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Linear Relations for Multiple Zeta Values

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Abstract

To look into algebraic relations for zeta values, it is useful to introduce multiple zeta values. Multiple zeta values can both be represented by a series and by an integral. From this integral representation, we derive the Duality theorem, which gives equalities between multiple zeta values. Furthermore, it is possible to compare the multiplication of two multiple zeta values given by the different representations. By doing so, the finite double shuffle relation occurs. To generalize this relation, we introduce the so-called stuffle and shuffle product on a non-commutative polynomial algebra. Furthermore, we explore other linear relations for multiple zeta values. One of them is the integral series identity, which is conjectured to imply all other linear relations for multiple zeta values. We derive Hoffman's relation and the restricted sum formula from this identity to support this conjecture. Finally, we look into the dimension and basis for the spaces spanned by multiple zeta values for a fixed weight. The dimension and basis for these spaces are conjectured by Zagier's conjecture and Hoffman's conjecture, respectively. We work out some examples of Brown's theorem, which partially proves Hoffman's conjecture.

Contents

1 Introduction

In 1859 German mathematician Bernard Riemann gave new insights regarding the Riemann zeta function in his book "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" [Riemann, 1859]. While the function is named after him, he was not the first person to explore this function. Swiss mathematician Leonhard Euler wrote about this function before Riemann did. The formal definition of this function can be written as follows.

Definition 1.1. *The Riemann zeta function is defined by the series*

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

and is convergent for $\Re(s) > 1$.

Moreover, [Ivic, 2012] proves in Theorem 1.2 that the Riemann zeta function admits an analytic continuation over the whole complex plane with a simple pole at $s = 1$ with residue 1. Furthermore, function evaluations are called (Riemann) zeta values. The zeta values for the even positive integers were studied by Euler. [Ivic, 2012] proves in Theorem 1.4 an explicit formula for them

$$\zeta(2k) = (-1)^{(k+1)} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, \qquad k \geq 1, \tag{1}$$

where the B_k's are Bernoulli numbers. [Gould, 1972] gives in equation (1) an explicit formula for these numbers, namely

$$B_k = \sum_{i=0}^{k} \frac{1}{i+1} \sum_{j=0}^{i} (-1)^j \binom{i}{j} j^k, \qquad k \geq 0.$$

This means that the zeta values for the even positive integers are irrational, since they are a multiple of π. We might question whether we know something similar for the odd positive integers. Unfortunately, mathematicians have not found a similar explicit formula for these zeta values. We do not even know whether they are irrational or not.

When looking at numbers we can categorize them as algebraic and transcendental over $\mathbb{Q}$.

Definition 1.2. *A number α is called algebraic over $\mathbb{Q}$ if there exists a nonzero polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. In the other case, α is called transcendental over $\mathbb{Q}$.*

German mathematician Ferdinand von Lindemann proved in 1882 that π is transcendental over $\mathbb{Q}$. Therefore the zeta values for the even positive integers are transcendental as well since they are a multiple of π. Mathematicians tried to see whether the zeta values for odd positive integers are transcendental or not, but nothing did succeed. However, a stronger statement is conjectured.

Conjecture 1.3 (Transcendence Conjecture)**.** *The set*

$$\mathcal{S} := \{\pi, \zeta(2n+1) | n \geq 1\}$$

is algebraically independent over $\mathbb{Q}$. This means that for any finite set $\{\alpha_1, \ldots, \alpha_k\} \subset \mathcal{S}$ for some $k \geq 1$ there does not exists a nonzero polynomial $f \in \mathbb{Q}[x_1, \ldots, x_k]$ such that $f(\alpha_1, \ldots, \alpha_k) = 0$.

Note that this statement implies that the numbers in the set $\mathcal{S}$ are then transcendental, by taking a singleton. The set $\mathcal{S}$ contains the number π such that it also includes all zeta values at the even positive integers. Exploring this conjecture, it might be helpful to define so-called multiple zeta values. They are defined by the series

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}},$$

where in this paper we restrict to the case in which we have integers $s_i \geq 1$ for $2 \leq i \leq k$ and $s_1 \geq 2$ (this last condition ensures convergence of the series). The product of two zeta values can be expressed in terms of a linear combination of multiple zeta values. For example

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

This identity was already found by Euler before the theory of multiple zeta values was defined. This identity is the motivation to find linear relations for multiple zeta values. The goal of this paper is to explore general linear relations for multiple zeta values. An important notion for this is the integral representation of multiple zeta values

$$\zeta(s_1, \ldots, s_k) = \int_{\Delta^u} \eta_0(t_1) \ldots \eta_0(t_{s_1-1})\eta_1(t_{s_1})\eta_0(t_{s_1+1}) \ldots \eta_1(t_u),$$

where $u = s_1 + \ldots + s_k$ (which is called to weight), Δ^u is the simplex given by

$$\Delta^u = \{(t_1, \ldots, t_u) \in \mathbb{R}^u | 0 < t_u < \ldots < t_1 < 1\},$$

and we have the differential forms $\eta_0(t) = \frac{dt}{t}$ and $\eta_1(t) = \frac{dt}{1-t}$. The different representations of the multiple zeta values (series and integral) help us in exploring the finite double shuffle relations, which compares the multiplication of multiple zeta values in these different representations. Another relation that uses these representations is the integral-series identity, which compares the different representations directly. It is conjectured that this identity implies all other linear relations for multiple zeta values. To support this conjecture, we want to derive the Hoffman's relation

$$\sum_{i=1}^{k} \zeta(s_1, \ldots, s_{i-1}, s_i + 1, s_{i+1}, \ldots, s_k) = \sum_{\substack{i=1, \\ s_i \geq 2}}^{k} \sum_{j=0}^{s_i-2} \zeta(s_1, \ldots, s_{i-1}, s_i - j, j + 1, s_{i+1}, \ldots, s_k)$$

for $s_i \geq 1$ for $2 \leq i \leq k$ and $s_1 \geq 2$. Furthermore we want to derive the restricted sum formula

$$\sum_{\substack{s_1+\ldots+s_p=k+p-1, \\ s_j \geq 1}} \zeta(s_1 + s - k - p + 1, s_2, \ldots, s_p) = \sum_{\substack{s_1+\ldots+s_k=s-p, \\ s_j \geq 1}} \zeta(s_1 + 1, s_2, \ldots, s_k, \underbrace{1, \ldots, 1}_{p-1 \text{ times}}).$$

To be able to derive these relations from the integral series identity, we need some algebra setup. Therefore we take the non-commutative algebra $\mathfrak{H} := \mathbb{Q}\langle x, y \rangle$. Computations in $\mathfrak{H}$ can be translated to multiple zeta values by the map

$$x^{s_1-1}yx^{s_2-1}y \ldots x^{s_k-1}y \mapsto \zeta(s_1, \ldots, s_k).$$

Another main part of this paper is to investigate the $\mathbb{Q}$-subvector space generated by all MZV of weight s

$$\mathcal{Z}_s = \langle \zeta(s_1, \ldots, s_k) | s_1 + \ldots + s_k = s \rangle_{\mathbb{Q}}.$$

In particular we want to investigate the basis and dimension for these spaces. It is conjectured that the dimension is given by the recursion formula

$$d_s = d_{s-2} + d_{s-3}, \qquad s \geq 3,$$

with initial values $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$ and the basis is given by the set

$$\{\zeta(s_1, \ldots, s_k) : s_i \in \{2, 3\}, 1 \leq i \leq k \text{ and } s_1 + \ldots + s_k = s\}.$$

Another goal of this paper is to gather enough linear relations to support these conjectures in some spaces.

Outline

In Section 2 we give a more detailed introduction to multiple zeta values. We see the motivation to define multiple zeta values and give the formal definition of multiple zeta values. Furthermore, we prove in which cases the multiple zeta values are convergent. Thereafter, we see how they can be represented by a series and by an integral. We work out the multiplication of zeta values in integral representation. To accomplish this, we introduce permutations that respect two disjoint subsets. Lastly, we see the first theorem about linear relations for multiple zeta values: the Duality theorem. In Section 2, we see that the different representations give different results when we look at multiplications of two zeta values. This is the motivation to compare the multiplications of the two different representations in Section 3. By doing so, we derive an example of the finite double shuffle relation. In this section, we generalize this relation. Therefore, we introduce the stuffle and shuffle product on a non-commutative polynomial algebra. We prove that these products are commutative. Moreover, we create a map from this non-commutative polynomial algebra to multiple zeta values. We prove that this map is an algebra homomorphism with respect to the shuffle and stuffle product. This algebra homomorphism proves the finite double shuffle relation. We work out some examples of the finite double shuffle relation. To explore more linear relations for multiple zeta values, we investigate the integral-series identity in Section 4. To understand this relation, we need an introduction to the circled stuffle product, star notation, 2-posets and Hasse diagrams. Terminology and notation for these concepts are explained by exploring many examples. To simplify the proof for the integral-series identity, we prove two lemmas, which tell us how we can represent the identity by a series and an integral. It is conjectured that this integral-series identity implies all other relations for multiple zeta values. To give some evidence for this conjecture, we derive some other relations from the integral-series identity. Finally, in Section 5 we explore spaces spanned by multiple zeta values. We mention Zagier's and Hoffman's conjecture, which are related to the dimension and basis for these spaces, respectively. We prove how Zagier's and Hoffman's conjecture are equivalent. Finally, we support these conjectures using linear relations.

