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A lower bound on the concentration of $\chi(G_{n,1/2})$

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Abstract

Graph theory is a field within discrete mathematics, which is concerned with graphs and their properties. A graph whose set of edges is determined randomly is called a random graph. Such graphs are researched in the field of random graph theory, which lies at an intersection of graph theory and probability theory. The central concept of this paper is the chromatic number. The chromatic number of a graph is the minimum number of colors which one needs to use to color the vertices of the graph in such a way that whenever a pair of vertices is connected by an edge, those two vertices are assigned different colors. Clearly, in case of a random graph, its chromatic number is a random variable. This thesis explains Annika Heckel's article "Non-concentration of the chromatic number of a random graph". The article provides a revolutionary result on the lower bound on the concentration of the chromatic number of the binomial random graph $G_{n,1/2}$. While the majority of research has been focused on finding an upper bound on the length of the interval where the chromatic number lies with high probability, Heckel is the first to provide a non-trivial lower bound. In this thesis, we provide an introduction into graph theory and random graph theory, followed by the history of research into the bounds on the chromatic number of random graphs. Finally, we walk the reader through Annika Heckel's proof and raise some open questions related to the chromatic number of random graphs.

Contents

1 Introduction

This paper is based on the article "Non-concentration of the chromatic number of a random graph" by Annika Heckel, published in 2021 in *Journal of the American Mathematical Society* [14]. We explain the results of that paper in a way which is understandable for bachelor students of mathematics in their final year.

Graph theory is a field within discrete mathematics, which is concerned with graphs: objects, which are made up of a set of vertices and a set of edges, which connect (some of) the vertices. It also explores properties of those graphs. The property we are especially interested in is the chromatic number. It is the smallest number of colors which need to be used to color the vertices of a graph in such a way that whenever two vertices are connected by an edge, those vertices are assigned different colors. Random graph theory is a particular area, which focuses on graphs whose set of edges is determined by a probability distribution. Thus, the properties such as the chromatic number are random variables and except for trivial cases, we do not know the outcomes. We can, however, analyze their probability distributions and determine probability bounds on them.

Annika Heckel's paper focuses on $G_{n,1/2}$ - a random graph on n vertices, where the probability that there is an edge between any pair of vertices is equal to $1/2$. She proves that there exists no sequence of intervals of length less than $n^{1/4-\epsilon}$, such that they contain the chromatic number of $G_{n,1/2}$ *with high probability.* In this paper, we explain what this statement means, provide sufficient (random) graph theory background and explain Heckel's ingenious argument.

2 Fundamentals

2.1 Basic definitions of graph theory

In this section we define some basic concepts of graph theory. The definitions are based on section I.1. in [3].

A graph G is an ordered pair of disjoint sets (V, E), where V is the set of vertices of G, and E is the set of edges. The set of vertices of graph G is commonly denoted with $V(G)$, and the set of edges with $E(G)$. Their cardinalities are denoted with $v(G)$ and $e(G)$, respectively. If it is clear from context which graph is being referred to, we will write V for $V(G)$ and E for $E(G)$ for simplicity. Let us notice that $E \subset \binom{V}{2}$, where $\binom{V}{2}$ is the set of all unordered pairs of vertices on the vertex set V. Clearly, the cardinality of $\binom{V}{2}$ is $\binom{v(G)}{2} =: N$.

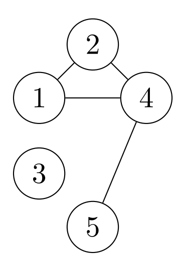

Figure 1: Graph G with $V = \{1, \dots, 5\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{4, 5\}\}$.

If for some $x, y \in V$ we have $\{x, y\} \in E$, then the vertices x and y are joined by an edge. If there exists an edge between two vertices, the vertices are called adjacent.

Subgraphs of G are formed by taking subsets of V and E. Namely, $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A particular type of a subgraph is an induced subgraph. G' is the induced subgraph of G on some set $V' \subset V$ if G' contains all edges of G that join vertices of V'. The induced subgraph of G on vertex set V' is denoted with $G[V']$.

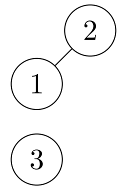

Figure 2: Induced subgraph of G on the subset $V' = \{1, 2, 3\}$.

A graph is called complete if all of its vertices are adjacent, i.e., if we have $\{x, y\} \in E$ whenever $x, y \in V$. Conversely, a graph is called empty if none of its vertices are adjacent, i.e., $E = \varnothing$. A complete subgraph is called a clique, while the vertex set of an empty subgraph is called an independent set (in some literature, for example in [17] , it is called a stable set). In figure 1, $\{1, 3, 5\}$ is an independent set, while the induced subgraph on $\{1, 2, 4\}$ is a clique.

Lastly, the graph $\bar{G}$ is the complement of G if $V(\bar{G}) = V(G) = V$ and $E(\bar{G}) = \binom{V}{2} \setminus E(G)$. In other words, for any two vertices $x, y \in V$ we have $\{x, y\} \in E(G) \iff \{x, y\} \notin E(\bar{G})$.

2.2 Introduction to random graphs

Random graph theory is an especially interesting discipline, which lies on an intersection of graph theory and probability theory. It is concerned with random graphs and their properties. The definitions in this section are based on section 1.1 in [17].

A random graph is a graph on a fixed set of vertices, whose set of edges is determined randomly by a probability distribution. In general, the set of vertices is $V = [n] = \{1, \ldots, n\}$. Two common models of random graphs are the binomial model and the uniform model.

In case of the binomial random graph, $G_{n,p}$, the vertex set is $[n]$ and the probability that there is an edge between any pair of vertices is p. One may understand this as deciding on the presence or absence of an edge by flipping a coin. Suppose that we have a coin such that the probability of heads is p, while the probability of tails is $q = 1 - p$. We choose a pair of vertices in $[n]$ and toss the coin. If it falls on heads, we draw an edge between the two vertices, otherwise we do not. Then, we repeat the process for each pair of vertices exactly once. This way, we obtain a binomial random graph. The probability of obtaining a particular graph G with $e(G)$ edges is

$$\mathbb{P}(G) = p^{e(G)} q^{\binom{n}{2} - e(G)}. \tag{1}$$

The uniform random graph, $G_{n,m}$, also has the vertex set $[n]$. However, the number of edges is equal to m, which is constant. The graph is produced by choosing m pairs of vertices, uniformly at random without replacement, and drawing an edge between each of the chosen pairs of vertices. The probability of obtaining a particular graph G is

$$\mathbb{P}(G) = \frac{1}{\binom{N}{m}}, \tag{2}$$

where $N = \binom{n}{2}$.

Annika Heckel's article focuses on $G_{n,1/2}$ - the binomial random graph on n vertices, with probability of each edge occurring in the graph equal to 1/2. Moreover, as a corollary, she asserts that the results she presents about the chromatic number of $G_{n,1/2}$ also hold for the uniform random graph with $m = \lfloor \frac{n^2}{4} \rfloor$.

2.3 Independent sets and vertex coloring

The central topic of this thesis is the chromatic number of a random graph. In this section, we introduce that concept and explore its connection with some other notions in (random) graph theory.

The chromatic number stems from vertex coloring of a graph. Although there exist other ways of coloring a graph, such as for example edge coloring, they are not relevant for this thesis. Hence, in this paper, the term coloring specifically refers to vertex coloring. A (vertex) coloring is an assignment of colors to vertices, such that no two adjacent vertices are assigned the same color [3]. More formally, we may say that a k-coloring of graph G is a map $\phi : V(G) \to [k]$, such that if $\{x, y\} \in E(G)$ then $\phi(x) \neq \phi(y)$. The chromatic number of graph G, denoted with $\chi(G)$, is the smallest number k, such that a k-coloring of the graph G is possible.

