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COMPARISON OF METHODS FOR THE COMPUTATION OF EUROPEAN OPTION PRICES

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Abstract

Financial mathematics is a growing field with a lot of scope. The use of differential equations, stochastic calculus, and probability theory concepts is rife in this field. To delve further into this interconnected field of theoretical mathematics and finance, we evaluate a mathematical model known as the Black-Scholes model. The model is derived using concepts from stochastic calculus as well as through reasoning of financial markets. The Black-Scholes model is represented by a partial differential equation and this is numerically analyzed using the finite difference method. A Feynman-Kac approach to find an exact solution to the differential equation is derived. The discretizations and Feynman-Kac solutions are numerically simulated and compared with the results of the numerical simulation of the analytical solution provided by Fischer Black and Myron Scholes. The comparison is carried out based on computational time, accuracy of results, and the stability of the methods used.

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1 Introduction

Financial mathematics is a growing field that encapsulates the intersection between theoretical mathematics and the financial world. Billions of (US) dollars worth of stocks and commodities are traded daily in the Foreign Exchance (stock) market [1]. Financial derivatives are financial instruments whose value depends on the price of the underlying asset. Options are a type of financial derivative which are traded on the stock market daily. Financial derivatives, and by association, options, represent a more risk-averse venture into the trading world. The evolution of the price of a stock over a period of time can be modelled by a geometric Brownian motion. It is interesting to observe that the motions governing the price movements of the stock market are familiar to mathematicians around the world, even with a lack of pre-existing knowledge of the market itself. In 1973, a financial mathematical model was developed. This model aids in evaluating the theoretical fair price of an option contract. This model is known as the Black-Scholes model [2] and is used till this date to evaluate theoretical prices of options. In this report, we will consider the standard Black-Scholes model with respect to European options.

This paper will be structured in the following way. Section 2 is composed of the preliminary knowledge and concepts required for the reader to successfully follow the remainder of the paper. Crucial concepts from stochastic calculus are discussed briefly. Section 3 introduces the reader to the Black-Scholes model and the exact solutions. The derivation of the model from a stochastic ordinary differential equation is briefly discussed. In section 4, the numerical discretization schemes of the Black-Scholes equation are derived. In section 5, the numerical analysis of the stochastic differential equation that governs the movements of the stock price is carried out. In section 6 the results of the simulations run for the analytical solutions and numerical discretizations are discussed. Finally, we conclude this report in section 7.

2 Preliminary Knowledge

In order for this report to be self contained and readable without too much extra reading required, there are several concepts and terms that must be introduced and described.

2.1 Financial terms

The models and equations used in this paper describe the motions of stocks and option pricing, which are primarily used in the financial industry. This section serves to provide readers with a brief overview of the financial concepts and markets that will be used in this report.

2.1.1 Options

The information presented in this subsection can be found in [3].
Options are essentially contracts that enable the holder of an option to either buy or sell an underlying asset within a given time frame. Whether the holder is entitled to buy an underlying asset or to sell it is determined by the type of contract they hold or have purchased. There are several types of options such as American, Asian, Barrier, European etc. Each type of option has different pricing theories, expiry times, and other factors [4]. For this paper, the focus will be on analyzing partial differential equations and models that govern the European options. A European option can only be exercised at the time of expiry, that is, the holder may only choose to buy or sell their option on the expiry date that was predetermined.

- Strike price: This is a fixed price at which the owner of an option can either sell or buy the contract at the time of expiry.

Whether the holder of an option has the right to buy or sell the underlying asset is determined by the type of options contract they purchase. The two contracts that will be considered here are:

- Call option: The owner of a call has the right, but not the obligation, to buy the underlying asset at the strike price.

- Put option: The owner of a put has the right, but not the obligation, to sell the underlying asset at the strike price.

There are multiple ways to determine the value of an option, each depending on different factors.

The example below is provided to give readers a brief intuition about the workings of options and why they might be beneficial in comparison to trading directly on the market. This example was developed using the help of examples provided in [5].

Call option example: Suppose a stock is currently worth \$100 and you have reason to think that the price of the stock will go up within the next 3 months (this results in a 90 day contract which must be set by the seller of the contract). With a call option, the buyer pays a premium, w for each share, in order to obtain the right to buy the stock at the strike price. Suppose the premium agreed upon is \$2 per share and the buyer purchases

100 shares. This means that the buyer pays a total premium of $\$2 \cdot 100 = \$200$. The strike price is \$100. In the case of European options, the buyer can only choose to exercise their options at the time of expiry, which is after a period of 90 days in this case. There are now two possible scenarios for the buyer at the time of expiry.

1. If the price of the stock has increased, for example, to \$120 then the buyer chooses to exercise their right to buy the option at the strike price. This means that they can buy the underlying stock at \$100 per share that was agreed upon at the beginning of the contract period. The buyer buys 100 shares of the stock at \$100, for a total of \$10000. The buyer can then go on to sell them on the market for \$12000. The total profit earned on this option would be $\$2000 - \$200 = \$1800$.

2. If the price of the stock falls below the strike price, for example, to \$80 then the buyer chooses not to exercise the option. This means that at the date of expiry, they let the stock expire. If the buyer chose to exercise the option at this stage they would make a net loss of \$2200. By letting the option expire, the buyer has a net loss only consisting of the total premium paid to purchase the option contracts. Hence the loss in this case would be $\$2 \cdot 100 = \$200$.

Options are frequently used by traders and investors to limit their risk. The buyer could directly buy 100 shares of a stock at \$100 per share. If the price of the stock increases to \$120, and the buyer chooses to sell the shares at this point, then they make a profit of \$2000. If the price of the stock falls down to \$80 and the price appears to be on a downward trend, then they may choose to sell at \$80 per share to prevent further losses in the future. This would result in a total loss of \$2000 for the buyer. Comparing this with the net profit and loss in the case of an options contract, we observe that the profit obtained will be higher when trading directly without the use of options contracts. However the loss incurred is significantly higher when trading without the use of options contracts, and more risk-averse traders may prefer to minimize their losses.

2.2 Mathematical concepts

2.2.1 Classification of partial differential equations

The following information can be found in [6].
A general second order partial differential equation for a function $u(t, x)$ is of the form:

$$L[u] = a(t,x)u_{tt} + b(t,x)u_{tx} + c(t,x)u_{xx} + d(t,x)u_t + e(t,x)u_x + f(t,x)u = g.$$

Here u_x, u_{xx} represent the partial derivative of u with respect to x and the second partial derivative of u with respect to x respectively. Similar reasoning can be applied for the remaining terms. It is assumed that atleast one of $a(t,x), b(t,x), c(t,x)$ is not zero. The type of partial differential equation is determined by the discriminant of the equation, which is written as:

$$\Delta = b(t,x)^2 - 4a(t,x)c(t,x). \tag{1}$$

At a point (t, x) the second order partial differential equation is called **parabolic** if $\Delta(t,x) = 0$ but $a(t,x)^2 + b(t,x)^2 + c(t,x)^2 \neq 0$.

2.2.2 Brownian motion

A **standard Brownian motion** is a stochastic process characterized by a family of random variables, $\{B_t | t \in [0, \infty)\}$. The Brownian motion must satisfy the following conditions:

- $B_{t_0} = 0$.
- Every increment is independent. This means that $\forall t_u, t_v \in \mathbb{R}$, we have that $B_{t_u} - B_{t_v}$ is independent of any past values of B_{t_x} where $t_x < t_v$. For increments to be classified as independent, we must have that $B_{t_u} - B_{t_v}$ and $B_{t_s} - B_{t_r}$ are independent, for any $t_s, t_r, t_u, t_v \in \mathbb{R}$ with $0 \leq t_s < t_r < t_u < t_v$.
- B_t is continuous in time on the interval $[0, \infty)$.
- $\forall t_u > 0$, B_{t_u} is normally distributed with mean 0 and variance t. [7](p. 640).