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2 Multiple Zeta Values

This section is mostly based on sections 1.2 and 1.5 of [Gil and Fresán, 2017]. Furthermore, the notation for the integral representation coincides with the notation from [Zagier, 1994] on page 510.

2.1 Series Representation

When studying Conjecture 1.3, we have to look at the algebraic relations of the zeta values. While exploring these relations, we might face multiplication of two zeta values. So take $s_1, s_2 \in \mathbb{C}$, then

$$\begin{aligned}
\zeta(s_1)\zeta(s_2) &= \left(\sum_{n_1 \geq 1} \frac{1}{n_1^{s_1}}\right)\left(\sum_{n_2 \geq 1} \frac{1}{n_2^{s_2}}\right) \\
&= \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} \frac{1}{n_1^{s_1}} \frac{1}{n_2^{s_2}} \\
&= \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} \\
&= \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2 > n_1 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_1 = n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} \\
&= \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2 > n_1 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n \geq 1} \frac{1}{n^{s_1 + s_2}},
\end{aligned}$$

where in the fourth equality we decompose the sum into three parts, when n_1 is equal to n_2, when n_1 is strictly greater than n_2 and vice versa. By doing so, we cover all values n_1 and n_2 can take. By looking at these computations it is reasonable to set the following function. For $s_1, \dots, s_k \in \mathbb{C}$ with $k \geq 1$, define

$$\zeta(s_1, \dots, s_k) := \sum_{n_1 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}. \tag{2}$$

In that case we can, in the above computation where we have the case $k = 2$, reduce to the abstract form

$$\begin{aligned}
\zeta(s_1)\zeta(s_2) &= \sum_{n_1 > n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2 > n_1 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n \geq 1} \frac{1}{n^{s_1 + s_2}} \\
&= \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).
\end{aligned}$$

Likewise, for the Riemann zeta function, the convergence of this function is not guaranteed. From now on we restrict to an input of positive integers, i.e. $s_1, \dots, s_k \in \mathbb{Z}_{>0}$. This simplifies the function and it helps by looking at the convergence.

Definition 2.1. *Let $s_1, \dots, s_k \in \mathbb{Z}$, then $\boldsymbol{s} := (s_1, \dots, s_k)$ is called a multi-index. $\boldsymbol{s}$ is called a positive multi-index if $s_i \geq 1$ for all $1 \leq i \leq k$ and is called an admissible multi-index if it is positive and $s_1 \geq 2$. The weight of a multi-index is defined to be $wt(\boldsymbol{s}) := s_1 + \dots + s_k$ and the length to be $l(\boldsymbol{s}) := k$. Additionally, $wt(\emptyset) = 0 = l(\emptyset)$, where $\emptyset$ denotes the empty multi-index.*

Theorem 2.2. *If $\boldsymbol{s} = (s_1, \dots, s_k)$ is an admissible multi-index, then $\zeta(\boldsymbol{s})$ is finite.*

To prove this theorem, we need a result about the logarithm compared to the square root. This lemma is self-written to improve the proof of Theorem 2.2 that follows the proof of Lemma 1.23 in [Gil and Fresán, 2017].

Lemma 2.3. *For all integers $k \geq 1$, there exists an integer N such that $(1+\log n)^{k-1} < \sqrt{n}$ for all $n \geq N$.*

Proof. We notice that $\lim_{n\to\infty} \log n = \infty$. So

$$\lim_{n\to\infty} \frac{\log(1+\log n)}{\log n} = \frac{\infty}{\infty}.$$

Therefore we can use L'Hopital's rule on this limit.

$$\begin{aligned}\lim_{n\to\infty} \frac{\log(1+\log n)}{\log n} &= \lim_{n\to\infty} \frac{\frac{d}{dn}\log(1+\log n)}{\frac{d}{dn}\log n} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n}\frac{1}{1+\log n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{1+\log n} \\ &= 0.\end{aligned}$$

Fix $k \geq 1$ to be an integer. Then by scalar multiplication we have

$$\begin{aligned}0 &= 2(k-1)\lim_{n\to\infty} \frac{\log(1+\log n)}{\log n} \\ &= \lim_{n\to\infty} \frac{(k-1)\log(1+\log n)}{\frac{1}{2}\log n} \\ &= \lim_{n\to\infty} \frac{\log((1+\log n)^{k-1})}{\log \sqrt{n}},\end{aligned}$$

where we used logarithm rules. By this limit we have that for sufficient big N

$$\begin{aligned}\frac{\log((1+\log n)^{k-1})}{\log\sqrt{n}} &< 1, \quad n \geq N, \\ \log((1+\log n)^{k-1}) &< \log\sqrt{n}, \quad n \geq N, \\ e^{\log((1+\log n)^{k-1})} &< e^{\log\sqrt{n}}, \quad n \geq N, \\ (1+\log n)^{k-1} &< \sqrt{n}, \quad n \geq N.\end{aligned}$$

Since k was taken arbitrarily, the lemma has been proven. □

Proof. (*Theorem 2.2*) We have for any admissible multi-index $\boldsymbol{s} = (s_1, \ldots, s_k)$ of length k that the weight $(s_1 + \ldots + s_k)$ is the smallest by taking $\boldsymbol{s} = (2, \underbrace{1, \ldots, 1}_{k-1 \text{ times}})$ (we need the 2 in the first entry, because we need an admissible multi-index). In that case we have that the fraction in the series is the biggest compared to any other multi-index, i.e. for any admissible multi-index $\boldsymbol{s} = (s_1, \ldots, s_k)$ we have

$$\frac{1}{n_1^{s_1} \ldots n_k^{s_k}} \leq \frac{1}{n_1^2 n_2 \ldots n_k}$$

for all $n_i \geq 1$ with $1 \leq i \leq k$. So

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}} \leq \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^2 n_2 \ldots n_k} = \zeta(2, \underbrace{1, \ldots, 1}_{k-1 \text{ times}}).$$

Therefore, it is enough to show that $\zeta(2, \underbrace{1, \ldots, 1}_{k-1 \text{ times}})$ is finite. Therefore notice that

$$\sum_{i=2}^{n} \frac{1}{i} \leq \int_1^n \frac{1}{x} dx = \log n.$$

This holds since the sum on the left is the sum of the rectangles on the interval $[1, n]$ of width 1 and with a height on the right side of the rectangle of $\frac{1}{i}$ for $i \in \{2, \ldots, n\}$. Therefore the rectangles fit under the curve $\frac{1}{x}$, so the sum of the rectangles are less than the integral on $[1, n]$. Adding 1 on both sides, we obtain

$$\sum_{i=1}^{n} \frac{1}{i} \leq 1 + \log n. \tag{3}$$

Then

$$\begin{aligned}
\zeta(2, 1, \ldots, 1) &= \sum_{n > n_2 > \ldots > n_k \geq 1} \frac{1}{n^2 n_2 \ldots n_k} \\
&\leq \sum_{n > n_2, \ldots, n_k \geq 1} \frac{1}{n^2 n_2 \ldots n_k} \\
&= \sum_{n \geq 1} \frac{1}{n^2} \left(\sum_{n > n_2, \ldots, n_k \geq 1} \frac{1}{n_2 \ldots n_k} \right) \\
&= \sum_{n \geq 1} \frac{1}{n^2} \left(\sum_{i=1}^{n} \frac{1}{i} \right)^{k-1} \\
&\leq \sum_{n \geq 1} \frac{(1 + \log n)^{k-1}}{n^2},
\end{aligned}$$

where the first inequality follows from the fact that we add extra terms by saying that the n_i's may be equal, greater or smaller than n_{i+1} for $2 \leq i \leq k-1$. Furthermore, we used rules for multiplication of (infinite) sums and equation (3). Using Lemma 2.3, there exists an N such that

$$\begin{aligned}
\zeta(2, 1, \ldots, 1) &\leq \sum_{N \geq n \geq 1} \frac{(1 + \log n)^{k-1}}{n^2} + \sum_{n \geq N+1} \frac{(1 + \log n)^{k-1}}{n^2} \\
&\leq \sum_{N \geq n \geq 1} \frac{(1 + \log n)^{k-1}}{n^2} + \sum_{n \geq N+1} \frac{\sqrt{n}}{n^2} \\
&= \sum_{N \geq n \geq 1} \frac{(1 + \log n)^{k-1}}{n^2} + \sum_{n \geq N+1} \frac{1}{n^{\frac{3}{2}}} \\
&< \infty.
\end{aligned}$$

This is finite since the first term is a finite sum and the second term converges by Definition 1.1. Hence we proved the theorem. □

Finally, we are ready to define so-called multiple zeta values (abbreviated as MZV).

