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The knot complement and its homotopy

Bachelor's Project Mathematics

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Contents

Introduction

What is a knot? How can we study it topologically? Those are the questions that this thesis seeks to answer. We study the knot complement as topological space and apply techniques found in algebraic topology.

To start of, this thesis introduces the reader to knots and equivalence of knots. We talk about knot invariants and in particular, the knot complement, a topological space that is studied in detail in this thesis.

Secondly, we study Seifert surfaces. The existence of Seifert surfaces is proven using Seifert's algorithm. Then, we use the genus of the Seifert surface to compute its fundamental group.

In the third chapter, the reader is introduced to the infinite cyclic cover of the knot complement. We first use the homology of the knot complement to prove the existence of the infinite cyclic cover, and then construct it explicitly by cutting the knot complement along a Seifert surface.

Fourthly, we zoom in to a specific type of knot, called the fibred knot. The reader is first introduced to some theory on fibre bundles, which we use to compute the commutator subgroup of the knot complement for fibred knots.

To finish of this thesis, we construct a space that is homotopy-equivalent to the knot complement and somewhat easier to grasp. We study the homology of the infinite cyclic cover of this space and give ideas on how this space can be used to give the homology group a $\mathbb{Z}[t^{\pm}]$-module structure.

Many of the techniques used in this thesis are techniques from algebraic topology. As of writing this thesis, the University of Groningen does not have a bachelor course on algebraic topology. Therefore many bachelor students may struggle reading this thesis. In case the reader is interested in learning about algebraic topology, the author recommends reading chapters 11-14 of [4] and chapter 13 of [6].

CHAPTER 1

Knot theory basics

The aim of this chapter is to provide the reader with the basic definitions in knot theory. To be specific, we define what a knot is and how they are classified. Furthermore, knot invariants are defined and it is proven that the knot complement is a knot invariant.

1. Definition of a knot

In order to define knots, we first recall the definition of an embedding.

Definition 1.1. Let X and Y be topological spaces. An embedding of X in Y is a continuous map $f : X \to Y$ such that the restriction $f : X \to f(X)$ is a homeomorphism.

Sometimes, the notation $X \hookrightarrow Y$ may be used to express that X is embedded in Y, without needing to give the map a name.

It is clear that an embedding is injective. Note however that not every continuous injective map is an embedding, as the inverse of the restriction may not be continuous.

The definition of an embedding is enough to give the definition of a knot.

Definition 1.2. A *knot* is an embedding $k : S^1 \hookrightarrow S^3$. The image of this map is also called a knot and is also denoted k.

The ambiguity of this notation is not an issue, as it is always clear from context whether k refers to an embedding $S^1 \hookrightarrow S^3$ or the image of that embedding.

One should recall that S^3 is the one point compactification of $\mathbb{R}^3$. Therefore S^3 can be viewed as $\mathbb{R}^3 \cup \{\infty\}$. Consequently, by thinking of knots as embeddings $S^1 \hookrightarrow \mathbb{R}^3$ with a point at infinity, it becomes much easier to visualise and draw them.

Example 1.3 (Unknot and trefoil knot). We give some examples of knots. Firstly, there is the trivial embedding $S^1 \hookrightarrow S^3$, as seen in the left drawing of figure 1.1. This knot is called the *unknot*.

Secondly, there is the embedding $S^1 \hookrightarrow S^3$ that is drawn on the right in figure 1.1. This knot is called the *trefoil knot*.

As with many objects in mathematics, we need a method to classify knots. The image of a knot is not of use of us here, as every image of a knot is homeomorphic with the circle S^1. So instead, we need to take the surrounding space into account. This leads us to the following definition:

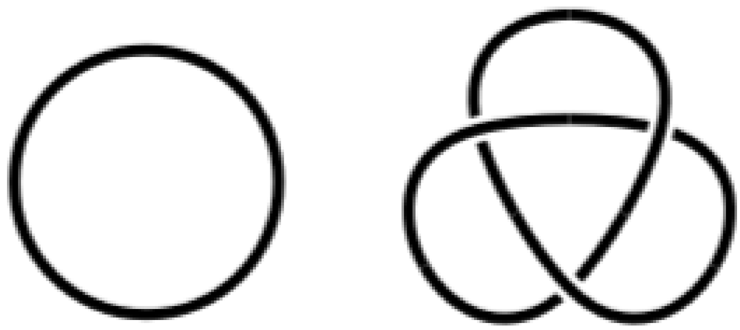


FIGURE 1.1. The unknot (left) and the trefoil knot (right).

DEFINITION 1.4. Two knots $k_1 : S^1 \hookrightarrow S^3$ and $k_2 : S^1 \hookrightarrow S^3$ are *equivalent* if there exists an orientation-preserving homeomorphism $h : S^3 \xrightarrow{\sim} S^3$ that carries k_1 into k_2, i.e. $h \circ k_1 = k_2$.

It is clear that the above definition is an equivalence relation.

The definition of a knot presented above lacks a property that will be used extensively in future chapters. To be specific, we require our knots to have a tubular neighbourhood, which is a neighbourhood that is homeomorphic with $S^1 \times D^2$. An example of a knot that doesn't have this property is the infinitely nested knot presented in figure 1.2, as no tube can be formed around the (limit) point L.

Despite appearing as if you can unravel the knot from the right side, it can be shown that this knot is not equivalent to the unknot.

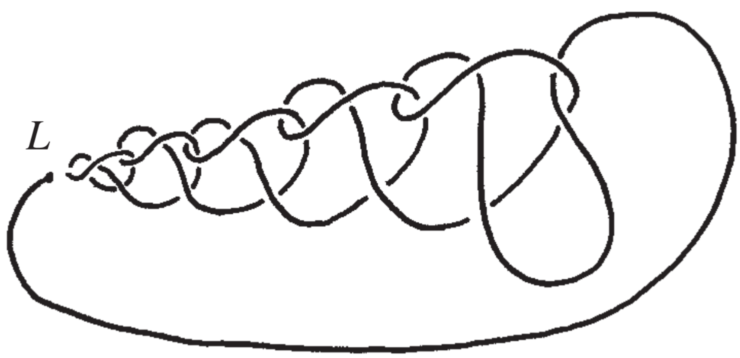


FIGURE 1.2. An knot with infinitely nested crossings, from [2].

Examples of knots that clearly have a tubular neighbourhood are those that satisfy the following:

DEFINITION 1.5. A *polygonal knot* is a knot whose image is the union of a finite number of line segments.

It is clear that the knots presented in figure 1.1 are polygonal knots. These knots turn out to be precisely the ones we are interested in.

THEOREM 1.6. *Let k be a knot. The following statements are equivalent:*

(1) *The knot k is equivalent to a polygonal knot;*

(2) *there exists a neighbourhood k that is homeomorphic to $S^1 \times D^2$. This neighbourhood is called a tubular neighbourhood of k, denoted $V(k)$.*

Proof. This theorem is not proven in this thesis. A proof of a stronger version of this theorem is given in [8]. ☺

Definition 1.7. A knot is called *tame* if it satisfies the equivalent conditions (1) and (2) in theorem 1.6. A knot that is not tame is called *wild*.

From now on, all knots in this thesis are assumed to be tame.

2. Knot invariants

The concept of equivalence of knots has been presented in definition 1.4. This definition is quite straightforward and knot theorists have developed a method to show that two knots are equivalent involving the use of drawings. This method is called *equivalence by Raidemaister moves* and an in-depth explanation can be found in the first chapter of [2].

On the contrary, a problem that is omnipresent in knot theory is showing two knot are *not* equivalent. It is difficult to directly show no orientation-preserving homeomorphism of S^3 exists that carries one knot to another. For instance, it is tough to show that the knots in figure 1.1 are not equivalent. Therefore knot theorists turn to *knot invariants* instead to show two knots are not equivalent.

Definition 1.8. A *knot invariant* is a map

$$\{\text{knots } k : S^1 \hookrightarrow S^3\}/(\text{equivalence}) \longrightarrow Z,$$

where Z is any set. An knot invariant is called *complete* if it is injective.

So a knot invariant assigns to any knot a quantity (in a very broad sense) that does not change under equivalence of knots. The main focus of knot theory is to find knot invariants and compute them. As of the publication of this thesis, no easily computable complete knot invariant exists. Finding such an invariant would be the pinnacle achievement of knot theory.

The focus of this thesis is on the knot complement, defined below.

Definition 1.9. Let $k : S^1 \hookrightarrow S^3$ be a knot. The space $S^3 \setminus \text{Im } k$ is called the *knot complement of k*.

The knot complement turns out to be a knot invariant, up to homeomorphisms (denoted $\cong$).

Proposition 1.10. *The following map is a knot invariant:*

$$C : \{\text{knots } k : S^1 \hookrightarrow S^3\}/(\text{equivalence}) \longrightarrow \{S^3 \setminus \text{Im } k \mid k : S^1 \hookrightarrow S^3\}/\cong$$
$$k \longmapsto S^3 \setminus \text{Im } k.$$

Proof. We need to show that C is well-defined.

Let $k_1 : S^1 \hookrightarrow S^3$ and $k_2 : S^1 \hookrightarrow S^3$ be two equivalent knots and $h : S^3 \xrightarrow{\sim} S^3$ an orientation-preserving homeomorphism such that $h \circ k_1 = k_2$. It is clear that $h(\mathrm{Im}\ k_1) = \mathrm{Im}\ k_2$, and therefore the restriction

$$\begin{aligned} h|_{S^3 \setminus \mathrm{Im}\ k_1} : S^3 \setminus \mathrm{Im}\ k_1 &\longrightarrow S^3 \setminus \mathrm{Im}\ k_2 \\ x &\longmapsto h(x) \end{aligned}$$

is a homeomorphism. We conclude that C is well-defined. ☺

A noteworthy fact about the knot complement is that it is a *complete invariant*. A proof of this can be found in [5]. As mentioned before, no known easily computable knot invariants exists. Indeed, it is tough to determine whether the knot complements of two knots are homeomorphic. Instead, we choose to focus on computing topological invariants of the knot complement, which is done in the remainder of this thesis.

CHAPTER 2

Seifert surfaces

Before studying topological invariants of the knot complement, we need to take a detour to the theory of Seifert surfaces. These surfaces will be required to construct a covering map of the knot complement in chapter 3, called the *infinite cyclic cover of the knot complement.*

In this chapter, we discuss Seifert surfaces and their existence, as well as compute their fundamental group in terms of their genus.

1. Existence of Seifert Surfaces

DEFINITION 2.1. A *Seifert surface* of a knot is an orientable surface with boundary equal to the knot.

The existence of Seifert surfaces is non-trivial, but does turn out to be guaranteed. An algorithm called *Seifert's algorithm* allows us to construct a Seifert surface explicitely for any knot.

THEOREM 2.2 (Seifert's algorithm). *Every knot admits a Seifert surface.*

PROOF. Let k be a knot. We will construct an orientable surface S satisfying $\partial S = k$.

First, choose an orientation and a knot diagram for k. Then, at each crossing of the knot in the diagram, alter k as shown in figure 2.1.

