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Master Thesis Mathematics

Random Polytopes with Vertices on the Boundary of a Ball or a Cube

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Abstract

Let $X_1, ..., X_N$ be independent points that are uniformly distributed on the boundary of a compact convex set P and let P_N be the convex hull of those points. This thesis gives an extensive proof of the following two (already existing) theorems. If $P = B^d$, which is the d-dimensional unit ball, then $\mathbb{E}[V_d(B^d) - V_d(P_N)] = O(N^{-\frac{2}{d-1}})$ as $N \to \infty$. If $P = C := [0,1]^3$, then the expected number of facets of the convex hull P_N is $\mathbb{E}f_2(P_N) = c \ln N(1 + O((\ln N)^{-1}))$ as $N \to \infty$, with some $c > 0$ independent of C.

Notation

$\lVert \cdot \rVert$	Euclidean norm on $\mathbb{R}^d$
$\mathbb{1}(\cdot)$	indicator function
B^d	unit ball of $\mathbb{R}^d$
$[x_1, ..., x_j]$	convex hull of the points $x_1, ..., x_j$
Δ_q	q-dimensional volume of a convex hull
$\mathbb{E}[\cdot]$	expectation of a random variable
f_k	number of k-dimensional faces
$\Gamma(\cdot)$	gamma function
$\mathcal{H}^k$	k-dimensional Hausdorff measure
κ_d	volume of B^d
λ_d	Lebesgue measure on $\mathbb{R}^d$
$[N]$	set of integers $\{1, 2, ..., N\}$
$\mathbb{N}$	$\{1, 2, 3, ...\}$
ω_d	surface area of $\mathbb{S}^{d-1}$
$\mathbb{R}$	$= (-\infty, \infty)$ is the real line
$\mathbb{R}_+$	$= [0, \infty)$ is the non-negative real half-line
$\mathbb{R}^d$	Euclidean space of dimension $d \in \mathbb{N}$
$\mathbb{S}^{d-1}$	unit sphere of $\mathbb{R}^d$
$\mathcal{S}_n$	set of all permutations of $\{1, ..., n\}$
σ	spherical Lebesgue measure of $\mathbb{S}^{d-1}$
V_d	d-dimensional volume
$A(d, q)$	perimetrization of all q-dimensional affine subspace of $\mathbb{R}^d$

Contents

Chapter 1

Introduction

Fix a dimension $d \geq 2$ and let $X_1, ..., X_N$ be independent points that are uniformly distributed on the boundary of a compact convex set P. The convex hull of these points is denoted by P_N.

Much research has been done about points that are distributed *in the interior* of compact convex sets and functionals of that. It started with the two papers by Rényi and Sulanke [2],[3]. They have found the expected area, perimeter, and number of vertices of P_N. This was followed by many other papers, generalizing this and other results to higher dimensions. For example, when the boundary of P is sufficiently smooth and the points are chosen from the interior, it follows from the papers by Bárány [7] and Böröczky, Hoffmann and Hug [11] that $\mathbb{E}[V_d(P) - V_d(P_N)] = c_d \Omega(P) V_d(P)^{\frac{2}{d+1}} N^{-\frac{2}{d+1}}(1 + o(1))$, where $\Omega(P)$ is the affine surface area of P and c_d is a constant only depending on d.

Results for points on the boundary of P are much less known. In this thesis we want to find the value of $\mathbb{E}[V_d(P) - V_d(P_N)]$ as well, but for points that are chosen *on the boundary* of P. We still assume that P has a smooth boundary. The archetype of sets with a smooth boundary is the ball. Therefore, in the first part of this thesis, we consider the d-dimensional unit ball B^d. Let $P = B^d$, meaning that the points $X_1, ..., X_N$ live on the sphere $\mathbb{S}^{d-1}$. Furthermore, let V_d be the d-dimensional volume measure. Explicit results for fixed N cannot be expected, so we investigate the asymptotics as $N \to \infty$. As the number of points N goes to infinity, the volume of the convex hull P_N approaches the volume of the ball, which means that the difference in volume goes to zero. More specifically, we want to prove that

$$\mathbb{E}[V_d(B^d) - V_d(P_N)] = O(N^{-\frac{2}{d-1}}) \quad \text{as } N \to \infty. \tag{1.1}$$

This theorem has been proven in the paper by Müller [5]. However, the proof is rather brief. The first goal of this thesis to give an extensive proof of this theorem using a different method than Müller. Furthermore, this theorem implies the following two special cases. If the points $X_1, ..., X_N$ are distributed uniformly on $\mathbb{S}^1$, then

$$\mathbb{E}[V_2(B^2) - V_2(P_N)] = O(N^{-2}) \quad \text{as } N \to \infty. \tag{1.2}$$

and if the points $X_1, ..., X_N$ are distributed uniformly on $\mathbb{S}^2$, then

$$\mathbb{E}[V_3(B^3) - V_3(P_N)] = O(N^{-1}) \quad \text{as } N \to \infty. \tag{1.3}$$

We will prove both special cases as a warm up for the proof of the general theorem in dimension d.

It is much harder to find results when assuming that P is a polytope. Another functional that has been investigated is the expected number of ℓ-dimensional faces $\mathbb{E} f_\ell(P_N)$ of the convex hull P_N when the points are chosen in the interior of a polytope P. The work of Reitzner [10] shows that if P is a polytope, then $\mathbb{E}f_\ell(P_N) = c_{d,\ell}\text{flag}(P)(\ln N)^{d-1}(1+o(1))$ for $\ell \in \{0,...,d-1\}$, where flag(P) is the number of flags of a polytope P. A flag [12] is a sequence $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ of i-dimensional faces F_i of P. In this thesis we want to find the value of $\mathbb{E}f_\ell(P_N)$ as well, but for points that are chosen *on the boundary* of a polytope P and $\ell = d-1$. The archetype of polytopes is the unit cube $C = [0,1]^d$. To make things not too complicated we take $d = 3$. Therefore, in the second part of this thesis, we consider the 3-dimensional unit cube $C = [0,1]^3$. Now let $P = C$, so that the points $X_1, ..., X_N$ are uniformly distributed on the boundary of the cube and the convex hull of these points is denoted by P_N. We are interested in the expected number of facets $\mathbb{E}f_2(P_N)$ of the convex hull P_N. More specifically, we want to prove that

$$\mathbb{E}f_2(P_N) = c\ln N(1+O((\ln N)^{-1})) \quad \text{as } N \to \infty, \tag{1.4}$$

with some $c > 0$ independent of C. A more general version of this theorem is proven in the pre-print of Reitzner, Schütt and Werner [18]. They prove this theorem for simple polytopes of which the cube C is an example. Proving this theorem for the cube is the second and last goal of this thesis.

This thesis is structured as follows. In Chapter 2, the background material that is necessary for proving the theorems is given. In Chapter 3, we will prove first Equations (1.2) and (1.3) followed by the prove of Equation (1.1). Lastly, Chapter 4 gives the proof of Equation (1.4).

Chapter 2

Preliminaries

2.1 Geometry

Following the notation in Schneider and Weil [12], we write for the volume of the d-dimensional unit ball,

$$\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$$

and the surface area of the unit sphere $\mathbb{S}^{d-1}$,

$$\omega_d = d\kappa_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

Furthermore, we will use the identity

$$b_{dq} = \frac{\omega_{d-q+1}\cdots\omega_d}{\omega_1\cdots\omega_q} \implies b_{(d+1)(d-1)} = \frac{\omega_d\omega_{d+1}}{\omega_1\omega_2} = \frac{\omega_d\omega_{d+1}}{4\pi},\ b_{(d-1)(d-1)} = 1.$$

2.1.1 Polytopes

If V is a set of *vertices* or *points*, then the line connecting two distinct points $x, y \in V$ is called an *edge* and it is denoted by $[x, y]$. From [13], a set $A \in \mathbb{R}^d$ is *convex* if for any two points $x, y \in A$, the segment $[x, y] \in A$, thus if $(1-\lambda)x + \lambda y \in A$ for $x, y \in A$, $0 \leq \lambda \leq 1$. The *convex hull* of A is the smallest convex set that contains A and it is denoted by $[A]$. The convex hull of finitely many points is called a *polytope*. There are two examples of a convex hull in Figures 2.1 and 2.2. Note that in Figure 2.2, due to the smoothness of the boundary, each point that is chosen on the boundary of the set is also included in the convex hull. In these figures, the set A is a set of points $\{X_1, ..., X_n\}$, so the convex hull is denoted by $[A] = [X_1, ..., X_n]$. If the points $X_1, ..., X_n$ are random variables, then the convex hull of these points is a *random polytope*. A d-dimensional polytope, or a *d-polytope*, has k-dimensional faces for $k = 0, ..., d-1$. The 0-dimensional faces are the vertices, 1-dimensional faces are the edges and the $(d-1)$-dimensional faces are called the *facets*. An n-polytope is called *simple* if each of its vertices is contained in exactly n facets. A *k-simplex* is a k-polytope that is the convex hull of $k+1$ points. That means that the $k+1$ points must be linearly independent or else the polytope would not be k-dimensional. For example, a 0-simplex is a point, a 1-simplex is a line, a 2-simplex is a triangle (and it can never be a rectangle), a 3-simplex is a tetrahedron, etcetera.

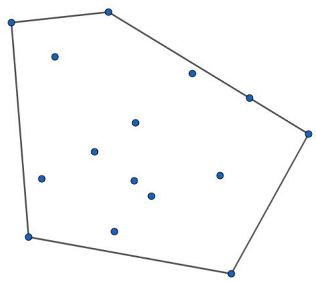


Figure 2.1: Convex hull of some points in $\mathbb{R}^2$.

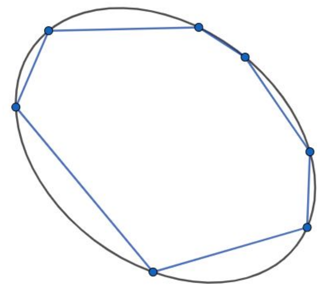


Figure 2.2: Convex hull of points on the boundary of a smooth convex set.

2.1.2 Affine Geometry

From [4], an *affine subspace* W of $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ such that $\overline{W} = \{w - y : w \in W\}$ is a linear subspace of $\mathbb{R}^d$ for a fixed point $y \in \mathbb{R}^d$. If an affine subspace of $\mathbb{R}^d$ has dimension $d-1$, it is called an *affine hyperplane*. In practice, this means that we can write the equation for an affine hyperplane as

$$a_1x_1 + a_2x_2 + \cdots + a_dx_d = a_{d+1}, \quad a_1, ..., a_{d+1} \in \mathbb{R},$$

where not all of the $a_1, ..., a_d$ can be zero. We can also say that an affine subspace is obtained by shifting a linear subspace by a fixed vector. An *affine transformation* maps points to points, lines to lines, planes to planes. As a result of this, parallelism is preserved. Actions that are allowed in an affine transformation are translation, rotation, scaling and shearing.

2.2 Integral Geometry

There are three specific measures that we will introduce in this section. For the first two, we use the clear explanation that is given in Last and Penrose [15]. Fix a number $d \in \mathbb{N}$ and consider the Euclidean space $\mathbb{R}^d$ with norm $\|\cdot\|$. For any subset $S \subset \mathbb{R}^d$, the *diameter* of S is defined as $\operatorname{diam}(S) = \sup\{|x-y| : x, y \in S\}$. The *Lebesgue measure* λ_d is the unique measure satisfying $\lambda_d([0,1]^d) = 1$. The volume of the unit ball B^d is denoted by $\kappa_d = \lambda_d(B^d)$.

We want to assign an m-dimensional measure to an m-dimensional subset of $\mathbb{R}^d$ for $m < d$, but that is not straightforward. Using the previously mentioned definition, we can define a measure that can deal with these sets. For $B \subset \mathbb{R}^d$ with $m < d$, we define the m-dimensional *Hausdorff measure* $\mathcal{H}^m(B)$ by the following process. For small δ, cover B efficiently by countably many sets B_j with $\operatorname{diam}(B_j) \leq \delta$, add up all the $\alpha_m(\frac{\operatorname{diam}(B_j)}{2})^m$, and take the limit as $\delta \to 0$. That is,

$$\mathcal{H}^m(B) = \lim_{\delta \to 0} \inf_{\substack{B \subset \cup B_j \\ \operatorname{diam}(B_j) \leq \delta}} \sum_{j=1}^{\infty} \alpha_m \left(\frac{\operatorname{diam}(B_j)}{2} \right)^m,$$

where the infimum is taken over all countable collections $B_1, B_2, ... \subset \mathbb{R}^d$. When $m = d$, the Hausdorff measure is equal to the Lebesgue measure. When $m = 0$, the Hausdorff measure $\mathcal{H}^0$ is the counting measure. The volume functional V_n on a compact set K is defined as the restriction of the n-dimensional Hausdorff measure $\mathcal{H}^n$ to K.

Lastly, we use the book by Conway [16] to learn about the Haar measure. Let G be a locally compact group. Then there is, up to a multiplicative constant, a unique positive Borel measure μ on G such that

1. if U is a nonempty open subset of G, then $\mu(U) > 0$.
2. if S is any Borel subset of G and $x \in G$, then $\mu(S) = \mu(Sx)$.

The measure μ is called the *left-Haar measure* for G. The second condition means that this measure is *left-translation invariant*. If the group G is a group under addition, then the Haar measure is both left and right-translation invariant, because addition is commutative. When a measure is left and right-translation invariant, we call it the *Haar measure* for G. For example, consider the group $(\mathbb{R}, +)$ and let $[a, b] = S \subset \mathbb{R}$. Denote $c + S = \{c + s : s \in S\}$, then

$$\mu(S) = \int_S \lambda(\mathrm{d}x) = b - a = (b + c) - (a + c) = \int_{c+S} \lambda(\mathrm{d}x) = \mu(c + S) = \mu(S + c).$$

This means that the Haar measure of $(\mathbb{R}, +)$ is the restriction of the Lebesgue measure to subsets of $\mathbb{R}$.

2.2.1 Blaschke-Petkantschin formula

A theorem of major importance for this thesis is a Blaschke-Petkantschin formula. This formula gives a decomposition of d copies of the space $\mathbb{S}^{d-1}$ into d-dimensional affine subspaces of $\mathbb{S}^{d-1}$. We will see that this setting fits perfectly in the problems we will be investigating. Let's clarify the notation beforehand. The $(d-1)$-dimensional volume of the convex hull of d points $x_1, ..., x_d$ is denoted by $\Delta_{d-1}(x_1, ..., x_d)$. Furthermore, the space $A(d, q)$ denotes the parameterization of all q-dimensional affine subspaces of $\mathbb{R}^d$ for $q \in \{0, 1, ..., d\}$. It is a locally compact space with respect to the group of Euclidean motions. This group comprises arbitrary combinations of translations and rotations. We will see later that this is exactly what we need to do with our subspaces: we are going to shift the elements of $A(d, q)$ to the origin so that they become linear subspaces. The corresponding q-dimensional Haar measure μ_q is normalized such that

$$\mu_q(\{H \in A(d, q) : H \cap B^d \neq \emptyset\}) = \kappa_{d-q}.$$

We will state here the Blaschke-Petkantschin formula for points on a sphere in the same form as in [14, Proposition 3]:

Theorem 2.1 (Blaschke-Petkantschin for points on a sphere)**.** *Let $f : (\mathbb{S}^{d-1})^d \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\begin{aligned}
&\int_{(\mathbb{S}^{d-1})^d} f(x_1, ..., x_d) \mathcal{H}^{d(d-1)}_{(\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1, ..., x_d)) \\
&= \frac{w_d}{2}(d-1)! \int_{A(d,d-1)} \int_{(H \cap \mathbb{S}^{d-1})^d} f(x_1, ..., x_d) \\
&\qquad \times \Delta_{d-1}(x_1, ..., x_d)(1 - h^2)^{-\frac{d}{2}} \mathcal{H}^{d(d-2)}_{(H \cap \mathbb{S}^{d-1})^d}(\mathrm{d}(x_1, ..., x_d)) \mu_{d-1}(\mathrm{d}H),
\end{aligned}$$

where h denotes the distance from H to the origin.

The proof of a more general theorem can be found in Zähle [6, Theorem 1]. The purpose of this theorem is to apply a geometric transformation to the original domain of the integration, since in many applications, the integration over the original domain cannot be executed directly.

From the Blaschke-Petkantschin formula in d dimensions, we can readily derive the formula for 2 and 3 dimensions.

Corollary 2.2. *Let $f : (\mathbb{S}^1)^2 \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\begin{aligned}&\int_{(\mathbb{S}^1)^2} f(x_1,x_2)\mathcal{H}^2_{(\mathbb{S}^1)^2}(\mathrm{d}(x_1,x_2))\\&= \frac{w_2}{2}\int_{A(2,1)}\int_{(H\cap\mathbb{S}^1)^2} f(x_1,x_2)\Delta_1(x_1,x_2)(1-h^2)^{-1}\mathcal{H}^0_{(H\cap\mathbb{S}^1)^2}(\mathrm{d}(x_1,x_2))\mu_1(\mathrm{d}H),\end{aligned}$$

where h denotes the distance from H to the origin.

Corollary 2.3. *Let $f : (\mathbb{S}^{d-1})^d \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\begin{aligned}&\int_{(\mathbb{S}^2)^3} f(x_1,x_2,x_3)\mathcal{H}^6_{(\mathbb{S}^2)^3}(\mathrm{d}(x_1,x_2,x_3))\\&= w_3\int_{A(3,2)}\int_{(H\cap\mathbb{S}^2)^3} f(x_1,x_2,x_3)\Delta_{d-1}(x_1,x_2,x_3)(1-h^2)^{-\frac{3}{2}}\mathcal{H}^3_{(H\cap\mathbb{S}^2)^3}(\mathrm{d}(x_1,x_2,x_3))\mu_2(\mathrm{d}H),\end{aligned}$$

where h denotes the distance from H to the origin.

We have found a formula that changes the integration over d point on $\mathbb{S}^{d-1}$ to the integration over the intersection of $\mathbb{S}^{d-1}$ with an affine hyperplane and over all affine hyperplanes. We still can make this domain of integration easier to deal with. It is not straightforward to integrate over the space $A(d,d-1)$, so we decompose this domain into two domains. A rigorous proof of a more general result is given by Schneider and Weil [12, Theorem 13.2.12], but we will give an intuitive explanation here. An element $H \in A(d,d-1)$ is uniquely determined by the normal vector $u \in \mathbb{S}^{d-1}$ and the distance $h \in (-\infty,\infty)$ from the origin to H. This gives $H = u^\perp + h$. This doesn't change the value of the measure μ_{d-1}, because it is invariant under translation. The hyperplane H is intersecting the sphere $\mathbb{S}^{d-1}$ which is symmetric, so instead of taking $h \in (-\infty,\infty)$, we can take $h \in [0,\infty)$ and multiply the result by 2. Furthermore, for $h \in (1,\infty)$, we have $H\cap\mathbb{S}^{d-1} = \emptyset$, hence we restrict to $h \in [0,1]$. Lastly, $u \in \mathbb{S}^{d-1}$ has Hausdorff measure ω_d, so we divide by the normalization constant $\frac{1}{\omega_d}$. This comes down to the following theorem.

Theorem 2.4. *The Haar measure μ_{d-1} satisfies*

$$\int_{A(d,d-1)} f(H)\mu_{d-1}(\mathrm{d}H) = \frac{2}{\omega_d}\int_{\mathbb{S}^{d-1}}\int_0^1 f(H(u,h))\mathrm{d}h\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$

for every measurable function $f \geq 0$ on $A(d,d-1)$.

This theorem is for all dimensions $d \in \{2,3,...\}$. We will need it in later sections for 2 and 3 dimensions. These special cases can be found readily.

Corollary 2.5. *The Haar measure μ_1 satisfies*

$$\int_{A(2,1)} f(H)\mu_1(\mathrm{d}H) = \frac{2}{\omega_2}\int_{\mathbb{S}^1}\int_0^1 f(H(u,h))\mathrm{d}h\mathcal{H}^1_{\mathbb{S}^1}(\mathrm{d}u)$$

for every measurable function $f \geq 0$ on $A(2,1)$.

Corollary 2.6. *The Haar measure μ_2 satisfies*

$$\int_{A(3,2)} f(H)\mu_2(\mathrm{d}H) = \frac{2}{\omega_3}\int_{\mathbb{S}^2}\int_0^1 f(H(u,h))\mathrm{d}h\mathcal{H}^2_{\mathbb{S}^2}(\mathrm{d}u)$$

for every measurable function $f \geq 0$ on $A(3,2)$.

2.2.2 Other Theorems

As a preparation for the upcoming calculations, another theorem will be presented. This theorem is useful when a univariate function is integrated over a $(d-1)$-dimensional sphere as will be done in Chapters 3.2 and 3.3. A similar strategy as in Theorem 2.1 is used: instead of integrating over $\mathbb{S}^{d-1}$, integrate over lower-dimensional spherical slices. This theorem is originally stated in Axler et al [17, A.4], but here it is used in the same form as in [14, Proposition 4].

Theorem 2.7. *Let $f : \mathbb{S}^{d-1} \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int_{\mathbb{S}^{d-1}} f(x)\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}x) = \int_{-1}^{1} (1-t^2)^{\frac{d-3}{2}} \int_{\mathbb{S}^{d-2}} f(t, \sqrt{1-t^2}y)\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}y)\mathrm{d}t.$$

The variable $x = x_1, ..., x_d \in \mathbb{S}^{d-1}$ is decomposed into the variables t and y with $t = x_1$ and $y = x_2, ..., x_d$.

Another important theorem for our purposes is taken from Schneider and Weil [12, Theorem 8.2.3]. The theorem contains two statements, but only the second one will be used. Furthermore, we restrict to the case where $k = 2$ and $q = d$, since those are the relevant settings here. Up to a normalising constant, it gives the second moment of the volume of a random simplex with vertices on a sphere. It states the following:

Theorem 2.8. *For integers $d \geq 1$, $1 \leq q \leq d$, $k \geq 0$,*

$$\begin{aligned} S(d,q,k) :&= \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} \Delta_d(u_0, ..., u_d)^2 \sigma(\mathrm{d}u_0) \cdots \sigma(\mathrm{d}u_d) \\ &= \frac{1}{(d!)^2} \omega_{d+2}^{d+1} \frac{\kappa_{d^2+d-2}}{\kappa_{d(d+1)}} \frac{b_{dd}}{b_{(d+2)d}} \end{aligned}$$

with constants that are given in the beginning of this chapter.

Lastly, Wendel [1] proved a theorem about points on the surface of a d-dimensional sphere, just like in our problem.

Theorem 2.9. *Let N points be scattered at random on the surface of the unit sphere $\mathbb{S}^{d-1}$. The probability that all points lie on some hemisphere is equal to $N \cdot 2^{-N+1}$.*

2.2.3 Tools for the cube

We will need Theorem 1 from Zähle [6] once more, but now for an application to polytopes. We use the version of this theorem that is stated in Reitzner, Schütt and Werner [18] and we adapt it so that it applies to 3-dimensional polytopes:

Lemma 2.10. *For a polytope P, let $g(x_1, x_2, x_3)$ be a continuous function. Then there is a constant β such that*

$$\begin{aligned} &\int\limits_{(\partial P)^3_{\neq}} g(x_1, x_2, x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ &= \beta^{-1} \int\limits_{\mathbb{S}^2} \int\limits_{\mathbb{R}} \int\limits_{(\partial P \cap H)^3_{\neq}} g(x_1, x_2, x_3)\lambda_2([x_1, x_2, x_3]) J(T_{x_1}, H)^{-1} J(T_{x_2}, H)^{-1} J(T_{x_3}, H)^{-1} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}h\mathrm{d}u \end{aligned}$$

with $\mathrm{d}x$, $\mathrm{d}u$, $\mathrm{d}h$ denoting integration with respect to the Hausdorff measure on the respective range of integration.

Here, T_{x_i} denotes the tangent hyperplane at x_i to ∂P. The point x_i is located on a facet F of ∂P, so T_{x_i} is the hyperplane containing the facet F. Furthermore, $H = H(x_1, x_2, x_3)$ is the affine hull of the points x_1, x_2, x_3. Then $J(T_{x_i}, H)$ is the length of the orthogonal projection of a unit interval in $T_{x_i} \cap (T_{x_i} \cap H)^\perp$ onto $H^\perp$. This equal to $\sin \sphericalangle(T_{x_i}^\perp, H^\perp)$. If T_{x_i} and H are parallel, then $J(T_{x_i}, H) = 0$.

2.3 Laplace's method

Integrals of the form $I(N) = \int_a^b f(x)e^{-Ng(x)}\mathrm{d}x$ can be approximated using Laplace's approximation. Here, the functions $f(x)$ and $g(x)$ are real, continuous on $[a, b]$ and $g(x)$ is positive there. The assumptions and statement of this approximation are given in Theorem 1 in Wong [9]. We will use this theorem to approximate the integral $I(N)$. We assume that

1. $g(x)$ has one minimum in $[a, b]$ which occurs at either $x = a$ or $x = b$,
2. If the minimum occurs at $x = a$, then $g(x) = g(a) + c_1(x - a)^\mu + O((x - a)^{\mu+1})$ as $x \to a^+$.
   If the minimum occurs at $x = b$, then $g(x) = g(b) + c_1(b - x)^\mu + O((b - x)^{\mu+1})$ as $x \to b^-$.
   In both cases, $\mu > 0$ and $c_1 \neq 0$.
3. If the minimum occurs at $x = a$, then $f(x) = c_2(x - a)^{\alpha-1} + O((x - a)^\alpha)$ as $x \to a^+$.
   If the minimum occurs at $x = b$, then $f(x) = c_2(b - x)^{\alpha-1} + O((b - x)^\alpha)$ as $x \to b^-$.
   In both cases, $\alpha > 0$ and $c_2 \neq 0$.

Then the theorem states that the integral $I(N) = \int_a^b f(x)e^{-Ng(x)}\mathrm{d}x$ can be approximated by

$$I(N) = e^{-Ng(m)} \left(\Gamma\Big(\frac{\alpha}{\mu}\Big) \frac{c_0}{N^{\alpha/\mu}} + O(N^{-(\alpha+1)/\mu}) \right), \tag{2.1}$$

as $N \to \infty$, where $m := \arg\min_{[a,b]} g(x)$ so that it is equal to either a or b. The coefficient c_0 is given by

$$c_0 = \frac{c_2}{\mu c_1^{\alpha/\mu}}.$$

Chapter 3

Sphere

Each section of this chapter deals with the same question, but for different dimensions. First, we will treat the problem in 2 and 3 dimensions. Later, this will be generalized to d dimensions. Actually, the results for the 2 and 3-dimensional case both follow from the d-dimensional case. However, the first two sections function as a warm-up for the last section. Like introduced in Chapter 1, we will proof Equation (1.2) in Chapter 3.1, Equation (1.3) in Chapter 3.2 and Equation (1.1) in Chapter 3.3.

3.1 The 2-dimensional case

When N points are chosen on the circle $\mathbb{S}^1$, the convex hull of those points always consist of N points. Therefore, as N increases, the convex hull of the points will approach the shape of the circle. See Figure 3.1. The area enclosed by $\mathbb{S}^1$ is the area of the unit disk B^2, which is equal to π, so the area of the convex hull of N points on $\mathbb{S}^1$ goes to π as N goes to infinity.

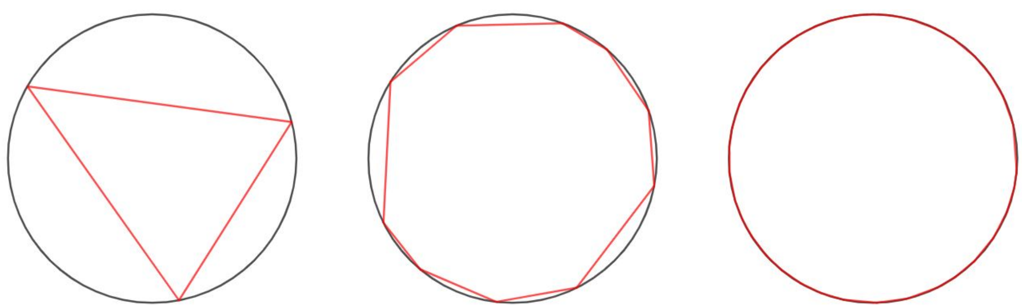


Figure 3.1: Convex hull of points on $\mathbb{S}^1$ for different values of N.