Definition 2.4. *For an admissible multi-index $\boldsymbol{s} = (s_1, \ldots, s_k)$ the MZV are defined as $\zeta(\boldsymbol{s})$ and in addition $\zeta(\emptyset) = 1$. The weight and length of a MZV is defined to be the weight and length of the corresponding multi-index, respectively.*

Example 1. So we have the relation for (multiple) zeta values

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

Let us take $s_1 = s_2 = 2$, then

$$\zeta(2)^2 = \zeta(2,2) + \zeta(2,2) + \zeta(4) = 2\zeta(2,2) + \zeta(4).$$

Hence using formula (1) for $\zeta(2), \zeta(4)$ we get

$$\zeta(2,2) = \frac{1}{2}\left[\zeta(2)^2 - \zeta(4)\right] = \frac{1}{2}\left[\left(\frac{\pi^2}{6}\right)^2 - \frac{\pi^4}{90}\right] = \frac{\pi^4}{120}.$$

In terms of infinite sum this means that

$$\begin{aligned}
\frac{\pi^4}{120} &= \zeta(2,2) \\
&= \sum_{n>m\geq 1} \frac{1}{n^2 m^2} \\
&= \left(\frac{1}{2^2 1^2} + \frac{1}{3^2 1^2} + \frac{1}{4^2 1^2} + \ldots\right) + \left(\frac{1}{3^2 2^2} + \frac{1}{4^2 2^2} + \frac{1}{5^2 2^2} + \ldots\right) \\
&\quad + \left(\frac{1}{4^2 3^2} + \frac{1}{5^2 3^2} + \frac{1}{6^2 3^2} + \ldots\right) + \ldots.
\end{aligned}$$

■

2.2 Integral Representation

We have seen a series representation for the MZV (see equation (2)). But there is another way to represent MZV, this is done by an integral representation. Therefore we need some notation. Let $k \geq 1$ be an integer, then we define the simplex

$$\Delta^k := \{(t_1, \ldots, t_k) \in \mathbb{R}^k | 0 < t_k < \ldots < t_1 < 1\}.$$

An integral of a function $g : \mathbb{R}^k \to \mathbb{R}$ over this simplex can be read as follow

$$\begin{aligned}
\int_{\Delta^k} g(t_1, \ldots, t_k) dt_k \ldots dt_1 &= \int_{0<t_k<\ldots<t_1<1} g(t_1, \ldots, t_k) dt_k \ldots dt_1 \\
&= \int_0^1 \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{k-1}} g(t_1, \ldots, t_k) dt_k \ldots dt_1.
\end{aligned}$$

Furthermore, let $\varepsilon_i \in \{0, 1\}$ for $1 \leq i \leq k$. We define the integral

$$\lambda(\varepsilon_1, \ldots, \varepsilon_k) := \int_{\Delta^k} \eta_{\varepsilon_1}(t_1) \ldots \eta_{\varepsilon_k}(t_k) = \int_{0<t_k<\ldots<t_1<1} \eta_{\varepsilon_1}(t_1) \ldots \eta_{\varepsilon_k}(t_k),$$

where we have the differential form

$$\eta_\varepsilon(t) = \begin{cases} \frac{dt}{1-t}, & \varepsilon = 1 \\ \frac{dt}{t}, & \varepsilon = 0 \end{cases}.$$

Remark 1. One thing we should mention is that the order of the ε_i's in λ determine the order of the t_i's in the simplex Δ^k and vice versa. To justify this we define the following function. For any permutation $\sigma \in \mathcal{S}_k$, i.e. permutation of the set $\{1, \ldots, k\}$, we define the function $\widetilde{\sigma} : \mathbb{R}^k \to \mathbb{R}^k$ by the rule

$$\widetilde{\sigma}(t_1, \ldots, t_k) := (t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(k)}). \tag{4}$$

This means that the function $\widetilde{\sigma}$ can be applied to Δ^k for some integer $k \geq 1$. Then by construction of λ, for any permutation $\sigma \in \mathcal{S}_k$ we have

$$\begin{aligned} \lambda(\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(k)}) &= \int_{0<t_k<\ldots<t_1<1} \eta_{\varepsilon_{\sigma(1)}}(t_1) \ldots \eta_{\varepsilon_{\sigma(k)}}(t_k) \\ &= \int_{0<t_{\sigma^{-1}(k)}<\ldots<t_{\sigma^{-1}(1)}<1} \eta_{\varepsilon_1}(t_1) \ldots \eta_{\varepsilon_k}(t_k) \\ &= \int_{\widetilde{\sigma}(\Delta^k)} \eta_{\varepsilon_1}(t_1) \ldots \eta_{\varepsilon_k}(t_k). \end{aligned}$$

Hence we see that the order of the ε_i's in λ determines the order of the t_i's in the simplex Δ^k, by this permutation σ. ♦

[Zagier, 1994] sketches on page 510 the proof of the integral representation, here we work the proof out.

Theorem 2.5. *Let $\boldsymbol{s} = (s_1, \ldots, s_k)$ be an admissible multi-index, then*

$$\zeta(\boldsymbol{s}) = \lambda(\underbrace{0, \ldots, 0}_{s_1 - 1 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{s_2 - 1 \text{ times}}, 1, \ldots, \underbrace{0, \ldots, 0}_{s_k - 1 \text{ times}}, 1).$$

Proof. Let $S := (\underbrace{0, \ldots, 0}_{s_1 - 1 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{s_2 - 1 \text{ times}}, 1, \ldots, \underbrace{0, \ldots, 0}_{s_k - 1 \text{ times}}, 1)$ and $u := wt(\boldsymbol{s})$. Then by construction of the function λ we have

$$\begin{aligned} \lambda(S) &= \int_{\Delta^u} \Big(\eta_0(t_1) \ldots \eta_0(t_{s_1-1})\eta_1(t_{s_1})\Big) \ldots \Big(\eta_0(t_{u-s_k+1}) \ldots \eta_0(t_{u-1})\eta_1(t_u)\Big) \\ &= \int_{\Delta^u} \left(\frac{dt_1}{t_1} \ldots \frac{dt_{s_1-1}}{t_{s_1-1}} \frac{dt_{s_1}}{1-t_{s_1}}\right) \ldots \left(\frac{dt_{u-s_k+1}}{t_{u-s_k+1}} \ldots \frac{dt_{u-1}}{t_{u-1}} \frac{dt_u}{1-t_u}\right). \end{aligned}$$

So by construction of this simplex, we need to determine each integral from the inside to the outside. So we start with the integral over t_u. Since $0 < t_u < 1$, we can use geometric series and derive that

$$\frac{1}{1-t_u} = \sum_{n_k \geq 1} t_u^{n_k - 1}$$

Then

$$\int_0^{t_{u-1}} \frac{dt_u}{1-t_u} = \sum_{n_k \geq 1} \int_0^{t_{u-1}} t_u^{n_k-1} dt_u = \sum_{n_k \geq 1} \frac{t_{u-1}^{n_k}}{n_k}.$$

So we are ready to take the integral over t_{u-1} we get

$$\begin{aligned}\int_0^{t_{u-2}} \frac{1}{t_{u-1}} \left(\int_0^{t_{u-1}} \frac{dt_u}{1-t_u}\right) dt_{u-1} &= \int_0^{t_{u-2}} \frac{1}{t_{u-1}} \left(\sum_{n_k \geq 1} \frac{t_{u-1}^{n_k}}{n_k}\right) dt_{u-1} \\ &= \sum_{n_k \geq 1} \frac{1}{n_k} \int_0^{t_{u-2}} t_{u-1}^{n_k-1} dt_{u-1} \\ &= \sum_{n_k \geq 1} \frac{t_{u-2}^{n_k}}{n_k^2}.\end{aligned}$$

Then we can proceed to the integral over t_{u-2}. But we can reason that we have the same computation exactly $s_k - 2$ times more, since we have $s_k - 1$ zeros is S and hence the same function for $\eta_{\varepsilon_i}(t_i)$ for $u - s_k + 1 \leq i \leq u - 1$ (call this step $\star$). Therefore

$$\int_{0<t_u<\ldots<t_{u-s_k+1}<t_{u-s_k}} \frac{dt_{u-s_k+1}}{t_{u-s_k+1}} \cdots \frac{dt_{u-1}}{t_{u-1}} \frac{dt_u}{1-t_u} = \sum_{n_k \geq 1} \frac{t_{u-s_k}^{n_k}}{n_k^{s_k}}.$$

