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Scalable Stability Properties of Networks of Linear Systems

Abstract: In vehicle platooning, string stability guarantees that the effect of a sudden disturbance remains bounded as it propagates through a platoon of arbitrary length. By modelling such a platoon as a network of scalar linear systems, this thesis aims to find necessary and sufficient conditions for string stability. In particular, various networks adhering to a predecessor following structure are considered. To deal with external disturbances affecting the network, the notion of disturbance string stability is used instead. By recognizing the structure of a Taylor series and geometric series in the system solutions and exploiting them appropriately, it is found that the conditions for string stability and disturbance string stability are identical. To verify the results, numerical simulations are shown.

Keywords: Vehicle platooning, string stability, exponential stability, linear system, network, disturbances, positive system.

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1 Introduction

As automation becomes more advanced, one of the technologies that has become more prominent is that of vehicle platooning. In vehicle platooning (see Figure 1), one deals with a string of N self-driving vehicles, driving in a formation behind one another. At the front of the platoon is a leader vehicle, serving as an autonomous system, which determines the velocity of each vehicle in the platoon. The remaining vehicles in the platoon then follow the leader vehicle and are, consequently, called follower vehicles. Each vehicle in the platoon can be seen as a system belonging to a network, i.e. the platoon. Through wireless communication, these vehicles exchange information between one another, which is then used to update their own velocities. This is done with the goal of maintaining a desired intervehicular distance for all time between all the vehicles. As the reaction time of these vehicles is much better than that of a human driver, one can make this intervehicular distance very small, leading to increased traffic flow. Additional benefits from vehicle platooning include increased traffic safety, lower fuel consumption and lower carbon emissions [7].

Things get interesting when a sudden disturbance occurs to the leader vehicle. For example, due to a traffic jam, one could think of the leader vehicle suddenly having to brake, disrupting its own acceleration and velocity. Since all the vehicles in the platoon have to maintain a given desired intervehicular distance, this will result in all of the remaining vehicles braking as well. In other

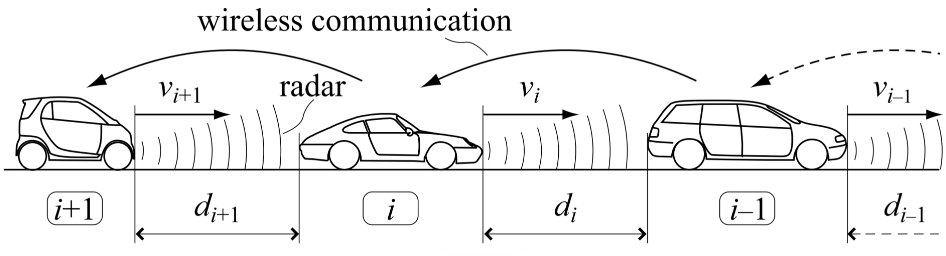


Figure 1: Vehicles in a platoon [13].

words, when a sudden disturbance is introduced to the platoon, its effect will propagate through the string of vehicles. In order to deal with this situation, what we initially want is for the effect of this disturbance to disappear over time with each vehicle. To achieve that, we require *exponential stability* of the platoon. Exponential stability is the property guaranteeing that the effect of the disturbance on each vehicle will disappear over time which, in turn, will lead to the desired behavior of each vehicle being restored. In this case, this would mean that each vehicle retrieves its original velocity before the disturbance was introduced.

However, this solves only half of the problem. Exponential stability does not help us in mitigating the rate at which the disturbance propagates through the network. As such, exponential stability allows for the effect of the disturbance on each vehicle to grow as it propagates through the string of vehicles. This is made clear in Figure 2 (b). Namely, notice how each trajectory tends to the origin as time grows, but the peak deviation from the origin grows larger with each subsequent vehicle in the platoon. In other words, each vehicle receives a larger disturbance effect than the previous vehicle, although the effect of the disturbance vanishes over time. As a consequence, one should be careful to make the platoon larger by adding more vehicles, as this can lead to undesired outcomes like car crashes between the vehicles near the end of the platoon. This is unfavorable, as there is now a limitation on how long the string of vehicles is allowed to be. Naturally, one would like to be able to freely add and remove vehicles without having to worry too much about the effect of a sudden disturbance propagating through the string of vehicles. This is where the stronger notion of *string stability* comes into play. In addition to restoring the desired behavior of

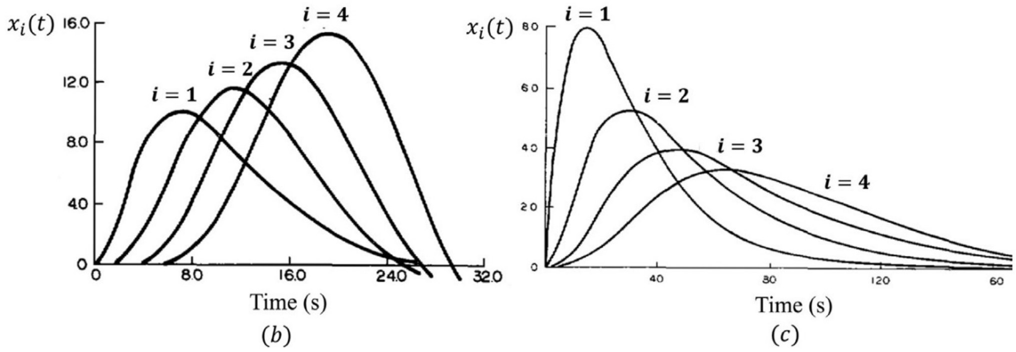


Figure 2: Exponentially stable platoon (left) vs string stable platoon (right) [12].

each vehicle after a disturbance has been introduced (in other words, exponential stability), one would like to bound the effect of the disturbance over all vehicles in the platoon, no matter how long this string of vehicles is. If this is possible, then we will have the freedom to freely add and remove vehicles, as the effect of the disturbance will never exceed a certain bound, independently of the size of the platoon. An example of this can be seen in Figure 2 (c). Namely, notice how these peaks are now getting smaller per vehicle in the platoon, meaning that the effect of the disturbance decreases as it propagates through the network. When it is possible to add and remove vehicles without losing string stability, we say that the network of systems, i.e. the platoon, is *scalable*.

Achieving string stability of a network ultimately comes down to modeling each vehicle appropriately, by choosing the appropriate values in the system dynamics of each vehicle. What then constitutes "appropriate" depends on the *information flow topology* (IFT) of the network, i.e. how all of the systems in a network exchange information between one another. It is important to note that there

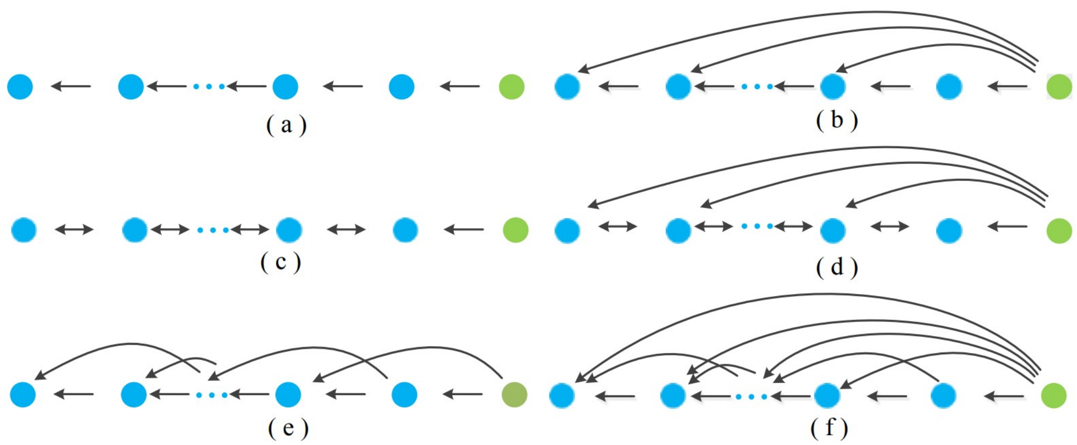


Figure 3: Examples of IFTs for platoons. (a) PF; (b) PLF; (c) BD; (d) BLD; (e) 2PF; (f) 2PLF [10].

are many ways in which information can be exchanged between systems. In order to mathematically represent these IFTs, one can employ the use of graph theory. Namely, a network of systems can be viewed as a graph, where each system in the network is a vertex and the edges connecting these vertices represent the information flows. In particular, one distinguishes between *unidirectional* and *bidirectional* information flows. In the unidirectional case, the network graph is directed, whereas in the bidirectional case, the network graph is undirected. Hybrid information flows, with both uni- and bidirectional information flows, are also possible (see Figure 3 (d)). Moreover, within these two distinctive information flows, one can choose to receive information from just one or several vehicles in the platoon. Figure 3 gives a few examples of how this can be done. The only thing that needs to be fixed, however, is that the leader vehicle receives no input whatsoever, as it is an autonomous system in the network. Since each IFT gives rise to a different network structure, the conditions for string stability will change as well. As such, each IFT poses its own problem to be solved.

Much research on string stability and vehicle platooning has been done throughout the years, with the original vehicle platoon control problem dating all the way back to 1966 in Levine's and Athans' paper "On the optimal error regulation of a string of moving vehicles" [9]. A first definition of string stability, however, was not established until 1974 by Chu [3], after which many different definitions have sprung forth. These various definitions depend on the IFT and the type of disturbance that affects the platoon, of which [6] states the four most common ones, and the choice of definition greatly impacts the analysis method used for string stability problems. For example, in unidirectional IFTs, it is extremely common to use a frequency-domain approach and define string stability in terms of the transfer functions of the vehicles. This gives rise to the frequency-domain based definition of *strong string stability* [11] and its various modified versions, like that of *eventual string stability* [8]. Although convenient to work with, using a frequency domain has several limitations. For example, it assumes linearity of the platoon network and only works for a select type of disturbances. In order to resolve the issue of linearity, the time-domain based notions of $\mathcal{L}_p$ string stability and $\mathcal{L}_\infty$ string stability [14] can then be used instead. In an attempt to generalize the definition of string stability to a platoon with as few limitations as possible, [6] recommends the definition of *input-to-state string stability*, to be used for all purposes, as the formal definition of string stability. It is important to note, however, that as of yet no single common definition of string stability has been established. Instead, the definition will vary depending on the problem that is to be solved.

Each of the platoon formations in Figure 3 has been extensively studied in the literature. Since the various definitions of string stability depend on the domain used, one can group the analysis methods into a time-domain analysis method and a frequency-domain analysis method. In particular, the frequency-domain analysis methods rely on the use of transfer functions, which are used for linear platoon formations, whereas time-domain analysis methods are used for nonlinear platoons instead. A frequently used time-domain method is that of Lyapunov techniques, where a suitable Lyapunov function is constructed and its properties used to prove string stability of a platoon (e.g. [1]). For results on unidirectional IFTs, we refer to [11], [4], [2] and [12]. For bidirectional IFTs we refer to [15]. Finally, as a general starting point for string stability, we recommend [6].

In this thesis, necessary and sufficient conditions for string stability of networks consisting of scalar linear systems will be found. In analogy with the motivation of vehicle platooning, we will be pretending that each system in the network is a vehicle in a platoon. As such, several of the problems as posed in Figure 3 will be solved. In particular, we will restrict ourselves to the unidirectional case only, meaning that the bidirectional (leader) following (BF/BLF) formations in Figure 3 (c) and Figure 3 (d) will not be treated in this thesis. It is important to note that this means that the goal in this thesis is not to realistically model the network of systems as a platoon of vehicles. Rather, the results in this thesis will hold for generalized networks of scalar linear systems, following IFTs primarily used in vehicle platooning. To emphasize this, throughout this thesis, we will always talk about a network of systems instead of a platoon of vehicles. However, to facilitate the interpretation of the obtained results, several connections with vehicle platooning will be made.

The contents of this thesis are structured as follows. In Section 2, a time-domain based definition of string stability will be provided, which is the definition that we will work with throughout this thesis. After this, we will cover networks of the predecessor following (PF) topology and the predecessor-leader following (PLF) topology (see, respectively, Figure 3 (a) and (b)) and conditions for string stability of these two networks will be found. In Section 3, an external disturbance will be added to the PF and PLF networks, giving rise to the definition of *disturbance string stability*. Necessary and sufficient conditions for disturbance string stability of PF and PLF networks will then be found. In Section 4, an additional predecessor will be considered in the input, leading to the 2PF and 2PLF problem (see Figure 3 (e) and (f), respectively). At the end of Section 4, we will consider arbitrary predecessors, leading to the rPF problem. For this problem, only sufficient conditions will be found. In Section 5, the results that have been obtained will be verified by means of numerical simulations. Finally, Section 6 summarizes everything that has been done in this thesis and provides a brief discussion on potential future research topics.

2 Single Predecessor Following Problems

2.1 Defining String Stability

Throughout this thesis, we will consider networks consisting of scalar linear systems of the form

$$\Sigma_i : \dot{x}_i(t) = -ax_i(t) + B_i u_i(t), \quad i = 1, \dots, N, \tag{1}$$

with state $x_i(t) \in \mathbb{R}$, input $u_i(t) \in \mathbb{R}^{m_i}$, $a \in \mathbb{R}$ and $B_i \in \mathbb{R}^{1\times m_i}$. Since we only consider unidirectional information flows, the input term $B_i u_i(t)$ will always be some linear combination of the states of the previous systems in the network. Namely, since the leader system Σ_1 takes on the role of a leader vehicle, which is an autonomous system and therefore cannot receive any inputs, the only types of unidirectional information flows possible are the ones where each system receives an input from one or more of its predecessors. As a result, the network itself will always be an autonomous system, even if each system in the network is not an autonomous system itself (with the exception of the leader system Σ_1). Explicitly, the network of systems is given by

$$\Sigma : \dot{x}(t) = Ax(t), \tag{2}$$

where $x(t) \in \mathbb{R}^N$ satisfies $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_N(t) \end{bmatrix}^\top$ and $A \in \mathbb{R}^{N\times N}$. In order to define string stability of the network (2), we first require (2) to be *exponentially stable*.

Definition 2.1. *The autonomous system* (2) *is said to be exponentially stable if there exists real numbers $K > 0$ and $\mu > 0$ such that*

$$\max_{i=1,\dots,N} |x_i(t)| \leq Ke^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|,$$

for all initial conditions $x_i(0) = x_{i,0}$, where $x_i(t)$ are the components of the resulting state trajectories.

In other words, each state trajectory $x_i(t)$ converges to the origin in an exponential manner. In particular, for fixed N, we can employ the same exponential bound over all systems in the network.

We recall the well-known result that an autonomous system is exponentially stable if and only if the eigenvalues of A have negative real part. For the one-dimensional case $N = 1$, we obtain $A = -a$. We therefore we require that $a > 0$ such that $-a < 0$, as this ensures the state trajectory $x(t) = e^{-at}x(0)$ converges to the origin. For $N \geq 2$, the matrix A will always be a lower triangular matrix, whose eigenvalues are given on its diagonal entries. This is because, again, in unidirectional information flows with an autonomous leader system, each system can only receive inputs from one or more of its predecessors. For example, in the simplest case, one can set $B_i u_i(t) = bx_{i-1}(t)$ for all $i = 2, \dots, N$, where $b \in \mathbb{R}$. For $i = 1$, there is no input present i.e. $B_1 u_1(t) = 0$. The network will then look like

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -a & & & & \\ b & -a & & & \\ & b & -a & & \\ & & \ddots & \ddots & \\ & & & b & -a \end{bmatrix}}_{A} x(t).$$

Indeed, one can see that the matrix A is lower triangular. This particular network adheres to a predecessor following topology and will be extensively studied in Section 2.2. As $a_{i,i} = -a$ for all $i = 1, \dots, N$, the network (2) will always be exponentially stable whenever $a > 0$. Hence, throughout this thesis, we require $a > 0$ on each system no matter the network size and no matter the unidirectional information flow topology considered.

In the context of vehicle platooning, we can interpret the systems (1) as the system dynamics for each vehicle in the platoon, whereas the network (2) can be seen as the platoon of vehicles. In order to find conditions for string stability of the network (2), we need to extend Definition 2.1. Throughout this thesis, the following definition of string stability will be employed.

Definition 2.2. *Consider the systems as in* (1) *and the network of systems* (2)*. The network* (2) *is said to be string stable if there exists real numbers $K > 0$ and $\mu > 0$ such that*

$$\max_{i=1,\dots,N} |x_i(t)| \leq Ke^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|,$$

for all initial conditions $x_i(0) = x_{i,0}$ and for all $N \in \mathbb{N}$.

Before continuing, it is important to note the "for all $N \in \mathbb{N}$" part. Omitting this part means that the above definition reduces to that of exponential stability, which is not what we are looking for. What Definition 2.2 states is that we wish to apply the same exponential bound to all systems in the network, no matter how large this network is. This implies scalability of the network, since we can then make the network as large as we wish without having to change bounds. It is then also easy to see why exponential stability is a necessary requirement for string stability. Namely, if we have a string stable network, then we can apply the exponential bound to any network of fixed size N of the same IFT, which implies exponential stability.

2.2 Predecessor Following Problem

As mentioned before, throughout this thesis we will pretend that each system in the network is a vehicle in a platoon. We assign the role of leader vehicle to the leader system Σ_1, which means it receives no inputs, making it an autonomous system. Moreover, we assume that the systems $\Sigma_2, \dots, \Sigma_N$ are follower vehicles, which means they do receive inputs from other systems in the network. The first problem we will consider is the predecessor following (PF) problem. In this problem, the network adheres to a PF topology, as in Figure 3 (a). Each system receives an input from its predecessor and sends its state to its successor. What this means is that for the j'th system Σ_j, we set $u_j(t) = x_{j-1}(t)$ for all $j = 2, \dots, N$. Since the leader system Σ_1 receives no inputs, we set $u_1(t) \equiv 0$. The network of systems is then given by

$$\begin{aligned} &\Sigma_1 : \dot{x}_1(t) = -ax_1(t), \\ &\Sigma_j : \dot{x}_j(t) = -ax_j(t) + bx_{j-1}(t), \quad j = 2, \dots, N, \end{aligned} \tag{3}$$

where $a, b \in \mathbb{R}$ and $a > 0$.

In order to find sufficient conditions for string stability, we will employ the following strategy. By finding a closed-form solution of each system in the network, we will be able to bound this solution by an expression which will hold for any network size $N \in \mathbb{N}$. This then immediately implies string stability, since this bound will also hold for the maximum over all the solutions. The strategy of proving necessity will be slightly different. The following lemma gives us the closed-form solution of each system in (3).

Lemma 2.3. *Consider the PF network* (3) *for given initial conditions $x_i(0) = x_{i,0}$. Then, for each $N \in \mathbb{N}$, the solution is given by*

$$x_N(t) = e^{-at} \left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} \right). \tag{4}$$

Proof. We compute the solution $x_N(t)$ explicitly. Since for the input we have $u_N(t) = x_{N-1}(t)$, the solution $x_N(t)$ is given by

$$x_N(t) = e^{-at}x_{N,0} + \int_0^t e^{-a(t-\tau)}bx_{N-1}(\tau)d\tau. \tag{5}$$

The result will be shown through induction. To this end, consider the base case $N = 1$. The solution is given by

$$x_1(t) = e^{-at}x_{1,0} = e^{-at}\left(\sum_{i=0}^{0} \frac{b^j x_{1-i,0}t^i}{i!}\right).$$

Hence, the base case holds. Assume now that the result holds for arbitrary $N = m$. The use of (4) and (5) gives

$$\begin{aligned} x_{m+1}(t) &= e^{-at}\left(x_{m+1,0} + \int_0^t \sum_{i=0}^{m-1} \frac{b^{i+1}x_{m-i,0}\tau^i}{i!}d\tau\right) \\ &= e^{-at}\left(x_{m+1,0} + \sum_{i=0}^{m-1} \frac{b^{i+1}x_{m-i,0}t^{i+1}}{(i+1)!}\right). \end{aligned}$$

Rearranging indices in the summation term and collecting all the terms, we obtain

$$\begin{aligned} x_{m+1}(t) &= e^{-at}\left(x_{m+1,0} + \sum_{i=1}^{m} \frac{b^i x_{m+1-i,0}t^i}{i!}\right) \\ &= e^{-at}\left(\sum_{i=0}^{m} \frac{b^i x_{m+1-i,0}t^i}{i!}\right). \end{aligned}$$

Therefore, the result (4) holds for $N = m + 1$ as well. The statement now follows by induction. □

Setting the initial conditions equal to one, we can recognize the Taylor series of the exponential function e^{bt} centered at $t = 0$ in (4). This observation will be used several times when trying to find conditions for string stability.

Now that we have an explicit closed-form solution of the trajectory for each system in the network, we have enough information to provide a sufficient condition for string stability as per Definition 2.2. In order to prove necessity, the following lemma will be used.

Lemma 2.4. *Let $m, b, K \in \mathbb{R}$ with $K > 0$. Assume that*

$$e^{mt}\sum_{i=0}^{N} \frac{b^i t^i}{i!} \leq K,$$

for all $N \in \mathbb{N}$ and for all $t \geq 0$. Then,

$$e^{(m+b)t} \leq K,$$

for all $t \geq 0$.

