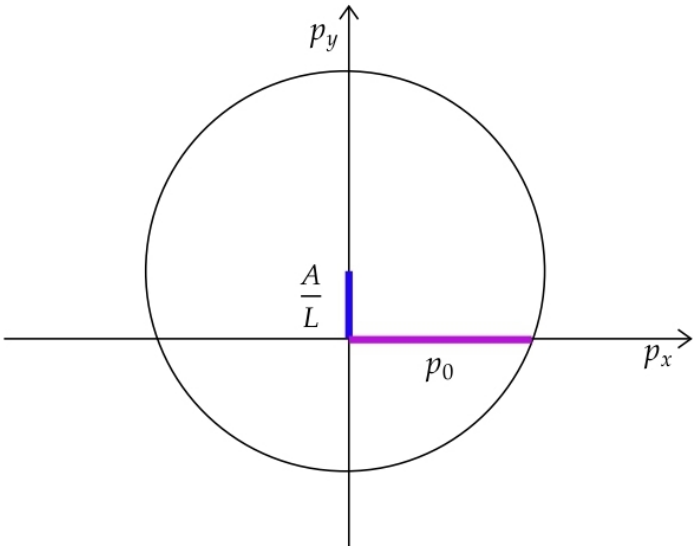


Figure 3: *Every bound state in the Kepler-Coulomb system can be described in momentum space as perfect circles. The center of the circle is always a distance* $\frac{A}{L}$ *in the* p_y *direction away from the origin. From Equation 16 it becomes clear that the intersection of the hodograph with the* p_x *axis, denoted symbolically by* p_0*, is directly related to the energy of the state.*

k. Secondly, Equation 15 reveals that the eccentricity of a position space orbit is dependent on the magnitude of the LRL-vector and the interaction strength. Combining these observations together it becomes possible to conclude that the same energy state can be obtained by a family of distinct spatial orbits. The desired physical interpretation of the hidden symmetry is directly connected to the operation that links these energy degenerate orbits.

The explicit form of the symmetry operation linking the previously found energy degenerate states can actually be determined through more sophisticated approaches [11]. In one approach the infinitesimal variation in both position and momentum, corresponding to the hidden symmetry, is obtained by computing the Poisson brackets of the respective quantity with the LRL-vector. Then it is shown that these infinitesimal transformations are equivalent to a variation in the eccentricity, from which one can conclude that the symmetry generated by the LRL-vector is identical to an infinitesimal rotation. Alternatively, one can work with finite transformations by using more graphical arguments. By means of stereographic projection one is able to project the hodographs on the surface of a four-dimensional sphere. These projected circles then become great circles on this hypersphere. This allows the inference that transforming between energy degenerate hodographs is achieved by the same operation that links great circles on this four-dimensional sphere, which is given by a four-dimensional rotation. Finally, it has become possible to conclude that the hidden symmetry of the Kepler-Coulomb problem is the same as SO(4) symmetry.

Within this chapter the characteristics of the Kepler-Coulomb system has been scrutinised by describing the orbital motion in different perspectives. The position and phase space descriptions revealed the existence of an additional constant of motion, an invariant vector quantity. To find the hidden symmetry associated to this LRL-vector the problem was reformulated to its momentum representation, which revealed that the system is invariant under four-dimensional rotations. The discovery of an additional constant of motion could imply the existence of other hidden symmetries.

Although the LRL-vector was relatively easy to obtain, in the case that other additional constants of motion exist their derivation need not be straightforward or easy. Ideally, one would like to make use of the results obtained in the Kepler-Coulomb problem and use it to investigate other systems. Fortunately, the two-dimensional harmonic oscillator system is known to be related to the Kepler-Coulomb problem. To attain more understanding about hidden symmetries the next chapter will investigate the connection between the Kepler-Coulomb and the harmonic oscillator systems.

3 Kepler-Coulomb and the two-dimensional harmonic oscillator

3.1 Complex notation

The Kepler-Coulomb and the harmonic oscillator systems are not only very important models used throughout the entirety of physics, but they also happen to be very closely related. By describing the planar orbital motion in Cartesian coordinates and reimagining the y-axis as an imaginary axis, all vectors $\vec{r} = (x, y)$ can be rewritten compactly as a single complex number $z = x + iy$. In this alternative notation a mapping between the two systems can be constructed. This connection becomes more clear by the inspection of the variational principle in its Maupertuis' form [12]. This principle links the action of a system to the following integral

$$S = \int_\gamma ds\sqrt{2m}\sqrt{E - V},$$

where $ds = vdt$ is the distance in generalised coordinates. For the particular case where the orbital motion is in a plane described by the complex coordinate z this equation transforms into

$$S = \int_\gamma |dz|\sqrt{2m}\sqrt{E - V(|z|)}. \tag{18}$$

If we choose the potential V to be proportional to a harmonic oscillator potential and define a new variable $w = z^2$, then the potential can be expressed as

$$V(|z|) = |z|^2 = \frac{1}{4}\left(\left|\frac{dw}{dz}\right|\right)^2.$$

Substituting this form of the potential into Equation 18 returns an expression for the action of a harmonic oscillator

$$S = \int_\gamma |dz|\sqrt{2m}\sqrt{E - \frac{1}{4}\left(\left|\frac{dw}{dz}\right|\right)^2}. \tag{19}$$

Equation 19 is very interesting, as it allows the integration variable to be changed when the differential $|\frac{dw}{dz}|$ is taken outside of the square root

$$S = \int_\gamma \frac{1}{2}|dw|\sqrt{2m}\sqrt{(4\left(\left|\frac{dz}{dw}\right|\right)^2)E - 1}. \tag{20}$$

The differential $|\frac{dz}{dw}|$ can be evaluated explicitly by means of the definition of w

$$\left|\frac{dw}{dz}\right| = 2|z|$$

$$\left|\frac{dz}{dw}\right| = \left(\left|\frac{dw}{dz}\right|\right)^{-1} = \frac{1}{2|z|} \tag{21}$$

and by putting Equation 21 into Equation 20 it reads

$$S = \int_\gamma \frac{1}{2}|dw|\sqrt{2m}\sqrt{\frac{E}{|w|} - 1}. \tag{22}$$

This equation is very fascinating, as it can be reinterpreted as another variational principle equation of the form

$$S' = \int_\gamma |dw|\sqrt{2m}\sqrt{E' - V'}, \tag{23}$$

where by comparison with Equation 22 one can determine that $E' < 0$ and $V' = -\frac{E}{|w|}$. These familiar characteristics are exhibited only by the bound states of the Kepler-Coulomb potential, this proves that the complex mapping $w = z^2$ connects the harmonic oscillator to the Kepler-Coulomb system [12]. Moreover, as no information is lost during the derivation, the mapping could be applied backwards. Consequently, by rewriting the orbits in terms of complex coordinates an invertible mapping between both systems can be obtained.

3.2 The constraints imposed by the Kepler-Coulomb system

An interesting consequence of this mapping is that it demands the orbits in the harmonic oscillator to be ellipses just like in the Kepler-Coulomb case [12]. This can be proven by defining a circle in the complex plane

$$\zeta = re^{i\phi}$$

and defining the relation

$$z = \zeta + \frac{1}{\zeta}. \tag{24}$$

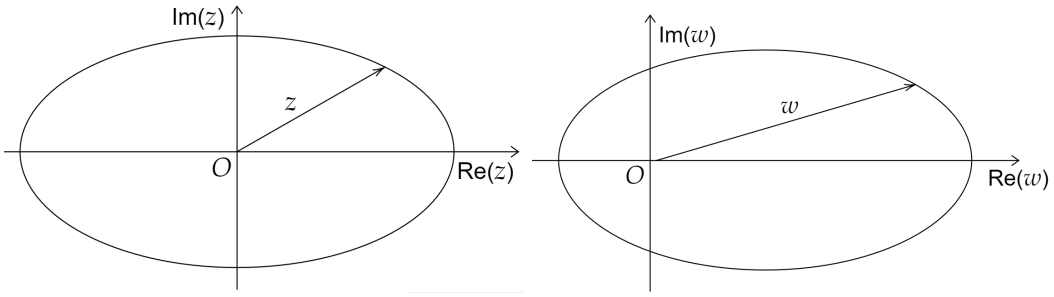
As a result z will describe ellipses with specific characteristics. First of all, from its definition the origin, which is equivalent to the center of attraction, is at the geometrical center of the ellipse. In addition to that it has a the semi-major axis $a = r + \frac{1}{r}$, a semi-minor axis $b = r - \frac{1}{r}$, and their foci are located at $x = \pm 2$. Applying the complex mapping to these ellipses then gives rise to

$$w = z^2 = \zeta^2 + \frac{1}{\zeta^2} + 2. \tag{25}$$

Since the square of a circle ζ is also a circle, it can be concluded that the square of an ellipse is another ellipse. Moreover, the constant term in Equation 25 reveals that the center of attraction is repositioned as a result of the mapping, moving it to the leftmost focus of the ellipse. The ellipses described by w can be thought of as the Kepler-Coulomb orbits and are illustrated in Figure 4b. The ellipses described by z can be interpreted as the harmonic oscillator orbits, these are showcased in Figure 4a. More generally, any ellipse given by $z = a\cos(\phi) + ib\sin(\phi)$ with its center of attraction at the geometrical center will be mapped onto an ellipse where this center of attraction is located at the leftmost focus [12].