The next integral over t_{u-s_k} becomes a bit different, because we come to the point that there is a 1 in S. Then, by using geometric series again

$$\begin{aligned}&\int_0^{t_{u-s_k-1}} \frac{1}{1-t_{u-s_k}} \left(\int_{0<t_u<\ldots<t_{u-s_k+1}<t_{u-s_k}} \frac{dt_{u-s_k+1}}{t_{u-s_k+1}} \cdots \frac{dt_{u-1}}{t_{u-1}} \frac{dt_u}{1-t_u}\right) dt_{u-s_k} \\ &= \int_0^{t_{u-s_k-1}} \frac{1}{1-t_{u-s_k}} \left(\sum_{n_k \geq 1} \frac{t_{u-s_k}^{n_k}}{n_k^{s_k}}\right) dt_{u-s_k} \\ &= \int_0^{t_{u-s_k-1}} \sum_{n_{k-1} \geq 1} t_{u-s_k}^{n_{k-1}-1} \left(\sum_{n_k \geq 1} \frac{t_{u-s_k}^{n_k}}{n_k^{s_k}}\right) dt_{u-s_k} \\ &= \int_0^{t_{u-s_k-1}} \sum_{n_{k-1}, n_k \geq 1} \frac{t_{u-s_k}^{n_k+n_{k-1}-1}}{n_k^{s_k}} dt_{u-s_k} \\ &= \sum_{n_{k-1}, n_k \geq 1} \frac{1}{n_k^{s_k}} \int_0^{t_{u-s_k-1}} t_{u-s_k}^{n_k+n_{k-1}-1} dt_{u-s_k} \\ &= \sum_{n_{k-1}, n_k \geq 1} \frac{t_{u-s_k-1}^{n_k+n_{k-1}}}{n_k^{s_k}(n_k+n_{k-1})}.\end{aligned}$$

For the following integrals we move back to step $\star$. Then repeating this process again and again and notice that the last integral has upper bound 1, we see that we end up with something like

$$\begin{aligned}\lambda(S) &= \sum_{n_k, n_{k-1}, \ldots, n_1 \geq 1} \frac{1}{n_k^{s_k}(n_k+n_{k-1})^{s_{k-1}} \ldots (n_k + \ldots + n_1)^{s_1}} \\ &= \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}} \\ &= \zeta(s_1, \ldots, s_k).\end{aligned}$$

□

A theorem that follows directly from this integral representation is the Duality theorem, which gives equalities between MZV. [Zagier, 1994] gives on page 510 a sketch for the proof of this statement. We work out the proof in more detail.

Theorem 2.6 (Duality Theorem)**.** *Let $\varepsilon_i \in \{0,1\}$ for $1 \leq i \leq k$, we have*

$$\lambda(\varepsilon_1, \dots, \varepsilon_k) = \lambda(1-\varepsilon_k, \dots, 1-\varepsilon_1).$$

Proof. We set the change of variables by $1 - t'_{k+1-i} = t_i$ for $1 \leq i \leq k$. If we write a function

$$\varphi(t'_1, \dots, t'_k) = (1-t'_k, \dots, 1-t'_1).$$

Then we have

$$(t_1, \dots, t_k) = \varphi(t'_1, \dots, t'_k).$$

Furthermore, we define the function

$$\gamma_\varepsilon(t) = \begin{cases} \frac{1}{1-t}, & \varepsilon = 1 \\ \frac{1}{t}, & \varepsilon = 0 \end{cases}.$$

Note that by the change of variables we have

$$\begin{aligned} \gamma_{\varepsilon_i}(t_i) &= \begin{cases} \frac{1}{1-t_i}, & \varepsilon_i = 1 \\ \frac{1}{t_i}, & \varepsilon_i = 0 \end{cases} \\ &= \begin{cases} \frac{1}{t'_{k+1-i}}, & \varepsilon_i = 1 \\ \frac{1}{1-t'_{k+1-i}}, & \varepsilon_i = 0 \end{cases} \\ &= \gamma_{1-\varepsilon_i}(t'_{k+1-i}). \end{aligned}$$

Furthermore, we can write

$$\eta_\varepsilon(t) = \gamma_\varepsilon(t)dt.$$

So by using this all and by using change of variables we have

$$\begin{aligned} \lambda(\varepsilon_1, \dots, \varepsilon_k) &= \int_{\Delta^k} \eta_{\varepsilon_1}(t_1) \dots \eta_{\varepsilon_k}(t_k) \\ &= \int_{0<t_k<\dots<t_1<1} \gamma_{\varepsilon_1}(t_1) \dots \gamma_{\varepsilon_k}(t_k) dt_1 \dots dt_k \\ &= \int_{0<1-t'_1<\dots<1-t'_k<1} \gamma_{1-\varepsilon_1}(t'_k) \dots \gamma_{1-\varepsilon_k}(t'_1) \left| det\Big(D\varphi(t'_1, \dots, t'_k)\Big) \right| dt'_1 \dots dt'_k \\ &= \int_{0<t'_k<\dots<t'_1<1} \gamma_{1-\varepsilon_1}(t'_k) \dots \gamma_{1-\varepsilon_k}(t'_1) \left| det\Big(D\varphi(t'_1, \dots, t'_k)\Big) \right| dt'_1 \dots dt'_k, \end{aligned}$$

where $D\varphi$ is the Jacobian matrix of φ, which is needed by such a change of variables. We can calculate that

$$D\varphi(t'_1, \dots, t'_k) = \begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

So $|det(D\varphi(t'_1,\dots,t'_k))| = |(-1)^k| = 1$. Therefore

$$\begin{aligned}
\lambda(\varepsilon_1,\dots,\varepsilon_k) &= \int_{0<t'_k<\dots<t'_1<1} \gamma_{1-\varepsilon_1}(t'_k)\dots\gamma_{1-\varepsilon_k}(t'_1)|D\varphi(t'_1,\dots,t'_k)|dt'_1\dots dt'_k \\
&= \int_{0<t'_k<\dots<t'_1<1} \gamma_{1-\varepsilon_1}(t'_k)\dots\gamma_{1-\varepsilon_k}(t'_1)dt'_1\dots dt'_k \\
&= \int_{0<t'_k<\dots<t'_1<1} \eta_{1-\varepsilon_1}(t'_k)\dots\eta_{1-\varepsilon_k}(t'_1) \\
&= \lambda(1-\varepsilon_k,\dots,1-\varepsilon_1).
\end{aligned}$$

□

Example 2. We have the following equalities between MZV if we use the Duality theorem.

$$\zeta(4,1) = \lambda(0,0,0,1,1) = \lambda(0,0,1,1,1) = \zeta(3,1,1),$$

$$\zeta(3,2) = \lambda(0,0,1,0,1) = \lambda(0,1,0,1,1) = \zeta(2,2,1),$$

$$\zeta(2,3) = \lambda(0,1,0,0,1) = \lambda(0,1,1,0,1) = \zeta(2,1,2),$$

$$\zeta(2,3,1) = \lambda(0,1,0,0,1,1) = \lambda(0,0,1,1,0,1) = \zeta(3,1,2).$$

■

The next result does not come from any literature but is a useful result in this paper.

Corollary 2.7. *Let $\boldsymbol{s} = (s_1,\dots,s_k)$ be multi-index such that $s_i \geq 2$ for all $1 \leq i \leq k$, then*

$$\zeta(\boldsymbol{s}) = \zeta(2, \underbrace{1,\dots,1}_{s_k-2\ times}, 2, \underbrace{1,\dots,1}_{s_2-2\ times}, \dots, 2, \underbrace{1,\dots,1}_{s_1-2\ times}).$$

Proof. We have by the integral representation and by the Duality theorem that

$$\begin{aligned}
\zeta(\boldsymbol{s}) &= \lambda(\underbrace{0,\dots,0}_{s_1-1\text{ times}}, 1, \underbrace{0,\dots,0}_{s_2-1\text{ times}}, 1, \dots, \underbrace{0,\dots,0}_{s_k-1\text{ times}}, 1) \\
&= \lambda(0, \underbrace{1,\dots,1}_{s_k-1\text{ times}}, 0, \underbrace{1,\dots,1}_{s_{k-1}-1\text{ times}}, \dots, 0, \underbrace{1,\dots,1}_{s_1-1\text{ times}}) \\
&= \lambda(0,1, \underbrace{1,\dots,1}_{s_k-2\text{ times}}, 0,1, \underbrace{1,\dots,1}_{s_{k-1}-2\text{ times}}, \dots, 0,1, \underbrace{1,\dots,1}_{s_1-2\text{ times}}) \\
&= \zeta(2, \underbrace{1,\dots,1}_{s_k-2\text{ times}}, 2, \underbrace{1,\dots,1}_{s_2-2\text{ times}}, \dots, 2, \underbrace{1,\dots,1}_{s_1-2\text{ times}}).
\end{aligned}$$

□

Likewise, for the multiplication of zeta values represented by a series, it might be interesting to look at the multiplication of two zeta values with this representation at hand. We need some terminology to accomplish this.

Definition 2.8. *Let $s_1, s_2 \geq 1$, we say that a permutation σ of the set $\{1, \dots, s_1 + s_2\}$ is a permutation of type (s_1, s_2) if*

$$\sigma(1) < \sigma(2) < \ldots < \sigma(s_1), \qquad \sigma(s_1+1) < \sigma(s_1+2) < \ldots < \sigma(s_1+s_2).$$

We denote the set of all permutations of type (s_1, s_2) by $\mathcal{S}(s_1, s_2)$.