Proof. Fix an arbitrary $t = t_* \geq 0$. By assumption, we have for all $N \in \mathbb{N}$

$$e^{mt_*}\sum_{i=0}^{N} \frac{b^i t_*^i}{i!} \leq K.$$

Since e^{mt_*} is positive, we can divide this term out to obtain

$$\sum_{i=0}^{N} \frac{b^i t_*^i}{i!} \leq K e^{-mt_*} =: \tilde{K}. \tag{6}$$

The goal is now to show that this implies that $e^{bt_*} \leq \tilde{K}$. Assume for contradiction that $e^{bt_*} > \tilde{K}$ and set $\delta := e^{bt_*} - \tilde{K} > 0$. Using (6), we then have for all $N \in \mathbb{N}$

$$e^{bt_*} - \sum_{i=0}^{N} \frac{b^i t_*^i}{i!} \geq \delta. \tag{7}$$

Since $\lim_{N\to\infty} \sum_{i=0}^{N} \frac{b^i t_*^i}{i!} = e^{bt_*}$, we have by definition that for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that, whenever $N \geq N_\epsilon$, we have

$$\left| e^{bt_*} - \sum_{i=0}^{N} \frac{b^i t_*^i}{i!} \right| < \epsilon.$$

In particular, pick $\epsilon = \delta$. Then, there exists $N^* \in \mathbb{N}$ such that, whenever $N \geq N^*$, we have

$$\left| e^{bt_*} - \sum_{i=0}^{N} \frac{b^i t_*^i}{i!} \right| < \delta.$$

But this contradicts what we have in (7). Hence, we conclude that $e^{bt_*} \leq \tilde{K}$. Multiplying both sides by e^{mt_*}, we have $e^{(m+b)t_*} \leq K$. Since t_* was arbitrary, this holds for all $t \geq 0$. Hence, the result follows. □

We now have enough information to find necessary and sufficient conditions for string stability of the PF network (3). The first main result is given by the following theorem.

Theorem 2.5. *Consider the PF network* (3) *with $a > 0$. There exists real numbers $K, \mu > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,*

$$\max_{i=1,\dots,N} |x_i(t)| \leq K e^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|,$$

if and only if $|b| < a$.

Proof. ($\Longleftarrow$) By Lemma 2.3, we have

$$x_N(t) = e^{-at} \left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} \right).$$

The goal now is to bound the above for all $N \in \mathbb{N}$. We use the fact that each initial condition $x_{i,0}$ can by bounded by the maximum in absolute value over all initial conditions. Moreover, since b can be negative, we will apply the triangle inequality and the basic rules for absolute values to ensure that each term in the summation stays positive for all $t \geq 0$. That way, adding more terms to the summation will only make it larger, since we are only adding positive terms. This leads to

$$\begin{aligned} x_N(t) &\leq e^{-at}\left|\sum_{i=0}^{N-1}\frac{b^i x_{N-i,0}t^i}{i!}\right| \\ &\leq e^{-at}\left(\sum_{i=0}^{N-1}\frac{|b|^i|x_{N-i,0}|t^i}{i!}\right) \\ &\leq \max_{i=1,\dots,N}|x_{i,0}|e^{-at}\left(\sum_{i=0}^{N-1}\frac{|b|^i t^i}{i!}\right) \\ &\leq \max_{i=1,\dots,N}|x_{i,0}|e^{-at}\left(\sum_{i=0}^{\infty}\frac{|b|^i t^i}{i!}\right), \end{aligned}$$

which holds for all $N \in \mathbb{N}$. We recognize the infinite series in the last inequality to be the Taylor expansion of the exponential function $e^{|b|t}$ centered at $t = 0$. In other words, $\sum_{i=0}^{\infty}\frac{|b|^i t^i}{i!} = e^{|b|t}$. Substituting this result gives

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N}|x_{i,0}|e^{-at}e^{|b|t} \\ &= \max_{i=1,\dots,N}|x_{i,0}|e^{-(a-|b|)t}. \end{aligned}$$

This holds for all $N \in \mathbb{N}$, which implies

$$\max_{i=1,\dots,N}|x_i(t)| \leq \max_{i=1,\dots,N}|x_{i,0}|e^{-(a-|b|)t},$$

for all $N \in \mathbb{N}$. Since $|b| < a$ by assumption, we have $a - |b| > 0$. Setting $\mu = a - |b|$ and $K = 1$, the result follows.

($\implies$) By assumption, there exists real numbers $K, \mu > 0$ such that

$$\max_{i=1,\dots,N}|x_i(t)| \leq Ke^{-\mu t}\max_{i=1,\dots,N}|x_{i,0}|,$$

for all initial conditions $x_{i,0}$. If $b \geq 0$, pick $x_{i,0} = 1$ for all $i = 1, \dots, N$. If $b < 0$, pick $x_{N-j,0} = 1$ when j is even and $x_{N-j,0} = -1$ when j is odd. Then, using Lemma 2.3, the solution for all $N \in \mathbb{N}$ and all $t \geq 0$ for these particular initial conditions is given by

$$x_N(t) = e^{-at}\sum_{i=0}^{N-1}\frac{|b|^i t^i}{i!} \leq Ke^{-\mu t},$$

where the above inequality follows from the assumption of string stability. To show that $|b| < a$, assume for contradiction that $|b| \geq a$. Then, for all $t \geq 0$, we have that $e^{-|b|t} \leq e^{-at}$. Using this fact and multiplying both sides of the inequality by $e^{\mu t}$, we have

$$e^{(\mu-|b|)t}\sum_{i=0}^{N-1}\frac{|b|^i t^i}{i!} \leq K,$$

which holds for all $N \in \mathbb{N}$. By Lemma 2.4, this implies

$$e^{\mu t} \leq K,$$

where we cancel out the $|b|$ terms in the exponential function. However, this is a contradiction as $\mu > 0$ implies $e^{\mu t}$ is monotonically increasing and unbounded from above. In particular, there exists $t_* \geq 0$ such that for all $t \geq t_*$ we have $e^{\mu t} > K$. Therefore, we must have $|b| < a$. This shows the result. □

In order to interpret these results, we will consider the context of vehicle platooning again. Recall that the goal of string stability is to bound the effect of a sudden disturbance propagating through the platoon regardless of the platoon size. When this disturbance affects the leader vehicle, it will disrupt the state trajectory of this vehicle. The leader vehicle then provides the effect of this disturbance to the next vehicle by means of an input. However, since $|b| < a$, the succeeding vehicle in line will receive a smaller disturbance (say, $|b|$) than that which the leader vehicle experiences (namely, a). This, in turn, will mean that the state trajectory of the succeeding vehicle will not be as disrupted as that of the leader vehicle. This same behavior will continue down the string of vehicles, where each vehicle will experience less of a disrupted state trajectory than its predecessor. In other words, the effect of the disturbance decreases as it propagates along the string of vehicles. In particular, the larger $\mu = a - |b|$ is, the faster each state trajectory will converge to zero i.e. the faster each vehicle will restore its original state trajectory, for example its desired velocity. Hence, when designing each system in the network, one should choose $|b|$ to be as small as possible to optimize string stability.

It is important to note that Theorem 2.5 requires that each system in the network has the same system parameters. Namely, for each system Σ_i we set $B_i = b$ with state parameter $-a$. When each system has the same parameters in its system dynamics, we say that the network is *homogeneous*. On the other hand, if the parameters vary per system, we say that the network is *heterogeneous*. The following corollary shows that once we relax this condition of homogeneity in the input term by allowing for different values of b per system, the necessary condition for string stability will be lost.

Corollary 2.6. *Consider the heterogeneous PF-network given by*

$$\begin{aligned}\Sigma_1 &: \dot{x}_1(t) = -ax_1(t),\\ \Sigma_j &: \dot{x}_j(t) = -ax_j(t) + b_{j-1}x_{j-1}(t), \quad j = 2, \dots, N,\end{aligned}$$

where $a > 0$ and $b_j \in \mathbb{R}$ for all $j = 1, \dots, N-1$. If $\max|b_j| < a$, then there exists real numbers $K, \mu > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,

$$\max_{i=1,\dots,N} |x_i(t)| \le Ke^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|.$$

Proof. Set $b := \max|b_j|$. Analogous to the proofs in Lemma 2.3 and Theorem 2.5, we find

$$x_N(t) = e^{-at}\left(x_{N,0} + b_{N-1}x_{N-1,0}t + \frac{b_{N-1}b_{N-2}}{2}x_{N-2,0}t^2 + \cdots + \frac{b_{N-1}b_{N-2}\dots b_2 b_1}{(N-1)!}x_{1,0}t^{N-1}\right).$$

Applying the triangle inequality to the above, pulling out $\max|x_{i,0}|$ and using that $b_j \le b$ for all $j = 1, \dots, N-1$, we obtain

$$\begin{aligned}x_N(t) &\le \max_{i=1,\dots,N} |x_{i,0}| e^{-at}\left(\sum_{i=0}^{N-1} \frac{b^i t^i}{i!}\right)\\ &\le \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-b)t},\end{aligned}$$

for all $N \in \mathbb{N}$. Set $K = 1$ and $\mu = a - \max|b_j|$. Since $\max|b_j| < a$ by assumption, we have $\mu > 0$. This shows the result. □

Note that in the above corollary we only allow variations in the input parameters, while the state parameters, $-a$, remain the same throughout the network. The case where the PF network is fully heterogeneous, meaning we allow different values of both a and b, will be treated in Section 4.2.

To show why the converse does not hold, consider the following counterexample. Set $a = 1$ and set

$b_1 = 2$, $b_j = 0.1$ for all $j = 2, \dots, N-1$. Finally, for the initial conditions, pick $x_{i,0} = 1$ for all $i = 1, \dots, N$. Plugging in these values into the solution $x_N(t)$, we obtain

$$\begin{aligned} x_N(t) &= e^{-t}\left(1 + 0.1t + \frac{0.01t^2}{2} + \dots + 2 \cdot \frac{0.1^{N-2}t^{N-1}}{(N-1)!}\right) \\ &= e^{-t}\left(1 + 0.1t + \frac{0.01t^2}{2} + \dots + 20 \cdot \frac{0.1^{N-1}t^{N-1}}{(N-1)!}\right) \\ &\leq e^{-t}\left(e^{0.1t} + 20e^{0.1t}\right) \\ &= 21e^{-0.9t}, \end{aligned}$$

for all $N \in \mathbb{N}$. Intuitively, this result should make sense. Even if the effect of the disturbance is initially amplified per additional vehicle as it propagates through the string (i.e. $|b_i| \geq a$), eventually there will be a vehicle in the platoon where each vehicle, from that point on, will receive a smaller disturbance than the one coming before it. In other words, from that point on, the dynamics of those vehicles satisfy $|b_i| < a$. Hence, despite the initial amplification of the disturbance, eventually this effect will start to decrease per vehicle at some point in the string and so string stability can still be achieved.

2.3 Predecessor-Leader Following Problem

We will now focus our attention to the predecessor-leader following (PLF) problem (see Figure 3 (b)). The system dynamics are a direct extension of that of (3). Each system still receives an input from its predecessor, but in addition, each system receives an input from the leader system Σ_1. This means that there are now two inputs to be considered instead of one. The dynamics of each system are now given by

$$\begin{aligned} \Sigma_1 &: \dot{x}_1(t) = -ax_1(t), \\ \Sigma_2 &: \dot{x}_2(t) = -ax_2(t) + bx_1(t), \\ \Sigma_3 &: \dot{x}_j(t) = -ax_j(t) + bx_{j-1}(t) + cx_1(t), \quad j = 3, \dots, N, \end{aligned} \tag{8}$$

where $a, b, c \in \mathbb{R}$ and $a > 0$. Notice how the $cx_1(t)$-term is absent in the second system. This is because the second system Σ_2 already receives an input from the leader system Σ_1. The approach to finding conditions for string stability is analogous to the PF problem. We will need the closed-form solutions of (8), which will turn out to be nothing but a slight modification of Lemma 2.3.

Lemma 2.7. *Consider the PLF network* (8) *for given initial conditions $x_i(0) = x_{i,0}$. Then, for all $N \geq 3$, the solution is given by*

$$x_N(t) = e^{-at}\left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} + \sum_{k=1}^{N-2} \frac{cx_{1,0} b^{k-1} t^k}{k!}\right).$$

For $N = 1, 2$, the solution is given by

$$x_N(t) = e^{-at}\left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!}\right).$$

Proof. For $N = 1, 2$, the system dynamics reduce to those in Lemma 2.3. Hence, the result immediately follows from Lemma 2.3. We prove this statement by induction for $N \geq 3$. Setting $B = \begin{bmatrix} b & c \end{bmatrix}$ and $u_N(t) = \begin{bmatrix} x_{N-1}(t) & x_1(t) \end{bmatrix}^\top$, each solution is given by

$$x_N(t) = e^{-at}x_{N,0} + \int_0^t e^{-a(t-\tau)} B u_N(\tau) d\tau.$$

For the base case $N = 3$ we have

$$\begin{aligned} x_3(t) &= e^{-at}x_{3,0} + \int_0^t e^{-a(t-\tau)}(bx_2(\tau) + cx_1(\tau))d\tau \\ &= e^{-at}\left(x_{3,0} + \int_0^t cx_{1,0} + bx_{2,0} + b^2x_{1,0}\tau d\tau\right) \\ &= e^{-at}\left(x_{3,0} + cx_{1,0}t + bx_{2,0}t + \frac{b^2x_{1,0}t^2}{2}\right) = e^{-at}\left(\sum_{i=0}^{2}\frac{b^ix_{3-i,0}t^i}{i!} + \sum_{k=1}^{1}\frac{cx_{1,0}b^{k-1}t^k}{k!}\right). \end{aligned}$$

So the base case holds. Assume now that the statement holds for arbitrary $N = m$. We note that each solution contains two sums. In particular, the first sum is exactly that of Lemma 2.3. Hence, we only need to show that the second sum holds. We compute

$$\begin{aligned} x_{m+1}(t) &= e^{-at}\left(x_{m+1,0} + \int_0^t \sum_{j=0}^{m-1}\frac{b^{j+1}x_{m-j,0}\tau^j}{j!} + \sum_{k=1}^{m-2}\frac{cx_{1,0}b^k\tau^k}{k!} + cx_{1,0}d\tau\right) \\ &= e^{-at}\left(\sum_{j=0}^{m}\frac{b^jx_{m+1-j,0}t^j}{j!} + \sum_{k=1}^{m-2}\frac{cx_{1,0}b^kt^{k+1}}{(k+1)!} + cx_{1,0}t\right). \end{aligned}$$

Rearranging indices in the second sum and collecting terms, we obtain

$$\begin{aligned} x_{m+1}(t) &= e^{-at}\left(\sum_{i=0}^{m}\frac{b^ix_{m+1-i,0}t^i}{i!} + \sum_{k=2}^{m-1}\frac{cx_{1,0}b^{k-1}t^k}{k!} + cx_{1,0}t\right) \\ &= e^{-at}\left(\sum_{i=0}^{m}\frac{b^ix_{m+1-i,0}t^i}{i!} + \sum_{k=1}^{m-1}\frac{cx_{1,0}b^{k-1}t^k}{k!}\right). \end{aligned}$$

Hence, the result holds for $N = m + 1$. Since m was arbitrary, the statement holds for all $N \geq 3$. This proves the statement. □

Lemma 2.7 will help us in proving sufficiency in the following theorem. In order to prove necessity, it turns out we can employ Lemma 2.4. The following theorem solves the PLF problem for homogeneous networks.

Theorem 2.8. *Consider the PLF network* (8)*. Then, there exists real numbers* $K, \mu > 0$ *such that, for all initial conditions* $x_i(0) = x_{i,0}$ *and all* $N \in \mathbb{N}$,

$$\max_{i=1,\dots,N}|x_i(t)| \leq Ke^{-\mu t}\max_{i=1,\dots,N}|x_{i,0}|,$$

if and only if $|b| < a$.

Proof. ($\Longleftarrow$) The solution $x_N(t)$ of each system is given by Lemma 2.7. The idea of the proof is analogous to that of Theorem 2.5. Applying the triangle inequality, we have for all $N \in \mathbb{N}$

$$\begin{aligned} x_N(t) &\leq e^{-at}\left(\left|\sum_{i=0}^{N-1}\frac{b^ix_{N-i,0}t^i}{i!}\right| + \left|\sum_{k=1}^{N-2}\frac{cx_{1,0}b^{k-1}t^k}{k!}\right|\right) \\ &\leq e^{-at}\max_{i=1,\dots,N}|x_{i,0}|\left(\sum_{i=0}^{N-1}\frac{|b|^it^i}{i!} + \sum_{k=0}^{N-2}\frac{|c||b|^{k-1}t^k}{k!}\right) \\ &= e^{-at}\max_{i=1,\dots,N}|x_{i,0}|\left(\sum_{i=0}^{N-1}\frac{|b|^it^i}{i!} + \frac{|c|}{|b|}\sum_{k=0}^{N-2}\frac{|b|^kt^k}{k!}\right). \end{aligned}$$

Letting $N \longrightarrow \infty$, we see that both sums converge to the Taylor series of $e^{|b|t}$ centered at $t = 0$. Moreover, since each term is positive, the above equality can be bounded by this exponential function. This implies

$$\begin{aligned} x_N(t) &\leq e^{-at} \max_{i=1,\ldots,N} |x_{i,0}| \left(e^{|b|t} + \frac{|c|}{|b|} e^{|b|t} \right) \\ &= e^{-(a-|b|)t} \max_{i=1,\ldots,N} |x_{i,0}| \left(1 + \frac{|c|}{|b|} \right), \end{aligned}$$

for all $N \in \mathbb{N}$, which also means

$$\max_{i=1,\ldots,N} |x(t)| \leq e^{-(a-|b|)t} \max_{i=1,\ldots,N} |x_{i,0}| \left(1 + \frac{|c|}{|b|} \right), \tag{9}$$

for all $N \in \mathbb{N}$. Set $\mu = a - |b|$, $K = 1 + \frac{|c|}{|b|}$. Since $a - |b| > 0$ by assumption, we have that $\mu > 0$. This proves the statement.

($\Longrightarrow$) By assumption there exists $K, \mu > 0$ such that

$$x_N(t) = e^{-at} \left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} + \frac{c}{b} \sum_{k=1}^{N-2} \frac{x_{1,0} b^k t^k}{k!} \right) \leq K e^{-\mu t} \max_{i=1,\ldots,N} |x_{i,0}|,$$

for all initial conditions $x_{i,0}$ and any $N \in \mathbb{N}$. First, choose $x_{1,0} = 0$. Then, for any $N \in \mathbb{N}$, the solution becomes

$$x_N(t) = e^{-at} \sum_{i=0}^{N-2} \frac{b^i x_{N-i,0} t^i}{i!}.$$

Next, if $b \geq 0$, pick $x_{i,0} = 1$ for all $i = 2, \ldots, N$. If $b < 0$, pick $x_{N-j,0} = 1$ when j is even and $x_{N-j,0} = -1$ when j is odd for all remaining initial conditions. Then, by assumption, we have for all $N \in \mathbb{N}$

$$e^{-at} \sum_{i=0}^{N-2} \frac{|b|^i t^i}{i!} \leq K e^{-\mu t}.$$

Hence, the problem reduces to that of the proof in Theorem 2.5. Therefore, it immediately follows that $|b| < a$. This shows the result. □

It is interesting to note that there is no restriction on the value of c to achieve string stability. This does not mean that the value of c has no effect on the exponential bound of the network, as the value of K depends on c. Rather, it means that we are free to choose any value for c without losing string stability. On the other hand, we do not have this freedom for the choice of b.

To give an interpretation as to why the presence of c does not change the conditions for string stability, recall the interpretation of the condition $|b| < a$. In the context of vehicle platooning, finding conditions for string stability can be phrased as the question: how should each vehicle send the disturbance it is experiencing to its successor? The answer is $|b| < a$. Namely, each vehicle sends a smaller disturbance to its successor than what it is experiencing. In this case, though, each vehicle in the platoon now also receives the same disturbance $|c|$ from the leader vehicle. As such, the vehicles do not have any influence in how they send this disturbance $|c|$ to their successors, as each vehicle will experience the same disturbance $|c|$ nonetheless.

However, this does not mean any choice c is equally effective in constructing an exponential bound.

This is because in the context of vehicle platooning, K physically represents the maximum effect that a sudden disturbance can have on each vehicle in the platoon. In order to make this clear, consider a platoon of arbitrary length satisfying $\max |x_{i,0}| = 1$. If this platoon is string stable, by Definition 2.2 there exists $K, \mu > 0$ such that

$$\max_{i=1,\ldots,N} |x_i(t)| \leq Ke^{-\mu t},$$

for all $t \geq 0$ and all $N \in \mathbb{N}$. One can see that since the right hand side is an exponentially decaying bound, it attains its maximum at $t = 0$, which means that the maximum value each system will attain will never exceed K. Naturally, one would like for this K to then be as small as possible. In this case, that would mean that we would ideally want $|c| << |b|$ such that $\frac{|c|}{|b|} << 1$. As will be seen in the Examples section, the presence of $|c|$ is responsible for the initial peaks that each system experiences before its state trajectory converges to the origin, which is absent when $|c| = 0$ i.e. in a PF network with the same initial conditions.

One might observe that the bound for the PLF formation is larger than that of the PF formation. Namely, in both the PLF and PF formation we found that $\mu = a - |b|$, but in the PF formation we obtained $K = 1$ whereas for the PLF formation we found $K = 1 + \frac{|c|}{|b|} > 1$. This is because in the PLF formation each system receives inputs from not one but two systems. As established in the PF problem, the parameters b and c in the input represent the disturbance that each vehicle in a platoon provides to its respective receiver. As each vehicle in the PLF formation receives two inputs instead of one, it receives a larger total disturbance than a vehicle in the PF formation, which receives only one input. This, naturally, negatively affects the string stability bound, resulting in a larger value for K.