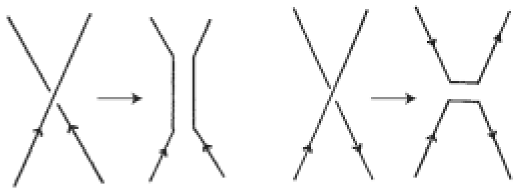


FIGURE 2.1. The creation of Seifert cycles, from [2].

After this, we end up with a disjoint union of oriented simple closed curves. These curves are called *Seifert cycles.* Recall that an oriented simple closed curve is the boundary of an oriented surface. For each Seifert cycle, choose such a surface and embed them into S^3 such that their boundaries are the Seifert cycles, while keeping the surfaces disjoint. The surface can be kept disjoint by lifting them up or down to create a three-dimensional stack of surfaces. These surfaces are call *Seifert cells*

Lastly, we undo the process of 'cutting' the knot in figure 2.1 by merging the Seifert cells together. This is done by adding a half-twisted strip at each position that there used to be a crossing, see figure 2.2. We let the half-twists cross in the same way as the original crossing, creating a connected surface S satisfying $\partial S = k$.

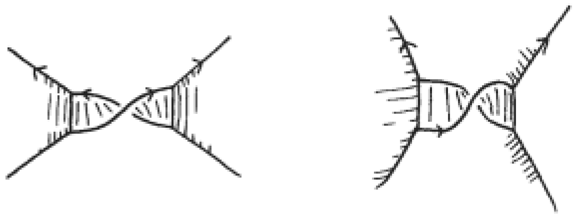


FIGURE 2.2. Twisted bands merging the surfaces, from [2].

The Seifert cells are orientable and due to the twisted bands from figure 2.2 we find that S itself is also orientable. We conclude that S is a Seifert surface of k. ☺

To further clarify the algorithm presented above, a Seifert surface of the so-called 'figure eight' knot is constructed in figure 2.3.

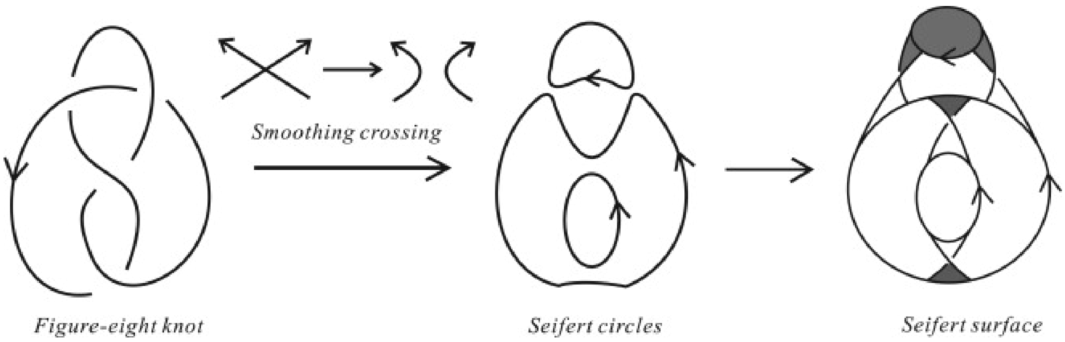


FIGURE 2.3. Applying Seifert's algorithm to the figure eight knot, from [3].

A fact that may strike interest into the reader is that Seifert surfaces are not unique. Every knot has infinitely many non-homeomorphic Seifert surfaces. This statement is easily proven using topological surgery, but this is not done in this thesis.

2. Genus of a surface

In this section, we discuss the concept of the *genus of the surface*. When defining the genus, it is important to distinguish between surfaces with- and without boundary.

2.1. Surfaces without boundary. The genus is defined as follows for orientable surfaces (without boundary):

DEFINITION 2.3. Let S be a connected and orientable surface. Then the *genus* of S is the maximum number of simple closed curves that S can be cut along without the reulting surface being disconnected.

When talking about surfaces, it is useful to consider the classification of surfaces. This classification is a useful aid in finding the fundamental group of Seifert surfaces in the next section.

PROPOSITION 2.4 (Classification of surfaces). *Any connected surface is homeomorphic with one of*

(1) *the 2-sphere;*
(2) *the torus;*
(3) *the projective plane,*

or connected sums of these surfaces.

Recall that the connected sum of two connected surfaces S and T is obtained by removing an open disc in S and T and then identifying their respective boundaries. Note that the connected sum of two spheres is a single sphere, and the connected sum of two tori is a torus with two holes, see figure 2.4.

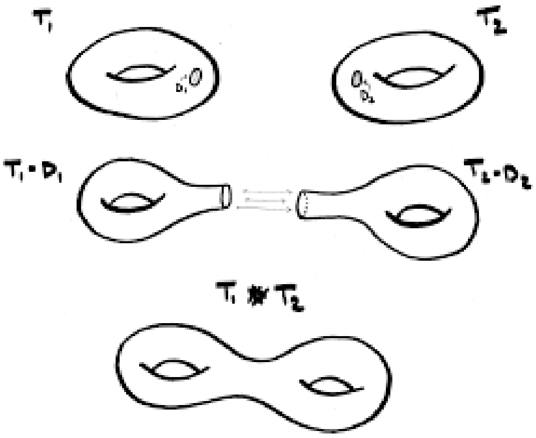


FIGURE 2.4. The connected sum of two tori is a torus with two holes.

Since the projective plane is non-orientable, we can classify the connected orientable surfaces as tori with n holes, where the sphere is the torus with 0 holes.

With this information, we can find the genus of all connected orientable surfaces:

PROPOSITION 2.5. *A torus with n holes has genus n.*

PROOF. There are two different simple closed curves that can be cut out of a torus without the resulting surface being disconnected. One of them yields a cylinder, and the other an annulus, which are homeomorphic. No simple closed curve can be taken out of these surface without making the result disconnected.

By performing either of these cuts at every hole of the torus with n holes, we obtain a connected sum of n cylinders. No more cuts can be done to this, without making disconnecting the space, so we conclude that the torus with n holes has genus n. ☺

2.2. Surfaces with boundary. We can view surfaces with boundary as closed surfaces with discs taken out of them. To be specific, S is said to be a *connected surface with n boundary* if for some connected surface C,

$$S = C \setminus \left(\bigsqcup_{i=1}^{n} D^2 \right).$$

Here D^2 is the open disc. We can now define the genus of a surface with boundary:

DEFINITION 2.6. Let $S = C \setminus (\bigsqcup_{i=1}^{n} D^2)$ be a surface with boundary. Then the *genus* of S is the genus of C.

It should be noted that defining the genus of a surface with boundary as the number of simple closed curves that can be cut from the surface, without disconnecting it, would result in an equivalent definition. This fact is not shown in this thesis.

One example of a connected and orientable surface with boundary is a Seifert surface. The genus of the Seifert surface is of interest to us, but it turns out that not all Seifert surfaces of a knot have the same genus. Therefore, we define the concept of the genus of a knot as follows:

DEFINITION 2.7. The *genus of a knot* is the minimal genus of its Seifert surfaces.

3. Fundamental group of Seifert Surfaces

Consider a knot k of genus g. Let S be a Seifert surface of k of genus g. The boundary of S is k, so since k has 1 connected component we see that S is a surface with 1 boundary. As S is orientable, the classification of oriented surfaces tells us that S is homeomorphic with a torus with g holes and 1 disc taken out of it. Now that we can grasp the Seifert surface more easily, we can compute its fundamental group.

PROPOSITION 2.8. *The fundamental group of a connected and oriented surface with genus g and 1 boundary is $\mathcal{F}_{2g}$, the free group with $2g$ generators.*

PROOF. Let S be a connected and oriented surface with genus g and 1 boundary. Then by the classification of surfaces we find that S is a torus with g holes and a disc taken out of it. This space can be retracted to the bouquet of $2g$ circles, see figure 2.5. The fundamental group of the bouquet with $2g$ circles is $\mathcal{F}_{2g}$. This completes the proof. ☺

COROLLARY 2.9. *The fundamental group of a Seifert surface of genus g is $\mathcal{F}_{2g}$.*

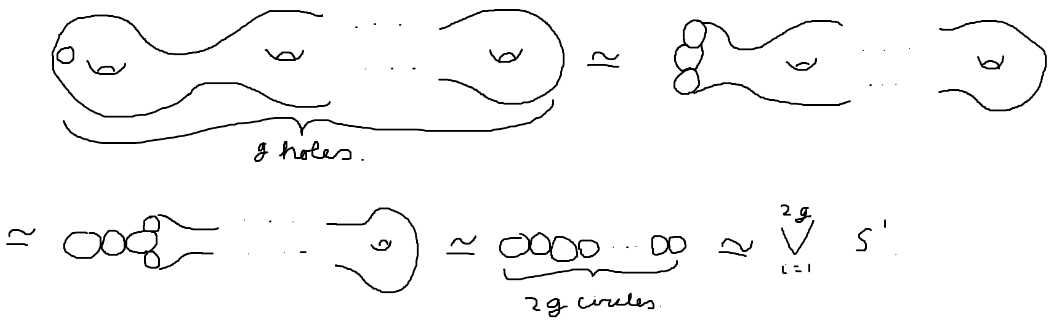


FIGURE 2.5. The torus with n holes and a disc 1 boundary can be retracted to the bouquet of $2g$ circles.

CHAPTER 3

Cyclic coverings

In this chapter we start studying the knot complement. If k is a knot with tubular neighbourhood $V = V(k)$, then we denote the knot complement by C. By knot complement we mean one of the spaces $S^3 \setminus k$, $S^3 \setminus V$, $\overline{S^3 \setminus k}$ or $\overline{S^3 \setminus V}$. Even though these spaces are not homeomorphic, they are homotopy-equivalent. Seeing as we are studying the homology groups and fundamental group of the knot complement, this does not turn out to be a problem.

The fundamental group of the knot complement is of great interest to knot theorists, as it is a very powerful invariant. However, computing this group explicitly turns out to be a difficult task that has yet to be overcome. Therefore, knot theorists prefer to study invariants of this group, such as the first homology group or the commutator subgroup.

The aim of this chapter is to compute the homology of the knot complement, and explictely construct a space that has the commutator subgroup as its fundamental group. This space is studied in-depth in future chapters.

It should be noted that the knot complement is assumed to be a connected 3-manifold. This is non-trivial, but is not proven in this thesis.

1. Homology of the knot complement

In this section, the homology of the knot complement is computed. The following tool is required to compute this homology.

THEOREM 3.1 (Mayer-Vietoris). *Let X be a topological space and $U_1, U_2 \subset X$ open subspaces such that $U_1 \cup U_2 = X$. Consider the group homomorphisms induced by the inclusion maps:*

$$\begin{tikzcd}
 & H_p(U_1) \arrow[dr, "k_\star"] & \\
H_p(U_1 \cap U_2) \arrow[ur, "i_\star"] \arrow[dr, "j_\star"'] & & H_p(X) \\
 & H_p(U_2) \arrow[ur, "l_\star"'] &
\end{tikzcd}$$

Also consider the group homomorphism $\partial_\star : H_p(X) \to H_p(U_1 \cap U_2)$ given by $\partial_\star[c] = \partial_\star[c_1 - c_2] = [\partial c_1] = [\partial c_2]$, where c_1 and c_2 are p-chains in U_1 and U_2 respectively. Recall that any cycle in X can be written in this way.