Another important notion is that of an independent set. As previously mentioned in section 2.1, a subset of vertices is an independent (or stable) set if the induced subgraph on these vertices is empty. The cardinality of the largest independent subset of vertices of G is called the independence number (or the stability number) of G, and is denoted with $\alpha(G)$ [17].

The opposite notion of the stability number is the clique number of G. The clique number of G is the cardinality of the vertex set of the largest clique in G. Let us notice that if some set V' forms a clique in G, then it forms a stable set in the complement of G [4].

Of course, in case of a random graph, the chromatic number is a random variable. There are several remarkable results on its concentration and its asymptotic value. Some of them will be mentioned in section 3.1.

Intuition implies that there is a relationship between the chromatic number and the stability number of a graph. All vertices which form a stable set can be assigned the same color, as none of them are adjacent. Conversely, the vertices of a complete subgraph must all be assigned a different color as there is an edge joining every single pair of them. If graph G has multiple independent sets of size $\alpha(G)$ and they are disjoint, ideally each of those sets should make up a color class for optimal coloring (i.e., to minimize the number of colors used). This connection between the stability number and the chromatic number of a graph will play an important role in the main proof.

2.4 Asymptotics

A major part of known results in random graph theory explores what happens to a particular property of $G_{n,p}$ asymptotically, i.e. as $n \to \infty$. This is also the case for Annika Heckel's article. Thus, it is important to define some notions related to asymptotics and introduce some frequently used notation.

If for a sequence of events $(E_n)_{n \in \mathbb{N}}$ we have $\mathbb{P}(E_n) \to 1$ as $n \to \infty$, we say that the sequence holds *with high probability* [14]. In some literature, for example [17], the term *asymptotically almost surely* is used instead.

Moreover, the article frequently uses the $o(.)$, $O(.)$, and $\Theta(.)$ notation. The following definitions come from [17].

- $a_n = o(b_n)$ means that for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $|a_n| < \epsilon b_n$ whenever $n \geq n_\epsilon$. In other words, $a_n/b_n \to 0$ as $n \to \infty$.
- $a_n = O(b_n)$ means there exists a constant $C \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $a_n < Cb_n$ whenever $n \geq n_0$. Notably, we have $\log n = O(n)$.
- $a_n = \Theta(b_n)$ means that that there exist constants $C, c > 0$ and n_0 such that $cb_n \leq a_n \leq Cb_n$ whenever $n \geq n_0$. In other words, there exist constants, such that, for a sufficiently large n, a_n is bounded above and below by b_n multiplied by those constants.

Moreover, we say that a_n and b_n are asymptotically equivalent if $\frac{a_n}{b_n} \to 1$. We denote it with $a_n \sim b_n$.

3 Discussion of A. Heckel's article

3.1 Context

In 1947, Paul Erdös laid foundation for the later development of graph theory. His paper "Some Remarks on the Theory of Graphs" [9] is recognized as the first instance of use of probabilistic methods to solve graph theory problems. He later developed this notion in [7] and [8]. Works of Erdös and Rényi [11][10], where they presented the uniform random graph model, are considered to be the birth of random graph theory. The binomial model was introduced by Gilbert in 1959 [12].

One of the earliest results about the chromatic number of random graphs is from Grimmett and McDiarmid [13]: in their 1975 paper, they established its order of magnitude. Thirteen years later, Bollobás proved that almost every random graph had chromatic number $(\frac{1}{2} + o(1)) \log\left(\frac{1}{1-p}\right) \frac{n}{\log n}$ [5]. Several improvements to that result have been made later, for example by McDiarmid [20].

Regarding the concentration of the chromatic number, the vast majority of results provides upper bounds on the length of intervals containing $\chi(G_{n,p})$; in other words, they prove that the chromatic number is highly concentrated around a few consecutive values. Moreover, a lot of research has been focused on the case $p = p(n) \to 0$ as $n \to \infty$. It has been known since 1991, due to the work of Łuczak [18], that for $p < n^{-5/6-\epsilon}$ the chromatic number is concentrated around two values with high probability. Subsequently, Alon and Krivelevich set the upper bound at two consecutive values for $p < n^{-1/2-\epsilon}$ in 1997 [1] .

Annika Heckel published her paper "Non-concentration of the chromatic number of a random graph" [14] in 2020. In it she finds and proves a lower bound on the concentration of the chromatic number of $G_{n,1/2}$. With her paper, she addressed two under-researched areas. Firstly, very little was known about the length of the interval containing $\chi(G_{n,p})$ for the case when p was constant. The work of Shamir and Spencer [21] from 1987 showed that for any sequence $p = p(n)$ the length of the intervals should be about $\sqrt{n}$. Later, this was improved by Alon to $\frac{\sqrt{n}}{\log n}$ for $p = 1/2$ (chapter 7.9, exercise 3 in [2]). Secondly, Annika Heckel's paper became the first one to provide a lower bound on the length of the interval (i.e., the non-concentration of the chromatic number), in contrast to many papers providing upper bounds.

3.2 Main assertion and outline of the proof

Below we present the main result of Annika Heckel's paper [14]. Proving it will be the objective of the remainder of chapter 3.

Theorem 3.1. *For any constant $c < \frac{1}{4}$, there is no sequence of intervals of length n^c which contain $\chi(G_{n,\frac{1}{2}})$ with high probability.*

In other words, the chromatic number of $G_{n,1/2}$ is not concentrated with high probability on fewer than $n^{1/4-\epsilon}$ consecutive values, for some $\epsilon > 0$.

Suppose that $[s_n, t_n]$ is a sequence of intervals, which contains the chromatic number of $G_{n,1/2}$ with high probability. Theorem 2 in [14] (first published in [15]) states that with high probability

$$\chi(G_{n,1/2}) = \frac{n}{2\log_2 n - 2\log_2\log_2 n - 2} + o\left(\frac{n}{\log^2 n}\right). \tag{3}$$

This gives us an interval of length $o(n/\log^2 n)$, which contains the chromatic number of $G_{n,1/2}$ with high probability. Let us set

$$s_n = f(n) + o\left(\frac{n}{\log^2 n}\right), \quad f(n) = \frac{n}{2\log_2 n - 2\log_2\log_2 n - 2}. \tag{4}$$

Let us define the length of the interval as $l_n = t_n - s_n$. In order to prove theorem 3.1, we must show that there exists some n^* such that $l_{n^*} > (n^*)^c$, where $c \in (0, 1/4)$.

We will outline the steps of the proof below. The proof is based on Annika Heckel's original argument, with the level of details adapted for the level of bachelor students of mathematics. Each of the steps below is dedicated a separate section.

1. Estimation of the number of independent *a-sets*. An a-set is a set of vertices of cardinality a, where a is roughly equal to $\alpha(G_{n,1/2})$ (this will be formally derived in section 3.3.1). We show that the number of independent a-sets, X_a, is approximately Poisson distributed with mean $\mu = E[X_a] = \binom{n}{a}\left(\frac{1}{2}\right)^{\binom{a}{2}}$.

2. Comparison of $G_{n,1/2}$ and $G_{n',1/2}$, where n' is slightly larger than n. We define $r = \lfloor\sqrt{\mu}\rfloor$ and take $n' = n + ar$. We investigate how the distributions of X_a and X'_a (the number of independent a-sets in $G_{n',1/2}$) differ. We discover the stability numbers of those two random graphs are very close, and that the expected values of X_a and X'_a only differ by $o(1)$. Thus, X_a and X'_a are almost identically distributed.