Let $X_1, ..., X_N$ be independent random points distributed uniformly on $\mathbb{S}^1$. The convex hull of these points will be denoted by $[X_1, ..., X_N] =: P_N$. The surface between the circle and the convex hull is called the missing surface and it will thus go to zero, but it is not evident how fast this goes. The goal of this section is to find this rate of convergence. This can be found by calculating the expected value of the missing surface between $\mathbb{S}^1$ and P_N, formally written as $\mathbb{E}[V_2(B^2) - V_2(P_N)]$. We will prove the following theorem in this section.

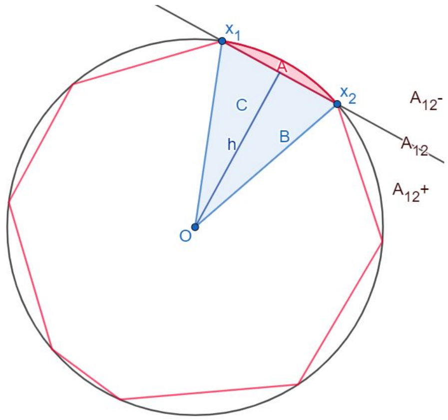


Figure 3.2: Missing surface in red created by the edge $[x_1, x_2]$. Here, origin is contained in P_N.

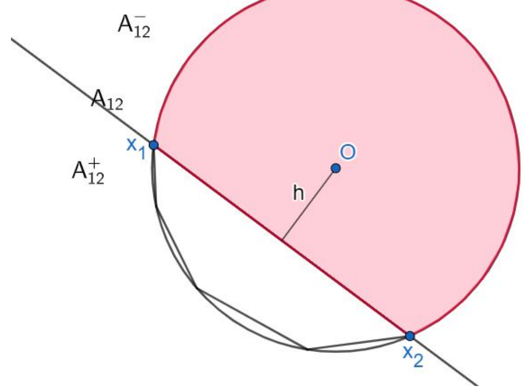


Figure 3.3: Situation when origin is not contained in P_N.

Theorem 3.1. *If P_N is the convex hull of N independent and uniformly distributed random points on $\mathbb{S}^1$, then*

$$\mathbb{E}[V_2(B^2) - V_2(P_N)] = O(N^{-2}).$$

Proof. In Figure 3.2, the area A is the missing surface created by two points x_1 and x_2. When N points are drawn on the circle, there are N areas like area A. However, x_1 and x_2 create a missing surface between $\mathbb{S}^1$ and P_N *only if* $[x_1, x_2]$ is an edge of P_N.

Before doing any calculations, we will define some events to ease notation. Let $\mathcal{F}_k(P_N)$ be the set of k-dimensional faces of P_N, hence $\mathcal{F}_1(P_N)$ is the set of all edges of P_N. We want to specify which points form an edge of P_N. This is described by the event $F_{ij} = \{[X_i, X_j] \in \mathcal{F}_1(P_N)\}$. Furthermore, if the event F_{ij} holds, the area enclosed by the edge $[X_i, X_j]$ and the circle is part of the missing surface.

Assuming that F_{ij} holds, define A_{ij} as the line spanned by the edge $[X_i, X_j]$. Let A_{ij}^+ be the halfplane bounded by A_{ij} that contains P_N and let A_{ij}^- be the other halfplane, so $P_N \cap A_{ij}^+ = P_N$ and $P_N \cap A_{ij}^- = \emptyset$. This is also pictured in Figure 3.2 for x_1 and x_2. Then the missing surface created by the edge $[X_i, X_j]$ is given by $B^2 \cap A_{ij}^- := M_{ij}$. For example, the red area in Figure 3.2 would be called M_{12}. The total missing surface between $\mathbb{S}^1$ and P_N is found by adding the missing surfaces created by the facets of P_N.

$$\mathbb{E}[V_2(B^2) - V_2(P_N)] = \sum_{1 \leq i \leq j \leq N} \mathbb{E}[V_2(M_{ij})\mathbb{1}(F_{ij})].$$

The summation is done over all ordered pairs of distinct and increasing indices, hence there are $\binom{N}{2}$ of them. The points $X_1, ..., X_N$ are independent and identically distributed, which means that for all $1 \leq i \leq j \leq N$ the summands are equal. Therefore, we can restrict ourselves to one such pair, e.g. $(i, j) = (1, 2)$, and multiply the result with the number of pairs, $\binom{N}{2}$:

$$\mathbb{E}[V_2(B^2) - V_2(P_N)] = \binom{N}{2}\mathbb{E}[V_2(M_{12})\mathbb{1}(F_{12})]. \tag{3.1}$$

In Figure 3.2, the polytope P_N contains the origin, so the shape of each M_{ij} is the same. It can happen that the origin is not contained in the polytope P_N, even tough it is with low probability.

An example of this situation is given in Figure 3.3, where we see that M_{12} is given by the red area which has a different shape than we have seen before. Hence, the area M_{12} cannot be described by a universal formula. However, the situation in Figure 3.3 happens only when $0 \notin P_N$, which happens with very low probability.

The area M_{12} is either the smaller or larger part of the circle. Let $\hat{M}_{12}$ be the smaller area that is created by the edge $[x_1, x_2]$. That means that if $0 \in P_N$, then $M_{12} = \hat{M}_{12}$ and if $0 \notin P_N$, then $M_{12} \neq \hat{M}_{12}$. We want to use $\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})]$ instead of the expected value that is in Equation (3.1), but we have to make sure that we do not lose too much precision. Therefore, let's see what the difference between these expected values is.

$$
\begin{aligned}
0 &\leq \mathbb{E}[V_2(M_{12})\mathbb{1}(F_{12})] - \mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] \\
&= \mathbb{E}[\big(V_2(M_{12}) - V_2(\hat{M}_{12})\big)\mathbb{1}(F_{12})] \\
&\leq \mathbb{E}[V_2(M_{12}) - V_2(\hat{M}_{12})] \\
&\leq \mathbb{E}[\mathbb{1}(M_{12} \neq \hat{M}_{12})\kappa_2] \\
&= \kappa_2 \mathbb{P}(M_{12} \neq \hat{M}_{12}) \\
&\leq \kappa_2 \mathbb{P}(0 \notin P_N)
\end{aligned}
$$

The probability $\mathbb{P}(0 \notin P_N)$ is the probability that all points are located in one hemisphere. The value of this probability is given in Theorem 2.9. Using that result, we get

$$
0 \leq \mathbb{E}[V_2(M_{12})\mathbb{1}(F_{12})] - \mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] \leq \frac{\kappa_2 N}{2^{N-1}} = O\Big(\frac{N}{2^{N-1}}\Big). \tag{3.2}
$$

Equations (3.1) and (3.2) give

$$
\mathbb{E}[V_2(B^2) - V_2(P_N)] \leq \binom{N}{2}\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] + \binom{N}{2} O\Big(\frac{N}{2^{N-1}}\Big). \tag{3.3}
$$

The difference is indeed small, but in order to see if it is small enough, we have to find the remaining expected value. That is what we will do in the rest of this section.

The points $X_1, ..., X_N$ are uniformly distributed on the circle, so

$$
\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] = \int_{(\mathbb{S}^1)^2} V_2(\hat{M}_{12})\mathbb{P}(F_{12}) \frac{\mathcal{H}^2_{(\mathbb{S}^1)^2}}{(2\pi)^2} \mathrm{d}(x_1, x_2). \tag{3.4}
$$

Here, $\frac{1}{(2\pi)^2}$ is the normalization constant since the Hausdorff measure of the sphere $\mathbb{S}^1$ is equal to 2π and we are integrating both x_1 and x_2 over the sphere.

We will take a closer look at $V_2(\hat{M}_{12})$ and $\mathbb{P}(F_{12})$. The value of $V_2(\hat{M}_{12})$ is given by the surface of the red area in Figure 3.2. It is found by subtracting the isosceles triangle formed by the blue area from the sector formed by the union of the red and blue areas. The distance from the edge $[X_1, X_2]$ to the origin will be denoted by $h := h_{X_1,X_2}$ and it depends hence on the choice of X_1 and X_2. The blue area is an isosceles triangle and we assume it has height h. Then the surface of the blue area is equal to $h\sqrt{1-h^2}$. We can also find the area of the union of the blue and red area, which is a sector of the circle. Assuming that the angle $\angle x_1 O x_2 = \alpha$, the area of the sector is equal to $\frac{\alpha}{2\pi}\pi = \frac{\alpha}{2}$. Note that $h = \cos(\frac{\alpha}{2})$, so in terms of h, the surface of the union of the blue and red area is equal to $\cos^{-1}(h)$. Hence, when the origin is contained in the polytope P_N, the area of $\hat{M}_{12}$ is equal to

$$
V_2(\hat{M}_{12}) = \cos^{-1}(h) - h\sqrt{1-h^2}.
$$

Now for $\mathbb{P}(F_{12})$, the probability that $[X_1, X_2]$ is an edge of P_N is equal to the probability that $X_3, ..., X_N$ is not on the arc between X_1 and X_2. Since we assume that the origin is in P_N, we take the smaller arc between X_1 and X_2. The points $X_3, ..., X_N$ are all chosen independently, so this can be generalized using a random variable X that is uniformly distributed on the circle as well:

$$\begin{aligned}\mathbb{P}(F_{12}) &= \mathbb{P}([X_1, X_2] \in \mathcal{F}_1(P_N)) \\ &= \mathbb{P}(X_3, ..., X_N \text{ is not on } \operatorname{arc}(X_1, X_2)) \\ &= \mathbb{P}(X \text{ is not on } \operatorname{arc}(X_1, X_2))^{N-2} \\ &= \left(\frac{2\pi - 2\cos^{-1}(h)}{2\pi}\right)^{N-2} \\ &= \left(\frac{\pi - \cos^{-1}(h)}{\pi}\right)^{N-2}.\end{aligned}$$

Substitute the expressions for $V_2(\hat{M}_{12})$ and $\mathbb{P}(F_{12})$ into Equation (3.4):

$$\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] = \frac{1}{4\pi^2}\int_{(\mathbb{S}^1)^2} (\cos^{-1}(h) - h\sqrt{1-h^2}) \left(\frac{\pi - \cos^{-1}(h)}{\pi}\right)^{N-2} \mathcal{H}^2_{(\mathbb{S}^1)^2}(\mathrm{d}(x_1, x_2))$$

and recall that $h := h_{X_1, X_2}$ depends on the choice of X_1 and X_2.

Since we are investigating a volume that will go to zero, any constants in front of our equations that do not depend on N will not make a difference for our final result, so we will denote every constant by c. Hence, the value of c can differ from occurrence to occurrence.

The 2-dimensional Blaschke–Petkantschin formula in Corollary 2.2 can be applied to the last equation, which gives

$$\begin{aligned}\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] &= c\int_{A(2,1)}\int_{(H\cap\mathbb{S}^1)^2} \left(\cos^{-1}(h) - h\sqrt{1-h^2}\right)\left(\frac{\pi - \cos^{-1}(h)}{\pi}\right)^{N-2} \\ &\quad \times \Delta_1(x_1, x_2)\frac{1}{1-h^2}\mathcal{H}^0_{(H\cap\mathbb{S}^1)^2}(\mathrm{d}(x_1, x_2))\mu_1(\mathrm{d}H) \\ &= c\int_{A(2,1)} \left(\cos^{-1}(h) - h\sqrt{1-h^2}\right)\frac{1}{1-h^2}\left(\frac{\pi - \cos^{-1}(h)}{\pi}\right)^{N-2} \\ &\quad \times \int_{(H\cap\mathbb{S}^1)^2} \Delta_1(x_1, x_2)\mathcal{H}^0_{(H\cap\mathbb{S}^1)^2}(\mathrm{d}(x_1, x_2))\mu_1(\mathrm{d}H).\end{aligned}$$

The factor $\Delta_1(x_1, x_2)$ is the 1-dimensional surface of the convex hull of x_1 and x_2, which is simply distance from x_1 to x_2. The points x_1 and x_2 are chosen from $H \cap \mathbb{S}^1$. A circle intersected with a line that goes through the circle results in just two points. So there are only four configurations of x_1, x_2 in this integration: $(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2)$. Therefore, the integral changes to a sum. The length between two equal points is zero and the length between two different points is equal to $2\sqrt{1-h^2}$. Therefore the integral with respect to (x_1, x_2) can be written as

$$\int_{(H\cap\mathbb{S}^1)^2} \Delta_1(x_1, x_2)\mathcal{H}^0_{(H\cap\mathbb{S}^1)^2}(\mathrm{d}(x_1, x_2)) = \sum_{\substack{(x_1,x_1),(x_1,x_2),\\(x_2,x_1),(x_2,x_2)}} \Delta_1(x_1, x_2) = 4\sqrt{1-h^2}.$$

Plugging this back in gives

$$\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] = c\int_{A(2,1)} \left(\cos^{-1}(h) - h\sqrt{1-h^2}\right)\frac{1}{1-h^2}\left(\frac{\pi - \cos^{-1}(h)}{\pi}\right)^{N-2}\sqrt{1-h^2}\,\mu_1(\mathrm{d}H).$$

The domain of the integral, $A(2,1)$, can be decomposed into the linear hyperplane parallel to H and the distance between H and the origin, as is done in Corollary 2.5.

$$\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] = c\int_{\mathbb{S}^1}\int_0^1 \left(\frac{\cos^{-1}(h)}{\sqrt{1-h^2}} - h\right)\left(\frac{\pi - \cos^{-1}(h)}{\pi}\right)^{N-2} \mathrm{d}h\mathcal{H}^1_{\mathbb{S}^1}(\mathrm{d}u).$$

The h in the last expression is the same h as in Figure 3.2. The integrand is independent of u, so the integration over $\mathbb{S}^1$ is equal to $\omega_2 = 2\pi$:

$$\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] = c\int_0^1 \left(\frac{\cos^{-1}(h)}{\sqrt{1-h^2}} - h\right)\left(\frac{\pi - \cos^{-1}(h)}{\pi}\right)^{N-2} \mathrm{d}h$$

For reasons that will become clear later, we will make the substitution $h = \sqrt{1-s^2}$ with $\mathrm{d}h = \frac{s}{\sqrt{1-s^2}}\mathrm{d}s$. This gives

$$\begin{aligned}
\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] &= c\int_0^1 \left(\frac{\cos^{-1}(\sqrt{1-s^2})}{s} - \sqrt{1-s^2}\right)\left(\frac{\pi - \cos^{-1}(\sqrt{1-s^2})}{\pi}\right)^{N-2} \frac{s}{\sqrt{1-s^2}}\mathrm{d}h \\
&= c\int_0^1 \left(\frac{\cos^{-1}(\sqrt{1-s^2})}{\sqrt{1-s^2}} - s\right)\left(\frac{\pi - \cos^{-1}(\sqrt{1-s^2})}{\pi}\right)^{N-2} \mathrm{d}h \\
&= c\int_0^1 f(s)e^{-(N-2)g(s)}\mathrm{d}s. \qquad (3.5)
\end{aligned}$$

where $f(s) = \frac{\cos^{-1}(\sqrt{1-s^2})}{\sqrt{1-s^2}} - s$ and $g(s) = -\ln\left(\frac{\pi - \cos^{-1}(\sqrt{1-s^2})}{\pi}\right)$. The maximum of the function $g(s)$ is at $s = 0$. This last integral can be computed using Laplace's method which is explained in Chapter 2.3. This method uses three assumptions which are also listed in that chapter. The minimum of the function $g(s)$ occurs at $s = 0$, so the first assumption is satisfied. We can satisfy the second and third assumptions by finding the Taylor expansions of the functions f and g around $s = 0$.

$$\begin{aligned}
f(s) &= f(0) + f'(0)s + \frac{f''(0)}{2}s^2 + \frac{f'''(0)s^3}{6} + O(s^4) \\
&= \frac{4}{6}s^3 + O(s^5) \\
g(s) &= g(0) + g'(0)s + O(s^2) \\
&= \frac{s}{\pi} + O(s^2)
\end{aligned}$$

This gives for $f(h)$: $c_2 = \frac{4}{6}$ and $\alpha = 4$ and for $g(h)$: $c_1 = \frac{1}{\pi}$ and $\mu = 1$. We find the coefficient $c_0 = \frac{4/6}{(1/\pi)^4} = \frac{4\pi^4}{6}$. If we didn't make the substitution $s = \sqrt{1-h^2}$, then we would get for the function g that $g(h) = -\ln(\frac{\pi - \cos^{-1} h}{\pi})$. However, for $g(h)$ at its minimum $h = 1$ it holds that, $g(1) = 0$ and $g'(1) = -\infty$, so it has no Taylor expansion around that point. Hence, the substitution was necessary. We can approximate Equation (3.5) using Equation (2.1).

$$\begin{aligned}
\mathbb{E}[V_2(\hat{M}_{12})\mathbb{1}(F_{12})] &= c\left(\Gamma(4)\frac{4\pi^4}{6N^4} + O(N^{-5})\right) \\
&= c\frac{1}{N^4}(1 + O(N^{-1})). \qquad (3.6)
\end{aligned}$$

Now we go back to the formula in Equation (3.3) and plug in the results of Equation (3.6). We find that

$$\mathbb{E}[V_2(B^2) - V_2(P_N)] \leq \binom{N}{2}c\frac{1}{N^4}(1 + O(N^{-1})) + \binom{N}{2}O\Big(\frac{N}{2^{N-1}}\Big).$$

We can write for the binomial: $\binom{N}{2} = \frac{N(N-1)}{2} = \frac{N^2}{2} - \frac{N}{2} = \frac{N^2}{2}(1 + O(N^{-1}))$. Then

$$\begin{aligned}\mathbb{E}[V_2(B^2) - V_2(P_N)] &\leq \frac{N^2}{2} c \frac{1}{N^4}(1 + O(N^{-1})) + O\Big(\frac{N^3}{2^{N-1}}\Big) \\ &= c\frac{1}{N^2}(1 + O(N^{-1})) + O\Big(\frac{N^3}{2^{N-1}}\Big) \\ &= O(N^{-2}).\end{aligned}$$

which is the statement of Theorem 3.1. □

3.2 The 3-dimensional case

In this section, we will answer the same question as in previous section, but now for the sphere $\mathbb{S}^2$. However, we use a different method in this 3-dimensional case compared to the previous section. In the previous section, we added the local missing volumes of the facets of the convex hull, but in this section we approximate the volume of the convex hull directly. Therefore, it is important to keep track of the constants that we find along the way, so we cannot generalize it to one constant c like we did in the previous section.

Let $X_1, ..., X_N$ be independent random points distributed uniformly on $\mathbb{S}^2$, which is the boundary of the ball B^3. The convex hull of these points will be denoted by $[X_1, ..., X_N] =: P_N$. As N goes to infinity, the convex hull of the points will tend to cover the whole ball. Hence the 3-dimensional volume of the convex hull approaches the volume of the ball. The difference of the volumes of the ball and the convex hull is called the missing volume and it will thus go to zero. The goal is to find how fast the missing volume goes to zero. We do this by finding the volume of P_N and subtract it from the volume of the sphere. Since the volume of the sphere is known to be $\frac{4\pi}{3}$, the only expected value that will be calculated is the volume of the convex hull: $\mathbb{E}[V_3(P_N)]$. When this number is found, we have found the expected missing volume

$$\mathbb{E}[V_3(B^3) - V_3(P_N)].$$

3.2.1 Exact formula

There are two cases that we need to be aware of. How can the volume of the convex hull be found when $0 \in P_N$ and when $0 \notin P_N$? Again, let $\mathcal{F}_k(P_N)$ be the set of k-dimensional faces of P_N, hence $\mathcal{F}_2(P_N)$ is the set of all facets of P_N. With probability one, the facets of P_N are triangles. Suppose that $0 \in P_N$. In this case, the union of volumes int $[0, F]$, for $F \in \mathcal{F}_2(P_N)$, form the polytope P_N:

$$\text{int } P_N = \text{int } [X_1, ..., X_N] = \bigcup_{F \in \mathcal{F}_2(P_N)} \text{int } [0, F]. \tag{3.7}$$

Recall that $[0, F]$ is the convex hull of 0 and the points that form the facet F. Suppose that $0 \notin P_N$. Then the convex hull P_N is located in only one half of the sphere $\mathbb{S}^2$. Now the union in Equation 3.7 doesn't apply, because it is strictly larger than int P_N. Since the origin is not in P_N, there are facets $G \in \mathcal{F}_2(P_N)$ such that int $[0, G] \cap \text{int } P_N = \emptyset$. Define the set of these kind of facets by

$$\mathcal{F}_- = \{F \in \mathcal{F}_2(P_N) \mid \text{int } [0, F] \cap \text{int } (P_N) = \emptyset\}$$

and its complement by

$$\mathcal{F}_+ = \mathcal{F}_2(P_N) \backslash \mathcal{F}_-.$$

If we now take the union $\bigcup_{F \in \mathcal{F}_+} \text{int } [0, F]$, this is still strictly larger than int P_N, because the origin is included in all of the volumes. This excessive volume is exactly equal to $\bigcup_{F \in \mathcal{F}_-} \text{int } [0, F]$. Therefore,

$$\text{int } P_N = \bigcup_{F \in \mathcal{F}_+} \text{int } [0, F] \backslash \bigcup_{F \in \mathcal{F}_-} \text{int } [0, F].$$

Note that this formula also works when $0 \in P_N$, since then $\mathcal{F}_- = \emptyset$ and $\mathcal{F}_+ = \mathcal{F}$ which gives the same result as given before. In terms of volume, we have

$$\begin{aligned} V_3(P_N) &= V_3(\bigcup_{F \in \mathcal{F}_+} [0, F]) - V_3(\bigcup_{F \in \mathcal{F}_-} [0, F]) \\ &= \sum_{F \in \mathcal{F}_+} V_3([0, F]) - \sum_{F \in \mathcal{F}_-} V_3([0, F]), \end{aligned}$$

where the last equality holds because for both sums, the $[0, F]$ have pairwise disjoint interiors. Now the computation of the expected value of the convex hull P_N can be started:

$$\begin{aligned}\mathbb{E}[V_3(P_N)] &= \mathbb{E}[\sum_{F\in\mathcal{F}_+} V_3([0,F]) - \sum_{F\in\mathcal{F}_-} V_3([0,F])] \\ &= \mathbb{E}[\sum_{F\in\mathcal{F}_2(P_N)} V_3([0,F])\varepsilon_F]\end{aligned}$$

where $\varepsilon_F = \begin{cases} +1 & \text{if } F\in\mathcal{F}_+, \\ -1 & \text{if } F\in\mathcal{F}_-, \\ 0 & \text{otherwise.}\end{cases}$

The facets $F \in \mathcal{F}_2(P_N)$ are almost surely simplices, so they consist of three points X_i, X_j, X_k. Instead of only summing over the facets of P_N, we can sum over all triples (i, j, k) and restrict to the triples that are a facet of P_N. To that end, define the event $F_{ijk} = \{[X_i, X_j, X_k] \in \mathcal{F}_2(P_N)\}$ as the event that (X_i, X_j, X_k) forms a facet of P_N. Then

$$\begin{aligned}\mathbb{E}[V_3(P_N)] &= \mathbb{E}[\sum_{1\le i<j<k\le N} V_3([0,X_i,X_j,X_k])\varepsilon_{ijk}\mathbb{1}(F_{ijk})] \\ &= \sum_{1\le i<j<k\le N} \mathbb{E}[V_3([0,X_i,X_j,X_k])\varepsilon_{ijk}\mathbb{1}(F_{ijk})] \end{aligned} \tag{3.8}$$

where now $\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (X_i,X_j,X_k)\in\mathcal{F}_+, \\ -1 & \text{if } (X_i,X_j,X_k)\in\mathcal{F}_-, \\ 0 & \text{otherwise.}\end{cases}$

The sum can be taken out of the expectation in the last step because the points $X_1, ..., X_N$ are independent and identically distributed. Because of the same reason, the summands in Equation (3.8) are equal for all triples (i, j, k). Therefore, we might as well fix one such triple, for example $(1, 2, 3)$, and multiply with the number of triples, $\binom{N}{3}$.

$$\mathbb{E}[V_3(P_N)] = \binom{N}{3}\mathbb{E}[V_3([0,X_1,X_2,X_3])\varepsilon_{123}\mathbb{1}(F_{123})].$$

The points X_1, X_2, X_3 are uniformly distributed on the sphere $\mathbb{S}^2$, so

$$\begin{aligned}\mathbb{E}[V_3(P_N)] &= \binom{N}{3}\int_{(\mathbb{S}^2)^3} V_3([0,x_1,x_2,x_3])\mathbb{E}[\varepsilon_{123}\mathbb{1}(F_{123})]\frac{\mathcal{H}^6_{(\mathbb{S}^2)^3}}{(4\pi)^3}(\mathrm{d}(x_1,x_2,x_3)) \\ &= \frac{1}{64\pi^3}\binom{N}{3}\int_{(\mathbb{S}^2)^3} V_3([0,x_1,x_2,x_3])\mathbb{E}[\varepsilon_{123}\mathbb{1}(F_{123})]\mathcal{H}^6_{(\mathbb{S}^2)^3}(\mathrm{d}(x_1,x_2,x_3)).\end{aligned}$$

Here, $\frac{1}{(4\pi)^3}$ is the normalization constant since the Hausdorff measure of the sphere $\mathbb{S}^2$ is equal to 4π and we are integrating x_1, x_2 and x_3 over the sphere. The 3-dimensional Blaschke–Petkantschin formula in Corollary 2.3 can be applied to the last equation:

$$\begin{aligned}\mathbb{E}[V_3(P_N)] = \frac{\omega_3}{64\pi^3}\binom{N}{3}\int_{A(3,2)}\int_{(H\cap\mathbb{S}^2)^3} & V_3([0,x_1,x_2,x_3])\mathbb{E}[\varepsilon_{123}\mathbb{1}(F_{123})] \\ & \times \Delta_2(x_1,x_2,x_3)\frac{1}{\sqrt{1-h^2}^3}\mathcal{H}^3_{(H\cap\mathbb{S}^2)^3}(\mathrm{d}(x_1,x_2,x_3))\mu_2(\mathrm{d}H).\end{aligned}$$

The domain of the outer integral is the space of affine subspaces of $\mathbb{R}^3$, $A(3, 2)$. This integral will be split into two integrals: an integral over the linear hyperplane parallel to H and then an integral over the distance between H and the origin, as is done in Corollary 2.6.

$$\begin{aligned}\mathbb{E}[V_3(P_N)] = \frac{\omega_3}{64\pi^3}\frac{2}{\omega_3}\binom{N}{3}\int_{\mathbb{S}^2}\int_0^1\int_{(H\cap\mathbb{S}^2)^3} V_3([0,x_1,x_2,x_3])\mathbb{E}[\varepsilon_{123}\mathbb{1}(F_{123})]\\ \times \Delta_2(x_1,x_2,x_3)\frac{1}{\sqrt{1-h^2}^3}\mathcal{H}^3_{(H\cap\mathbb{S}^2)^3}(\mathrm{d}(x_1,x_2,x_3))\mathrm{d}h\mathcal{H}^2_{\mathbb{S}^2}(\mathrm{d}u).\end{aligned}$$

Thus, u is the direction of the hyperplane H and h is the distance from H to the origin. This integrand is independent of u, so the outer integral can be calculated explicitly:

$$\begin{aligned}\mathbb{E}[V_3(P_N)] = \frac{\omega_3}{32\pi^3}\binom{N}{3}\int_0^1\int_{(H\cap\mathbb{S}^2)^3} V_3([0,x_1,x_2,x_3])\mathbb{E}[\varepsilon_{123}\mathbb{1}(F_{123})]\\ \times \Delta_2(x_1,x_2,x_3)\frac{1}{\sqrt{1-h^2}^3}\mathcal{H}^3_{(H\cap\mathbb{S}^2)^3}(\mathrm{d}(x_1,x_2,x_3))\mathrm{d}h.\end{aligned} \tag{3.9}$$