The following sequence of groups is exact:

$$\cdots \longrightarrow H_{p+1}(U_1) \oplus H_{p+1}(U_2) \xrightarrow{k_\star - l_\star} H_{p+1}(X) \xrightarrow{\partial_\star}$$

$$\hookrightarrow H_p(U_1 \cap U_2) \xrightarrow{i_\star \oplus j_\star} H_p(U_1) \oplus H_p(U_2) \xrightarrow{k_\star - l_\star} H_p(X) \xrightarrow{\partial_\star}$$

$$\hookrightarrow H_{p-1}(U_1 \cap U_2) \xrightarrow{i_\star \oplus j_\star} H_{p-1}(U_1) \oplus H_{p-1}(U_2) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_0(U_1) \oplus H_0(U_2) \xrightarrow{k_\star - l_\star} H_0(X) \longrightarrow 0$$

Proof. This theorem is assumed to be prior knowledge to the reader. A proof can be found in chapter 13 of [6]. ☺

The Mayer-Vietoris theorem is used not only to proof the following theorem, but also to proof several theorems in chapter 5. Therefore it is imperative that the reader has a good understanding of this theorem.

Proposition 3.2 (Homology of the knot complement). *Let k be a knot, $V = V(k)$ a tubular neighbourhood and $C = S^3 \setminus k$ the corresponding knot complement, then*

$$H_p(C) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 1, \\ 0 & \text{if } p > 1. \end{cases}$$

Proof. It is assumed without proof that the knot complement is a connected 3-manifold. Therefore C is path-connected and $H_0(C) = \mathbb{Z}$.

The Mayer-Vietoris theorem is used to find $H_p(C)$ for $p > 0$. In this case, let $X = S^3$, $U_1 = C$ and $U_2 = V$. Since V is homeomorphic with a solid torus, it's homotopy-equivalent with the circle S^1. The intersection $U_1 \cap U_2 = C \cap V$ is a solid torus with a circle taken out of it, i.e. homeomorphic with $S^1 \times (D^2 \setminus \{*\})$. The space $D^2 \setminus \{*\}$ is homotopy-equivalent with S^1, so $U_1 \cap U_2$ is is homotopy-equivalent with the torus T.

Recall the following results from algebraic topology:

$$H_p(S^1) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 1, \\ 0 & \text{if } p > 1, \end{cases}$$

$$H_p(S^3) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 3, \\ 0 & \text{if } p = 1, 2 \text{ or } p > 3, \end{cases}$$

$$H_p(T) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } p = 1, \\ 0 & \text{if } p > 2. \end{cases}$$

Applying Mayer-Vietoris yields the following exact sequence for $p > 3$:

$$H_p(C \cap V) \longrightarrow H_p(C) \oplus H_p(V) \longrightarrow H_p(S^3)$$

Since $H_p(T) = H_p(S^1) = H_p(S^3) = 0$, we find that $H_p(C) = 0$ for $p > 3$. In addition, there is the following exact sequence at the bottom of the sequence:

$$
\begin{tikzcd}
H_3(C \cap V) \arrow[r] & H_3(C) \oplus H_3(V) \arrow[r] & H_3(S^3) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[pos=0.75, below]{\partial_\star} ([yshift=-1.5ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_2(C \cap V) \arrow[r, "i_\star \oplus j_\star"] & H_2(C) \oplus H_2(V) \arrow[r, "k_\star - l_\star"] & H_2(S^3) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1.5ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
H_1(C \cap V) \arrow[r] & H_1(C) \oplus H_1(V) \arrow[r] & H_1(S^3)
\end{tikzcd}
$$

As $H_1(S^3) = H_2(S^3) = 0$ we find that $H_1(C \cap V) = H_1(C) \oplus H_1(V)$, so $\mathbb{Z} \oplus \mathbb{Z} = H_1(C) \oplus \mathbb{Z}$. Thus $H_1(C) = 0$.

If we view $C \cap V$ as the torus by the homotopy-equivalence, then any 2-cycle on the torus is the boundary of a 3-chain in S^3. Because our knot is tame, this 3-chain can be made so that it does not intersect the knot anywhere. Therefore the inclusion $H_2(C \cap V) \to H_2(C)$ is trivial so $\operatorname{Im} i_\star \oplus j_\star = 0$. From the homomorphism theorem we get $(H_2(C) \oplus H_2(V))/\ker(k_\star - l_\star) = H_2(S^3)$ so by exactness $H_2(C) \oplus H_2(V) = H_2(C) = 0$.

Since $\ker i_\star \oplus j_\star = \mathbb{Z}$ we find that $\partial_\star$ is surjective, so since $H_3(S_3) = H_2(C \cap V) = \mathbb{Z}$ we find that $\ker \partial_\star = 0$. Furthermore, $H_3(C \cap V) = 0$ thus $H_3(C \cap V) \to H_3(C) \oplus H_3(V)$ is trivial. We conclude by exactness that $H_3(C) \oplus H_3(V) = 0$ and therefore $H_3(C) = 0$. ☺

2. Existence of cyclic coverings

In the previous section, it was shown that the first homology group of the knot complement is independent of the knot, and is always infinite cyclic. In this section, this is used to prove the existence of a cyclic covering of the knot complement of which the fundamental group is equal to the commutator subgroup of the fundamental group of the knot complement.

We first recall the notion of a regular covering and the Galois Correspondence of covering maps.

Definition 3.3. Let $p : Y \to X$ be a covering with Y connected and X locally path-connected. The covering p is called *regular* if it is a G-covering for some group G.

Theorem 3.4 (Galois Correspondence). *Let X be a topological space that is* connected, locally path-connected *and* semi-locally simply connected. *Let $\mathcal{S}$ be the set of pointed regular coverings $p : (Y, y) \to (X, x)$, up to isomorphism. Let $\mathcal{P}$ be the set of subgroups of $\pi_1(X, x)$. The map*

$$
\begin{aligned}
\mathcal{S} &\longrightarrow \mathcal{P} \\
p &\longmapsto p_\star(\pi_1(Y, y))
\end{aligned}
$$

is a bijection. This bijection is called the Galois Correspondence.

A covering $p \in \mathcal{S}$ is regular *if and only if its corresponding subgroup of $\pi_1(X, x)$ is* normal. *If this is the case, then $p : (Y, y) \to (X, x)$ is a $\pi_1(X, x)/p_\star(\pi_1(Y, y))$-covering.*

Proof. This theorem is assumed to be prior knowledge to the reader. A proof can be found in chapter 13 of [4]. ☺

The existence of the cyclic coverings described below is an immediate consequence of the Galois Correspondence.

Proposition 3.5 (Existence of cyclic coverings). *There exists a unique regular $\mathbb{Z}$-covering of the knot complement $p_\infty : C_\infty \to C$ that satisfies*

$$p_{\infty\star}(\pi_1(C_\infty)) \cong [\pi_1(C), \pi_1(C)].$$

For $n \in \mathbb{Z}_{\geq 2}$ there exists a unique regular $\mathbb{Z}/n\mathbb{Z}$-covering of the knot complement $p_n : C_n \to C$ that satisfies

$$p_{n\star}(\pi_1(C_n)) \cong n\mathbb{Z} \oplus [\pi_1(C), \pi_1(C)].$$

Proof. Recall that the commutator subgroup $[\pi_1(C), \pi_1(C)]$ of $\pi_1(C)$ is a normal subgroup. Furthermore, recall from Hurewicz' Theorem that $H_1(C) \cong \pi_1(C)/[\pi_1(C), \pi_1(C)]$ and that $H_1(C) \cong \mathbb{Z}$ by proposition 3.2. Therefore we find by the Galois Correspondence (theorem 3.4) that there exists a unique regular $\mathbb{Z}$-covering $p_\infty : C_\infty \to C$ that satisfies

$$p_{\infty\star}(\pi_1(C_\infty)) \cong [\pi_1(C), \pi_1(C)].$$

By further quotienting $H_1(C)$ to $\mathbb{Z}/n\mathbb{Z}$ we find the $\mathbb{Z}/n\mathbb{Z}$-covering, again by the Galois Correspondence. ☺

These coverings are significant enough in this thesis to be given their own name:

Definition 3.6. The $\mathbb{Z}$-covering of proposisition 3.5 is called the *infinite cyclic covering* and the $\mathbb{Z}/n\mathbb{Z}$-covering is called the *finite (n-fold) cyclic covering*.

3. Cutting along a surface

A technique that is used to construct the cyclic covering of the knot complement is called *cutting a 3-manifold along a surface*. The most intuitive way to cut a 3-manifold M along a surface S in M would be to consider the subspace $M \setminus S$. However, this space is not closed in M. This turns out to be problematic when constructing the cyclic cover. Therefore the technique below is used to cut along a surface instead.

Definition 3.7 (Cutting a 3-manifold along a surface). Let M be a 3-manifold and S an oriented surface in M. Consider a neighbourhood U around S such that $U \cong S \times [-1, 1]$. Then $U \setminus S = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $U_1, U_2 \cong S \times (0, 1]$.

Let M'_0, U'_1 and U'_2 be homeomorphic copies of $\overline{M \setminus U}$, $\overline{U_1}$ and $\overline{U_2}$ respectively, with homeomorphisms $f_0 : \overline{M \setminus U} \xrightarrow{\sim} M'_0$ and $f_i : \overline{U_i} \xrightarrow{\sim} U'_i$ for $i \in \{1, 2\}$.

The space M' is obtained from the disjoint union $M'_0 \sqcup U'_1 \sqcup U'_2$ by identifying $f_0(x)$ with $f_i(x)$ when $x \in \overline{M \setminus U} \cap \overline{U_i} = \partial(M \setminus U) \cap \partial U_i$ $(i \in \{1, 2\})$. The space M' is a 3-manifold and is called *the space obtained by cutting M along S*.

First of all, it should be noted that the space constructed above is homeomorphic with $\overline{M \setminus U}$, which is closed in M, solving our problem mentioned earlier

Additionally, Through the homeomorphisms f_i there is a natural map

$$i : M_0' \sqcup U_1' \sqcup U_2' \longrightarrow M$$
$$x \longmapsto \begin{cases} f_0(x) & \text{if } x \in M_0' \\ f_1(x) & \text{if } x \in U_1' \\ f_2(x) & \text{if } x \in U_2'. \end{cases}$$

In turn, this map induces a natural map $\iota : M' \to M$ called the *identification map*, not to be confused with the quotient map $M_0' \sqcup U_1' \sqcup U_2' \to M'$.

As an example, we can cut the solid torusy $S^1 \times D^2$ along a disc D^2 to obtain a solid cylinder $[0,1] \times D^2$ with two copies of the disc D^2 as boundary. This can be seen in figure 3.1. It should be noted that the identification map maps points in the red discs of the solid cylinder to the corresponding points in the red disc of the solid torus.