In other words, a permutation of type (s_1, s_2) respects the ordering of two disjoint subsets.

Example 3. Let us determine $\mathcal{S}(2,2)$. We require that

$$\sigma(1) < \sigma(2), \qquad \sigma(3) < \sigma(4).$$

This means that 1 has to be sent to a lower number than 2 is, and similarly, 3 must be sent to a lower number than 4 is. Therefore it can be seen that

$$\mathcal{S}(2,2) = \{id, (123), (23), (13)(24), (243), (1243)\}.$$

■

For $\sigma \in \mathcal{S}(s_1, s_2)$ (see Definition 2.8) we can make sense of $\widetilde{\sigma}(\Delta^{s_1+s_2})$ (see equation 4).

Example 4. In Example 3 we have determine $\mathcal{S}(2,2)$, let us determine $\widetilde{\sigma}(\Delta^4)$ for each $\sigma \in \mathcal{S}(2,2)$.

- For $\sigma = id$ we have $\widetilde{\sigma}(\Delta^4) = (t_1, t_2, t_3, t_4)$.
- For $\sigma = (123)$ we have $\widetilde{\sigma}(\Delta^4) = (t_3, t_1, t_2, t_4)$.
- For $\sigma = (23)$ we have $\widetilde{\sigma}(\Delta^4) = (t_1, t_3, t_2, t_4)$.
- For $\sigma = (13)(24)$ we have $\widetilde{\sigma}(\Delta^4) = (t_3, t_4, t_1, t_2)$.
- For $\sigma = (243)$ we have $\widetilde{\sigma}(\Delta^4) = (t_1, t_3, t_4, t_2)$.
- For $\sigma = (1243)$ we have $\widetilde{\sigma}(\Delta^4) = (t_3, t_1, t_4, t_2)$.

■

With this in hand we can see how multiplication happens. We follow the proof of Proposition 1.123 in [Gil and Fresán, 2017].

Theorem 2.9. *Let $s_1, s_2 \geq 2$, then*

$$\zeta(s_1)\zeta(s_2) = \sum_{\sigma \in \mathcal{S}(s_1,s_2)} \int_{\widetilde{\sigma}(\Delta^{s_1+s_2})} \frac{dt_1}{t_1} \cdots \frac{dt_{s_1-1}}{t_{s_1-1}} \frac{dt_{s_1}}{1-t_{s_1}} \frac{dt_{s_1+1}}{t_{s_1+1}} \cdots \frac{dt_{s_1+s_2-1}}{t_{s_1+s_2-1}} \frac{dt_{s_1+s_2}}{1-t_{s_1+s_2}}. \tag{5}$$

Proof. By using the representation of $\zeta(s_1)$ and $\zeta(s_2)$ from Theorem 2.5 we have

$$\begin{aligned}
\zeta(s_1)\zeta(s_2) &= \lambda(\underbrace{0,\dots,0}_{s_1-1\text{ times}}, 1)\lambda(\underbrace{0,\dots,0}_{s_2-1\text{ times}}, 1) \\
&= \left(\int_{\Delta^{s_1}} \frac{dt_1}{t_1} \cdots \frac{dt_{s_1-1}}{t_{s_1-1}} \frac{dt_{s_1}}{1-t_{s_1}}\right)\left(\int_{\Delta^{s_2}} \frac{du_1}{u_1} \cdots \frac{du_{s_2-1}}{u_{s_2-1}} \frac{du_{s_2}}{1-u_{s_2}}\right) \\
&= \int_{\Delta^{s_1}\times\Delta^{s_2}} \frac{dt_1}{t_1} \cdots \frac{dt_{s_1-1}}{t_{s_1-1}} \frac{dt_{s_1}}{1-t_{s_1}} \frac{du_1}{u_1} \cdots \frac{du_{s_2-1}}{u_{s_2-1}} \frac{du_{s_2}}{1-u_{s_2}},
\end{aligned}$$

where

$$\Delta^{s_1} \times \Delta^{s_2} = \{(t_1, \ldots, t_{s_1}) \in \mathbb{R}^{s_1} | 0 < t_{s_1} < \ldots < t_1 < 1\}$$
$$\times \{(u_1, \ldots, u_{s_2}) \in \mathbb{R}^{s_2} | 0 < u_{s_2} < \ldots < u_1 < 1\}.$$

By renaming $u_i = t_{s_1+i}$ for $1 \leq i \leq s_2$, we can write this multiplication as all $(s_1 + s_2)-$tuples in which we go from bigger to smaller numbers (similar to Δ^k for some integer $k \geq 1$) such that $0 < t_{s_1} < \ldots < t_1 < 1$ and $0 < t_{s_1+1} < \ldots < t_{s_1+s_2} < 1$ hold. Denote $\mathcal{X}$ to be the set that contains all the cases in which $t_i = t_j$ for some $1 \leq i \leq s_1$ and some $s_1 + 1 \leq j \leq s_1 + s_2$. The integral over the set $\mathcal{X}$ can be ignored, because in those cases the simplex is of lower dimension and so the integral over this simplex is zero. So the remaining $(s_1 + s_2)-$tuples in this multiplication are all $(s_1 + s_2)-$tuples such that the inequalities $0 < t_{s_1} < \ldots < t_1 < 1$ and $0 < t_{s_1+1} < \ldots < t_{s_1+s_2} < 1$ hold from these two disjoint sets. Note that this is equivalent as permuting the t_i's for $1 \leq i \leq s_1 + s_2$ in $\Delta^{s_1+s_2}$, but respecting the two disjoint subsets. Therefore by using the definition of $\sigma \in \mathcal{S}(s_1, s_2)$ and function $\widetilde{\sigma}$ we have

$$\Delta^{s_1} \times \Delta^{s_2} = \left(\bigcup_{\sigma \in \mathcal{S}(s_1,s_2)} \widetilde{\sigma}(\Delta^{s_1+s_2}) \right) \cup \mathcal{X}. \tag{6}$$

Hence we obtain exactly equation (5) by using the fact that the integral over $\mathcal{X}$ is zero. □

Example 5. We see in Theorem 2.9 that if we choose $s_1 = s_2 = 2$ in equation (5) and with Example 4 in hand we have

$$\begin{aligned}
\zeta(2)^2 &= \sum_{\sigma \in \mathcal{S}(2,2)} \int_{\widetilde{\sigma}(\Delta^4)} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} \frac{dt_4}{1-t_4} \\
&= \Bigg(\int_{0<t_4<t_3<t_2<t_1<1} + \int_{0<t_4<t_2<t_1<t_3<1} + \int_{0<t_4<t_2<t_3<t_1<1} \\
&\quad + \int_{0<t_2<t_1<t_4<t_3<1} + \int_{0<t_2<t_4<t_3<t_1<1} + \int_{0<t_2<t_4<t_1<t_3<1} \Bigg) \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} \frac{dt_4}{1-t_4} \\
&= \lambda(0,1,0,1) + \lambda(0,0,1,1) + \lambda(0,0,1,1) + \lambda(0,1,0,1) + \lambda(0,0,1,1) + \lambda(0,0,1,1) \\
&= \zeta(2,2) + \zeta(3,1) + \zeta(3,1) + \zeta(2,2) + \zeta(3,1) + \zeta(3,1) \\
&= 2\zeta(2,2) + 4\zeta(3,1).
\end{aligned}$$

Note that we use here that the order of the simplex determines the order of the 0 and 1 in λ as explained in Remark 1. ■

3 Finite Double Shuffle Relation

In Example 1 we see that

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4). \tag{7}$$

This product of two zeta values is an example of a so-called stuffle product and this relation of MZV is called a stuffle relation. The stuffle product generalizes the product of MZV in series representation. Furthermore, in Example 5 we see that

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1). \tag{8}$$

This product of two zeta values is an example of a so-called shuffle product and this relation of MZV is called a shuffle relation. The shuffle product generalizes the product of MZV in integral representation. We can combine equations (7) and (8), then we get

$$\begin{aligned} 2\zeta(2,2)+\zeta(4) &= 2\zeta(2,2)+4\zeta(3,1), \\ \zeta(4) &= 4\zeta(3,1). \end{aligned} \tag{9}$$

This last equation is an example of a so-called finite double shuffle relation. The finite double shuffle relation occurs when we combine the stuffle and shuffle products, i.e. when we compare the multiplication of MZV for the different representations. This finite double shuffle relation contains many linear combinations for MZV. So the question arises of whether we can generalize this concept?

3.1 Algebra Setup

We need some terminology to generalize the finite double shuffle relation. Therefore we make use of a non-commutative polynomial algebra, some sub-algebras and products. Because it is fundamental for studying MZV, this algebra setup can be found in many papers about MZV. See Section 4 in [Zudilin, 2003] or Section 1.6 in [Gil and Fresán, 2017].