The integrand contains two expressions that can be evaluated further. The polytope $[0,x_1,x_2,x_3]$ is a triangular pyramid with base $[x_1,x_2,x_3]$ and a peak at 0 of height h, so its volume is given by

$$\begin{aligned}V_3([0,x_1,x_2,x_3]) &= \frac{1}{3}\times V_2([0,x_1,x_2,x_3])\times \mathrm{h}\\ &= \frac{h}{3}\Delta_2(x_1,x_2,x_3).\end{aligned} \tag{3.10}$$

Furthermore,

$$\begin{aligned}\mathbb{E}[\varepsilon_{123}\mathbb{1}(F_{123})] &= \mathbb{P}(\varepsilon_{123}=+1,F_{123}) - \mathbb{P}(\varepsilon_{123}=-1,F_{123})\\ &= \mathbb{P}(X_4,...,X_N \text{ are below hyperplane spanned by } X_1,X_2,X_3)\\ &\quad - \mathbb{P}(X_4,...,X_N \text{ are above hyperplane spanned by } X_1,X_2,X_3)\end{aligned}$$

Again, since $X_4,...,X_N$ are independent, the probabilities of the individual points can be multiplied. Furthermore, since $X_4,...,X_N$ are identically distributed, each of them can be replaced by a random variable X that is uniformly distributed on $\mathbb{S}^2$ as well. Therefore

$$\begin{aligned}\mathbb{E}[\varepsilon_{123}\mathbb{1}(F_{123})] &= \mathbb{P}(X \text{ is below hyperplane spanned by } X_1,X_2,X_3)^{N-3}\\ &\quad - \mathbb{P}(X \text{ is above hyperplane spanned by } X_1,X_2,X_3)^{N-3}.\end{aligned} \tag{3.11}$$

These probabilities can be expressed in an integral form and reduced to a shorter expression with the help of the slice integration formula (Theorem 2.7). For the first probability we find

$$\begin{aligned}&\mathbb{P}(X \text{ is below hyperplane spanned by } X_1,X_2,X_3)^{N-3}\\ &= \left(\int_{\mathbb{S}^2}\mathbb{1}(x \text{ is below hyperplane spanned by } x_1,x_2,x_3)\frac{\mathcal{H}^2_{\mathbb{S}^2}}{\omega_3}\mathrm{d}x\right)^{N-3}\\ &= \left(\frac{1}{\omega_3}\int_{-1}^1\int_{\mathbb{S}^1}\mathbb{1}(t<h)\mathcal{H}^1_{\mathbb{S}^1}(\mathrm{d}y)\mathrm{d}t\right)^{N-3} = \left(\frac{\omega_2}{\omega_3}\int_{-1}^1\mathbb{1}(t<h)\mathrm{d}t\right)^{N-3}\\ &= \left(\frac{\omega_2}{\omega_3}\int_{-1}^h\mathrm{d}t\right)^{N-3} = \left(\frac{1+h}{2}\right)^{N-3}.\end{aligned} \tag{3.12}$$

Similarly, we find for the second probability

$$\begin{aligned}
&\mathbb{P}(X \text{ is above hyperplane spanned by } X_1, X_2, X_3)^{N-3} \\
&= \left(\int_{\mathbb{S}^2} \mathbb{1}(x \text{ is above hyperplane spanned by } x_1, x_2, x_3) \frac{\mathcal{H}^2_{\mathbb{S}^2}}{\omega_3} \mathrm{d}x \right)^{N-3} \\
&= \left(\frac{1}{\omega_3} \int_{-1}^{1} \int_{\mathbb{S}^1} \mathbb{1}(t > h) \mathcal{H}^1_{\mathbb{S}^1}(\mathrm{d}y) \mathrm{d}t \right)^{N-3} = \left(\frac{\omega_2}{\omega_3} \int_{-1}^{1} \mathbb{1}(t > h) \mathrm{d}t \right)^{N-3} \\
&= \left(\frac{\omega_2}{\omega_3} \int_{h}^{1} \mathrm{d}t \right)^{N-3} = \left(\frac{1-h}{2} \right)^{N-3}
\end{aligned} \tag{3.13}$$

Equations (3.9) to (3.13) result in

$$\begin{aligned}
\mathbb{E}[V_3(P_N)] &= \frac{\omega_3}{96\pi^3} \binom{N}{3} \int_0^1 \int_{(H\cap\mathbb{S}^2)^3} \left(\left(\frac{1+h}{2} \right)^{N-3} - \left(\frac{1-h}{2} \right)^{N-3} \right) \\
&\quad \times \Delta_2(x_1, x_2, x_3)^2 \frac{h}{\sqrt{1-h^2}^3} \mathcal{H}^3_{(H\cap\mathbb{S}^2)^3}(\mathrm{d}(x_1, x_2, x_3)) \mathrm{d}h \\
&= \frac{\omega_3}{96\pi^3} \binom{N}{3} \int_0^1 \left(\left(\frac{1+h}{2} \right)^{N-3} - \left(\frac{1-h}{2} \right)^{N-3} \right) \frac{h}{\sqrt{1-h^2}^3} \\
&\quad \times \int_{(H\cap\mathbb{S}^2)^3} \Delta_2(x_1, x_2, x_3)^2 \mathcal{H}^3_{(H\cap\mathbb{S}^2)^3}(\mathrm{d}(x_1, x_2, x_3) \mathrm{d}h
\end{aligned} \tag{3.14}$$

The inner integral in the last equation can be evaluated using Theorem 2.8, but a transformation of variables is needed in order to do it correctly. In the current situation, the points x_1, x_2, x_3 are located on any circle obtained by intersecting a hyperplane H with $\mathbb{S}^2$. The hyperplane is at distance h from the origin of $\mathbb{S}^2$, so the points x_1, x_2, x_3 are on a circle of radius $\sqrt{1-h^2}$. Theorem 2.8 requires that the points lie on the unit sphere, so the following transformation is used:

$$\begin{aligned}
x_1 &= hu + w_1\sqrt{1-h^2} \\
x_2 &= hu + w_2\sqrt{1-h^2} \\
x_3 &= hu + w_3\sqrt{1-h^2},
\end{aligned}$$

where the points w_1, w_2, w_3 are points on $\mathbb{S}^2 \cap u^\perp$, which is the unit sphere parallel to H. The transformation is visualized in Figure 3.4. The 2-dimensional volume changes in the following way:

$$\Delta_2(x_1, x_2, x_3) = \Delta_2(hu + w_1\sqrt{1-h^2}, hu + w_2\sqrt{1-h^2}, hu + w_3\sqrt{1-h^2}).$$

The 2-dimensional volume is translation invariant, giving

$$\Delta_2(x_1, x_2, x_3) = \Delta_2(w_1\sqrt{1-h^2}, w_2\sqrt{1-h^2}, w_3\sqrt{1-h^2}).$$

This volume is also homogeneous of degree 2, so we get

$$\Delta_2(x_1, x_2, x_3) = (1-h^2)\Delta_2(w_1, w_2, w_3).$$

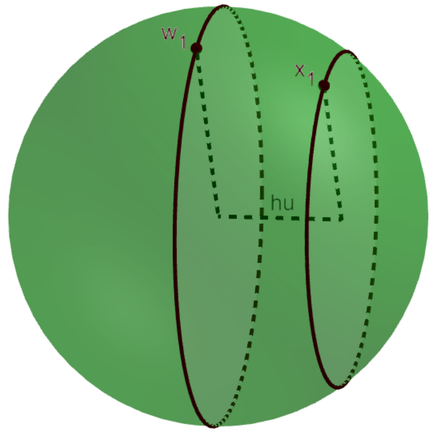


Figure 3.4: Transformation from x_1 to w_1.

Applying this transformation gives:

$$\begin{aligned}
&\int_{(H\cap\mathbb{S}^2)^3} \Delta_2(x_1,x_2,x_3)^2 \mathcal{H}^3_{(H\cap\mathbb{S}^2)^3}(\mathrm{d}(x_1,x_2,x_3)) \\
&= \int_{(u^\perp\cap\mathbb{S}^2)^3} (1-h^2)^2 \Delta_2(w_1,w_2,w_3)^2 \sqrt{1-h^2}^3 \mathcal{H}^3_{(u^\perp\cap\mathbb{S}^2)^3}(\mathrm{d}(w_1,w_2,w_3)) \\
&= (1-h^2)^{\frac{7}{2}} \int_{(\mathbb{S}^1)^3} \Delta_2(w_1,w_2,w_3)^2 \mathcal{H}^3_{(\mathbb{S}^1)^3}(\mathrm{d}(w_1,w_2,w_3)) \\
&= (1-h^2)^{\frac{7}{2}} \frac{\omega_4^3\kappa_4 b_{2,2}}{4\kappa_6 b_{4,2}},
\end{aligned}$$

where in the last line Theorem 2.8 is applied using $k=q=d=2$. This expression can be inserted in Equation (3.14):

$$\begin{aligned}
\mathbb{E}[V_3(P_N)] &= \frac{\omega_3}{96\pi^3}\binom{N}{3}\int_0^1 \left(\left(\frac{1+h}{2}\right)^{N-3} - \left(\frac{1-h}{2}\right)^{N-3}\right) \frac{h}{\sqrt{1-h^2}^3}(1-h^2)^{\frac{7}{2}} \frac{\omega_4^3\kappa_4 b_{2,2}}{4\kappa_6 b_{4,2}}\mathrm{d}h \\
&= \frac{\omega_3\omega_4^3\kappa_4 b_{2,2}}{48\pi^3\kappa_6 b_{4,2}}\binom{N}{3}\int_0^1 \left(\left(\frac{1+h}{2}\right)^N \frac{h(1-h^2)^2}{(1+h)^3} - \left(\frac{1-h}{2}\right)^N \frac{h(1-h^2)^2}{(1-h)^3}\right)\mathrm{d}h \qquad (3.15)
\end{aligned}$$

The integral that is left here, can be written in a more compact form.

$$\begin{aligned}
I(N) &= \int_0^1 \left(\left(\frac{1+h}{2}\right)^N \frac{h(1-h^2)^2}{(1+h)^3} - \left(\frac{1-h}{2}\right)^N \frac{h(1-h^2)^2}{(1-h)^3}\right)\mathrm{d}h \\
&= \int_0^1 \left(\frac{1+h}{2}\right)^N \frac{h(1-h^2)^2}{(1+h)^3}\mathrm{d}h - \int_0^1 \left(\frac{1-h}{2}\right)^N \frac{h(1-h^2)^2}{(1-h)^3}\mathrm{d}h
\end{aligned}$$

Note that the two integrands are symmetric to each other in $h=0$. If we mirror the first integrand in $h=0$ and change the boundaries accordingly we can combine the two integrals to one integral:

$$\begin{aligned}
I(N) &= \int_0^1 \left(\frac{1+h}{2}\right)^N \frac{h(1-h^2)^2}{(1+h)^3}\mathrm{d}h - \int_{-1}^0 \left(\frac{1+h}{2}\right)^N \frac{-h(1-h^2)^2}{(1+h)^3}\mathrm{d}h \\
&= \int_{-1}^1 \left(\frac{1+h}{2}\right)^N \frac{h(1-h^2)^2}{(1+h)^3}\mathrm{d}h.
\end{aligned}$$

Furthermore, the constant in Equation (3.15) can also be given explicitly using the definitions in Chapter 2.1. We find that

$$\frac{\omega_3\omega_4^3\kappa_4 b_{2,2}}{48\pi^3\kappa_6 b_{4,2}} = \frac{4\pi\cdot 8\pi^6\cdot\frac{\pi^2}{2}\cdot 1}{48\pi^3\cdot\frac{\pi^3}{6}\cdot\frac{\omega_3\omega_4}{\omega_1\omega_2}} = \frac{4\pi\cdot 8\pi^6\cdot\frac{\pi^2}{2}\cdot 1}{48\pi^3\cdot\frac{\pi^3}{6}\cdot\frac{4\pi\cdot 2\pi^2}{2\cdot 2\pi}} = \pi.$$

Equation (3.15) becomes

$$\mathbb{E}[V_3(P_N)] = \pi\binom{N}{3}\int_{-1}^{1}\left(\frac{1+h}{2}\right)^N \frac{h(1-h^2)^2}{(1+h)^3}\mathrm{d}h. \tag{3.16}$$

This is the most exact answer we can give for finding the volume of the convex hull P_N. The integral cannot be calculated directly, so we will need to approximate it in order to find an answer. That will be done in the next section.

3.2.2 Approximation

The goal of this section is to find an approximation of the integral

$$I(N) = \int_{-1}^{1}\left(\frac{1+h}{2}\right)^N \frac{h(1-h^2)^2}{(1+h)^3}\mathrm{d}h$$

for $N\to\infty$. If we take $f(h) = \frac{h(1-h^2)^2}{(1-h)^3}$ and $g(h) = -\ln(\frac{1+h}{2})$, then our integral has the same form as in the Laplace method, explained in Chapter 2.3. Hence, we will use that method here to approximate the integral $I(N)$. The minimum of the function $g(h)$ is at $h=1$, so the first assumption of the Laplace method is satisfied. For the second and third assumption we need the Taylor series of $f(h)$ and $g(h)$ at $h=1$:

$$f(h) = \frac{1}{2}(1-h)^2 + O((1-h)^3)$$
$$g(h) = \frac{1}{2}(1-h) + O((1-h)^2).$$

This gives for $f(h)$: $c_2 = \frac{1}{2}$ and $\alpha = 3$ and for $g(h)$: $c_1 = \frac{1}{2}$ and $\mu = 1$. The value of c_0 is given by:

$$c_0 = \frac{c_2}{\mu c_1^{\alpha/\mu}} = \frac{0.5}{(0.5)^3} = 4.$$

Now we have all the values we need to approximate the integral $I(N)$ using Equation (2.1). Hence,

$$\begin{aligned} I(N) &= \Gamma(3)\frac{c_0}{N^3} + O(N^{-4}) \\ &= \frac{8}{N^3} + O(N^{-4}) \\ &= \frac{8}{N^3}(1+O(N^{-1})). \end{aligned}$$

With this approximation of the integral $I(N)$, we can complete Equation (3.16):

$$\mathbb{E}[V_3(P_N)] = \pi\binom{N}{3}\frac{8}{N^3}(1+O(N^{-1})).$$

We can write for the binomial: $\binom{N}{3} = \frac{N(N-1)(N-2)}{6} = \frac{N^3}{6}(1+\frac{3}{N}+\frac{2}{N^2}) = \frac{N^3}{6}(1+O(N^{-1}))$. Thus,

$$\begin{aligned} \mathbb{E}[V_3(P_N)] &= \pi\frac{N^3}{6}(1+O(N^{-1}))\frac{8}{N^3}(1+O(N^{-1})) \\ &= \frac{4}{3}\pi\left(1+O(N^{-1})\right). \end{aligned}$$

Hence, the volume of the convex hull of the random points $X_1, ..., X_N$ goes to $\frac{4}{3}\pi$ at rate N^{-1}. Recall that $V_3(B^3) = \kappa_3 = \frac{4}{3}\pi$. Consequently, the missing volume between the sphere $\mathbb{S}^2$ and the convex hull P_N goes to zero at rate N^{-1} as well:

$$\begin{aligned}\mathbb{E}[V_3(B^3) - V_3(P_N)] &= \frac{4}{3}\pi - \frac{4}{3}\pi \cdot \big(1 - O(N^{-1})\big)\\ &= O(N^{-1}),\end{aligned}$$

Altogether, we found in the previous section that the missing surface between the sphere $\mathbb{S}^1$ and the convex hull goes to zero at rate N^{-2} and in this section the missing surface between the sphere $\mathbb{S}^2$ and the convex hull goes to zero at rate N^{-1}. We now have a feeling of how to deal with this problem, so we are ready to extend this problem to the d-dimensional sphere $\mathbb{S}^{d-1}$.

3.3 The d-dimensional case

In the previous chapters, the missing volumes were found for the spheres $\mathbb{S}^1$ and $\mathbb{S}^2$. This set up can be generalized to d-dimensions, in which case random points live on $\mathbb{S}^{d-1}$ for $d \in \{2, 3, 4, ...\}$. The method we are going to use here is the similar to the previous section, but we will come across some more challenges here. When this generalization is completed, the values $d = 2$ and $d = 3$ can be filled in to make a comparison with the results of Chapters 3.1 and 3.2. The same method as in the previous section will be used here as well. That is, we will approximate the volume of the convex hull directly.

3.3.1 Exact formula

Let $N \in \mathbb{N}$ and let $X_1, ..., X_N$ be independent random points distributed uniformly on $\mathbb{S}^{d-1}$. The goal is to find the expected missing volume between the sphere $\mathbb{S}^{d-1}$ and the convex hull of the points $X_1, ..., X_N$. To ease notation, define $P_N = [X_1, ..., X_N]$. Hence, we want to find the value of

$$\mathbb{E}[V_d(B^d) - V_d(P_N)]. \tag{3.17}$$

As the numbers of points N goes to infinity, the shape of the convex hull approaches the shape of the sphere, so it is to be expected that the missing volume goes to zero. However, it is not evident how fast this goes. This will be the main focus of our investigation. The volume of the ball B^d is known. Recall from Chapter 2.1 that

$$\kappa_d = V_d(B^d) = \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})}.$$

It remains to find the expected value of the d-dimensional volume of the convex hull:

$$\mathbb{E}[V_d(P_N)].$$

The boundary of the convex hull P_N consists of $(d-1)$-dimensional facets F. With probability 1, those facets contain d points, since otherwise an event of the form $\{x_{d+1} \in \text{affine-hull}(x_1, ..., x_d)\}$ would be satisfied, but such event has probability measure zero. Define $\mathcal{F}_k(P_N)$ to be the set of k-dimensional faces of P_N, hence $\mathcal{F}_{d-1}(P_N)$ is the set of all facets of P_N.

We need to define a way to calculate the volume of P_N. Depending on how the N points are distributed on the sphere $\mathbb{S}^{d-1}$, the convex hull P_N either or not contains the origin. Distinguish between these cases of P_N. First, suppose that $0 \in P_N$. In this case, the union of the volumes int $[0, F]$, for $F \in \mathcal{F}_{d-1}(P_N)$, form the convex hull P_N. Therefore,

$$\text{int } P_N = \bigcup_{F \in \mathcal{F}_{d-1}(P_N)} \text{int } [0, F]. \tag{3.18}$$

Recall that $[0, F]$ is the convex hull of 0 and the points that form the facet F.

Secondly, suppose that $0 \notin P_N$. Then the union in Equation (3.18) is strictly larger than int P_N. In this case, there are facets G such that int $[0, G] \cap \text{int } (P_N) = \emptyset$, so the contribution of those facets to the union need to be left out *and* subtracted from the union in Equation (3.18). These facets will be collected in a set. Define

$$\mathcal{F}_- = \{F \in \mathcal{F}_{d-1}(P_N) \mid \text{int } [0, F] \cap \text{int } P_N = \emptyset\}$$

and

$$\mathcal{F}_+ = \mathcal{F}_{d-1}(P_N) \backslash \mathcal{F}_-.$$

Therefore,

$$\text{int } P_N = \bigcup_{F\in\mathcal{F}_+} \text{int } [0,F] \setminus \bigcup_{F\in\mathcal{F}_-} \text{int } [0,F].$$

Note that this formula also works when $0 \in P_N$, since then $\mathcal{F}_- = \emptyset$ and $\mathcal{F}_+ = \mathcal{F}_{d-1}(P_N)$ which gives again Equation (3.18). In terms of volume, it implies that

$$\begin{aligned} V_d(P_N) &= V_d(\bigcup_{F\in\mathcal{F}_+}[0,F]) - V_d(\bigcup_{F\in\mathcal{F}_-}[0,F]) \\ &= \sum_{F\in\mathcal{F}_+} V_d([0,F]) - \sum_{F\in\mathcal{F}_-} V_d([0,F]), \end{aligned}$$

where the last equality holds because for both sums, the simplices involved have pairwise disjoint interiors. Now the computation of the expected value of the convex hull P_N can be started:

$$\begin{aligned} \mathbb{E}[V_d(P_N)] &= \mathbb{E}[\sum_{F\in\mathcal{F}_+} V_d([0,F]) - \sum_{F\in\mathcal{F}_-} V_d([0,F])] \\ &= \mathbb{E}[\sum_{F\in\mathcal{F}_+} V_d([0,F])\varepsilon_F] \end{aligned}$$

where $\varepsilon_F = \begin{cases} +1 & \text{if } F\in\mathcal{F}_+, \\ -1 & \text{if } F\in\mathcal{F}_-, \\ 0 & \text{otherwise.} \end{cases}$

Define the event $F_{i_1,\dots,i_d} = \{[X_{i_1},\dots,X_{i_d}] \in \mathcal{F}_{d-1}(P_N)\}$. Instead of only summing over the facets of P_N, we can sum over all d-tuples $(i_1,\dots,i_d)$ with distinct and increasing indices and restrict to the event $F_{i_1,\dots,i_d}$. This results in the following equation:

$$\begin{aligned} \mathbb{E}[V_d(P_N)] &= \mathbb{E}[\sum_{1\le i_1<\dots<i_d\le N} V_d([0,X_{i_1},\dots,X_{i_d}])\varepsilon_{i_1,\dots,i_d}\mathbb{1}(F_{i_1,\dots,i_d})] \\ &= \sum_{1\le i_1<\dots<i_d\le N} \mathbb{E}[V_d([0,X_{i_1},\dots,X_{i_d}])\varepsilon_{i_1,\dots,i_d}\mathbb{1}(F_{i_1,\dots,i_d})], \end{aligned} \tag{3.19}$$

where the last equation holds because the points $X_1,\dots,X_N$ are independent and identically distributed. The summation is done over all d-tuples of d distinct and increasing indices $i_1,\dots,i_d \in [N]$, hence there are $\binom{N}{d}$ of them. Since the points are drawn independently, the expected value in Equation (3.19) is the same for each d-tuple $(i_1,\dots,i_d)$. Therefore, we can restrict ourselves to one such d-tuple, e.g. $(1,\dots,d)$, and multiply the result with the number of d-tuples $\binom{N}{d}$. That gives the following equation:

$$\mathbb{E}[V_d(P_N)] = \binom{N}{d}\mathbb{E}[V_d([0,X_1,\dots,X_d])\varepsilon_{1,\dots,d}\mathbb{1}(F_{1,\dots,d})].$$

Recall that the points are uniformly distributed on the sphere. Therefore

$$\mathbb{E}[V_d(P_N)] = \binom{N}{d}\int_{(\mathbb{S}^{d-1})^d} V_d([0,x_1,\dots,x_d])\mathbb{E}[\varepsilon_{1,\dots,d}\mathbb{1}(F_{1,\dots,d})]\frac{\mathcal{H}^{(d-1)d}_{(\mathbb{S}^{d-1})^d}}{\omega_d^d}(\mathrm{d}(x_1,\dots,x_d)).$$

The Blaschke–Petkantschin formula for points on the sphere (Theorem 2.1) can be applied to the latter equation:

$$\begin{aligned} \mathbb{E}[V_d(P_N)] = \frac{\omega_d(d-1)!}{2\omega_d^d}\binom{N}{d}\int_{A(d,d-1)}\int_{(H\cap\mathbb{S}^{d-1})^d} & V_d([0,x_1,\dots,x_d])\mathbb{E}[\varepsilon_{1,\dots,d}\mathbb{1}(F_{1,\dots,d})] \\ & \times \Delta_{d-1}(x_1,\dots,x_d)(1-h^2)^{-\frac{d}{2}}\mathcal{H}^{d(d-2)}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d))\mu_{d-1}(\mathrm{d}H). \end{aligned}$$

To make further computations easier, the hyperplanes $H \in A(d, d-1)$ will be split into two components: the linear hyperplane parallel to H and the distance from the origin to H, as is done in Theorem 2.4.

$$\mathbb{E}[V_d(P_N)] = \frac{(d-1)!}{2\omega_d^{d-1}} \frac{2}{\omega_d} \binom{N}{d} \int_{\mathbb{S}^{d-1}} \int_0^1 \int_{(H\cap\mathbb{S}^{d-1})^d} V_d([0, x_1, ..., x_d]) \mathbb{E}[\varepsilon_{1,...,d} \mathbb{1}(F_{1,...,d})] \times \Delta_{d-1}(x_1, ..., x_d)(1-h^2)^{-\frac{d}{2}} \mathcal{H}^{(d-2)d}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1, ..., x_d))\mathrm{d}h\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

The integrand is independent of u, so the value of the outer integral is simply ω_d:

$$\mathbb{E}[V_d(P_N)] = \frac{(d-1)!}{\omega_d^{d-1}} \binom{N}{d} \int_0^1 \int_{(H\cap\mathbb{S}^{d-1})^d} V_d([0, x_1, ..., x_d]) \mathbb{E}[\varepsilon_{1,...,d} \mathbb{1}(F_{1,...,d})] \times \Delta_{d-1}(x_1, ..., x_d)(1-h^2)^{-\frac{d}{2}} \mathcal{H}^{(d-2)d}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1, ..., x_d))\mathrm{d}h. \tag{3.20}$$

The convex hull $[0, x_1, ..., x_d]$ is a d-dimensional hyperpyramid with $[x_1, ..., x_d]$ as a base and 0 as a peak of height h. As described in Mathai [8] the volume of $[0, x_1, ..., x_d]$ can be given in terms of the volume of the base and the height:

$$\begin{aligned} V_d([0, x_1, ..., x_d]) &= \frac{1}{d} \times V_{d-1}([x_1, ..., x_d]) \times \mathrm{h} \\ &= \frac{h}{d} \Delta_{d-1}(x_1, ..., x_d). \end{aligned} \tag{3.21}$$

The expected value in the integrand can also be evaluated further:

$$\begin{aligned} \mathbb{E}[\varepsilon_{1,...,d} \mathbb{1}(F_{1,...,d})] &= \mathbb{P}(\varepsilon_{1,...,d} = +1, F_{1,...,d}) - \mathbb{P}(\varepsilon_{1,...,d} = -1, F_{1,...,d}) \\ &= \mathbb{P}(X_{d+1}, ..., X_N \text{ are below hyperplane spanned by } x_1, ..., x_d) \\ &\quad - \mathbb{P}(X_{d+1}, ..., X_N \text{ are above hyperplane spanned by } x_1, ..., x_d) \end{aligned}$$

Again, since $X_{d+1}, ..., X_N$ are independent, the probabilities of the individual points can be multiplied. Furthermore, since the points $X_{d+1}, ..., X_N$ are identically distributed, each of them can be replaced by one particular point X that is uniformly distributed on $\mathbb{S}^{d-1}$ as well. Therefore

$$\begin{aligned} \mathbb{E}[\varepsilon_{1,...,d} \mathbb{1}(F_{1,...,d})] &= \mathbb{P}(X \text{ is below hyperplane spanned by } x_1, ..., x_d)^{N-d} \\ &\quad - \mathbb{P}(X \text{ is above hyperplane spanned by } x_1, ..., x_d)^{N-d}. \end{aligned}$$

These probabilities can be expressed in an integral form and reduced to a shorter expression with the help of the slice integration formula (Theorem 2.7). For the first probability we find

$$\begin{aligned} &\mathbb{P}(X \text{ is below hyperplane spanned by } x_1, ..., x_d)^{N-d} \\ &= \left(\int_{\mathbb{S}^{d-1}} \mathbb{1}(x \text{ is below hyperplane spanned by } x_1, ..., x_d) \frac{\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}}{\omega_d} \mathrm{d}x \right)^{N-d} \\ &= \left(\frac{1}{\omega_d} \int_{-1}^1 (1-t^2)^{\frac{d-3}{2}} \int_{\mathbb{S}^{d-2}} \mathbb{1}(t < h) \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}y)\mathrm{d}t \right)^{N-d} \\ &= \left(\frac{\omega_{d-1}}{\omega_d} \int_{-1}^1 (1-t^2)^{\frac{d-3}{2}} \mathbb{1}(t < h)\mathrm{d}t \right)^{N-d} \\ &= \left(\frac{\omega_{d-1}}{\omega_d} \int_{-1}^h (1-t^2)^{\frac{d-3}{2}} \mathrm{d}t \right)^{N-d} \\ &=: S(h)^{N-d}. \end{aligned}$$