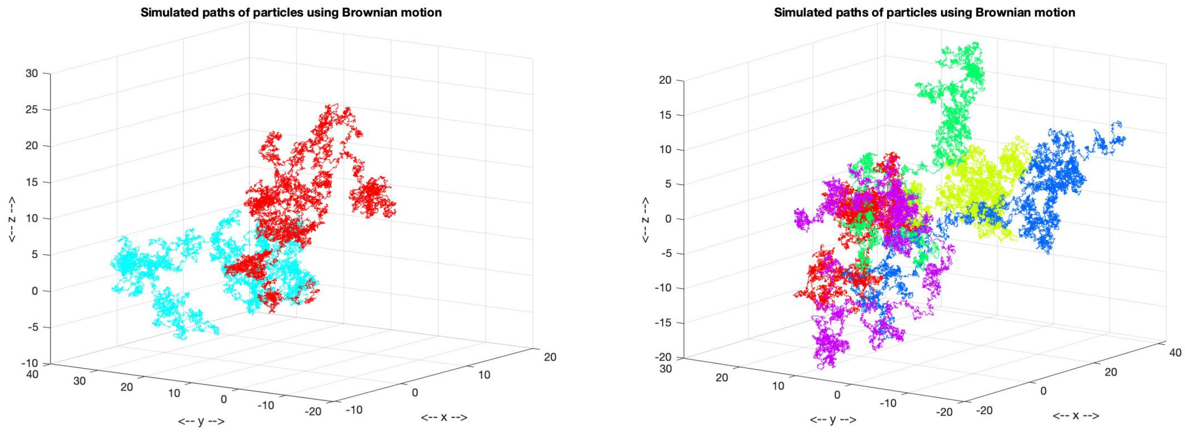


Figure 1: Brownian motions of particles

Figure 1 shows the simulated paths of particles following a Brownian motion. On the left figure, there are two particles being simulated, and on the right figure there are five particles being simulated. Each color represents a different particle and the path represented by that color is the motion of the particle. If the simulation is run again, a different graph will be produced due to the randomness of the particle movements.

2.3 Stochastic Calculus

The movements of the price of a stock can be described using the geometric Brownian motion, which is a stochastic differential equation [2]. The knowledge of some concepts in stochastic calculus is required in order to derive an analytical solution for the Black-Scholes equation.

Definition 1: An n-dimensional **Ito process** satisfies

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dB_t,$$

where B_t is a Brownian motion. This can also be written as

$$x_t = x_0 + \int_0^t \mu(x_s, s)ds + \int_0^t \sigma(x_s, s)dB_s \text{ [8].}$$

Definition 2: A random variable X follows a **Brownian motion with drift** if it satisfies

$$dx = a\ dt + b\ dB_t.$$

where B_t is a Brownian motion. [8]

Definition 3: A stochastic process is said to follow a **geometric Brownian motion** if it satisfies the following stochastic differential equation:

$$ds_t = \mu s_t dt + \sigma s_t dB_t \text{ [8].} \tag{2}$$

Definition 4: Ω is the set of all possible outcomes of an experiment. A set function is a real-valued function defined on some class of subsets of Ω [7]. A set function $\mathbb{Q}$ on a field $\mathcal{F}$ is known as a probability measure if it satisfies the following conditions [7]:

- $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$.
- $P(\phi) = 0$ and $P(\phi) = 1$, where ϕ represents the null set.
- If $A_1, A_2, \ldots$ is a disjoint sequence of $\mathcal{F}$-sets and if $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$, then

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k).$$

Lemma 1 (Ito's Lemma): Let x_t be an Ito process which satisfies the following stochastic differential equation (SDE), $ds_t = \mu(x_t, t)dt + \sigma(x_t, t)dB_t$. If B_t is a standard Brownian motion, i.e without any drift, and $f(t, x)$ is a $\mathcal{C}^2$ function, then $f(t, x_t)$ is also an Ito process whose differential is given by:

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu(x_t, t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x_t, t)\right) dt + \sigma(x_t, t)\frac{\partial f}{\partial x}dB_t \text{ [8].} \tag{3}$$

3 Black Scholes model

This section primarily uses information that can be found in [2, 8]. The assumptions about the model, stock price movements, and the analytical solution are all obtained using the aforementioned papers.
There are several variables that will be used in this section and subsequent sections. The table below provides an overview of these variables.

Variable	**Meaning**
K	Strike price
s_t	Price of stock at time t
p	Payoff function
r	Interest rate
μ	Mean rate of return
σ	Volatility of the stock

Table 1: Variables commonly used in the Black-Scholes model.

In financial mathematics, the movement of the price of an underlying asset (stocks, contracts etc) is governed by the following equation:

$$\frac{dS}{S} = \mu dt + \sigma dB, \tag{4}$$

$$\implies dS = \mu S dt + \sigma S dB.$$

where S denotes the price of the stock, μ is the interest rate. σ is the volatility of the stock price, and B is the Brownian motion which is a stochastic variable.

In the formulation of the model, Black and Scholes made some assumptions about the conditions of the market that is considered. The assumptions are as follows:

- The interest rate of the market is known and constant through time.
- The stock price follows a random walk in continuous time. The stock price also follows a lognormal distribution[1]. The variance of the rate of return on the stock is constant.
- The stock does not pay any dividends.
- The option is of the "European" type. This means the option can only be exercised at the time of maturity.
- There are no transaction costs involved in the purchasing and selling of the option.
- Short selling is permitted and does not have penalties [2].

These assumptions ensure that the value of the option only depends on the price of the underlying asset, in our case this is the stock, and the choice of constants for the known variables. When we consider an asset with risk free returns, i.e the theoretical rate of return

[1] For a detailed explanation of the lognormal distribution, please refer to [9]
[2] For an introduction and in-depth explanation of the mechanics of short selling, please refer to [10]

of an investment with zero risk, we denote μ as r [8].

The Black-Scholes partial differential equation can be derived from (4). There are various stochastic and financial methods used in the derivation. We will briefly mention the structure of the proof without delving into detail. The interested reader can refer to [2, 11, 8] for a detailed proof. Using Ito's Lemma, (4) can be written out in a form similar to that of (3). More precisely,

$$df = \left(\frac{\partial f}{\partial t} + \mu s_t \frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s_t^2 \frac{\partial^2 f}{\partial s^2} \right) dt + \sigma s_t \frac{\partial f}{\partial s} dB_t.$$

Financial theories such as self-financing portfolios and delta-hedging are used to develop the dynamics of the options and obtaining a risk-less portfolio. Using these aforementioned theories the Black-Scholes PDE can be derived.

The Black-Scholes equation can be represented by the following PDE:

$$\frac{\partial p}{\partial t} + rs\frac{\partial p}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 p}{\partial s^2} - rp = 0. \tag{5}$$

Using knowledge from section 2.2.1, we observe that:

$$\begin{aligned} c(t,s) &= \frac{\sigma^2 s^2}{2}, \\ d(t,s) &= 1, \\ e(t,s) &= rs, \\ f(t,s) &= -r. \end{aligned}$$

and $a(t,s), b(t,s), g$ are equal to zero. The discriminant of (5) is then equal to:

$$\Delta = 0 - 4 \cdot 0 \cdot \frac{\sigma^2 s^2}{2}, \text{ and}$$
$$a^2 + b^2 + c^2 = 0 + 0 + \left(\frac{\sigma^2 s^2}{2}\right) \neq 0,$$

when σ and s are not equal to zero. Hence, the Black-Scholes equation for European options is a parabolic partial differential equation.

3.1 Feynmann-Kac formula

Definition 3: Suppose we have a parabolic partial differential equation of the form:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) + \mu(x,t)\frac{\partial u}{\partial x}(x,t) + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} - V(x,t)u(x,t) + f(x,t) = 0, \\ u(x,T) = \phi(x,T), \end{aligned} \tag{6}$$

which is defined for all $x \in \mathbb{R}$ and $t \in [0,T]$. Then the **Feynmann-kac formula** tells us that the solution of this equation can be written as:

$$u(x,t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^r V(X_\tau,\tau)d\tau} f(X_r,r)dr + e^{-\int_t^T V(X_\tau,\tau)d\tau}\Phi(X_T)|X_t = x\right], \tag{7}$$

such that X is an Ito process under the probability measure $\mathbb{Q}$ [12].

Looking at (5), one can observe the similarity between that equation and (6). From (6), if we define x to be s, u to be p, $\mu(x,t)$ as rs, $\sigma(x,t)$ as $\sigma^2 s^2$, $V(x,t)$ as r, and $f(x,t)$ as 0, we obtain (5). It has already been established that the Black-Scholes PDE is a parabolic equation and hence we can apply the Feynmann-Kac formula. We assume that

$$p(s,T) = \phi(s,T).$$

The Feynman-Kac formula then gives us

$$p(s,t) = e^{-r(T-t)}\mathbb{E}_s[\phi(s_T)], \tag{8}$$

where

$$s_T = s_t + \int_t^T rs_\tau \, d\tau + \int_t^T \sigma s_\tau \, dW_\tau.$$

(8) can be interpreted as follows. The payoff value for a given stock price is computed over a period of time, with the final time being the time of maturity. The only price that we consider here is the payoff value at the time of maturity. This process is repeated a sufficient number of times and then the expectation of these values are taken. In our particular case, this is considered to be the average of all the values of $\phi(s_T)$ found. The average found is multiplied by a discounting factor which is dependent on the interest rate. The final payoff value we obtain, $p(s,t)$, is the expected profit the owner of an option will have at time t.