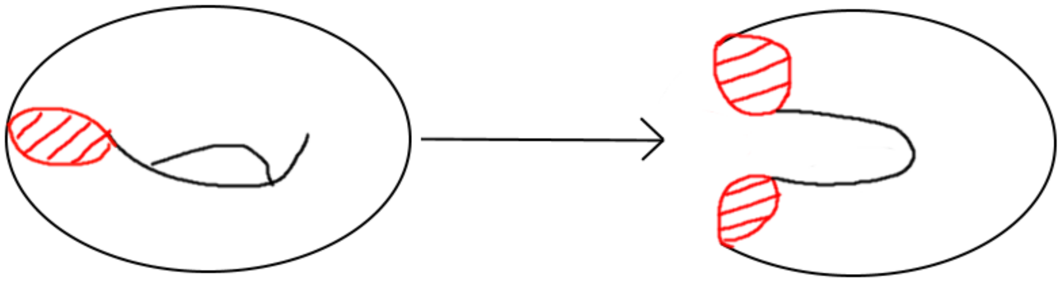


FIGURE 3.1. Cutting a solid torus along a disc.

4. Construction of the cyclic covering

In this section, we construct the cyclic coverings of the knot complement.

Let k be a knot, let $V = V(k)$ be a tubular neighbourhood of k and let S' be a Seifert surface of k. Furthermore, let $C = \overline{S^3 \setminus V}$ be the knot complement and let $S = S' \cap C$. Lastly, define $\lambda = \partial S' \cap V$, which is a simple closed curve along the boundary of V.

Now cut C along S to obtain the 3-manifold $C^\star$. The boundary of $C^\star$ is a connected surface that consists of an annulus that is obtained by cutting the torus ∂V along λ, and two disjoint parts S^+ and S^- that are homeomorphic to S. A local overview of this can be seen in figure 3.2.

Let $r : S^+ \xrightarrow{\sim} S^-$ be the homeomorphism that maps a point of S^+ to the point of S^- which corresponds to the same point of S. You can think of this as drawing vertical lines between S^+ and S^- in figure 3.2 and sending points in S^+ along those lines to the corresponding point in S^-.

Take homeomorphic copies $(C_i^*)_{i\in\mathbb{Z}}$ of $C^\star$ with homeomorphisms $(h_i : C^\star \xrightarrow{\sim} C_i^\star)_{i\in\mathbb{Z}}$. The space C_∞ is obtained from the disjoint union $\bigsqcup_{i\in\mathbb{Z}} C_i^\star$ by identifying $h_i(x)$ and $h_{i+1}(r(x))$ when $x \in S^+$ and $i \in \mathbb{Z}$. Furthermore, the space

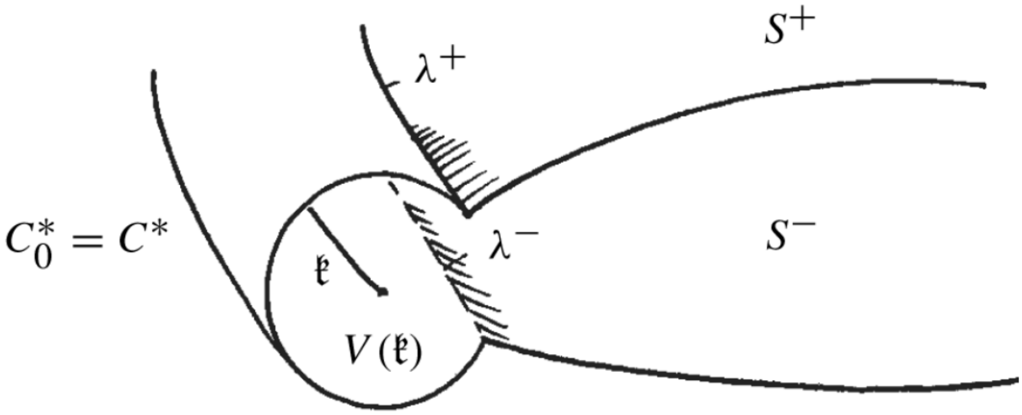


FIGURE 3.2. Local overview of cutting the knot complement along a Seifert surface, from [2].

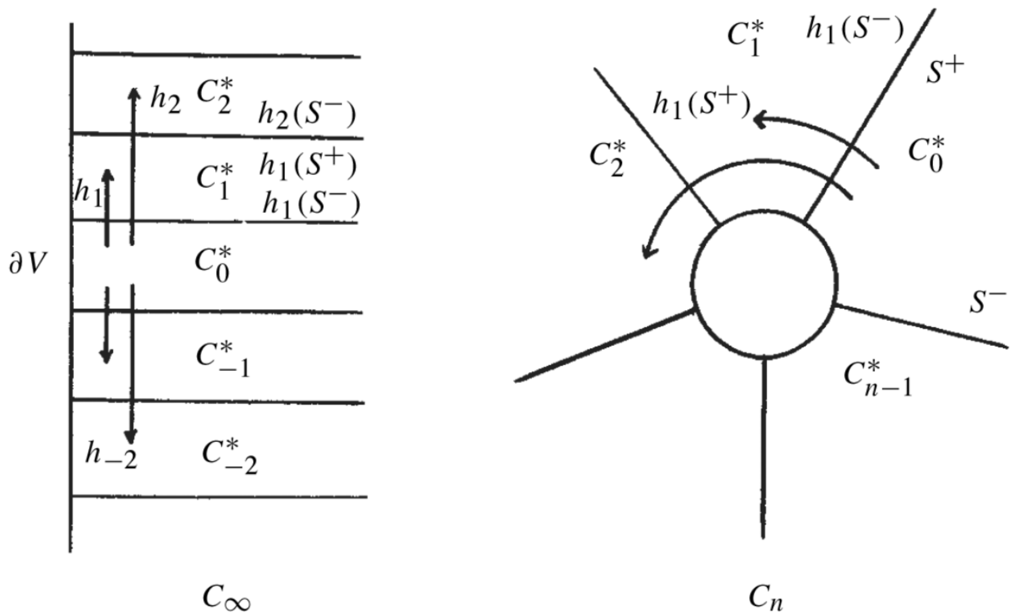


FIGURE 3.3. Moving between layers in C_∞ and C_n, from [2].

C_n is obtained from $\bigsqcup_{i=0}^{n-1} C_i^\star$ by identifying $h_i(x)$ with $h_{i+1}(r(x))$ and $h_{n-1}(x)$ with $h_0(r(x))$ for $i \in \{1, 2, \ldots, n-1\}$ and when $x \in S^+$.

The spaces C_n and C_∞ are stacks of $C^\star$, where one can move from one layer up or down through S^+ or S^- respectively. In C_n, going up through S^+ in $C_{n-1}^\star$ puts you in $C_0^\star$. To illustrate this moving between layers, see figure 3.3.

The covering maps can now be introduced. Let $\iota : C^\star \to C$ be the identification map. We define the maps

$$\begin{aligned} p_\infty : C_\infty &\longrightarrow C \\ x &\longmapsto \iota(h_i^{-1}(x)) \quad \text{if } x \in C_i^\star \end{aligned}$$

and

$$\begin{aligned} p_n : C_n &\longrightarrow C \\ x &\longmapsto \iota(h_i^{-1}(x)) \quad \text{if } x \in C_i^\star. \end{aligned}$$

It is clear that these maps are coverings of the knot complement. It remains to be shown that they are the cyclic coverings of the knot complement.

Consider the map

$$\begin{aligned} t : C_\infty &\longrightarrow C_\infty \\ x &\longmapsto h_{i+1}(h_i^{-1}(x)) \quad \text{if } x \in C_i^\star. \end{aligned}$$

The map t moves a point up a layer in C_∞. We have an even action of $\mathbb{Z}$ on C_∞ through $z \cdot x := t^z(x)$, for $z \in \mathbb{Z}$ and $x \in C_\infty$. Since the orbits of $\mathbb{Z}$ are equal to the fibres of p_∞, we find that p_∞ is a $\mathbb{Z}$-covering. Since C_∞ is connected and C is locally path-connected, p_∞ is in fact a regular $\mathbb{Z}$-covering. Hence p_∞ is *the* infinite cyclic cover of the knot complement from definition 3.6.

The proof that p_n is the n-fold cyclic covering of the knot complement is analogous.

CHAPTER 4

Fibred knots

The focus of this chapter is on a specific type of knots, called *fibred knots*. As the name suggests, fibred knots have something to do with fibre bundles. The first section introduces fibre bundles and the pullback bundle and the second section relates this pullback bundle to homotopy. In the last section, we are ready to introduce fibred knots and prove the main theorem on fibred knots, which has to do with the commutator subgroup of the fundamental group of the knot complement.

1. Fibre bundles

A fibre bundle is defined as follows:

DEFINITION 4.1. A *fibre bundle* is a continuous surjection $\pi : E \to B$ with a fibre F satisfying the following property:

For every $p \in B$ there exists an open neighbourhood U and a homeomorphism $\varphi : \pi^{-1}(U) \xrightarrow{\sim} U \times F$ such that the following diagram commutes ($U \times F \to U$ is the projection map):

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl] \\ U & \end{tikzcd}$$

The duplet (U, φ) is called the *local trivialisation of* p. From the above property it follows that $\pi^{-1}(\{p\}) \cong F$ for all $p \in B$.

The space B is called the *base space* of the bundle, E the *total space*. The map π is called the *projection* of the bundle.

A fibre bundle $\pi : E \to B$ is called *trivial* if there is a homeomorphism $\psi : B \times F \xrightarrow{\sim} E$ such that $\pi \circ \psi$ is the projection onto B.

To aid in the readers understanding of fibre bundle, we give some examples of fibre bundles.

EXAMPLE 4.2. The cylinder and the Möbius strip are both fibre bundles over the circle with fibre $[0, 1]$, see figure 4.1. Note that the cylinder is a trivial fibre over the circle as the cylinder is homeomorphic with $S^1 \times [0, 1]$. On the contrary, the Möbius strip is a non-trivial fibre over the circle.

Another example of a fibre bundle that strikes our interest is a covering map.

PROPOSITION 4.3. *Let $p : Y \to X$ be a map. The following two statements are equivalent:*

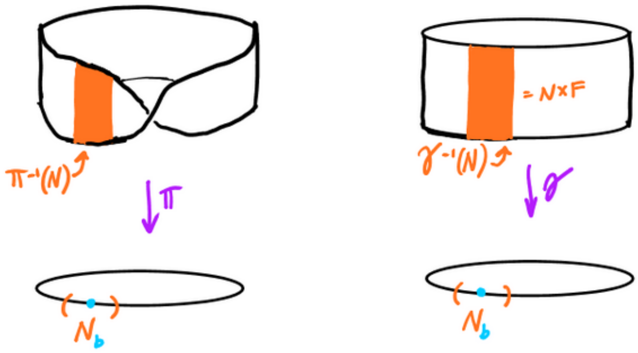


FIGURE 4.1. The Möbius strip and the cylinder as fibre bundles over the circle, from [7].

(1) *p is a covering map with homeomorphic fibres;*
(2) *p is a fibre bundle with a discrete fibre.*

In the above statement, discrete fibre means that the fibre has the discrete topology. Recall that a covering over a connected space has homeomorphic fibres, so this extra condition in (1) is usually satisfied.

PROOF OF PROPOSITION 4.3. (1) $\implies$ (2): Let $p : Y \to X$ be a covering with homeomorphic fibres and let $x \in X$. We prove that p is a fibre bundle with fibre $p^{-1}(\{x\})$.