Apart from that, the complex mapping is valuable as it allows the quantities of both systems to be expressed in terms of their counterparts. For instance, by recognising that the difference between the total and potential energy is equivalent to the kinetic energy, the term $E - V$ becomes

$$E - V = T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\phi}^2,$$



(a) *Within the complex notation the harmonic oscillator orbits can be are described using a single complex number z. Within the harmonic oscillator, the center of attraction O is always located at the geometrical center of the ellipse. Upon mapping to the Kepler-Coulomb system O will be relocated towards the left as is required by Equation 25.*

(b) *As was demonstrated by studying the harmonic oscillator action, the complex number w, where $w = z^2$, can be used to describe the Kepler-Coulomb orbits. For the Kepler-Coulomb system, the center of attraction O is always at the leftmost focus of the orbit. When the mapping is inverted O will reposition itself at the geometrical center.*

Figure 4

where $|\vec{v}| = |\frac{d\vec{r}}{dt}| = \frac{d|z|}{dt}$ and I is the moment of inertia. Since a free particle rotating around a fixed axis has $I = mr^2$ and the conservation of angular momentum imposes that $\dot{\phi}^2 = \frac{L^2}{m^2 r^4}$, the kinetic energy can be expressed as

$$E - V = \frac{m}{2}\left(\frac{d|z|}{dt}\right)^2 + \frac{L^2}{2m|z|^2}. \tag{26}$$

Upon substitution of Equation 26 into Equation 18 the variational principle becomes

$$S = \int_\gamma |dz| \sqrt{m^2\left(\frac{d|z|}{dt}\right)^2 + \frac{L^2}{|z|^2}}. \tag{27}$$

The connection to the Kepler-Coulomb system can be made by multiplying the previous equation by one and recalling that

$$\left|\frac{dw}{dt}\right| = \left|\frac{d}{dt}(z \cdot z)\right| = 2|z|\frac{d|z|}{dt}$$

$$\frac{d|z|}{dt} = \frac{1}{2|z|}\frac{dw}{dt}.$$

$$S = \int_\gamma |dw| \frac{1}{2\sqrt{|w|}} \sqrt{m^2\left(\frac{1}{2\sqrt{|w|}}\frac{dw}{dt}\right)^2 + \frac{L^2}{|w|}}.$$

$$S = \int_\gamma \frac{1}{2}|dw| \sqrt{m^2\left(\frac{1}{2|w|}\frac{dw}{dt}\right)^2 + \frac{L^2}{|w|^2}}.$$

To make the previous equation resemble the form shown in Equation 27, the time derivative in the square root can be rewritten into

$$S = \int_\gamma \frac{1}{2}|dw| \sqrt{m^2\left(\frac{dw}{d\tau}\right)^2 + \frac{L^2}{|w|^2}},$$

where $d\tau = 2|w|dt$. This observation reveals that the mapping induces a rescaling of the time coordinate, providing the relationship between the time coordinates of the systems. Similar to the variational principle, the angular momentum equation can also be utilised to shine some light on how the system quantities are related. By the conservation of angular momentum, obtained by rewriting Equation 4

$$L_K = m|w|^2 \frac{d(\phi_K)}{d\tau} = m|w|^2 \cdot \frac{1}{2|w|} \frac{d(\phi_K)}{dt} = m|z|^2 \frac{d(\frac{\phi_K}{2})}{dt} = L_H,$$

where the subscripts are used to differentiate between the systems. This demonstrates that in addition to the time coordinate also the rotation angle ϕ is rescaled by the mapping.

3.3 The LRL-vector and the constants of motion

Being able to express the quantities of the two-dimensional harmonic oscillator in terms of their counterparts of the Kepler-Coulomb system provides insight into the traits of the former. As both systems are similar in the sense that they share having periodic closed planar orbits, the amount of conserved quantities should be the same. Before continuing to the rest of the analysis, it is important to highlight some choices taken in this paper. Despite the fact that the two-dimensional harmonic oscillator can be fully described in two spatial dimensions, it has been chosen to describe the system in three spatial dimensions. This is to preserve the vector nature of the angular momentum, which always points perpendicular to the plane of the orbits. In principle, this is unnecessary, but the author believes that this description is beneficial for the analyses of this paper. In the Kepler-Coulomb system the additional constant of motion was identified, via Equation 11, to be a vector pointing from the center of attraction to the periapsis. However, in the harmonic oscillator the origin does not reside at one of the foci of the ellipse. In this system the center of attraction is located at its geometric center, the point where the two axes of symmetry intersect. Therefore, the orbit does not have a single point that is closest to the origin. This would make the concept of another conserved vector such as $\vec{A}$ in this system hard to visualise. However, it turns out that the vector described in Equation 12 is still related to the hidden conserved quantity when used together with the complex mapping.

To exhibit the connection between the LRL-vector $\vec{A}_K$ and the hidden conserved quantity of the harmonic oscillator, Equation 12 must be rewritten in terms of its the mapped components. To simplify the notation of the upcoming derivation let

$$w = x + iy \qquad p = p_x + ip_y, \tag{28}$$

which imply that

$$p \times L = L(p_y - ip_x) = -iLp.$$

Utilising these definitions Equation 12 simplifies into

$$\vec{A}_K = -iL_K p_K - \kappa \frac{w}{|w|}, \tag{29}$$

where κ is a constant and the subscripts are used to highlight that these quantities are from the Kepler-Coulomb system. The angular momentum is defined as the outer product of the position and momentum vector

$$L_K = (xp_y - yp_x).$$

This expression can be rewritten by utilising the definition of w, given in Equation 28, and recognising that $p_{x_i} = m\frac{dx_i}{d\tau}$. The angular momentum then becomes

$$L_K = \frac{m}{2i}\left(w^*\frac{dw}{d\tau} - w\frac{dw^*}{d\tau}\right). \tag{30}$$

Equation 30 can also be reformulated in terms of z by making use of

$$\frac{dw}{d\tau} = \frac{1}{2|z|^2}\frac{d}{dt}(z^2) = \frac{1}{z^*}\frac{dz}{dt},$$

which results in

$$L_K = \frac{m}{2i}\left(z^*\frac{dz}{dt} - z\frac{dz^*}{dt}\right) = L_H. \tag{31}$$

Upon comparison of Equation 30 and Equation 31 one is able to conclude that the structure is preserved regardless whether complex variable z or w is chosen. For the second term in Equation 29 it turns out that taking the second derivative of w with respect to τ is very illuminating. In the following, dots are used to represent derivatives with respect to t

$$\frac{d^2w}{d\tau^2} = \frac{z^*\ddot{z} - \dot{z}\dot{z^*}}{2z(z^*)^3} = \frac{z^*(\ddot{z}+z) - (|\dot{z}|^2 + |z|^2)}{2z(z^*)^3}. \tag{32}$$

Fortunately this expression can be simplified by recalling both the definition of the total energy and the Newtonian equations of motion of the harmonic oscillator [12]. For simplifying the relation the angular frequency has been set to one, then these two relations respectively reveal that

$$E_H \equiv \frac{1}{2}|\dot{z}|^2 + \frac{1}{2}|z|^2 \tag{33}$$

and

$$\frac{d^2z}{dt^2} + z = 0. \tag{34}$$

Consequently, Equation 32 can be reformulated into

$$\frac{d^2w}{d\tau^2} = \frac{-E_H}{z(z^*)^3} = -E_H\frac{w}{|w|^3}.$$

This identity is the second derivative of the complex position coordinate, in other words, it can be interpreted as an acceleration. By utilising Newton's second law this can be directly related to the inverse square law force

$$\vec{F} = m\vec{a} \rightarrow -k\frac{w}{|w|^3} = m\frac{d^2w}{d\tau^2} = -mE_H\frac{w}{|w|^3}.$$

By comparing Equation 29 with Equation 12 the relation $\kappa = mk$ becomes apparent and thus one concludes that $\kappa = m^2E_H$. With Equation 31 and $\kappa = m^2E_H$ it now becomes possible to rewrite Equation 29 in terms of mapped components

$$\vec{A}_K = -\frac{-m}{2}(z^*\dot{z} - z\dot{z^*})\left(m\frac{\dot{z}}{z^*}\right) - m^2E_H\frac{w}{|w|},$$

which can be rewritten using some algebra combined with both Equation 33 and Equation 34 to obtain

$$\vec{A}_K = -m^2\left(\frac{1}{2}z^2 + \frac{1}{2}\dot{z}^2\right) \equiv -\vec{A}_H. \tag{35}$$

At last the LRL-vector has been expressed in terms of z and $\dot{z}$. As the time derivative of Equation 34 can be expressed in terms of the equations of motion, given in Equation 34, it must vanish. Consequently, the mapped LRL-vector $\vec{A}_H$ can be thought of as the hidden constant of motion of the harmonic oscillator.