Similarly, we find for the second probability

$$
\begin{aligned}
&\mathbb{P}(X \text{ is above hyperplane spanned by } x_1,...,x_d)^{N-d}\\
&= \left(\int_{\mathbb{S}^{d-1}} \mathbb{1}(x \text{ is above hyperplane spanned by } x_1,...,x_d)\frac{\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}}{\omega_d}\mathrm{d}x\right)^{N-d}\\
&= \left(\frac{1}{\omega_d}\int_{-1}^{1}(1-t^2)^{\frac{d-3}{2}}\int_{\mathbb{S}^{d-2}}\mathbb{1}(t>h)\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}y)\mathrm{d}t\right)^{N-d}\\
&= \left(\frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1}(1-t^2)^{\frac{d-3}{2}}\mathbb{1}(t>h)\mathrm{d}t\right)^{N-d}\\
&= \left(\frac{\omega_{d-1}}{\omega_d}\int_{h}^{1}(1-t^2)^{\frac{d-3}{2}}\mathrm{d}t\right)^{N-d}\\
&=: T(h)^{N-d}.
\end{aligned}
$$

Note that $T(h) + S(h) = 1$. Putting this together gives

$$\mathbb{E}[\varepsilon_{1,...,d}\mathbb{1}(F_{1,...,d})] = S(h)^{N-d} - T(h)^{N-d}. \tag{3.22}$$

Postponing the treatment of $S(h)^{N-d} - T(h)^{N-d}$, Equation (3.20) can be completed using Equations (3.21) and (3.22):

$$
\begin{aligned}
\mathbb{E}[V_d(P_N)] = \frac{(d-1)!}{\omega_d^{d-1}}\binom{N}{d}\int_0^1\int_{(H\cap\mathbb{S}^{d-1})^d}\frac{h}{d}\Delta_{d-1}(x_1,...,x_d)\left(S(h)^{N-d} - T(h)^{N-d}\right)\\
\times\,\Delta_{d-1}(x_1,...,x_d)(1-h^2)^{-\frac{d}{2}}\mathcal{H}^{(d-2)d}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,...,x_d))\mathrm{d}h.
\end{aligned}
$$

We pull out all the terms of the inner integral that only depend on h:

$$
\begin{aligned}
\mathbb{E}[V_d(P_N)] = \frac{(d-1)!}{d\omega_d^{d-1}}\binom{N}{d}\int_0^1 h(1-h^2)^{-\frac{d}{2}}\left(S(h)^{N-d} - T(h)^{N-d}\right)\\
\int_{(H\cap\mathbb{S}^{d-1})^d}\Delta_{d-1}(x_1,...,x_d)^2\mathcal{H}^{(d-2)d}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,...,x_d))\mathrm{d}h.
\end{aligned} \tag{3.23}
$$

The inner integral in the latter equation can be evaluated using Theorem 2.8, but a transformation of variables is needed in order to do it correctly. In the current situation, the points $x_1,...,x_d$ are located on a circle obtained by intersecting a hyperplane H with $\mathbb{S}^{d-1}$. The hyperplane is at distance h from the origin, so the points $x_1,...,x_d$ are on a $(d-2)$-dimensional sphere of radius $\sqrt{1-h^2}$. Theorem 2.8 requires that the points lie on the unit sphere, so the following transformations are used:

$$
\begin{aligned}
x_1 &= hu + w_1\sqrt{1-h^2},\\
&\vdots\\
x_d &= hu + w_d\sqrt{1-h^2},
\end{aligned}
$$

where $u \in \mathbb{S}^{d-1}$ is the unit vector orthogonal to H and the points $w_1,...,w_d$ are points on $\mathbb{S}^{d-1}\cap u^\perp$. Note that the intersection of $\mathbb{S}^{d-1}$ with a linear hyperplane results in a sphere of one dimension lower: $\mathbb{S}^{d-1}\cap u^\perp = \mathbb{S}^{d-2}$. The transformation is visualized in 3 dimensions in Figure 3.4. The transformation should also be applied to the $(d-1)$-dimensional volume:

$$\Delta_{d-1}(x_1,...,x_d) = \Delta_{d-1}(hu + w_1\sqrt{1-h^2},...,hu + w_d\sqrt{1-h^2}).$$

This volume is translation invariant, so

$$\Delta_{d-1}(x_1, ..., x_d) = \Delta_{d-1}(w_1\sqrt{1-h^2}, ..., w_d\sqrt{1-h^2}).$$

Furthermore, it is homogeneous, giving

$$\Delta_{d-1}(x_1, ..., x_d) = (1-h^2)^{\frac{d-1}{2}}\Delta_{d-1}(w_1, ..., w_d).$$

Applying this transformation to the inner integral in Equation (3.23) gives:

$$\begin{aligned}
&\int_{(H\cap\mathbb{S}^{d-1})^d} \Delta_{d-1}(x_1, ..., x_d)^2 \mathcal{H}^{(d-2)d}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1, ..., x_d)) \\
&= \int_{(u^\perp\cap\mathbb{S}^{d-1})^d} (1-h^2)^{d-1}\Delta_{d-1}(w_1, ..., w_d)^2 \sqrt{1-h^2}^{d(d-2)} \mathcal{H}^{d(d-2)}_{(u^\perp\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(w_1, ..., w_d)) \\
&= (1-h^2)^{\frac{d^2-2}{2}} \int_{(\mathbb{S}^{d-2})^d} \Delta_{d-1}(w_1, ..., w_d)^2 \mathcal{H}^{d(d-2)}_{(\mathbb{S}^{d-2})^d}(\mathrm{d}(w_1, ..., w_d)) \\
&= (1-h^2)^{\frac{d^2-2}{2}} \frac{\omega_{d+1}^d}{((d-1)!)^2} \frac{\kappa_{d^2-d-2}}{\kappa_{d(d-1)}} \frac{b_{(d-1)(d-1)}}{b_{(d+1)(d-1)}},
\end{aligned} \tag{3.24}$$

where in the last line Theorem 2.8 is applied. Equation (3.24) can be written into Equation (3.23):

$$\begin{aligned}
\mathbb{E}[V_d(P_N)] &= \frac{(d-1)!}{d\omega_d^{d-1}} \binom{N}{d} \int_0^1 h(1-h^2)^{-\frac{d}{2}} \left(S(h)^{N-d} - T(h)^{N-d}\right) \\
&\qquad \times (1-h^2)^{\frac{d^2-2}{2}} \frac{\omega_{d+1}^d \kappa_{d^2-d-2} b_{(d-1)(d-1)}}{((d-1)!)^2 \kappa_{d(d-1)} b_{(d+1)(d-1)}} \mathrm{d}h \\
&= \frac{\omega_{d+1}^d \kappa_{d^2-d-2} b_{(d-1)(d-1)}}{d\omega_d^{d-1}(d-1)!\kappa_{d(d-1)} b_{(d+1)(d-1)}} \binom{N}{d} \int_0^1 h(1-h^2)^{\frac{d^2-d-2}{2}} \left(S(h)^{N-d} - T(h)^{N-d}\right) \mathrm{d}h.
\end{aligned}$$

The constant in the last equation is very elaborate, but can be given explicitly. The definitions of ω_d, κ_d and b_{dq} are given in Chapter 2.1. Using these definitions, we get $\frac{b_{(d-1)(d-1)}}{b_{(d+1)(d-1)}} = \frac{4\pi}{\omega_d\omega_{d+1}}$ and $\frac{\kappa_{d^2-d-2}}{\kappa_{d(d-1)}} = \frac{\Gamma(\frac{d^2-d+2}{2})}{\pi\Gamma(\frac{d^2-d}{2})} = \frac{(\frac{d^2-d}{2})!}{\pi(\frac{d^2-d-2}{2})!}$. Since $\frac{d^2-d}{2}$ and $\frac{d^2-d-2}{2}$ are always integers with unit difference, it follows that $\frac{\kappa_{d^2-d-2}}{\kappa_{d(d-1)}} = \frac{d(d-1)}{2\pi}$. This gives

$$\frac{\omega_{d+1}^d \kappa_{d^2-d-2} b_{(d-1)(d-1)}}{d\omega_d^{d-1}(d-1)!\kappa_{d(d-1)} b_{(d+1)(d-1)}} = \frac{2\omega_{d+1}^{d-1}}{\omega_d^d(d-2)!}.$$

Writing out the definitions of ω_d and ω_{d+1} doesn't give a nicer fraction, so we leave them like this for now. This gives the following, simpler expression:

$$\mathbb{E}[V_d(P_N)] = \frac{2\omega_{d+1}^{d-1}}{\omega_d^d(d-2)!} \binom{N}{d} \int_0^1 h(1-h^2)^{\frac{d^2-d-2}{2}} \left(S(h)^{N-d} - T(h)^{N-d}\right) \mathrm{d}h \tag{3.25}$$

The functions $S(h)$ and $T(h)$ are related in the following way, using the symmetry of their integrands: $S(-h) = \frac{\omega_{d-1}}{\omega_d} \int_{-1}^{-h} (1-t^2)^{\frac{d-3}{2}} \mathrm{d}t = \frac{\omega_{d-1}}{\omega_d} \int_h^1 (1-t^2)^{\frac{d-3}{2}} \mathrm{d}t = T(h)$. This relation will be used to write the integral more efficiently:

$$\begin{aligned}
&\int_0^1 h(1-h^2)^{\frac{d^2-d-2}{2}} \left(S(h)^{N-d} - T(h)^{N-d}\right) \mathrm{d}h \\
&= \int_0^1 h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{N-d}\mathrm{d}h - \int_0^1 h(1-h^2)^{\frac{d^2-d-2}{2}} S(-h)^{N-d}\mathrm{d}h \\
&= \int_0^1 h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{N-d}\mathrm{d}h + \int_{-1}^0 h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{N-d}\mathrm{d}h \\
&= \int_{-1}^1 h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{N-d}\mathrm{d}h.
\end{aligned}$$

Equation (3.25) becomes a bit shorter using this trick:

$$\mathbb{E}[V_d(P_N)] = \frac{2\omega_{d+1}^{d-1}}{\omega_d^d (d-2)!} \binom{N}{d} \int_{-1}^1 h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{N-d}\mathrm{d}h. \tag{3.26}$$

This is the most exact formula we can find for the volume of the convex hull P_N. The integral cannot be calculated directly, so we will approximate it in the next section.

3.3.2 Approximation

Only one integral is left to be calculated in Equation (3.26), namely

$$I(N) = \int_{-1}^1 h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{N-d}\mathrm{d}h.$$

This integral is of the same form as described in Laplace's method (Chapter 2.3). Indeed, we can rewrite the integrand as follows:

$$\begin{aligned}
h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{N-d} &= \frac{h(1-h^2)^{\frac{d^2-d-2}{2}}}{S(h)^d} e^{-N\cdot(-\ln(S(h)))} \\
&= f(h)e^{-Ng(h)},
\end{aligned}$$

where $f(h) = h(1-h^2)^{\frac{d^2-d-2}{2}} S(h)^{-d}$ and $g(h) = -\ln(S(h))$. Now we can check the three assumptions of Laplace's method in Chapter 2.3. The minimum of the function $g(h)$ is at $h_0 = 1$, so the first assumption is satisfied. It may seem that we can just find the Taylor expansion of the functions f and g around h_0 in order to satisfy the second and third assumptions. However, the Taylor expansions of these functions do not exist because of the fixed but unknown value of d in the exponents in f and g. Therefore, we have to take a different route. First, the integral in the function $S(h)$ will be evaluated using the substitution $u = 1 - t^2$:

$$\begin{aligned}
S(h) &= S(1) - \int_h^1 \frac{\omega_{d-1}}{\omega_d}(1-t^2)^{\frac{d-3}{2}} \mathrm{d}t \\
&= S(1) - \int_0^{1-h^2} \frac{\omega_{d-1}}{\omega_d} u^{\frac{d-3}{2}} \frac{\mathrm{d}u}{2\sqrt{1-u}} \\
&= S(1) - \int_0^{1-h^2} \frac{\omega_{d-1}}{2\omega_d} u^{\frac{d-3}{2}} \mathrm{d}u \times \Delta(h),
\end{aligned}$$

with $1 \leq \Delta(h) = \frac{1}{\sqrt{1-u}} \leq \frac{1}{h}$. It will turn out to be useful to define $\Delta(h)$ like this, since the value of h will get close to one, hence $\Delta(h)$ will also be close to one. Now the integration is straightforward:

$$S(h) = S(1) - \frac{\omega_{d-1}}{\omega_d (d-1)}(1-h^2)^{\frac{d-1}{2}} \Delta(h).$$

Note that $S(1) = \int_{-1}^{1} \frac{\omega_{d-1}}{\omega_d}(1-t^2)^{\frac{d-3}{2}} \mathrm{d}t = 1$. Then the function g becomes

$$\begin{aligned} g(h) &= -\ln\left(1 - \frac{\omega_{d-1}}{\omega_d(d-1)}(1-h^2)^{\frac{d-1}{2}}\Delta(h)\right) \\ &= \frac{\omega_{d-1}}{\omega_d(d-1)}(1-h^2)^{\frac{d-1}{2}}\Delta(h)(1+o(1)) \end{aligned}$$

The function $g(h)$ should be expressed in terms of $1-h$ in order to satisfy the second condition, so the following Taylor expansion around $h_0 = 1$ is used: $(1-h^2)^{\frac{d-1}{2}} = (2(1-h)+o(1))^{\frac{d-1}{2}} = 2^{\frac{d-1}{2}}(1-h)^{\frac{d-1}{2}}(1+o(1))^{\frac{d-1}{2}}$. Substitute this into the function g:

$$g(h) = \frac{\omega_{d-1}}{\omega_d(d-1)} 2^{\frac{d-1}{2}}(1-h)^{\frac{d-1}{2}}(1+o(1)),$$

giving $c_1 = \frac{\omega_{d-1}}{\omega_d(d-1)}2^{\frac{d-1}{2}}$ and $\mu = \frac{d-1}{2}$. As the value of h gets close to 1, the value of $S(h)$ gets close to 1 as well, hence the function f can be dealt with as follows:

$$\begin{aligned} f(h) &= \frac{h(1-h^2)^{\frac{d^2-d-2}{2}}}{S(h)^d} \\ &= (1-h^2)^{\frac{d^2-d-2}{2}}(1+o(1)) \\ &= \big(2(1-h)+o(1)\big)^{\frac{d^2-d-2}{2}}(1+o(1)) \\ &= 2^{\frac{d^2-d-2}{2}}(1-h)^{\frac{d^2-d-2}{2}}(1+o(1)) \end{aligned}$$

Then the constants are $c_2 = 2^{\frac{d^2-d-2}{2}}$ and $\alpha = \frac{d^2-d}{2} = \frac{d(d-1)}{2}$. Using Equation (2.1), the value of the integral is approximated by:

$$\begin{aligned} \int_{-1}^{1} \frac{h(1-h^2)^{\frac{d^2-d-2}{2}}}{S(h)^d} S(h)^N \mathrm{d}h &= \Gamma\Big(\frac{\alpha}{\mu}\Big)\frac{c_0}{N^{\alpha/\mu}} + O(N^{-(1+\alpha)/\mu}) \\ &= \Gamma(d)\frac{c_0}{N^d} + O(N^{-(2/(d-1)+d)}), \end{aligned} \tag{3.27}$$

where $c_0 = \frac{c_2}{\mu c_1^{\alpha/\mu}} = \frac{2^{\frac{d^2-d-2}{2}}}{\frac{d-1}{2}(\frac{\omega_{d-1}}{\omega_d(d-1)}2^{\frac{d-1}{2}})^d} = \frac{\omega_d^d(d-1)^{d-1}}{\omega_{d-1}^d}$. Substitute Equation (3.27) into Equation (3.25):

$$\begin{aligned} \mathbb{E}[V_d(P_N)] &= \frac{2\omega_{d+1}^{d-1}}{\omega_d^d(d-2)!}\binom{N}{d}\left(\Gamma(d)\frac{c_0}{N^d} + O(N^{-(d+\frac{2}{d-1})})\right) \\ &= \frac{2\omega_{d+1}^{d-1}}{\omega_d^d(d-2)!}\frac{N^d}{d!}(1+O(N^{-1}))\left(\Gamma(d)\frac{c_0}{N^d} + O(N^{-(d+\frac{2}{d-1})})\right) \\ &= \frac{2\omega_{d+1}^{d-1}\Gamma(d)c_0}{\omega_d^d(d-2)!d!}(1+O(N^{-\frac{2}{d-1}})). \end{aligned}$$

The value of the constant will be found in steps. First, the values of $\Gamma(d)$ and c_0 will be filled in:

$$\begin{aligned} \mathbb{E}[V_d(P_N)] &= \frac{2\omega_{d+1}^{d-1}(d-1)!\omega_d^d(d-1)^{d-1}}{\omega_d^d(d-2)!d!\omega_{d-1}^d}(1+O(N^{-\frac{2}{d-1}})) \\ &= \frac{2\omega_{d+1}^d(d-1)^d}{\omega_{d-1}^d\omega_{d+1}d!}(1+O(N^{-\frac{2}{d-1}})). \end{aligned}$$

We can simplify the factor $\frac{\omega_{d+1}}{\omega_{d-1}}$ by

$$\frac{\omega_{d+1}}{\omega_{d-1}} = \frac{2\pi^{\frac{d+1}{2}}\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}\Gamma(\frac{d+1}{2})} = \frac{\pi\Gamma(\frac{d-1}{2})}{\frac{d-1}{2}\Gamma(\frac{d-1}{2})} = \frac{2\pi}{d-1}.$$

Use this simplification and fill in the value of ω_{d+1}:

$$\begin{aligned}\mathbb{E}[V_d(P_N)] &= \frac{2(2\pi)^d\Gamma(\frac{d+1}{2})(d-1)^d}{(d-1)^d 2\pi^{\frac{d+1}{2}} d!}(1+O(N^{-\frac{2}{d-1}})) \\ &= \frac{2^d\pi^{\frac{d-1}{2}}\Gamma(\frac{d+1}{2})}{d!}(1+O(N^{-\frac{2}{d-1}})).\end{aligned}$$

Furthermore, we prove in Appendix A that $\Gamma(\frac{d}{2}+\frac{1}{2}) = \frac{d!}{2^d\Gamma(\frac{d}{2}+1)}\sqrt{\pi}$ for $d \in \mathbb{N}$. This can be plugged in:

$$\begin{aligned}\mathbb{E}[V_d(P_N)] &= \frac{2^d\pi^{\frac{d-1}{2}} d!\pi^{\frac{1}{2}}}{2^d\Gamma(\frac{d}{2}+1)d!}(1+O(N^{-\frac{2}{d-1}})) \\ &= \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}(1+O(N^{-\frac{2}{d-1}})) \\ &= \kappa_d(1+O(N^{-\frac{2}{d-1}})).\end{aligned}$$

The goal of this chapter was to find the expected missing volume between the surface of the sphere $\mathbb{S}^{d-1}$ and the convex hull of the points $X_1, ..., X_N$ as given in Equation (3.17). The volume of the former is given by definition, namely $V_d(B^d) = \kappa_d$ and the latter was also found, namely $\mathbb{E}[V_d(P_N)] = \kappa_d(1+O(N^{-\frac{2}{d-1}}))$. As a result:

$$\begin{aligned}\mathbb{E}[V_d(B^d) - V_d(P_N)] &= \kappa_d - \kappa_d(1+O(N^{-\frac{2}{d-1}})) \\ &= O(N^{-\frac{2}{d-1}}).\end{aligned}$$

Hence, the missing volume between the sphere $\mathbb{S}^{d-1}$ and the convex hull of the points $X_1, ..., X_N$ goes to zero at rate $N^{-\frac{2}{d-1}}$. As a final check, this result can be compared to the findings of Chapters 3.1 and 3.2. Filling in $d = 2$ gives N^{-2} and filling in $d = 3$ gives N^{-1} which indeed agrees with the findings of the respective chapters.

Chapter 4

Cube

The calculations involving spheres are completed and our focus will shift to the 3-dimensional cube. In the pre-print of Reitzner, Schütt and Werner [18], the following theorem is proved:

Theorem 4.1. *Let $n \geq 2$ and choose N uniform random points on the boundary of a simple polytope P in $\mathbb{R}^n$. Let P_N be the convex hull of these points. For the expected number of facets of the random polytope P_N, we have*

$$\mathbb{E}f_{n-1}(P_N) = c_n f_0(P)(\ln N)^{n-2}(1 + O((\ln N)^{-1})),$$

with some $c_n > 0$ independent of P.

The theorem assumes that P is a simple polytope. The 3-dimensional unit cube $C = [0,1]^3$ is a simple polytope. Therefore, we should be able to prove this theorem for C in this chapter. Hence, we will prove the following theorem which is adapted from Theorem 4.2:

Theorem 4.2. *Choose N uniform random points on the boundary of the simple polytope $C = [0,1]^3$. Let C_N be the convex hull of these points. For the expected number of facets of the random polytope C_N, we have*

$$\mathbb{E}f_2(C_N) = c \ln N(1 + O((\ln N)^{-1})),$$

with some $c > 0$ independent of C.

Throughout this chapter, c is a generic constant whose precise value may differ from occurrence to occurrence. The goal of this chapter is to prove Theorem 4.2. The prove is rather long and involved, so we divide the proof into several sections. Section 4.1 contains the body of the proof of this theorem. Tools that are needed to prove this theorem are given in Sections 4.2, 4.3 and 4.7 with additional proofs in Sections 4.4, 4.5 and 4.6.

4.1 The number of facets

To begin, let $X_1, ..., X_N$ be uniformly distributed points on the boundary ∂C of the cube. The convex hull of these points is denoted by $C_N = [X_1, ..., X_N]$. Figure 4.1 shows an example of how these points are distributed and what the convex hull looks like, focusing only on three sides of the cube. The expression $\mathbb{E}f_2(C_N)$ gives the expected number of facets of the convex hull C_N and that is the quantity that we want to find in this chapter. Similar to the definitions given before, let $\mathcal{F}_k(C_N)$ be the set of k-dimensional faces of C_N and let its cardinality be denoted by $|\mathcal{F}_k(C_N)| = f_k(C_N)$. Since we are interested in $\mathbb{E}f_2(C_N)$, the focus of this chapter is on the set of the facets of C_N, $\mathcal{F}_2(C_N)$, and its cardinality, $f_2(C_N)$.

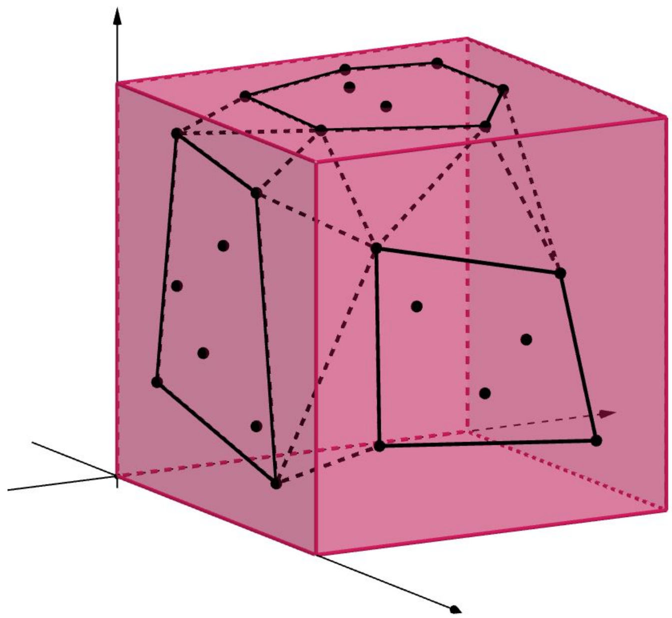


Figure 4.1: Convex hull of points distributed uniformly on three sides of the boundary of the cube.

Let $F = [X_1, ..., X_k] \in \mathcal{F}_2(C_N)$. The set $\mathcal{F}_2(C_N)$ consists of two kinds of facets.

1. $F \in \partial C$. These are facets that are contained in ∂C. This occurs when $X_1, ..., X_k$ live on the same facet of ∂C. There are only 6 facets in ∂C, so the number of facets that are contained in ∂C is bounded by 6. See Figure 4.1.
2. $F \notin \partial C$. These are facets that are not contained in ∂C. This occurs when $X_1, ..., X_k$ live on different facets of ∂C and almost surely they are simplices. Hence, $k = 3$. See Figure 4.1.