Let $z \in X$, then there exists an open neighbourhood U of z such that

$$p^{-1}(U) = \bigsqcup_{y \in p^{-1}(\{z\})} V_y \quad \text{with } V_y \text{ open},$$

and the restriction $p|_{V_y} : V_y \to U$ is a homeomorphism. Let $\psi : p^{-1}(\{z\}) \xrightarrow{\sim} p^{-1}(\{x\})$ be a homeomorphism. Define the map

$$\begin{aligned} \varphi : p^{-1}(U) &\xrightarrow{\sim} U \times p^{-1}(\{x\}) \\ w &\longmapsto (p(w), \psi(y)) \quad \text{if } w \in V_y. \end{aligned}$$

Then φ is a homeomorphism such that the following diagram (with $U \times p^{-1}(\{x\}) \to U$ projection) commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p"'] & U \times p^{-1}(\{x\}) \arrow[dl] \\ U & \end{tikzcd}$$

We conclude that $p : Y \to X$ is a fibre bundle with fibre $p^{-1}(\{x\})$.

(2) $\implies$ (1): Let $p : Y \to X$ be a fibre bundle with discrete bundle F. Let $x \in X$, then there exists an open neighbourhood U of x and a homeomorphism

$\varphi : p^{-1}(U) \xrightarrow{\sim} U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p"'] & U \times F \arrow[dl] \\ U & \end{tikzcd}$$

Since F has the discrete topology, the subspace $U \times \{f\} \subset U \times F$ is open for all $f \in F$. Also note that the projection $U \times \{f\} \xrightarrow{\sim} U$ is a homeomorphism. We conclude that

$$p^{-1}(U) = \bigsqcup_{f \in F} \varphi^{-1}(U \times \{f\}),$$

so $p^{-1}(U)$ is a disjoint union of opens such that the restriction $p|_{\varphi^{-1}(U \times \{f\})}$ is a homeomorphism, hence p is a covering map with homeomorphic fibres (each fibre is homeomorphic to F). ☺

We don't need a lot of theory on fibre bundles in this thesis, but there is one more definition that we require, namely that of the pullback bundle.

DEFINITION 4.4 (pullback bundle). Let $\pi : E \to B$ be a fibre bundle with fibre F and let $f : B' \to B$ be a continuous map. Consider the space

$$f^\star E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}.$$

Let $\pi' : f^\star E \to B'$ be the projection onto the first coordinate, and $h : f^\star E \to E$ the projection onto the second coordinate. Then π' is a fibre bundle with fibre F and the following diagram commutes:

$$\begin{tikzcd} f^\star E \arrow[r, "h"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

The fibre bundle $\pi' : f^\star E \to B'$ (with fibre F) is called the *pullback bundle of π along f.*

If (U, φ) is a local trivialisation of (a point in) E, then $(f^{-1}(U), \psi)$ is a local trivialisation of (a point in) B'. Here ψ is given by

$$\begin{aligned} \psi : \pi'^{-1}(f^{-1}(U)) &\longrightarrow f^{-1}(U) \times F \\ (b', e) &\longmapsto (b', \mathrm{proj}_2(\varphi(e))), \end{aligned}$$

with proj_2 the projection onto the second coordinate. So $\pi' : f^\star E \to B'$ is in fact a fibre bundle with fibre F.

2. Homotopy invariance of the pullback bundle

The following theorem is the relation between fibre bundles and homotopy. This is called the homotopy-invariance of the lifting property or the homotopy-lifting property of the fibre bundle. Despite being of great importance to prove the main result of the theorem on fibred knots, this theorem of is not proven in this thesis.

THEOREM 4.5. *Let $\pi : E \to B$ be a fibre bundle. Let $f, g : B' \to B$ be two homotopic maps, then their pullbacks are homeomorphic:*

$$f^\star E \cong g^\star E.$$

PROOF. This theorem is proven in chapter 11 of [10]. ☺

The following corollary of this theorem is used to prove the main theorem on fibred knots.

COROLLARY 4.6. *A fibre bundle over a contractible space is trivial.*

PROOF. Let $\pi : E \to B$ be a fibre bundle with B contractible and with fibre F. Let $\mathrm{id}_B : B \to B$ be the identity and $f : B \to B$ a constant map.
By the definition of the pullback we find that

$$\begin{aligned} \mathrm{id}_B^\star E &= \{(b, e) \in B \times E \mid \mathrm{id}_B(b) = \pi(e)\} \\ &= \bigsqcup_{b \in B} \{b\} \times \pi^{-1}(\{b\}) \\ &\cong E. \end{aligned}$$

The homeomorphism in the last step is

$$\begin{aligned} \bigsqcup_{b \in B} \{b\} \times \pi^{-1}(\{b\}) &\longrightarrow E \\ (b, e) &\longmapsto e. \end{aligned}$$

Furthermore, we find that

$$\begin{aligned} f^\star E &= \{(b, e) \in B \times E \mid f(b) = \pi(E)\} \\ &= B \times \pi^{-1}(f(B)) \\ &\cong B \times F. \end{aligned}$$

By theorem 4.5 we find that there is a homeomorphism $\psi : B \times F \xrightarrow{\sim} \mathrm{id}_B^\star E$ and that the following diagram commutes:

$$\begin{tikzcd} B \times F \arrow[r, "\mathrm{id}'_B \circ \psi"] \arrow[d] & E \arrow[d, "\pi"] \\ B \arrow[r, "\mathrm{id}_B"] & B. \end{tikzcd}$$

Therefore we conclude that the fibre bundle π is trivial. ☺

3. Fibred knots and the commutator subgroup

We are ready to define fibred knots.

DEFINITION 4.7. Let k be a knot with complement C and S a Seifert surface of k of genus g. The knot k is a *fibred knot* if there is a fibre bundle $\pi : C \to S^1$ with fibre S.

It turns out to be difficult to prove that a knot is fibred. The simplest example of a fibred knot is the unknot.

Proposition 4.8. *The unknot is a fibred knot.*

Proof. We construct a fibre bundle. However, we first require to view the unknot in an alternative way. Recall that S^3 is the one point compactification of $\mathbb{R}^3$. Furthermore, recall that there is a homeomorphism $S^1 \xrightarrow{\sim} \mathbb{R} \cup \{\infty\}$ called the stereographic projection. The stereographic projection maps every point x of S^1 to the point on the real line that intersects the line through x and the north pole of S^1. The north pole is mapped to ∞, see figure 4.2.

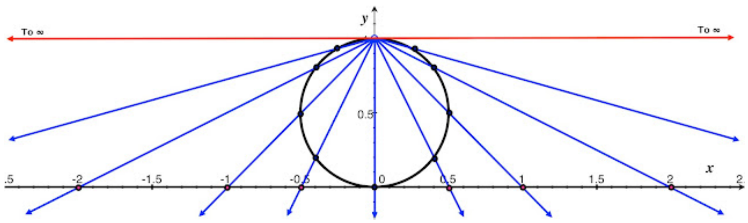


Figure 4.2. The stereographic projection of S^1.

Therefore there is a knot $S^1 \overset{k}{\hookrightarrow} S^3$ equivalent tot the unknot such that

$$\text{Im } k = \{(0,0,z) \mid z \in \mathbb{R}\} \cup \{\infty\} (\cong S^1).$$

Therefore the knot complement $C = \mathcal{S}^3 \setminus \text{Im } k$ is $\mathbb{R}^3$ minus the z-axis.

A Seifert surface of this unknot is

$$S = \{(x,0,z) \mid x \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}\} \subset S^3.$$

The space C is homeomorphic with $\mathbb{R} \times (\mathbb{C} \setminus \{0\})$. Elements of C are written as elements of $\mathbb{R} \times (\mathbb{C} \setminus \{0\})$ from now on. By writing $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$, the following fibre bundle is constructed:

$$\begin{aligned} \pi : C &\longrightarrow S^1 \\ (x,z) &\longmapsto \frac{z}{\|z\|}. \end{aligned}$$

In addition, there is a homeomorphism

$$\begin{aligned} \varphi : C &\xrightarrow{\sim} S^1 \times S \\ (x,z) &\longmapsto \left(\frac{z}{\|z\|}, (\|z\|, 0, x) \right). \end{aligned}$$

This homeomorphism leads to a global trivialisation (S^1, φ) because the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(S^1) \arrow[r, "\varphi"] \arrow[d, "\pi"] & S^1 \times S \arrow[dl] \\ S^1 & \end{tikzcd}$$

☺

In the previous section, we prepared the proof of the main theorem on fibred knots, presented below.

THEOREM 4.9. *Let k be a fibred knot with complement C and as fibre a Seifert surface of genus g. Then*

$$[\pi_1(C), \pi_1(C)] \cong \mathcal{F}_{2g}.$$

PROOF. Let k be a fibred knot with complement C. Let S be a Seifert surface of genus g and $\pi : C \to S^1$ a fibre bundle with fibre S.

Consider the universal covering $u : \mathbb{R} \to S^1$ of S^1 given by the quotient $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Since $\mathbb{Z}$ acts even on $\mathbb{R}$, the universal cover u is a regular $\mathbb{Z}$-covering. By proposition 4.3 u is a fibre bundle.

By taking the pullback along both of these fibre bundles, we obtain the spaces

$$u^\star C = \{(r, c) \in \mathbb{R} \times C \mid u(r) = \pi(c)\}$$

and

$$\pi^\star \mathbb{R} = \{(c, r) \in C \times \mathbb{R} \mid \pi(c) = u(r)\}.$$

It is clear that $u^\star C \cong \pi^\star \mathbb{R}$. The pullback fibre $u^\star C \to \mathbb{R}$ is a fibre bundle with fibre S. Since $\mathbb{R}$ is contractible, the bundle is trivial by corollary 4.6. So $u^\star C \cong \mathbb{R} \times S$.

In addition, the pullback fibre $\pi^\star \mathbb{R} \to C$ is a $\mathbb{Z}$-covering by proposition 4.3 as $\mathbb{Z}$ has the discrete topology. Since $\pi^\star \mathbb{R} \cong \mathbb{R} \times S$ is connected, the fibre is a regular $\mathbb{Z}$-covering of C. This means that $\pi^\star \mathbb{R}$ is the infinite cyclic cover space of C.

Consequently, the infinite cyclic cover is homeomorphic with $\mathbb{R} \times S$, so homotopy-equivalent with S. By proposition 3.5 and corollary 2.9 we find

$$[\pi_1(C), \pi_1(C)] \cong \mathcal{F}_{2g},$$

concluding the proof. ☺

A remarkable fact about fibred knots, is that the previous theorem has a converse. Said converse is not proven in this thesis, but a proof can be found in [9]. Below you can find the theorem in its stronger form.

THEOREM 4.10. *Let k be a knot of genus g with complement C. The following are equivalent:*

1. *The knot k is fibred;*
2. *The commutator subgroup $[\pi_1(C), \pi_1(C)]$ is finitely generated;*
3. *The commutator subgroup $[\pi_1(C), \pi_1(C)]$ is isomorphic with $\mathcal{F}_{2g}$.*

CHAPTER 5

A different way to study the knot complement

In this chapter, we present a space called the *knot tube* that is homotopy-equivalent to the knot complement. Therefore we can study the fundamental group of the knot tube, rather than that of the knot complement. Computing this fundamental group is still a difficult task, but studying the infinite cyclic covering of the knot tube is significantly easier.

In the first two sections, we construct the knot tube. In the following two sections, we construct the infinite cyclic covering of the knot tube and compute its first homology group.