Aside from that, Equation 35 becomes more illuminating when the complex coordinate z is expanded into its real and imaginary components. As $\vec{A}_H$ is a constant of motion, both the real and imaginary parts of the mapped vector need to be separately invariant. Combining this property with the conservation of energy, it is imposed that within the harmonic oscillator system the energy along the real and imaginary axes must also be individually conserved. From this point on there will be no further discussion on complex coordinate w, as a result it has been decided to follow the convention introduced in [12] and stop denoting the real and complex parts of w using x and y. Concretely, this means that from now on instead of Equation 28 the following definitions will be used

$$z = x + iy \qquad p = p_x + ip_y.$$

Using these identities Equation 35 can be expanded into

$$\frac{\vec{A}_H}{m^2} = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}(p_x^2 + p_y^2) + i(xy + p_x p_y), \tag{36}$$

revealing a very special structure. Upon closer inspection, it turns out that the real and imaginary components of $\vec{A}_H$ can be associated with a symmetric tensor. This tensor has its origin in the quantum mechanical description of the harmonic oscillator and has a similar role as the LRL-vector in the Kepler-Coulomb system [4]. This means that it not only completely specifies the orbital motion, but it also has special constraints that relates it to all of the other conserved quantities of the system. More about the algebraic properties of the tensor itself and how one can reveal that the harmonic oscillator obeys SU(2) symmetry can be found in [4] [12] respectively. The tensor is defined by the following relation

$$A_{ij} = \frac{1}{2}(p_i p_j + r_i r_j), \qquad i, j = 1, 2, 3, \tag{37}$$

where as before the angular frequency has been taken as one. Upon comparison of Equation 37 and Equation 36 it can be immediately recognised that

$$\mathrm{Im}(\vec{A}_H) = 2A_{12} = 2A_{21},$$

showing that $\vec{A}_H$ and A_{ij} are related. Further scrutinizing their structures reveals that $\mathrm{Re}(\vec{A}_H)$ can also be expressed by the tensor as

$$\mathrm{Re}(\vec{A}_H) = A_{11} - A_{22}.$$

Therefore, the analysis has shown that the additional constants of motion of both the Kepler and harmonic oscillator systems are related through the complex mapping, allowing one to determine the other by knowing either one.

This goal of this chapter is to show the relation between the Kepler-Coulomb system and the harmonic oscillator. Just like the procedure in the previous chapter, the analysis starts with expressing the orbital motion of the Kepler-Coulomb problem in an alternative method. By reimagining

the orbital plane as a complex plane, vectors have been rewritten in terms of complex numbers. This complex notation allowed the construction of a mapping between the Kepler-Coulomb problem and the harmonic oscillator. The connection becomes more apparent by studying the action of the harmonic oscillator. Upon expressing the kinetic energy and angular momenta in their counterparts, it is revealed that system characteristics such as the time and angle coordinates will be rescaled due to the mapping. Aside from that, mapping the orbits themselves allow one to conclude that the two-dimensional harmonic oscillator admits a hidden symmetry. Using the newly obtained relations, the corresponding conserved quantity is determined by mapping the components of the LRL-vector, which is concluded to be the additional constant of motion by looking at its time derivative. It has now become obvious that, by taking advantage of a mapping between systems, the LRL-vector can be employed to locate other hidden symmetries. Ideally one would like to generalise this approach to other systems. However, not any mapping can be utilised for this purpose. From the phase space analysis in section 2.2, it has become clear that only systems with closed orbits for all bound states could permit the existence of a hidden symmetry. Thus the next chapter will explore the constraints of these closed orbit potentials, revealing how many systems the LRL-vector could be mapped into.

4 Bertrand's theorem: beyond the Kepler-Coulomb and the harmonic oscillator

4.1 The closed orbit condition

As shown in the previous chapter, being able to express a system in terms of another system can be very helpful, especially when the latter has been explored thoroughly. For instance, this allows for the acquisition of new insights by connecting them to well-understood concepts such as the LRL symmetry of the Kepler-Coulomb system. Naturally, this approach is more effective when the systems share similar properties. For example, in order to make use of the aforementioned hidden symmetry, the mapped system needs to have bound states that are described by closed planar orbits, just like the states in the Kepler-Coulomb system. Before the set of mappings toward and from the Kepler-Coulomb system could be sought out, it is required to know what potentials are capable of creating such orbits in the first place.

Self-replicating orbits are special and have to obey additional constraints. Although all bound states will have an orbit, generally, these orbits do not need to repeat themselves. Just like a position in a plane can be expressed by the position coordinates r and ϕ, orbital motion can be separated into a radial component and an angular component. First of all, to have periodic orbits both these components have to be periodic themselves. In addition to this, if one demands that these orbits are closed as well, then the radial and angular components are required to synchronise after a certain period of time. What this last requirement means concretely, is that an integer number of radial periods NT_r must equal another integer number of angular periods MT_ϕ [5], where both $N > 0$ and $M > 0$. Instead of attempting to express both periods in units of time, it turns out that they are more readily expressed in terms of angles. Due to the fact that the potential is central, the periodic time evolution of the radial velocity can be described by simple harmonic motion. What this means is it will continuously go back and forth between the the radial extremes at a specific rate. As a result, a constant angle can be defined between these extreme values with the center of attraction. This angle, referred to as the apsidal angle ϕ_A, is then equal to one half of the radial period. The definition of the aspidal angle permits the closed orbit condition to be summarised mathematically as

$$2N\phi_A = 2M\pi \rightarrow \phi_A = \frac{\pi}{R}, \tag{38}$$

where $R = \frac{N}{M}$ is a rational number. Consequently, the closed orbit condition is the same statement as demanding that the apsidal angle is rational multiple of π.

4.2 Classical derivation

For the sake of determining the potentials that respect the closed orbit condition, given in Equation 38, one must solve the orbit equation. This equation can be obtained by taking advantage of the familiar energy and angular momentum conservation laws. Conservation of angular momentum introduces two constraints into the system. First of all, the orbits traversed by the particle are constraint to be within a plane, which justify the usage of polar coordinates. Secondly, the potentials are only allowed to be central, depending only on the distance r. The Lagrangian of a system satisfying these conditions is given by

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

When this Lagrangian is examined more closely, it can be noticed to be very similar to Equation 3 found in Section 2.2. In the exact same approach as was performed in that section, the orbit equation can be determined by combining the two conservation laws to be

$$\frac{dr}{d\phi} = \frac{mr^2}{L}\sqrt{\frac{2}{m}\left(E - \left[\frac{L^2}{2\mu r^2} + V(r)\right]\right)},$$

which shares the identical structure to Equation 7. To solve the differential equation it is prudent to introduce a change of variables

$$u = \frac{1}{r} \qquad dr = -r^2 du.$$

This would then give rise to the following expression

$$\frac{du}{d\phi} = \frac{-m}{L}\sqrt{\frac{2}{m}\left(E - \left[\frac{L^2u^2}{2\mu} + V(\frac{1}{u})\right]\right)},$$

which can be rearranged into

$$\frac{m^*}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{m^*u^2}{2} + V(\frac{1}{u}) = E, \tag{39}$$

where $m^* = \frac{L^2}{m}$ can be thought of as a "effective mass". Equation 39 is reminiscent of the total energy equation of a harmonic oscillator given by

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E.$$

By comparison one concludes that the solution to the rewritten orbit equation is known whenever the effective potential energy term is of the form

$$V_{eff} = \frac{m^*u^2}{2} + V(\frac{1}{u}) \propto \frac{1}{2}ku^2. \tag{40}$$

For these set of effective potential energies the orbit equation simply describes a harmonic oscillator, which solutions are already known to be

$$x = A\cos(\omega t) \tag{41}$$

where $\omega = \sqrt{\frac{k}{m}}$. In order to make use of this knowledge to simplify the orbit equation, it is crucial to perform a Taylor expansion of V_{eff} around a circular orbit.