Definition 3.1. *Let $X := \{x_1, \ldots, x_n\}$ be a set. We say that the x_i's are letters and X is an alphabet. We define $\mathbb{Q}\langle X\rangle$ to be the non-commutative polynomial algebra. This algebra consists of a normal addition operator $+$ and a concatenation product $\cdot$ defined as*

$$x_{i_1}\ldots x_{i_p}\cdot x_{j_1}\ldots x_{j_q} = x_{i_1}\ldots x_{i_p}x_{j_1}\ldots x_{j_q}, \qquad x_{i_k}, x_{j_l} \in X \text{ for } 1 \le k \le p, 1 \le l \le q,$$

where the concatenation product $\cdot$ is non-commutative, i.e. $x_i \cdot x_j \neq x_j \cdot x_i$ for all $x_i, x_j \in X$. Elements formed under the concatenation product are called words and $\mathbf{1}$ is the empty word (unit element of the algebra). So elements of $\mathbb{Q}\langle X\rangle$ can be generalized as linear combination of words, i.e. any $\omega \in \mathbb{Q}\langle X\rangle$ is of the form

$$\omega = \sum_{k=0}^{\infty} \sum_{i_1,\ldots,i_k \in \{1,\ldots,n\}} a_{i_1,\ldots,i_k} x_{i_1}\ldots x_{i_k},$$

where $a_{i_1,\ldots,i_k} \in \mathbb{Q}$ for all $k \ge 0$.

Remark 2. Note that $\mathbb{Q}[X] \neq \mathbb{Q}\langle X\rangle$. When we use the [,] brackets we talk about a commutative polynomial algebra. ♦

We define $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$, i.e. non-commutative polynomial algebra in two letters. Furthermore, we define

$$\mathfrak{H}^1 := \mathbb{Q}\mathbf{1} + \mathfrak{H}y, \qquad \mathfrak{H}^0 := \mathbb{Q}\mathbf{1} + x\mathfrak{H}y.$$

Note that

$$\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}.$$

Moreover, we include $\mathbb{Q}\mathbf{1}$ in the sum such that the empty word is included in $\mathfrak{H}^0, \mathfrak{H}^1$. By doing this they become sub-algebras of $\mathfrak{H}$. To find a basis for these sub-algebras, define the words $z_n := x^{n-1}y \in \mathfrak{H}$ for $n \ge 1$. By construction, we have that any word in $\mathfrak{H}^1$ ends with a y and any word in $\mathfrak{H}^0$ starts with a x and ends with a y. Then we can reason, as [Li and Qin, 2016] explains in Section 2.1, that the words

$$z_{n_1}\ldots z_{n_j}, \qquad n_i \ge 1 \text{ for } 1 \le i \le j$$

form a basis for $\mathfrak{H}^1$ and the words

$$z_{n_1} \dots z_{n_j}, \qquad n_1 \geq 2, n_i \geq 1 \text{ for } 2 \leq i \leq j$$

form a basis for $\mathfrak{H}^0$.

By the map from words to multi-index defined by

$$z_{s_1} \dots z_{s_k} \mapsto (s_1, \dots, s_k), \tag{10}$$

we notice that words in $\mathfrak{H}^1$ and multi-indexes in $\mathbb{Z}_{>0}^k$ for any integer $k \geq 1$ are in bijection. This means we can interchange words $\mathfrak{H}^1$ and multi-indexes anytime.

Remark 3. Take a word $\omega = z_{s_1} \dots z_{s_k} \in \mathfrak{H}^1$. We notice that this can be written as

$$x^{s_1-1}y \dots x^{s_k-1}y.$$

For each z we have one y. Therefore the amount of y's in the word is equal to k because we have k amount of z's. We know k is the length of the multi-index, so the number of y in a word determines the length of the corresponding multi-index. We say the word ω is of length k. Likewise, for each z_{s_i} we have precisely s_i amount of letters x, y, therefore the amount of letters in the word is equal to $s_1 + \dots + s_k$. We know this is the weight of the multi-index, so the number of letters in a word determines the weight of the corresponding multi-index. We say ω is of weight $s_1 + \dots + s_k$. ♦

Since any word $z_{s_1} \dots z_{s_k}$ in $\mathfrak{H}^1$ satisfies that $s_i \geq 1$ for $1 \leq i \leq k$, we have that the multi-index $(s_1, \dots, s_k)$ satisfies the same, i.e. the multi-index is positive. Likewise, any word $z_{s_1} \dots z_{s_k}$ in $\mathfrak{H}^0$ satisfies that $s_i \geq 1$ for $2 \leq i \leq k$ and $s_1 \geq 2$, we have that the multi-index $(s_1, \dots, s_k)$ satisfies the same, i.e. the multi-index is admissible. We say that any word corresponding to an admissible (resp. positive) multi-index is an admissible (resp. positive) word. Then we can say that $\mathfrak{H}^1$ consists of the linear combination of positive words and $\mathfrak{H}^0$ consists of the linear combination of admissible words. With this in hand we set the map $Z : \mathfrak{H}^0 \to \mathbb{R}$ defined on the basis elements we saw above by

$$Z(z_{s_1} \dots z_{s_k}) := \zeta(s_1, \dots, s_k) \tag{11}$$

and we extend this $\mathbb{Q}$-linearly. In addition we set $Z(\mathbf{1}) = 1$. This additional condition coincides with the fact that we stated in Definition 2.4 that $\zeta(\emptyset) = 1$. Since the domain is $\mathfrak{H}^0$, we have admissible words, and so $\zeta(s_1, \dots, s_k)$ is finite. Furthermore, this function is well-defined since words from $\mathfrak{H}^0$ and admissible multi-indexes are in bijection.

Remark 4. Take the word $\omega = z_{s_1} \dots z_{s_k}$. Then by this function and by the integral representation from Theorem 2.5 we have that

$$Z(\omega) = \zeta(s_1, \dots, s_k) = \lambda(\underbrace{0, \dots, 0}_{s_1 - 1 \text{ times}}, 1, \underbrace{0, \dots, 0}_{s_2 - 1 \text{ times}}, 1, \dots, \underbrace{0, \dots, 0}_{s_k - 1 \text{ times}}, 1).$$

Writing the word explicit in Z we have

$$Z(x^{s_1-1}y \dots x^{s_k-1}y) = \lambda(\underbrace{0, \dots, 0}_{s_1 - 1 \text{ times}}, 1, \underbrace{0, \dots, 0}_{s_2 - 1 \text{ times}}, 1, \dots, \underbrace{0, \dots, 0}_{s_k - 1 \text{ times}}, 1).$$

We see that a x and y correspond to a 0 and a 1 in λ, respectively. Therefore the order of the letters x, y determine the order of the $0, 1$. So by Remark 1 the letters also determine the order of the t_i's in the simplex Δ^u and vice versa, where u is the weight. This is important to notice for later. ♦

3.2 Stuffle Relation

We define an operator on $\mathfrak{H}^1$ to generalize the stuffle relation. This was firstly done by Hoffman in [Hoffman, 1997]. Other reference is [Li and Qin, 2018].

Definition 3.2. *The stuffle product on $\mathfrak{H}^1$, denoted by $*$, is the unique distributive operator that recursively is defined by the following axioms*

T1. $\mathbf{1} * \omega = \omega * \mathbf{1} = \omega, \quad \omega \in \mathfrak{H}^1,$

T2. $z_k\omega * z_l\nu = z_k(\omega * z_l\nu) + z_l(z_k\omega * \nu) + z_{k+l}(\omega * \nu), \quad \omega, \nu \in \mathfrak{H}^1, k, l \geq 1.$

We denote the algebra with the stuffle product by $\mathfrak{H}^1_*$.

Since $\mathfrak{H}^0 \subset \mathfrak{H}^1$, we have that this stuffle product is defined on $\mathfrak{H}^0$ as well, we denote it by $\mathfrak{H}^0_*$.

Theorem 3.3. *The stuffle product is commutative on $\mathfrak{H}^1_*$.*

Proof. Since the stuffle product satisfies the distributive law (see Definition 3.2), it is enough to show $\omega * \nu = \nu * \omega$ for words $\omega, \nu \in \mathfrak{H}^1$, which are not linear combinations of words. We define them by $\omega = z_{s_1}\omega'$ and $\nu = z_{r_1}\nu'$ for $\omega', \nu' \in \mathfrak{H}^1$ and $s_1, r_1 \geq 1$. We prove it by induction on $u := l(\boldsymbol{s}) + l(\boldsymbol{r})$, where $\boldsymbol{s}$ is the multi-index corresponding to ω and $\boldsymbol{r}$ the multi-index corresponding to ν. If $u = 0$, then $\omega = \nu = \mathbf{1}$ and the equality follows immediately by construction of the stuffle product. For the induction step we assume that $*$ is commutative for u' with $u' < u$. Then

$$\begin{aligned}
\omega * \nu &= z_{s_1}\omega' * z_{r_1}\nu' \\
&= z_{s_1}\left(\omega' * z_{r_1}\nu'\right) + z_{r_1}\left(z_{s_1}\omega' * \nu'\right) + z_{s_1+r_1}\left(\omega' * \nu'\right) \\
&= z_{s_1}\left(z_{r_1}\nu' * \omega'\right) + z_{r_1}\left(\nu' * z_{s_1}\omega'\right) + z_{s_1+r_1}\left(\nu' * \omega'\right) \\
&= z_{r_1}\nu' * z_{s_1}\omega' \\
&= \nu * \omega.
\end{aligned}$$

We could use the induction hypothesis since the length in the stuffle products $\omega' * z_{r_1}\nu'$, $z_{s_1}\omega' * \nu'$, $\omega' * \nu'$ is less than u. $\square$

In this way, $\mathfrak{H}^1_*$ and $\mathfrak{H}^0_*$ become commutative sub-algebras of $\mathfrak{H}$. We would like to prove that Z is an algebra homomorphism with respect to the stuffle product on $\mathfrak{H}^0_*$. We follow sections 3 and 4 in [Hoffman, 1997], but the proofs are written more explicitly.