As mentioned earlier, there is no restriction on the choice of c to obtain string stability of a PLF network. A natural question to ask is then whether we can extend the result of Theorem 2.8 to networks with heterogeneous inputs. In other words, we would like to extend the result of Corollary 2.6 to the PLF problem as well. In the final result of this section, we will show that allowing for different values of b and c means we can still choose the leader input term freely without losing string stability. On the other hand, just like in Corollary 2.6, necessity will be lost.

Corollary 2.9. *Consider the heterogeneous PLF-network given by*

$$\begin{aligned}
&\Sigma_1 : \dot{x}_1(t) = -ax_1(t), \\
&\Sigma_2 : \dot{x}_2(t) = -ax_2(t) + b_1 x_1(t), \\
&\Sigma_3 : \dot{x}_j(t) = -ax_j(t) + b_{j-1}x_{j-1}(t) + c_{j-2}x_1(t), \quad j = 3, \ldots, N,
\end{aligned}$$

where $a > 0$, $b_k \in \mathbb{R}$ *for all* $k = 1, \ldots, N-1$ *and* $c_m \in \mathbb{R}$ *for all* $m = 1, \ldots, N-2$. *If* $\max |b_k| < a$, *then there exists real numbers* $K, \mu > 0$ *such that, for all initial conditions* $x_i(0) = x_{i,0}$ *and all* $N \in \mathbb{N}$,

$$\max_{i=1,\ldots,N} |x_i(t)| \leq Ke^{-\mu t} \max_{i=1,\ldots,N} |x_{i,0,}|.$$

Proof. Set $b := \max |b_k|$ and $c := \max |c_m|$. Analogous to Corollary 2.6, it can be shown that

$$x_N(t) = \hat{x}_N(t) + \tilde{x}_N(t),$$

where

$$\hat{x}_N(t) = e^{-at}\left(x_{N,0} + b_{N-1}x_{N-1,0}t + \frac{b_{N-1}b_{N-2}}{2}x_{N-2,0}t^2 + \cdots + \frac{b_{N-1}b_{N-2}\ldots b_2 b_1}{(N-1)!}x_{1,0}t^{N-1}\right),$$

and

$$\tilde{x}_N(t) = e^{-at}\left(c_{N-2}x_{1,0}t + \tfrac{b_{N-1}c_{N-3}}{2}x_{1,0}t^2 + \tfrac{b_{N-1}b_{N-2}c_{N-4}}{6}x_{1,0}t^3 + \cdots + \tfrac{b_{N-1}b_{N-2}\ldots b_4 b_3 c_1}{(N-2)!}x_{1,0}t^{N-2}\right).$$

By Corollary 2.6, we have

$$\hat{x}_N(t) \leq \max_{i=1,\ldots,N} |x_{i,0}| e^{-(a-b)t}.$$

Similarly, by applying the triangle inequality to $\tilde{x}_N(t)$ and following the approach in Corollary 2.6, we obtain

$$\begin{aligned}\tilde{x}_N(t) &\leq \max_{i=1,\ldots,N} |x_{i,0}| e^{-at} \sum_{i=1}^{N-2} c \frac{b^{i-1} t^i}{i!} \\ &\leq \frac{c}{b} \max_{i=1,\ldots,N} |x_{i,0}| e^{-(a-b)t},\end{aligned}$$

which means that the solution $x_N(t)$ can be bounded as

$$x_N(t) \leq \max_{i=1,\ldots,N} |x_{i,0}| \left(1 + \frac{c}{b}\right) e^{-(a-b)t}.$$

Since this holds for all $N \in \mathbb{N}$, this implies

$$\max_{i=1,\ldots,N} |x_i(t)| \leq \max_{i=1,\ldots,N} |x_{i,0}| \left(1 + \frac{c}{b}\right) e^{-(a-b)t}.$$

Hence, setting $K = 1 + \frac{\max |c_m|}{\max |b_k|}$ and $\mu = a - \max |b_k|$ gives the result.
To show why the converse does not hold, set $a = 1$, $b_1 = 2$, $b_p = 0.1$ for $p = 2, \ldots, N-1$, $c_1 = 2$ and $c_q = 1$ for $q = 2, \ldots, N-2$. Finally, set $x_{i,0} = 1$ for all $i = 1, \ldots, N$. It can then be shown that

$$\begin{aligned}x_N(t) &\leq 21 e^{-0.9t} + e^{-t} \left(\sum_{i=1}^{N-3} \frac{0.1^{i-1} t^i}{i!} + 2 \frac{0.1^{N-3} t^{N-2}}{(N-2)} \right) \\ &\leq 21 e^{-0.9t} + \frac{1}{0.1} e^{-0.9t} + \frac{2}{0.1} e^{-0.9t} \\ &= 51 e^{-0.9t},\end{aligned}$$

for all $N \in \mathbb{N}$. Hence, again, we obtain $\max |b_k| > a$ while string stability is still achieved. This concludes the counterexample. □

3 Networks with Disturbances

3.1 PF with Disturbances

In this section we will be revisiting the PF and PLF problems. The PF network (3) represents an ideal case scenario. Namely, it assumes that there is no external unknown disturbance acting on the network for all time. In many real life settings, however, this disturbance is in fact present. In vehicle platooning, for example, one can think of this disturbance as a strong wind perpetually affecting the position of each vehicle in the platoon. As a result, this disturbance will have to be taken into account somehow. Factoring this disturbance into the PF network, the network can be modeled as

$$\begin{aligned} \Sigma_1 &: \dot{x}_1(t) = -ax_1(t) + d_1(t), \\ \Sigma_j &: \dot{x}_j(t) = -ax_j(t) + bx_{j-1}(t) + d_j(t), \quad j = 2, \ldots, N, \end{aligned} \tag{10}$$

where $a, b \in \mathbb{R}$ with $a > 0$ and $d_i(t)$ is an unknown, time-dependent disturbance acting on the system Σ_i. The total network of systems for the PF formation is now given by

$$\Sigma : \dot{x}(t) = Ax(t) + d(t), \tag{11}$$

where $a_{i,i} = -a$, $a_{i+1,i} = b$ and $d(t) = \begin{bmatrix} d_1(t) & d_2(t) & \ldots & d_N(t) \end{bmatrix}^\top$. Now that each system in the network contains an additional unknown term $d_i(t)$, it begs the question of whether string stability can still be achieved. Naturally, if $d_i(t) \equiv 0$, the problem reduces to that of Theorem 2.5. Otherwise, solving this problem is not so obvious. Especially as these external disturbances are unknown, their effect will have to be factored into achieving string stability. Definition 2.2 will turn out to be too inflexible for this problem. However, if the disturbances are all bounded, then we can resort to the following definition instead.

Definition 3.1. *Consider the systems as in* (10) *and the network of systems* (11). *The network* (11) *is said to be disturbance string stable if there exists real numbers $K > 0$, $\mu > 0$ and $\gamma > 0$ such that*

$$\max_{i=1,\ldots,N} |x_i(t)| \leq Ke^{-\mu t} \max_{i=1,\ldots,N} |x_{i,0}| + \gamma \max_{i=1,\ldots,N} \sup_t |d_i(t)|,$$

for all initial conditions $x_i(0) = x_{i,0}$, all disturbances $d_i(t)$ and all $N \in \mathbb{N}$.

Recall that in order to obtain string stability as in Definition 2.2, it is necessary that the network is exponentially stable. That is, the state trajectory of each system converges to the equilibrium point $x(t) = 0$ in an exponential manner as time goes to infinity. Definition 3.1, on the other hand, states that as $t \longrightarrow \infty$, we have $|x_i(t)| \leq M := \gamma \max \sup |d_i(t)|$ for all $i = 1, \ldots, N$ and all $N \in \mathbb{N}$. What this means is that it is no longer necessarily true that each state trajectory converges to the origin. Instead, the best we can wish for is input-to-state stability of each system in the network. Each state trajectory $x_i(t)$ may not converge to zero, but at least we can still ensure that the state trajectory of each system does not leave the strip of radius M centered around the origin, no matter how large the network of systems is.

In order to achieve disturbance string stability, the same general technique as in Theorem 2.5 and Theorem 2.8 will be employed. Namely, the closed form solution of each system in (10) will be used to create an exponential bound. However, in this case, the presence of the disturbances complicates the problem. In order to deal with the disturbances, the following lemmas will be needed.

Lemma 3.2. *Let $x : [0, \infty) \longrightarrow [0, \infty)$ and $y : [0, \infty) \longrightarrow \mathbb{R}$ be continuous functions. Let $a, b \geq 0$ be real numbers satisfying $a \leq b$. If $\sup |y(t)|$ exists, then*

$$\int_a^b x(t)y(t)dt \leq \sup_{t \in [a,b]} |y(t)| \int_a^b x(t)dt.$$

Proof. Consider a partition of $[a, b]$ given by $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Define $\Delta t = \frac{b-a}{n}$ and assume each subinterval $[t_{i-1}, t_i]$ has length Δt. Finally, let t_i^* be a sampling point in the interval $[t_{i-1}, t_i]$. Then, by definition of the definite integral, we have

$$\int_a^b x(t)y(t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} x(t_i^*)y(t_i^*)\Delta t.$$

Applying the triangle inequality to the sum on the right hand side and noting that $x(t) \geq 0$ for all $t \geq 0$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} x(t_i^*)y(t_i^*)\Delta t &\leq \lim_{n\to\infty} \sum_{i=1}^{n} x(t_i^*)|y(t_i^*)|\Delta t \\ &\leq \sup_{t\in[a,b]} |y(t)| \lim_{n\to\infty} \sum_{i=1}^{n} x(t_i^*)\Delta t \\ &= \sup_{t\in[a,b]} |y(t)| \int_a^b x(t)dt.\end{aligned}$$

Hence, the result follows. □

In order to obtain the solution $x_N(t)$, we will need to integrate the unknown disturbances. Since these integrals cannot be computed, due to these disturbances being unknown functions, we can still apply Lemma 3.2 to pull out $\sup |d_j(t)|$ for each disturbance. This already gives us the supremum part of the disturbance bound in Definition 3.1. To deal with the remaining integrands, the following lemma will be employed.

Lemma 3.3. *Let $a \in \mathbb{R}$ and $k \in \mathbb{N}$. Define*

$$I_k(x)(t) := \int_0^t \int_0^{t_{k-1}} \cdots \int_0^{t_1} x(\tau)\ d\tau dt_1 \ldots dt_{k-1}.$$

In other words, we integrate a function x k times. Then,

$$I_k(e^{at}) = \sum_{i=0}^{k-1} -\frac{t^i}{i! a^{k-i}} + \frac{1}{a^k} e^{at},$$

for all $k \in \mathbb{N}$.

Proof. We will proceed by induction. Consider the base case $k = 1$. We have

$$\begin{aligned}I_1(e^{at}) &= \int_0^t e^{a\tau} d\tau \\ &= \frac{1}{a}e^{at} - \frac{1}{a} = \sum_{i=0}^{0} -\frac{t^i}{a^{1-i}} + \frac{1}{a}e^{at}.\end{aligned}$$

Hence, the base holds. Assume now that the result holds for arbitrary $k = n$. We will show the result holds for $k = n + 1$. we have

$$I_{n+1}(e^{at}) = \int_0^t I_n(e^{a\tau})d\tau.$$

Plugging in the induction hypothesis, we obtain

$$\begin{aligned} I_{n+1}(e^{at}) &= \int_0^t \sum_{i=0}^{n-1} -\frac{\tau^i}{i!a^{n-i}} + \frac{1}{a^n} e^{a\tau} d\tau \\ &= \sum_{i=0}^{n-1} -\frac{t^{i+1}}{(i+1)!a^{n-i}} + \frac{1}{a^{n+1}} e^{at} - \frac{1}{a^{n+1}}. \end{aligned}$$

Rearranging indices, this is the same as

$$\begin{aligned} I_{n+1}(e^{at}) &= \sum_{i=1}^{n} -\frac{t^i}{i!a^{n+1-i}} + \frac{1}{a^{n+1}} e^{at} - \frac{1}{a^{n+1}} \\ &= \sum_{i=0}^{n} -\frac{t^i}{i!a^{n+1-i}} + \frac{1}{a^{n+1}} e^{at}. \end{aligned}$$

Hence, $k = n + 1$ holds as well. The statement then follows by induction. □

Using the above lemma, we can now find a closed-form expression for the solution of each system in the disturbed PF network (10).

Lemma 3.4. *Consider the network* (10) *for given initial conditions* $x_i(0) = x_{i,0}$. *Then, for each* $N \in \mathbb{N}$, *the solution is given by*

$$x_N(t) = e^{-at} \left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} + \sum_{j=0}^{N-1} b^j I_{j+1}(e^{at} d_{N-j})(t) \right).$$

Proof. First we note that each solution is given by

$$x_N(t) = e^{-at} \left(x_{N,0} + \int_0^t e^{a\tau} b x_{N-1}(\tau) + e^{a\tau} d_N(\tau) d\tau \right), \tag{12}$$

which can be obtained by taking $B = \begin{bmatrix} b & 1 \end{bmatrix}$ and $u_N(t) = \begin{bmatrix} x_{N-1}(t) & d_N(t) \end{bmatrix}^\top$ and multiplying out both matrices. Note that we can take the disturbance $d_N(t)$ as part of the input. We will proceed by induction. Consider the base case $N = 1$. Then, the solution is given by

$$\begin{aligned} x_1(t) &= e^{-at} \left(x_{1,0} + \int_0^t e^{a\tau} d_1(\tau) d\tau \right) \\ &= e^{-at} \left(\sum_{i=0}^{0} \frac{b^i x_{1,0} t^i}{i!} + \sum_{j=0}^{0} b^j I_{j+1}(e^{at} d_{1-j})(t) \right). \end{aligned}$$

Hence, the base case holds. Assume now that the statement holds for arbitrary $N = k$. We will show it holds for $N = k + 1$. Plugging the induction hypothesis into (12), we get

$$x_{k+1}(t) = e^{-at} \left(x_{k+1,0} + \int_0^t b \sum_{i=0}^{k-1} \frac{b^i x_{k-i,0} \tau^i}{i!} + b \sum_{j=0}^{k-1} b^j I_{j+1}(e^{a\tau} d_{k-j})(\tau) + e^{a\tau} d_{k+1}(\tau) d\tau \right).$$

We can split the above integral into three integrals: one integral for each term of the integrand. Doing so, we note that we can directly apply Lemma 2.3 to the first two terms in parentheses above.

Hence, we only need to show that the last two terms hold too. Applying the definition of $I_k(x)(t)$, this yields

$$x_{k+1}(t) = e^{-at}\left(\sum_{i=0}^{k} \frac{b^i x_{k-i,0} t^i}{i!} + \sum_{j=0}^{k-1} b^{j+1} I_{j+2}(e^{at}d_{k-j})(t) + \int_0^t e^{a\tau} d_{k+1}(\tau) d\tau\right).$$

Finally, rearranging indices in the second sum and noting that

$$\int_0^t e^{a\tau} d_{k+1}(\tau) d\tau = I_1(e^{at}d_{k+1})(t),$$

we obtain

$$\begin{aligned} x_{k+1}(t) &= e^{-at}\left(\sum_{i=0}^{k} \frac{b^i x_{k-i,0} t^i}{i!} + \sum_{j=1}^{k} b^j I_{j+1}(e^{at}d_{k+1-j})(t) + I_1(e^{at}d_{k+1})(t)\right) \\ &= e^{-at}\left(\sum_{i=0}^{k} \frac{b^i x_{k-i,0} \tau^i}{i!} + \sum_{j=0}^{k} b^j I_{j+1}(e^{at}d_{k+1-j})(t)\right). \end{aligned}$$

Hence, $N = k+1$ holds as well. Since $N = k$ was arbitrary, the result holds for all $N \in \mathbb{N}$. This shows the result. □

Using these three lemmas, finding necessary and sufficient conditions for disturbance string stability will turn out to be very simple. We can now state the main result.

Theorem 3.5. *Consider the disturbed PF network* (10)*. Assume that each disturbance $d_i(t)$ is bounded i.e.* $\sup|d_i(t)|$ *exists. Then, there exists real numbers $K, \mu, \gamma > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$, all disturbances $d_i(t)$ and all $N \in \mathbb{N}$,*

$$\max_{i=1,\dots,N} |x_i(t)| \le Ke^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}| + \gamma \max_{i=1,\dots,N} \sup_t |d_i(t)|,$$

if and only if $|b| < a$.

Proof. ($\implies$) By assumption,

$$\max_{i=1,\dots,N} |x_i(t)| \le Ke^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}| + \gamma \max_{i=1,\dots,N} \sup_t |d_i(t)|,$$

for all initial conditions $x_{i,0}$, all disturbances $d_i(t)$ and all $N \in \mathbb{N}$. Pick $d_i(t) \equiv 0$ for all $i = 1, \dots, N$. Then, the problem reduces to that of Theorem 2.5. Hence, it immediately follows that $|b| < a$.
($\impliedby$) By Lemma 3.4, the solution for all $N \in \mathbb{N}$ is given by

$$x_N(t) = e^{-at} \sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} + e^{-at} \sum_{j=0}^{N-1} b^j I_{j+1}(e^{at} d_{N-j})(t).$$

We can use the proof of Theorem 2.5 to bound the first term on the right hand side. Hence, we only need to concern ourselves with the second term. Using Lemma 3.2 and applying the triangle inequality, we can then say

$$\begin{aligned} x_N(t) &\le \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + e^{-at} \sum_{j=0}^{N-1} |b|^j I_{j+1}(|e^{at}||d_{N-j}|)(t) \\ &\le \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + e^{-at} \sum_{j=0}^{N-1} |b|^j \sup_t |d_{N-j}(t)| I_{j+1}(e^{at}). \end{aligned}$$

Next, we apply Lemma 3.3, which yields

$$x_N(t) \leq \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + e^{-at} \sum_{j=0}^{N-1} |b|^j \sup_t |d_{N-j}(t)| \left(\sum_{k=0}^{j} -\frac{t^k}{k! a^{j+1-k}} + \frac{1}{a^{j+1}} e^{at} \right).$$

We note that all the terms above, except for the minus term in the sum in parentheses, are positive for all $t \geq 0$. Since this minus term is negative, it immediately follows that

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + e^{-at} \sum_{j=0}^{N-1} |b|^j \sup_t |d_{N-j}(t)| \frac{1}{a^{j+1}} e^{at} \\ &= \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \frac{1}{a} \sum_{j=0}^{N-1} \sup_t |d_{N-j}(t)| \frac{|b|^j}{a^j} \\ &\leq \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \frac{1}{a} \max_{i=1,\dots,N} \sup_t |d_i(t)| \sum_{j=0}^{N-1} \left(\frac{|b|}{a} \right)^j. \end{aligned}$$

We recognize the geometric series. Since $|b| < a$, we know that the series converges as $N \longrightarrow \infty$. It follows that

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \frac{1}{a} \max_{i=1,\dots,N} \sup_t |d_i(t)| \sum_{j=0}^{\infty} \left(\frac{|b|}{a} \right)^j \\ &= \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \frac{1}{a} \max_{i=1,\dots,N} \sup_t |d_i(t)| \frac{1}{1 - \frac{|b|}{a}} \\ &= \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \frac{1}{a-|b|} \max_{i=1,\dots,N} \sup_t |d_i(t)|, \end{aligned}$$

where, in the second equality, we used the convergence formula for the geometric series $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$ for $|r| < 1$. Since the above inequality holds for any $N \in \mathbb{N}$, it follows that

$$\max_{i=1,\dots,N} |x_i(t)| \leq \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \frac{1}{a-|b|} \max_{i=1,\dots,N} \sup_t |d_i(t)|,$$

for all $N \in \mathbb{N}$. Set $K = 1$, $\mu = a - |b|$ and $\gamma = \frac{1}{a-|b|}$. Since $|b| < a$ by assumption, we have $\mu, \gamma > 0$. This shows the result. □

It is interesting to note that Theorem 3.5 tells us that the presence of a disturbance does not complicate matters when it comes to designing each system in the network to achieve disturbance string stability. Moreover, it is interesting to note that the the optimal choice for $|b|$ in the design process does not change either. Whereas in Theorem 2.5 one would like to pick $|b|$ to be as small as possible to maximize $\mu = a - |b|$, in this case the choice of $|b|$ will also affect the extent to which the external disturbance is present. Namely, $\gamma = \frac{1}{a-|b|}$ depends on b. One can see, however, that $\gamma = \frac{1}{\mu}$ and so maximizing μ means minimizing γ. Since we want to make the disturbance string stability bound as small as possible, we want γ to be as small as possible as well. Hence, choosing $|b|$ to be as small as possible will optimize convergence speed of the bound and minimize the effect of the disturbance on the network.