Circular orbits are the bound states where the radial separation of the particle to the center of attraction does not change. This becomes more illuminating when studied graphically. When the effective potential of the Kepler-Coulomb problem and the radial separation are plotted alongside each other, the result will be similar to what is shown in Figure 5. As the horizontal and vertical axes represent the magnitudes of the radii and the energies respectively, the radii that are available to each bound state can be read by studying the shape of the potential energy curve. If the effective potential is multi-valued for any energy E, then the radii associated to these two points will represent the range of the allowed radii of the bound state. For a circular orbit in particular, it is not only required that the bound state only has access to a single value for the radius, but the state has to remain bound by the system. There is only one point in Figure 5 that satisfy these criteria. The circular orbit is located at the minimum of the curve denoted by radius r_1 and energy E_4. In general circular potentials can be found for any central potential at the global maxima and global minima, but only global minima are considered stable orbits. This is due to the fact that the bound states at the maxima could become free particle states when the smallest amounts of energy are introduced into the system.

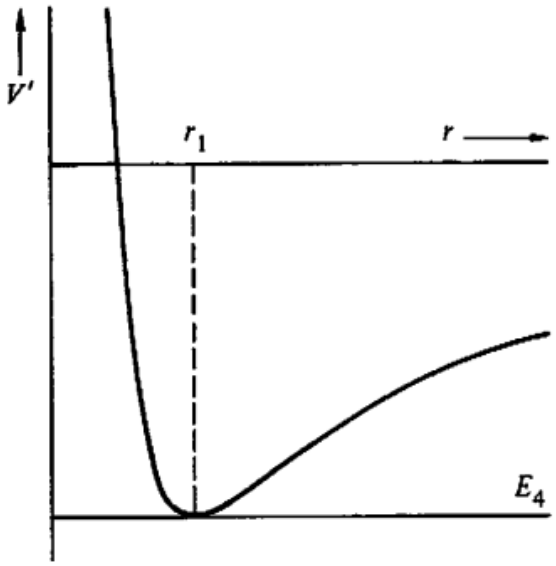


Figure 5: *Here the magnitude of the Kepler-Coulomb potential has been plotted against the displacement from the center of attraction. The range of radii available to each bound state can be determined by looking the energies where the potential curve is multi-valued. A particle in the minimum of the potential has energy E_4 and an unchanging radial distance r_1, this is a bound state with a circular orbit. This figure has been taken from [8].*

By Taylor expanding the effective potential energy near these circular orbits the expansion sim-

plifies. As these orbits are near the extreme values of the potential curve, the linear term in the Taylor expansion vanishes. Aside from that, the higher order correction terms inside the expansion can also be safely neglected by constraining the radii to be close enough to the circular orbits. If the circular orbit is located at the radial distance $r_0 = \frac{1}{u_0}$, then the Taylor expansion becomes

$$V_{eff}(u) = V_{eff}(u_0) + \frac{1}{2}(u - u_0)^2 V''_{eff}(u_0), \tag{42}$$

where V''_{eff} denotes the second derivative with respect to u. Substituting Equation 42 into Equation 39 then results into

$$\frac{m^*}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}(u - u_0)^2 V''_{eff}(u_0) = E - V_{eff}(u_0),$$

which becomes more illuminating by introducing the change of variable

$$\rho = u - u_0 \qquad du = d\rho$$

and allows the orbit equation to be rewritten into

$$\frac{1}{2}m^*\left(\frac{d\rho}{d\phi}\right)^2 + \frac{1}{2}V''_{eff}(u_0)\rho^2 = E - V_{eff}(u_0) \equiv E'. \tag{43}$$

Since Equation 43 is just a harmonic oscillator equation, the solution is given by $\rho = A\cos(\omega\phi)$ where

$$\omega = \sqrt{\frac{V''_{eff}}{m^*}}.$$

Using the definition for V_{eff}, provided in Equation 40, the second derivative can be determined explicitly to give rise to the following relations

$$V'_{eff} = m^* u - \frac{1}{u^2}\frac{dV}{dr} = 0$$

$$m^* = \frac{1}{u^3}\frac{dV}{dr}$$

$$V''_{eff} = m^* + \frac{2}{u^3}\frac{dV}{dr} + \frac{1}{u^4}\frac{d^2V}{dr^2} = 3r^3V' + r^4V''.$$

Finally, the angular frequency corresponding to Equation 43 can be determined to be

$$\omega = \sqrt{\frac{3V' + rV''}{V'}}. \tag{44}$$

As the general solution to the harmonic oscillator is given by a cosine function, given in Equation 41, its angular frequency must be real. Otherwise the trigonometric function will transform into its non-periodic hyperbolic counterpart. In addition to that, as the apsidal angle is defined as the angle between the radial extremes and the period of a cosine is defined as 2π, the following relation is true

$$\phi_A = \frac{\pi}{\omega}.$$

Therefore, one can conclude that the angular frequency ω must be rational to respect the closed orbit condition, given in Equation 38. These two conditions are only met when $\frac{3V'+rV''}{V'}$ is equal to a positive rational and real number C. This means that

$$\frac{3V' + rV''}{V'} = C$$

$$rV'' = (C-3)V',$$

which reveals that the only central potentials that are allowed are of the form

$$V = \frac{a}{C-2} r^{C-2},$$

where a is an integration constant. When the power is rewritten into a more familiar constant $n = C - 2$, the closed potentials that satisfy the closed orbit condition are given by

$$V = \frac{a}{n} r^n, \tag{45}$$

which reveals that classically only single power laws in r are capable of hosting hidden symmetries. Moreover, it also becomes possible to rewrite the closed orbit constraint in terms of the angular frequency

$$\omega = \sqrt{n+2}, \tag{46}$$

which need to be rational to have closed orbits for all bound states.

At first glance neither Equation 45 or Equation 46 seem to forbid the existence of other physical systems with hidden symmetries. Yet, none of these presented alternatives have been observed in nature. The answer to this apparent paradox lies within the orbit equation itself. It turns out that merely obeying the closed orbit condition is not sufficient. In order for systems to have closed orbits for all bound states in classical mechanics, the set of central potentials must also satisfy the orbit equation. All of the power law potentials can be neatly divided among two groups separated by the sign of the power in r. For the $n < 0$ case it is beneficial to introduce a new positive and real constant $s = -n > 0$. Substituting this new definition into the potential energy then gives rise to

$$V = \frac{-a}{s} r^{-s}. \tag{47}$$

Now that the explicit form of the potential energy is obtained in Equation 47 it can be introduced into the initial orbit equation found in Equation 39

$$\frac{m^*}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{m^* u^2}{2} - \frac{a}{s} u^s = E.$$

By dividing this newfound expression by u^s allows it to be written into

$$\frac{m^*}{2} u^{2-s}\left(\frac{1}{u}\frac{du}{d\phi}\right)^2 + \frac{m^*}{2} u^{2-s} - \frac{a}{s} = E u^{-s}.$$

Just like the previous differential equations this one becomes more manageable when a change of variables is introduced

$$x^2 = u^{2-s} \qquad 2x = \frac{d}{dx}(u^{2-s}) = \frac{du}{dx}(2-s)u^{1-s},$$

which can be rearranged into the relation

$$\frac{2}{2-s}\frac{dx}{x} = \frac{du}{u}.$$

After a simple substitution the following will be obtained

$$\frac{m^*}{2}\left(\frac{2}{2-s}\right)^2\left(\frac{dx}{d\phi}\right)^2 + \frac{m^*}{2} x^2 = E u^{-s} + \frac{a}{s}. \tag{48}$$

Equation 48 starts to resemble a harmonic oscillator, however it still contains the undesired obviously not constant term Eu^{-s}. Fortunately, this problematic term can be eradicated by taking a specific limit [3] [5]. As the discussion only concerns bound states, it can be safely concluded that the energy of the bound state may never exceed the potential energy. From inspecting the potential energy provided in Equation 47, it becomes obvious that the potential tends to vanish for large radii, regardless of the actual values of both a and s. Phrased differently, this means that bound states must obey $E < 0$. Since the potential energy does not depend on the energy E, one is allowed to study its behaviour at the boundary where E approaches zero [5]. This assumption is actually a bit sketchy from the perspective of the potential, but more details on that will follow in the discussion section.