It is useful to have a set that describes the facets of the second kind, that is, facets in $\mathcal{F}_2(C_N)$ that are not contained in ∂C. These facets are formed by points $X_1, X_2, X_3 \in \partial C$ that are not all chosen from the same facet of ∂C. To this end, write $(\partial C)^3_{\neq}$ for the set of all triples $(X_1, X_2, X_3) \in \partial C$ such that not all X_i live on the same facet of ∂C. Since the number of facets of the first kind mentioned above is bounded, we restrict ourselves to the facets of the second kind. Hence from now on we assume that all facets of $\mathcal{F}_2(C_N)$ that we are dealing with are in the set $(\partial C)^3_{\neq}$. The goal is to find the number of facets of C_N, so we need to find a way to identify the points in $(\partial C)^3_{\neq}$ that form a facet of C_N. To do this, the convexity of C_N will be exploited. Let $I = \{i_1, i_2, i_3\} \subset [N]$ be an indexing set of distinct points i_1, i_2, i_3 and take $(X_{i_1}, X_{i_2}, X_{i_3}) \in (\partial C)^3_{\neq}$. The convex hull $[X_{i_1}, X_{i_2}, X_{i_3}]$ forms a facet of C_N if its affine hull does not intersect the convex hull of the remaining points $[\{X_j\}_{j \notin I}]$. This is described by the intersection of the following events:

$$E_I = \{\mathrm{aff}[\{X_i\}_{i \in I}] \cap [\{X_j\}_{j \notin I}] = \emptyset\} \quad \text{and} \quad F_I = \{(X_i)_{i \in I} \in (\partial C)^3_{\neq}\}$$

These two events identify all simplicial facets in $\mathcal{F}_2(C_N)$, which corresponds to the facets of the second kind. We can write

$$\begin{aligned} \mathbb{E} f_2(C_N) &= \mathbb{E} \sum_{I \subset [N], |I|=3} \mathbb{1}(E_I \cap F_I) + O(1) \\ &= \sum_{I \subset [N], |I|=3} \mathbb{E}\mathbb{1}(E_I \cap F_I) + O(1), \end{aligned}$$

where the $O(1)$-term comes from the facets of the first kind, i.e., the facets of C_N that are in ∂C which is bounded by 6. The points $X_1, ..., X_N$ are independent and identically distributed, so the summands in the last equation are the same for each indexing set I. Therefore, we can fix one indexing set I, e.g. $I = \{1, 2, 3\}$, and multiply by the number of indexing sets, $\binom{N}{3}$:

$$\mathbb{E}f_2(C_N) = \binom{N}{3}\mathbb{E}\mathbb{1}(E_{123} \cap F_{123}) + O(1),$$

with $E_{123} = \{\text{aff}[X_1, X_2, X_3] \cap [X_4, ..., X_N] = \emptyset\}$ and $F_{123} = \{(X_1, X_2, X_3) \in (\partial C)^3_{\neq}\}$. The affine hull $\text{aff}[X_1, X_2, X_3]$ is a recurring object, so we simplify notation by setting

$$H := \text{aff}[X_1, X_2, X_3].$$

We will first look at the probability of the event E_{123}. If the event E_{123} holds, i.e., if the points X_1, X_2, X_3 form a facet of C_N, then their affine hull H is a supporting hyperplane of the random polytope C_N. This hyperplane can be represented by $H = H(h, u) = \{x : \langle x, u\rangle = h\}$, where $u_{X_1,X_2,X_3} =: u$ is the unit outer normal vector of the facet $[X_1, X_2, X_3]$ and $h_{X_1,X_2,X_3} =: h$ is chosen such that $H(h, u)$ coincides with $\text{aff}([X_1, X_2, X_3])$. Consequently, the halfspace $H_- = H_-(h, u) = \{x : \langle x, u\rangle \leq h\}$, which is bounded by H, contains the random polytope C_N. The probability that E_{123} occurs is equal to the probability that $X_4, ..., X_N$ are contained in H_-. For one point X_i with $i = 4, ..., N$, this is given by the proportion of space that H_- takes in ∂C. Hence for $N - 3$ points this is given by

$$\left(\frac{\lambda_2(\partial C \cap H_-)}{\lambda_2(\partial C)}\right)^{N-3} = \left(1 - \frac{1}{6}\lambda_2(\partial C \cap H_+)\right)^{N-3},$$

where $H_+ = H_+(h, u) = \{x : \langle x, u\rangle \geq h\}$ is the complement of H_-. This gives the following expression:

$$\mathbb{E}f_2(C_N) = \binom{N}{3}\mathbb{E}\left(\left(1 - \frac{1}{6}\lambda_2(\partial C \cap H_+)\right)^{N-3}\mathbb{1}(F_{123})\right) + O(1).$$

The unit outer normal vector u that appears in $H_+(h, u)$ can have any direction. All vectors u have length 1, so they live on the sphere $\mathbb{S}^2$. Using the 8 vertices of the cube C, we will separate the sphere $\mathbb{S}^2$ into 8 parts. Denote by $H(C, u)$ a supporting hyperplane with normal u, supporting C in a vertex of C. The normal cone of vertex v in C is defined as

$$\mathcal{N}(v, C) = \{u \in \mathbb{R}^3 \backslash \{0\} : v \in H(C, u) \cup \{0\}\}.$$

Each vertex has a normal cone that corresponds to exactly one octant of $\mathbb{R}^3$. For example, the normal cone of the vertex $(0, 1, 1)$ equals the octant of $\mathbb{R}^3$ with $(-, +, +)$ coordinates and the normal cone of the vertex $(1, 0, 1)$ equals the octant of $\mathbb{R}^3$ with $(+, -, +)$ coordinates. All normal cones together cover $\mathbb{R}^3$. The boundaries of the cones are covered twice, but they have measure zero. Therefore, with probability one, the unit normal vector u of a random facet is contained in the interior of exactly one of the normal cones $\mathcal{N}(v, C)$ of the vertices $v \in \mathcal{F}_0(C)$. Hence

$$\mathbb{E}f_2(C_N) = \sum_{v\in\mathcal{F}_0(C)} \binom{N}{3}\mathbb{E}\left(\left(1 - \frac{1}{6}\lambda_2(\partial C \cap H_+)\right)^{N-3}\mathbb{1}(u \in \mathcal{N}(v, C), F_{123})\right) + O(1).$$

By symmetry of the cube, all summands are equal, so we might as well fix one vertex, e.g. $v = 0$, and multiply the result with the number of vertices:

$$\mathbb{E}f_2(C_N) = f_0(C)\binom{N}{3}\mathbb{E}\left(\left(1 - \frac{1}{6}\lambda_2(\partial C \cap H_+)\right)^{N-3}\mathbb{1}(u \in \mathcal{N}(0, C), F_{123})\right) + O(1).$$

The points X_1, X_2, X_3 appear in the event F_{123} and they are uniformly distributed on ∂C restricted to $(\partial C)^3_{\neq}$, so

$$\mathbb{E}f_2(C_N) = f_0(C)\binom{N}{3}\iiint\limits_{(\partial C)^3_{\neq}}\Big(1-\frac{1}{6}\lambda_2(\partial C\cap H_+)\Big)^{N-3}\mathbb{1}(u\in\mathcal{N}(0,C))\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3+O(1),$$

where $\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3$ denote integration with respect to the Hausdorff measure on $(\partial C)^3_{\neq}$. The vector u and distance h depend on the choice of x_1, x_2, x_3, so the integration is only taken over x_1, x_2, x_3. The last equation allows for an application of Lemma 2.10. This gives

$$\begin{aligned}\mathbb{E}f_2(C_N) =&cf_0(C)\binom{N}{3}\int\limits_{\mathbb{S}^2}\int\limits_{\mathbb{R}}\iiint\limits_{(\partial C\cap H)^3_{\neq}}\Big(1-\frac{1}{6}\lambda_2(\partial C\cap H_+)\Big)^{N-3}\mathbb{1}(u\in\mathcal{N}(0,C))\\ &\lambda_2([x_1,x_2,x_3])\big(J(T_{x_1},H)J(T_{x_2},H)J(T_{x_3},H)\big)^{-1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}h\mathrm{d}u+O(1).\end{aligned}$$

Now, the vector u and distance h no longer depend on the choice of x_1, x_2, x_3. Instead, the direction u and distance h are chosen first and then the points x_1, x_2, x_3 are chosen from $(\partial C\cap H(h,u))^3_{\neq}$. For the moment, we keep the factors $J(T_{x_i}, H)$ in this form. We will make it more explicit in further sections.

Moving on, we can make the expression for $\mathcal{N}(0,C)$ more precise. The normal cone of C at the origin is equal to the octant of $\mathbb{R}^3$ where all coordinates have negative value, so $\mathcal{N}(0,C) = -\mathbb{R}^3_+$. The range of integration of u is $\mathbb{S}^2$, so the condition $\mathbb{1}(u\in\mathcal{N}(0,C)) = \mathbb{1}(u\in-\mathbb{R}^3_+)$ can be taken into account by changing the range of integration of u to $\mathbb{S}^2\cap-\mathbb{R}^3_+ = -\mathbb{S}^2_+ = \{u\in\mathbb{S}^2 : u_1,u_2,u_3\le 0\}$:

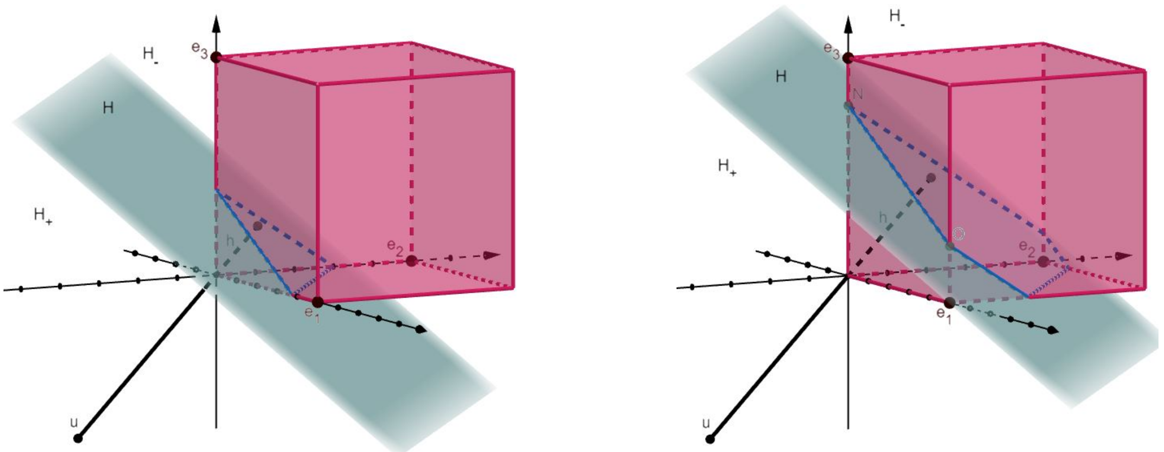
$$\begin{aligned}\mathbb{E}f_2(C_N) =&c\binom{N}{3}\int\limits_{-\mathbb{S}^2_+}\int\limits_{\mathbb{R}}\iiint\limits_{(\partial C\cap H)^3_{\neq}}\Big(1-\frac{1}{6}\lambda_2(\partial C\cap H_+)\Big)^{N-3}\\ &\lambda_2([x_1,x_2,x_3])\big(J(T_{x_1},H)J(T_{x_2},H)J(T_{x_3},H)\big)^{-1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}h\mathrm{d}u+O(1).\end{aligned}$$

Fix a vector $u\in-\mathbb{S}^2_+$. If h is close enough to zero, the hyperplane H_- contains all unit vectors e_1, e_2, e_3. This happens when

$$\max_{i=1,2,3} u_i \le h \le 0.$$

Figure 4.2a shows an example of this case and Figure 4.2b shows what happens when this is not satisfied. The blue lines indicate the intersection $\partial C\cap H(h,u)$. Note that $h\le 0$, since $u\in-\mathbb{S}^2_+$. The integral over $\mathbb{R}$ will be split into two parts: $\max u_i\le h\le 0$ and $-\infty\le h\le\max u_i$. The values $0\le h\le\infty$ will not be considered, because H does not intersect the cube C in that case and the integral will be zero. The expected number of facets is

$$\begin{aligned}\mathbb{E}f_2(C_N) =&c\binom{N}{3}\int\limits_{-\mathbb{S}^2_+}\int\limits_{\max u_i}^{0}\Big(1-\frac{1}{6}\lambda_2(\partial C\cap H_+)\Big)^{N-3}\\ &\iiint\limits_{(\partial C\cap H)^3_{\neq}}\lambda_2([x_1,x_2,x_3])\big(J(T_{x_1},H)J(T_{x_2},H)J(T_{x_3},H)\big)^{-1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}h\mathrm{d}u\\ &+c\binom{N}{3}\int\limits_{-\mathbb{S}^2_+}\int\limits_{-\infty}^{\max u_i}\Big(1-\frac{1}{6}\lambda_2(\partial C\cap H_+)\Big)^{N-3}\\ &\iiint\limits_{(\partial C\cap H)^3_{\neq}}\lambda_2([x_1,x_2,x_3])\big(J(T_{x_1},H)J(T_{x_2},H)J(T_{x_3},H)\big)^{-1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}h\mathrm{d}u+O(1).\end{aligned}$$



(a) $H(h,u)$ with $u = (-0.4, -0.5, -0.77)$ and $h = -0.3$. With these settings, the halfspace H_- contains the unit vectors e_1, e_2 and e_3.

(b) $H(h,u)$ with $u = (-0.4, -0.5, -0.77)$ and $h = -0.6$. With these settings, the halfspace H_- only contains the unit vector e_3.

Figure 4.2: Hyperplane $H(h,u)$ intersected with the cube C for different values of h. The blue lines indicate the intersection $\partial C \cap H(h,u)$.

The values for u_1, u_2, u_3 and h are negative, so we can substitute $u \mapsto -u$ and $h \mapsto -h$. The hyperplane $H(h,u)$ remains unchanged, but due to the multiplication with a negative number, the inequality sign flips, hence the halfspaces H_+ and H_- switch places. Furthermore, as can be seen in Figure 4.2a, for $0 \le h \le \min u_i$, the hyperplane $H(h,u)$ intersects ∂C only in the facets that are in $\mathbb{R}^3_+$, so $\partial C \cap H = \partial \mathbb{R}^3_+ \cap H$. We will use the more convenient formula

$$\mathbb{E} f_2(C_N) = c\binom{N}{3}(I_1 + I_2) + O(1), \tag{4.1}$$

where

$$I_1 := \int\limits_{\mathbb{S}^2_+} \int\limits_0^{\min u_i} \Big(1 - \frac{1}{6}\lambda_2(\partial\mathbb{R}^3_+ \cap H_-)\Big)^{N-3} \iiint\limits_{(\partial\mathbb{R}^3_+ \cap H)^3_{\neq}} \lambda_2([x_1,x_2,x_3])\big(J(T_{x_1},H)J(T_{x_2},H)J(T_{x_3},H)\big)^{-1} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}h\mathrm{d}u \tag{4.2}$$

and

$$I_2 := \int\limits_{\mathbb{S}^2_+} \int\limits_{\min u_i}^{\infty} \Big(1 - \frac{1}{6}\lambda_2(\partial C \cap H_-)\Big)^{N-3} \iiint\limits_{(\partial C \cap H)^3_{\neq}} \lambda_2([x_1,x_2,x_3])\big(J(T_{x_1},H)J(T_{x_2},H)J(T_{x_3},H)\big)^{-1} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}h\mathrm{d}u. \tag{4.3}$$

The integrals I_1 and I_2 will be investigated in the rest of this chapter. In Lemma 4.3 of Section 4.2, we will see that for $0 \le h \le \min u_i$, we can write the inner triple integral of I_1 as

$$\iiint\limits_{(\partial C \cap H)^3_{\neq}} \lambda_2([x_1,x_2,x_3])\big(J(T_{x_1},H)J(T_{x_2},H)J(T_{x_3},H)\big)^{-1} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = \frac{h^5}{2\sqrt{6}} \sum_{\boldsymbol{f} \in \{1,2,3\}^3_{\neq}} \frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4}\mathcal{E}_{\boldsymbol{f}},$$

where $\mathcal{E}_{\boldsymbol{f}}$ are positive constants independent of u. Then the integral I_1 becomes

$$I_1 = \frac{1}{2\sqrt{6}} \sum_{\boldsymbol{f}\in\{1,2,3\}^3_{\neq}} \mathcal{E}_{\boldsymbol{f}} \int\limits_{\mathbb{S}^2_+} \int\limits_0^{\min u_i} \Big(1 - \frac{1}{6}\lambda_2(\partial C \cap H_-)\Big)^{N-3} h^5 \frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4} \mathrm{d}h\mathrm{d}u.$$

In Section 4.3, we investigate what happens with this last expression for I_1 when $N \to \infty$, namely in Lemma 4.4 we show that the asymptotics of I_1 are

$$I_1 = cN^{-3}\ln N(1 + O((\ln N)^{-1})), \tag{4.4}$$

with some constant $c > 0$ as $N \to \infty$.

In Lemma 4.9 in Section 4.7 we show that $I_2 = O(N^{-3})$, which means that I_1 is dominating over I_2. Substituting these asymptotics of I_1 and I_2 into the expression for the expected number of facets in Equation (4.1) gives

$$\begin{aligned}\mathbb{E}f_2(C_N) &= c\binom{N}{3}\Big(cN^{-3}\ln N(1 + O((\ln N)^{-1})) + O(N^{-3})\Big) + O(1)\\ &= c\ln N(1 + O((\ln N)^{-1})),\end{aligned}$$

with some $c > 0$ which is Theorem 4.2 for the cube $C = [0, 1]^3$.

4.2 Expected volume of a facet

In the remaining of this chapter, the integral I_1 in Equation (4.2) will be investigated. In Section 4.3, the asymptotics of I_1 will be established. In the current section, we will look at only a part of I_1, namely the triple integral

$$\mathcal{E}(h, u) = \iiint\limits_{(\partial\mathbb{R}^3_+\cap H)^3_{\neq}} \lambda_2([x_1, x_2, x_3])\big(J(T_{x_1}, H)J(T_{x_2}, H)J(T_{x_3}, H)\big)^{-1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3,$$

which is the first moment of the volume of random simplex in $\partial\mathbb{R}^n_+ \cap H(h, u)$. The points x_i are chosen according to the weight functions $J(T_{x_i}, H)^{-1}$. We will prove the following lemma in this section:

Lemma 4.3. *There are constants $\mathcal{E}_{\boldsymbol{f}} > 0$ independent of u, such that*

$$\mathcal{E}(h, u) = \frac{h^5}{2\sqrt{6}} \sum_{\boldsymbol{f}\in\{1,2,3\}^3_{\neq}} \frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4}\mathcal{E}_{\boldsymbol{f}}.$$

Proof. It will be useful to distinguish between the different facets of ∂C, so we will introduce some new notation. Since we are taking points from $\partial\mathbb{R}^3_+ \cap H$, there are only three facets that we have to consider here. Let $F_{\hat{k}}$ be the facet of C that is spanned by the vectors e_i and e_j for distinct i, j, k. It follows that e_k is normal vector of $F_{\hat{k}}$. For example, in Figure 4.2, the facet $F_{\hat{3}}$ would be the "bottom" of the cube and e_3 is the vector normal to $F_{\hat{3}}$.

We will make the factors $J(T_{x_\ell}, H)$ explicit now. As explained in Chapter 2.2.3, for a point x_ℓ on a facet F, the hyperplane T_{x_ℓ} contains the facet F. Since the points x_1, x_2, x_3 are not all in the same facet of ∂C, the hyperplanes T_{x_ℓ} and H are not parallel. Then, for a point x_ℓ on the facet $F_{\hat{k}}$, the weight function $J(T_{x_\ell}, H)$ is equal to $\sin\sphericalangle(e_k, u)$. This is equal to the norm of the vector $v = u|_{F_{\hat{k}}}$ created by the orthogonal projection of the vector u onto the facet $F_{\hat{k}}$. The coordinates of this vector are $v_i = u_i$, $v_j = u_j$ and $v_k = 0$. Hence $\|v\| = \sqrt{u_i^2 + u_j^2} = \sqrt{1 - u_k^2}$, since we know that $\|u\| = 1$.

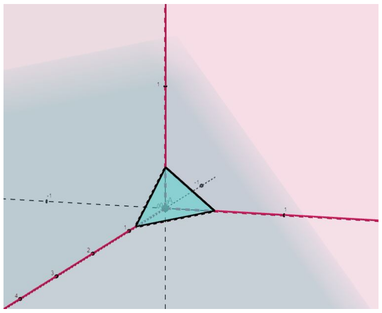


Figure 4.3: $\mathbb{R}^3_+ \cap H(h,u)$ with $h = 0.3, u = (0.3, 0.6, 0.74)$

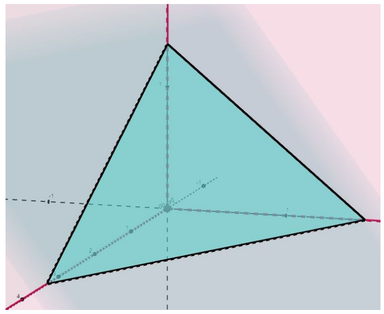


Figure 4.4: $\mathbb{R}^3_+ \cap H(1,u)$ with $u = (0.3, 0.6, 0.74)$

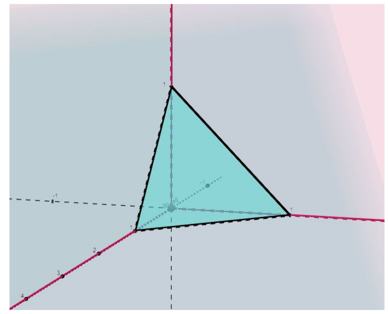


Figure 4.5: $\mathbb{R}^3_+ \cap H(1,\mathbf{1})$

Putting this together gives

$$J(T_{x_\ell}, H) = \sqrt{1 - u_k^2}, \tag{4.5}$$

which is independent of h.

The points x_1, x_2, x_3 are taken from $(\partial\mathbb{R}^3_+ \cap H)^3_{\neq}$, where $H = H(h,u)$. Now we make the substitution $x_\ell = hy_\ell$ with $y_\ell \in H(1,u)$. The 2-dimensional volume is homogeneous, so

$$\lambda_2([x_1, x_2, x_3]) = h^2 \lambda_2([y_1, y_2, y_3])$$

and since x_ℓ are in the 1-dimensional planes $(\partial\mathbb{R}^3_+ \cap H)^3_{\neq}$, we have $\mathrm{d}x_\ell = h\mathrm{d}y_\ell$. The transformation we made here is pictured in Figures 4.3 and 4.4. This results in the following integral:

$$\begin{aligned}\mathcal{E}(h,u) &= h^5 \iiint\limits_{(\partial\mathbb{R}^3_+ \cap H(1,u))^3_{\neq}} \lambda_2([y_1, y_2, y_3])\big(J(T_{x_1}, H)J(T_{x_2}, H)J(T_{x_3}, H)\big)^{-1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 \\ &= h^5\mathcal{E}(1,u).\end{aligned} \tag{4.6}$$

We want to evaluate $\mathcal{E}(1,u)$. The points y_1, y_2, y_3 are chosen from $(\partial\mathbb{R}^3_+ \cap H(1,u))^3_{\neq}$ where the facets $F_{\hat{1}}, F_{\hat{2}}, F_{\hat{3}}$ are located in. There are multiple ways to pick three points from these three facets such that the points are not all chosen from the same facet. We will make a distinction of these cases. Define $\{1,2,3\}^3_{\neq}$ as the set of triples where not all entries are the same, e.g. $\{1,2,2\}$, but not $\{3,3,3\}$. For an element $\boldsymbol{f} \in \{1,2,3\}^3_{\neq}$ we condition on the events $y_i \in F_{\hat{f_i}}$ for $i = 1,2,3$. We can do this for every element and sum over the results. Recalling 4.5, we obtain

$$\begin{aligned}\mathcal{E}(1,u) = &\sum_{\boldsymbol{f} \in \{1,2,3\}^3_{\neq}} \sqrt{(1-u_{f_1}^2)(1-u_{f_2}^2)(1-u_{f_3}^2)}^{\,-1} \\ &\times \iiint\limits_{(\partial\mathbb{R}^3_+ \cap H(1,u))^3_{\neq}} \lambda_2([y_1, y_2, y_3])\mathbb{1}(y_1 \in F_{\hat{f}_1})\mathbb{1}(y_2 \in F_{\hat{f}_2})\mathbb{1}(y_3 \in F_{\hat{f}_3})\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3.\end{aligned} \tag{4.7}$$

The hyperplane $H(1,u)$ is given by the equation $u_1x + u_2y + u_3z = 1$ for $x, y, z \in \mathbb{R}$, restricted to $u_1^2 + u_2^2 + u_3^2 = 1$. It meets the x-axis when $y = z = 0$, in which case $u_1 x = 1$, so $x = \frac{1}{u_1}$. In general, the hyperplane $H(1,u)$ meets the coordinate axis in the points $\frac{1}{u_i}e_i$. The intersection $\mathbb{R}^3_+ \cap H(1,u)$ forms a triangle between the points $(\frac{1}{u_1}, 0, 0), (0, \frac{1}{u_2}, 0), (0, 0, \frac{1}{u_3})$. See Figure 4.4 for an example. The area of this triangle is calculated in Appendix B and it is equal to $\frac{1}{2u_1u_2u_3}$. We have assumed that

$H(1,u)$ is the hyperplane with normal vector u. However, we can also obtain the vector u from a transformation of the vector $\mathbf{1} = (1,1,1)$ using the affine map

$$A = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}.$$

This map transforms $H(1,\mathbf{1})$ into $H(1,u)$, so it is natural to make the substitution $y = A^{-1}z$, where $z \in H(1,\mathbf{1})$. This transformation is pictured in Figures 4.4 and 4.5. The intersection $\mathbb{R}^3_+ \cap H(1,\mathbf{1})$ forms a triangle between the unit vectors e_1, e_2, e_3. The area of this triangle is calculated in Appendix B and it is equal to $\frac{\sqrt{3}}{2}$.

The map A^{-1} scales triangles $\mathbb{R}^3_+ \cap H(1,u)$ and $\mathbb{R}^3_+ \cap H(1,\mathbf{1})$ with a factor $\frac{1}{u_1u_2u_3\sqrt{3}}$. By setting $z_i = Ay_i$, the factor $\lambda_2([y_1,y_2,y_3])$ in the integrand in Equation (4.7) can be rewritten as

$$\lambda_2([y_1,y_2,y_3]) = \lambda_2([A^{-1}z_1, A^{-1}z_2, A^{-1}z_3]) = \frac{1}{u_1u_2u_3\sqrt{3}}\lambda_2([z_1,z_2,z_3]). \tag{4.8}$$

It remains to find the Jacobian of the map A. The points y_1, y_2, y_3 are chosen from $\partial\mathbb{R}^3_+ \cap H(1,u)$. Each edge of this simplex lies in one facet $F_{\hat{i}}$. We know that the coordinates of the simplex are $(\frac{1}{u_1},0,0), (0,\frac{1}{u_2},0), (0,0,\frac{1}{u_3})$, so the length of the edge of $\partial\mathbb{R}^3_+ \cap H(1,u)$ in $F_{\hat{i}}$ is equal to

$$\lambda_2(\partial\mathbb{R}^3_+ \cap H(1,u) \cap F_{\hat{i}}) = \sqrt{\frac{1}{u_j^2} + \frac{1}{u_k^2}} = \frac{1}{u_ju_k}\sqrt{u_k^2 + u_j^2} = \frac{1}{u_ju_k}\sqrt{1-u_i^2}.$$

The points z_1, z_2, z_3 are chosen from $\partial\mathbb{R}^3_+ \cap H(1,\mathbf{1})$. The length of an edge of $\partial\mathbb{R}^3_+ \cap H(1,\mathbf{1})$ in $F_{\hat{i}}$ is equal to the distance from e_j to e_k, which is

$$\lambda_2(\partial\mathbb{R}^3_+ \cap H(1,\mathbf{1}) \cap F_{\hat{i}}) = \sqrt{2}.$$

Comparing the length of an edge before and after the transformation shows that the Jacobian in $F_{\hat{f}_i}$ of the map A is equal to

$$\begin{aligned} \frac{\lambda_2(\partial\mathbb{R}^3_+ \cap H(1,u) \cap F_{\hat{f}_i})}{\lambda_2(\partial\mathbb{R}^3_+ \cap H(1,\mathbf{1}) \cap F_{\hat{f}_i})} &= \sqrt{\frac{1-u_{f_i}^2}{2}} \prod_{\ell \neq f_i} \frac{1}{u_\ell} \mathbb{1}(z_i \in F_{\hat{f}_i}) \\ &= \sqrt{\frac{1-u_{f_i}^2}{2}} \frac{u_{f_i}}{u_1u_2u_3} \mathbb{1}(z_i \in F_{\hat{f}_i}). \end{aligned}$$

Combining these Jacobians with Equations (4.7) and (4.8) gives

$$\begin{aligned} \mathcal{E}(1,u) = &\sum_{\boldsymbol{f} \in \{1,2,3\}^3_{\neq}} \sqrt{(1-u_{f_1}^2)(1-u_{f_2}^2)(1-u_{f_3}^2)}^{\,-1} \\ &\times \iiint_{(\partial\mathbb{R}^3_+ \cap H(1,\mathbf{1}))^3_{\neq}} \frac{1}{u_1u_2u_3\sqrt{3}} \lambda_2([z_1,z_2,z_3]) \sqrt{\frac{1-u_{f_1}^2}{2}} \sqrt{\frac{1-u_{f_2}^2}{2}} \sqrt{\frac{1-u_{f_3}^2}{2}} \\ &\times \frac{u_{f_1}}{u_1u_2u_3} \frac{u_{f_2}}{u_1u_2u_3} \frac{u_{f_3}}{u_1u_2u_3} \mathbb{1}(z_1 \in F_{\hat{f}_1}) \mathbb{1}(z_2 \in F_{\hat{f}_2}) \mathbb{1}(z_3 \in F_{\hat{f}_3}) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3. \end{aligned}$$

We can simplify this expression further.

$$\begin{aligned}\mathcal{E}(1,u) &= \sum_{\boldsymbol{f}\in\{1,2,3\}^3_{\neq}} \frac{1}{2\sqrt{6}}\frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4} \\ &\quad\times \iiint\limits_{(\partial\mathbb{R}^3_+\cap H(1,\mathbf{1}))^3_{\neq}} \lambda_2([z_1,z_2,z_3])\mathbb{1}(z_1\in F_{\hat{f}_1})\mathbb{1}(z_2\in F_{\hat{f}_2})\mathbb{1}(z_3\in F_{\hat{f}_3})\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\ &=: \sum_{\boldsymbol{f}\in\{1,2,3\}^3_{\neq}} \frac{1}{2\sqrt{6}}\frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4}\mathcal{E}_{\boldsymbol{f}},\end{aligned}$$

where $\mathcal{E}_{\boldsymbol{f}}$ is a constant independent of u. Together with Equation (4.6) we find that

$$\begin{aligned}\mathcal{E}(h,u) &= h^5\mathcal{E}(1,u) \\ &= \frac{h^5}{2\sqrt{6}}\sum_{\boldsymbol{f}\in\{1,2,3\}^3_{\neq}}\frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4}\mathcal{E}_{\boldsymbol{f}},\end{aligned}$$

which is Lemma 4.3. □

4.3 Asymptotics of I_1

The purpose of this section is to show that the asymptotics of I_1 in Equation (4.4) hold. The first mention of I_1 is in Equation (4.2), which can be updated using Lemma 4.3. This gives

$$I_1 = c\sum_{\boldsymbol{f}\in\{1,2,3\}^3_{\neq}}\mathcal{E}_{\boldsymbol{f}}\int\limits_{\mathbb{S}^2_+}\int\limits_0^{\min u_i}\Big(1-\frac{1}{6}\lambda_2(\partial\mathbb{R}^3_+\cap H_-)\Big)^{N-3}h^5\frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4}\mathrm{d}h\mathrm{d}u. \tag{4.9}$$

We want to find the asymptotics of I_1 in this section. This is stated in the following lemma:

Lemma 4.4. *Consider I_1 as in Equation (4.9). The asymptotics are given by*

$$I_1 = cN^{-3}\ln N(1+O((\ln N)^{-1})).$$

This lemma will be proven in this section.