3. Coupling of the distributions of conditional random graphs. We construct random graphs H and H' on $[n]$ and $[n']$, respectively, conditional on some typical values of X_a and X'_a in such a way, that H is an induced subgraph of H', and their difference can be partitioned into r independent a-sets. We show that $\chi(H') \in [s_{n'}, t_{n'}]$ and $\chi(H) \in [s_n, t_n]$ with significant probability. Thanks to that, we can make conclusions about the relationship of the interval endpoints, which are deterministic variables. We conclude that $s_{n'} \le \chi(H') \le \chi(H) + r \le t_n + r$, which implies $l_n \ge s_{n'} - s_n - r$.

4. Beating the error term. If we ignore the error term $o(n/\log^2 n)$ and take $s_n \approx f(n)$, we find that $l_n \ge \Theta(r(n)/\log n)$. The error term $o(n/\log^2 n)$ is much larger than $\Theta(r(n)/\log n)$, thus, we need to find a way to tackle it. We apply previously established results to an appropriately chosen finite sequence of integers $(n_i)_i$ and find a lower bound on the sum of corresponding

intervals $l_{n_i} = l_i$. Hence, we find that there is some index n^* such that $l_{n^*} > (n^*)^c$, which concludes the proof.

3.3 Proof

3.3.1 The distribution of X_a

The first step of the proof is approximating the distribution of X_a - the number of independent sets of cardinality a, where a is roughly equal to the stability number of $G_{n,1/2}$. Before discussing how a is defined, let us explain the link between the clique number and the stability number of a random graph.

Let Y_r be the number of cliques of size r, where r is roughly the clique number. Let us consider a random graph $G_{n,p}$, and let V' be a subset of vertices, such that $|V'| = k \leq n$. Then, following from equation 1, we have

$$\mathbb{P}(\text{subgraph on } V' \text{ is complete}) = p^{\binom{k}{2}}, \ \mathbb{P}(\text{subgraph on } V' \text{ is empty}) = q^{\binom{k}{2}}.$$

In this case, we have $p = q = \frac{1}{2}$, so all graphs on $[n]$ are equiprobable. This also applies to its subgraphs. Hence, we can make conclusions about the distribution of X_a based on information about the distribution of Y_r. Thus, the derivations about the stability number and the distribution of X_a will be analogous to those found in sections 11.1 and 11.2 in [4], which concern the clique number and the distribution of Y_r.

As found previously, the probability that a subgraph on k vertices is empty is $(1/2)^{\binom{k}{2}}$. There are $\binom{n}{k}$ possible choices of the k vertices, thus we have

$$\mathbb{E}[X_k] = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \tag{5}$$

Due to the combination factor, the function $\mathbb{E}[X_k]$ has a sharp drop towards zero past a certain value of k [19]. We are searching for a value of k, such $\mathbb{E}[X_k]$ is much larger than 0 and $\mathbb{E}[X_{k+1}]$ is close to 0. In other words, we are searching for the largest k, such that we expect that the random graph G will have an independent set of size k. This happens around $\alpha_0 + o(1)$, where α_0 is as follows from [6]:

$$\alpha_0 = \alpha_0(n) = 2\log_2 n - \log_2 \log_2 n + 2\log_2\left(\frac{e}{2}\right) + 1 \tag{6}$$

This can be verified by plugging α_0 into

$$f(k) = (2\pi)^{-1/2} n^{n+1/2}(n-k)^{-n+k-1/2}k^{-k-1/2}\left(\frac{1}{2}\right)^{k(k-1)/2},$$

which is just the expression from equation 5 with $\binom{n}{k}$ substituted with its Stirling approximation [4]. As $\mathbb{E}[X_k]$ approaches 0 very quickly for $k > \alpha_0 + o(1)$, we know that the stability number is $\lfloor \alpha_0 + o(1) \rfloor$. For most graphs we actually have $\alpha(G_{n,1/2}) = \lfloor \alpha_0 \rfloor =: a$ [14]. Thus, we have defined a.

Similarly to Y_r, the number of a-sets in the random graph $G_{n,1/2}$ is approximately Poisson distributed with mean μ, where μ is as follows:

$$\mu = \mu(n) = \binom{n}{a}\left(\frac{1}{2}\right)^{\binom{a}{2}} = n^x \tag{7}$$

for some $x = x(n)$, such that $o(1) \leq x(n) \leq 1 + o(1)$.

3.3.2 Comparing $G_{n,1/2}$ and $G_{n',1/2}$

The next step is the comparison of $G_{n,1/2}$ and $G_{n',1/2}$, where n' is slightly larger than n. We define $n' = n + ar$, and choose $r = \lfloor n^{x/2} \rfloor = \lfloor \sqrt{\mu} \rfloor$. Note that r is roughly equal to the standard deviation of X_a. Now, let X'_a be the number of a-sets on $G_{n',1/2}$, $\mu' = \mu(n')$, $a' = a(n')$ and $\alpha'_0 = \alpha_0(n')$. We will show that the distributions of X_a and X'_a are almost identical. First, we find a relationship between α_0 and α'_0.

$$\begin{aligned}
\alpha'_0 &= 2\log_2 n' - 2\log_2\log_2 n' + 2\log_2\left(\frac{e}{2}\right) + 1 && (8)\\
&= 2\log_2(n+ar) - 2\log_2\log_2(n+ar) + 2\log_2\left(\frac{e}{2}\right) + 1 && (9)\\
&= 2\log_2\left(n\left(1+\frac{ar}{n}\right)\right) - 2\log_2\log_2\left(n\left(1+\frac{ar}{n}\right)\right) + 2\log_2\left(\frac{e}{2}\right) + 1 && (10)\\
&= 2\log_2 n + 2\log_2\left(1+\frac{ar}{n}\right) - 2\log_2\log_2 n - 2\log_2\log_2\left(1+\frac{ar}{n}\right) + 2\log_2\left(\frac{e}{2}\right) + 1 && (11)\\
&= \alpha_0 + 2\log_2\left(1+\frac{ar}{n}\right) - 2\log_2\log_2\left(1+\frac{ar}{n}\right) && (12)\\
&= \alpha_0 + O\left(\log_2\left(1+\frac{ar}{n}\right)\right) && (13)\\
&= \alpha_0 + O\left(\frac{\log(1+\frac{ar}{n})}{\log 2}\right) && (14)\\
&= \alpha_0 + O\left(\frac{ar}{n}\right) && (15)\\
&= \alpha_0 + o(1) && (16)
\end{aligned}$$

where the last line is because we have $a = O(\log n)$ and $r = O(n^{x/2})$, so

$$\lim_{n\to\infty} \frac{ar}{n} = 0. \tag{17}$$

Now, let us compare μ and μ'. First, let us note that we have

$$\begin{aligned}
\prod_{i=0}^{a-1} \frac{n'-i}{n-i} &= \frac{n'(n'-1)\dots(n'-a+2)(n'-a+1)}{n(n-1)\dots(n-a+2)(n-a+1)} && (18)\\
&= \frac{n'(n'-1)\dots(n'-a+2)(n'-a+1)}{n(n-1)\dots(n-a+2)(n-a+1)} \cdot \frac{(n'-a)!}{(n-a)!} \cdot \frac{(n-a)!}{(n'-a)!} && (19)\\
&= \frac{n'!}{(n'-a)!} \cdot \frac{(n-a)!}{n!}. && (20)
\end{aligned}$$

Moreover, if $i = 0, \dots, a-1$, then

$$\frac{n'-i}{n-i} = \frac{n+ar-i}{n-i} = 1 + \frac{ar}{n-i} = 1 + O\left(\frac{ar}{n}\right). \tag{21}$$