When this specific limit $E \to 0^-$ is taken Equation 48 becomes

$$\frac{m^*}{2}\left(\frac{2}{2-s}\right)^2\left(\frac{dx}{d\phi}\right)^2 + \frac{m^*}{2}x^2 = \frac{a}{s},$$

which has the same structure as a harmonic oscillator equation. Within this interpretation the angular frequency can be determined to be

$$\omega_s = \sqrt{\frac{m^*}{m^*(\frac{2}{2-s})^2}} = \frac{2-s}{2}.$$

At this moment it is important to recognise that orbits need to be physical. In other words, despite the fact that the general solution to the harmonic oscillator, given in Equation 41, allows for negative values, in order to make sense radii are not allowed to be less than zero. The consequence of this phenomena is perhaps best explained graphically. In general, simple harmonic motion can be thought of as a particle rolling down the quadratic potential curve. After leaving its initial position it will pass the potential minima at the origin and head toward the reflection of its initial position in the vertical axis. After that it will return to its initial position by retracing the identical path and start the cycle anew. This procedure is illustrated in Figure 6a. Due to the fact that the orbits are not allowed to have negative radii, the harmonic motion will be slightly different. As a result of this constraint the particle is not allowed to pass the origin. Instead it will reflect the particle back towards the initial position and finishes its cycle much earlier. This is shown in Figure 6b. As the constraint halves the traversed path length, the angular frequency associated to this cycle must be twice as large as the general case. Concretely, this constraint implies that the angular frequency associated to the physical system is given by

$$\omega_{-n} = 2\omega_s = 2 - s = n + 2.$$

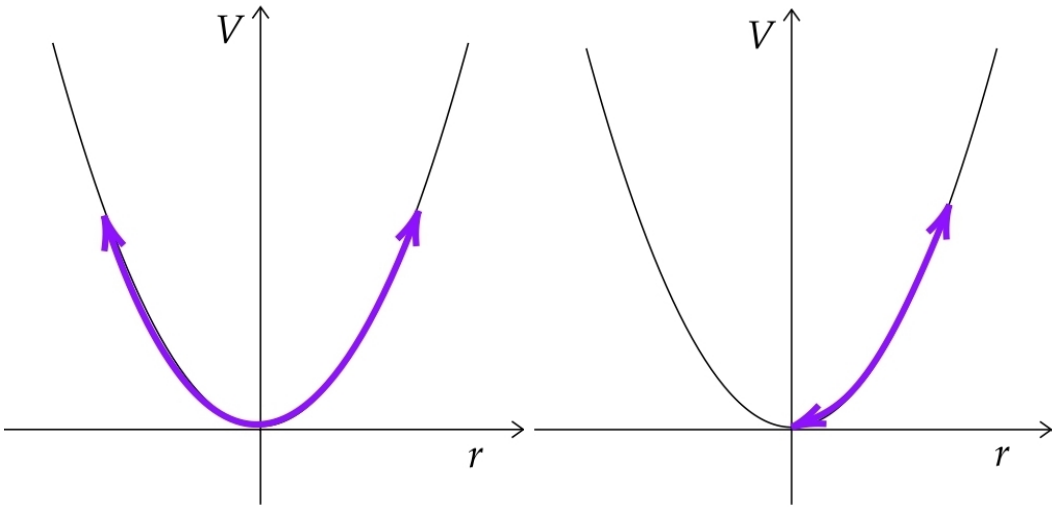
Then comparing this expression to the closed orbit condition, given in Equation 46, results in the observation that valid solutions require that

$$\sqrt{n+2} = n + 2.$$

From all the power law potentials with negative powers only the Kepler-Coulomb potential is capable of satisfying this constraint, excluding all other negative values of n. As the characteristics of this system has already been explored thoroughly in the second chapter, it will not be studied again in this section.

For the other case, when $n > 0$, Equation 39 can be treated in an alternative way. By substitution of the form of the potential given in Equation 45 it becomes

$$\frac{m^*}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{m^*u^2}{2} + \frac{a}{n}u^{-n} = E. \tag{49}$$



(a) *Harmonic motion can in general be interpreted graphically as a particle rolling down the potential landscape of the harmonic oscillator* $V = \frac{1}{2}mr^2$. *The period of the motion is defined as the time it takes to return to this initial position after rolling towards the origin. In principle, the system allows for negative values, so the particle is freely allowed to move in the parabolic curve.*

(b) *Harmonic motion described in Equation 4.2 has to be physical, thus additionally requiring that* $r \geq 0$. *This means that only half of the harmonic potential* $V = \frac{1}{2}m^*x^2$ *is available to the particle. Consequently, after the particle rolls towards the origin it bounces back without going into the negative values. Since the traversed path is only half of the general case, the angular frequency is actually twice as large.*

Figure 6

Just as the $n < 0$ case, Equation 49 becomes more comprehensible when the appropriate limits are chosen. At small radii, thus large u, the contribution of the centrifugal term dominates. On the other hand, at large radii the central potential in Equation 45 is the dominant contribution. Both the very small and very large values of r correspond to the same rapid increase in the effective potential energy. As a result, at these specific regions of r Equation 49 simplifies. In order to allow one to study both the arbitrary small and arbitrary large radii, one only needs to take a single limit. In the limit where the total energy E tends to infinity, all values of the radii are accessible for discussion. Similarly to the $n < 0$ case, this limit is also a bit slippery for the same reason. Details and justification will be given in the discussion section. First of all, in the small radius region the relation becomes

$$\frac{m^*}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{m^*u^2}{2} = E,$$

which implies that

$$\omega_{r\to 0^+} = \sqrt{\frac{m^*}{m^*}} = 1.$$

Once again, by demanding that the orbits are physical one imposes that $r > 0$, as a result the angular frequency of the physical system becomes

$$\omega_{n>0} = 2\omega_{r\to 0^+} = 2.$$

Comparing this expression with the closed orbit condition directly gives rise to

$$\omega_{n>0} = \omega \to 2 = \sqrt{2+n}, \tag{50}$$

which only solution is $n = 2$. From this can be concluded that for small radii in the high energy limit, only the harmonic oscillator is capable of meeting this criteria, immediately disqualifying all other positive powers n.

In order to check the validity of the harmonic oscillator solution, the other extreme case must also be considered. For radii that become very large Equation 49 becomes

$$\frac{m^*}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{a}{n}u^{-n} = E.$$

Substituting the earlier found solution $n = 2$ transforms the orbit equation once again into a harmonic oscillator. To finalise the analysis for the $n = 2$ solution the angular momentum is considered. As demonstrated in section 2.2, the angular momentum equation of the harmonic oscillator can be obtained from the Euler-Lagrange equations of motion. This expression for the angular momentum of the harmonic oscillator happens to be equal to the relation shown in Equation 4. By rewriting the conservation of angular momentum one can obtain

$$-\frac{du}{dr} = \frac{1}{r^2} = \frac{m}{L}\frac{d\phi}{dt},$$

which is the same as

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{m}{L}\frac{dr}{dt}\right)^2.$$

Therefore, for the specific case $n = 2$ Equation 49 reduces into

$$\frac{m^*}{2}\left(\frac{m}{L}\frac{dr}{dt}\right)^2 + \frac{m^*}{2r^2} + \frac{a}{2}r^2 = E,$$

which can be simplified by recalling the definition of the "effective mass" given by

$$m^* = \frac{L^2}{m}$$

into

$$\frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{a}{2}r^2 = E. \tag{51}$$

Finally, Equation 51 be rewritten for the last time by recalling the conservation of angular momentum imposes that

$$\frac{L^2}{2mr^2} = \frac{m}{2}r^2\dot{\phi}^2$$

and the velocity in polar coordinates is defined as

$$v^2 = (\dot{r}^2 + r^2\dot{\phi}^2)$$

to give rise to

$$\frac{1}{2}mv^2 + \frac{1}{2}ar^2 = E. \tag{52}$$

Equation 52 can be recognised as the familiar textbook harmonic oscillator equation with the substitution $a = k$. As a result the solution $n = 2$ is a valid for all radii. As a side note, notice that due to the similar structure of the angular momentum conservation, the last analysis could also be applied to the Kepler-Coulomb system. This is done through simply changing the magnitude of the power n and recognising that in this case $\frac{a}{2} = \frac{q_1 q_2}{4\pi\epsilon_0}$.

This section has been dedicated to studying the constraints of the orbit equation and how these impositions relate to the closed orbit condition found in the previous section. Similarly to was demonstrated for the Kepler-Coulomb problem, through the conservation of energy and angular momentum an orbit equation can be constructed for any arbitrary central potential. The structure of this newfound equation is reminiscent of the harmonic oscillator energy. To take advantage of this, the effective potential was Taylor expanded around circular orbits. This allowed the usage of the definition of the angular frequency, which determined the structure of the central potentials. Together with the closed orbit condition one is able to conclude, that only single power laws of r with rational angular frequencies can give rise to closed orbits for all bound states. However, it turns out that closed orbit condition by itself is insufficient. By substituting the power law solution, the orbit equation can be solved to obtain another set of angular frequencies. For the central potentials to have closed orbits for all bound states, it is required that the angular frequencies of the orbit equation agrees with the closed orbit condition. Within classical mechanics only two solutions for n are able to satisfy this criterion. Thus, the only two systems in classical mechanics that give rise to hidden symmetries are the Kepler-Coulomb and the harmonic oscillator. However, so far the discussion has been purely classical. It is widely known that classical mechanics is unable to describe reality in its entirety, giving rise to false predictions under specific circumstances. Therefore, the final conclusion obtained in this section may not be valid when one leaves the realm of classical mechanics. The next section is dedicated to study the effects of special relativistic corrections on the existence of hidden symmetries, to check whether the classical conclusions remain valid.