3.2 Boundary conditions for the model

The formulation of the boundary conditions can be found in [2, 8].
In the following subsection we will replace $p(s,t)$ with $C(s,t)$ when referring to a call option and with $P(s,t)$ when referring to a put option.
For an option of the "call" type, we have the following final condition,

$$C(s,T) = \max\{s-K, 0\}, \tag{9}$$

and the following boundary conditions

$$\begin{aligned} C(0,t) &= 0, \\ C(s_{\text{end}},t) &= s_{\text{end}} - Ke^{-r(T-t)}. \end{aligned} \tag{10}$$

For an option of the "put" type, we have the following final condition

$$P(s,T) = \max\{K-s, 0\}, \tag{11}$$

and the following boundary conditions

$$\begin{aligned} P(0,t) &= Ke^{-r(T-t)}, \\ P(s_{\text{end}},t) &= 0. \end{aligned} \tag{12}$$

3.3 Analytical solution

In the case of European options, it is possible to find the exact solution of (5). This has been done by Black and Scholes in their original paper. From the previous section, it can be observed that a call option has different boundary conditions when compared to a put option. Hence there will be two forms of exact solutions, one with respect to the call option, and one with respect to the put option.
With regards to a call option, the following are defined:

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned} \tag{13}$$

In the following equation $N(d)$ represents the cumulative normal density function. The cumulative normal density function is as follows:

$$N(z) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{z} exp\left(-\frac{y^2}{2}\right) dy.$$

Using the equations for d_1, d_2, the exact solution for the call option is found to be:

$$C(s,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2). \tag{14}$$

With regards to a put option, d_1, d_2 are defined the same as in (13). Using these, the exact solution for the put option is found to be:

$$P(s,t) = -SN(-d_1) + Ke^{-rT}N(-d_2), \tag{15}$$

where, $N(-d) = 1 - N(d)$.

4 Numerical methods

The following subsection contains brief explanations about the discretization methods that will be employed in this paper. Throughout the discretization, it is assumed that the grid points have equidistant spacing.

4.1 Discretization

An in-depth explanation for the derivation of the finite difference schemes can be found in [13]. Assume that f is a continuous function on a chosen interval. Using Taylor's expansion, we have that

$$\begin{aligned} f(x_i + \Delta x) &= f(x_i) + \Delta x \frac{df}{dx}(x_i) + \frac{\Delta x^2}{2}\frac{d^2 f}{dx^2}(x_i) + \dots \\ \Rightarrow \frac{df}{dx}(x_i) &= \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} + \mathcal{O}(\Delta x). \end{aligned} \tag{16}$$

$$\begin{aligned} f(x_i - \Delta x) &= f(x_i) - \Delta x \frac{df}{dx}(x_i) + \frac{\Delta x^2}{2}\frac{d^2 f}{dx^2}(x_i) + \dots \\ \Rightarrow \frac{df}{dx}(x_i) &= \frac{f(x_i) - f(x_i - \Delta x)}{\Delta x} + \mathcal{O}(\Delta x). \end{aligned} \tag{17}$$

(16) is known as the forward difference scheme, and (17) is known as the backwards difference scheme. Subtracting (17) from (16) we get a second order accurate discretization for the first derivative.

$$\frac{df}{dx}(x_i) = \frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2). \tag{18}$$

(18) is known as the central difference scheme.

4.2 Backward Euler method

We discretize (5) with respect to the Backward Euler method. This is an implicit method. Recall that we assume the grid points always have equidistant spacing. First, assume that we split the x-axis, which represents the price of the underlying asset, into M intervals. Then $\Delta S = \frac{1}{M}$, and the x_j^{th} point can be found by $x_j = j\Delta S$. We also assume that the y-axis, which represents time, is split into N intervals, which are equidistant. Then $\Delta t = \frac{1}{N}$. The value of x_j at time n is represented as x_j^n.

Note that P_j^n represents P when analyzed at the j^{th} step on the x-axis and the n^{th} time step.

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} = rP_j^n - \frac{\sigma^2 (j\Delta s)^2}{2}\frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{\Delta s^2} - r(j\Delta s)\frac{P_{j+1}^n - P_{j-1}^n}{2\Delta s}.$$

Rearranging this, we get

$$P_j^{n+1} = P_j^n + \Delta t \left[rP_j - \frac{\sigma^2 (j\Delta s)^2}{2}\frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{\Delta s^2} - r(j\Delta s)\frac{P_{j+1}^n - P_{j-1}^n}{2\Delta s} \right]. \tag{19}$$

We further rearrange (19) in order to collect the terms with respect to $(j+1), j, (j-1)$.

$$\begin{aligned} P_j^{n+1} = P_{j+1}^n \left(-\frac{\sigma^2 (j\Delta s)^2 \Delta t}{2\Delta s^2} - \frac{r(j\Delta s)\Delta t}{2\Delta s} \right) + P_j^n \left(1 + r\Delta t + \frac{\sigma^2 (j\Delta s)^2 \Delta t}{2\Delta s^2} \right) \\ + P_{j-1}^n \left(\frac{r(j\Delta s)\Delta t}{2\Delta s} - \frac{\sigma^2 (j\Delta s)^2 \Delta t}{2\Delta s^2} \right). \end{aligned} \tag{20}$$

This can be further simplified as follows:

$$\begin{aligned} P_j^{n+1} = P_{j+1}^n \left(-\frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2} \right) + P_j^n \left(1 + r\Delta t + \frac{\sigma^2 j^2 \Delta t}{2} \right) \\ + P_{j-1}^n \left(\frac{rj\Delta t}{2} - \frac{\sigma^2 j^2 \Delta t}{2} \right) \end{aligned} \tag{21}$$

The discretization found above can be written in matrix notation. Let

$$a_j = \left(-\frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2} \right), \tag{22}$$

$$b_j = \left(1 + r\Delta t + \frac{\sigma^2 j^2 \Delta t}{2} \right), \tag{23}$$

$$c_j = \left(\frac{rj\Delta t}{2} - \frac{\sigma^2 j^2 \Delta t}{2} \right). \tag{24}$$

Using the coefficients above and (21), a matrix notation can be developed for the implicit discretization.

$$\mathbf{A} = \begin{bmatrix} b_1 & a_1 & & & & & \\ c_2 & b_2 & a_2 & & & & \\ & c_j & b_j & a_j & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & c_{M-2} & b_{M-2} & a_{M-2} \\ & & & & & c_{M-1} & b_{M-1} \end{bmatrix}, \tag{25}$$

$$\mathbf{P}^n = \begin{bmatrix} P_1^n \\ P_2^n \\ \vdots \\ P_{M-2}^n \\ P_{M-1}^n \end{bmatrix}. \tag{26}$$

We need to account for boundary conditions of our model. To do this, we define the following vector:

$$\mathbf{b}^n = \begin{bmatrix} c_1 P_0^n \\ 0 \\ \vdots \\ 0 \\ a_{M-1} P_M^n \end{bmatrix} \tag{27}$$

Using these notations, the implicit method discretization can now be written as:

$$\mathbf{A}\mathbf{P}^n = \mathbf{P}^{n+1} - \mathbf{b}^n \text{ for n = N-1, N-2, } \ldots, 0. \tag{28}$$

The implicit method is unconditionally stable [14].