1. The metro station

Before constructing the knot tube, we need to define the following space, which is used to construct the knot tube in the following section.

DEFINITION 5.1. Let X_1, X_2 and X_3 be three homeomorphic copies of the square $[0,1] \times [0,1]$ with homeomorphisms $f_i : [0,1] \times [0,1] \xrightarrow{\sim} X_i$ for $i \in \{1,2,3\}$.

The *metro station* is obtained from the disjoint union $X_1 \sqcup X_2 \sqcup X_3$ by identifying the corresponding points in the following sets:

(1) Identify $f_1([0,1] \times \{0\})$ with $f_2([0,1] \times \{0\}))$ and $f_1([0,1] \times \{1\})$ with $f_2([0,1] \times \{1\})$;
(2) identify $f_3(\{0\} \times [0,1])$ with $f_2(\{0\} \times [0,1])$ and $f_3(\{1\} \times [0,1])$ with $f_2(\{1\} \times [0,1])$.

A drawing of the metro station can be found in figure 5.1.

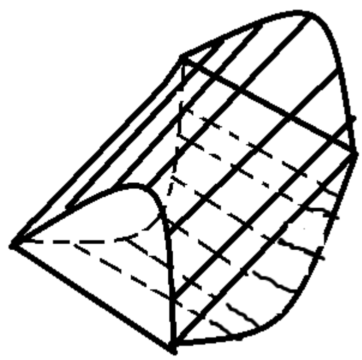


FIGURE 5.1. The metro station. Only the boundary is drawn of the middle square X_2.

By means of Mayer-Vietoris (theorem 3.1), the first homology group of the metro station can be found.

Proposition 5.2. *Let M be the metro station, then $H_1(M) \cong \mathbb{Z}^2$.*

Proof. Let M be the metro station as constructed in definition 5.1. Since M is path connected we have $H_0(M) \cong \mathbb{Z}$. Let

$$U = M \setminus (f_2([0,1] \times \{0\}) \cup f_2(\{0\} \times [0,1])$$

and

$$V = M \setminus (f_2([0,1] \times \{1\}) \cup f_2(\{1\} \times [0,1])).$$

Using figure 5.1 it can be verified that U and V are contractible, and that $U \cap V$ is homotopy-equivalent to the discrete space with three points. Applying Mayer-Vietoris with this decomposition yields the following exact sequence:

$$0 \longrightarrow H_1(M) \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

From this exact sequence we deduce that $H_1(M) \cong \mathbb{Z}^2$. ☺

2. The knot complement as a tube

In this section, we display a new way of viewing the knot complement. Under homotopy-equivalence, many parts of the knot complement can be retracted. The resulting space, called the knot tube, is obtained by gluing together metro stations from definition 5.1. Before constructing this space, we need two definitions from graph theory, that the reader may be unfamiliar with.

Definition 5.3. Given a compact graph embedded in a 2-manifold, a *face* of the graph is a connected component of the complement of the graph.

Definition 5.4. Consider a compact and connected graph that is embedded in a 2-manifold. The *dual graph* of this graph is the graph that has a vertex in each face and an edge between every pair of vertices of which the connected components share an edge. See figure 5.2.

We are now ready to construct the knot tube.

Let k be a knot that passes through infinity (this means that $\infty \in k$ when viewing S^3 as $\mathbb{R}^3 \cup \{\infty\}$). It is intuitive that any knot is equivalent to such a knot. While constructing the space, we use the trefoil knot as example. The trefoil knot can be viewed as a knot that passes through infinity as in figure 5.3a. There is a natural way to view a knot as a graph in S^2, by seeing each crossing as a vertex and the lines connecting the crossings as edges. Consider the dual graph of the knot graph, as seen in figure 5.3b. If k has crossings $c_1, c_2, \ldots, c_n$, then the dual graph has $n+1$ faces. With the exception of the outer face, each face contains one of the crossings of k. Furthermore, the four ends of these crossings each go to one of the four edges of the face. At each

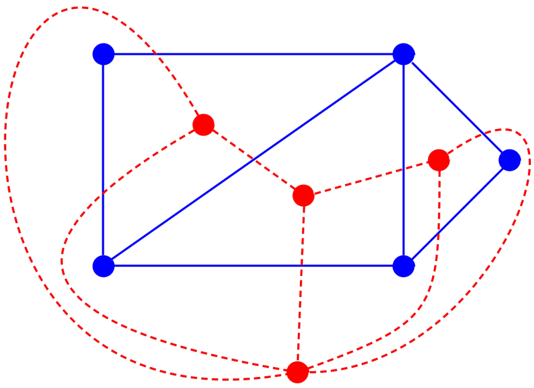
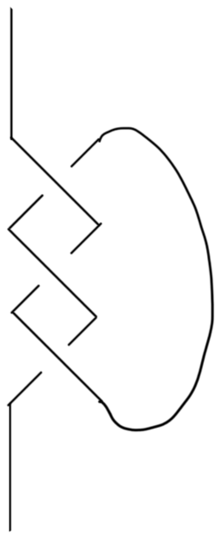
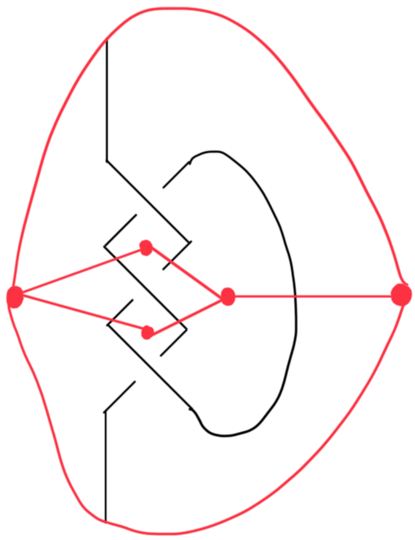


FIGURE 5.2. The red graph is the dual graph of the blue graph.



(A) The trefoil knot as a knot that passes through infinity.



(B) The dual graph of the trefoil graph, given in red.

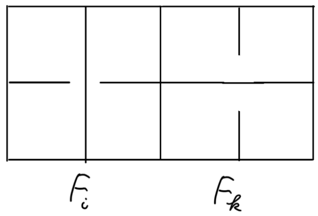
FIGURE 5.3. A different way to view the trefoil knot.

crossing c_i, $i \in \{1, 2, \ldots, n\}$, let F_i be the face of the dual graph containing c_i. This face is homeomorphic with the square $[0, 1] \times [0, 1]$.

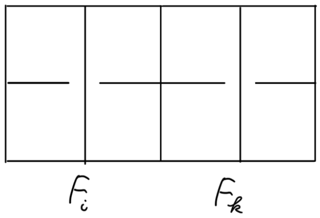
Let M be the metro station from definition 5.1 constructed by gluing together the squares X_1, X_2 and X_3 with homeomorphisms $f_j : [0, 1]\times[0, 1] \xrightarrow{\sim} X_j$ ($j \in \{1, 2, 3\}$). Let $M_1, M_2, \ldots, M_n$ be homeomorphic copies of the metro station M with homeomorphisms $h_i : M_i \xrightarrow{\sim} M$.

We now make identifications in the disjoint union $\bigsqcup_{i=1}^{n} M_i$. For every pair of faces F_i and F_k that are connected to each other (i.e. next to each other), identify the corresponding points in M_i and M_k given in figure 5.4 in a natural way.

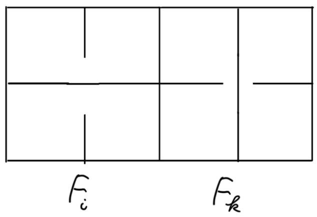
The space obtained from this identification process is called *the knot tube*. Making the necessary identifications to the metro stations of the trefoil knot



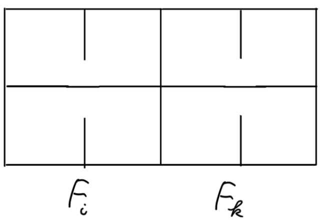
Identify $h_i(f_2([0,1] \times \{1\})$ with $h_k(f_1(\{0\} \times [0,1]);$ identify $h_i(f_3([0,1] \times \{1\})$ with $h_k(f_2(\{0\} \times [0,1]).$



Identify $h_i(f_2([0,1] \times \{1\})$ with $h_k(f_2([0,1] \times \{0\});$ identify $h_i(f_3([0,1] \times \{1\})$ with $h_k(f_3([0,1] \times \{0\}).$



Identify $h_i(f_1(\{1\} \times [0,1])$ with $h_k(f_2([0,1] \times \{0\});$ identify $h_i(f_2(\{1\} \times [0,1])$ with $h_k(f_3([0,1] \times \{0\}).$



Identify $h_i(f_1(\{1\} \times [0,1])$ with $h_k(f_2(\{1\} \times [0,1]);$ identify $h_i(f_2(\{0\} \times [0,1])$ with $h_k(f_3(\{0\} \times [0,1]).$

FIGURE 5.4. The identification process of the metro stations.

yields the space that can be found in figure 5.5. In this figure, all the adjacent metro stations are connected by the identifications shown in figure 5.4. Furthermore, the trefoil knot is still drawn in this figure as a visual aid, but the knot itself is not a part of the knot tube. It should also be noted that the top and bottom sides of the top and bottom metro stations are also connected together, as their corresponding crossings are connected through the point at infinity.

3. Homotopy-equivalence of the knot complement and the knot tube

As mentioned in previous sections, the knot tube is homotopy-equivalent to the knot complement. This section seeks to provide an explicit homotopy-equivalence between these two spaces.

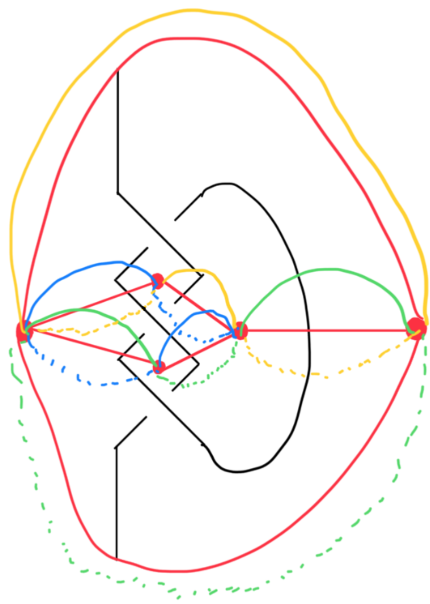


FIGURE 5.5. The knot tube of the trefoil knot.

However, before constructing the homotopy-equivalence, we introduce the following theorem that is used to create a decomposition of the knot complement.