4.3 Special relativistic extension

In order to generalise the classical theorem to include special relativistic effects, one has to remember that the kinetic energy of these systems become more complicated. For a relativistic particle in presence of a continuous central potential the Lagrangian is given by

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - V(r).$$

As the ultimate goal of this derivation is to find closed orbits for all bound states, once again the conservation of energy and angular momentum is imposed. Similar to its classical counterpart, the conservation of angular momentum is synonymous with demanding the orbits to be confined within a plane, making it is most convenient to work in polar coordinates $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$. As demonstrated in section 2.2, the angular momentum equation is given by

$$L = \frac{d\mathcal{L}}{d\dot{\phi}} = \frac{mr^2\dot{\phi}}{\sqrt{1 - \frac{v^2}{c^2}}} = mr^2\gamma\dot{\phi},$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the familiar Lorentz factor. Recalling the relativistic definition of the three-momentum

$$\vec{p} = m\gamma\vec{v} = p_r\hat{r} + p_\phi\hat{\phi}$$

and expanding v to polar coordinates reveals that

$$p_r = m\gamma\dot{r} \qquad p_\phi = mr\gamma\dot{\phi} = \frac{L}{r}.$$

By taking advantage of the fact that the structure of the momentum components are very similar, the radial momentum can be expressed in terms of its angular counterpart

$$\frac{p_r}{p_\phi} = \frac{\dot{r}}{r\dot{\phi}} = \frac{dr}{rd\phi} \rightarrow p_r = p_\phi \frac{dr}{rd\phi}.$$

When one recalls that p_ϕ is directly related to the magnitude of the angular momentum L, it becomes possible to simplify p_r into

$$p_r = \frac{L}{r^2}\frac{dr}{d\phi}. \tag{53}$$

The relativistic of the orbit equation is, just like its classical equivalent, determined using the conservation of energy

$$E = \sqrt{p^2c^2 + m^2c^4} + V(r),$$

which can be rearranged, using the relation $p^2 = p_r^2 + p_\phi^2$, to give rise to

$$(E - V)^2 = \left(\frac{L}{r^2}\frac{dr}{d\phi}\right)^2 c^2 + \left(\frac{L}{r}\right)^2 c^2 + m^2c^4. \tag{54}$$

Equation 54 becomes more manageable under the familiar change of variables

$$u = \frac{1}{r} \qquad -r^2du = dr$$

resulting in the following relation

$$(E - V)^2 = \left(-L\frac{du}{d\phi}\right)^2 c^2 + (Lu)^2c^2 + m^2c^4,$$

which allows itself to be rewritten into

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{L^2c^2}((E - V)^2 - m^2c^4). \tag{55}$$

Equation 55 is still not very illuminating, however it can still be simplified by differentiating with respect to $\frac{d}{d\phi}$

$$2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2\frac{du}{d\phi}u = \frac{1}{L^2c^2}\left(2\frac{du}{d\phi}(E - V)\frac{dV}{du}\right),$$

which upon dividing by the common factor becomes

$$\frac{d^2u}{d\phi^2} + u = \frac{(E - V)}{L^2c^2}\frac{dV}{du}. \tag{56}$$

Equation 56 can be thought of as the proper relativistic generalisation of the classical orbit equation. This becomes more obvious when the total energy equation is rewritten into the form

$$E = m\gamma c^2 + V \rightarrow (E - V) = m\gamma c^2,$$

transforming the relativistic orbit equation into

$$\frac{d^2u}{d\phi^2} + u = \frac{m\gamma}{L^2}\frac{dV}{du}.$$

Finally, upon realisation that returning to classical mechanics is equivalent to taking the limit where $\gamma \to 1$, the last equation can be integrated to regain Equation 39.

Although the relativistic orbit equation has been found explicitly in Equation 56, the non-linear equation is not informative in its current form. Fortunately, just like its classical counterpart, the orbit equation simplifies by considering circular orbits. First of all, the notation is simplified by defining

$$J(u) \equiv \frac{(E-V)}{L^2c^2}\frac{dV}{du}. \tag{57}$$

Secondly, when this J is Taylor expanded around a circular orbit located at $u = u_0$, it can be rewritten into

$$J(u) = J(u_0) + (u-u_0)J'(u_0) + \frac{(u-u_0)^2}{2}J''(u_0) + ...$$

From the Taylor expansion can immediately concluded that the constant term $J(u_0)$ must equal u_0. This can be inferred from explicitly taking the limit $u \to u_0$. Consequently, all terms in the Taylor series except for $J(u_0)$ will vanish. In addition to that, in order for the left hand side to describe a circular orbit, the radius and therefore also u may not vary with ϕ. This solution is only included when the linear u term and the constant term $J(u_0)$ vanish together. Secondly, by only considering values of u near this circular orbit, the higher order terms in the Taylor expansion can be safely neglected. Combining these findings together give rise to

$$\frac{d^2u}{d\phi^2} + u = J(u_0) + (u-u_0)J'(u_0),$$

which can be rewritten with the change of variables

$$x = u - u_0 \qquad du = dx$$

into

$$\frac{d^2x}{d\phi^2} + (1 - J'(u_0))x = 0. \tag{58}$$

Upon closer inspection, Equation 58 can be recognised as the equations of motion for a harmonic oscillator with an angular frequency $\zeta^2 = (1 - J'(u_0))$.

By interpreting Equation 58 as a harmonic oscillator equation, additional constraints can be determined. From the general solution of a harmonic oscillator, given in Equation 41, it can be deduced that to have stable orbits the angular frequency needs to be real. Otherwise, the trigonometric solutions transform into their aperiodic hyperbolic counterparts. Aside from that, it can also be inferred that this value has to be a constant. This can be understood by first looking at the closed orbit condition introduced in Equation 38. Even in the relativistic case an apsidal angle ϕ_A can be constructed, which is demanded to be a rational multiple of π. As the set of rational numbers is not continuous, ϕ_A is only allowed to take specific values. Furthermore, this rational multiple of π is not allowed to vary between closed orbits. This is due to the fact that the radii r are continuous. As a result, if an orbit with radius r and another orbit with radius $r + \delta r$, where $\delta r \ll r$, have different angular frequencies, then problems arise when the limit is taken where $\delta r \to 0$ [9]. The orbits will tend towards each other, but as the set of allowed angular frequencies is not continuous the angular

frequency must change discontinuously. Consequently, orbits can not be closed at this discontinuity, this implies that ζ^2 must be both constant and positive.

The angular frequency ζ combined with its accompanying constraints allow for the formulation of a new differential equation

$$1 - J'(u_0) = \zeta^2,$$

which can be rephrased by using Equation 57 into

$$\left(\frac{dV}{du}\right)^2 + (V - E)\frac{d^2V}{du^2} = (1 - \zeta^2)L^2c^2. \tag{59}$$

Unlike Equation 56 which has a non-relativistic analogue, given by Equation 39, a differential equation due to the constraints on the angular frequency, such as Equation 59, was never found in the classical derivation. This can be made more obvious by rewriting the relation in terms of the Lorentz factor by recalling that $\frac{(E-V)}{mc^2} = \gamma$, this results in the following equation [9]

$$\gamma\frac{d^2\gamma}{du^2} + \left(\frac{d\gamma}{du}\right)^2 = \frac{(1 - \zeta^2)L^2}{mc^2},$$

which is not well-defined in the classical limit. It is believed that the justification of this phenomenon can be found within the orbit equation [9]. Although the classical orbits are only dependent on the radial separation, as the right hand side of the relativistic counterpart, the function $J(u)$, can be written in terms of γ, it is also dependent on the particle velocity. This discrepancy does not allow for a well defined classical limit for Equation 59.