First, we will find the value of the area $\lambda_2(\partial\mathbb{R}^3_+\cap H_-)$. The hyperplane $H(h,u)$ is given by the equation $u_1x+u_2y+u_3z=h$ for $x,y,z\in\mathbb{R}$. It meets the x-axis when $y=z=0$ in which case $x=\frac{h}{u_1}$ and similarly for the y-axis and z-axis. Consequently, $H(h,u)$ meets the coordinate axis in the points $t_ie_i=\frac{h}{u_i}e_i$ for $i=1,2,3$. The intersection $\partial\mathbb{R}^3_+\cap H_-$ consists of three right triangles, each formed by the origin and two of the intersection points with the coordinate axis. Therefore, $\lambda_2(\partial\mathbb{R}^3_+\cap H_-)=\frac{1}{2}\frac{h}{u_1}\frac{h}{u_2}+\frac{1}{2}\frac{h}{u_1}\frac{h}{u_3}+\frac{1}{2}\frac{h}{u_2}\frac{h}{u_3}=\frac{h^2(u_1+u_2+u_3)}{2u_1u_2u_3}$. Plugging this in gives

$$I_1 = c\sum_{\boldsymbol{f}\in\{1,2,3\}^3_{\neq}}\mathcal{E}_{\boldsymbol{f}}\int\limits_{\mathbb{S}^2_+}\int\limits_0^{\min u_i}\left(1-\frac{h^2(u_1+u_2+u_3)}{12u_1u_2u_3}\right)^{N-3}h^5\frac{u_{f_1}u_{f_2}u_{f_3}}{(u_1u_2u_3)^4}\mathrm{d}h\mathrm{d}u. \tag{4.10}$$

We saw before that the hyperplane H meets the coordinate axis in the points $t_ie_i=\frac{h}{u_i}e_i$ for $i=1,2,3$. The value of h is in the interval $[0,\min_{i=1,2,3}u_i]$, so $t_i\in[0,1]$ for all $i=1,2,3$. Therefore, instead of integrating over the set of hyperplanes defined by h and u, we can integrate with respect to t_1,t_2,t_3

which are the intersections of the hyperplane $H(h,u)$ with the coordinate axis. The goal is to make the substitution $t_i = \frac{h}{u_i}$. Before we can do that, we have to rewrite the integrand in Equation (4.10). Set $m_i = \sum_j \mathbb{1}(f_j = i)$. The numbers m_i count how many of the points y_1, y_2, y_3 are taken from facet $F_{\hat{i}}$ in the set $\boldsymbol{f}$. Note that $m_1 + m_2 + m_3 = 3$, because we will always choose 3 points in total. As a result, $u_{f_1} u_{f_2} u_{f_3} = u_1^{m_1} u_2^{m_2} u_3^{m_3}$, so it holds that

$$\frac{u_{f_1} u_{f_2} u_{f_3}}{(u_1 u_2 u_3)^4} = \frac{1}{u_1^{4-m_1}} \frac{1}{u_2^{4-m_2}} \frac{1}{u_3^{4-m_3}}.$$

We also want to incorporate the h^5 in the fraction with u_i. Using that $\sum_i m_i = 3$, we can write

$$h^5 \frac{u_{f_1} u_{f_2} u_{f_3}}{(u_1 u_2 u_3)^4} = h^5 \frac{1}{u_1^{4-m_1}} \frac{1}{u_2^{4-m_2}} \frac{1}{u_3^{4-m_3}} = h^{-4} \left(\frac{h}{u_1}\right)^{4-m_1} \left(\frac{h}{u_2}\right)^{4-m_2} \left(\frac{h}{u_3}\right)^{4-m_3}.$$

Plug this back in to Equation (4.10):

$$\begin{aligned} I_1 = c \sum_{\boldsymbol{f} \in \{1,2,3\}^3_{\neq}} \mathcal{E}_{\boldsymbol{f}} \int_{\mathbb{S}^2_+} \int_0^{\min u_i} &\left(1 - \frac{h^2}{12 u_1 u_2} - \frac{h^2}{12 u_1 u_3} - \frac{h^2}{12 u_2 u_3}\right)^{N-3} \\ &\times h^{-4} \left(\frac{h}{u_1}\right)^{4-m_1} \left(\frac{h}{u_2}\right)^{4-m_2} \left(\frac{h}{u_3}\right)^{4-m_3} \mathrm{d}h \mathrm{d}u. \end{aligned} \tag{4.11}$$

We will make some steps towards the substitution $t_i = \frac{h}{u_i}$. First, substitute $r = h^{-1}$ to get $\mathrm{d}h = -r^{-2}\mathrm{d}r$. This gives $h^{-4}\mathrm{d}h\mathrm{d}u = -r^4 r^{-2}\mathrm{d}r\mathrm{d}u = -r^2\mathrm{d}r\mathrm{d}u$. The variables r, u define a system of polar coordinates (r,u). Pass this to the Cartesian coordinate system, which is achieved by $r^2\mathrm{d}r\mathrm{d}u = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$. The final substitution we will make is $x_i = \frac{1}{t_i}$ with $\mathrm{d}x_i = -t_i^{-2}\mathrm{d}t_i$. Finally, we have

$$h^{-4}\mathrm{d}h\mathrm{d}u = -\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = (t_1 t_2 t_3)^{-2}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3$$

with $h^{-1}u_i = r u_i = x_i = t_i^{-1}$. We started with two variables h and u, where h determines the distance from the origin to the hyperplane H and u determines the direction of H. The resulting hyperplane H intersects the axes in the points $\frac{h}{u_1}e_1, \frac{h}{u_2}e_2, \frac{h}{u_3}e_3$. This gives rise to the new coordinate system $(\frac{h}{u_1}, \frac{h}{u_2}, \frac{h}{u_3}) = (t_1, t_2, t_3)$ where the t_i indicate where H intersects the i-th axis. Since $\frac{h}{u_i} \in [0,1]$, it holds that $t_i \in [0,1]$. Applying this substitution to Equation (4.11) gives

$$\begin{aligned} I_1 &= c \sum_{\boldsymbol{f} \in \{1,2,3\}^3_{\neq}} \mathcal{E}_{\boldsymbol{f}} \int_0^1 \int_0^1 \int_0^1 \left(1 - \frac{1}{12}(t_1 t_2 + t_1 t_3 + t_2 t_3)\right)^{N-3} t_1^{2-m_1} t_2^{2-m_2} t_3^{2-m_3} \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3. \\ &= c \sum_{\boldsymbol{f} \in \{1,2,3\}^3_{\neq}} \mathcal{E}_{\boldsymbol{f}} J(\boldsymbol{m} - \boldsymbol{1}) \end{aligned} \tag{4.12}$$

and here we define

$$J(\boldsymbol{l}) := \int_0^1 \int_0^1 \int_0^1 \left(1 - \frac{1}{12}(t_1 t_2 + t_1 t_3 + t_2 t_3)\right)^{N-3} t_1^{1-l_1} t_2^{1-l_2} t_3^{1-l_3} \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3, \tag{4.13}$$

where $l_1, l_2, l_3 \in \{-1, 0, 1\}$. We will spend the rest of this chapter on computing this last integral. We denote $\boldsymbol{l} = (l_1, l_2, l_3)$. The integrand of the integral $J(\boldsymbol{l})$ is symmetric in the variables t_1, t_2, t_3, so choosing $\boldsymbol{l} = (-1, 0, 1)$ or permutations of that will all result in the same value for $J(\boldsymbol{l})$. Intuitively this makes sense, because these choices of $\boldsymbol{l}$ all come down to taking one point from one facet and two points from another facet, which is symmetric in the facets. Hence we can say that

$$J(-1,0,1) = J(-1,1,0) = J(0,-1,1) = J(0,1,-1) = J(1,0,-1) = J(1,-1,0).$$

The final possible choice is $\boldsymbol{l} = (0,0,0)$, which is its own category. There are no more ways of choosing $\boldsymbol{l}$. We can split the summation in Equation (4.12) according to these categories. Then

$$I_1 = cJ(-1,0,1) + cJ(0,0,0). \tag{4.14}$$

We only have to find $J(\boldsymbol{l})$ for $\boldsymbol{l} = (-1,0,1)$ and $\boldsymbol{l} = (0,0,0)$.

Lemmas 4.5, 4.6 and 4.7 deal with the integral $J(\boldsymbol{l})$ in Equation (4.13), to make it useful for our purposes. The proofs of these lemmas are rather technical, so in order to keep the contents of this section easy-to-follow, they are postponed to separate sections. After that, Lemma 4.8 proves the crucial asymptotics.

Lemma 4.5. *Let $\mathcal{S}_3$ be the set of all permutations of $\{1,2,3\}$ and let $f : (0,\infty)^3 \to (0,\infty)^3$ be defined by*

$$f_j(x) = \prod_{i \neq j} x_i \qquad j = 1,2,3.$$

1. *The inverse function to f is $g : (0,\infty)^3 \to (0,\infty)^3$ given by*

$$g_i(x) = \frac{\sqrt{x_1 x_2 x_3}}{x_i}$$

2. *f maps the open set $(0,1)^3$ bijectively onto $S = \{y \in (0,1)^3 : y_1 y_2 y_3 < y_i^2 \; \forall i\}$.*
3. *$A := \{x \in (0,\beta)^3 : x_1 x_2 x_3 < \beta x_i^2\} = \bigcup_{\pi \in \mathcal{S}_3} \{(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) : x \in M\} =: B$ for $\beta > 0$ and where the set M is defined as $M = \{x \in (0,\infty)^3 : x_3 \leq x_2 \leq x_1, \beta x_3 > x_1 x_2\}$.*

Proof. Section 4.4. ∎

Using this lemma, we can prove the following lemma regarding the function $J(\boldsymbol{l})$.

Lemma 4.6. *Let $\boldsymbol{l} \in \{(-1,0,1),(0,0,0)\}$. Then we have*

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}}}_{\sqrt{s_1 s_2 s_3} \leq s_i \sqrt{\frac{12}{N-3}} \, \forall i} \left(1 - \frac{s_1+s_2+s_3}{N-3}\right)^{N-3} s_1^{l_1} s_2^{l_2} s_3^{l_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.$$

Proof. Section 4.5. ∎

We have almost reached the optimal form of $J(\boldsymbol{l})$. Until now, the number of points N appears in the range of integration of all s_i. However, using Lemmas 4.5 and 4.6 as tools, we change this to smaller intervals. Recall that $\mathcal{S}_3$ is the set of all permutations of $\{1,2,3\}$.

Lemma 4.7. *Let $\boldsymbol{l} \in \{(-1,0,1),(0,0,0)\}$. Then we have*

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \sum_{\pi \in \mathcal{S}_3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_1+s_2+s_3}{N-3}\right)^{N-3} s_1^{l_{\pi(1)}} s_2^{l_{\pi(2)}} s_3^{l_{\pi(3)}} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.$$

Proof. Section 4.6. ∎

In light of Lemma 4.7, we introduce integrals of the type

$$\mathcal{S}(\boldsymbol{q}) = \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} s_1^{q_1} s_2^{q_2} s_3^{q_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1, \tag{4.15}$$

where $\boldsymbol{q} := (q_1, q_2, q_3) = (l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)})$. The role of $\boldsymbol{q}$ is the same as the role of $\boldsymbol{l}$ in Lemmas 4.6 and 4.7. Namely, $\boldsymbol{q} + 1$ represents how many points are taken from each facet. However, the integral in Equation (4.15) is not symmetric in the variables s_1, s_2, s_3, so the order of q_1, q_2, q_3 *does* matter, in contrary to l_1, l_2, l_3. We can find the asymptotics of $\mathcal{S}(\boldsymbol{q})$ for different values of $\boldsymbol{q}$. This will be done in the lemma below. It has a lengthy proof, but since its results are crucial in finding the asymptotics of I_1, the proof is given here.

Lemma 4.8. *Assume that* $\boldsymbol{q} \in \{(-1, 0, 1), (-1, 1, 0), (0, -1, 1), (0, 1, -1), (1, 0, -1), (1, -1, 0), (0, 0, 0)\}$. *Then there is a constant* $c_{\boldsymbol{q}} \geq 0$ *such that*

$$\mathcal{S}(\boldsymbol{q}) = c_{\boldsymbol{q}} \ln N + O(1)$$

as $N \to \infty$. *More precisely:*

1. *If* $q_3 = -1$, *then*

$$\mathcal{S}(q_1, q_2, -1) + \mathcal{S}(q_2, q_1, -1) = \ln N + O(1) \tag{4.16}$$

2. *If* $q_3 > -1$, *then* $c_{\boldsymbol{q}} = 0$, *so*

$$\mathcal{S}(\boldsymbol{q}) = O(1).$$

We start with proving item 1 of this lemma. The proof of item 2 follows immediately after.

Proof of item 1 in Lemma 4.8. We assume that $q_3 = -1$. It follows that one of q_1 and q_2 is 0 and the other one is 1 and for now it does not matter which one is which. Recall the formula of $\mathcal{S}(\boldsymbol{q})$ in (4.15). The formula of $\mathcal{S}(q_1, q_2, -1)$ reduces to

$$\mathcal{S}(q_1, q_2, -1) = \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} \frac{s_1^{q_1} s_2^{q_2}}{s_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1. \tag{4.17}$$

This integration will be done by dissecting the current range of integration along the sets $J_1 = \{s_3 \leq 1\}$ and $J_2 = \{s_3 \geq 1\}$. Then the expression for $\mathcal{S}(q_1, q_2, -1)$ can be written as

$$\begin{aligned} \mathcal{S}(q_1, q_2, -1) = & \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\{\frac{12 s_1 s_2}{N-3} \leq s_3\} \cap J_1} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} \frac{s_1^{q_1} s_2^{q_2}}{s_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\ & + \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\{\frac{12 s_1 s_2}{N-3} \leq s_3\} \cap J_2} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} \frac{s_1^{q_1} s_2^{q_2}}{s_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\ =: & \, \mathcal{S}_1(q_1, q_2, -1) + \mathcal{S}_2(q_1, q_2, -1). \end{aligned} \tag{4.18}$$

We will first deal with the computation of $\mathcal{S}_1(q_1, q_2, -1)$. This is done by bounding the term $1 - \frac{s_1+s_2+s_3}{N-3}$ from above and below. For the upper bound, we use the trivial inequality $1 - \frac{s_1+s_2+s_3}{N-3} \leq 1 - \frac{s_1+s_2}{N-3}$. Now we can also found an upper bound for the integrand of $\mathcal{S}_1(q_1, q_2, -1)$. For this, we focus on the inner integral of $\mathcal{S}_1(q_1, q_2, -1)$. This is the integral with respect to $\mathrm{d}s_3$ and its upper bound is found as follows.

$$\begin{aligned}
\mathcal{S}_{1,q_3} &= \int\limits_{\left[\frac{12 s_1 s_2}{N-3}, s_2\right] \cap J_1} \left(1 - \frac{s_1+s_2+s_3}{N-3}\right)^{N-3} \frac{1}{s_3} \mathrm{d}s_3 \\
&\leq \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} \int\limits_{\left[\frac{12 s_1 s_2}{N-3}, s_2\right] \cap J_1} \frac{1}{s_3} \mathrm{d}s_3 \\
&= \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} \Big[\ln(s_3)\Big]_{\frac{12 s_1 s_2}{N-3}}^{s_2} \\
&= \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} (-\ln(s_1) + \ln(N-3) - \ln(12))
\end{aligned}$$

which is an upper bound for $\mathcal{S}_{1,q_3}$. For the lower bound of $1 - \frac{s_1+s_2+s_3}{N-3}$ we use that $s_2 \leq s_1 \leq \frac{N-3}{12} \implies \frac{s_1+s_2}{N-3} \leq \frac{2s_1}{N-3} \leq \frac{1}{2}$. Keeping this in mind, we find:

$$\begin{aligned}
(1 - \frac{s_1+s_2}{N-3})(1 - \frac{2s_3}{N-3}) &= 1 - \frac{s_1+s_2}{N-3} - \frac{2s_3}{N-3} + \frac{2s_3(s_1+s_2)}{(N-3)^2} \\
&= 1 - \frac{s_1+s_2}{N-3} - \frac{s_3}{N-3}\left(2 - \frac{2(s_1+s_2)}{N-3}\right) \\
&\leq 1 - \frac{s_1+s_2+s_3}{N-3}.
\end{aligned}$$

We will use the elementary inequality $(1-y)^k \geq 1 - ky$ when $y \leq 1$ and that $[\frac{12 s_1 s_2}{N-3}, s_2] \in [0, 1]$. Now we can also find a lower bound for $\mathcal{S}_{1,q_3}$.

$$\begin{aligned}
\mathcal{S}_{1,q_3} &\geq \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} \int\limits_{\left[\frac{12 s_1 s_2}{N-3}, s_2\right] \cap J_1} (1 - \frac{2s_3}{N-3})^{N-3} \frac{1}{s_3} \mathrm{d}s_3 \\
&\geq \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} \int\limits_{\left[\frac{12 s_1 s_2}{N-3}, s_2\right] \cap J_1} (1 - 2s_3) \frac{1}{s_3} \mathrm{d}s_3 \\
&= \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} \Big[\ln(s_3) - 2s_3\Big]_{\frac{12 s_1 s_2}{N-3}}^{s_2} \\
&= \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} (-\ln(s_1) + \ln(N-3) - \ln(12) - 2(s_2 - \frac{12 s_1 s_2}{N-3})) \\
&\geq \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} (-\ln(s_1) + \ln(N-3) - \ln(12) - 2).
\end{aligned}$$

These bounds can be summarized in one expression using an error term:

$$\begin{aligned}
\mathcal{S}_{1,q_3} &= \left(1 - \frac{s_1+s_2}{N-3}\right)^{N-3} (-\ln(s_1) + \ln(N-3) + E), \\
&\text{where } -\ln(12) - 2 \leq E \leq -\ln(12).
\end{aligned}$$

Plugging this back into the equation for $\mathcal{S}_1(q_1, q_2, -1)$ gives

$$\mathcal{S}_1(q_1, q_2, -1) = \int_0^{\frac{N-3}{12}} \int_0^{s_1} \left(1 - \frac{s_1 + s_2}{N-3}\right)^{N-3} (-\ln(s_1) + \ln(N-3) + E) s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 \tag{4.19}$$

The integrand in the last expression contains a summation of three terms, hence these terms can be separated to three double integrals. The integral with the $\ln(N-3)$-term is the dominating one, which we will handle after bounding the two other integrals. First, using the identity $(1 + \frac{x}{N})^N \leq e^x$, we get

$$\begin{aligned}
\left| \int_0^{\frac{N-3}{12}} \int_0^{s_1} \left(1 - \frac{s_1 + s_2}{N-3}\right)^{N-3} (-\ln(s_1)) s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 \right| &\leq \int_0^{\infty} \int_0^{\infty} e^{-s_1 - s_2} |\ln(s_1)| s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 \\
&= \int_0^{\infty} e^{-s_1} |\ln(s_1)| s_1^{q_1} \mathrm{d}s_1 \int_0^{\infty} e^{-s_2} s_2^{q_2} \mathrm{d}s_2 \\
&= k_1 \Gamma(q_2 + 1) = O(1),
\end{aligned}$$

where k_1 is a finite constant since integrals of the form $\int_0^\infty e^{-x} x^k |\ln(x)| \mathrm{d}x$ are convergent. Second,

$$\begin{aligned}
\left| \int_0^{\frac{N-3}{12}} \int_0^{s_1} \left(1 - \frac{s_1 + s_2}{N-3}\right)^{N-3} E s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 \right| &\leq \int_0^{\frac{N-3}{12}} \int_0^{s_1} \left(1 - \frac{s_1 + s_2}{N-3}\right)^{N-3} (\ln(12) + 2) s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 \\
&\leq (\ln(12) + 2) \int_0^{\infty} \int_0^{\infty} e^{-s_1 - s_2} s_1^{q_1} s_2^{q_2} \mathrm{d}s_1 \mathrm{d}s_2 \\
&= (\ln(12) + 2) \Gamma(q_1 + 1) \Gamma(q_2 + 1) = O(1).
\end{aligned}$$

The integrands containing the terms $-\ln(s_1)$ and E are of order $O(1)$ and the term containing $\ln(N-3)$ remains to be bounded. Equation (4.19) becomes

$$\mathcal{S}_1(q_1, q_2, -1) = \ln(N-3) \int_0^{\frac{N-3}{12}} \int_0^{s_1} \left(1 - \frac{s_1 + s_2}{N-3}\right)^{N-3} s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 + O(1), \tag{4.20}$$

We have used before that $\frac{s_1+s_2}{N-3} < 1$. Note that for $|t| < 1$ it holds that $e^t(1-t) \geq (1+t)(1-t) = 1 - t^2$ and $(1 - t^2)^m \geq 1 - mt^2$. With this given, we find the following inequalities:

$$\begin{aligned}
0 \leq e^{-(s_1+s_2)} - \left(1 - \frac{s_1 + s_2}{N-3}\right)^{N-3} &\leq e^{-(s_1+s_2)} - e^{-(s_1+s_2)} \left(1 - \frac{(s_1 + s_2)^2}{(N-3)^2}\right)^{N-3} \\
\leq e^{-(s_1+s_2)} - e^{-(s_1+s_2)} \left(1 - \frac{(s_1 + s_2)^2}{N-3}\right) &= e^{-(s_1+s_2)} \frac{(s_1 + s_2)^2}{N-3},
\end{aligned}$$

which implies that $\left(1-\frac{s_1+s_2}{N-3}\right)^{N-3} = e^{-(s_1+s_2)}(1+O(\frac{(s_1+s_2)^2}{N}))$. Using this in Equation (4.20) gives

$$\begin{aligned}
\mathcal{S}_1(q_1,q_2,-1) &= \ln(N-3)\int_0^{\frac{N-3}{12}}\int_0^{s_1} e^{-(s_1+s_2)}(1+O(N^{-1}(s_1+s_2)^2))s_1^{q_1}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1 + O(1)\\
&= \ln(N-3)\int_0^{\frac{N-3}{12}}\int_0^{s_1} e^{-(s_1+s_2)}s_1^{q_1}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1 + O(N^{-1}) + O(1)\\
&= \ln(N-3)\int_0^{\infty}\int_0^{\infty} \mathbb{1}(s_2\le s_1)e^{-s_1-s_2}s_1^{q_1}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1\\
&\quad - \ln(N-3)\int_{\frac{N-3}{12}}^{\infty}\int_0^{s_1} e^{-s_1-s_2}s_1^{q_1}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1 + O(1).
\end{aligned}$$

The integral in the last line is of order $O(N^{q_1}e^{-\frac{N-3}{12}})$ which is shown by the following computations:

$$\begin{aligned}
&\int_{\frac{N-3}{12}}^{\infty}\int_0^{s_1} e^{-s_1-s_2}s_1^{q_1}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1 = \int_{\frac{N-3}{12}}^{\infty} e^{-s_1}s_1^{q_1}\int_0^{s_1} e^{-s_2}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1 \le \int_{\frac{N-3}{12}}^{\infty} e^{-s_1}s_1^{q_1}\int_0^{\infty} e^{-s_2}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1\\
&= \Gamma(q_2+1)\int_{\frac{N-3}{12}}^{\infty} e^{-s_1}s_1^{q_1}\mathrm{d}s_1 = \begin{cases}\Gamma(q_2+1)\big[-e^{-s_1}(s_1+1)\big]_{\frac{N-3}{12}}^{\infty} & \text{if } q_1=1\\ \Gamma(q_2+1)\big[-e^{-s_1}\big]_{\frac{N-3}{12}}^{\infty} & \text{if } q_1=0\end{cases}\\
&= \begin{cases}\Gamma(q_2+1)e^{-\frac{N-3}{12}}(\frac{N-3}{12}+1) & \text{if } q_1=1\\ \Gamma(q_2+1)e^{-\frac{N-3}{12}} & \text{if } q_1=0\end{cases} \qquad = O(N^{q_1}e^{-\frac{N-3}{12}}).
\end{aligned}$$

Hence,

$$\mathcal{S}_1(q_1,q_2,-1) = \ln(N-3)\int_0^{\infty}\int_0^{\infty} \mathbb{1}(s_2\le s_1)e^{-s_1-s_2}s_1^{q_1}s_2^{q_2}\mathrm{d}s_2\mathrm{d}s_1 + O(1) \tag{4.21}$$

which we recall is the integral $\mathcal{S}(q_1,q_2,q_3)$ given in Equation (4.17) with the additional restriction that $s_3\le 1$.