THEOREM 5.5 (Alexander-Schoenflies). *Let $i : S^2 \hookrightarrow S^3$ be a piecewise linear embedding. Then there are closed balls B_1 and B_2 such that*

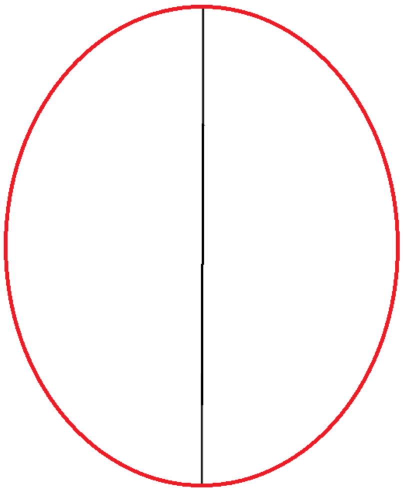
$$S^3 = B_1 \cup B_2 \quad \textit{and} \quad i(S^2) = B_1 \cap B_2 = \partial B_1 = \partial B_2.$$

PROOF. A proof of this theorem can be found in [1]. ☺

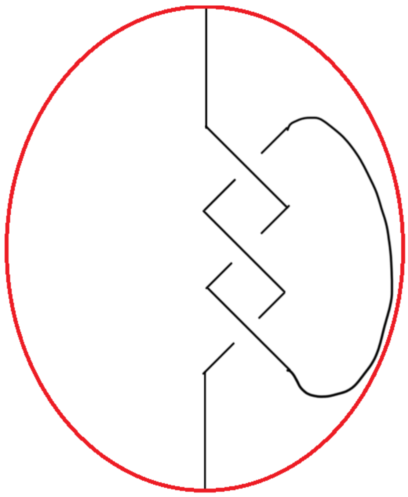
The term *piecewise linear* may be unfamiliar to the reader. This thesis does not provide an explanation of this, but a good explanation can be found in [8]. Nevertheless, the requirement that the embedding $S^2 \hookrightarrow S^3$ is piecewise linear does not hamper any of the arguments presented here.

The Alexander-Schoenflies theorem provides us with a new way to view the 3-sphere, namely as two closed balls whose boundaries are identified with each other. This decomposition of S^3 is used to construct the homotopy-equivalence.

Let k be a knot that passes through infinity and let $C = S^3 \setminus k$ be its complement. As before, the trefoil is used as example and passes through infinity as shown in figure 5.3a. The knot k can be separated into two parts: the knotted part which has all the crossings in it, and the line through infinity that connects the knotted part to itself. Let B_1 be a closed ball with a line taken out of it; let $B_2 \subset C$ be a closed ball around the knotted part of k. See figure 5.6. The knot complement C can be constructed by identifying the the boundaries of the closed balls at the corresponding points, making sure that the knotted parts are connected together.



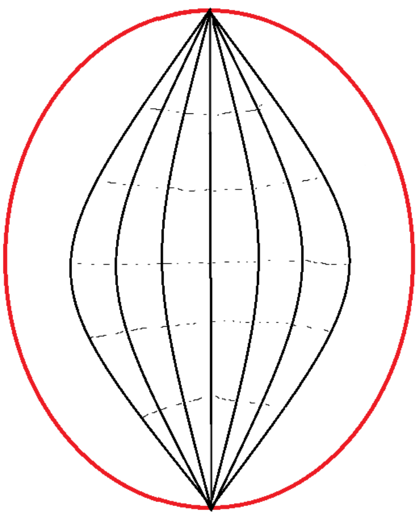
(A) The closed ball with a line taken out, denoted B_1.



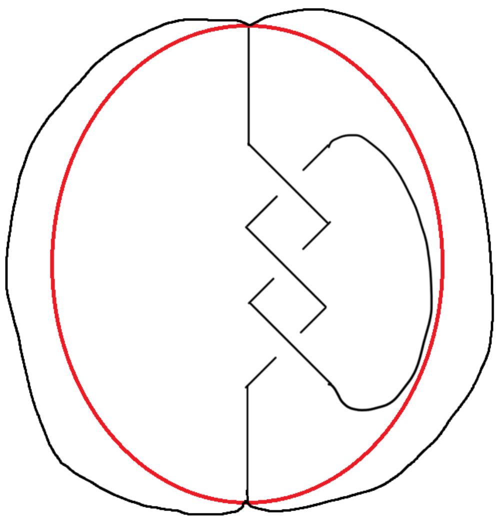
(B) The closed ball around the knotted part of k, denoted B_2.

FIGURE 5.6. The closed balls B_1 and B_2.

The missing line from B_1 can be 'thickened up' under homotopy-equivalence to attain the space in figure 5.7a. As a result, gluing together B_1 and B_2 yields the space given in figure 5.7b. This space is homotopy-equivalent, even homeomorphic, with B_2.



(A) The thickened up line in B_1.



(B) B_1 and B_2 glued together.

FIGURE 5.7. The thickening and identification.

In summary, the knot complement is homotopy-equivalent to B_2. It's time to introduce the dual graph used to construct the knot tube. When constructing the dual graph, we make sure that the vertices corresponding to the outer faces of the graph are placed on the boundary of B_2 and that the edges connecting these vertices are also on the boundary of B_2. In this manner, the boundary of the ball in figure 5.6b corresponds with the outer edges of the dual graph in figure 5.3b.

To contract B_2 to the knot tube, first recall that the punctured disc $D^2 \setminus \{*\}$ can be retracted to the circle S^1. At each of the faces of the dual graph, B_2 is a closed ball with two lines taken out of it that together form a crossing, see 5.8. By splitting this ball up into an upper and lower half, separated

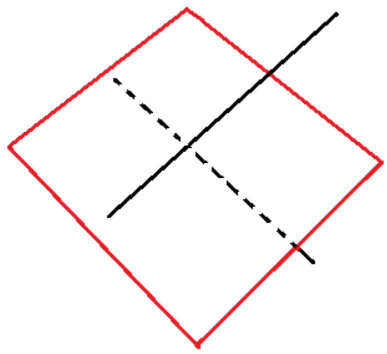


FIGURE 5.8. B_2 in a neighbourhood of each face.

by the face of the dual graph, we obtain two spaces homeomorphic with the punctured solid cylinder $[0,1] \times D^2 \setminus \{*\}$. Contracting both halves to cylinders (so homeomorphic with $[0,1] \times S^1$) gives us the upper and lower part of the metro station from figure 5.1. Applying this process to all the faces of the dual graph shows that the B_2 is homotopy-equivalent to the knot tube, completing the proof.

4. The cyclic covering of the metro station

The metro station M admits an infinite cyclic covering. It can be constructed as follows:

For all $i \in \mathbb{Z}$, let X_i, Y_i and Z_i be homeomorphic copies of the square $[0,1] \times [0,1]$ with homeomorphisms

$$f_i : [0,1] \times [0,1] \xrightarrow{\sim} X_i$$
$$g_i : [0,1] \times [0,1] \xrightarrow{\sim} Y_i$$
$$h_i : [0,1] \times [0,1] \xrightarrow{\sim} Z_i$$

respectively. The infinite cyclic covering space C is obtained from the disjoint union $(\bigsqcup_{i\in\mathbb{Z}} X_i) \sqcup (\bigsqcup_{i\in\mathbb{Z}} Y_i) \sqcup (\bigsqcup_{i\in\mathbb{Z}} Z_i)$ by identifying the corresponding points in the following sets:

(1) Identify $g_i([0,1] \times \{0\})$ with $f_i([0,1] \times \{0\})$ and $g_i([0,1] \times \{1\})$ with $f_{i+1}([0,1] \times \{1\})$;
(2) identify $h_i(\{0\} \times [0,1])$ with $f_i(\{0\} \times [0,1])$ and $h_i(\{1\} \times [0,1])$ with $f_{i+1}(\{1\} \times [0,1])$.

Let $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}_3$ be the squares from definition 5.1 with homeomorphisms $\tilde{f}_j : [0,1] \times [0,1] \xrightarrow{\sim} \tilde{X}_j$ $(j \in \{1,2,3\})$. The covering map $p : C \to M$ is given by

$$
\begin{aligned}
p : C &\longrightarrow M \\
x &\longmapsto \begin{cases} \tilde{f}_2(f_i^{-1}(x)) & \text{if } x \in X_i \\ \tilde{f}_1(g_i^{-1}(x)) & \text{if } x \in Y_i \\ \tilde{f}_3(h_i^{-1}(x)) & \text{if } x \in Z_i. \end{cases}
\end{aligned}
$$

Consider the map

$$
\begin{aligned}
t : C &\xrightarrow{\sim} C \\
x &\longmapsto \begin{cases} f_{i+1}(f_i^{-1}(x)) & \text{if } x \in X_i \\ g_{i+1}(g_i^{-1}(x)) & \text{if } x \in Y_i \\ h_{i+1}(h_i^{-1}(x)) & \text{if } x \in Z_i. \end{cases}
\end{aligned}
$$

The map t sends elements of X_i, Y_i and Z_i to the corresponding elements in X_{i+1}, Y_{i+1} and Z_{i+1} respectively. This induces a natural even $\mathbb{Z}$-action on C that is compatible with p. Therefore p is a $\mathbb{Z}$-covering, as was required.

The infinite cyclic cover is a double staircase, at each level X_i, you can 'move up' to X_{i+1} via Y_i or Z_i. Stepping down is done similarly. These steps up and down are the lifts of the two loops in the metro station.

One can think of the infinite cyclic covering of the metro stations as 'folding open' the upper and lower squares of infinitely many metro stations and connecting them together in such a way to form a staircase in two directions. To visualise this, one can consider two staircases as given on the left in figure

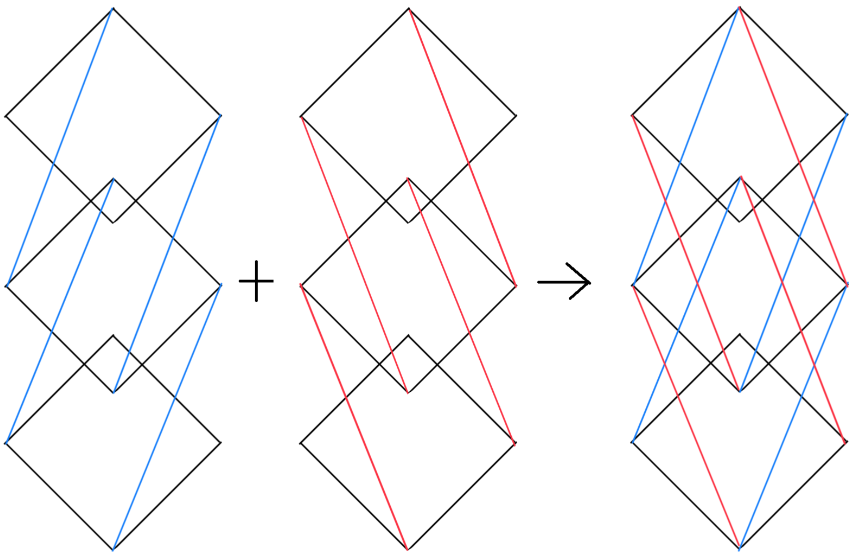


FIGURE 5.9. The infinite cyclic cover as two merged staircases

5.9 and then merge them together by identifying the corresponding points in the black squares. This way we obtain the space given on the right in figure 5.9. Only the boundary of the covering is drawn in this figure to improve clarity. The squares X_i are given in black, Y_i in red and Z_i in blue. It appears as if the blue and red squares intersect, but topologically they don't.