Finally, as $r = O(n^{x/2})$ and $a = O(\log n)$, we have

$$O\left(\frac{ar}{n}\right) = O\left(\frac{(\log n)^2}{n^{1-x/2}}\right) = o(1), \tag{22}$$

because $x < 1$. Combining the results above, we obtain

$$\mu' = \binom{n'}{a}\left(\frac{1}{2}\right)^{\binom{a}{2}} = \frac{n'!}{a!(n'-a)!}\left(\frac{1}{2}\right)^{\binom{a}{2}} \tag{23}$$

$$= \frac{n!}{(n-a)!}\frac{(n-a)!}{n!}\frac{n'!}{a!(n'-a)!}\left(\frac{1}{2}\right)^{\binom{a}{2}} \tag{24}$$

$$= \frac{n!}{(n-a)!a!}\frac{(n-a)!}{n!}\frac{n'!}{(n'-a)!}\left(\frac{1}{2}\right)^{\binom{a}{2}} \tag{25}$$

$$= \mu\frac{n'!}{(n'-a)!}\cdot\frac{(n-a)!}{n!} \tag{26}$$

$$= \mu\prod_{i=0}^{a-1}\frac{n'-i}{n-i} \tag{27}$$

$$= \mu\prod_{i=0}^{a-1}\left(1+O\left(\frac{ar}{n}\right)\right) \tag{28}$$

$$= \mu\left(1+O\left(\frac{ar}{n}\right)\right)^{a} = \mu\left(1+O\left(\frac{ra^2}{n}\right)\right) = \mu + o(1). \tag{29}$$

Hence, the parameter of X_a and X'_a differs only by some term which approaches 0. We conclude that these two random variables are almost identically distributed.

3.3.3 Coupling of distributions

In this section, we will make use of conditional distributions of random graphs of n and n' vertices to make conclusions about the relationship between $[s_n, t_n]$ and $[s_{n'}, t_{n'}]$. Note that s_n and $s_{n'}$ are deterministic variables (i.e., determined entirely by a function, not random). This will be important at the end of this section.

First, let us show that if we condition on some typical values of X_a and X'_a, and on the events that all independent a-sets of conditional G and G' are disjoint, then the chromatic numbers of the conditional graphs still lie in the intervals typical for (unconditional) G and G' with a significant probability. This notion is expressed more formally in the lemma below. This is lemma 8 from [14].

Lemma 3.2. *Let $G \sim G_{n,1/2}$ and $G' \sim G_{n',1/2}$. Let $\mathcal{E}$ and $\mathcal{E}'$ be the events that all independent a-sets in G and G' are disjoint, respectively. Then, if n is large enough, there is an integer $A = A(n) \in [\frac{1}{2}n^x, 2n^x]$ such that*

$$\mathbb{P}(\chi(G) \in [s_n, t_n] \mid \{X_a = A\} \cap \mathcal{E}) > \frac{3}{4} \tag{30}$$

$$\mathbb{P}(\chi(G') \in [s_{n'}, t_{n'}] \mid \{X'_a = A + r\} \cap \mathcal{E}') > \frac{3}{4} \tag{31}$$

Before proving the lemma, let us make (and prove) the following claim.

Claim 1. *Events $\mathcal{E}$ and $\mathcal{E}'$ hold with high probability.*

Proof. We will prove that $\mathcal{E}$ holds with high probability, i.e., that all a-sets in $[n]$ are disjoint with high probability. The proof for $\mathcal{E}'$ is analogous as $\mu' = \mu + o(1)$.

The proof is based on the proof of theorem 4.5.1 in [2]. Let $A_S = \{\text{S is an independent } a\text{-set}\}$. Note that X_a can be expressed as follows:

$$X_a = \sum_{|S|=a} \mathbf{1}_{A_S}. \tag{32}$$

Let us fix an independent a-set S. Then, for any randomly chosen set T, the events A_S, A_T are not independent if and only if the sets S and T share i vertices, where i is at least 2 and strictly less than a (if both S and T are a-sets and share a vertices, they are the same set). Hence, let us define

$$\Delta^* = \mathbb{P}(A_T \mid A_S \cap \{A_S, A_T \text{ are not independent}\}) = \sum_{i=2}^{a-1} \binom{a}{i}\binom{n-a}{a-i} 2^{\binom{i}{2}-\binom{a}{2}}. \tag{33}$$

Now, let us divide Δ^* by $\mathbb{E}[X_a]$.

$$\frac{\Delta^*}{\mathbb{E}[X_a]} = \frac{\sum_{i=2}^{a-1} \binom{a}{i}\binom{n-a}{a-i} 2^{\binom{i}{2}-\binom{a}{2}}}{\binom{n}{a}\left(\frac{1}{2}\right)^{\binom{a}{2}}} = \sum_{i=2}^{a-1} g(i), \tag{34}$$

where

$$g(i) = \frac{\binom{a}{i}\binom{n-a}{a-i}}{\binom{n}{a}} 2^{\binom{i}{2}}. \tag{35}$$

Note that $\frac{\binom{a}{i}\binom{n-a}{a-i}}{\binom{n}{a}}$ is the probability that a randomly chosen a-set T shares i vertices with S, while $2^{\binom{i}{2}}$ is the factor by which $\mathbb{P}(A_T)$ increases if that is the case. Hence, we have

$$\frac{\Delta^*}{\mathbb{E}[X_a]} \geq \mathbb{P}(T \cap S \neq \varnothing) \geq \mathbb{P}(\{T \cap S \neq \varnothing\} \cap A_T) \tag{36}$$

$$= \mathbb{P}(T \text{ is an independent } a\text{-set not disjoint from } S). \tag{37}$$

Thus, in order to prove the statement, it suffices to prove that $\sum_{i=1}^{a-1} g(i) = o(1)$.

Calculations which can be found in [2] show that $g(2) \leq o(n^{-1})$, and $g(a-1) \leq o(n^{-1})$. Moreover, $g(i)$ for other values of i (i.e. $i = 3, \ldots, a-2$) are also negligible and so is their sum. Thus, we have

$$\mathbb{P}(\{T \cap S \neq \varnothing\} \cap A_T) \leq \sum_{i=2}^{a-1} g(i) \leq o(1). \tag{38}$$

From that we conclude that any randomly chosen independent a-set T is disjoint from other independent a-sets with high probability. □

Having proved that events $\mathcal{E}$ and $\mathcal{E}'$ occur with high probability, we may prove lemma 3.2.

Proof of lemma 3.2. First, let us notice that for any events A, B, C we have $\mathbb{P}(A \mid B \cap C) \geq \mathbb{P}(A \cap B \mid C)$. This is because

$$\mathbb{P}(A \mid B \cap C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} \geq \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \mathbb{P}(A \cap B \mid C). \tag{39}$$

Let us now define the following events:

$$\mathcal{F} = \{\chi(G) \in [s_n, t_n]\} \cap \mathcal{E} \tag{40}$$

$$\mathcal{F}' = \{\chi(G) \in [s_{n'}, t_{n'}]\} \cap \mathcal{E}'. \tag{41}$$

Due to inequality 39, in order to prove the theorem, it suffices to show that there exists some integer A such that $\mathbb{P}(\mathcal{F} \mid X_a = A) > 3/4$ and $\mathbb{P}(\mathcal{F}' \mid X'_a = A + r) > 3/4$. The intervals $[s_n, t_n]$ are defined in such a way that they contain $\chi(G_{n,1/2})$ with high probability. Moreover, by claim 1, events $\mathcal{E}, \mathcal{E}'$ hold with high probability. Thus, $\mathcal{F}$ and $\mathcal{F}'$ also hold with high probability.