Define for a word $\omega = z_{s_1} \dots z_{s_k} \in \mathfrak{H}^1$ and for an integer $p \geq k + 1 > 0$ the map

$$\phi_p(\omega) = \sum_{p \geq n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}}$$

and in addition $\phi_p(\mathbf{1}) = 1$. Note that ϕ_p is finite since it is a finite sum. An immediate consequence is that

$$\lim_{p \to \infty} \phi_p(\omega) = Z(\omega), \quad \omega \in \mathfrak{H}^0.$$

This limit exists since we work with admissible words in $\mathfrak{H}^0$. We prove a result that is useful for the rest of this section.

Lemma 3.4. *For $\omega \in \mathfrak{H}^1$ and integers $p, s_1 \geq 1$ we have*

$$\phi_p(z_{s_1}\omega) = \sum_{p \geq n_1 > 1} \frac{1}{n_1^{s_1}} \phi_{n_1 - 1}(\omega).$$

Proof. Write $\omega := z_{s_2} \dots z_{s_k}$, then

$$\begin{aligned}
\phi_p(z_{s_1}\omega) &= \sum_{p \geq n_1 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}} \\
&= \sum_{p \geq n_1 > 1} \frac{1}{n_1^{s_1}} \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_2^{s_2} \dots n_k^{s_k}} \\
&= \sum_{p \geq n_1 > 1} \frac{1}{n_1^{s_1}} \sum_{n_1 - 1 \geq n_2 > \dots > n_k \geq 1} \frac{1}{n_2^{s_2} \dots n_k^{s_k}} \\
&= \sum_{p \geq n_1 > 1} \frac{1}{n_1^{s_1}} \phi_{n_1 - 1}(\omega).
\end{aligned}$$

□

[Hoffman, 1997] proves in Theorem 3.2 that ϕ_p respect the the stuffle product on $\mathfrak{H}^1_*$.

Lemma 3.5. *For $\omega, \nu \in \mathfrak{H}^1_*$ and for integer $p \geq 1$ we have*

$$\phi_p(\omega * \nu) = \phi_p(\omega)\phi_p(\nu). \tag{12}$$

Proof. Since the stuffle product satisfies the distributive law (see Definition 3.2), it is enough to show $\phi_p(\omega * \nu) = \phi_p(\omega)\phi_p(\nu)$ for words $\omega, \nu \in \mathfrak{H}^1_*$ defined by $\omega = z_{s_1} z_{s_2} \dots z_{s_k}$ and $\nu = z_{r_1} z_{r_2} \dots z_{r_h}$. We prove it by induction on $l(s_1, \dots, s_k) + l(r_1, \dots, r_h) = k + h$. If $k + h = 0$, then $\omega = \nu = \mathbf{1}$ and the equality follows immediately by construction of the stuffle product and ϕ_p. We write $\omega = z_{s_1}\omega'$ and $\nu = z_{r_1}\nu'$ for words $\omega', \nu' \in \mathfrak{H}^1_*$ defined by $\omega' = z_{s_2} \dots z_{s_k}$ and $\nu' = z_{r_2} \dots z_{r_h}$. For the induction step we assume that equation (12) holds for all u with $u < k + h$. Then

$$\begin{aligned}
\phi_p(\omega * \nu) &= \phi_p(z_{s_1}\omega' * z_{r_1}\nu') \\
&= \phi_p\left(z_{s_1}(\omega' * z_{r_1}\nu') + z_{r_1}(z_{s_1}\omega' * \nu') + z_{s_1 + r_1}(\omega' * \nu')\right) \\
&= \phi_p(z_{s_1}(\omega' * z_{r_1}\nu')) + \phi_p(z_{r_1}(z_{s_1}\omega' * \nu')) + \phi_p(z_{s_1 + r_1}(\omega' * \nu')) \\
&= \sum_{p \geq n > 1} \frac{1}{n^{s_1}} \phi_{n-1}(\omega' * z_{r_1}\nu') + \sum_{p \geq m > 1} \frac{1}{m^{r_1}} \phi_{m-1}(z_{s_1}\omega' * \nu') \\
&\quad + \sum_{p \geq l > 1} \frac{1}{l^{s_1 + r_1}} \phi_{l-1}(\omega' * \nu') \\
&= \sum_{p \geq n > 1} \frac{1}{n^{s_1}} \phi_{n-1}(\omega')\phi_{n-1}(z_{r_1}\nu') + \sum_{p \geq m > 1} \frac{1}{m^{r_1}} \phi_{m-1}(z_{s_1}\omega')\phi_{m-1}(\nu') \\
&\quad + \sum_{p \geq l > 1} \frac{1}{l^{s_1 + r_1}} \phi_{l-1}(\omega')\phi_{l-1}(\nu'),
\end{aligned}$$

where we used the stuffle product and Lemma 3.4. Moreover, we could use the induction hypothesis since the length in the stuffle products $\omega' * z_{r_1}\nu'$, $z_{s_1}\omega' * \nu'$, $\omega' * \nu'$ is less than $k + h$.

On the other side, we have

$$\begin{aligned}
\phi_p(\omega)\phi_p(\nu) &= \phi_p(z_{s_1}\dots z_{s_k})\phi_p(z_{r_1}\dots z_{r_h}) \\
&= \left(\sum_{p\geq n>n_2>\dots>n_k\geq 1} \frac{1}{n^{s_1}n_2^{s_2}\dots n_k^{s_k}}\right)\left(\sum_{p\geq m>m_2>\dots>m_h\geq 1} \frac{1}{m^{r_1}m_2^{r_2}\dots m_h^{r_h}}\right) \\
&= \left(\sum_{\substack{p\geq n>n_2>\dots>n_k\geq 1,\\ p\geq m>m_2>\dots>m_h\geq 1}} \frac{1}{n^{s_1}n_2^{s_2}\dots n_k^{s_k}m^{r_1}m_2^{r_2}\dots m_h^{r_h}}\right) \\
&= \sum_{p\geq n>\dots>m>\dots>1} \frac{1}{n^{s_1}}\frac{1}{n_2^{s_2}\dots n_k^{s_k}m^{r_1}m_2^{r_2}\dots m_h^{r_h}} \\
&\quad + \sum_{p\geq m>\dots n>\dots>1} \frac{1}{m^{r_1}}\frac{1}{n^{s_1}n_2^{s_2}\dots n_k^{s_k}m_2^{r_2}\dots m_h^{r_h}} \\
&\quad + \sum_{p\geq l=n=m>\dots>1} \frac{1}{l^{s_1+r_1}}\frac{1}{n_2^{s_2}\dots n_k^{s_k}m_2^{r_2}\dots m_h^{r_h}},
\end{aligned}$$

where in the last equation we decomposed the sum into sums where $n > m$, $n < m$ and $n = m$. The other n_i for $2 \leq i \leq k$ and m_j for $2 \leq j \leq h$ are somewhere on the dots, but stay smaller than n and m, respectively. Rewriting it further