4.3 Forward Euler Method

The assumptions made with respect to equidistant grids in the Backwards Euler section will be made in this section as well. The finite difference schemes for the spatial derivative remain similar to that of the implicit scheme.
Note that P_j^n represents P when analyzed at the j^{th} step on the x-axis and the n^{th} time step.

$$\frac{P_j^n - P_j^{n-1}}{\Delta t} = rP_j^n - r(j\Delta s)\frac{P_{j+1}^n - P_{j-1}^n}{2\Delta s} - \frac{\sigma^2 (j\Delta s)^2}{2}\frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{\Delta s^2}.$$

Rearranging this, the following is obtained:

$$P_j^{n-1} = P_j^n - \Delta t\left(rP_j^n - r(j\Delta s)\frac{P_{j+1}^n - P_{j-1}^n}{2\Delta s} - \frac{\sigma^2 (j\Delta s)^2}{2}\frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{\Delta s^2}\right).$$

The above can be further rearranged and simplified, similar to the simplification step used in (20), to obtain:

$$\begin{aligned} P_j^{n-1} = P_{j+1}^n \left(\frac{rj\Delta t}{2} + \frac{\sigma^2 j^2 \Delta t}{2}\right) + P_j^n \left(1 - r\Delta t - \sigma^2 j^2 \Delta t\right) \\ + P_{j-1}^n \left(\frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2}\right). \end{aligned} \tag{29}$$

Using the method outlined above, if we consider the discretization at time step $(n+1)$ instead of n, we get the following:

$$\begin{aligned} P_j^{n} = P_{j+1}^{n+1} \left(\frac{rj\Delta t}{2} + \frac{\sigma^2 j^2 \Delta t}{2}\right) + P_j^{n+1} \left(1 - r\Delta t - \sigma^2 j^2 \Delta t\right) \\ + P_{j-1}^{n+1} \left(\frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2}\right). \end{aligned} \tag{30}$$

Let

$$\alpha_j = \left(\frac{rj\Delta t}{2} + \frac{\sigma^2 j^2 \Delta t}{2}\right), \tag{31}$$

$$\beta_j = \left(1 - r\Delta t - \sigma^2 j^2 \Delta t\right), \tag{32}$$

$$\gamma_j = \left(\frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2}\right), \tag{33}$$

and let $\mathbf{P}^n$ be the same as (26). Using the above information, a matrix notation can be developed for the explicit discretization.

$$\mathbf{A} = \begin{bmatrix} \beta_1 & \alpha_1 & & & & & \\ \gamma_2 & \beta_2 & \alpha_2 & & & & \\ & \gamma_j & \beta_j & \alpha_j & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \gamma_{M-2} & \beta_{M-2} & \alpha_{M-2} \\ & & & & & \gamma_{M-1} & \beta_{M-1} \end{bmatrix}, \tag{34}$$

$$\mathbf{b}^{n+1} = \begin{bmatrix} \gamma_1 P_0^{n+1} \\ 0 \\ \vdots \\ 0 \\ \alpha_{M-1} P_M^n \end{bmatrix}. \tag{35}$$

$\mathbf{b}^n$ accounts for the boundary conditions of the model.
Using these notations, the explicit method discretization can now be written as:

$$\mathbf{P}^n = \mathbf{A}\mathbf{P}^{n+1} + \mathbf{b}^{n+1} \text{ for n = N-1, N-2, \ldots, 0.} \tag{36}$$

4.3.1 Stability analysis of the explicit method

For the forward Euler, a stability analysis must be carried out. Observe that linear operator for the forward euler method is of the form:

$$\mathcal{L}_h(P_j) = (a+b)P_{j+1} - 2bP_j + (b-a)P_{j-1}, \tag{37}$$

where,

$$\begin{aligned} a &= \frac{r}{2\Delta s}(j\Delta s), \\ b &= \frac{\sigma^2}{2\Delta s^2}(j\Delta s)^2. \end{aligned}$$

By plugging in, $P_j = e^{ij\phi}$ into (37), also known as the Fourier component, the following is obtained:

$$\begin{aligned} \mathcal{L}_h(e^{ij\phi}) &= (a+b)e^{i(j+1)\phi} - 2be^{ij\phi} + (b-a)e^{i(j-1)\phi} \\ &= (a+b)e^{ij\phi+i\phi} - 2be^{ij\phi} + (b-a)e^{ij\phi-i\phi} \\ &= e^{ij\phi}((a+b)e^{i\phi} - 2b + (b-a)e^{-i\phi}). \end{aligned} \tag{38}$$

To obtain the eigenvalue of the operators, the expression within the brackets must be solved.

$$\begin{aligned} &((a+b)e^{i\phi} - 2b + (b-a)e^{-i\phi}) = ae^{i\phi} + b(e^{i\phi} - 2 + e^{i\phi}) - ae^{-i\phi} \\ &= a(e^{i\phi} - e^{-i\phi}) - 4b\cdot\sin^2\left(\frac{\phi}{2}\right) = 2i\cdot a\cdot\sin(\phi) - 4b\cdot\sin^2\left(\frac{\phi}{2}\right). \end{aligned} \tag{39}$$

Hence (39) then becomes,

$$\mathcal{L}_h(e^{ij\phi}) = e^{ij\phi}\left(2i \cdot a \cdot \sin(\phi) - 4b \cdot \sin^2\left(\frac{\phi}{2}\right)\right). \tag{40}$$

This implies that the eigenvalue, λ, of the operator is the final expression obtained in (39). The forward euler method is defined by

$$P_j^{n+1} = P_j^n + \Delta t f(P_j^n, t_n) \text{ where } \frac{dP}{dt} = f(P,t).$$

Using the eigenvalues for the given f, it is known that for a general w_j:

$$w^{j+1} = w^j + \Delta t \lambda w^j = (1 + \Delta t \lambda) w^j.$$

In order for stability to be satisfied, one must have that $|1 + \Delta t\lambda| < 1$.

$$|1 + \Delta t\lambda| = \left|1 + \Delta t\left(-4b \cdot \sin^2\left(\frac{\phi}{2}\right) + i \cdot 2a \cdot \sin(\phi)\right)\right| < 1. \tag{41}$$

Let,

$$\begin{aligned} x &= 1 - 4b(\Delta t) \cdot \sin^2\left(\frac{\phi}{2}\right), \\ y &= 2a(\Delta t) \cdot \sin(\phi). \end{aligned}$$

Substituting these variables into (41), it can be rewritten as:

$$|x + iy| < 1. \tag{42}$$

Recall that for complex numbers,

$$|x + iy| = \sqrt{x^2 + y^2}.$$

$$x^2 = \left(1 - 4b(\Delta t) \cdot \sin^2\left(\frac{\phi}{2}\right)\right)^2 = 1 - 8b(\Delta t)\sin^2\left(\frac{\phi}{2}\right) + 16b^2(\Delta t)^2\sin^4\left(\frac{\phi}{2}\right),$$

$$y^2 = (2a(\Delta t) \cdot \sin(\phi))^2 = 4a^2(\Delta t)^2\sin^2(\phi).$$

Substituting the above expansions into (42),

$$\sqrt{x^2 + y^2} = \left[1 - 8b(\Delta t)\sin^2\left(\frac{\phi}{2}\right) + 16b^2(\Delta t)^2\sin^4\left(\frac{\phi}{2}\right) + 4a^2(\Delta t)^2\sin^2(\phi)\right]^{\frac{1}{2}} < 1, \tag{43}$$

$$\sqrt{x^2 + y^2} < 1 \Rightarrow x^2 + y^2 < 1.$$

The case of $x^2 + y^2 < -1$ is disregarded here because in our case $|\cdot|$ represents the magnitude of a number, which cannot be negative. Going back to (43), we find

$$\begin{aligned} &1 - 8b(\Delta t)\sin^2\left(\frac{\phi}{2}\right) + 16b^2(\Delta t)^2\sin^4\left(\frac{\phi}{2}\right) + 4a^2(\Delta t)^2\sin^2(\phi) < 1, \\ &\Rightarrow -8b(\Delta t)\sin^2\left(\frac{\phi}{2}\right) + 16b^2(\Delta t)^2\sin^4\left(\frac{\phi}{2}\right) + 4a^2(\Delta t)^2\sin^2(\phi) < 0. \end{aligned} \tag{44}$$