There are two ways Equation 59 can be approached. One method is by introducing a trial solution into the differential equation to study its characteristics. Aside from that, one can also search for general solutions by rewriting the differential equation in a familiar form. Inspection of the findings from the previous section suggest the trial solution

$$V_{trial} = -bu^\delta,$$

where $b > 0$. Substituting this trial solution into Equation 59 returns the following

$$\delta^2b^2u^{2(\delta-1)} + (\delta - 1)\delta b^2u^{2(\delta-1)} + (\delta - 1)\delta Ebu^{\delta-2} = (1 - \zeta^2)L^2c^2,$$

which can rearranged to obtain

$$(\delta^2b^2 + (\delta - 1)\delta b^2)u^{2(\delta-1)} + (\delta - 1)\delta Ebu^{\delta-2} = (1 - \zeta^2)L^2c^2. \tag{60}$$

The structure of Equation 60 simplifies by a lot for the specific case $\delta = 1$, corresponding to the Kepler-Coulomb potential. Performing this substitution will transform the equation into constraints on both the angular momentum and the angular frequency of the form

$$b^2 = (1 - \zeta^2)L^2c^2 \rightarrow L = \frac{b}{c\sqrt{1 - \zeta^2}}. \tag{61}$$

This not only reveals that ζ^2 is not allowed to be larger than one, in order to keep the angular momentum real, but also that the angular momentum must be larger than a specific lower bound $L > \frac{b}{c}$. Equation 60 does not seem to simplify for any other δ, seemingly excluding the harmonic

oscillator potential as a valid solution. Aside from this approach one can also try to find a general solution. It turns out that the left hand side of Equation 59 can be collected into a single term

$$\frac{d^2}{du^2}\left[\frac{(V-E)^2}{2}\right] = (1-\zeta^2)L^2c^2.$$

For a differential equation of this form, the general solution is known to be [9]

$$V(r) - E = -\sqrt{d(f+\frac{1}{r})^2 + g}, \tag{62}$$

where $d = (1-\zeta^2)L^2c^2$ and f, g are integration constants. The force accompanying this special potential is given by

$$\vec{F} = -\vec{\nabla}V = -\frac{d(f+\frac{1}{r})}{r^2\sqrt{d(f+\frac{1}{r})^2+g}}. \tag{63}$$

From the symmetry exhibited in Equation 63 it becomes very clear that the force simplifies under the special condition $g = 0$ into

$$\vec{F} = -\frac{\sqrt{d}}{r^2}.$$

This can be directly recognised as the inverse square force law belonging to the Kepler-Coulomb potential. Furthermore, it specifies the constant of the trial solution to be $b = \sqrt{d}$. However, for all potentials where $g \neq 0$ the force keeps its complex structure. Similar to the findings of the trial solution, closer inspection of Equation 63 does seem to exclude the harmonic oscillator as a solution. This would imply that Equation 59 excludes the harmonic oscillator as a solution regardless of the chosen approach. As the structure provided in Equation 63 is rather unfamiliar, this would suggest that from the currently known systems, only the Kepler-Coulomb potential could be considered as a valid solution. However, in light of Equation 61, even the Kepler-Coulomb system does not give rise to closed orbits for all bound states.

Using the obtained form of the potential given in Equation 62, it is possible to solve the relativistic orbit equation explicitly. Substituting the general solution of the potential into Equation 57 will give rise to

$$J(u) = \frac{1}{L^2c^2}(-\sqrt{d(f+u)^2+g})(-\frac{d(f+u)}{\sqrt{d(f+u)^2+g}}) = \frac{d}{L^2c^2}(f+u), \tag{64}$$

which implies a linear dependence of $J(u)$ on u. Although the linear functional dependence is very beneficial, simplifying Equation 56 in such a way that can be solved directly, it is also a observation that needs to be treated with care. More attention will be spend on this in the discussion section. Equation 64 can be substituted directly into Equation 56 to obtain

$$\frac{d^2u}{d\phi^2} + u = \frac{d}{L^2c^2}(f+u),$$

which can be solved using the method of undetermined coefficients to give rise to the solution [9]

$$\frac{1}{r} = \frac{1}{R}\cos(\zeta\phi) + \frac{f(1-\zeta^2)}{\zeta^2} \tag{65}$$

with

$$R = Lc\zeta[\frac{f^2L^2c^2(1-\zeta^2)}{\zeta^2} + g - m^2c^4]^{-\frac{1}{2}}$$

Interestingly, the structure of Equation 65 can be compared directly to Equation 14. Although the latter describes a constant ellipse, the former does not. This notion becomes more apparent when the findings are compared to an older derivation [10], which will be done in the discussion section.

Within this section Bertrand's theorem was generalised to include special relativistic corrections. At first, the methodology resembled its classical counterpart. Consisting of imposing the existence of the energy and angular momentum conservation laws and utilising them to construct an orbit equation. Its validity was then checked by taking the limit to classical mechanics, which can be successfully used to reproduce Equation 39. However, from here on the derivation starts the deviate. After trying to simplify the relativistic orbit equation, by performing a Taylor expansion close to circular orbits, a relativistic differential equation is obtained. This equation is special in the sense that both it has no classical analogue and provides important constraints to the problem. Through two alternative approaches this relativistic equation was explored, which gave rise to two important observations. Firstly, the harmonic oscillator system was excluded as a solution altogether. Secondly, only when the angular momentum is greater than a certain threshold, given by Equation 61, may the Kepler-Coulomb system have closed orbits. Not only has the relativistic derivation revealed that relativistic corrections have significant implications for the orbit equation, but it has also shown that within special relativity no well-known systems have closed orbits for all bound states. The next section will be dedicated to providing some extra detail on some important aspects of the presented analyses.

5 Discussion

In section 3.3 the equations of motion of the harmonic oscillator have been simplified by effectively ignoring the angular frequency. Although it is not rare in theoretical physics to simplify expressions by setting some dimensionful quantities, such as the mass [11] [12] and the angular frequency [12], to one, at times it might come over as rather artificial. The justification of this method seems to lie in the physical interpretation of the system. To the best of the author's knowledge, the system can always be rescaled such that these quantities are equal to one. While it has been helpful during the derivation, being required to set them to one does feel uncomfortable.

In section 4.2 a modernised version of Bertrand's theorem [5] is introduced. The beauty of this version lies in its relative simplicity, but the validity of some assumptions can be justifiably called into question. In this derivation Equation 48 is simplified by studying its characteristics under specific limits of the particle energy. The heart of the problem lies in the fact the effective potential has been expanded in a Taylor series. The expansion was made around a circular orbit to make sure that the first derivative vanishes. In addition to that, the higher order components of the expansion has been ignored, implicitly constraining the description only to radii close to this circular orbit. Yet, for both the cases $n > 0$ and $n < 0$ the performed limits can be rephrased as describing the behaviour of the system for very large or very small radii. This contradiction is not something that was included in Bertrand's original derivation [13], in which he works with extreme values of r that differ infinitesimally. Perhaps a less straightforward but more technically elegant method is demonstrated in [3], where for the $n > 0$ case a new scaled effective potential $v_{eff} = \frac{V_{eff}}{E}$ is defined. This scaled effective potential implies that, for energies that tend to infinity, the effective potential reduces to a harmonic potential with a vertical wall at the origin. This description is similar to the one shown in Figure 6b. Unfortunately both [3] and [5] do not seem to justify the limit $E \to 0^-$, seemingly implicitly assuming that the radii are sufficiently close the circular orbit. The best justification the author think of is the fact that one searches potentials that are valid over all radii. As the single power laws in r are the only central potentials that could give rise to closed

orbits near the potential minimum, the set of potentials that are valid for the entire radial spectrum must be within this set of potentials. Aside from that, by utilising the orbit equation in section 2.2 and recalling the complex mapping in section 3.1, one can conclude that both the Kepler-Coulomb and the harmonic oscillator do have closed orbits for all bound states. These observations do seem to be consistent with each other, suggesting that the used assumptions might not have been too far-fetched.

Although the linear dependence of $J(u)$, found in section 4.3, is very helpful, one must be careful to avoid circular reasoning. $J(u)$ was first Taylor expanded to obtain a more manageable u dependence. The expansion was chosen specifically to be around a circular orbit as this would allow the constant term $J(u_0)$ to be inferred. Aside from that, all terms with a quadratic or higher order dependence on $(u - u_0)$ have been neglected. These imposed constraints simplify the expressions enough to be solved analytically. However, one could also say that during this process a linear $J(u)$ solution has been constructed artificially. Consequently, it would not be strange that the solution to Equation 59, which follows from the substitution of this specific form of $J(u)$ into Equation 56, would give rise to a potential that ensures that $J(u)$ is linear. The fact that the neglected higher order expansion terms of $J(u)$ happen to vanish by themselves is suspicious, possibly arising due to the self-imposed restrictions.