Let $\mathcal{A}$ be the set of all values of A such that $\mathbb{P}(\mathcal{F} \mid X_a = A) > 3/4$. Then, as $\mathcal{F}$ holds with high probability, we have

$$\mathbb{P}(\mathcal{F} \mid X_a \in \mathcal{A}) + \mathbb{P}(\mathcal{F} \mid X_a \notin \mathcal{A}) = \mathbb{P}(\mathcal{F}) = 1 - o(1). \tag{42}$$

Note that $\mathbb{P}(\mathcal{F} \mid X_a \notin \mathcal{A}) \leq 1/4$, so $\mathbb{P}(\mathcal{F}^c \mid X_a \notin \mathcal{A}) \geq 1/4$. This way we obtain

$$o(1) = \mathbb{P}(\mathcal{F}^c) = \sum_{A \notin \mathcal{A}} \left(\mathbb{P}(\mathcal{F}^c \mid X_a = A)\mathbb{P}(X_a = A)\right) \geq \frac{1}{4}\mathbb{P}(X_a \notin \mathcal{A}). \tag{43}$$

Analogously, letting $\mathcal{A}'$ be the set of values of A such that $\mathbb{P}(\mathcal{F}' \mid X'_a = A + r) > 3/4$, we obtain

$$o(1) \geq \frac{1}{4}\mathbb{P}(X'_a \notin \mathcal{A}' + r). \tag{44}$$

Hence, with high probability $X_a \in \mathcal{A}$ and $X'_a \in \mathcal{A}' + r$. Due to an argument related to total variation distance, which is beyond the scope of this paper, we find $\mathcal{A} \cap \mathcal{A}' \cap [\frac{1}{2}\mu, 2\mu] \neq \varnothing$. We conclude that there exists a value of A which lies in $[\frac{1}{2}\mu, 2\mu]$ which ensures both $\mathbb{P}(\chi(G) \in [s_n, t_n] \mid \{X_a = A\} \cap \mathcal{E}) > \frac{3}{4}$ and $\mathbb{P}(\chi(G') \in [s_{n'}, t_{n'}] \mid \{X'_a = A + r\} \cap \mathcal{E}') > \frac{3}{4}$. □

Let us now define some conditional graphs, which later will help us make conclusions about $\chi(G_{n,1/2})$ and $\chi(G_{n',1/2})$. We will denote the distribution of graph $G_{n,p}$ conditioned on event $\mathcal{P}$ with $G_{n,p \mid \mathcal{P}}$. Now, we will need to construct two graphs, H and H', such that they fulfill the following criteria:

- H is a random graph with vertex set $[n]$ and edge probability $\frac{1}{2}$, such that $X_a = A$ and all of its independent a-sets are disjoint. H' is a random graph with vertex set $[n']$ and edge probability $\frac{1}{2}$, such that $X'_a = A + r$ and all of its independent a-sets are disjoint.
- H is an induced subgraph of H'.
- Their difference, i.e., the induced subgraph of H' on $\{n + 1, \ldots, n'\}$, can be partitioned into r independent a-sets.

Let $V = [n]$ and $V' = [n']$. In order to satisfy the first criterion, let us fix some arbitrary disjoint a-sets: $S_1, \ldots, S_r, \ldots, S_{r+A} \subset V'$. A of those sets must be contained in V. As the sets are arbitrary, and H is an induced subgraph of H', let us label them in such a way that $S_1, \ldots, S_r \subset V' \setminus V$ and $S_{r+1}, \ldots, S_{r+A} \subset V$. Finally, in order to satisfy the third criterion, there may not be any vertices in $V' \setminus V$ which are not contained in S_i for some $i \in [r]$. Thus, $V' \setminus V = \cup_{i=1}^{r} S_i$. A diagram portraying the desired structure of graphs H and H' is shown below.

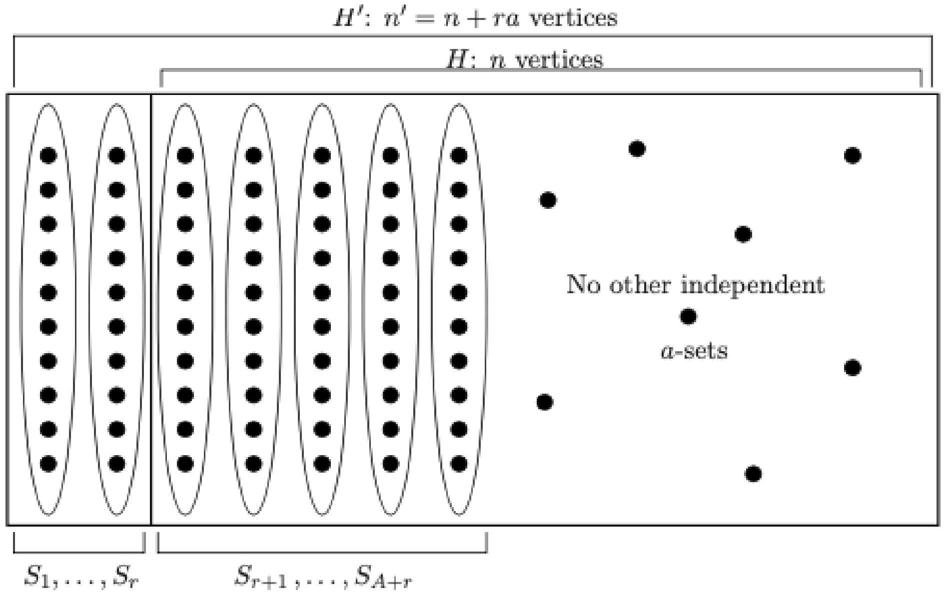

Figure 3: Simplified diagram showing the structure of graphs H and H' from [14]. The difference of the graphs consists entirely of r independent disjoint a-sets, $S_1, \dots, S_r$. The graph H contains independent disjoint a-sets $S_{r+1}, \dots, S_{r+A}$, and perhaps some other vertices, which do not form an independent a-set. For any pair of vertices which do not belong to the same a-set S_i, $i \in [A+r]$, the probability that there is an edge connecting them is $\frac{1}{2}$.

Now, we will define four events, such that random graphs H and H' conditioned on them will fulfill the three criteria above. The events are as follows:

- $\mathcal{D}_1$: The a-sets $S_1, \dots, S_r$ are independent.
- $\mathcal{D}_2$: The a-sets $S_{r+1}, \dots, S_{r+A}$ are independent.
- $\mathcal{U}_1$: There are no independent a-sets with at least one vertex in $V' \setminus V$, other than the a-sets $S_1, \dots, S_r$ (which may or may not be independent).
- $\mathcal{U}_2$: There are no independent a-sets completely contained in V, other than the a-sets $S_{r+1}, \dots, S_{r+A}$ (which may or may not be independent).

Note that the event $\mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1 \cap \mathcal{U}_2$ means that the sets $S_1, \dots, S_{r+A}$ are independent and no other independent a-sets in V' exist. Immediately we notice that we can obtain the required distributions of H, H' by setting $H' \sim G_{n,1/2 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1 \cap \mathcal{U}_2}$ and taking H to be the induced graph of H' on V: $H = H'[V]$.

Let us obtain $\hat{H}'$ by a random permutation of H', and $\hat{H} = \hat{H}'[V]$. As no edges are added or removed, $\hat{H}'$ still has exactly $A+r$ disjoint independent a-sets. Thus, we have $\hat{H}' \sim G_{n',1/2 \mid \{X'_a = A_r\} \cap \mathcal{E}'}$. Moreover, as only the vertex labels have been changed, $\hat{H}'$ has the same chromatic number as H'. Hence, from 3.2 we can easily conclude

$$\mathbb{P}(\chi(H') \in [s_{n'}, t_{n'}]) = \mathbb{P}(\chi(\hat{H}') \in [s_{n'}, t_{n'}]) = \mathbb{P}(\chi(G') \in [s_{n'}, t_{n'}] \mid \{X'_a = A + r\} \cap \mathcal{E}') > \frac{3}{4} \quad (45)$$

Unfortunately, as the vertices have been permutated, we do not necessarily have A independent

a-sets in $\hat{H}$. Thus, $\chi(H)$ may be different from $\chi(\hat{H})$. However, we may bind the probability that H has a certain property $\mathcal{B}$, which is invariant under vertex permutation, by the probability that $G_{n,1/2\mid\{X_a=A\}\cap\mathcal{E}}$ has the same property. For example, $\mathcal{B}$ may be the property that the chromatic number of H lies in a certain interval. The following is claim 2 in [14].