Looking at (44), it can be deduced that the only way this expression is less than zero is if the first term is greater than the second and third terms combined. This is because a and b are defined to be positive, $\sin^2$, $\sin^4$, and $(\Delta t)^2$ are positive as well. Hence,

$$
\begin{aligned}
&16b^2(\Delta t)^2\sin^4\left(\frac{\phi}{2}\right) + 4a^2(\Delta t)^2\sin^2(\phi) < 8b(\Delta t)\sin^2\left(\frac{\phi}{2}\right), \\
&\Rightarrow 4b^2(\Delta t)\sin^4\left(\frac{\phi}{2}\right) + (\Delta t)a^2\sin^2(\phi) < 2b\sin^2\left(\frac{\phi}{2}\right), \\
&\Rightarrow (\Delta t)\left(4b^2\sin^4\left(\frac{\phi}{2}\right) + a^2\sin^2(\phi)\right) < 2b\sin^2\left(\frac{\phi}{2}\right), \\
&\Rightarrow \Delta t < \frac{2b\sin^2\left(\frac{\phi}{2}\right)}{4b^2\sin^4\left(\frac{\phi}{2}\right) + a^2\sin^2(\phi)}, \\
&\qquad = \frac{2b\sin^2\left(\frac{\phi}{2}\right)}{4b^2\sin^4\left(\frac{\phi}{2}\right) + 4a^2\sin^2\left(\frac{\phi}{2}\right)\cos^2\left(\frac{\phi}{2}\right)}, \\
&\qquad = \frac{b}{2b^2\sin^2\left(\frac{\phi}{2}\right) + 2a^2\cos^2\left(\frac{\phi}{2}\right)}, \\
&\qquad = \frac{b}{2a^2 + 2(b^2 - a^2)\sin^2\left(\frac{\phi}{2}\right)}.
\end{aligned}
\tag{45}
$$

Hence we conclude that if $a < b$, then

$$\Delta t \leq \frac{1}{2b},$$

and if $b < a$, then

$$\Delta t \leq \frac{b}{2a^2}.$$

5 Stochastic ODE model

Another worthwhile analysis to carry out is that of the stochastic differential equation. This analysis can help give some insight as to why people are more prone to use the Black-Scholes model or the binomial option pricing method [15]. As shown in Section 3, the Black-Scholes model is derived from the following ODE:

$$dS = \mu S dt + \sigma S dB \tag{46}$$

The discrete time version of this equation can be respresented using the following information that can be deduced from previous knowledge.

$$\begin{aligned} dS &= S_{i+1} - S_i \\ dt &= t_{i+1} - t_i = \Delta t \\ dB &= Z\sqrt{\Delta t} \end{aligned}$$

where Z is a normally distributed random variable with mean 0 and variance 1. The discretization of the Brownian motion is found in [16, 17].

Using the discretizations that we have obtained above, we created two possible solutions to model the evolution of (46). For the two figures, the time grid was divided into equidistant intervals of varying sizes.

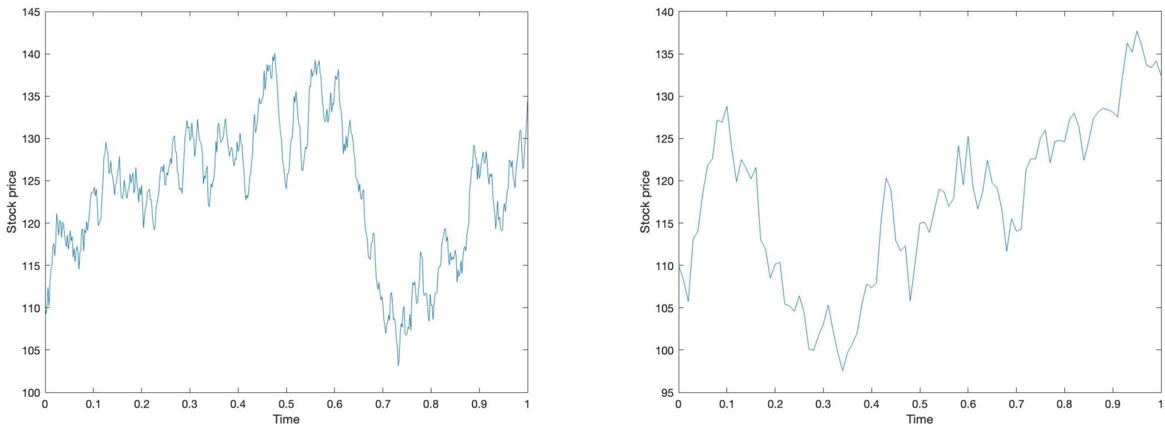


Figure 2: Some solutions of the stochastic differential equation.

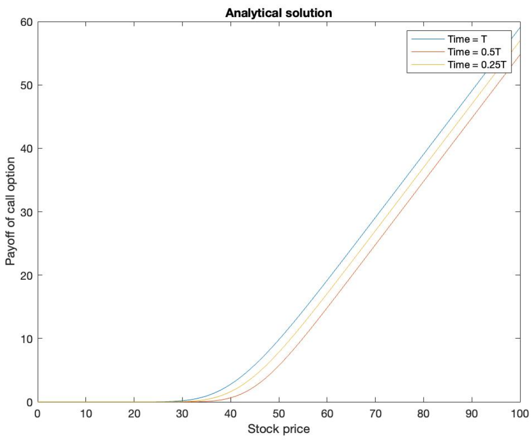
6 Results and discussion

The following subsections contain results from a numerical analysis of the analytical solutions and finite difference method discretizations described previously in the report.
For the numerical experiments, we assume that the time to maturity is one month, i.e, $T = 1$. We also assume that the constant rate of interest is 20%, i.e $r = 0.2$, and the strike price agreed upon is \$50, i.e $K = 50$. The values for σ will be varied. Similarly, the value for Δt must be varied for the Forward Euler scheme when we vary σ because a change in the latter will result in an appropriate change required in Δt due to the results found from the stability analysis. We divide the grid for the stock price into 100 equidistant intervals, hence ΔS will be taken as:

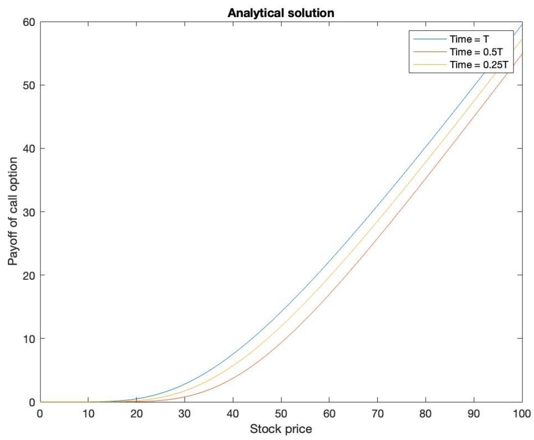
$$\Delta S = \frac{\text{Maximum value of S on the grid}}{\text{Number of intervals}} = \frac{S_{\max}}{100}$$

6.1 Analytical solution

We introduced two different analytical solutions in Section 3. One was derived in the Black-Scholes paper [2] and the other was obtained using the Feynman-Kac formula (8). In this subsection we present the results obtained by numerically analyzing both of these solutions and comparing the results.
The following graphs have been obtained using a numerical analysis of the analytical solution presented in [2].

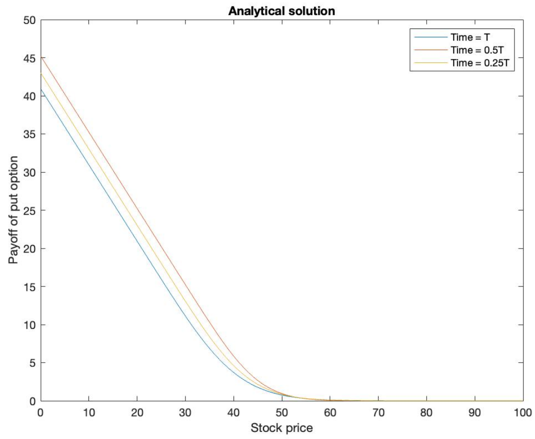


(a) Call solution when $\sigma = 0.2$.

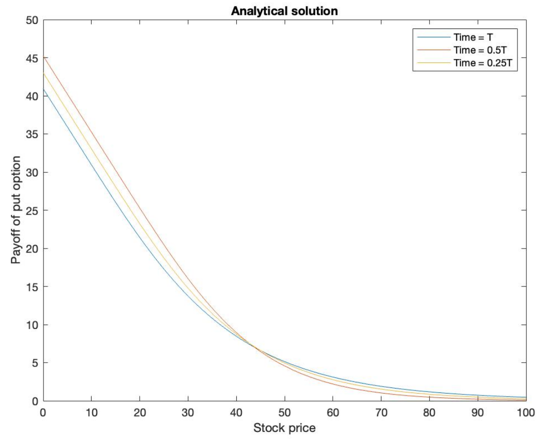


(b) Call solution when $\sigma = 0.5$.

Figure 3: Analytical solutions of the European call option.



(a) Put solution when $\sigma = 0.2$.