3.2 PLF with Disturbances

We would now like to extend the result given by Theorem 3.5 to networks adhering to the PLF topology. Recall that the dynamics of a PLF network are given by (8). In this case, we will consider a

PLF formation with an unknown, bounded disturbance $d_i(t)$ added to each system. The network (8) then becomes

$$\begin{aligned} \Sigma_1 &: \dot{x}_1(t) = -ax_1(t) + d_1(t), \\ \Sigma_2 &: \dot{x}_2(t) = -ax_2(t) + bx_1(t) + d_2(t), \\ \Sigma_j &: \dot{x}_j(t) = -ax_j(t) + bx_{j-1}(t) + cx_1(t) + d_j(t), \quad j = 3, \dots, N. \end{aligned} \tag{13}$$

where $a, b, c \in \mathbb{R}$ and $a > 0$ and $d_i(t)$ an unknown time dependent disturbance. As with the disturbed PF network, it is no longer feasible to make this disturbed PLF network string stable. The best we can opt for is disturbance string stability. The approach to solving this issue is very similar to that in Theorem 3.5. Again, we are trying to find a closed-form solution of each system and use the triangle inequality to appropriately bound each solution. This time, though, we will have to factor in the presence of an additional leader term $cx_1(t)$ as well. This does complicate things to some extent, but the general idea of solving this problem remains the same as in Theorem 3.5. This means that we first need to know the solution of each system in the network (13). The following lemma provides us with those closed-form solutions.

Lemma 3.6. *Consider the network* (13) *for given initial conditions* $x_i(0) = x_{i,0}$. *For* $N = 1, 2$, *the solution to each system is given by*

$$x_N(t) = e^{-at} \left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} + \sum_{j=0}^{N-1} b^j I_{j+1}(e^{at} d_{N-j})(t) \right).$$

For $N \geq 3$ *the solution to each system is given by*

$$\begin{aligned} x_N(t) = e^{-at} \Bigg(&\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} + \sum_{k=1}^{N-2} \frac{c x_{1,0} b^{k-1} t^k}{k!} + \sum_{j=0}^{N-1} b^j I_{j+1}(e^{at} d_{N-j})(t) \\ &+ \sum_{m=1}^{n-2} b^{m-1} c I_{m+1}(e^{at} d_1)(t) \Bigg). \end{aligned}$$

Proof. The proof is similar to that of Lemma 3.4. We proceed by induction. For the cases $N = 1, 2$, the system dynamics reduce to those in (10). The result then immediately follows from Lemma 3.4. The result will be shown for $N \geq 3$. To this end, consider the base case $N = 3$. By direct computation, we obtain

$$\begin{aligned} x_3(t) &= e^{-at} \Bigg(x_{3,0} + bx_{2,0}t + \frac{b^2}{2} x_{1,0} t^2 + b^2 I_3(e^{at} d_1)(t) + bI_2(e^{at} d_2)(t) + cx_{1,0}t + cI_2(e^{at} d_1)(t) \\ &\quad + I_1(e^{at} d_3)(t) \Bigg) \\ &= e^{-at} \left(\sum_{i=0}^{2} \frac{b^i x_{3-i,0} t^i}{i!} + \sum_{k=1}^{1} \frac{c x_{1,0} b^{k-1} t^k}{k!} + \sum_{j=0}^{2} b^j I_{j+1}(e^{at} d_{3-j})(t) + \sum_{m=1}^{1} b^{m-1} c I_{m+1}(e^{at} d_1)(t) \right). \end{aligned}$$

Hence, the base case holds. Assume now that the result holds for arbitrary $N = p$. We will show that the result holds for $N = p + 1$. Omitting time arguments, the solution is given by

$$x_{p+1} = e^{-at} \left(x_{p+1,0} + \int_0^t e^{a\tau} (bx_p + cx_1 + d_{p+1}) \, d\tau \right).$$

Noting that $x_1(t) = e^{-at}x_{1,0} + e^{-at}I_1(e^{at}d_1)(t)$ and plugging in the induction hypothesis, the above becomes

$$x_{p+1} = e^{-at}\left(x_{p+1,0} + \int_0^t \sum_{i=0}^{p-1} \frac{b^{i+1}x_{p-i,0}\tau^i}{i!} + \sum_{k=1}^{p-2} \frac{cx_{1,0}b^k\tau^k}{k!} + \sum_{j=0}^{p-1} b^{j+1}I_{j+1}(e^{a\tau}d_{p-j}) \right.$$
$$\left. + \sum_{m=1}^{p-2} cb^m I_{m+1}(e^{a\tau}d_1) + cx_{1,0} + cI_1(e^{a\tau}d_1) + e^{a\tau}d_{p+1}\ d\tau \right).$$

Collecting all the terms into their relevant summation, the above can be shortened to

$$x_{p+1} = e^{-at}\left(x_{p+1,0} + \int_0^t \sum_{i=0}^{p-1} \frac{b^{i+1}x_{p-i,0}\tau^i}{i!} + \sum_{k=0}^{p-2} \frac{cx_{1,0}b^k\tau^k}{k!} + \sum_{j=0}^{p} b^{j}I_{j}(e^{a\tau}d_{p-j+1}) \right.$$
$$\left. + \sum_{m=0}^{p-2} cb^m I_{m+1}(e^{a\tau}d_1)\ d\tau \right),$$

which, after integrating, becomes

$$x_{p+1} = e^{-at}\left(x_{p+1,0} + \sum_{i=0}^{p-1} \frac{b^{i+1}x_{p-i,0}t^{i+1}}{(i+1)!} + \sum_{k=0}^{p-2} \frac{cx_{1,0}b^k t^{k+1}}{(k+1)!} + \sum_{j=0}^{p} b^{j}I_{j+1}(e^{at}d_{p-j+1}) \right.$$
$$\left. + \sum_{m=0}^{p-2} cb^m I_{m+2}(e^{at}d_1) \right).$$

Finally, collecting all the remaining terms into their relevant summation and rearranging indices yields the final result

$$x_{p+1} = e^{-at}\left(\sum_{i=0}^{p} \frac{b^{i}x_{p+1-i,0}t^{i}}{i!} + \sum_{k=1}^{p-1} \frac{cx_{1,0}b^{k-1}t^{k}}{k!} + \sum_{j=0}^{p} b^{j}I_{j+1}(e^{at}d_{p-j+1}) + \sum_{m=1}^{p-1} cb^{m-1} I_{m+1}(e^{at}d_1) \right).$$

Hence, the result holds for $N = p+1$ as well. Since $N = p$ was arbitrary, the result holds for all $N \geq 3$. This shows the result. □

At first glance, the result in Lemma 3.6 looks quite daunting due to the presence of four different sums. Upon closer inspection, however, one can recognize that the solution can be split up into two parts, where one part is precisely the expression given in Lemma 3.4. This also means that when trying to find conditions for disturbance string stability of the disturbed PLF network (13), half the work will be done, since we can immediately employ Theorem 3.5 to this half of the solution. This leads us to the final result of this section.

Theorem 3.7. *Consider the disturbed PLF network* (13)*. Assume that each disturbance $d_i(t)$ is bounded i.e.* $\sup |d_i(t)|$ *exists. Then, there exists real numbers $K, \mu, \gamma > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$, all disturbances $d_i(t)$ and all $N \in \mathbb{N}$,*

$$\max_{i=1,\dots,N} |x_i(t)| \leq Ke^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}| + \gamma \max_{i=1,\dots,N} \sup_t |d_i(t)|,$$

if and only if $|b| < a$.

Proof. ($\Longrightarrow$) Set $d_i(t) \equiv 0$ for all $i = 1, \dots, N$. The problem then reduces to that of Theorem 2.8. It immediately follows that $|b| < a$.
($\Longleftarrow$) Assume $|b| < a$. Omitting time arguments, by Lemma 3.6 the solution is given by

$$x_N = e^{-at}\left(\sum_{i=0}^{N-1} \frac{b^i x_{N-i,0} t^i}{i!} + \sum_{j=0}^{N-1} b^j I_{j+1}(e^{at} d_{N-j})(t) + \sum_{k=1}^{N-2} \frac{c x_{1,0} b^{k-1} t^k}{k!} + \sum_{m=1}^{N-2} b^{m-1} c I_{m+1}(e^{at} d_1)\right),$$

which holds for all $N \geq 3$. Note that the above solution can be split into two parts. In particular, one can see that the first two summations are precisely those as in Lemma 3.4. After applying the triangle equality to the solution above, half the work will be done already by Theorem 3.5. After applying Theorem 2.8 as well, for all $N \in \mathbb{N}$, the solution then satisfies

$$x_N \leq \left(1 + \frac{|c|}{|b|}\right) \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \max_{i=1,\dots,N} \sup_t |d_i(t)| \frac{1}{a - |b|} + e^{-at}\left(\sum_{m=1}^{N-2} |b|^{m-1} |c| I_{m+1}(e^{at} d_1)\right).$$

We only need to deal with the summation term in the above inequality. Applying Lemma 3.3 and following the proof of Theorem 3.5, this term satisfies

$$\begin{aligned} e^{-at}\left(\sum_{m=1}^{N-2} |b|^{m-1} |c| I_{m+1}(e^{at} d_1)(t)\right) &\leq |c| e^{-at}\left(\sum_{m=1}^{N-2} |b|^{m-1} \sup_t |d_1(t)| I_{m+1}(e^{at})\right) \\ &\leq |c|\left(\sum_{m=1}^{N-2} |b|^{m-1} \sup_t |d_1(t)| \frac{1}{a^{m+1}}\right) \\ &= \frac{|c|}{a^2}\left(\sum_{m=0}^{N-3} \sup_t |d_1(t)| \frac{|b|^m}{a^m}\right), \end{aligned}$$

where in the last equality we pulled out the term a^2 to obtain the geometric series inside the parentheses. Since $|b| < a$ by assumption, the geometric series converges. Letting $N \longrightarrow \infty$, the above inequality still holds, which gives us

$$\begin{aligned} e^{-at}\left(\sum_{m=1}^{N-2} |b|^{m-1} |c| I_{m+1}(e^{at} d_1)(t)\right) &\leq \max_{i=1,\dots,N} \sup_t |d_i(t)| \frac{|c|}{a^2} \frac{1}{1 - \frac{|b|}{a}} \\ &= \max_{i=1,\dots,N} \sup_t |d_i(t)| \frac{|c|}{a(a - |b|)}. \end{aligned}$$

Plugging this inequality into the solution inequality, we obtain the final result

$$x_N(t) \leq \left(1 + \frac{|c|}{|b|}\right) \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \max_{i=1,\dots,N} \sup_t |d_i(t)| \frac{a + |c|}{a(a - |b|)},$$

which holds for all $N \in \mathbb{N}$, hence also

$$\max_{i=1,\dots,N} |x_i(t)| \leq \left(1 + \frac{|c|}{|b|}\right) \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-|b|)t} + \max_{i=1,\dots,N} \sup_t |d_i(t)| \frac{a + |c|}{a(a - |b|)},$$

for all $N \in \mathbb{N}$. Set $K = 1 + \frac{|c|}{|b|}$, $\mu = a - |b|$ and $\gamma = \frac{a+|c|}{a(a-|b|)}$. Since $|b| < a$ by assumption, $\gamma > 0$ and $\mu > 0$ holds. This proves the theorem. □

Note that setting $c = 0$ yields precisely the result of Theorem 3.5. Similar to the interpretation of the result in Theorem 3.5, the optimal choice for $|b|$ and $|c|$ does not change now that there is a disturbance present in the system dynamics. Theorem 2.8 told us to choose $|b|$ and $|c|$ to be as small as possible, in order to minimize K and maximize μ. In this case, just like in Theorem 3.5, this will also minimize γ. Hence, the optimal choice for $|b|$ and $|c|$ remains unchanged.

4 Two Predecessor Following Problems

In the previous sections, necessary and sufficient conditions for string stability and disturbance string stability of networks adhering to the PF and PLF topologies were found. Focusing on the PF topology, recall that each system receives an input from strictly its predecessor. A natural extension of the PF problem is to generalize the predecessor following problem to an r predecessor following (rPF) problem. In this generalization, each system receives inputs from its r predecessors. In this case, the i'th system can be modeled by

$$\Sigma_i : \dot{x}_i(t) = -ax_i(t) + Bu_i(t),$$

where $a > 0$, $B \in \mathbb{R}^{1\times r}$ and $u_i = \begin{bmatrix} x_{i-1}(t) & x_{i-2}(t) & \dots & x_{i-r}(t) \end{bmatrix}^\top$. Note that if $r = 1$, then the rPF problem reduces to that of the PF problem, which has been solved in Theorem 2.5. In this section, necessary and sufficient conditions will be found for string stability of rPF networks for the case $r = 2$ (see Figure 3 (f)). Explicitly, this means we will initially be focusing on networks of systems of the form

$$\begin{aligned} &\Sigma_1 : \dot{x}_1(t) = -ax_1(t), \\ &\Sigma_2 : \dot{x}_2(t) = -ax_2(t) + bx_1(t), \\ &\Sigma_j : \dot{x}_j(t) = -ax_j(t) + bx_{j-1}(t) + cx_{j-2}(t), \quad j = 3, \dots, N, \end{aligned} \tag{14}$$

where $a, b, c \in \mathbb{R}$ with $a > 0$. Finding conditions for string stability of this network will prove to be increasingly more complicated than the case $r = 1$. After solving the 2PF problem, the result will be extended to the 2PLF problem, in analogy with extending the PF problem to the PLF problem. At the end of this section, sufficient conditions for string stability of rPF networks will be found for arbitrary r.

4.1 Positive Systems

When finding conditions for string stability of an rPF network for the case $r = 1$, we made use of the solution of each system as given by Lemma 2.3. Unfortunately, for rPF networks with $r \geq 2$ predecessors, this approach becomes considerably more complex, making the approach unfeasible. Therefore, a different method will have to be employed. In this new method, we can bypass having to use the solution of each system in the first place and skip straight to the exponential bound, as long as certain conditions are satisfied. The most important condition is that we require that the network of systems (14) is a positive system. In order to define positivity, we first need to define *nonnegativity* of a vector.

Definition 4.1. *A vector $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$ is said to be nonnegative if $x_i \geq 0$ for all $i = 1, \dots, n$. If, in addition, x is time-dependent function, i.e. $x = x(t)$ with $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}^\top$, then x is nonnegative if $x_i(t) \geq 0$ for any $t \geq 0$ and all $i = 1, \dots, n$.*

We can now define positivity of a system. The following definition is a slight modification of the one given in [5].

Definition 4.2. *Consider the linear system*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{15}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$. The linear system (15) is said to be positive if, for any nonnegative initial state x_0 and any nonnegative input $u(t)$, its state trajectory $x(t)$ is nonnegative for all $t \geq 0$.

In other words, once the state trajectory $x(t)$ is nonnegative at some $t = t_*$ it will stay nonnegative for all $t \geq t_*$. Now that we have defined positivity of a network of systems, we need to be able to check whether our network is positive. After all, the approach that we will be using will not work when the network is not positive. In order to do that, the following theorem will be necessary.

Theorem 4.3. *Consider the linear system*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{16}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$. Then, (16) *is positive if and only if*

1. $b_{i,j} \geq 0$ *for all* $i = 1, \dots, n$ *and for all* $j = 1, \dots, r$.
2. $a_{i,j} \geq 0$ *for all* $i \neq j$.

Proof. The proof can be found in [5, pages 14-15]. □

4.2 2PF Problem

Now that we have some preliminary knowledge on positive systems, we can focus our attention on the main method of solving the 2PF problem. Recall that this means we will find conditions for string stability of the network of systems (14), which can be written in terms of a single, autonomous system as

$$\Sigma : \dot{x}(t) = \begin{bmatrix} -a & & & & & \\ b & -a & & & & \\ c & b & -a & & & \\ & \ddots & \ddots & \ddots & & \\ & & c & b & -a & \\ & & & c & b & -a \end{bmatrix} x(t), \tag{17}$$

where $a > 0$ and $b, c \geq 0$ are real numbers and $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_N(t) \end{bmatrix} \in \mathbb{R}^N$. Note that we now require that $b, c \geq 0$. As the network (17) is an autonomous system, meaning that the input term in Definition 4.2 is absent, we require that $a_{i,j} \geq 0$ for all $i \neq j$ as per Theorem 4.3. This amounts to picking b and c to be nonnegative. Note that this places a large restriction on the allowable values for b and c. For rPF networks of the case $r = 1$, Theorem 2.5 states that b is allowed to be negative, as long as $|b| < a$. Similarly, Theorem 3.5 allows for negative values of b as well. In this case, however, we are forced to restrict ourselves to the case where b and c are nonnegative instead. In exchange for this trade-off, we can make use of the following lemma, which will be crucial for the remainder of this section.

Lemma 4.4. *Consider the nonnegative functions $v_i : [0, \infty) \longrightarrow [0, \infty)$. Let $\mu > 0$ and $\sigma \in (0, 1)$ be real numbers. Define recursively $V_1(t) = v_1(t)$, $V_{n+1}(t) = v_{n+1}(t) + \sigma V_n(t)$ for all $n \geq 1$. If*

$$\dot{V}_k(t) \leq -\mu V_k(t),$$

for any $k \in \mathbb{N}$, then

$$\max_{i=1,\dots,k} |v_i(t)| \leq \frac{1}{1-\sigma} e^{-\mu t} \max_{i=1\dots,k} |v_{i,0}|,$$

for all initial conditions $v_i(0) = v_{i,0}$ and any $k \in \mathbb{N}$.

Proof. Denote $V_k(0) = V_{k,0}$. By assumption, we have for all $k \in \mathbb{N}$,

$$\dot{V}_k(t) \leq -\mu V_k(t) \implies V_k(t) \leq e^{-\mu t} V_{k,0}.$$

Expanding both sides of the inequality, this yields

$$\begin{aligned} & V_k(t) \leq e^{-\mu t} V_{k,0} \\ \iff & v_k(t) + \sigma V_{k-1}(t) \leq e^{-\mu t} \left(v_{k,0} + \sigma V_{k-1,0} \right). \end{aligned}$$

Since $V_k(t)$ is nonnegative for any $k \in \mathbb{N}$ and all $t \geq 0$, due to the nonnegativity of each $v_i(t)$, the above inequality implies

$$\begin{aligned} & v_k(t) \leq e^{-\mu t} \left(v_{k,0} + \sigma V_{k-1,0} \right) \\ \iff & v_k(t) \leq e^{-\mu t} \left(\sum_{i=0}^{k-1} \sigma^i v_{k-i,0} \right) \\ \implies & v_k(t) \leq \max_{i=1,\dots,k} |v_{i,0}| e^{-\mu t} \left(\sum_{i=0}^{k-1} \sigma^i \right), \end{aligned}$$

where the summation term is the direct result of expanding $V_{k-1}(t)$ and collecting the $v_{k,0}$ term inside the summation. Note that the above series is the geometric series. Since $\sigma \in (0,1)$, the series converges. Moreover, since σ is positive, adding more terms will only make the right hand side larger. This implies

$$\begin{aligned} & v_k(t) \leq \max_{i=1,\dots,k} |v_{i,0}| e^{-\mu t} \left(\sum_{i=0}^{\infty} \sigma^i \right) \\ \iff & v_k(t) \leq \frac{1}{1-\sigma} e^{-\mu t} \max_{i=1,\dots,k} |v_{i,0}|, \end{aligned}$$

for all $k \in \mathbb{N}$, which means that also

$$\max_{i=1,\dots,k} |v_i(t)| \leq \frac{1}{1-\sigma} e^{-\mu t} \max_{i=1,\dots,k} |v_{i,0}|.$$

This proves the statement. □

If we are able to apply Lemma 4.4 to the functions $v_i(t) = x_i(t)$, then we will immediately have bounded each solution as per Definition 2.2. In other words, Lemma 4.4 allows us to find conditions for string stability without having to explicitly compute each solution. However, Lemma 4.4 does place an additional restriction on us. Not only do we require the network (14) to be positive, but we also require that $x_{i,0} \geq 0$ for all $i = 1, \dots, N$. This is an additional restriction that Definition 4.2 does not require.

In order to apply Lemma 4.4, we need to be able to show the inequality $\dot{V}_k(t) \leq -\mu V_k(t)$ holds for our functions $v_k(t) = x_k(t)$. The following lemma gives us a closed-form expression of $\dot{V}_k(t)$ for this choice of $v_k(t)$, which will be of great use in finding sufficient conditions for string stability.

Lemma 4.5. *Consider the positive 2PF network* (14) *and the functions $V_k(t)$ and $v_k(t)$ as defined in Lemma* 4.4. *Set $v_i(t) = x_i(t)$. Let $q_n(z)$ be the $n-$th degree polynomial defined by $q_n(z) = -az^n + bz^{n-1} + cz^{n-2}$. If $n - i < 0$, for $i = 1, 2$, set $z^{n-i}(t) \equiv 0$. Finally, let $\sigma \in (0,1)$ be a real number. Then,*

$$\dot{V}_k(t) = q_0(\sigma) x_k(t) + q_1(\sigma) x_{k-1}(t) + \sum_{i=0}^{k-3} \sigma^i q_2(\sigma) x_{k-2-i}(t). \tag{18}$$

Proof. We proceed by induction. Consider the base case $k = 1$. We have

$$\dot{V}_1(t) = \dot{v}_1(t) = -ax_1(t) = q_0(\sigma)x_1(t).$$

Hence, the base case holds. Assume the statement holds for arbitrary $k = n$. Applying the expression for $V_k(t)$ and plugging in the induction hypothesis, we compute (omitting time arguments)

$$\begin{aligned}\dot{V}_{n+1} &= \dot{v}_{n+1} + \sigma\dot{V}_{n+1} \\ &= -ax_{n+1} + bx_n + cx_{n-1} + \sigma\big(q_0(\sigma)x_n + q_1(\sigma)x_{n-1}\big) + \sum_{i=0}^{n-3}\sigma^{i+1}q_2(\sigma)x_{n-2-i}.\end{aligned}$$

Note that $\sigma q_0(\sigma) = q_1(\sigma) - b$ and $\sigma q_1(\sigma) = q_2(\sigma) - c$. Substituting this result and cancelling out terms yields

$$\begin{aligned}\dot{V}_{n+1} &= -ax_{n+1} + bx_n + cx_{n-1} + (q_1(\sigma) - b)x_n + (q_2(\sigma) - c)x_{n-1} + \sum_{i=0}^{n-3}\sigma^{i+1}q_2(\sigma)x_{n-2-i} \\ &= q_0(\sigma)x_{n+1} + q_1(\sigma)x_n + q_2(\sigma)x_{n-1} + \sum_{i=0}^{n-3}\sigma^{i+1}q_2(\sigma)x_{n-2-i} \\ &= q_0(\sigma)x_{n+1} + q_1(\sigma)x_n + \sum_{i=0}^{n-2}\sigma^{i}q_2(\sigma)x_{n-1-i}.\end{aligned}$$

Hence, the result holds for $k = n + 1$. The statement now follows by induction. □

We now have enough information to find sufficient conditions for string stability of the 2PF network. This leads us to the following theorem.