Proposition 5.6. *Let C be the infinite cyclic cover of the metro station, then*

$$H_1(C) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}.$$

Proof. This proposition is proven using Mayer-Vietoris. Let U be a small open neighbourhood of $C \setminus (\bigcup_{i\in\mathbb{Z}} Y_i)$ and V a small open neighbourhood of $C \setminus (\bigcup_{i\in\mathbb{Z}} Z_i)$, such that U and V are homotopy-equivalent to $C \setminus (\bigcup_{i\in\mathbb{Z}} Y_i)$ and $C \setminus (\bigcup_{i\in\mathbb{Z}} Z_i)$ respectively. Both of these spaces are infinite (single) staircases, thus contractible. The intersection $U \cap V$ is homotopy-equivalent to $\bigcup_{i\in\mathbb{Z}} X_i$, which in turn is homotopy-equivalent to the countable discrete space D. Using that $H_0(D) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$, Mayer-Vietoris provides the exact sequence

$$0 \longrightarrow H_1(C) \longrightarrow \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

From this sequence we conclude that $H_1(C) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$. ☺

5. The cyclic covering of the knot tube

Since the knot tube is homotopy-equivalent with the knot complement, the knot tube has a unique (regular) infinite cyclic covering. The goal of this chapter is to construct this space and find its first homology group. This done by *gluing G-coverings*. The result we are proving is the following:

Theorem 5.7. *Let C be the infinite cyclic cover of a knot tube, then*

$$H_1(C) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}.$$

Recall from algebraic topology that if you glue together two spaces with G-coverings, then this induces a natural gluing map on the covering spaces such that we obtain a new G-covering of the glued space.

This *gluing of G-coverings* does require the glued spaces to both be *connected*, *locally path-connected*, and *semi-locally simply connected*. In addition, the G-covering needs to be *regular*. More information in gluing G-coverings can be found in chapter 14 of [4].

Another tool we require to prove this theorem is the following lemma that may be familiar to the reader.

Lemma 5.8. *Let the following sequence of five groups be exact:*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E.$$

Then this induces a short exact sequence of groups:

$$0 \longrightarrow B/\mathrm{Im}\, f \xrightarrow{g} C \xrightarrow{h} \ker i \longrightarrow 0.$$

Proof. The proof of this lemma is a straightforward application of the definition of an exact sequence. ☺

The knot tube is constructed by gluing together metro stations. The metro station and its infinite cyclic covering satisfy the requirements to apply the gluing of G-coverings, so we can glue together the infinite cyclic covering of the metro station, to obtain the infinite cyclic covering of the knot tube.

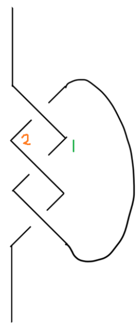


FIGURE 5.10. The trefoil knot with two numbered edges.

The infinite cyclic covering of the knot tube is constructed in steps. The metro stations are connected one edge at a time, and after each step we keep track of what happens to the infinite cyclic cover and its homology. To compute the first homology group we use Mayer-Vietoris (theorem 3.1).

To further clarify the steps for the reader, the infinite cyclic cover of the trefoil knot is constructed here. In particular, the trefoil knot as a knot that passes through infinity, as shown in figure 5.10. The construction is analogous for all other knots, but the process is easier to visualise when using an example. To further clarify the process, a drawing is presented of the cyclic cover at each step. Only two full layers of the cyclic cover are drawn to make it easier to visualise the identifications.

Two of the ‘edges’ between the crossing in figure 5.10 are numbered. Our proof commences by making the identifications required to connect edge 1. The resulting space is denoted C_1. The required identifications are shown in figure 5.11. Note that the bottom crossing is displayed on the left in figure 5.11 and the top crossing on the right. The first homology group of C_1 can easily be computed using Mayer-Vietoris. Let U be a small open neighbourhood around the left cover (meaning that it includes a small open around the identification line in the right cover), and let V be an open neighbourhood around the right cover. Then U and V are each homotopy-equivalent to the cyclic cover of the metro station, so

$$H_1(U) \oplus H_1(V) \cong \left(\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\right) \oplus \left(\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\right) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}.$$

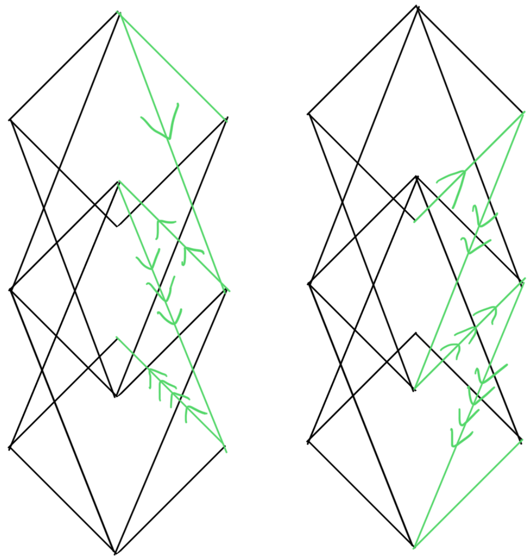


FIGURE 5.11. The glued cyclic covers, denoted C_1.

Furthermore, the intersection $U \cap V$ is homeomorphic with $\mathbb{R} \times (0,1)$ thus contractible. Applying Mayer-Vietoris with these opens yields the exact sequence

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \longrightarrow H_1(C_1) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

from which we deduce that $H_1(C_1) \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$.

Secondly, the identifications required for the edge labelled 2 in figure 5.10 are made. The resulting space is denoted C_2. These identifications can be seen in figure 5.12. Again, the first homology group of C_2 can be computed using Mayer-Vietoris. Let U be C_2 minus the identification line (this is open as the identification line is closed), and let V be a small open neighbourhood around the identification line. Then U is homotopy-equivalent to C_1 and V is homeomorphic with $\mathbb{R} \times (0,1)$ hence contractible. The intersection $U \cap V$ is homeomorphic with $\mathbb{R} \times (0,1) \sqcup \mathbb{R} \times (0,1)$, so $U \cap V$ is homotopy-equivalent with the discrete two-point space. The Mayer-Vietoris sequence of this decomposition is as follows:

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \longrightarrow H_1(C_3) \longrightarrow \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^2 \xrightarrow{g} \mathbb{Z} \longrightarrow 0.$$

By exactness, we get Im $g \cong \mathbb{Z}$ and hence $\ker g \cong \mathbb{Z}$ (by the homomorphism theorem). Also by exactness, deduce that Im $f \cong \mathbb{Z}$ and therefore $\ker f \cong \mathbb{Z}$. Now apply lemma 5.8 to the first five groups of the sequence to obtain the

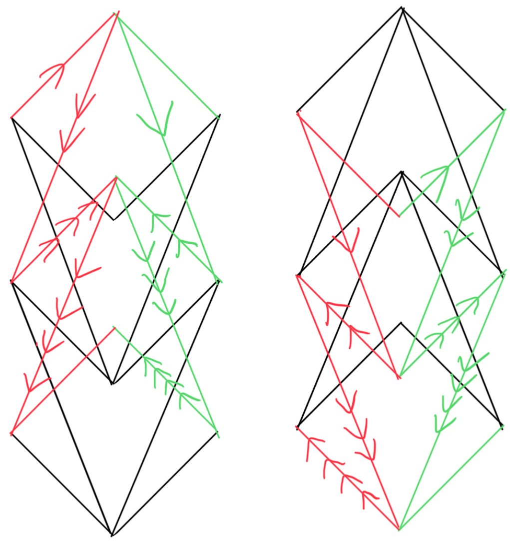


FIGURE 5.12. The glued cyclic covers, denoted C_2.

following short exact sequence:

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \longrightarrow H_1(C_3) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

All the groups in this sequence are abelian, and $\mathbb{Z}$ is a free abelian group. Therefore the sequence splits, and hence $H_1(C_3) \cong \left(\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}\right) \oplus \mathbb{Z} \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$.

For the remaining four edges of the trefoil knot, the proof that the first homology group remains $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$ is analogous. We are either connecting two disjoint covers, in which case we can use the proof used for edge 1 above; or there is a connecting of a cover to itself, in which case the proof used for edge 2 can be used.

In fact, this proof can be applied to any knot. Since knots are assumed to be tame, there is a finite number of identifications to be made and after every one of them the proofs above can be used to show that the first homology group is still $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$.

6. The homology of the cyclic cover as $\mathbb{Z}[t^{\pm}]$-module

In the previous section, it was shown that the first homology group of the infinite cyclic cover of the knot tube is $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}$. Despite being an invariant of the knot, this is a trivial invariant and therefore not very interesting. However, this group does have its uses in knot theory. By turning the first homology group into a module, a well-known invariant called the *Alexander polynomial* can be created. This invariant is not studied in this thesis, but a comprehensive overview can be found in [2]. Instead, we look at how the first homology group could be turned into such a module.

Let C_∞ be the infinite cyclic covering of the knot tube given in the previous section. Furthermore, consider the homeomorphism $t : C_\infty \xrightarrow{\sim} C_\infty$ that sends an element to the corresponding element one layer higher in the cyclic covering. Furthermore, consider the ring of Laurent polynomials $\mathbb{Z}[t^{\pm}]$, which is isomorphic with the polynomial ring $\mathbb{Z}[x, y]/(xy - 1)$. Then the induced isomorphism $t_\star : H_1(C_\infty) \xrightarrow{\sim} H_1(C_\infty)$ provides a natural way to make $H_1(C_\infty)$ a $\mathbb{Z}[t^\pm]$-module.

When using the infinite cyclic cover as constructed in section 4 of chapter 3, it is difficult to say something about $t_\star$. The idea behind the infinite cyclic cover constructed in the previous section, is that it is easier to see how $t_\star$ moves the generators of $H_1(C_\infty)$. However, despite the infinite cyclic cover being easier to visualise, it is still difficult to say what happens to any of the generators when applying $t_\star$.

Consequently, the knot tube has not yet proven itself to be very useful. There is also no existing literature on the knot tube, so no inspiration can be taken from that. In conclusion, the knot tube would have to be studied more thoroughly in order to make sense of what happens when applying $t_\star$.

Conclusion

This thesis covers a wide variety of topics in knot theory. The main results of the thesis are briefly summarised in this conclusion.

To start of, a constructive proof of the existence of Seifert surfaces is given. Then, we introduce the concept of the genus of a surface and use this to compute the fundamental group of a Seifert surface.

Secondly, a computation of the first homology group of the knot complement leads to a proof that the knot complement has a unique infinite cyclic covering. This infinite cyclic covering is then constructed by cutting the knot complement along a Seifert surface and stacking infinitely many copies of this space on top of each other.

Thirdly, this thesis contains an introduction to fibre bundles and basic theorems concerning them. Then this is used to find the commutator subgroup of the fundamental group of the knot complement in case the knot omits a fibre bundle to the circle.

To finish of the thesis, we provide a new way to look at the knot complement, up to homotopy-equivalence. This space is called the knot tube and is constructed by gluing together so-called metro stations. By constructing the infinite cyclic covering of the knot complement, we can compute the first homology group of the infinite cyclic coverings. This turns out to be $\bigoplus_{i\in\mathbb{Z}}\mathbb{Z}$ for all knots and is therefore a trivial invariant.

In the future, more research could be done on the knot tube. To be specific, more attempts could be made at properly describing the map $t_\star : H_1(C_\infty) \to H_1(C_\infty)$ so that in turn $H_1(C\infty)$ could be described as a $\mathbb{Z}[t^{\pm}]$-module. In theory, this should lead to a different way to find the Alexander polynomial of a knot. This new method may be useful when computing the Alexander polynomial of large knots.

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