We will do the same for the integral $\mathcal{S}(q_1,q_2,q_3)$ given in Equation (4.17), but with the additional restriction that $s_3\ge 1$. The goal is to show that $\mathcal{S}_2(q_1,q_2,-1)=O(1)$. This is more straightforward then the previous case. Since we assume that $s_3\ge 1$, it holds that $\frac{1}{s_3}\le 1$. Hence,

$$\begin{aligned}
|\mathcal{S}_2(q_1,q_2,-1)| &\le \left|\underbrace{\int_0^{\frac{N-3}{12}}\int_0^{s_1}\int_0^{s_2}}_{\{\frac{12s_1s_2}{N-3}\le s_3\}\cap J_2}\left(1-\frac{s_1+s_2+s_3}{N-3}\right)^{N-3}s_1^{q_1}s_2^{q_2}\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\right|\\
&\le \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(s_1+s_2+s_3)}s_1^{q_1}s_2^{q_2}\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 = \int_0^{\infty} e^{-s_1}s_1^{q_1}\mathrm{d}s_1\int_0^{\infty} e^{-s_2}s_2^{q_2}\mathrm{d}s_2\int_0^{\infty} e^{-s_3}\mathrm{d}s_3\\
&= \Gamma(q_1+1)\cdot\Gamma(q_2+1)\cdot 1 = O(1).
\end{aligned}$$

We established an expression for $\mathcal{S}_1(q_1, q_2, -1)$ in Equation (4.21) and we found that $\mathcal{S}_2(q_1, q_2, -1)$ is of order $O(1)$. Now we go back to Equation (4.18) and put these results together. Finally,

$$\mathcal{S}(q_1, q_2, -1) = \ln(N-3) \int_0^\infty \int_0^\infty \mathbb{1}(s_2 \leq s_1) e^{-s_1-s_2} s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 + O(1).$$

Recall that our goal was to prove Equation (4.16). To that end:

$$\begin{aligned}
\mathcal{S}(q_1, q_2, -1) + \mathcal{S}(q_2, q_1, -1) &= \ln(N-3) \int_0^\infty \int_0^\infty \mathbb{1}(s_2 \leq s_1) e^{-s_1-s_2} s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 \\
&\quad + \ln(N-3) \int_0^\infty \int_0^\infty \mathbb{1}(s_2 \leq s_1) e^{-s_1-s_2} s_1^{q_2} s_2^{q_1} \mathrm{d}s_2 \mathrm{d}s_1 + O(1) \\
&= \ln(N-3) \int_0^\infty \int_0^\infty e^{-s_1-s_2} s_1^{q_1} s_2^{q_2} \mathrm{d}s_2 \mathrm{d}s_1 + O(1) \\
&= \ln(N-3)\Gamma(q_1+1)\Gamma(q_2+1) + O(1).
\end{aligned}$$

There are two more details here. First, recall that one of q_1 and q_2 is equal to 0 and the other one is equal to 1. In either case, $\Gamma(q_1+1)\Gamma(q_2+1) = \Gamma(2)\Gamma(1) = 1$. Secondly,

$$\ln(N-3) = \ln(N \times \frac{N-3}{N}) = \ln(N) + \ln(\frac{N-3}{N}) = \ln(N) + \ln(1 - \frac{3}{N}) = \ln(N) + O(N^{-1}).$$

These two details give the desired result:

$$\mathcal{S}(q_1, q_2, -1) + \mathcal{S}(q_2, q_1, -1) = \ln(N) + O(1)$$

as Equation (4.16) states. □

Proof of item 2 in Lemma 4.8. Assume that $q_3 > -1$. In that case, the possible options of $\boldsymbol{q}$ are $\boldsymbol{q} \in \{(-1,0,1), (-1,1,0), (0,-1,1), (1,-1,0), (0,0,0)\}$. Note that $q_1 + q_2 + q_3 = 0$. We will use that $e^{-x} \leq 1$ for $x \geq 0$ in the upcoming calculations. The goal is to prove that $\mathcal{S}(\boldsymbol{q}) = O(1)$.

$$\begin{aligned}
|\mathcal{S}(\boldsymbol{q})| &= \left| \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_1+s_2+s_3}{N-3}\right)^{N-3} s_1^{q_1} s_2^{q_2} s_3^{q_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \right| \\
&\leq \int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2} e^{-s_1-s_2-s_3} s_1^{q_1} s_2^{q_2} s_3^{q_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\
&= \int_0^{\frac{N-3}{12}} e^{-s_1} s_1^{q_1} \int_0^{s_1} e^{-s_2} s_2^{q_2} \int_0^{s_2} e^{-s_3} s_3^{q_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\
&\leq \int_0^{\frac{N-3}{12}} e^{-s_1} s_1^{q_1} \int_0^{s_1} s_2^{q_2} \int_0^{s_2} s_3^{q_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.
\end{aligned}$$

Since $q_3 > -1$, the function $s_3^{q_3}$ is a polynomial. This gives

$$|\mathcal{S}(\boldsymbol{q})| = \int_0^{\frac{N-3}{12}} e^{-s_1} s_1^{q_1} \int_0^{s_1} \frac{1}{q_3+1} s_2^{q_2+q_3+1} \mathrm{d}s_2 \mathrm{d}s_1.$$

Like before, $q_2 + q_3 + 1 > -1$, so $s_2^{q_2+q_3+1}$ is a polynomial.

$$\begin{aligned}
|\mathcal{S}(\boldsymbol{q})| &= \int_0^{\frac{N-3}{12}} \frac{1}{(q_3+1)(q_2+q_3+2)} e^{-s_1} s_1^{q_1+q_2+q_3+2} \mathrm{d}s_1 \\
&= \frac{1}{(q_3+1)(q_2+q_3+2)} \int_0^{\frac{N-3}{12}} e^{-s_1} s_1^2 \mathrm{d}s_1 \\
&\leq \frac{1}{(q_3+1)(q_2+q_3+2)} \int_0^{\infty} e^{-s_1} s_1^2 \mathrm{d}s_1 \\
&= \frac{2}{(q_3+1)(q_2+q_3+2)} = O(1).
\end{aligned}$$

□

This concludes the proof of the two items in Lemma 4.8. It shows that the only asymptotically contributing terms are $\boldsymbol{q} \in \{(0,1,-1),(1,0,-1)\}$. Translating this to $\boldsymbol{m}$ tells us that choosing each point from a different facet results in a negligible number of facets compared to choosing two points from one facet and one from another. Recall that we are only considering the three facets that are adjacent to the origin of the cube C. If each point is chosen from a different facet of ∂C, then the corresponding facet of C_N is located in a corner of the cube C. In each corner there can only be one such facet. There are only 8 corners in the cube C, so it is not very surprising that we find that those facets give a very small contribution to the total number of facets. The facets of C_N that have two points in one facet of ∂C and the third point in another facet of ∂C are the only ones that contribute to the total number of facets. With these results, we can make our argument complete. The purpose of Lemmas 4.5, 4.6, 4.7 and 4.8 was to find the asymptotics of $J(\boldsymbol{l})$ in Equation (4.13). Using Lemma 4.7 we can write $J(\boldsymbol{l})$ as

$$\begin{aligned}
J(\boldsymbol{l}) &= \frac{c}{(N-3)^3} \sum_{\pi \in \mathcal{S}_3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_1+s_2+s_3}{N-3}\right)^{N-3} s_1^{l_{\pi(1)}} s_2^{l_{\pi(2)}} s_3^{l_{\pi(3)}} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\
&= \frac{c}{(N-3)^3} \sum_{\pi \in \mathcal{S}_3} \mathcal{S}(\boldsymbol{l}_\pi) \\
&= \frac{c}{(N-3)^3} \left(\mathcal{S}(l_1,l_2,l_3) + \mathcal{S}(l_2,l_1,l_3) + \mathcal{S}(l_1,l_3,l_2) + \mathcal{S}(l_3,l_1,l_2) + \mathcal{S}(l_2,l_3,l_1) + \mathcal{S}(l_3,l_2,l_1)\right).
\end{aligned} \tag{4.22}$$

Recall that we were investigating $J(\boldsymbol{l})$ to find the asymptotics of I_1 in Equation (4.14). Hence, we only have to evaluate Equation (4.22) for $\boldsymbol{l} = (-1,0,1)$ and $\boldsymbol{l} = (0,0,0)$.

- $\boldsymbol{l} = (-1,0,1)$. Filling this in gives

$$J(-1,0,1) = \frac{c}{(N-3)^3}\big(\mathcal{S}(-1,0,1) + \mathcal{S}(0,-1,1) + \mathcal{S}(-1,1,0) + \mathcal{S}(1,-1,0) + \mathcal{S}(0,1,-1) + \mathcal{S}(1,0,-1)\big).$$

  The terms $\mathcal{S}(-1,0,1)$, $\mathcal{S}(0,-1,1)$, $\mathcal{S}(-1,1,0)$, $\mathcal{S}(1,-1,0)$ allow for an application of item 2 in Lemma 4.8. The term $\mathcal{S}(0,1,-1) + \mathcal{S}(1,0,-1)$ allows for an application of item 1 in Lemma 4.8. Adding these terms gives

$$J(\boldsymbol{l}) = \frac{c}{(N-3)^3}(\ln N + O(1)) = \frac{c}{(N-3)^3}\ln N(1+O((\ln(N))^{-1})) = cN^{-3}\ln N(1+O((\ln(N))^{-1}).$$

- $\boldsymbol{l} = (0,0,0)$. Filling this in gives

$$J(0,0,0) = \frac{c}{(N-3)^3}\cdot 6 \cdot \mathcal{S}(0,0,0).$$

  We can apply item 2 of Lemma 4.8 to $\mathcal{S}(0,0,0)$:

$$J(0,0,0) = \frac{c}{(N-3)^3}O(1) = O(N^{-3}).$$

We have found the value of $J(\boldsymbol{l})$ for the different choices of $\boldsymbol{l}$. We needed this value of $J(\boldsymbol{l})$ to compute the value of the integral I_1 that was first stated in Equation (4.2). Later we found that I_1 could be written as in Equation (4.14), which is where the $J(\boldsymbol{l})$ came in. Now the final estimates of I_1 can be given. In the first line, Equation (4.14) is repeated.

$$\begin{aligned} I_1 &= cJ(-1,0,1) + cJ(0,0,0) \\ &= cN^{-3}\ln N(1+O((\ln(N))^{-1}) + O(N^{-3}) \\ &= cN^{-3}\ln N(1+O((\ln(N))^{-1}) \end{aligned}$$

which concludes the proof of Lemma 4.4.

4.4 Proof of Lemma 4.5

Recall the statement of Lemma 4.5. Let $f : (0,\infty)^3 \to (0,\infty)^3$ be defined by

$$f_j(x) = \prod_{i\neq j} x_i \qquad j = 1,2,3.$$

1. The inverse function to f is $g : (0,\infty)^3 \to (0,\infty)^3$ given by

$$g_i(y) = \frac{\sqrt{y_1y_2y_3}}{y_i}$$

2. f maps the open set $(0,1)^3$ bijectively onto $S = \{y \in (0,1)^3 : y_1y_2y_3 < y_i^2 \ \forall i\}$.
3. $A := \{x \in (0,\beta)^3 : x_1x_2x_3 < \beta x_i^2\} = \bigcup_{\pi\in\mathcal{S}_3}\{(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) : x \in M\} =: B$ for $\beta > 0$ and where the set M is defined as $M = \{x \in (0,\infty)^3 : x_3 \le x_2 \le x_1, \beta x_3 > x_1x_2\}$.

Proof. 1. For $j = 1,2,3$, $f_j(g(y)) = \prod_{i\neq j} g_i(y) = \prod_{i\neq j}\frac{\sqrt{y_1y_2y_3}}{y_i} = \frac{\sqrt{y_1y_2y_3}}{y_i}\frac{\sqrt{y_1y_2y_3}}{y_k} = \frac{y_1y_2y_3}{y_iy_k} = y_j$, where i and k are distinct and not equal to j. For $i = 1,2,3$, $g_i(f(x)) = g_i(x_2x_3, x_1x_3, x_1x_2) = \frac{\sqrt{x_2x_3x_1x_3x_1x_2}}{x_jx_k} = \frac{x_1x_2x_3}{x_jx_k} = x_i$ where j and k are distinct and not equal to i.

2. In order to prove the bijection, we will show that $f(x) \in S$ for $x \in (0,1)^3$ and $f^{-1}(y) = g(y) \in (0,1)^3$ for $y \in S$. In order to show that $f(x) \in S$ for all $x \in (0,1)^3$, we need to show that $f(x) = y \in (0,1)^3$ and that $y_1y_2y_3 < y_i^2$ for all $i = 1,2,3$. Indeed, $f_i(x) = x_jx_k \in (0,1)^3$ for all i, where j and k are distinct and not equal to i. Moreover, $y_1y_2y_3 = f_1(x)f_2(x)f_3(x) = x_2x_3x_1x_3x_1x_3 = x_1^2x_2^2x_3^2 < x_j^2x_k^2 = y_i^2$ for all i. Therefore, $f(x) \in S$ for $x \in (0,1)^3$. For $y \in S$ we want to show that $g(y) \in (0,1)^3$. As a property of the set S, $y_1y_2y_3 < y_i^2$, so $g_i(y) = \frac{y_1y_2y_3}{y_i} < \frac{y_i^2}{y_i} = y_i < 1$ for all i. Thus $g_i(y) \in (0,1)$ for all i, hence $g(y) \in (0,1)^3$.
3. We will show that $A \subset B$ and $A \supset B$ in order to show that $A = B$. First, let $x \in A$ and show that $x \in B$. There is a permutation $\pi \in \mathcal{S}_3$ such that $x_{\pi(3)} \leq x_{\pi(2)} \leq x_{\pi(1)}$ and it holds that $x_{\pi(3)}x_{\pi(2)}x_{\pi(1)} = x_1x_2x_3 < \beta x_i = \beta x_{\pi(j)}$ for some j such that $\pi(j) = i$. Using the condition of the set A, we know that $x_1x_2x_3 < \beta x_3^2$, hence $x_1x_2 < \beta x_3$. This shows that all conditions of the set M hold, hence $x \in B$. Now, let $x \in B$ and show that $x \in A$. If $(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \in B$, then $x' = (x_1', x_2', x_3') := (x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, x_{\pi^{-1}(3)}) \in M$. It suffices to show that $M \subset A$. For $x' \in M$ it holds that $x_3' \leq x_2' \leq x_1'$ and $\beta x_3' > x_1'x_2'$. In order to show that $x' \in A$, we need to show that $x_i' < \beta$ and $x_1'x_2'x_3' < \beta(x_i')^2$ for all $i = 1,2,3$. Indeed, $x_1'x_2' < \beta x_3'$ implies $x_1'x_2'x_3' < \beta(x_3')^2 < \beta(x_2')^2 < \beta(x_1')^2$ which implies $x_1'x_2'x_3' < \beta(x_i')^2$ for all i. This last inequality also shows that $x_i' < \beta$ for all i. Hence, $x' \in A$.

□

4.5 Proof of Lemma 4.6

Recall the statement of Lemma 4.6. Let $\boldsymbol{l} \in \{(-1,0,1),(0,0,0)\}$. Then we have

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \underbrace{\int_0^{\frac{N-3}{12}}\int_0^{\frac{N-3}{12}}\int_0^{\frac{N-3}{12}}}_{\sqrt{s_1s_2s_3} \leq s_i\sqrt{\frac{12}{N-3}}\ \forall i} \left(1 - \frac{s_1+s_2+s_3}{N-3}\right)^{N-3} s_1^{l_1}s_2^{l_2}s_3^{l_3}\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1.$$

Proof. We defined

$$J(\boldsymbol{l}) = \int_0^1\int_0^1\int_0^1 \left(1 - \frac{1}{12}(t_1t_2+t_1t_3+t_2t_3)\right)^{N-3} t_1^{1-l_1}t_2^{1-l_2}t_3^{1-l_3}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3.$$

We use the transformation of Lemma 4.5:

$$v_1 = t_2t_3,\ v_2 = t_1t_3,\ v_3 = t_1t_2 \qquad \text{and} \qquad t_i = \frac{\sqrt{v_1v_2v_3}}{v_i} \text{ for } i = 1,2,3.$$

Since $t_1, t_2, t_3 \in [0,1]$, it also holds that $v_1, v_2, v_3 \in [0,1]$. However, the reverse is not always true; if $v_1, v_2, v_3 \in [0,1]$, it does not always hold that $t_1, t_2, t_3 \in [0,1]$. Therefore, we have to condition the range of integration on $t_i = \frac{\sqrt{v_1v_2v_3}}{v_i} \leq 1 \implies \sqrt{v_1v_2v_3} \leq v_i$ for all i.

If $i \neq j$, then $\frac{\mathrm{d}t_i}{\mathrm{d}v_j} = \frac{\mathrm{d}}{\mathrm{d}v_j}\frac{\sqrt{v_1v_2v_3}}{v_i} = \frac{\sqrt{v_1v_2v_3}}{2v_jv_i}$ and if $i = j$, then $\frac{\mathrm{d}t_i}{\mathrm{d}v_j} = -\frac{\sqrt{v_1v_2v_3}}{2v_i^2}$. Using these derivatives, we can construct the Jacobian and compute its determinant:

$$J = \det\begin{bmatrix} -\frac{\sqrt{v_1v_2v_3}}{2v_1^2} & \frac{\sqrt{v_1v_2v_3}}{2v_1v_2} & \frac{\sqrt{v_1v_2v_3}}{2v_1v_3} \\ \frac{\sqrt{v_1v_2v_3}}{2v_1v_2} & -\frac{\sqrt{v_1v_2v_3}}{2v_2^2} & \frac{\sqrt{v_1v_2v_3}}{2v_2v_3} \\ \frac{\sqrt{v_1v_2v_3}}{2v_1v_3} & \frac{\sqrt{v_1v_2v_3}}{2v_2v_3} & -\frac{\sqrt{v_1v_2v_3}}{2v_3^2} \end{bmatrix} = \frac{(v_1v_2v_3)^{\frac{3}{2}}}{8}\det\begin{bmatrix} -\frac{1}{v_1^2} & \frac{1}{v_1v_2} & \frac{1}{v_1v_3} \\ \frac{1}{v_1v_2} & -\frac{1}{v_2^2} & \frac{1}{v_2v_3} \\ \frac{1}{v_1v_3} & \frac{1}{v_2v_3} & -\frac{1}{v_3^2} \end{bmatrix}$$

$$= \frac{(v_1v_2v_3)^{\frac{3}{2}}}{8}\frac{4}{v_1^2v_2^2v_3^2} = \frac{1}{2\sqrt{v_1v_2v_3}}.$$

Applying this transformation to $J(\boldsymbol{l})$ gives

$$J(\boldsymbol{l}) = \frac{1}{2} \underbrace{\int_0^1 \int_0^1 \int_0^1}_{\sqrt{v_1 v_2 v_3} \leq v_i \ \forall i} \left(1 - \frac{1}{12}(v_1 + v_2 + v_3)\right)^{N-3} v_1^{l_1} v_2^{l_2} v_3^{l_3} \mathrm{d}v_3 \mathrm{d}v_2 \mathrm{d}v_1$$

The last substitution will be $v_i = \frac{12 s_i}{N-3}$ with $\frac{\mathrm{d}v_i}{\mathrm{d}s_i} = \frac{12}{N-3}$, which gives

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}}}_{\sqrt{s_1 s_2 s_3} \leq s_i \sqrt{\frac{12}{N-3}} \ \forall i} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} s_1^{l_1} s_2^{l_2} s_3^{l_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.$$

□

4.6 Proof of Lemma 4.7

Recall the statement of Lemma 4.7. Let $\boldsymbol{l} \in \{(-1,0,1),(0,0,0)\}$. Then we have

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \sum_{\pi \in \mathcal{S}_3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} s_1^{l_{\pi(1)}} s_2^{l_{\pi(2)}} s_3^{l_{\pi(3)}} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.$$

Proof. In Lemma 4.6, we found that we can write

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}}}_{\sqrt{s_1 s_2 s_3} \leq s_i \sqrt{\frac{12}{N-3}} \ \forall i} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} s_1^{l_1} s_2^{l_2} s_3^{l_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.$$

The variables s_1, s_2, s_3 are taken from $[0, \frac{N-3}{12}]$ conditioned on $\sqrt{s_1 s_2 s_3} \leq s_i \sqrt{\frac{12}{N-3}} \ \forall i$. This can be described by the set $R = \{s \in (0, \frac{N-3}{12})^3 : \sqrt{s_1 s_2 s_3} \leq s_i \sqrt{\frac{12}{N-3}} \ \forall i\}$ defining $s = (s_1, s_2, s_3)$. Note that this set is the same as the set A in Lemma 4.5. This lemma shows that $A = B$, so we obtain $R = \bigcup_{\pi \in \mathcal{S}_3} \{(s_{\pi(1)}, s_{\pi(2)}, s_{\pi(3)}) : s \in M\}$, with $M = \{s \in (0,\infty)^3 : s_3 \leq s_2 \leq s_1, \frac{12}{N-3} s_3 > s_1 s_2\}$. Then the expression for $J(\boldsymbol{l})$ becomes

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \sum_{\pi \in \mathcal{S}_3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_{\pi(1)} + s_{\pi(2)} + s_{\pi(3)}}{N-3}\right)^{N-3} s_{\pi(1)}^{l_1} s_{\pi(2)}^{l_2} s_{\pi(3)}^{l_3} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.$$

It is obvious that $s_{\pi(1)} + s_{\pi(2)} + s_{\pi(3)} = s_1 + s_2 + s_3$. Furthermore, $s_{\pi(1)}^{l_1} s_{\pi(2)}^{l_2} s_{\pi(3)}^{l_3} = s_1^{l_{\pi^{-1}(1)}} s_2^{l_{\pi^{-1}(2)}} s_3^{l_{\pi^{-1}(3)}}$.

Since we sum over all $\pi \in \mathcal{S}_3$, we may replace π^{-1} by π in the last expression. Then,

$$J(\boldsymbol{l}) = \frac{864}{(N-3)^3} \sum_{\pi \in \mathcal{S}_3} \underbrace{\int_0^{\frac{N-3}{12}} \int_0^{s_1} \int_0^{s_2}}_{\frac{12 s_1 s_2}{N-3} \leq s_3} \left(1 - \frac{s_1 + s_2 + s_3}{N-3}\right)^{N-3} s_1^{l_{\pi(1)}} s_2^{l_{\pi(2)}} s_3^{l_{\pi(3)}} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1$$

as the lemma states. $\square$

4.7 Asymptotics of I_2

In this section, we will prove that the asymptotics of I_2 as defined in Equation (4.3) are $O(N^{-3})$. This is stated in the lemma below. We recall the formula for I_2 here.

$$I_2 := \int_{\mathbb{S}^2_+} \int_{\min u_i}^{\infty} \left(1 - \frac{1}{6}\lambda_2(\partial C \cap H_-)\right)^{N-3}$$
$$\iiint_{(\partial C \cap H)^3_{\neq}} \lambda_2([x_1, x_2, x_3]) \big(J(T_{x_1}, H) J(T_{x_2}, H) J(T_{x_3}, H)\big)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}h \mathrm{d}u.$$

Lemma 4.9. *Consider I_2 as in Equation (4.3). The asymptotics are given by*

$$I_2 = O(N^{-3}).$$

Proof. The diameter of the cube C is $\sqrt{3}$, so as soon as $h \geq \sqrt{3}$, we have $(\partial C \cap H) = \emptyset$ for all $u \in \mathbb{S}^2_+$. Therefore, we can restrict the integration with respect to h to the interval $[\min u_i, \sqrt{3}]$.

$$I_2 = \int_{\mathbb{S}^2_+} \int_{\min u_i}^{\sqrt{3}} \left(1 - \frac{1}{6}\lambda_2(\partial C \cap H_-)\right)^{N-3}$$
$$\iiint_{(\partial C \cap H)^3_{\neq}} \lambda_2([x_1, x_2, x_3]) \big(J(T_{x_1}, H) J(T_{x_2}, H) J(T_{x_3}, H)\big)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}h \mathrm{d}u.$$

Furthermore, we can upper bound $\lambda_2([x_1, x_2, x_3])$. The points x_1, x_2, x_3 are chosen from $(\partial C \cap H)^3_{\neq}$, so the convex hull $[x_1, x_2, x_3]$ is always contained in $C \cap H$. Hence,

$$I_2 \leq \int_{\mathbb{S}^2_+} \int_{\min u_i}^{\sqrt{3}} \left(1 - \frac{1}{6}\lambda_2(\partial C \cap H_-)\right)^{N-3}$$
$$\lambda_2(C \cap H) \iiint_{(\partial C \cap H)^3_{\neq}} \big(J(T_{x_1}, H) J(T_{x_2}, H) J(T_{x_3}, H)\big)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}h \mathrm{d}u. \tag{4.23}$$

Recall that the hyperplane $H = H(h, u)$ meets the coordinate axes in the points $\frac{h}{u_i} e_i$. We assume that $h \geq \min u_i$, so at least one of $\frac{h}{u_1}, \frac{h}{u_2}, \frac{h}{u_3}$ is larger than or equal to 1. Hence, the halfspace H_- contains at least one unit vector. This gives 3 possible situations:

1. When H_- contains one unit vector, it holds that

$$\begin{aligned} \text{either } & \frac{h}{u_1}, \frac{h}{u_2} \le 1, \frac{h}{u_3} \ge 1 \\ \text{or } & \frac{h}{u_1}, \frac{h}{u_3} \le 1, \frac{h}{u_2} \ge 1 \\ \text{or } & \frac{h}{u_2}, \frac{h}{u_3} \le 1, \frac{h}{u_1} \ge 1, \end{aligned}$$

   so there are $\binom{3}{2} = 3$ options.

2. When H_- contains two unit vectors, it holds that

$$\begin{aligned} \text{either } & \frac{h}{u_1} \le 1, \frac{h}{u_2}, \frac{h}{u_3} \ge 1 \\ \text{or } & \frac{h}{u_2} \le 1, \frac{h}{u_1}, \frac{h}{u_3} \ge 1 \\ \text{or } & \frac{h}{u_3} \le 1, \frac{h}{u_1}, \frac{h}{u_2} \ge 1, \end{aligned}$$

   so there are $\binom{3}{1} = 3$ options.

3. When H_- contains three unit vectors, it holds that

$$\frac{h}{u_1}, \frac{h}{u_2}, \frac{h}{u_3} \ge 1,$$

   so there is only $\binom{3}{0} = 1$ option.

In general, we can multiply by $\binom{3}{k}$ and assume that H_- contains $e_{k+1}, ..., e_3$, thus the points of intersection satisfy

$$\frac{h}{u_1}, ..., \frac{h}{u_k} \le 1 \text{ and } \frac{h}{u_{k+1}}, ..., \frac{h}{u_n} \ge 1,$$

with some $0 \le k \le 2$. We can split the integral in Equation (4.23) into three parts using $k = 0, 1, 2$.

$$\begin{aligned} I_2 \le \sum_{k=0}^{2} \binom{3}{k} \int\limits_{\mathbb{S}^2_+} \int\limits_{\substack{h \le u_1, ..., u_k \\ h \ge u_{k+1}, ..., u_3}} & \Big(1 - \frac{1}{6}\lambda_2(\partial C \cap H_-)\Big)^{N-3} \lambda_2(C \cap H) \\ & \iiint\limits_{(\partial C \cap H)^3_{\neq}} \big(J(T_{x_1}, H) J(T_{x_2}, H) J(T_{x_3}, H)\big)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}h \mathrm{d}u. \end{aligned}$$

The points $\frac{h}{u_1} e_1, ..., \frac{h}{u_k} e_k, e_{k+1}, ..., e_3$ are all in $\partial C \cap H_-$, hence

$$\begin{aligned} \lambda_2(\partial C \cap H_-) &\ge \lambda_2\Big(\big[\frac{h}{u_1} e_1, ..., \frac{h}{u_k} e_k, e_{k+1}, ..., e_3\big]\Big) \\ &= \frac{1}{2} \min\big(1, \frac{h}{u_1}\big) \min\big(1, \frac{h}{u_2}\big) + \frac{1}{2} \min\big(1, \frac{h}{u_1}\big) \min\big(1, \frac{h}{u_3}\big) + \frac{1}{2} \min\big(1, \frac{h}{u_2}\big) \min\big(1, \frac{h}{u_3}\big) \end{aligned}$$

For each value of k, we can make the lower bound of $\lambda_2(\partial C \cap H_-)$ more precise.

1. For $k = 0$, it holds that $\frac{h}{u_1}, \frac{h}{u_2}, \frac{h}{u_3} \ge 1$, so $\lambda_2(\partial C \cap H_-) \ge \frac{1}{2} \cdot 3 \ge \frac{1}{2}$.

2. For $k=1$, it holds that $\frac{h}{u_1} \leq 1, \frac{h}{u_2}, \frac{h}{u_3} \geq 1$, so $\lambda_2(\partial C \cap H_-) \geq \frac{1}{2}\frac{h}{u_1} + \frac{1}{2}\frac{h}{u_1} + \frac{1}{2} \geq \frac{1}{2}$.

3. For $k=2$, it holds that $\frac{h}{u_1}, \frac{h}{u_2} \leq 1, \frac{h}{u_3} \geq 1$, so $\lambda_2(\partial C \cap H_-) \geq \frac{1}{2}\frac{h}{u_1}\frac{h}{u_2} + \frac{1}{2}\frac{h}{u_1} + \frac{1}{2}\frac{h}{u_2} \geq \frac{1}{2}\frac{h}{u_1} + \frac{1}{2}\frac{h}{u_2}$.