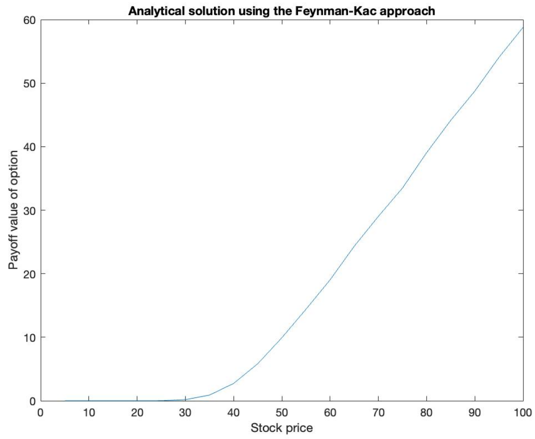
(b) Put solution when $\sigma = 0.5$.

Figure 4: Analytical solutions of the European put option.

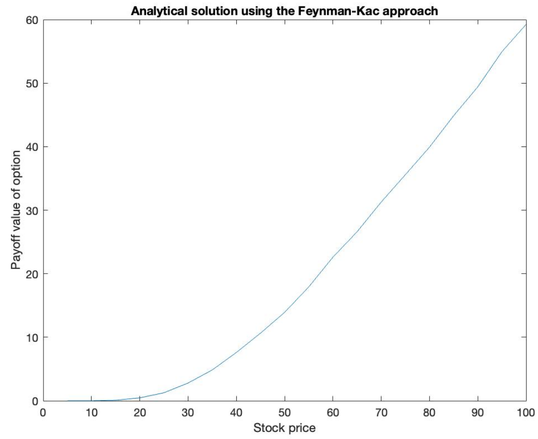
The graphs in Figure 3 and Figure 4 each contain three different lines. These lines represent the value of the payoff at different times. The line labelled as 0.5T shows what the value of the payoff of an option would theoretically be if we exercised the option at the halfway point of our time period. For example, if the maturity time is set as one month, then the line for 0.5T represents the theoretical payoff value expected in half a month. Similar reasoning can be applied for the case of 0.25T. In the case of European options, we are only interested in the value of the payoff at the time of maturity. In our case, this is when we take time to be equal to T. This is represented by the line labelled Time = T.

As volatility increases, the probability that the stock price will rise or fall increases, which in response will also increase the value of both call and put options. This is because the volatility represents the standard deviation of the return on the stocks. If there is a higher standard deviation, then there is a greater chance for the holder of the option to make a profit. We observe that while this also leads to an increase in the potential loss, in the case of options, the holder of the option can just choose to let the option expire. Hence, if we increase the volatility term, the σ^2 term in our discretization results in a larger possible payoff.

A numerical program was also developed to evaluate the solution found in (8) using the Feynman-Kac method. The following graphs have been obtained using the Feynman-kac formula, (8).



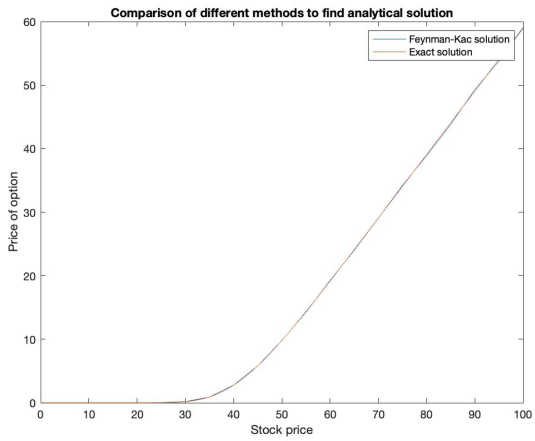
(a) Feynman-Kac call solution when $\sigma = 0.2$.



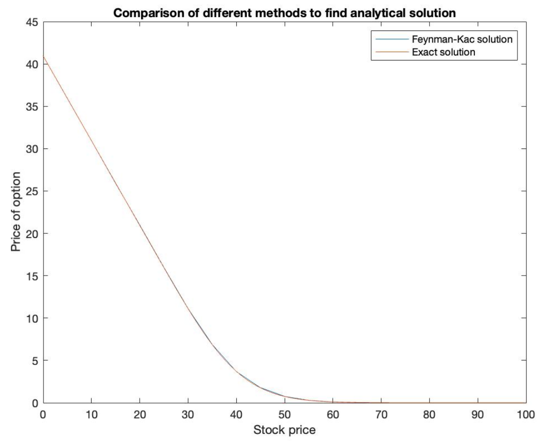
(b) Feynman-Kac call solution when $\sigma = 0.5$.

Figure 5: Analytical solutions of the European call option using the Feynman-Kac formula.

The final solution obtained at the time of expiry when $\sigma = 0.2$ was then compared with the solution obtained using the analytical solution given by Black and Scholes.



(a) Comparison for the call option solution.



(b) Comparison for the put option solution.

Figure 6: Comparing the solutions obtained when using the Feynman-Kac approach and the analytical approach highlighted in section 3.3.

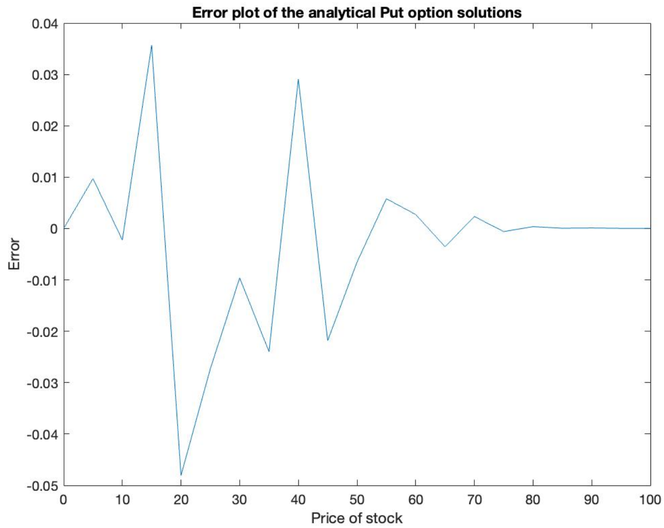


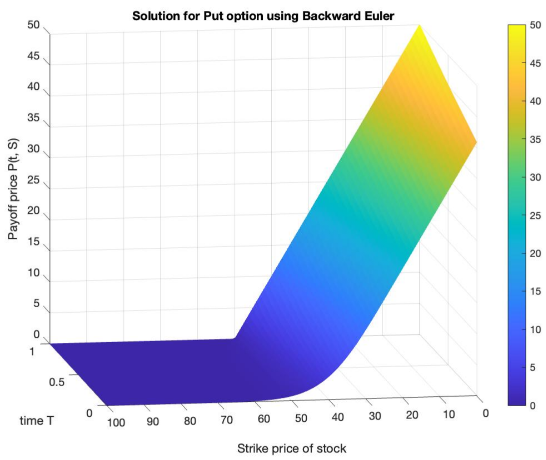
Figure 7: Error between the Feynman-Kac approach and the analytical approach highlighted in section 3.3.

6.2 Backward Euler

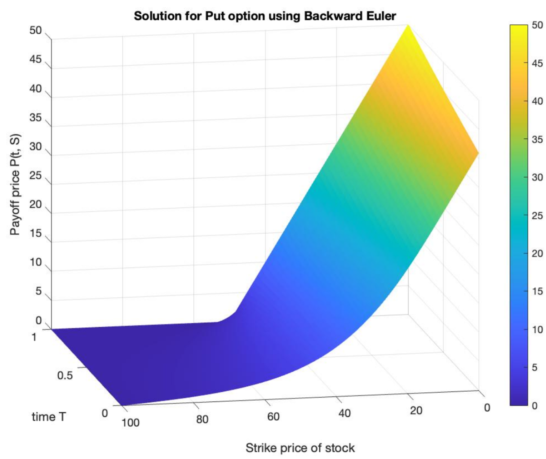
In this subsection we divide the grid for the time into 500 equidistant intervals, hence Δt will be taken as:

$$\Delta t = \frac{\text{Time to maturity - beginning time}}{500} = \frac{T}{500}.$$

We will always take the beginning time, i.e. the time that the two parties officially enter into an options contract, to be 0. The following graphs have been obtained using the discretization found in (28).



(a) Put solution when $\sigma = 0.2$.