Theorem 4.6 (Sufficiency of the 2PF problem)**.** *Consider the positive 2PF network* (14)*. Let $\mu > 0$ be a real number. Define the second-degree polynomial $p(z)$ by*

$$p(z) = (\mu - a)z^2 + bz + c.$$

If there exists a real number $\sigma \in (0, 1)$ such that $p(\sigma) \leq 0$, then there exists a real number $K > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,

$$\max_{i=1,\ldots,N} |x_i(t)| \leq Ke^{-\mu t} \max_{i=1,\ldots,N} |x_{i,0}|.$$

Proof. By assumption, there exists $\sigma \in (0, 1)$ such that $p(\sigma) \leq 0$. In order to prove the statement, we first need the following three inequalities. Firstly, by assumption we have

$$\begin{aligned}&(\mu - a)\sigma^2 + b\sigma + c \leq 0 \\ \iff\ & c + \sigma b - \sigma^2 a \leq -\sigma^2\mu.\end{aligned} \tag{19}$$

Secondly, since $c \geq 0$, it also holds that

$$(\mu - a)\sigma^2 + b\sigma \leq 0 \iff b - \sigma a \leq -\sigma\mu, \tag{20}$$

which can be obtained by dividing both sides by σ and moving terms around. Finally, since $b \geq 0$, this also means that

$$-\sigma a \leq -\sigma\mu \iff -a \leq -\mu. \tag{21}$$

Next, define $V_N(t)$ and $v_N(t)$ as in Lemma 4.4, where $N \in \mathbb{N}$. Setting $v_N(t) = x_N(t)$, it follows by Lemma 4.5 that

$$\dot{V}_N(t) = -ax_N(t) + (b - \sigma a)x_{N-1}(t) + \sum_{i=0}^{N-3} \sigma^i (c + \sigma b - \sigma^2 a) x_{N-2-i}(t),$$

for all $N \in \mathbb{N}$. Using the inequalities (19), (20) and (21) and recalling that $x_N(t) \geq 0$ for all $t \geq 0$ and all $N \in \mathbb{N}$, this implies

$$\dot{V}_N(t) \leq -\mu x_N(t) - \mu\sigma x_{N-1}(t) - \mu \sum_{i=0}^{N-3} \sigma^{i+2} x_{N-2-i}(t).$$

Collecting all the terms into one sum, the above inequality can be expressed as

$$\begin{aligned}\dot{V}_N(t) &\leq -\mu \sum_{i=0}^{N-1} \sigma^i x_{N-i}(t) \\ &= -\mu V_N(t),\end{aligned}$$

which holds for all $N \in \mathbb{N}$. By Lemma 4.4, it then immediately follows that

$$\max_{i=1,\dots,N} |x_i(t)| \leq \frac{1}{1-\sigma} e^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|,$$

for all $N \in \mathbb{N}$. Setting $K = \frac{1}{1-\sigma}$ gives the result. □

Now that sufficient conditions for string stability of the homogeneous 2PF network (14) have been found, a natural question to ask is whether this same method can be extended to the heterogeneous case. Fortunately, varying the parameters in each system does not complicate matters much. This result is given by the following corollary.

Corollary 4.7. *Consider the positive, heterogeneous 2PF network given by*

$$\begin{aligned}\Sigma_1 &: \dot{x}_1(t) = -a_1 x_1(t), \\ \Sigma_2 &: \dot{x}_2(t) = -a_2 x_2(t) + b_1 x_1(t), \\ \Sigma_j &: \dot{x}_j(t) = -a_j x_j(t) + b_{j-1} x_{j-1}(t) + c_{j-2} x_{j-2}(t), \quad j = 3, \dots, N,\end{aligned}$$

where $a_i > 0$ for all $i = 1, \dots, N$, $b_k \geq 0$ for all $k = 1, \dots, N-1$ and $c_m \geq 0$ for all $m = 1, \dots, N-2$ are real numbers. Let $p(z)$ be the second-degree polynomial defined by

$$p(z) = \left(\mu - \min_i |a_i|\right) z^2 + \max_k |b_k| z + \max_m |c_m|,$$

where $\mu > 0$ is a real number. If there exists a real number $\sigma \in (0,1)$ such that $p(\sigma) \leq 0$, then there exists a real number $K > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,

$$\max_{i=1,\dots,N} |x_i(t)| \leq K e^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|.$$

Proof. Analogous to Lemma 4.5, it can be shown that

$$\dot{V}_N(t) = -a_N x_N(t) + (b_{N-1} - \sigma a_{N-1}) x_{N-1}(t) + \sum_{l=0}^{N-3} \sigma^l (c_{N-2-l} + \sigma b_{N-2-l} + \sigma^2 a_{N-2-l}) x_{N-2-l}(t),$$

for all $N \in \mathbb{N}$, which implies

$$\begin{aligned}\dot{V}_N(t) &\leq -\min_i |a_i| x_N(t) + \left(\max_k |b_k| - \sigma \min_i |a_i|\right) x_{N-1}(t) \\ &+ \sum_{l=0}^{N-3} \sigma^l \left(\max_m |c_m| + \sigma \max_k |b_k| - \sigma^2 \min_i |a_i|\right) x_{N-2-l}(t),\end{aligned}$$

which, by assumption, then means that

$$\begin{aligned}\dot{V}_N(t) &\leq -\mu x_N(t) - \sigma\mu x_{N-1}(t) - \mu \sum_{l=0}^{N-3} \sigma^{l+2} x_{N-2-l}(t) \\ &= -\mu \sum_{l=0}^{N-1} \sigma^l x_{N-l}(t) \\ &= -\mu V_N(t).\end{aligned}$$

By Lemma 4.4, the result immediately follows, with $K = \frac{1}{1-\sigma}$. □

In order to find necessary conditions for string stability of homogeneous 2PF networks, we will be taking inspiration from Theorem 2.5. Recall the interpretation of the condition $|b| < a$ as per Theorem 2.5. Namely, when a sudden disturbance affects the leader vehicle, the effect of this disturbance propagates through the string of vehicles. Then, $|b| < a$ tells us that each vehicle provides a smaller disturbance (i.e. $|b|$) to its successor than what the vehicle itself is experiencing (which is a). However, in this case there are not one, but two inputs, namely $b \geq 0$ and $c \geq 0$. Analogous to the PF problem, one could then conjecture that a 2PF platoon is string stable if and only if $b + c < a$. This can similarly be interpreted as each vehicle receiving a smaller total sum of disturbances, $b + c$, than what the previous vehicle is experiencing, namely a. The following lemma will tell us how the polynomial defined in Theorem 4.6 partially confirms that this conjecture might be right.

Lemma 4.8. *Let $\mu > 0$ be a real number. Define the second-degree polynomial $p(z)$ by*

$$p(z) = (\mu - a)z^2 + bz + c,$$

where $a > 0$ and $b, c \geq 0$ are real numbers. Then, $b + c < a$ if and only if there exists a real number $\sigma \in (0, 1)$ such that $p(\sigma) \leq 0$.

Proof. ($\Longrightarrow$) Assume $b + c < a$. Then, there exists $\sigma \in (0, 1)$ such that $b + c = \sigma^2 a$. Moreover, $\sigma \in (0, 1)$ also implies that $\sigma b + c < b + c = \sigma^2 a$. Moving all terms to the left hand side, this is equivalent to saying that $-\sigma^2 a + \sigma b + c < 0$. Since the inequality is strict, there exists some $\mu > 0$ such that $(\mu - a)\sigma^2 + b\sigma + c \leq 0$ i.e. $p(\sigma) \leq 0$.
($\Longleftarrow$) Assume there exists $\sigma \in (0, 1)$ such that $p(z) \leq 0$. Then,

$$\begin{aligned}&(\mu - a)\sigma^2 + b\sigma + c \leq 0 \\ \Longrightarrow\ &-a\sigma^2 + b\sigma + c < 0 \\ \Longleftrightarrow\ &c + \sigma(b - a\sigma) < 0.\end{aligned}$$

Note that since $c \geq 0$, we must have that $\sigma(b - a\sigma) < 0$. Otherwise, the above inequality cannot hold. Since $\sigma \in (0, 1)$, this must also mean that $c + b - a\sigma < 0$ i.e. $c + b < \sigma a$. But since $\sigma a < a$, then we also have $c + b < a$. □

It is now easy to see that $c+b < a$ implies string stability of 2PF platoons. Namely, by Lemma 4.8, there exists $\sigma \in (0,1)$ such that that

$$p(\sigma) = (\mu - a)\sigma^2 + bz + c \leq 0.$$

The result then immediately follows from Theorem 4.6.

Before we continue with the 2PF problem, a quick remark needs to be made about Corollary 4.7. In the PF problem we were able to find sufficient conditions for heterogeneous networks with identical state parameters $-a$, which were given by Corollary 2.6. Note that this obstacle can now be overcome by Corollary 4.7. By setting $c_1 = c_2 = \ldots = c_{N-2} = 0$, Corollary 4.7 also gives us sufficient conditions for fully heterogeneous PF networks i.e. for networks where we allow a and b to vary per system. Namely, by Lemma 4.8 and Corollary 4.7, a fully heterogeneous PF network is string stable if $\max |b_k| \leq \min |a_j|$. If each state parameter $-a_j$ were the same, i.e. $-a_j = -a$, then we would indeed obtain Corollary 2.6 again. Note, however, that in this case we are dealing with a positive PF network. Hence, this only proves the case where the heterogeneous PF network has nonnegative input parameters, whereas the heterogeneous PF network in Corollary 2.6 does allow for negative input parameters as well.

Returning to the homogeneous 2PF problem, the inconvenience that comes with using the method employed so far is that it does not tell us how to pick an explicit value for μ and σ. Instead, assuming $b + c < a$ merely guarantees the existence of a $\mu > 0$ and $\sigma \in (0,1)$ such that $p(\sigma) \leq 0$. This is unfortunate, as we cannot determine an explicit bound and, hence, cannot say anything about the stability behavior of the solutions. Naturally, it would be nice if K was small and μ was big, such that the exponential bound converges to the origin fast, but this method does not tell us how to design each system in the network such that the choice for K and μ is optimized for string stability. In particular, if σ is close to one, then the term $K = \frac{1}{1-\sigma}$ blows up, which is unfavorable as the entire bound then blows up. Similarly, if μ is small, then the bound converges to the origin in a very slow manner, which is something we want to avoid as well. This was a problem that did not occur in Theorem 2.5 and Theorem 2.8, as the methods used there allowed us to find an explicit expression for the exponential bound.

Going back to the reasoning from Theorem 2.5, the PF problem made use of the closed-form solution given by Lemma 2.4. In particular, if we set $x_{i,0} = 1$ for all $i = 1, \ldots, N$, then by Lemma 2.3 the solution $x_N(t)$ in (4) converges to $e^{-(a-b)t}$ as $N \longrightarrow \infty$. The requirement $|b| < a$ is then necessary to ensure that the exponent in $e^{-(a-|b|)t}$ is negative. Combining this idea with Lemma 4.8, we may be able to say something about the solutions of the systems in (14) after all, which we will see is required to find necessary conditions for string stability of 2PF networks. This will also help us overcome the obstacle that Theorem 4.6 poses. Namely, having some explicit knowledge about the solution will allow us to determine how to pick the exponential bound efficiently, which is something that Theorem 4.6 could not do. However, before we can do that, we need to state an important property of the systems in (14).

Lemma 4.9. *Consider the positive 2PF network* (14) *with initial conditions* $x_i(0) = x_{i,0}$. *Set* $x_{i,0} = 1$ *for all* $i = 1, \ldots, N$. *Then,* $x_{k+1}(t) \geq x_k(t)$ *for all* $t \geq 0$ *and all* $k = 1, \ldots, N-1$.

Proof. Define the difference function $\epsilon_k(t) = x_{k+1}(t) - x_k(t)$. We claim that $\epsilon_k(t) \geq 0$ for all $k = 1, \ldots, N-1$ and all $t \geq 0$. This is then equivalent to showing that $x_{k+1}(t) \geq x_k(t)$ for all $t \geq 0$. The claim will be proven through induction. First, consider the base case. As each solution $x_j(t)$ depends on $x_{j-1}(t)$ and $x_{j-2}(t)$, the base case will consider the cases $k = 1$ and $k = 2$. Manually computing the solutions $x_1(t)$, $x_2(t)$ and $x_3(t)$, it can be shown that

$$\epsilon_1(t) = be^{-at}t,$$

and

$$\epsilon_2(t) = e^{-at}\left(ct + \frac{b^2}{2}t^2\right),$$

which are both nonnegative for all $t \geq 0$. Hence, the base case holds. Assume now that the statement holds for arbitrary $k = n$, such that it also holds for $k = n-1$. To show now is that the statement holds for $k = n+1$. We compute:

$$\begin{aligned}\epsilon_{n+1}(t) &= x_{n+2}(t) - x_{n+1}(t) \\ &= e^{-at}\left(\int_0^t e^{a\tau}(bx_{n+1}(\tau) + cx_n(\tau) - bx_n(\tau) - cx_{n-1}(\tau))d\tau\right).\end{aligned}$$

Grouping together the b terms and c terms, we recognize the difference functions $\epsilon_n(t)$ and $\epsilon_{n-1}(t)$. Substituting this result yields

$$\epsilon_{n+1}(t) = e^{-at}\left(\int_0^t e^{a\tau}(b\epsilon_n(\tau) + c\epsilon_{n-1}(\tau))d\tau\right).$$

Since $\epsilon_n(t) \geq 0$ and $\epsilon_{n-1}(t) \geq 0$ for all $t \geq 0$ by the induction hypothesis, each term in the integrand is nonnegative. This means the above integral will be nonnegative and therefore $\epsilon_{n+1}(t) \geq 0$ for all $t \geq 0$. This shows the result. □

Since, for all $x_{i,0} = 1$, the PF solution $x_N(t)$ in (4) converges to e^{bt} as $N \longrightarrow \infty$, we would like to show a similar result for the solutions of the systems in (14). As mentioned before, finding a closed-form expression of each solution is no longer viable. However, we are still able to do the next best thing, which is determining the limit the solutions of the systems convergence to. After all, the only reason we needed the closed-form expression (4) of the solutions in the PF problem was so that we could turn it into a Taylor series and find its limit as $N \longrightarrow \infty$. The following lemma will be an important tool in determining this as of yet unknown limit.

Lemma 4.10. *Consider the positive 2PF network* (14) *with initial conditions $x_i(0) = x_{i,0}$. Assume $x_{i,0} = 1$ and consider the transformation $z_N(t) = e^{at}x_N(t)$ for all $N \in \mathbb{N}$. Let $k \geq 0$ be even i.e. $k = 2m$ for some $m \in \mathbb{N}$. Then,*

$$\begin{aligned}\sum_{i=0}^{m-1}\frac{(b+c)^i t^i}{i!} &\leq z_{k-1}(t) \leq \sum_{i=0}^{k-1}\frac{(b+c)^i t^i}{i!}, \\ \sum_{i=0}^{m-1}\frac{(b+c)^i t^i}{i!} &\leq z_k(t) \leq \sum_{i=0}^{k-1}\frac{(b+c)^i t^i}{i!},\end{aligned}$$

for all $m \in \mathbb{N}$. In other words, each pair of solutions $z_{k-1}(t)$ and $z_k(t)$ can be bounded below and above by the same bound.

Proof. The statement will be proven by induction. Consider the base case $m = 1$. Then $k = 2$, so we compute $z_1(t)$ and $z_2(t)$. The two solutions are given by $z_1(t) = 1$ and $z_2(t) = 1 + bt$. Since $b, c \geq 0$ these two solutions satisfy

$$1 \leq z_j(t) \leq 1 + (b+c)t,$$

for $j = 1, 2$. Hence, the base case holds. Assume now that the statement holds for arbitrary $m = n$. To show is that the statement holds for $m = n + 1$. This means we have to show

$$\sum_{i=0}^{n} \frac{(b+c)^i t^i}{i!} \leq z_{k+1}(t) \leq \sum_{i=0}^{k+1} \frac{(b+c)^i t^i}{i!},$$
$$\sum_{i=0}^{n} \frac{(b+c)^i t^i}{i!} \leq z_{k+2}(t) \leq \sum_{i=0}^{k+1} \frac{(b+c)^i t^i}{i!}.$$

We first show that the lower bound holds for both solutions. The solution $z_{k+1}(t)$ is given by

$$z_{k+1}(t) = 1 + \int_0^t b z_k(\tau) + c z_{k-1}(\tau) d\tau.$$

Next, we can apply Lemma 4.9. Namely, since $x_k(t) \geq x_{k-1}(t)$, then also $z_k(t) \geq z_{k-1}(t)$. Plugging this into the above yields

$$z_{k+1}(t) \geq 1 + \int_0^t (b+c) z_{k-1}(\tau) d\tau.$$

The next step is to plug in the induction hypothesis $z_{k-1}(t) \geq \sum_{i=0}^{n-1} \frac{(b+c)^i t^i}{i!}$ which gives us

$$z_{k+1}(t) \geq 1 + \int_0^t \sum_{i=0}^{n-1} \frac{(b+c)^{i+1} \tau^i}{i!} d\tau.$$

After integrating, collecting all terms into the summation and rearranging indices, this becomes

$$z_{k+1}(t) \geq 1 + \sum_{i=0}^{n-1} \frac{(b+c)^{i+1} t^{i+1}}{(i+1)!}$$
$$= \sum_{i=0}^{n} \frac{(b+c)^i t^i}{i!}.$$

Finally, applying Lemma 4.9 again, we have $z_{k+2}(t) \geq z_{k+1}(t) \geq \sum_{i=0}^{n} \frac{(b+c)^i t^i}{i!}$. This shows that the lower bound holds for both solutions $z_{k+1}(t)$ and $z_{k+2}(t)$. To show that the upper bound holds, we proceed analogously. The solution $z_{k+2}(t)$ is given by

$$z_{k+2}(t) = 1 + \int_0^t b z_{k+1}(\tau) + c z_k(\tau) d\tau.$$

By applying Lemma 4.9, we have $z_k(t) \leq z_{k+1}(t)$. Applying this inequality to the expression above and writing out the solution $z_{k+1}(t)$ explicitly, we can bound $z_{k+2}(t)$ as

$$\begin{aligned} z_{k+2}(t) &\leq 1 + \int_0^t (b+c) z_{k+1}(\tau) d\tau \\ &= 1 + \int_0^t (b+c) \left(1 + \int_0^\tau b z_k(\hat{\tau}) + c z_{k-1}(\hat{\tau}) d\hat{\tau} \right) d\tau \\ &\leq 1 + \int_0^t (b+c) \left(1 + \int_0^\tau (b+c) z_{k-1}(\hat{\tau}) d\hat{\tau} \right) d\tau, \end{aligned}$$

where in the last inequality we applied Lemma 4.9 again in the second integrand. Next, we can use the induction hypothesis $z_{k-1}(t) \leq \sum_{i=0}^{k-1} \frac{(b+c)^i t^i}{i!}$. Plugging in the hypothesis into the above and computing the first integral yields

$$z_{k+2}(t) \leq 1 + \int_0^t (b+c) + \sum_{i=0}^{k-1} \frac{(b+c)^{i+2} \tau^{i+1}}{(i+1)!} d\tau,$$

which, after integrating again, gives us

$$z_{k+2}(t) \leq 1 + (b+c)t + \sum_{i=0}^{k-1} \frac{(b+c)^{i+2} t^{i+2}}{(i+2)!}.$$

Finally, collecting all terms into one summation and rearranging index gives us the final result

$$z_{k+2}(t) \leq \sum_{i=0}^{k+1} \frac{(b+c)^i t^i}{i!}.$$

Moreover, since by Lemma 4.9 we have $z_{k+1}(t) \leq z_{k+2}(t)$, the result holds for both solutions. This shows the result holds for $m = n+1$. Since $m = n$ was arbitrary, the result holds for all $m \in \mathbb{N}$. This proves the statement. □

Note that the upper and lower bound in Lemma 4.10 are both the truncated Taylor series of the exponential function $e^{(b+c)t}$ centered at $t = 0$. By applying the squeeze theorem to Lemma 4.10, it can immediately be determined what $z_k(t)$ converges to as $k \longrightarrow \infty$. Namely, as $k \longrightarrow \infty$ then so too does $m \longrightarrow \infty$ which means

$$\sum_{i=0}^{\infty} \frac{(b+c)^i t^i}{i!} \leq \lim_{k \to \infty} z_k(t) \leq \sum_{i=0}^{\infty} \frac{(b+c)^i t^i}{i!}.$$

Hence, by the squeeze theorem,

$$\lim_{k \to \infty} z_k(t) = e^{(b+c)t},$$

and since $x_k(t) = e^{-at} z_k(t)$, the solution $x_k(t)$ converges to $e^{(-a+b+c)t}$ as $k \longrightarrow \infty$. It is important to note that this only holds for the case where all initial conditions are set equal to one. Fortunately, this is all we require to know to find necessary and sufficient conditions for string stability of 2PF networks. Recall that Theorem 4.6 already provides us with sufficient conditions for string stability. Unfortunately, as mentioned earlier, this theorem is lackluster in the sense that it merely guarantees the existence of an exponential bound. It does not provide us with enough knowledge to design each system in the network optimally to optimize the exponential bound and consequently string stability. In the following theorem, this sufficiency condition will be proven again with the use of Lemma 4.10. This time, analogous to Theorem 2.5, we will be able to explicitly determine how to pick $K, \mu > 0$. In turn, this will allow us to optimally design each system in the 2PF network to optimize string stability. In addition to sufficiency, necessity will now also be proven.