When $k=0,1$, we find that $\lambda_2(\partial C \cap H_-) \geq \frac{1}{2}$, so $(1-\frac{1}{6}\lambda_2(\partial C \cap H_-))^{N-3} \leq (1-\frac{1}{12})^{N-3} \leq e^{-\frac{N-3}{12}}$. This exponential can be pulled out of the integral for $k=0,1$, so the remaining integral does not depend on N, which means it can be seen as a constant. This gives the asymptotics for $k=0,1$. Only the integral for $k=2$ remains to be bounded.

$$I_2 \leq \binom{3}{2} \int_{\mathbb{S}^2_+} \int_{\substack{h \leq u_1, u_2 \\ h \geq u_3}} \Big(1 - \frac{1}{12}\big(\frac{h}{u_1} + \frac{h}{u_2}\big)\Big)^{N-3} \lambda_2(C \cap H)$$
$$\iiint_{(\partial C \cap H)^3_{\neq}} \big(J(T_{x_1}, H) J(T_{x_2}, H) J(T_{x_3}, H)\big)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}h \mathrm{d}u + O(e^{-\frac{N-3}{12}}).$$

The intersection $C \cap H$ forms a quadrilateral that is located in the cube as represented in Figure 4.6. It is a triangle with the top cut off. When this top is not cut off, this triangle is formed by the points $\frac{h}{u_1}e_1, \frac{h}{u_2}e_2, \frac{h}{u_3}e_3$. In Appendix B, the area of such a triangle is calculated and it is equal to $\frac{h}{2u_1u_2u_3}$. The area of $C \cap H$ is bounded by the area of this triangle. Therefore, $\lambda_2(C \cap H) \leq \frac{h^2}{2u_1u_2u_3}$.

$$I_2 \leq 3 \int_{\mathbb{S}^2_+} \int_{\substack{h \leq u_1, u_2 \\ h \geq u_3}} \Big(1 - \frac{1}{12}\big(\frac{h}{u_1} + \frac{h}{u_2}\big)\Big)^{N-3} \frac{h^2}{2u_1u_2u_3}$$
$$\iiint_{(\partial C \cap H)^3_{\neq}} \big(J(T_{x_1}, H) J(T_{x_2}, H) J(T_{x_3}, H)\big)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}h \mathrm{d}u + O(e^{-\frac{N-3}{12}}). \tag{4.24}$$

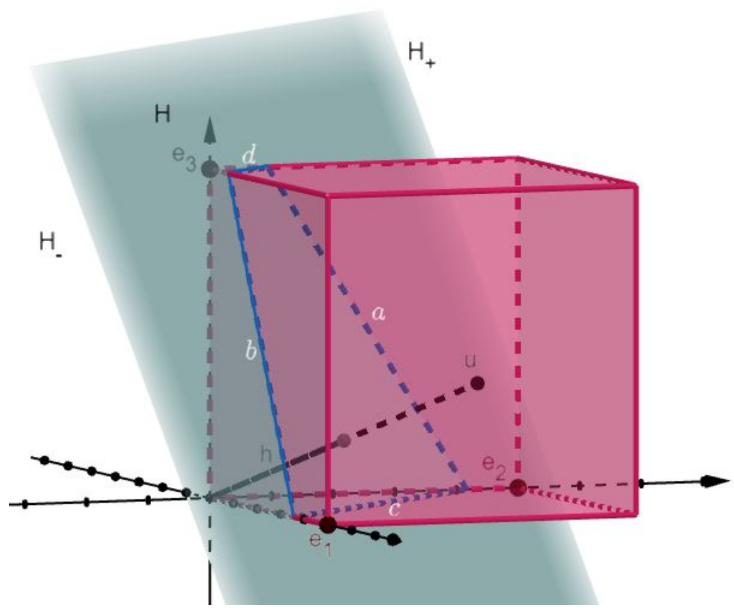


Figure 4.6: $u = (0.7, 0.6, 0.39)$, $h = 0.5$

We will deal with the inner triple integral over x_1, x_2, x_3. The only case that we have to consider now is $k=2$. An example of this situation is given in Figure 4.6, where it indeed holds that

$\frac{h}{u_1}, \frac{h}{u_2} \leq 1, \frac{h}{u_3} \geq 1$. Recall the notation of the facets of the cube that we introduced on page 35. In Figure 4.6, $F_{\hat{1}}$ denotes the "back" of the cube, $F_{\hat{2}}$ denotes the "left" side of the cube and $F_{\hat{3}}$ denotes the "bottom". The top of the cube is described by shifting the bottom of the cube with the vector e_3, so $F_{\hat{3}} + e_3$ denotes the "top" of the cube.

Since we assume that $\frac{h}{u_1}, \frac{h}{u_2} \leq 1, \frac{h}{u_3} \geq 1$, it is always the case that H intersects ∂C in the facets $F_{\hat{1}}, F_{\hat{2}}, F_{\hat{3}}, F_{\hat{3}} + e_3$ and that it does not intersect the remaining two facets (the front and right side), see Figure 4.6. Therefore, the intersection $\partial C \cap H$ is the boundary of a quadrilateral with one edge on each of the facets $F_{\hat{1}}, F_{\hat{2}}, F_{\hat{3}}, F_{\hat{3}} + e_3$. Let's determine the length of each edge.

- $\partial C \cap H \cap F_{\hat{1}}$ is a line between the points $e_3 + \frac{h-u_3}{u_2} e_2$ and $\frac{h}{u_2} e_2$ which has length $\frac{\sqrt{1-u_1^2}}{u_2}$. Indicated by a in Figure 4.6
- $\partial C \cap H \cap F_{\hat{2}}$ is a line between the points $e_3 + \frac{h-u_3}{u_1} e_1$ and $\frac{h}{u_1} e_1$ which has length $\frac{\sqrt{1-u_2^2}}{u_1}$. Indicated by b in Figure 4.6
- $\partial C \cap H \cap F_{\hat{3}}$ is a line between the points $\frac{h}{u_1} e_1$ and $\frac{h}{u_2} e_2$ which has length $\frac{h\sqrt{1-u_2^2}}{u_1 u_2}$. Indicated by c in Figure 4.6
- $\partial C \cap H \cap (F_{\hat{3}} + e_3)$ is a line between the points $e_3 + \frac{h-u_3}{u_1} e_1$ and $e_3 + \frac{h-u_3}{u_2} e_2$ which has length $\frac{(h-u_3)\sqrt{1-u_2^2}}{u_1 u_2} \leq \frac{h\sqrt{1-u_2^2}}{u_1 u_2}$. Indicated by d in Figure 4.6

Recall Equation (4.5). There we found that $J(T_x, H) = \sqrt{1-u_k^2}$ for $x \in F_{\hat{k}}$. Then for $f = 1, 2$,

$$\int_{\partial C \cap H \cap F_{\hat{f}}} J(T_x, H)^{-1} \mathrm{d}x = \left(1 - u_f^2\right)^{-\frac{1}{2}} \int_{\partial C \cap H \cap F_{\hat{f}}} \mathrm{d}x = \frac{1}{h} \cdot \prod_{\substack{i \leq 2 \\ i \neq f}} \frac{h}{u_i}$$

and for $f = 3$,

$$\int_{\partial C \cap H \cap F_{\hat{3}}} J(T_x, H)^{-1} \mathrm{d}x = \left(1 - u_f^2\right)^{-\frac{1}{2}} \int_{\partial C \cap H \cap F_{\hat{3}}} \mathrm{d}x = \frac{1}{h} \cdot \prod_{i \leq 2} \frac{h}{u_i}$$

$$\int_{\partial C \cap H \cap F_{\hat{3}} + e_3} J(T_x, H)^{-1} \mathrm{d}x \leq \frac{1}{h} \cdot \prod_{i \leq 2} \frac{h}{u_i}$$

Define a vector $\boldsymbol{f} \in \{1,2,3\}^3$, where $f_i = 1, 2$ denote that x_i is in $F_{\hat{f_i}}$ and $f_i = 3$ denotes that x_i is either in $F_{\hat{3}}$ or in $F_{\hat{3}} + e_3$. We require here that $m_f = \sum_{j=1}^{3} \mathbb{1}(f_j = f) \leq 2$ for $f = 1, 2$ and $m_1 + m_2 \leq 3$, since we cannot choose more than 3 points on the cube. This yields

$$\begin{aligned}
&\iiint\limits_{(\partial C \cap H)^3_{\neq}} \left(J(T_{x_1}, H) J(T_{x_2}, H) J(T_{x_3}, H)\right)^{-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\
&= \sum_{\boldsymbol{f} \in \{1,2,3\}^3} \prod_{j : f_j \leq 2} \left(\int_{\partial C \cap H \cap F_{\hat{f_j}}} J(T_{x_j}, H)^{-1} \mathrm{d}x_j \right) \\
&\qquad \times \prod_{j : f_j = 3} \left(\int_{\partial C \cap H \cap (F_{\hat{3}} \cup (F_{\hat{3}} + e_3))} J(T_{x_j}, H)^{-1} \mathrm{d}x_j \right) \\
&\leq \frac{h^3}{u_1^3 u_2^3} \sum_{\boldsymbol{f} \in \{1,2,3\}^3} \left(\frac{u_1}{h}\right)^{m_1} \left(\frac{u_2}{h}\right)^{m_2}.
\end{aligned}$$

This upper bound for the inner triple can be filled in in Equation (4.24).

$$I_2 \leq \frac{3}{2} \int_{\mathbb{S}^2_+} \int_{\substack{h \leq u_1, u_2 \\ h \geq u_3}} \Big(1 - \frac{1}{12}\big(\frac{h}{u_1} + \frac{h}{u_2}\big)\Big)^{N-3} h^{-4} \frac{h^9}{u_1^4 u_2^4 u_3} \sum_{\boldsymbol{f} \in \{1,2,3\}^3} \Big(\frac{u_1}{h}\Big)^{m_1} \Big(\frac{u_2}{h}\Big)^{m_2} \mathrm{d}h \mathrm{d}u + O(e^{-\frac{N-3}{12}}).$$

On page 39 we made the substitution $t_i = \frac{h}{u_i}$ with $h^{-4}\mathrm{d}h\mathrm{d}u = (t_1 t_2 t_3)^{-2} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3$ and we will apply it here as well.

$$\begin{aligned} I_2 &\leq \frac{3}{2} \iiint_{t_1, t_2 \leq 1, t_3 \geq 1} \Big(1 - \frac{1}{12}(t_1 + t_2)\Big)^{N-3} t_1^2 t_2^2 t_3^{-1} \sum_{\boldsymbol{f} \in \{1,2,3\}^3} t_1^{-m_1} t_2^{-m_2} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 + O(e^{-\frac{N-3}{12}}) \\ &= \frac{3}{2} \sum_{\boldsymbol{f} \in \{1,2,3\}^3} \iiint_{t_1, t_2 \leq 1, t_3 \geq 1} \Big(1 - \frac{1}{12}(t_1 + t_2)\Big)^{N-3} t_1^{2-m_1} t_2^{2-m_2} t_3^{-1} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 + O(e^{-\frac{N-3}{12}}). \end{aligned}$$

The integration with respect to t_3 is immediate since $t_3 = \frac{h}{u_3} \leq 1$ implies that $t_3^{-1} \geq 1$.

$$I_2 \leq \frac{3}{2} \sum_{\boldsymbol{f} \in \{1,2,3\}^3} \int_0^1 \int_0^1 \Big(1 - \frac{1}{12}(t_1 + t_2)\Big)^{N-3} t_1^{2-m_1} t_2^{2-m_2} \mathrm{d}t_1 \mathrm{d}t_2 + O(e^{-\frac{N-3}{12}}).$$

Finally, we will evaluate the value of this double integral. We start with substituting $t_i = \frac{12 s_i}{N-3}$

$$\begin{aligned} I &:= \int_0^1 \int_0^1 \Big(1 - \frac{1}{12}(t_1 + t_2)\Big)^{N-3} t_1^{2-m_1} t_2^{2-m_2} \mathrm{d}t_1 \mathrm{d}t_2 \\ &= \big(\frac{12}{N-3}\big)^{6-m_1-m_2} \int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}} \Big(1 - \frac{s_1 + s_2}{N-3}\Big)^{N-3} s_1^{2-m_1} s_2^{2-m_2} \mathrm{d}s_1 \mathrm{d}s_2. \end{aligned}$$

Recall that $m_i \in \{0, 1, 2\}$, so that $2 - m_i \in \{0, 1, 2\}$. Furthermore, we have assumed that $m_1 + m_2 \leq 3$, so $6 - m_1 - m_2 \geq 3$. That means that for large enough N,

$$\begin{aligned} I &\leq \big(\frac{12}{N-3}\big)^3 \int_0^{\frac{N-3}{12}} \int_0^{\frac{N-3}{12}} \Big(1 - \frac{s_1 + s_2}{N-3}\Big)^{N-3} s_1^{2-m_1} s_2^{2-m_2} \mathrm{d}s_1 \mathrm{d}s_2 \\ &\leq \big(\frac{12}{N-3}\big)^3 \int_0^{\infty} \int_0^{\infty} e^{-s_1 - s_2} s_1^{2-m_1} s_2^{2-m_2} \mathrm{d}s_1 \mathrm{d}s_2 \\ &= \big(\frac{12}{N-3}\big)^3 \int_0^{\infty} e^{-s_1} s_1^{2-m_1} \mathrm{d}s_1 \int_0^{\infty} e^{-s_2} s_2^{2-m_2} \mathrm{d}s_2 \\ &= \frac{c \cdot 12^3}{(N-3)^3} = O(N^{-3}). \end{aligned}$$

We conclude that

$$I_2 \leq \frac{3c}{2} \cdot O(N^{-3}) + O(e^{-\frac{N-3}{12}}) = O(N^{-3})$$

as was stated in Lemma 4.9. □

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Appendix A

We prove that $\Gamma(\frac{d}{2}+\frac{1}{2}) = \frac{d!}{2^d\Gamma(\frac{d}{2}+1)}\sqrt{\pi}$ for $d \in \mathbb{N}$.

We first prove another statement. Namely,

$$\Gamma(k+\frac{1}{2}) = \frac{(2k)!}{2^{2k}\Gamma(k+1)}\sqrt{\pi} \text{ for } k \in \mathbb{N}. \tag{A.1}$$

This is done as follows:

$$\begin{aligned}
\Gamma\Big(k+\frac{1}{2}\Big) &= \Big(k-1+\frac{1}{2}\Big)\Gamma\Big(k-1+\frac{1}{2}\Big) \\
&= \Big(k-1+\frac{1}{2}\Big)\Big(k-2+\frac{1}{2}\Big)\Gamma\Big(k-2+\frac{1}{2}\Big) \\
&= ... = \Big(k-1+\frac{1}{2}\Big)\Big(k-2+\frac{1}{2}\Big)\cdots\frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) \\
&= \Big(k-\frac{1}{2}\Big)\Big(k-\frac{3}{2}\Big)\cdots\frac{1}{2}\sqrt{\pi} \\
&= \frac{2^k(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{1}{2}}{2^k}\sqrt{\pi} \\
&= \frac{(2k-1)(2k-3)\cdots 1}{2^k}\sqrt{\pi} \\
&= \frac{(2k-1)(2k-2)(2k-3)(2k-4)\cdots 2\cdot 1}{2^k(2k-2)(2k-4)\cdots 2}\sqrt{\pi} \\
&= \frac{(2k-1)(2k-2)(2k-3)(2k-4)\cdots 2\cdot 1}{2^k 2^{k-1}(k-1)(k-2)\cdots 1}\sqrt{\pi} \\
&= \frac{(2k-1)!}{2^{2k-1}(k-1)!}\sqrt{\pi} \\
&= \frac{2k(2k-1)!}{2k2^{2k-1}(k-1)!}\sqrt{\pi} \\
&= \frac{(2k)!}{2^{2k}k!}\sqrt{\pi} \\
&= \frac{(2k)!}{2^{2k}\Gamma(k+1)}\sqrt{\pi}.
\end{aligned}$$

First, we assume that d is even, so $d = 2k$ for $k \in \mathbb{N}$. Then using Equation (A.1), we find

$$\begin{aligned}\Gamma\Big(\frac{d}{2} + \frac{1}{2}\Big) &= \Gamma\Big(\frac{2k}{2} + \frac{1}{2}\Big) \\ &= \Gamma\Big(k + \frac{1}{2}\Big) \\ &= \frac{(2k)!}{2^{2k}\Gamma(k+1)}\sqrt{\pi} \\ &= \frac{d!}{2^d\Gamma(\frac{d}{2}+1)}\sqrt{\pi}.\end{aligned}$$

This proves the statement for even d. Now we want to prove it for odd d. To that end, let $d = 2k+1$ for $k \in \mathbb{N}$. Then

$$\begin{aligned}\Gamma\Big(\frac{d}{2} + \frac{1}{2}\Big) &= \Gamma\Big(\frac{2k+1}{2} + \frac{1}{2}\Big) \\ &= \Gamma(k+1) \\ &= k!.\end{aligned}$$

Furthermore, again using $d = 2k+1$ for $k \in \mathbb{N}$, we get

$$\begin{aligned}\frac{d!}{2^d\Gamma(\frac{d}{2}+1)}\sqrt{\pi} &= \frac{(2k+1)!}{2^{2k+1}\Gamma(\frac{2k+2}{2}+\frac{1}{2})}\sqrt{\pi} \\ &= \frac{(2k+1)!}{2^{2k+1}\Gamma((k+1)+\frac{1}{2})}\sqrt{\pi}\end{aligned}$$

We can apply Equation (A.1) using $k+1$ to the last line. Then

$$\begin{aligned}\frac{d!}{2^d\Gamma(\frac{d}{2}+1)}\sqrt{\pi} &= \frac{(2k+1)!2^{2k+2}\Gamma(k+2)}{2^{2k+1}(2k+2)!\sqrt{\pi}}\sqrt{\pi} \\ &= \frac{2\cdot\Gamma(k+2)}{(2k+2)} \\ &= \frac{(k+1)!}{k+1} \\ &= k!.\end{aligned}$$

This proves the statement for odd d. Therefore,

$$\Gamma\Big(\frac{d}{2} + \frac{1}{2}\Big) = \frac{d!}{2^d\Gamma(\frac{d}{2}+1)}\sqrt{\pi}$$

for $d \in \mathbb{N}$.

Appendix B

The hyperplane $H(1,u)$ is given by the equation $u_1x + u_2y + u_3z = 1$ for $x, y, z \in \mathbb{R}$, restricted to $u_1^2 + u_2^2 + u_3^2 = 1$. It meets the x-axis when $y = z = 0$, in which case $u_1x = 1$, so $x = \frac{1}{u_1}$. In general, the hyperplane $H(1,u)$ meets the coordinate axis in the points $\frac{1}{u_i}e_i$. The intersection $\mathbb{R}^3_+ \cap H(1,u)$ is a triangle formed by the points $(\frac{1}{u_1}, 0, 0), (0, \frac{1}{u_2}, 0), (0, 0, \frac{1}{u_3})$. Define $P := (\frac{1}{u_1}, 0, 0)$, $Q := (0, \frac{1}{u_2}, 0)$ and $R := (0, 0, \frac{1}{u_3})$. The triangle that we are considering has edges defined by the vectors $PQ := P - Q$, $PR := P - R$ and $QR := Q - R$. The point $S := Q - P + R$ creates a parallelogram $PQRS$. The point S is also in the hyperplane $H(1,u)$. See Figure B.1 for a picture of this situation. The area of

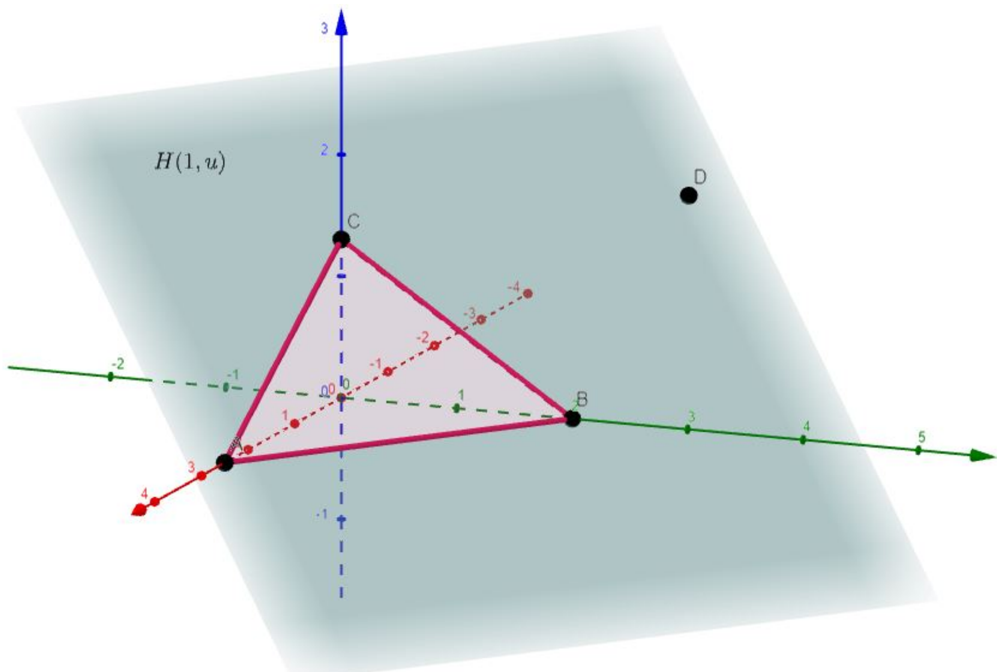

Figure B.1: Hyperplane $H(1,u)$ with vector u given by $u_1 = 0.4$, $u_2 = 0.5$, $u_3 = 0.77$ resulting in the points $P = (2.5, 0, 0)$, $Q = (0, 2, 0)$, $R = (0, 0, 1.3)$, $S = (-2.5, 2, 1.3)$.

the parallelogram $PQRS$ is given by

$$\begin{aligned}
\|PQ \times PR\| &= \| \begin{vmatrix} i & j & k \\ -\frac{1}{u_1} & \frac{1}{u_2} & 0 \\ -\frac{1}{u_1} & 0 & \frac{1}{u_3} \end{vmatrix} \| \\
&= \| \frac{1}{u_2 u_3} i + \frac{1}{u_1 u_3} j + \frac{1}{u_1 u_2} k \| \\
&= \| (\frac{1}{u_2 u_3}, \frac{1}{u_1 u_3}, \frac{1}{u_1 u_2}) \| \\
&= \sqrt{\frac{1}{u_2^2 u_3^2} + \frac{1}{u_1^2 u_3^2} + \frac{1}{u_1^2 u_2^2}} \\
&= \sqrt{\frac{u_1^2 + u_2^2 + u_3^2}{u_1^2 u_2^2 u_3^2}} \\
&= \frac{1}{u_1 u_2 u_3}.
\end{aligned}$$

The area of the parallelogram $PQRS$ is twice as big as the area of the triangle PQR, which is hence equal to $\frac{1}{2u_1u_2u_3}$.

The hyperplane $H(h,u)$ is given by the equation $u_1 x + u_2 y + u_3 z = h$ for $x, y, z \in \mathbb{R}$, restricted to $u_1^2 + u_2^2 + u_3^2 = 1$. It meets the x-axis when $y = z = 0$, in which case $u_1 x = h$, so $x = \frac{h}{u_1}$. In general, the hyperplane $H(h,u)$ meets the coordinate axis in the points $\frac{h}{u_i} e_i$. The intersection $\mathbb{R}^3_+ \cap H(h,u)$ is a triangle formed by the points $(\frac{h}{u_1}, 0, 0), (0, \frac{h}{u_2}, 0), (0, 0, \frac{h}{u_3})$. Define $P := (\frac{h}{u_1}, 0, 0)$, $Q := (0, \frac{h}{u_2}, 0)$ and $R := (0, 0, \frac{h}{u_3})$. The triangle that we are considering has edges defined by the vectors $PQ := P - Q$, $PR := P - R$ and $QR := Q - R$. The point $S := Q - P + R$ creates a parallelogram $PQRS$. The point S is also in the hyperplane $H(h,u)$. The area of the parallelogram $PQRS$ is given by

$$\begin{aligned}
\|PQ \times PR\| &= \| \begin{vmatrix} i & j & k \\ -\frac{h}{u_1} & \frac{h}{u_2} & 0 \\ -\frac{h}{u_1} & 0 & \frac{h}{u_3} \end{vmatrix} \| \\
&= \| \frac{h^2}{u_2 u_3} i + \frac{h^2}{u_1 u_3} j + \frac{h^2}{u_1 u_2} k \| \\
&= \| (\frac{h^2}{u_2 u_3}, \frac{h^2}{u_1 u_3}, \frac{h^2}{u_1 u_2}) \| \\
&= \sqrt{\frac{h^2}{u_2^2 u_3^2} + \frac{h^2}{u_1^2 u_3^2} + \frac{h^2}{u_1^2 u_2^2}} \\
&= h^2 \sqrt{\frac{u_1^2 + u_2^2 + u_3^2}{u_1^2 u_2^2 u_3^2}} \\
&= \frac{h^2}{u_1 u_2 u_3}.
\end{aligned}$$

The area of the parallelogram $PQRS$ is twice as big as the area of the triangle PQR, which is hence equal to $\frac{h^2}{2u_1u_2u_3}$.

The intersection $\mathbb{R}^3_+ \cap H(1, \mathbf{1})$ forms an equilateral triangle between the unit vectors e_1, e_2, e_3. Define $P = e_1$, $Q = e_2$, $R = e_3$. The point $S = (0.5, 0.5, 0)$ creates two rectangular triangles PSR and QSR. See Figure B.2. The distances between the vectors P, Q and R are all equal to $\sqrt{1+1+0} = \sqrt{2}$. Therefore, the distance between P and S is equal to $\frac{\sqrt{2}}{2}$. Using Pythagoras, the length of RS is equal to $\frac{\sqrt{3}}{\sqrt{2}}$. Then the area of the triangle PQR is equal to $\frac{1}{2} \cdot \sqrt{2} \cdot \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{2}$.

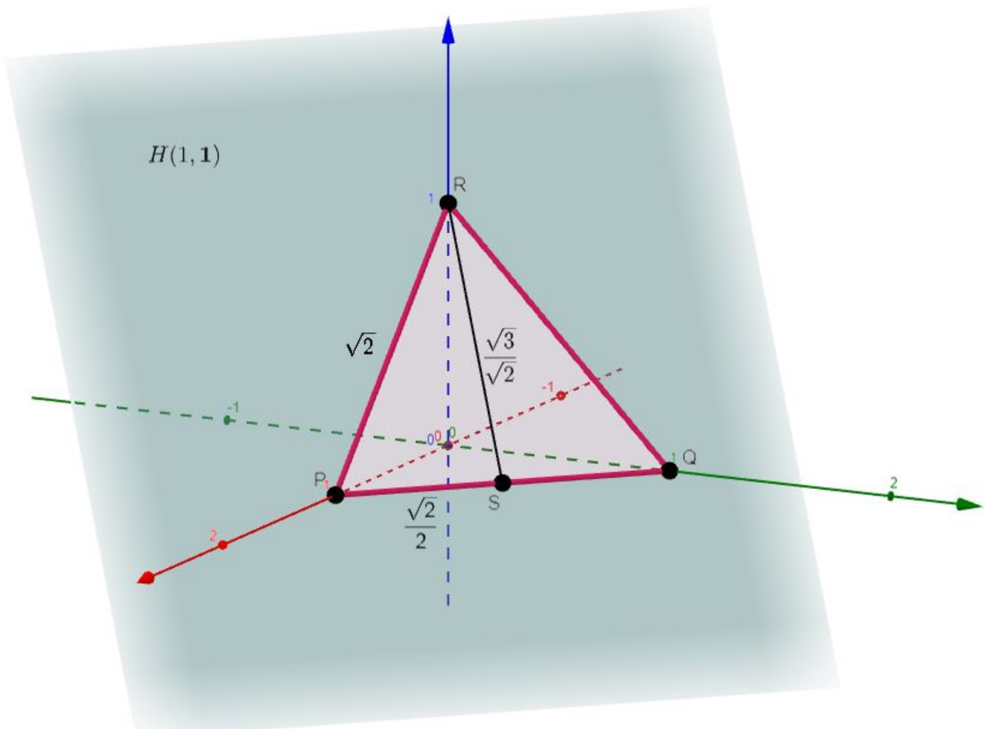

Figure B.2: Caption