Lemma 3.3. *Let $\mathcal{B}$ be an event for the set of graphs with vertex set V which is invariant under the permutation of vertex labels. Then*

$$\mathbb{P}(H \in \mathcal{B}) \leq (1+o(1))\,\mathbb{P}(G_{n,1/2\mid\{X_a=A\}\cap\mathcal{E}} \in \mathcal{B}). \tag{46}$$

In order to prove the lemma above, we will need the following result (based on lemma 9 in [14]):

Lemma 3.4. *Let Y be the number of independent a-sets with at least one vertex in $V' \setminus V$, other than the sets $S_1, \ldots, S_r$. Then, $\mathbb{E}[Y \mid \mathcal{D}_1 \cap \mathcal{D}_2] = o(1)$.*

The proof is a fairly straightforward, yet tedious calculation. Hence, we will only provide a sketch in this paper.

Sketch of proof. Let T be a set of vertices which counts towards Y. That means, T is an independent a-set with at least one vertex in $V' \setminus V$, and is not equal to any of the sets $S_1, \ldots, S_r$. We partition T into vertices which are included in $S_1, \ldots, S_{A+r}$, and remaining vertices. Thus, we have

$$T = \cup_{j=1}^{M} T_j \cup T_{\text{rest}}, \tag{47}$$

where $T_j \neq \varnothing$, $T_j \subset S_{i_j}$ for $i_j \in [A+r]$, and $T_{\text{rest}} \subset V$. Let us order the subscripts $i_1 < \cdots < i_M$, which we are allowed to do because $S_1, \ldots, S_{A+r}$ are arbitrary. In order to ensure that at least one vertex of T is in $V' \setminus V$, we need $i_1 \in [r]$. This also implies that $M \geq 1$. Moreover, as $|T| = a$, we have $M \leq a$.

Let $t_j := |T_j|$. As T_j is not empty for all $j \in [M]$, and T_j is not equal to S_{i_j}, we have

$$1 \leq t_j \leq a - 1. \tag{48}$$

Let $\mathcal{T}$ be a set of all possible pairs $(M, \mathbf{t})$, where $\mathbf{t} = (t_1, \ldots, t_M)$. Fix M and $\mathbf{t}$. Now let us find the number of possible sets T that correspond to $\mathbf{t}$ and M. We need to choose $i_1 \in [r]$, $\{i_2, \ldots, i_M\} \subset [A+r]$, t_j out of a vertices out of each S_{i_j}, and the remaining vertices of T_{rest}. Note that $|T_{\text{rest}}| = a - \sum_j t_j$. Thus, we obtain

$$r\binom{A+r}{M-1}\left(\prod_{j=1}^{M}\binom{a}{t_j}\right)\binom{n}{a-\sum_j t_j} \leq \frac{r}{A+r}\binom{n}{a}\prod_{j=1}^{M}\frac{A+r\binom{a}{t_j}a!}{(a-t_j)!(n-a)^{t_j}}. \tag{49}$$

The details on how the upper bound was obtained can be found in the proof of Lemma 9 in [14].

Now that we have found an upper bound on the number of potential sets T, let us find $\mathbb{P}(T \mid \mathcal{D}_1 \cap \mathcal{D}_2)$. As we condition on the events that the sets $S_1, \ldots, S_{A+r}$ are independent and by definition those sets are disjoint, we are guaranteed that the sets $T_1, \ldots, T_M$ are independent. Hence, we only have to consider the vertices of T_{rest}. This way, we obtain

$$\mathbb{P}(T \text{ is independent} \mid \mathcal{D}_1 \cap \mathcal{D}_2) = \left(\frac{1}{2}\right)^{\binom{a}{2} - \sum_j \binom{t_j}{2}} = 2^{\sum_j \binom{t_j}{2} - \binom{a}{2}}. \tag{50}$$

Multiplying the probability above by the upper bound on the number of possible sets T, and summing over all possible $(M, \mathbf{t}) \in \mathcal{T}$, we obtain an upper bound on the conditional expectation of Y:

$$\mathbb{E}[Y \mid \mathcal{D}_1 \cap \mathcal{D}_2] \leq \sum_{(M,\mathbf{t})\in\mathcal{T}} \left(2^{\sum_j \binom{t_j}{2} - \binom{a}{2}} \frac{r}{A+r} \binom{n}{a} \prod_{j=1}^{M} \frac{A + r\binom{a}{t_j} a!}{(a-t_j)!(n-a)^{t_j}} \right) = \frac{r\mu}{A+r} \sum_{(M,\mathbf{t})\in\mathcal{T}} \prod_{j=1}^{M} \sigma_{t_j}, \tag{51}$$

where

$$\sigma_t = \frac{(A+r)\binom{a}{t} a! 2^{\binom{t}{2}}}{(a-t)!(n-a)^t}. \tag{52}$$

By another set of calculations we obtain

$$\sigma_t \leq \sigma_1 = O^*(n^{x-1}), \tag{53}$$

where $f = O^*(g)$ means that there exist constants C, n_0 such that $|f(n)| \leq (\log n)^C g(n)$ whenever $n \geq n_0$. This way, using the fact that $1 \leq t_j \leq a-1$, we can further bound the conditional expectation of Y:

$$\mathbb{E}[Y \mid \mathcal{D}_1 \cap \mathcal{D}_2] \leq \frac{r\mu}{A+r} \sum_{(M,\mathbf{t})} \sigma_1^M \leq \frac{r\mu}{A+r} \sum_{M\geq 1} (a^M \sigma_1^M) = O^*(\frac{r\mu\sigma_1}{A+r}) = O^*(n^{3/2x-1}) = o(1). \tag{54}$$

□

Having shown that $\mathbb{E}[Y \mid \mathcal{D}_1 \cap \mathcal{D}_2]$, we may now prove lemma 3.3.

Proof of lemma 3.3. Let us first express the distribution of $G_{n,1/2 \mid \{X_a = A\}\cap\mathcal{E}}$ in terms of events $\mathcal{D}_1, \mathcal{D}_2, \mathcal{U}_1$ and $\mathcal{U}_2$. The conditional graph is conditioned upon having A independent a-sets, all of which all are disjoint. That is equivalent to conditioning on events $\mathcal{D}_2$ and $\mathcal{U}_2$, and then performing a random permutation of vertex labels. Hence, we have

$$\mathbb{P}(G_{n,1/2 \mid \{X_a=A\}\cap\mathcal{E}} \in \mathcal{B}) = \mathbb{P}(\mathcal{B} \mid \mathcal{D}_2 \cap \mathcal{U}_2), \tag{55}$$

as $\mathcal{B}$ is invariant under vertex permutation. As event $\mathcal{D}_1$ applies to vertex set $V' \setminus V$, so it is independent from $\mathcal{B}$, which by definition applies to vertex set V. Moreover, event $\mathcal{D}_1$ is also independent from $\mathcal{D}_2$ and $\mathcal{U}_2$. Hence, we have

$$\mathbb{P}(\mathcal{B} \mid \mathcal{D}_2 \cap \mathcal{U}_2) = \mathbb{P}(\mathcal{B} \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_2). \tag{56}$$