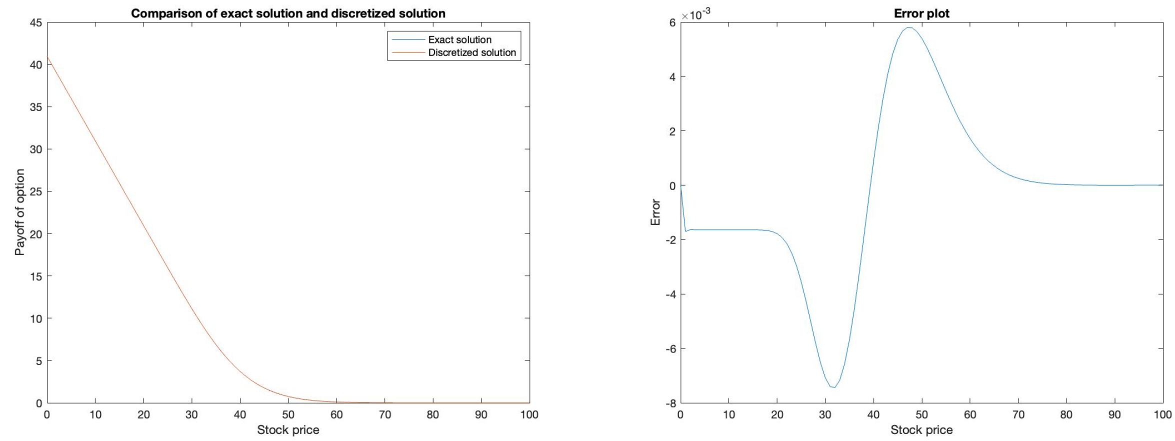


(b) Put solution when $\sigma = 0.5$.

Figure 8: Backward Euler solutions of the European put option with varying volatilities.

In this case, $T = 0$ represents the time of maturity. The mesh represents the value of the payoff at varying time steps and varying stock prices. While we are only concerned with

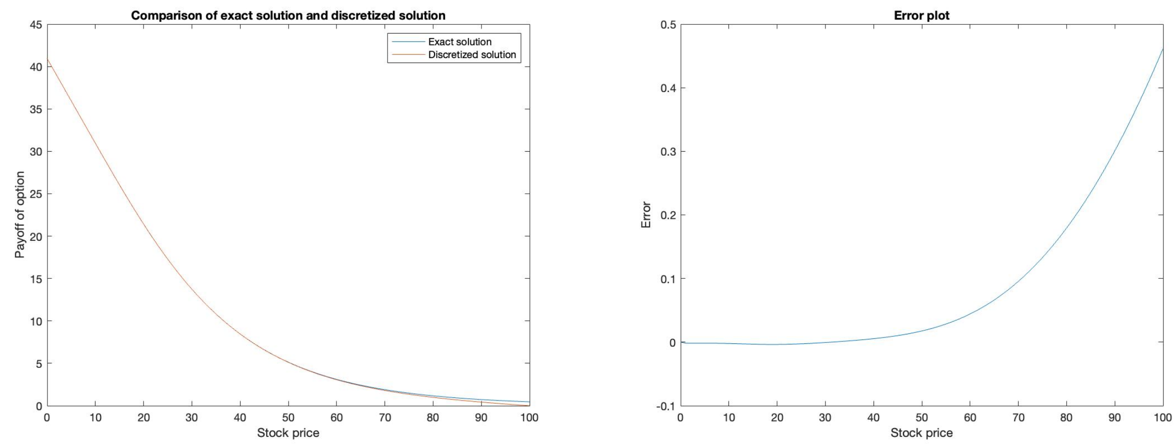
the value of the payoff at the time of maturity, it is interesting to observe the evolution of the solution over time.
The final solution obtained at the time of expiry for both σ's were compared with the solution obtained using the analytical solution in Section 3.3.



(a) Comparison of put solution when $\sigma = 0.2$.

(b) Error of the put solution when $\sigma = 0.2$.

Figure 9: Comparison of the solution of a put option, obtained using the Backward Euler method and the analytical solution.



(a) Comparison of put solution when $\sigma = 0.5$.

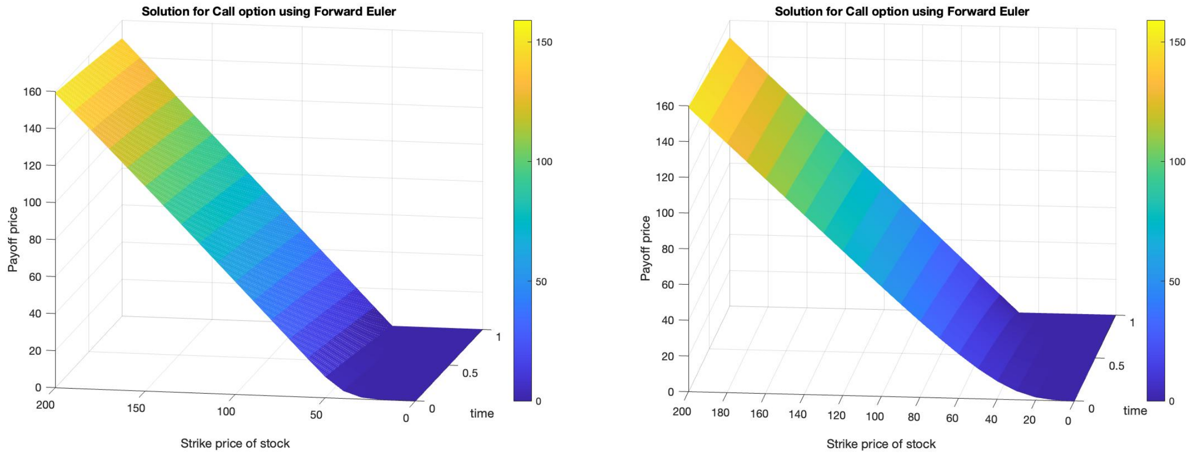
(b) Error of the put solution when $\sigma = 0.5$.

Figure 10: Comparison of the solution of a put option, obtained using the Backward Euler method and the analytical solution.

6.3 Forward Euler

We begin this subsection by evaluating the solution for $\sigma = 0.2$ first. In this case, Δt will remain the same as in Section 6.2. However when we evaluate the solution for $\sigma = 0.5$, the stability analysis requires changes to the value of Δt. That will be shown further in this subsection.

The following graphs have been obtained using the discretization found in (36).



(a) Call solution when $\sigma = 0.2$.

(b) Call solution when $\sigma = 0.5$.

Figure 11: Forward Euler solutions of the European call option.

When $\sigma = 0.5$, we encounter errors if we keep Δt the same as in the calculations for $\sigma = 0.2$. This can be attributed to the results of the stability analysis conducted previously. When increasing σ, the number of intervals to be considered for time must also be increased. In this case, we take $\Delta t = \frac{T}{2300}$.

In Figure 12 we highlight the behaviour of the solution if we don't comply with the conditions required for stability.

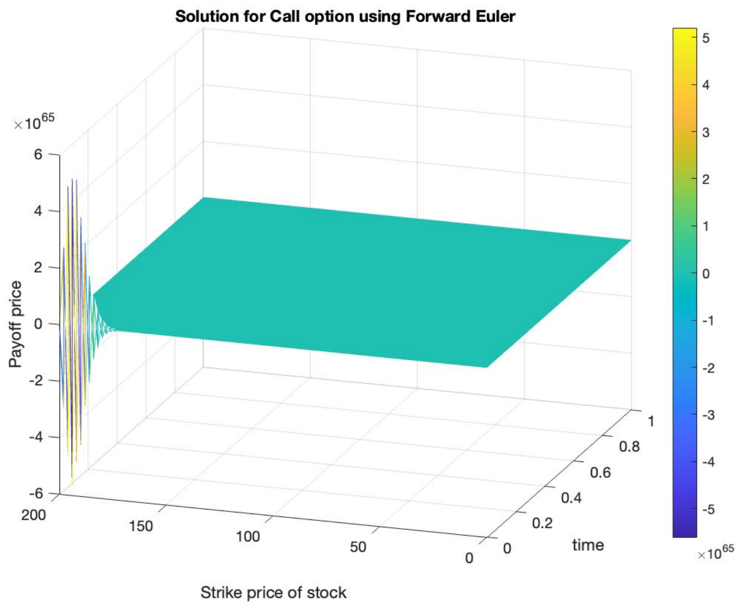
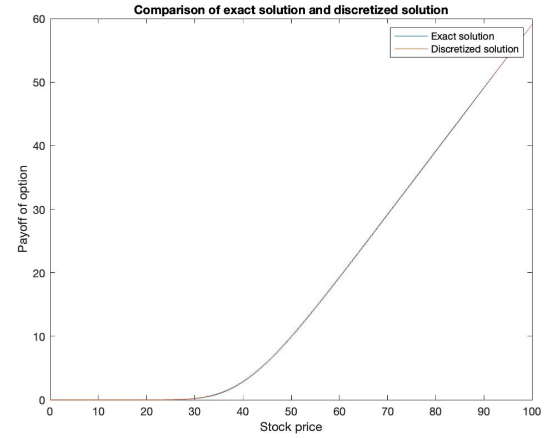
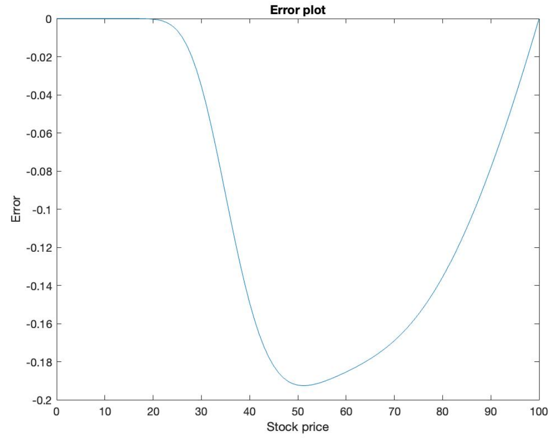