In section 4.3 the search for closed orbits was extended to include systems obeying special relativity. After finding the linear dependence of $J(u)$ Equation 56 could be solved analytically to find orbits described by Equation 65. This conclusion is actually very interesting when compared to an earlier result. In *The Classical Theory of Fields* [10] the same generalisation was performed specifically for a Kepler-Coulomb potential. In this book it has been shown that the Kepler-Coulomb potential may only have constant elliptical orbits when the constraint

$$\sqrt{1 - \frac{\alpha^2}{c^2 L^2}} = 1 \tag{66}$$

is satisfied. In Equation 66 α is interaction strength of the potential given by

$$V = \frac{\alpha}{r}.$$

The only way the constraint can be fulfilled is when the second term in the square root vanishes. This occurs only under three different conditions, namely when $\alpha = 0$, $L \to \infty$ or $c \to \infty$. Firstly, setting α to zero is equivalent to removing the potential energy in its entirety, in other words, getting rid of the particle interaction. In order to study the mechanics of the Kepler-Coulomb system this limit may not be taken, thus excluding this possibility. Secondly, requiring the angular momentum L to approach infinity is physically problematic. As this quantity is defined as the outer product of the position and momentum vector, if L approaches infinitely high values, then either one of the vectors will also have to tend to these large magnitudes. However, allowing that would make the physical system very difficult to understand. An infinitely large position vector can not possibly be thought of as a closed orbit and an infinitely large momentum is hard to conceptualise by itself. The last way to satisfy Equation 66 is by letting the speed of light to approach infinity. At first glance this might also seem strange, but in theoretical physics this limit is actually not that unfamiliar. Despite the fact that light does have a finite speed, by constraining systems to be and stay within classical mechanics, this upper limit will never be felt. As a result, within this specific realm of physics the speed of light can be considered to have an infinite large value. From the previous analysis it would seem that only classical systems qualify for possessing constant elliptical orbits.

The previous paragraph might confuse one into believing that the results found within section 4.3 and the *The Classical Theory of Fields* [10] disagree with each other, but that is simply false.

Although it has not been explicitly worked out in the latter, if the notion of closed elliptical orbits is let go in favor of closed orbits in general, then Equation 66 transforms into

$$\sqrt{1 - \frac{\alpha^2}{c^2 L^2}} = \zeta. \tag{67}$$

Just like in section 4.3, ζ found within Equation 67 symbolises a rational number. This relation can immediately be rewritten into

$$\frac{\alpha^2}{c^2 L^2} = 1 - \zeta^2$$
$$\frac{\alpha^2}{c^2(1-\zeta^2)} = L^2$$
$$L = \frac{\alpha}{c\sqrt{1-\zeta^2}}.$$

The final expression is identical to Equation 61, which was found when the Kepler-Coulomb potential was substituted as a trial solution to Equation 59. This observation confirms that both results are actually consistent with each other.

As was observed in section 4.3, the inclusion of the special relativistic corrections into Bertrand's theorem have given rise to a vastly different conclusion than its classical analogue. By comparing this relativistic extension with its classical counterpart, it becomes obvious that the former is still a relative new concept. If one would like to continue to explore hidden symmetries, then it might be most beneficial to either continue looking for alternative ways to generalise Bertrand's theorem or to explore the characteristics of its solutions. Exploring these subjects would not only allow one to better understand the validity of the presented derivation, but it could also inform about what systems or behaviour one should look for.

6 Conclusion

By scrutinising the characteristics of the Kepler-Coulomb system, the existence of a hidden symmetry was inferred. Using the Newtonian equations of motion and the conservation of angular momentum, the constant of motion associated to this hidden symmetry was shown to be the LRL-vector $\vec{A}$. From the existence of energy degenerate hodographs, the physical interpretation of the hidden symmetry is understood as the transformation between these energy degenerate states, which turns out to be equivalent to four-dimensional rotations. By expressing the Kepler-Coulomb problem in terms of complex numbers, a mapping to the two-dimensional harmonic oscillator can be constructed. This mapping allows one to rewrite the harmonic oscillator quantities in terms of their Kepler-Coulomb counterparts and vice versa. Allowing one to learn that the two-dimensional harmonic oscillator also contains a hidden symmetry. Upon mapping the LRL-vector it becomes clear that its components are related to a very special tensor, which demonstrates that the mapping is able to relate the additional constants of motion to each other. In order to determine whether other potentials could be studied in a similar fashion, the characteristics of closed orbits were researched. By studying a modernised version of Bertrand's theorem it has been revealed that within classical mechanics, only the Kepler-Coulomb and the harmonic oscillator systems are able to satisfy both the closed orbit condition and the classical orbit equation. As a result excluding the existence of other hidden symmetries. When one extends Bertrand's theorem to include special relativistic effects the final conclusion changes. The relativistic corrections change the orbits fundamentally, resulting in both the exclusion of the harmonic oscillator solution and introduces a restriction on the Kepler-Coulomb

solution. Consequently, it seems that no well-known potential is capable of having closed orbits for all bound states in special relativistic systems.

7 Acknowledgements

Throughout this bachelor's research project the author has received help from a number of people, this section will be dedicated to expressing my gratitude to them. First of all is my supervisor professor Diederik Roest. Our weekly meetings has played a crucial role in the creation of this paper. His vast experience did not only clear up my misunderstandings, but also served as a source of inspiration, which guidance allowed the research to explore so many different topics. Aside from him I was also assisted by PhD student MSc Dijs de Neeling. Despite the fact we have only met occasionally, our discussions has aided me in learning about both the harmonic oscillator and the generalisability of the LRL-vector to open systems. Lastly, I would like to spend some time to express my appreciation for my fellow bachelor student Roy Rodenburg. As a good friend of mine he offered indispensable feedback. Serving as a mirror to showcase both the strengths and weaknesses of my work.

References

[1] A. Alemi. *Laplace-Runge-Lenz Vector*. http://www.cds.caltech.edu/~marsden/wiki/uploads/projects/geomech/Alemicds205final.pdf. 2009.

[2] Uri Ben-Ya'acov. "Laplace–Runge–Lenz symmetry in general rotationally symmetric systems". In: *Journal of Mathematical Physics* 51.12 (2010), p. 122902. DOI: 10.1063/1.3520521. eprint: https://doi.org/10.1063/1.3520521. URL: https://doi.org/10.1063/1.3520521.

[3] Siu A. Chin. "A truly elementary proof of Bertrand's theorem". In: *American Journal of Physics* 83.4 (2015), pp. 320–323. DOI: 10.1119/1.4901974. eprint: https://doi.org/10.1119/1.4901974. URL: https://doi.org/10.1119/1.4901974.

[4] D. M. Fradkin. "Three-Dimensional Isotropic Harmonic Oscillator and SU3". In: *American Journal of Physics* 33.3 (1965), pp. 207–211. DOI: 10.1119/1.1971373. eprint: https://doi.org/10.1119/1.1971373. URL: https://doi.org/10.1119/1.1971373.

[5] Jared Galbraith and Jacob Williams. "An Even Simpler "Truly Elementary" Proof of Bertrand's Theorem". In: *Journal of Undergraduate Reports in Physics* 29.1 (2019), p. 100005. DOI: 10.1063/1.5129245. eprint: https://doi.org/10.1063/1.5129245. URL: https://doi.org/10.1063/1.5129245.

[6] H. Goldstein. "More on the prehistory of the Laplace or Runge–Lenz vector". In: (1976).

[7] H. Goldstein. "Prehistory of the"Runge–Lenz"vector". In: (1975).

[8] Herbert Goldstein, Charles Poole, and John Safko. *Classical Mechanics*. 3rd ed. Addison Wesley, 2001.

[9] Prashant Kumar and Kaushik Bhattacharya. "Possible potentials responsible for stable circular relativistic orbits". In: *European Journal of Physics* 32.4 (May 2011), pp. 895–903. DOI: 10.1088/0143-0807/32/4/005. URL: https://doi.org/10.1088%5C%2F0143-0807%5C%2F32%5C%2F4%5C%2F005.

[10] L. D. Landau. *The classical theory of fields (4th ed., Ser. Course of theoretical physics*. Elsevier Science, 1980.

[11] Bernard M. Nabet and Barak Kol. *Leading anomalies, the drift Hamiltonian and the relativistic two-body system*. 2014. DOI: 10.48550/ARXIV.1408.2628. URL: https://arxiv.org/abs/1408.2628.

[12] S. Palmisano et al. "The classical Lenz vector and the two-dimensional quantum harmonic oscillator". In: *Physics Open* 3 (2020), p. 100021. ISSN: 2666-0326. DOI: https://doi.org/10.1016/j.physo.2020.100021. URL: https://www.sciencedirect.com/science/article/pii/S2666032620300089.

[13] F. C. Santos, V. Soares, and A. C. Tort. "An English translation o Bertrand's theorem". In: (2007). DOI: 10.48550/ARXIV.0704.2396. URL: https://arxiv.org/abs/0704.2396.

[14] R. Shankar. *Principles of quantum mechanics*. Springer, 1994.

[15] David Tong. *Classical Dynamics University of Cambridge Part II Mathematical Tripos*. 2015. URL: http://www.damtp.cam.ac.uk/user/tong/dynamics.html.