Now, let us consider the distribution of H. As determined previously, we have $H' \sim G_{n,1/2|\mathcal{D}_1\cap\mathcal{D}_2\cap\mathcal{U}_1\cap\mathcal{U}_2}$, and H is an induced subgraph of H' on vertex set V. Hence, we have

$$\mathbb{P}(H \in \mathcal{B}) = \mathbb{P}(\mathcal{B} \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1 \cap \mathcal{U}_2) = \frac{\mathbb{P}(\mathcal{B} \cap \mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_2)}{\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1)} \leq \frac{\mathbb{P}(\mathcal{B} \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1)}{\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1)}. \tag{57}$$

This comes down to

$$\mathbb{P}(H \in \mathcal{B}) \leq \frac{1}{\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1)} \mathbb{P}(G_{n,1/2 \mid \{X_a=A\}\cap\mathcal{E}} \in \mathcal{B}), \tag{58}$$

so all we need to show to prove the statement is

$$\frac{1}{\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1)} = 1 + o(1), \tag{59}$$

or equivalently,

$$\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1) = 1 - o(1). \tag{60}$$

This is because if the inverse of $\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1)$ is equal to one plus some term which approaches zero, then $\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_1)$ must be slightly smaller than one, and approaching one.

For reasons related to measure theory, which are beyond the scope of this paper, we have

$$\mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{U}_2) \geq \mathbb{P}(\mathcal{U}_1 \mid \mathcal{D}_1 \cap \mathcal{D}_2). \tag{61}$$

This is a result of applying Harris' lemma, which requires knowledge of product probability spaces. Readers of this paper who have the required background are encouraged to verify the inequality above.

Now, in order to prove the lemma, it is sufficient to prove that $\mathbb{P}(\mathcal{U}_1^c \mid \mathcal{D}_1 \cap \mathcal{D}_2) = o(1)$. Let Y be defined the same as in lemma 3.4. Note that the event $\mathcal{U}_1^c$ means that there is at least one set which counts towards Y (i.e., an independent a-set with at least one vertex in $V' \setminus V$, other than $S_1, \ldots, S_r$). Hence, the event $\mathcal{U}_1^c$ occurs if and only if $Y \geq 1$. As a result, $\mathbb{E}[Y \mid \mathcal{D}_1 \cap \mathcal{D}_2] = o(1)$ implies $\mathbb{P}(\mathcal{U}_1^c \mid \mathcal{D}_1 \cap \mathcal{D}_2) = o(1)$, which concludes the proof. □

Let us take $\mathcal{B} = \{\chi(H) \notin [s_n, t_n]\}$. The chromatic number is invariant under vertex permutations, so we may apply lemma 3.3 to this event. Then, we have

$$\mathbb{P}(H \in \mathcal{B}) \leq (1 + o(1)) \cdot \mathbb{P}(\chi(G_{n,1/2 \mid \{X_a = A\} \cap \mathcal{E}}) \notin [s_n, t_n]) < (1 + o(1)) \cdot \frac{1}{4} \tag{62}$$

and hence, for a large enough n,

$$\mathbb{P}(H \notin \mathcal{B}) = \mathbb{P}(\chi(H) \in [s_n, t_n]) > \frac{3}{4} \cdot (1 + o(1)) > \frac{1}{2}. \tag{63}$$

We know that $V' \setminus V$ is made up of r disjoint independent sets, so we need to use no more than r colors to color the difference of H and H'. Hence, we have

$$\chi(H') \leq \chi(H) + r. \tag{64}$$

From inequality 45 we know that $\chi(H')$ is greater than $s_{n'}$ with probability greater than 3/4. Similarly, by 63, for a large n the probability that $\chi(H)$ is smaller than t_n is at least 1/2. Combining that with inequality 64 we obtain

$$s_{n'} \leq \chi(H') \leq \chi(H) + r \leq t_n + r, \tag{65}$$

which holds at least with probability 1/4. However, the intervals $[s_n, t_n]$ and $[s_{n'}, t_{n'}]$, as defined in equation 4, are deterministic. Hence, we may disregard the probability and conclude that we simply have

$$s_{n'} \leq t_n + r. \tag{66}$$

Thus, subtracting s_n from both sides, we may conclude the following about l_n:

$$l_n = t_n - s_n \geq s_{n'} - s_n - r. \tag{67}$$

Because of that, we may form the following conclusion (based on lemma 10 in [14]):

Lemma 3.5. *For any $\epsilon \in (0, \frac{1}{4})$ there exists N_ϵ, such that if $n \geq N_\epsilon$ and $\mu(n) = n^{x(n)}$ with $\epsilon \leq x(n) \leq \frac{1}{2}$, then*

$$l_n \geq s_{n'} - s_n - r,$$

where $r = r(n) = \lfloor n^{x(n)/2} \rfloor$ and $n' = n + a(n)r$.

3.3.4 Beating the error term

In the previous section, we have concluded that $l_n \geq s_{n'} - s_n - r$. Recall from equation 4 that $s_n = f(n) + o(n/(\log^2 n))$. For a moment, let us ignore the error term and take $s_n \approx f(n)$. Then, we have

$$s_{n'} - s_n - r \approx f(n') - f(n) - r, \tag{68}$$

where $r = r(n)$. By a calculation which can be found in the appendix of [14], we obtain

$$f(n') - f(n) = r + \frac{(1-x)r}{a} + o\left(\frac{r}{a}\right) = r + (1 - x + o(1))\,\frac{r}{a} > r + \frac{r}{2a}. \tag{69}$$

Combining that with inequality 67, we obtain

$$l_n > \frac{r}{2a} = \Theta\left(\frac{r}{\log n}\right). \tag{70}$$

Note that the error term $o(n/(\log^2 n))$ is much larger than the order of $r/2a$. Thus, ignoring the error term will not suffice and we need to find a way to beat it. In order to do that, we will apply the results above to a finite sequence of numbers $(n_i)_i$. First, we will choose n_1 such that $x(n_1)$ fits in a chosen narrow range. Then, we will inductively define subsequent terms. Finally, we will choose an appropriate final term $n_{i_{max}}$.

Let us now recall how we defined $x = x(n)$, as it will be important to define the sequence $(n_i)_i$; $x(n)$ is a sequence of values which satisfies:

- $x \in [o(1), 1 + o(1)]$;
- $x = \alpha_0 - \lfloor \alpha_0 \rfloor + o(1)$;
- $\mu = n^x$, where $\mu = \mathbb{E}[X_a]$.

Note that if a graph has more vertices, we expect that there will appear more independent a-sets. Thus, for $m \geq n$ we have $x(m) \geq x(n)$.

To define n_1, we will need the following lemma (lemma 4 from [14]:

Lemma 3.6. *Let $0 \leq c_1 < c_2 \leq 1$ and $N > 0$. There is an integer $n \geq N$ such that $x(n) \in (c_1, c_2)$.*

Let us take a constant $c \in (0, \frac{1}{4})$, and define $\epsilon = \frac{1}{4}(\frac{1}{4} - c) < 1/16$. Taking $c_1 = \frac{1}{2} - 4\epsilon$ and $c_2 = \frac{1}{2} - 3\epsilon$, we know by lemma 3.6 that there exists some arbitrarily large n_1, such that

$$\frac{1}{2} - 4\epsilon < x(n_1) < \frac{1}{2} - 3\epsilon. \tag{71}$$

Now, we will define the subsequent terms of the sequence. Let $a = a(n_1), x_i = x(n_i)$ and $r_i = \lfloor n_i^{x_i/2} \rfloor$. Then

$$n_{i+1} = n_i + a r_i, \tag{72}$$

for $i = 1, \ldots, i_{max}$.