Figure 12: Solution for $\sigma = 0.5$ using Forward Euler when we use 2200 intervals.

The final solution obtained at the time of expiry for both σ's were compared with the solution obtained using the analytical solution in Section 3.3.

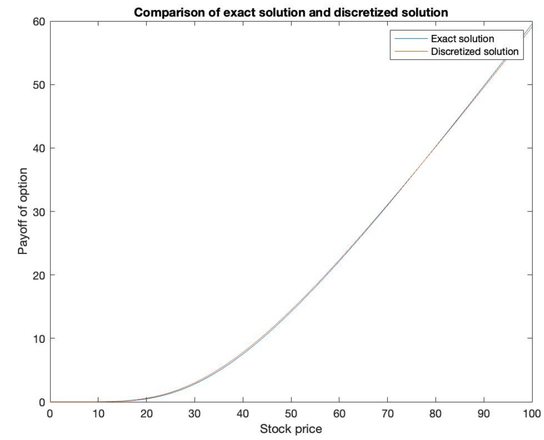


(a) Comparison of call solution when $\sigma = 0.2$

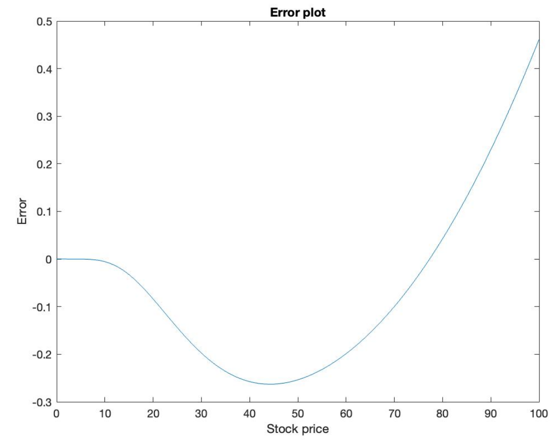


(b) Error of the call solution when $\sigma = 0.2$

Figure 13: Comparison of the solution of a European call option, obtained using the Forward Euler method and the analytical solution.



(a) Comparison of call solution when $\sigma = 0.5$



(b) Error of the call solution when $\sigma = 0.5$

Figure 14: Comparison of the solution of a European call option, obtained using the Forward Euler method and the analytical solution.

6.4 Discussion

The table below shows the computational time required by each method for varying sigmas. In order for a fair comparison in the case of $\sigma = 0.5$, the time grid for the Backward Euler method was divided into 2300 intervals, despite the unconditional stability of this method. The time grid for the analytical solution was also divided into 2300 intervals. All other variables were kept constant throughout these experiments.

	FE Computational time	BE Computational time	Analytical solution
$\sigma = 0.2$	0.012179s	0.034740s	0.872303s
$\sigma = 0.5$	0.016074s	0.070330s	3.381104s

Table 2: Computational times for the two numerical methods and the analytical solution.

From the computation times we observe that the Forward Euler method is the quickest method. The Backward Euler method takes longer than the Forward Euler which is to be expected because it is solving an implicit equation [18]. The analytical solution takes significantly longer to compute in comparison to the finite difference methods used. This could be due to the fact that we are evaluating this in the syms environment or evaluating a stochastic process at every interval.

While we have only focused on varying sigma in our experiments, from the graphs given in the previous subsections and some theory we can deduce that varying other factors will also affect the value of our payoff. In the case of call options, higher stock prices lead to higher payoff values whereas for put options a higher stock price leads to lower payoff values. Similarly, a different choice for the strike price will affect our final payoff value solutions. Increasing the interest leads to a higher payoff values for call options and lower payoff values for put options. Figure 15 shows the analytical solution when we consider the interest rate to be 0.5 and $\sigma = 0.2$. All other variables remain the same. Comparing this graph with the one in Figure 3, we observe that we obtain higher payoff values when the interest rate is higher.

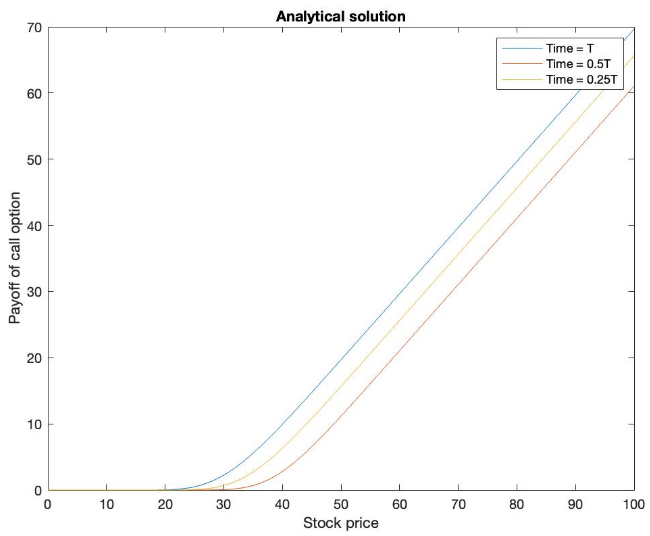


Figure 15: Analytical solution for European call option when $r = 0.5$ and $\sigma = 0.2$

We observe that the error for the put solution when $\sigma = 0.2$ is less than the error for the call solution when $\sigma = 0.2$. When $\sigma = 0.5$, the errors for the Backward Euler method and the Forward Euler method are both similar. In the error graphs for the Backward and Forward Euler methods we notice that towards $S_{\max}$ the error appears to be increasing rapidly. This can be attributed to the fact that there is no boundary condition limitation imposed on the analytical solution but we have boundary conditions imposed in the finite difference methods. This means that as we increase our grid, the error on the boundary can grow larger.

In the model presented in this report, we assumed ideal model conditions with a constant interest rate and constant volatility. Realistically, volatility and rate of return on assets do not stay constant. [19] is a graph showcasing the variation in the expectation of the volatility in the market. Analyzing the Black-Scholes model for changing variables will be more complicated when using the Forward Euler method because the stability of the time step is dependent on some of these variables. Section 6.9 in [8] concerns the introduction of jump diffusion models which account for changing volatility's and also volatility's that are a function of the price of the underlying stock at a given time.

7 Conclusion

In this report we have analysed numerical methods pertaining to the analytical solution of the Black Scholes equation and further considered the finite difference method. We obtained the computational times and errors of each of these mechanisms and were directly compared with the analytical scheme.

In the discussion of results we clearly observed that solving the equation analytically takes significantly longer than using the finite difference method. However, the two finite difference schemes that were discussed are not without their drawbacks. We noticed that Forward Euler scheme is computationally superior to other methods but the method is unstable. We can utilise the Backward Euler method to counteract this instability as the scheme is unconditionally stable. There is a trade-off as the Backward Euler scheme takes computationally longer but we believe it to be a worthwhile compromise due to the added overall stability. Thus, the Backward Euler is the recommended scheme if we choose to analyse the Black-Scholes Model for European options, or account for constantly changing volatility and interest rates.

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