Theorem 4.11. *Consider the positive 2PF network* (14). *Let $\hat{x}_n(t)$ be the solution $x_n(t)$ with all initial conditions set equal to one. Define the transformation $z_N(t) = e^{at} \hat{x}_N(t)$ for all $N \in \mathbb{N}$. Then, there exists real numbers $K, \mu > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,*

$$\max_{i=1,\dots,N} |x_i(t)| \leq K e^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|,$$

if and only if $b + c < a$.

Proof. ($\Longleftarrow$) Let $N \in \mathbb{N}$ be arbitrary. Then, the solution $x_N(t)$ is given by

$$x_N(t) = e^{-at}\left(x_{N,0} + \int_0^t e^{a\tau}(bx_{N-1}(\tau) + cx_{N-2}(\tau))d\tau\right).$$

Since each solution depends on one or more initial condition, we can replace these initial conditions with the maximum in absolute value over all initial conditions and pull this maximum term outside the solution. In turn, all the initial conditions inside the parentheses will now be set equal to one. This means the above can be bounded by

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N} |x_{i,0}| e^{-at}\left(1 + \int_0^t e^{a\tau}(b\hat{x}_{N-1}(\tau) + c\hat{x}_{N-2}(\tau))d\tau\right) \\ &= \max_{i=1,\dots,N} |x_{i,0}|\hat{x}_N(t). \end{aligned}$$

Note that $\hat{x}_N(t) = e^{-at}z_N(t)$ since $z_N(t) = e^{at}\hat{x}_N(t)$ by definition. Moreover, since N is arbitrary, N is either even or odd. In particular, there exists $m \in \mathbb{N}$ such that $N = 2m$ or $N = 2m-1$. Substituting $\hat{x}_N(t) = e^{-at}z_N(t)$ and applying Lemma 4.10, the solution $x_N(t)$ can be bounded by

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N} |x_{i,0}| e^{-at} z_N(t) \\ &\leq \max_{i=1,\dots,N} |x_{i,0}| e^{-at}\left(\sum_{i=0}^{2m-1} \frac{(b+c)^i t^i}{i!}\right). \end{aligned}$$

Note that since $b, c \geq 0$, each term in the series is positive. This means adding more terms to this series will only make it larger. In particular, the above inequality also holds when $m \longrightarrow \infty$, but then the series converges to $e^{(b+c)t}$. Substituting this result gives us

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N} |x_{i,0}| e^{-at} e^{(b+c)t} \\ &= \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-b-c)t}, \end{aligned}$$

which holds for all $N \in \mathbb{N}$ and any initial condition $x_{i,0}$. Hence, we also have

$$\max_{i=1,\dots,N} |x_i(t)| \leq \max_{i=1,\dots,N} |x_{i,0}| e^{-(a-b-c)t},$$

for all $N \in \mathbb{N}$. Set $K = 1$ and $\mu = a - b - c$. Since $b + c < a$ by assumption, we have that $\mu > 0$. Hence, the network is string stable.

($\Longrightarrow$) Assume that there exists $K, \mu > 0$ such that

$$\max_{i=1,\dots,N} |x_i(t)| \leq K e^{-\mu t} \max_{i=1,\dots,N} |x_{i,0}|,$$

for all $N \in \mathbb{N}$ and all initial conditions $x_{i,0}$. Set $x_{i,0} = 1$ for all $i = 1, \dots, N$. Then, by assumption, the solution satisfies

$$x_N(t) \leq K e^{-\mu t},$$

for all $N \in \mathbb{N}$. By applying the squeeze theorem to Lemma 4.10, $x_N(t)$ converges to $e^{(-a+b+c)t}$ as $N \longrightarrow \infty$. Analogous to Lemma 2.4, this then implies

$$e^{(-a+b+c)t} \leq K e^{-\mu t}.$$

Assume now that $b + c \geq a$ for contradiction. Then, $e^{-(b+c)t} \leq e^{-at}$, which implies

$$1 \leq Ke^{-\mu t},$$

which holds for all $t \geq 0$. Since $\mu > 0$, the right hand side decays to the origin. This means that there exists $t^* \geq 0$ such that $1 > Ke^{-\mu t^*}$, which is a contradiction as $1 \leq Ke^{-\mu t}$ for all $t \geq 0$ by assumption. Hence, we have $b + c < a$. This shows the result. □

Note that Theorem 4.11 gives us several advantages that Theorem 4.6 does not provide us. Aside from the obvious necessary condition, we can now explicitly determine how to design each system in the network to optimize string stability. In addition, Theorem 4.11 gives us a better K value than Theorem 4.6. Namely, according to Theorem 4.6, $K = \frac{1}{1-\sigma}$ for some $\sigma \in (0, 1)$. However, this means that no matter the value of σ, we would always have $K > 1$. On the other hand, Theorem 4.11 tells us that by choosing $\mu = a - b - c$, we can set $K = 1$. Analogous to Theorem 2.5, we wish to make $\mu = a - b - c$ as large as possible to optimize string stability. This can be achieved by choosing both b and c to be as close to zero as possible, such that μ is close to $a > 0$. Having solved the 2PF problem in its entirety now, we wish to focus our attention on the 2PLF problem.

4.3 2PLF Problem

The 2PLF problem is an extension of the PLF problem, whose results can be found in Theorem 2.8. We wish to study the network given by the systems

$$\begin{aligned}
&\Sigma_1 : \dot{x}_1(t) = -ax_1(t), \\
&\Sigma_2 : \dot{x}_2(t) = -ax_2(t) + bx_1(t), \\
&\Sigma_3 : \dot{x}_3(t) = -ax_3(t) + bx_2(t) + cx_1(t), \\
&\Sigma_i : \dot{x}_i(t) = -ax_i(t) + bx_{i-1}(t) + cx_{i-2}(t) + px_1(t), \quad i = 4, \dots, N,
\end{aligned} \tag{22}$$

where $a > 0$ and $b, c, p \in \mathbb{R}$. Just like in the 2PF problem, it is no longer viable to find a closed-form expression of the solution of each system in the above network. Fortunately, analogous to the 2PF problem, restricting ourselves to the case where the 2PLF network (22) is a positive system will make things considerably easier. The network (22) can be written as a single autonomous system as

$$\Sigma : \dot{x}(t) = \begin{bmatrix} -a & & & & & & \\ b & -a & & & & & \\ c & b & -a & & & & \\ p & c & b & -a & & & \\ p & & c & b & -a & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ p & & & & c & b & -a \end{bmatrix} x(t), \tag{23}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_N(t) \end{bmatrix} \in \mathbb{R}^N$. To ensure that (23) is a positive system, we require $b, c, p \geq 0$ by Theorem 4.3. Hence, when solving the 2PLF problem in this section, we will restrict ourselves to the case where $b, c, p \geq 0$.

The key difference in solving the 2PLF problem, is that in the 2PF problem positivity of the 2PF network was used so Lemma 4.4 could be applied. In this particular problem, it seems that Lemma 4.4 cannot be of much help. Fortunately, we can still employ similar methods to those in Theorem 4.11 to find necessary and sufficient conditions for string stability. As we will see, we will only need to bound the transformed solution $z_k(t) = e^{at}x_k(t)$ from above to find necessary and sufficient conditions for string stability of the 2PLF network (22). This is in contrast to the 2PF problem, where we explicitly had to find the limit for $z_k(t)$ as $k \longrightarrow \infty$ through employing the squeeze theorem. As we will see, we can use Lemma 4.10 for this problem as well. This simplifies the problem significantly. Moreover, we will also see that extending the 2PF problem to the 2PLF problem will be very similar to extending the PF problem to the PLF problem. As a first step to solving the 2PLF problem, the following lemma tells us how to bound the transformed solution $z_k(t)$ from above.

Lemma 4.12. *Consider the positive 2PLF network* (22) *with initial conditions $x_i(0) = x_{i,0}$. Assume $x_{i,0} = 1$ and consider the transformation $z_N(t) = e^{at}x_N(t)$ for all $N \in \mathbb{N}$. Let $k \geq 0$ be even i.e. $k = 2m$ where $m \in \mathbb{N}$. Then,*

$$z_{k-1}(t) \leq \sum_{i=0}^{k-1} \frac{(b+c)^i t^i}{i!},$$
$$z_k(t) \leq \sum_{i=0}^{k-1} \frac{(b+c)^i t^i}{i!},$$

for $m = 1$. For $m \geq 2$, each pair of solutions satisfies

$$z_{k-1}(t) \leq \sum_{i=0}^{k-1} \frac{(b+c)^i t^i}{i!} + \sum_{j=1}^{k-3} p\frac{(b+c)^{j-1} t^j}{j!},$$
$$z_k(t) \leq \sum_{i=0}^{k-1} \frac{(b+c)^i t^i}{i!} + \sum_{j=1}^{k-3} p\frac{(b+c)^{j-1} t^j}{j!}.$$

Proof. For the case $m = 1$ the problem reduces to that of Lemma 4.10. Hence, the result will be shown for $m \geq 2$. The proof is analogous to Lemma 4.10. We proceed by induction. First, consider the base case $m = 2$. Then, this means $k = 4$. Explicitly computing the solutions $z_3(t)$ and $z_4(t)$, we find $z_3(t) = 1 + (b+c)t + \frac{b^2}{2}t^2$ and $z_4(t) = 1 + (b+c)t + \left(\frac{b^2}{2} + bc\right)t^2 + \frac{b^3}{6}t^3 + pt$. This means that

$$z_l(t) \leq \sum_{i=0}^{3} \frac{(b+c)^i t^i}{i!} + \sum_{j=1}^{1} p\frac{(b+c)^{j-1} t^j}{j!},$$

for $l = 3, 4$. Hence, the base case holds. Assume that the statement holds for arbitrary $m = n \geq 2$. Analogous to Lemma 4.9, it can be shown that $z_{k+1}(t) \geq z_k(t)$ for all $k \in \mathbb{N}$ and for all $t \geq 0$. Computing the solution $z_{k+2}(t)$ and plugging in this observation, we obtain

$$\begin{aligned} z_{k+2}(t) &= 1 + \int_0^t b x_{k+1}(\tau) + c z_k(\tau) + p z_1(\tau) d\tau \\ &\leq 1 + pt + \int_0^t (b+c) z_{k+1}(\tau) d\tau. \end{aligned}$$

Next, we can plug in the formula for $z_{k+1}(t)$ and use the fact that $z_{k-1}(t) \leq z_k(t)$. After some computations, this gives us

$$\begin{aligned} z_{k+2}(t) &\leq 1 + pt + \int_0^t (b+c)\left(1 + \int_0^\tau (b+c) x_k(\hat{\tau}) d\hat{\tau} + p\tau\right) d\tau \\ &= 1 + pt + \int_0^t \left((b+c) + \int_0^\tau (b+c)^2 x_k(\hat{\tau}) d\hat{\tau} + p(b+c)\tau\right) d\tau. \end{aligned}$$

Plugging in the induction hypothesis and integrating, the above can also be written as

$$z_{k+2}(t) \leq 1 + pt + \int_0^t \left((b+c) + \sum_{i=0}^{k-1} \frac{(b+c)^{i+2}\tau^{i+1}}{(i+1)!} + \sum_{j=1}^{k-3} p\frac{(b+c)^{j+1}\tau^{j+1}}{(j+1)!} + p(b+c)\tau \right) d\tau$$

$$= 1 + pt + (b+c)t + \sum_{i=0}^{k-1} \frac{(b+c)^{i+2}t^{i+2}}{(i+2)!} + \sum_{j=1}^{k-3} p\frac{(b+c)^{j+1}t^{j+2}}{(j+2)!} + p\frac{(b+c)t^2}{2}.$$

Rearranging indices and collecting all the terms into their respective summation, the above can be more compactly stated as

$$z_{k+2}(t) \leq \sum_{i=0}^{k+1} \frac{(b+c)^i t^i}{i!} + \sum_{j=1}^{k-1} p\frac{(b+c)^{j-1}t^j}{j!}.$$

Since $z_{k+1}(t) \leq z_{k+2}(t)$, the result holds for both solutions $z_{k+1}(t)$ and $z_{k+2}(t)$. Hence, the result holds for $m = n+1$. The statement now follows by induction. □

Note that if we let $k \longrightarrow \infty$, then the upper bound converges. Namely, by multiplying the second series by $\frac{b+c}{b+c}$, both series in the bound now have the shape of the Taylor series of $e^{(b+c)t}$. However, the second series starts at $j = 1$. Since we want to let this series start at $j = 0$, we can subtract one such that we are adding zero. This ensures that the second series is now an appropriate Taylor series as well which converges to an existing limit. Explicitly writing this out, we obtain

$$\sum_{i=0}^{\infty} \frac{(b+c)^i t^i}{i!} + \frac{p}{b+c}\left(\sum_{j=0}^{\infty} \frac{(b+c)^j t^j}{j!} - 1 \right) = \left(1 + \frac{p}{b+c}\right) e^{(b+c)t} - \frac{p}{b+c}.$$

We can now find necessary and sufficient conditions for string stability of 2PLF networks. The proof will be analogous to that of Theorem 2.8 and Theorem 4.11.

Theorem 4.13. *Consider the positive 2PLF network* (22). *Let $\hat{x}_n(t)$ be the solution $x_n(t)$ with all initial conditions set equal to one. Define the transformation $z_N(t) = e^{at}\hat{x}_N(t)$ for all $N \in \mathbb{N}$. Then, there exists real numbers $K, \mu > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,*

$$\max_{i=1,\ldots,N} |x_i(t)| \leq Ke^{-\mu t} \max_{i=1,\ldots,N} |x_{i,0}|,$$

if and only if $b + c < a$.

Proof. ($\Longleftarrow$) Let $N \in \mathbb{N}$ be arbitrary. Then,

$$x_N(t) = e^{-at}\left(x_{N,0} + \int_0^t e^{a\tau}(bx_{N-1}(\tau) + cx_{N-2}(\tau) + px_1(\tau))d\tau \right).$$

Since each solution depends on one or more initial condition, we can replace these initial conditions by the maximum in absolute value over all initial condition and pull this maximum term outside the solution. In turn, all the initial conditions inside the parentheses will now be set equal to one. This means the above can be bounded by

$$x_N(t) \leq \max_{i=1,\ldots,N} |x_{i,0}| e^{-at} \left(1 + \int_0^t e^{a\tau}(b\hat{x}_{N-1}(\tau) + c\hat{x}_{N-2}(\tau) + p\hat{x}_1(\tau))d\tau \right)$$

$$= \max_{i=1,\ldots,N} |x_{i,0}| e^{-at}\hat{x}_N(t).$$

Note that $\hat{x}_N(t) = e^{-at}z_N(t)$ since $z_N(t) = e^{at}\hat{x}_N(t)$ by definition. Moreover, note that since N is arbitrary, N is either even or odd. In particular, there exists $m \in \mathbb{N}$ such that $N = 2m$ or $N = 2m-1$. Substituting $z_N(t)$ and applying Lemma 4.12, the solution $x_N(t)$ satisfies

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N} |x_{i,0}|e^{-at}z_N(t) \\ &\leq \max_{i=1,\dots,N} |x_{i,0}|e^{-at}\left(\sum_{i=0}^{2m-1}\frac{(b+c)^i t^i}{i!} + \sum_{j=1}^{2m-3} p\frac{(b+c)^{j-1}t^j}{j!}\right). \end{aligned}$$

Since $b, c \geq 0$, each term in the series is positive, so adding more terms to this series will only make the right hand side larger. In particular, the above inequality also holds when $m \longrightarrow \infty$, but then the series converges to $\left(1+\frac{p}{b+c}\right)e^{(b+c)t} - \frac{p}{b+c}$. Substituting this result gives us

$$\begin{aligned} x_N(t) &\leq \max_{i=1,\dots,N} |x_{i,0}|e^{-at}\left(\left(1+\frac{p}{b+c}\right)e^{(b+c)t} - \frac{p}{b+c}\right) \\ &\leq \max_{i=1,\dots,N} |x_{i,0}|e^{-at}\left(1+\frac{p}{b+c}\right)e^{(b+c)t} \\ &= \left(1+\frac{p}{b+c}\right)e^{-(a-b-c)t}\max_{i=1,\dots,N}|x_{i,0}|, \end{aligned}$$

which holds for all $N \in \mathbb{N}$ and all initial conditions. In the second line we used the fact that $\frac{p}{b+c} \geq 0$. Since the above inequality holds for all $N \in \mathbb{N}$, we also have

$$\max_{i=1,\dots,N}|x_i(t)| \leq \left(1+\frac{p}{b+c}\right)e^{-(a-b-c)t}\max_{i=1,\dots,N}|x_{i,0}|,$$

for all $N \in \mathbb{N}$. Set $K = 1 + \frac{p}{b+c}$ and $\mu = a - b - c$. Since $b + c < a$ by assumption, we have that $\mu > 0$. Hence, the network is string stable.
($\Longrightarrow$) Assume that there exists real numbers $K, \mu > 0$ such that

$$\max_{i=1,\dots,N}|x_i(t)| \leq Ke^{-\mu t}\max_{i=1,\dots,N}|x_{i,0}|,$$

for all $N \in \mathbb{N}$ and all initial conditions $x_{i,0}$. Set $x_{1,0} = 0$ and set $x_{i,0} = 1$ for all $i = 2, \dots, N$. Then, by assumption, each solution satisfies

$$x_n(t) \leq Ke^{-\mu t},$$

for all $n = 1, \dots, N$ and all $N \in \mathbb{N}$. Note that since $x_{1,0} = 0$, we have $x_1(t) = e^{-at}x_{1,0} = 0$ which means

$$x_n(t) = e^{-at}\left(1 + \int_0^t e^{a\tau}(bx_{n-1}(\tau) + cx_{n-1}(\tau))d\tau\right),$$

and

$$\dot{x}_n(t) = -ax_n(t) + bx_{n-1}(t) + cx_{n-2}(t).$$

In other the words, the dependency on $px_1(t)$ is fully removed in the system dynamics. Without loss of generality, we can then relabel the indices of the solutions by setting $x_n(t) = x_{n-1}(t)$ for all $n = 2, \dots, N$ to obtain an $N-1$ dimensional network with a 2PF formation. The problem then immediately reduces to Theorem 4.11. Hence, it follows that $b + c < a$. □

4.4 rPF Problem

In this final subsection we aim to generalize the 2PF problem to the rPF problem where $r \in \mathbb{N}$. In other words, we would like to find conditions for string stability of positive networks of the form

$$\begin{aligned} \Sigma_1 &: \dot{x}_1(t) = -ax_1(t), \\ \Sigma_i &: \dot{x}_i(t) = -ax_i(t) + b_1x_{i-1}(t) + b_2x_{i-2}(t) + \cdots + b_rx_{i-r}(t) \\ &\qquad\quad = -ax_i(t) + \sum_{m=1}^{r} b_m x_{i-m}(t), \quad i = 2, \ldots, N, \end{aligned} \tag{24}$$

where $a > 0$ and $b_j \geq 0$ for all $j = 1, \ldots, r$ are real numbers. If $k \leq 0$, set $x_k(t) \equiv 0$. Fortunately, this problem follows the exact same method as in the first part of the previous section. We would like to apply Lemma 4.4 to the network (24), which means that we will need to generalize Lemma 4.5 for arbitrary r, after which a generalization of Theorem 4.6 almost immediately follows. To this end, we will first generalize Lemma 4.5. This means sufficient conditions for the rPF problem can be found with relative ease. Due to the complexity of this problem, however, necessity will not be covered.