Finally, we will find i_{max}. Let M be the largest integer such that for any $n_1 \leq n \leq M$ we have

$$\alpha_0(n) < a(n_1) + \frac{1}{2} - 2\epsilon. \tag{73}$$

Then, i_{max} is the largest index, such that $n_i \leq M$. Thus, all the terms of the sequence fulfill inequality 73.

Let us now formulate the following lemma about the sequence $(n_i)_{i=1}^{i_{max}}$.

Lemma 3.7. *Let* $(n_i)_{i=1}^{i_{max}}$ *be a sequence, defined inductively as*

$$n_{i+1} = n_i + ar_i, \quad i = 1, \ldots, i_{max} - 1 \tag{74}$$

where $a = a(n_1)$*,* $x_i = x(n_i)$ *and* $r_i = \lfloor n_i^{x_i/2} \rfloor$*. Then, there exists an index* n^**, such that* $l_{n^*} > (n^*)^c$*.*

Note that for a fixed i, n_i and n_{i+1} correspond to previously used n and n', respectively. Thus, proving the lemma above, also proves the main assertion.

Again, we will only provide a sketch of the proof, as it consists of straightforward calculations. However, it is important to understand the asymptotic order of M and $n_{i_{max}}$. Let us start by noticing that $M - n_1 = \Theta(n_1)$. This is due to the fact that $\alpha_0(n) \sim 2\log_2(n)$, and hence also $a(n) \sim 2\log_2(n)$. Thus, we have

$$M - n_1 \sim 2^{\alpha_0(M)/2} - 2^{a(n_1)/2} < 2^{\frac{1}{2}(a(n_1)+1/2-2\epsilon)} - 2^{\frac{1}{2}a(n_1)} = 2^{\frac{1}{2}a(n_1)}(2^{1/4-\epsilon}-1) \sim n_1(2^{1/4-\epsilon}-1), \tag{75}$$

where $C = 2^{1/4-\epsilon} - 1 > 0$ because $\epsilon < 1/16$. Moreover, as M is defined as the largest integer satisfying inequality 73, $M - n_1$ is also bounded away from 0. From this result it also follows that $n_{i_{max}} - n_1 = \Theta(n_1)$, and thus $n_{i_{max}} = \Theta(n_1)$.

Sketch of proof of lemma 3.7. First, let us find a lower bound on the sum of the l_i's;

$$\sum_{i=1}^{i_{max}-1} l_i \geq \sum_{i=1}^{i_{max}-1} (s_{i+1} - s_i - r_i) = \sum_{i=1}^{i_{max}-1} (s_{i+1} - s_i) - \sum_{i=1}^{i_{max}-1} r_i = s_{i_{max}} - s_1 - \sum_{i=1}^{i_{max}-1} r_i. \tag{76}$$

Via a series of straightforward calculations on page 11, Annika Heckel in [14] finds more specific lower bounds, namely

$$s_{i_{max}} - s_1 > \sum_{i=1}^{i_{max}-1} \left(r_i + \frac{r_i}{2a} \right) + o\left(\frac{n_1}{\log^2 n_1} \right) \tag{77}$$

and

$$\sum_{i=1}^{i_{max}-1} \frac{r_i}{a} = \Theta\left(\frac{n_1}{\log^2 n_1} \right). \tag{78}$$

Combining those, we obtain

$$\sum_{i=1}^{i_{max}-1} l_i > \sum_{i=1}^{i_{max}-1} \frac{r_i}{2a} + o\left(\frac{n_1}{\log^2 n_1} \right) = \sum_{i=1}^{i_{max}-1} \frac{r_i}{2a} + \sum_{i=1}^{i_{max}-1} \frac{r_i}{a} \geq \sum_{i=1}^{i_{max}-1} \frac{r_i}{3a} \tag{79}$$

In order for the inequality above to hold, there must be at least one index i^*, such that for $n^* = n_{i^*}$ we have

$$l_{n^*} > \frac{r_{i^*}}{3}. \tag{80}$$

In fact, the terms of this sum may be distributed in various ways. In order for the inequality 79 to hold, we may have several shorter intervals which add up to a larger sum (e.g., $l_i > \frac{r_i}{10a}$), or we may have one very long interval such that $l_i > r_i$. Suppose the latter is the case. Recall that $r_{i^*} = \lfloor (n^*)^{x(n^*)/2} \rfloor \geq \lfloor (n^*)^{x(n_1)/2} \rfloor$ as $n^* \geq n_1$. Thus, we obtain

$$l_{n^*} > (n^*)^{x(n_1)/2} > (n^*)^{\frac{1}{2}(\frac{1}{2}-4\epsilon)} = (n^*)^{\frac{1}{4}-4\epsilon} = (n^*)^c. \tag{81}$$

Hence, we have proven that there exists some integer n^*, such that $l_{n^*} > (n^*)^c$, where $c \in (0, 1/4)$.
□

This concludes the proof of the main result, theorem 3.1.

4 Further research

After becoming familiar with Annika Heckel's research, several directions of research appear as a natural next step. Primarily, the paper in question is the first piece of research that provides a lower bound on the length of the interval containing the chromatic number with high probability. After Heckel has found the lower bound for $p = 1/2$, it appears intriguing to explore whether or not such a bound exists for other values of p. In general, the majority of research on the chromatic number of random graphs has focused on small values of p and especially on $p = p(n) \to 0$ as $n \to \infty$. We know that for those values of p, $\chi(G_{n,p})$ tends to have an extremely narrow concentration. Nevertheless, it is interesting to find out if a lower bound on that concentration exists. However, much less is known about graphs where p does not approach 0. One might be interested in researching the case where p is a constant other than $1/2$, or where $p = p(n)$ approaches a value other than 0, for example $p(n) \to 1$ as $n \to \infty$.

Another direction that research could focus on is the uniform random graph. This was briefly touched upon in [14], where Annika Heckel asserts that the conclusions of the article are true for $G_{n,m}$ with $m = \lfloor n^2/4 \rfloor$, as it corresponds to the binomial random graph $G_{n,1/2}$. Similarly to the binomial graph, one might focus on the case that m is constant, or m depends on n and approaches a certain value as $n \to \infty$. Almost all research on the chromatic number of random graphs focuses on binomial random graphs, thus results on uniform random graphs could be particularly insightful. Moreover, exploring the correspondence between $G_{n,p}$ and $G_{n,m}$ for particular values of p and m could aid the discovery of new bounds regarding $G_{n,p}$ via the knowledge about $G_{n,m}$ and vice versa.

One might also wonder if the chromatic number of $G_{n,p}$ follows any known probability distribution (at least for some values of p) and if so, try to find that distribution. Perhaps as $n \to \infty$, the distribution of $\chi(G_{n,p})$ approaches some extremal distribution, which could be used to approximate confidence intervals on its value for a large n.

Annika Heckel herself has chosen to search for a more accurate lower bound on the length of the interval containing the chromatic number. In the 2021 preprint which she wrote together with Oliver Riordan, they claim the width of the interval containing $\chi(G_{n,1/2})$ is at least $n^{1/2-o(1)}$ [16]. As mentioned previously, Alon had established the upper bound on the length of the interval at $\frac{\sqrt{n}}{\log n}$ for $p = 1/2$ [2]. This means that Heckel and Riordan have managed to match the lower bound to the uppper bound, up to the error term.

Many concepts in graph theory may be researched via their connection with other concepts. For example, in this case we exploited the opposition of the stability number and the clique number. This certainly facilitates the finding of new results, as existing results may be applied to complements of graphs to make conclusions about related concepts. Despite that, although graph theory has been a dynamic area of research ever since its foundation in the 1950's, there are still many open questions in this field.

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