Lemma 4.14. *Consider the positive rPF network (24) with $r \geq 1$ and $b_j \geq 0$ for all $j = 1, \ldots, r$. Define the functions $v_k(t)$ and $V_k(t)$ as in Lemma 4.4. Set $v_k(t) = x_k(t)$ for all $k \in \mathbb{N}$ and let $\sigma \in (0, 1)$ be a real number. Let $q_n(z)$ be the n-th degree polynomial defined by*

$$q_n(z) = -az^n + b_1z^{n-1} + b_2z^{n-2} + \cdots + b_{r-1}z^{n-r+1} + b_rz^{n-r}.$$

If $n - i < 0$ for $i = 1, \ldots, r$, set $z^{n-i} \equiv 0$. Then,

$$\dot{V}_k(t) = \sum_{j=0}^{r} q_j(\sigma)x_{k-j}(t) + \sum_{i=1}^{k-1-r} \sigma^i q_r(\sigma)x_{k-r-i}(t).$$

Proof. The proof is a generalization of that of Lemma 4.5. We proceed by induction. Consider the base case $k = 1$. Then, $\dot{V}_k(t) = -ax_1(t) = q_0(\sigma)x_1(t)$. Hence, the base case holds. Assume now that the statement holds for arbitrary $k = n$. We will show that the statement holds for $k = n + 1$ as well. Analogous to Lemma 4.5 and omitting time arguments, we proceed. We obtain

$$\begin{aligned} \dot{V}_{n+1} &= \dot{v}_{n+1} + \sigma\dot{V}_n \\ &= -ax_{n+1} + \sum_{m=1}^{r} b_m x_{n-m+1} + \sigma \sum_{j=0}^{r} q_j(\sigma)x_{n-j} + \sum_{i=1}^{n-1-r} \sigma^{i+1} q_r(\sigma)x_{n-r-i}. \end{aligned}$$

Note that for $j = 0, \ldots, r$ we have $\sigma q_j(\sigma) = q_{j+1}(\sigma) - b_{j+1}$. Rewriting the above, substituting the result and collecting terms, the above can be written as

$$\begin{aligned} \dot{V}_{n+1} &= -ax_{n+1} + \sum_{j=0}^{r-1} \Big(b_{j+1}x_{n-j} + \sigma q_j(\sigma)x_{n-j}\Big) + \sigma q_r(\sigma)x_{n-r} + \sum_{i=1}^{n-1-r} \sigma^{i+1} q_r(\sigma)x_{n-r-i} \\ &= -ax_{n+1} + \sum_{j=0}^{r-1} q_{j+1}(\sigma)x_{n-j} + \sum_{i=0}^{n-1-r} \sigma^{i+1} q_r(\sigma)x_{n-r-i} \\ &= \sum_{j=0}^{r} q_j(\sigma)x_{n+1-j} + \sum_{i=1}^{n-r} \sigma^i q_r(\sigma)x_{n+1-r-i}. \end{aligned}$$

Hence, the statement holds for $k = n + 1$ as well. Since $k = n$ was arbitrary, the statement holds for all $k \in \mathbb{N}$. □

Now that we have found a generalization of Lemma 4.5 for arbitrary r, we can state the main result. The following theorem will provide us with sufficient conditions for string stability of the rPF formation for any $r \in \mathbb{N}$. In essence, the following theorem is just a generalization of Theorem 4.6.

Theorem 4.15. *Consider the positive rPF network* (24). *Let $p_r(z)$ be the r-th degree polynomial defined by*

$$p_r(z) = (\mu - a)z^r + b_1 z^{r-1} + b_2 z^{r-2} + \cdots + b_{r-1} z + b_r, \tag{25}$$

where $\mu > 0$ is a real number. If there exists a real number $\sigma \in (0, 1)$ such that $p_r(\sigma) \leq 0$, then there exists a real number $K > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,

$$\max_{i=1,\ldots,N} |x_i(t)| \leq K e^{-\mu t} \max_{i=1,\ldots,N} |x_{i,0}|.$$

Proof. We proceed in the same way as in Theorem 4.6. Note that since $p_r(\sigma) \leq 0$ by assumption, this also means $p_j(\sigma) \leq 0$ for all $j = 0, \ldots, r$. This can be easily verified by using the assumption $p_r(\sigma) \leq 0$, subtracting the constant coefficient (which is allowed as each coefficient in the polynomial is nonnegative) and dividing by r. Repeatedly applying this procedure will then show $p_j(\sigma) \leq 0$ for all $j = 0, \ldots, r$. Next, we apply Lemma 4.14, which states

$$\dot{V}_N(t) = \sum_{j=0}^{r} q_j(\sigma) x_{N-j}(t) + \sum_{i=1}^{N-1-r} \sigma^i q_r(\sigma) x_{N-r-i}(t). \tag{26}$$

In order to bound this expression and use Lemma 4.4, we need

$$\begin{aligned} & q_j(\sigma) \leq -\mu\sigma^j \\ \iff & q_j(\sigma) + \mu\sigma^j \leq 0 \\ \iff & p_j(\sigma) \leq 0, \end{aligned}$$

for all $j = 0, \ldots, r$. Since $p_j(\sigma) \leq 0$ for all $j = 0, \ldots, r$ by assumption, the above inequality holds. Hence, we have $q_j(\sigma) \leq -\mu\sigma^j$ for all $j = 0, \ldots, r$. Substituting this result in (26), it follows that

$$\begin{aligned} \dot{V}_N(t) &\leq \sum_{j=0}^{r} -\mu\sigma^j x_{N-j}(t) + \sum_{i=1}^{N-1-r} -\mu\sigma^{r+i} x_{N-r-i}(t) \\ &= \sum_{j=0}^{N-1} -\mu\sigma^j x_{N-j}(t) \\ &\leq -\mu V_N(t), \end{aligned}$$

which holds for all $N \in \mathbb{N}$. By Lemma 4.4, the result directly follows, yielding $K = \frac{1}{1-\sigma}$. □

As this theorem is a generalization of Theorem 4.6, the same issues arise as in the 2PF problem. Namely, this theorem does not give us an explicit bound i.e. it does not tell us how to find $\sigma \in (0, 1)$ and how to find $\mu > 0$. As mentioned before, this is bad, because if σ is close to one, then $K = \frac{1}{1-\sigma}$ blows up. It also does not tell us how to choose each b_j in the input term to optimize string stability or even guarantee the existence of $\sigma \in (0, 1)$ and $\mu > 0$ to begin with. In the 2PF problem, Lemma 4.8 gave us conditions to ensure the existence of σ and μ, namely the requirement that $b + c < a$. This result can be generalized in the rPF problem, leaving us with at least the tools to know how to design each system appropriately to obtain string stability of rPF networks.

Lemma 4.16. *Consider the positive rPF network (24). Let $p_r(z)$ be the r-th degree polynomial defined by*

$$p_r(z) = (\mu - a)z^r + b_1 z^{r-1} + b_2 z^{r-2} + \cdots + b_{r-1}z + b_r, \tag{27}$$

where $\mu > 0$ is a real number. Then, there exists a real number $\sigma \in (0,1)$ such that $p_r(\sigma) \leq 0$ if and only if $\sum_{i=1}^r b_i < a$.

Proof. The proof is a generalization of Lemma 4.8.
($\Longleftarrow$) Assume $\sum_{i=1}^r b_i < a$. There exists $\sigma \in (0,1)$ such that $\sum_{i=1}^r b_i = \sigma^r a$, which means

$$\sum_{i=1}^r \sigma^{r-i} b_i < \sigma^r a,$$

i.e.

$$\sum_{i=1}^r \sigma^{r-i} b_i - \sigma^r a < 0.$$

Since the inequality is strict, there exists $\mu > 0$ such that

$$(\mu - a)\sigma^r + \sum_{i=1}^r \sigma^{r-i} b_i \leq 0,$$

or equivalently, $p_r(\sigma) \leq 0$.
($\Longrightarrow$) Assume there exists $\sigma \in (0,1)$ such that $p(\sigma) \leq 0$. Then, since $\mu > 0$ by assumption, this implies

$$-a\sigma^n + \sum_{i=1}^r \sigma^{r-i} b_i < 0,$$

which is equivalent to

$$b_r + \sigma\left(-a\sigma^{r-1} + \sum_{i=1}^{r-1} \sigma^{r-1-i} b_i\right) < 0.$$

Since $b_r \geq 0$, the term in parentheses in the above inequality is negative. Moreover, since $\sigma \in (0,1)$, the above implies

$$b_r - a\sigma^{r-1} + \sum_{i=1}^{r-1} \sigma^{r-1-i} b_i < 0.$$

Repeatedly applying the same procedure, inductively one obtains that $\sum_{i=1}^r b_i - a\sigma < 0$, which implies $\sum_{i=1}^r b_i < a$. This shows the result. □

We now have enough knowledge to optimally design each system in the rPF network (24) such that string stability is achieved and optimized. It can easily be seen now that $\sum_{i=1}^r b_i < a$ implies string stability. For good measure, we will explicitly state the result. This will be the final result of this thesis.

Corollary 4.17. *Consider the positive rPF network (24). Then, if $\sum_{i=1}^r b_i < a$, there exists real numbers $K, \mu > 0$ such that, for all initial conditions $x_i(0) = x_{i,0}$ and all $N \in \mathbb{N}$,*

$$\max_{i=1,\ldots,N} |x_i(t)| \leq K e^{-\mu t} \max_{i=1,\ldots,N} |x_{i,0}|.$$

Proof. By Lemma 4.16, there exists $\sigma \in (0,1)$ such that

$$p_r(\sigma) = (\mu - a)\sigma^r + b_1\sigma^{r-1} + b_2\sigma^{r-2} + \cdots + b_{r-1}\sigma + b_r \leq 0.$$

By Theorem 4.15, the result immediately follows with $K = \frac{1}{1-\sigma}$. □

Note that the same interpretation applies as in the 2PF and PF problems. Namely, each vehicle in a platoon receives a smaller total amount of disturbances, $\sum_{i=1}^{r} b_i$, than what the previous vehicle experiences, a.

5 Examples

This section serves to show the validity of the results obtained in this thesis by running simulations for various values of the system parameters and various network sizes. In particular, these examples aim to show two things. First of all, they aim to show that the string stability bounds are in fact valid. That is, given N systems in a network, the state trajectory of each system can be bounded by this exponential bound. Secondly, these examples aim to show the scalability property of the networks for the same bound. In other words, by considering increasingly large values of N, the same bound can always be applied. This would in turn then show that the string stability bound does not depend on N.

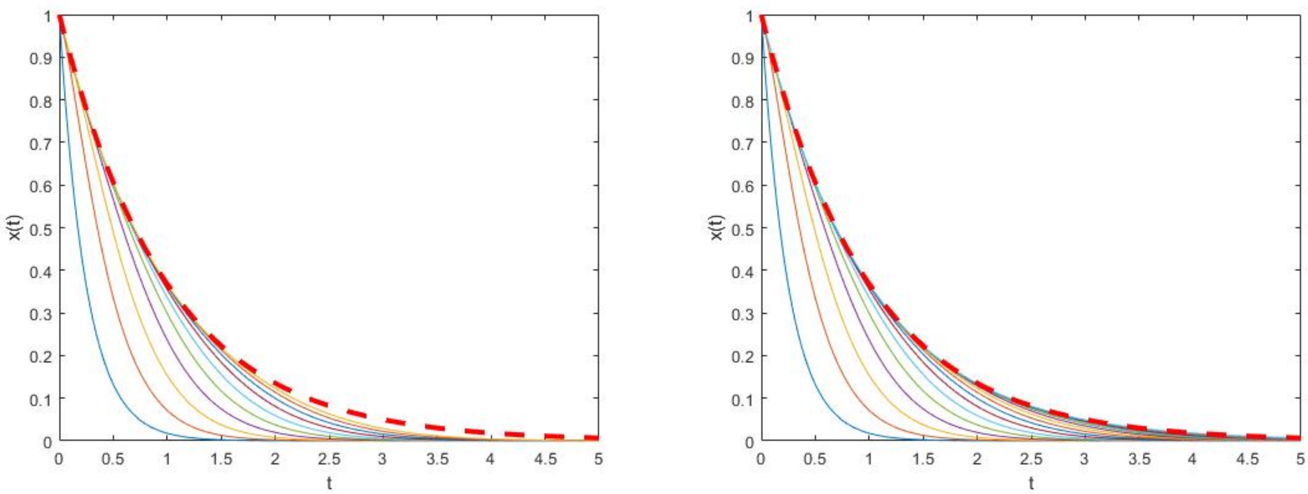
In the following examples, the thick red dashed lines represent the string stability bound, whereas all other lines represent the state trajectories of each system in the given network. For simplicity, we will set $x_{i,0} = 1$ for all $i = 1, \ldots, N$ unless otherwise specified. All the solutions have been computed using the forward Euler method. As the results for heterogeneous networks were weaker versions of their homogeneous counterparts, only homogeneous networks will be considered here.

5.1 Predecessor Following

We start off with the simplest case. By Theorem 2.5, string stability is achieved whenever $|b| < a$. Choose $a = 4$ and $b = 3$. By Theorem 2.5, we must have

$$x_i(t) \leq e^{-t},$$

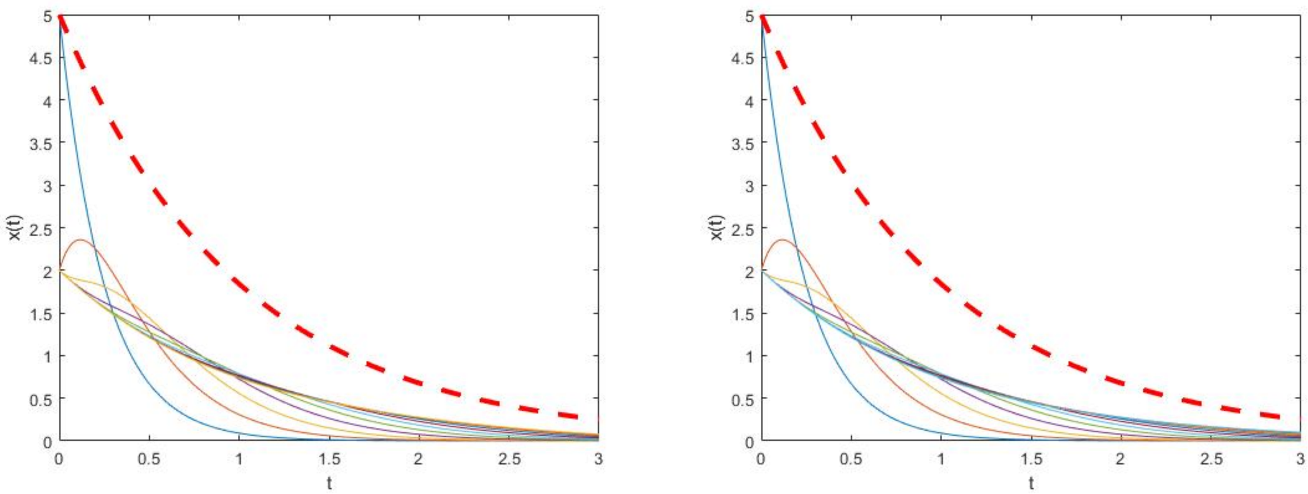
for all $i = 1, \ldots, N$. Figure 4 shows the result for $N = 10$ and $N = 1000$.



(a) $N = 10$ (b) $N = 1000$

Figure 4: PF string stability for $a = 4$ and $b = 3$.

One can clearly see that none of the state trajectories exceed the bound given by the red dashed line at any point. We can therefore conclude that the network of systems is string stable for $a = 4$ and $b = 3$. What if we changed the initial conditions? Again set $a = 4$ and $b = 3$, but this time set $x_{1,0} = 5$ and $x_{j,0} = 2$ for all $j = 2, \ldots, N$. Then, $x_i(t) \leq 5e^{-t}$ for all $i = 1, \ldots, N$. The results can be seen in Figure 5.



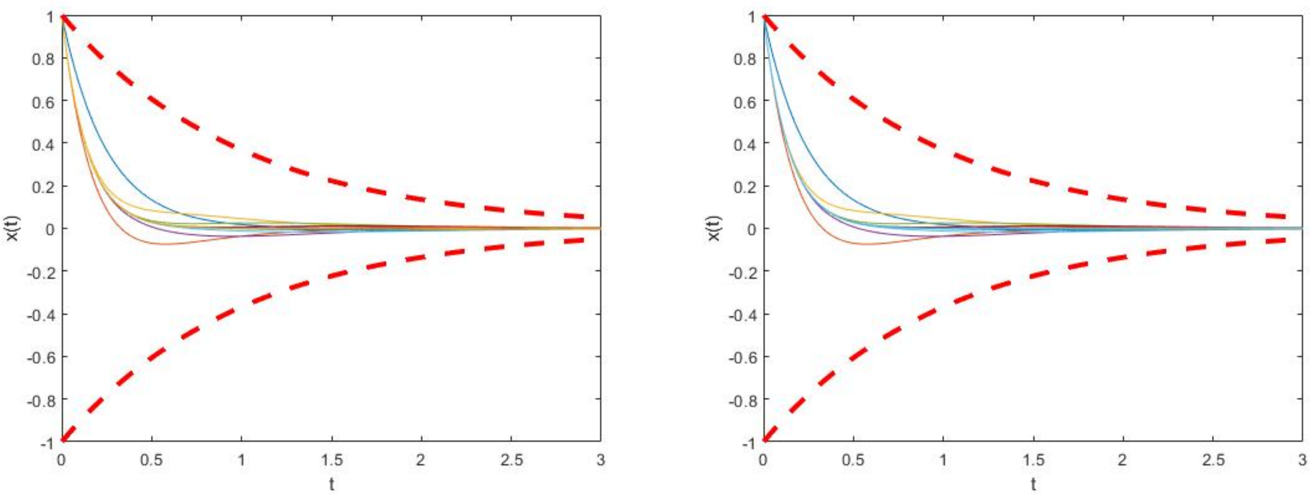
(a) $N = 10$ (b) $N = 1000$

Figure 5: PF string stability for $a = 4$, $b = 3$ and $\max |x_{i,0}| = 5$.

Despite the shapes of the solutions changing now that we vary the initial conditions, none of these state trajectories ever cross the exponential bound. Hence again, we achieve string stability. Similarly, the requirement $|b| < a$ means that negative values for b are allowed as well. This will no longer make the network a positive system and so each state trajectory can attain negative values as well. Setting $a = 4$ and $b = -3$, the state trajectories can be bounded by

$$-e^{-t} \leq x_i(t) \leq e^{-t},$$

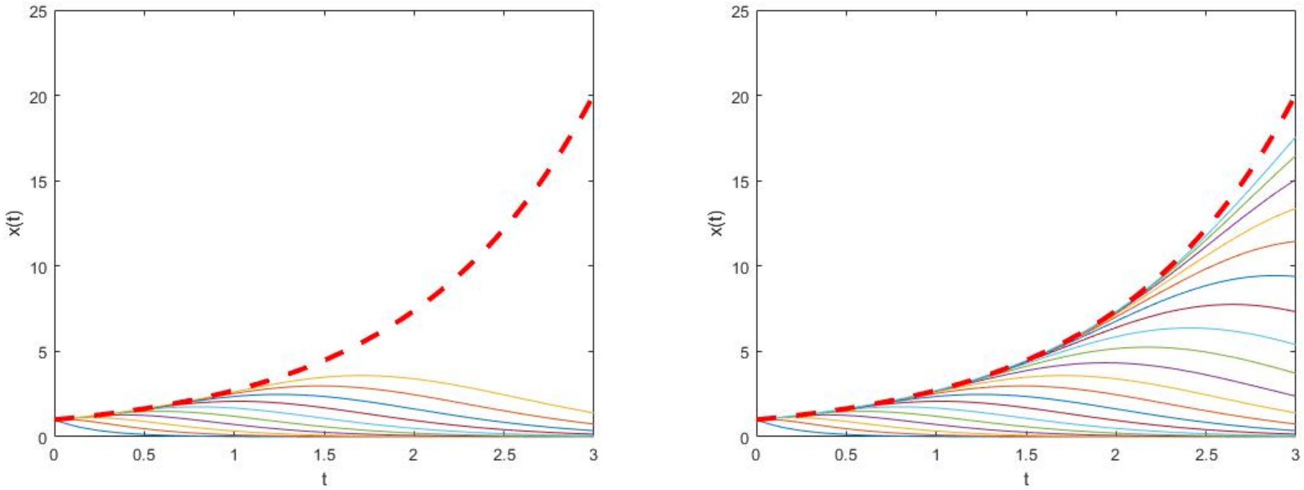
for all $i = 1, \dots, N$. Figure 6 shows the result for $N = 10$ and $N = 1000$.



(a) $N = 10$ (b) $N = 1000$

Figure 6: PF string stability for $a = 4$ and $b = -3$.

Again, it can be seen that string stability is achieved. Hence, we can indeed choose the input parameters to be negative as well. On the other hand, Theorem 2.5 tells us that if $|b| \geq a$, then we no longer have string stability. Setting $a = 4$ and $b = 5$, Figure 7 shows the solutions and the bounds for $N = 10$ and $N = 20$.



(a) $N = 10$ (b) $N = 20$

Figure 7: PF string unstability for $a = 4$ and $b = 5$.

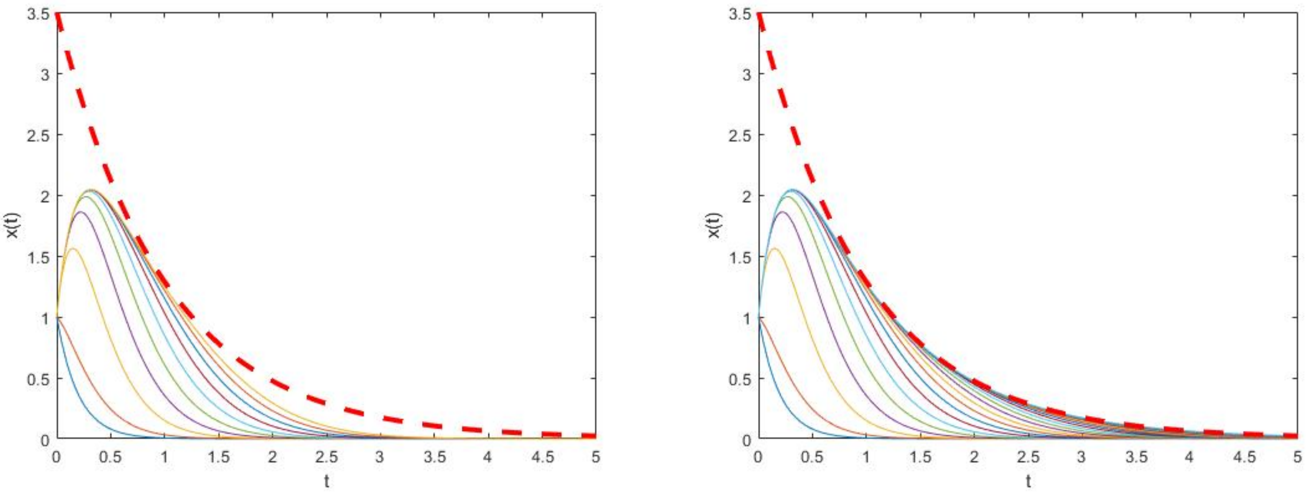
Notice how each solution converges to the origin over time. This shows that the network is exponentially stable. However, it can be seen that there is no exponentially decaying bound possible. Namely, as $N \longrightarrow \infty$, so too does the largest peak over all solutions shoot off to infinity. Hence, in this case, string stability is not possible, which is exactly in accordance with Theorem 2.5.

5.2 Predecessor-Leader Following

According to Theorem 2.8, string stability can be achieved as long as $|b| < a$. The value of c does not influence string stability, but it does influence the bound since $K = 1 + \frac{|c|}{|b|}$. Pick $a = 5$, $b = 4$, $c = 10$. By Theorem 2.8, the solutions satisfy

$$x_i(t) \leq \frac{7}{2}e^{-t},$$

for all $i = 1, \ldots, N$. Figure 8 shows the result for $N = 10$ and $N = 1000$.



(a) $N = 10$ (b) $N = 1000$

Figure 8: PLF string stability for $a = 5$, $b = 4$ and $c = 10$.

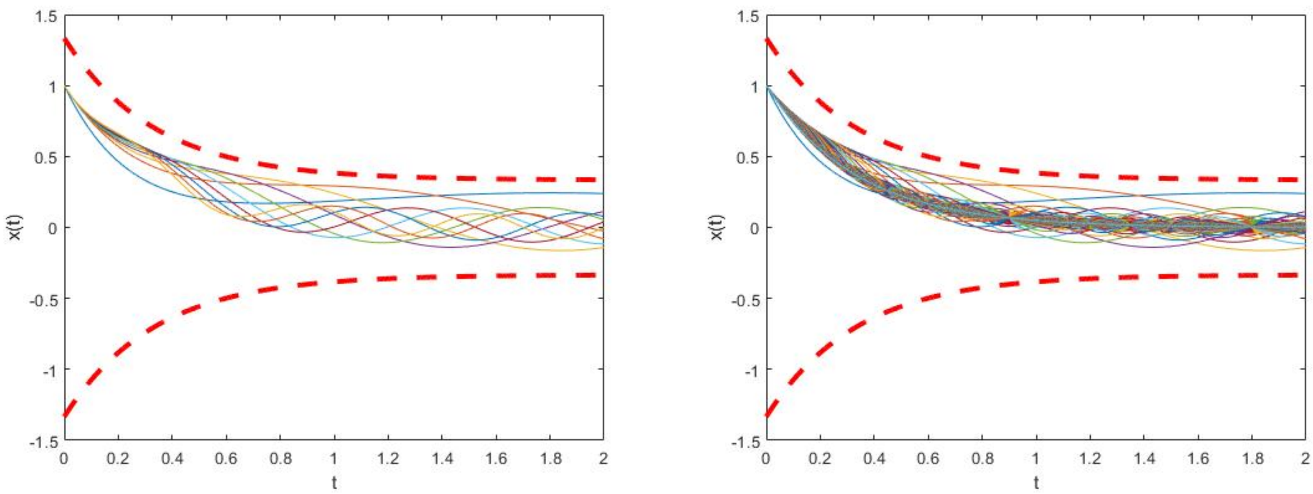
Note the peaks of each state trajectory. These arise due to the presence of c, causing the K term in the exponential bound to blow up to compensate for these peaks. In a string unstable network, these peaks would grow with each additional increase in network size. However, in this case, due to string stability, these peaks will remain bounded no matter the network size.

5.3 Predecessor Following with Disturbances

Recall that due to the presence of disturbances, we can no longer achieve string stability of the system. Instead, we will have to resort to disturbance string stability as in Definition 3.1. Per Theorem 3.5, we require that $|b| < a$. Moreover, we require that each disturbance $d_i(t)$ is bounded. Choose $a = 4$, $b = 1$ and $d_j(t) = \sin(jt)$ for all $j = 1, \ldots, N$. By Theorem 3.5, each solution $x_i(t)$ satisfies

$$-\frac{1}{3} - e^{-3t} \leq x_i(t) \leq e^{-3t} + \frac{1}{3},$$

for all $t \geq 0$ and all $i = 1, \ldots, N$. Figure 9 shows the result for $N = 10$ and $N = 1000$.



(a) $N = 10$ (b) $N = 1000$

Figure 9: PF disturbance string stability for $a = 4$, $b = 1$ and $d_j(t) = \sin(jt)$.

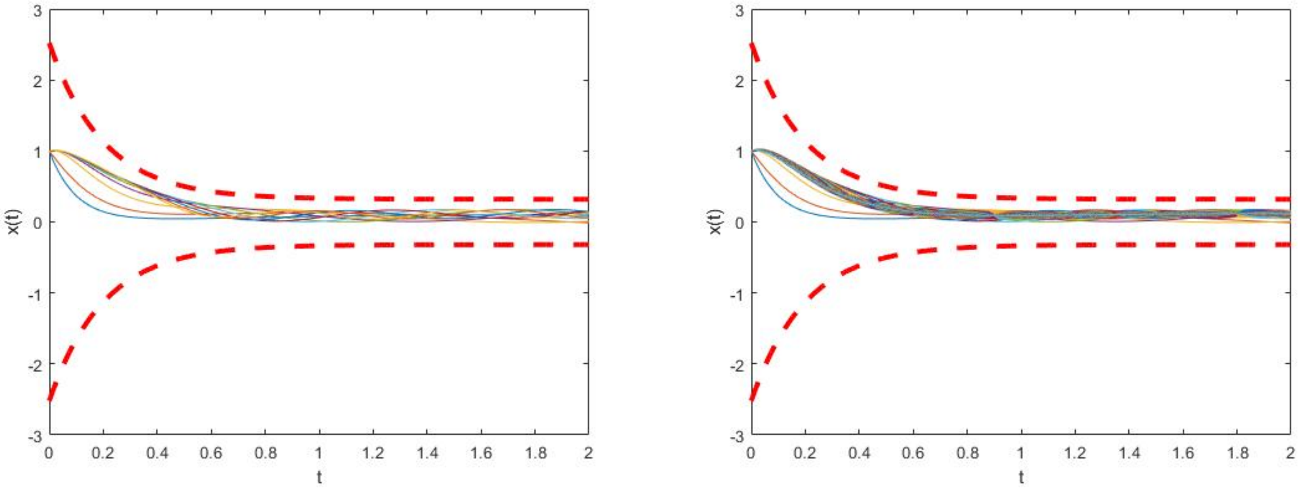
As can be seen, the state trajectories do not converge to the origin. Rather, each solution seems to oscillate in some small interval around the origin for all time. This reinforces the idea that while exponential stability is not possible, it is still possible to trap each state trajectory in the strip of radius $\gamma \max \sup |d_i(t)| = \frac{1}{3}$ (for sufficiently large $t \geq 0$). Hence, disturbance string stability is achieved, whereas string stability itself is not feasible.

5.4 Predecessor-Leader Following with Disturbances

Similar to the PF problem with disturbances, we need to settle for disturbance string stability. According to Theorem 3.7, this is achieved whenever $|b| < a$. The value of c does not matter to obtain string stability, but the choice for c does affect the shape of the bound, as $K = 1 + \frac{|c|}{|b|}$. In this example, pick $a = 10$, $b = 5$, $c = 6$ and set $d_j(t) = \sin(jt)$ for all $j = 1, \ldots, N$. Then, each solution $x_i(t)$ satisfies

$$-\frac{11}{5}e^{-5t} + \frac{8}{25} \leq x_i(t) \leq \frac{11}{5}e^{-5t} + \frac{8}{25},$$

for all $i = 1, \dots N$. Figure 10 shows the result for $N = 10$ and $N = 1000$.



(a) $N = 10$ (b) $N = 1000$

Figure 10: PLF disturbance string stability for $a = 10$, $b = 5$, $c = 6$ and $d_j(t) = \sin(jt)$.

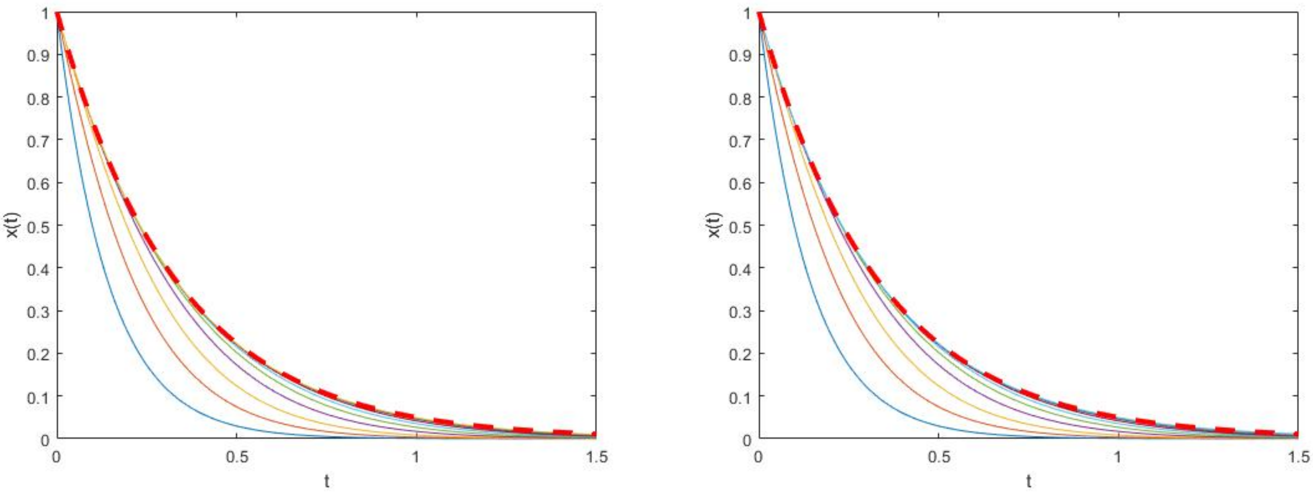
As can be seen, each state trajectory $x_i(t)$ stays trapped in the strip given by the dashed lines. Hence, in this case too, disturbance string stability is indeed achieved.

5.5 Two Predecessor Following

We solved the 2PF problem for the case where the 2PF networks were positive systems. Hence, we can only pick $b, c \geq 0$. By Theorem 4.11, we require $b + c < a$. Pick $a = 7$, $b = 3$, $c = 1$. Then,

$$x_i(t) \leq e^{-3t},$$

for all $i = 1, \dots, N$. Figure 11 shows the result.



(a) $N = 10$ (b) $N = 1000$

Figure 11: 2PF string stability for $a = 7$, $b = 3$ and $c = 1$.

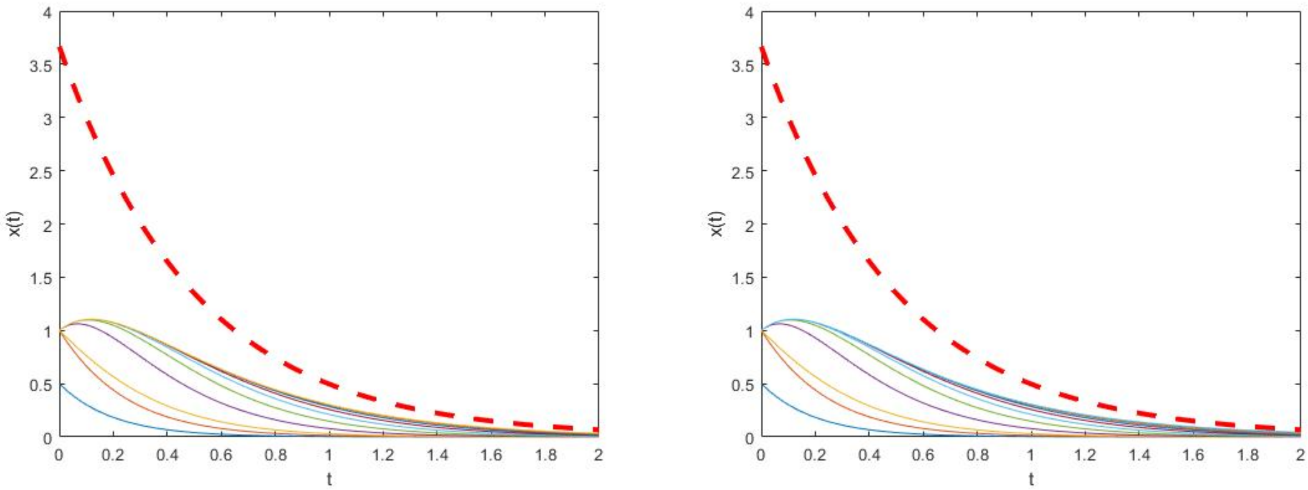
We can see that string stability is indeed achieved. Comparing the graphs in Figure 11 to the PF graphs in Figure 4, one can see that the general shapes of the state trajectories do not change once a second predecessor is considered.

5.6 Two Predecessor-Leader Following

The 2PLF problem was solved with the assumption that the 2PLF network was a positive system. This means that we can only pick $b, c, p \geq 0$. Per Theorem 4.13, we require that $b+c < a$. The choice of p does not matter to ensure string stability, but since $K = 1 + \frac{p}{b+c}$, the shape of the exponential bound will change depending on the choice of p. Pick $a = 5$, $b = 2$, $c = 1$ and $p = 8$. Moreover, for the initial conditions, set $x_{1,0} = \frac{1}{2}$ and $x_{j,0} = 1$ for all $j = 2, \ldots, N$. Then we should have

$$x_i(t) \leq \frac{11}{3} e^{-2t},$$

for all $i = 1, \ldots, N$. Figure 12 shows the result.



(a) $N = 10$ (b) $N = 1000$

Figure 12: 2PLF string stability for $a = 5$, $b = 2$, $c = 1$, $p = 8$ and $x_{1,0} = \frac{1}{2}$.

As one can see, the network is indeed string stable. Analogous to the PLF example in Figure 8, we notice the bounded peaks in the beginning. However, in Figure 12 these peaks are much smaller than those in Figure 8. This is not due to the addition of a second predecessor in the 2PLF formation, but rather due to the initial condition $x_{1,0} = 0.5$ causing a dampening in the peaks.

6 Conclusion

In this thesis, necessary and sufficient conditions for string stability of networks consisting of scalar linear systems were found. We started off by defining string stability of a network. This definition is nothing more than a stronger version of exponential stability, where the exponential bound of the network is independent of the network size. As the conditions for string stability change based on the structure of the network, each information flow topology poses its own separate problem in trying to find conditions for string stability. Studying string stability is worthwhile as it has applications in vehicle platooning. Namely, having a string stable platoon means that we can add as many vehicles to the platoon as we want, while ensuring that the effect of a disturbance stays bounded as it propagates through the string.

We first looked at the predecessor following (PF) and predecessor-leader following (PLF) problems. By recognizing that the solutions to the systems in both networks contained the structure of a Taylor series, necessary and sufficient conditions for both problems could be found with relative ease. In particular, the conditions found for the PLF problem turned out to be the exact same as that in the PF problem, although the exponential bound was structured slightly differently. However, necessary and sufficient conditions were only guaranteed under the premise that the network was homogeneous. Once we started to consider heterogeneous networks instead, necessity was lost.

After solving these two problems, we considered the PF and PLF problems with an additional unknown disturbance factored into each system of the network. Due to the presence of this unknown disturbance, string stability was no longer viable, which motivated the use of the notion of disturbance string stability. As it turned out, the conditions for string stability in both the PF and PLF problem remained the same, meaning that the presence of a disturbance did not change how to optimally model each system in the network to achieve string stability.

In the third and final section, we looked at the 2PF and 2PLF problems, which were direct extensions of the PF and PLF problems. Due to the complexity of a second predecessor in the system input, we restricted ourselves to the case where the networks were positive. This allowed us to fully bypass having to use the solution to each system and to immediately find sufficient conditions for string stability of the 2PF problem. Unfortunately, this result was lackluster, as it did not explicitly give us an exponential bound, but merely guaranteed its existence. Fortunately, this result did allow us to find necessary and sufficient conditions for the 2PF and 2PLF problems, where the initial issue was now resolved. In an attempt to generalize the 2PF problem to arbitrary predecessors, we finally turned ourselves to the rPF problem. Due to the complexity of that problem, only sufficient conditions were found. To verify all the results in this thesis, simulations were shown.

The contents in this thesis are far from complete. As only sufficient conditions for the rPF problem were found, a natural extension would be to find necessary conditions for string stability, such that the rPF problem is solved in its entirety. In particular, it would be interesting to see if the methods used in the 2PF and 2PLF problems could be generalized to prove necessity and sufficiency in the rPF and rPLF problems. Moreover, as the conditions for string stability in the PF and PLF problems did not change with the presence of an external disturbance, a future topic of research would be to find out if these conditions remain unchanged when an external disturbance is added to the 2PF and 2PLF problems. If that were indeed the case, then one could conjecture that disturbances do not influence the conditions for string stability in the rPF and rPLF problems. Once all these problems have been solved, the final step would be to generalize these results to higher dimensional linear systems. Otherwise, one could choose to continue to work with scalar linear systems and look at bidirectional information flow topologies instead.

References

[1] B. Besselink and K. H. Johansson. String stability and a delay-based spacing policy for vehicle platoons subject to disturbances. *IEEE Transactions on Automatic Control*, 62(9):4376–4391, 2017.

[2] B. Besselink and S. Knorn. Scalable input-to-state stability for performance analysis of large-scale networks. *IEEE Control Systems Letters*, 2(3):507–512, 2018.

[3] K. Chu. Decentralized control of high-speed vehicular strings. *Transportation Science*, 8(4):361–384, 1974.

[4] W. B. Dunbar and D. S. Caveney. Distributed receding horizon control of vehicle platoons: Stability and string stability. *IEEE Transactions on Automatic Control*, 57(3):620–633, 2012.

[5] L. Farina and S. Rinaldi. *Positive Linear Systems*. John Wiley & Sons, 2000.

[6] S. Feng, Y. Zhang, S. E. Li, Z. Cao, H. X. Liu, and L. Li. String stability for vehicular platoon control: Definitions and analysis methods. *Annual Reviews in Control*, 47:81–97, 2019.

[7] R. Janssen, H. Zwijnenberg, I. Blankers, and J. de Kruijff. Truck platooning driving the future of transportation. Technical report, TNO, February 2015.

[8] M. Khatir and E. Davidson. Bounded stability and eventual string stability of a large platoon of vehicles using non-identical controllers. In *2004 43rd IEEE Conference on Decision and Control (CDC)*, pages 1111–1116 Vol.1, 2004.

[9] W. Levine and M. Athans. On the optimal error regulation of a string of moving vehicles. *IEEE Transactions on Automatic Control*, 11(3):355–361, 1966.

[10] S. E. Li, Y. Zheng, K. Li, and J. Wang. An overview of vehicular platoon control under the four-component framework. In *2015 IEEE Intelligent Vehicles Symposium (IV)*, pages 286–291, 2015.

[11] G. Naus, R. Vugts, J. Ploeg, M. Molengraft, van de, and M. Steinbuch. String-stable cacc design and experimental validation, a frequency-domain approach. *IEEE Transactions on Vehicular Technology*, 59(9):4268–4279, 2010.

[12] L. E. Peppard. String stability of relative-motion pid vehicle control systems. *IEEE Transactions on Automatic Control*, 19(5):579–581, 1974.

[13] J. Ploeg, N. van de Wouw, and H. Nijmeijer. String stability of vehicle platoons. In *Book of Abstracts 30th Benelux meeting on Systems and Control*. Universiteit Gent, 2011.

[14] J. Ploeg, N. van de Wouw, and H. Nijmeijer. Lp string stability of cascaded systems: Application to vehicle platooning. *IEEE Transactions on Control Systems Technology*, 22(2):786–793, 2014.

[15] D. Yanakiev and I. Kanellakopoulos. A simplified framework for string stability analysis in AHS. *IFAC Proceedings Volumes*, 29(1):7873–7878, 1996. 13th World Congress of IFAC, 1996, San Francisco USA, 30 June - 5 July.