$$\begin{aligned}
\phi_p(\omega)\phi_p(\nu) = &\sum_{p\geq n>\dots>m>\dots>1} \frac{1}{n^{s_1}}\frac{1}{n_2^{s_2}\dots n_k^{s_k}m^{r_1}m_2^{r_2}\dots m_h^{r_h}} \\
&+ \sum_{p\geq m>\dots n>\dots>1} \frac{1}{m^{r_1}}\frac{1}{n^{s_1}n_2^{s_2}\dots n_k^{s_k}m_2^{r_2}\dots m_h^{r_h}} \\
&+ \sum_{p\geq l=n=m>\dots>1} \frac{1}{l^{s_1+r_1}}\frac{1}{n_2^{s_2}\dots n_k^{s_k}m_2^{r_2}\dots m_h^{r_h}} \\
= &\sum_{p\geq n>1}\frac{1}{n^{s_1}}\left(\sum_{n-1\geq n_2>\dots>n_k\geq 1}\frac{1}{n_2^{s_2}\dots n_k^{s_k}}\right)\left(\sum_{n-1\geq m>m_2>\dots>m_h\geq 1}\frac{1}{m^{r_1}m_2^{r_2}\dots m_h^{r_h}}\right) \\
&+ \sum_{p\geq m>1}\frac{1}{m^{r_1}}\left(\sum_{m-1\geq n>n_2>\dots>n_k\geq 1}\frac{1}{n^{s_1}n_2^{s_2}\dots n_k^{s_k}}\right)\left(\sum_{m-1\geq m_2>\dots>m_h\geq 1}\frac{1}{m_2^{r_2}\dots m_h^{r_h}}\right) \\
&+ \sum_{p\geq l>1}\frac{1}{l^{s_1+r_1}}\left(\sum_{l-1\geq n_2>\dots>n_k\geq 1}\frac{1}{n_2^{s_2}\dots n_k^{s_k}}\right)\left(\sum_{l-1\geq m_2>\dots>m_h\geq 1}\frac{1}{m_2^{r_2}\dots m_h^{r_h}}\right) \\
= &\sum_{p\geq n>1}\frac{1}{n^{s_1}}\phi_{n-1}(\omega')\phi_{n-1}(z_{r_1}\nu') + \sum_{p\geq m>1}\frac{1}{m^{r_1}}\phi_{m-1}(z_{s_1}\omega')\phi_{m-1}(\nu') \\
&+ \sum_{p\geq l>1}\frac{1}{l^{s_1+r_1}}\phi_{l-1}(\omega')\phi_{l-1}(\nu'),
\end{aligned}$$

where we used rules of multiplication of series. Comparing $\phi_p(\omega * \nu)$ and $\phi_p(\omega)\phi_p(\nu)$, we see that we obtain equation (12). Hence we proved the theorem by induction. □

[Hoffman, 1997] proves in Theorem 4.1 that this Lemma can be used to prove that Z is an algebra homorphism with respect to the stuffle product on $\mathfrak{H}^0_*$.

Theorem 3.6. *The map* $Z : \mathfrak{H}^0_* \to \mathbb{R}$ *is an algebra homorphism on the algebra* $\mathfrak{H}^0_*$*, that is*

A1 $Z(\mathbf{1}) = 1$,

A2 $Z(a\omega) = aZ(\omega), \quad \omega \in \mathfrak{H}^0_*, a \in \mathbb{Q}$,

A3 $Z(\omega + \nu) = Z(\omega) + Z(\nu), \quad \omega, \nu \in \mathfrak{H}^0_*$,

A4 $Z(\omega * \nu) = Z(\omega)Z(\nu), \quad \omega, \nu \in \mathfrak{H}^0_*$.

Proof. A1 is proved by construction and A2, A3 are true since we extend Z $\mathbb{Q}$-linear on the rule (11). So we are left with A4. Since the stuffle product satisfies the distributive law (see Definition 3.2), it is enough to show $Z(\omega * \nu) = Z(\omega)Z(\nu)$ for words $\omega, \nu \in \mathfrak{H}^0_*$. Therefore we use the properties of ϕ_p.

$$\begin{aligned} \phi_p(\omega * \nu) &= \phi_p(\omega)\phi_p(\nu), \\ \lim_{p\to\infty} \phi_p(\omega * \nu) &= \lim_{p\to\infty} \phi_p(\omega)\phi_p(\nu), \\ Z(\omega * \nu) &= Z(\omega)Z(\nu). \end{aligned}$$

Limits are defined since we work with admissible words in $\mathfrak{H}^0_*$. Hence we proved the theorem. □

We can not say Z is an algebra homomorphism on $\mathfrak{H}^1_*$, because we can not take the limit if $\omega, \nu \in \mathfrak{H}^1_*$ since the limit might not exist.

3.3 Shuffle Relation

Similarly, we define an operator on $\mathfrak{H}^1$ to generalize the shuffle relation. For reference see [Li and Qin, 2017].

Definition 3.7. *The shuffle product on* $\mathfrak{H}^1$*, denoted by* $\sqcup\!\sqcup$*, is the unique distributive operator that recursively is defined by the following axioms*

H1. $\mathbf{1} \sqcup\!\sqcup \omega = \omega \sqcup\!\sqcup \mathbf{1} = \omega, \quad \omega \in \mathfrak{H}^1$,

H2. $c_1\omega \sqcup\!\sqcup c_2\nu = c_1(\omega \sqcup\!\sqcup c_2\nu) + c_2(c_1\omega \sqcup\!\sqcup \nu), \quad \omega, \nu \in \mathfrak{H}^1, c_1, c_2 \in \{x, y\}$.

We denote the algebra with the shuffle product by $\mathfrak{H}^1_{\sqcup\!\sqcup}$.

Since $\mathfrak{H}^0 \subset \mathfrak{H}^1$, we have that this shuffle product is defined on $\mathfrak{H}^0$ as well, we denote it by $\mathfrak{H}^0_{\sqcup\!\sqcup}$.

Theorem 3.8. *The shuffle product is commutative on* $\mathfrak{H}^1_{\sqcup\!\sqcup}$.

Proof. Since the shuffle product satisfies the distributive law (see Definition 3.7), it is enough to show $\omega \sqcup\!\sqcup \nu = \nu \sqcup\!\sqcup \omega$ for words $\omega, \nu \in \mathfrak{H}^1_{\sqcup\!\sqcup}$, which are not linear combinations of words. We define them by $\omega = c_1\omega'$ and $\nu = c_2\nu'$ for $c_1, c_2 \in \{x, y\}$. We prove it by induction on $u := wt(\boldsymbol{s}) + wt(\boldsymbol{r})$, where $\boldsymbol{s}$ is the multi-index corresponding to ω and $\boldsymbol{r}$ the multi-index corresponding to ν. If $u = 0$, then $\omega = \nu = \mathbf{1}$ and the equality follows immediately by construction of the shuffle product.

For the induction step we assume that ⧢ is commutative for u' with $u' < u$. Then

$$\begin{aligned} \omega ⧢ \nu &= c_1\omega' ⧢ c_2\nu' \\ &= c_1 \left(\omega' ⧢ c_2\nu'\right) + c_2 \left(c_1\omega' ⧢ \nu'\right) \\ &= c_1 \left(c_2\nu' ⧢ \omega'\right) + c_2 \left(\nu' ⧢ c_1\omega'\right) \\ &= c_2\nu' ⧢ c_1\omega' \\ &= \nu ⧢ \omega. \end{aligned}$$

We could use the induction hypothesis since the weight in the shuffle products $\omega' ⧢ c_2\nu'$, $c_1\omega' ⧢ \nu'$, $\omega' ⧢ \nu'$ is less than u. □

In this way, $\mathfrak{H}^1_{⧢}$ and $\mathfrak{H}^0_{⧢}$ become commutative sub-algebras of $\mathfrak{H}$.

We would like to prove, similar to the stuffle product, that Z is an algebra homomorphism with respect to the shuffle product on $\mathfrak{H}^0_{⧢}$. Therefore, let us look closer at the shuffle product. [Gil and Fresán, 2017] shows in Proposition 1.151 the following result.

Lemma 3.9. *Define the two words $c_1 \dots c_p$ and $c_{p+1} \dots c_{p+q}$ with $c_i \in \{x, y\}$ for $1 \le i \le p+q$. The shuffle product of these two words can be written as*

$$c_1 \dots c_p ⧢ c_{p+1} \dots c_{p+q} = \sum_{\sigma \in \mathcal{S}(p,q)} c_{\sigma^{-1}(1)} \dots c_{\sigma^{-1}(p+q)}. \tag{13}$$

Proof. We notice by the definition of the shuffle product that if we compute

$$c_1 \dots c_p ⧢ c_{p+1} \dots c_{p+q},$$

we get a linear combination of words in which we permute the c_i's. Since we take one letter outside the shuffle product each time, we can say that we have only two conditions on these words:

- the letter c_i stands left with respect to the letter c_{i+1} for $1 \le i \le p-1$,
- the letter c_j stands left with respect to the letter c_{j+1} for $p+1 \le j \le p+q-1$.

This means that the order of the indexes $1, \dots, p$ and $p+1, \dots, p+q$ are respected. With this being the definition of the permutations of type (p, q) from Definition 2.8, we can see that we obtain equation (13). □

The proof of the following result is self-written. However, another proof can be found in Theorem 4.1 of [Hoffman and Ohno, 2003].

Theorem 3.10. *The map $Z : \mathfrak{H}^0_{⧢} \to \mathbb{R}$ is an algebra homomorphism on the algebra $\mathfrak{H}^0_{⧢}$, that is*

A1 $Z(\mathbf{1}) = 1$,

A2 $Z(a\omega) = aZ(\omega), \quad \omega \in \mathfrak{H}^0_{⧢}, a \in \mathbb{Q}$,

A3 $Z(\omega + \nu) = Z(\omega) + Z(\nu), \quad \omega, \nu \in \mathfrak{H}^0_{⧢}$,

A4 $Z(\omega ⧢ \nu) = Z(\omega)Z(\nu), \quad \omega, \nu \in \mathfrak{H}^0_{⧢}$.

Proof. A1 is proved by construction and A2,A3 are true since we extend Z $\mathbb{Q}$-linear on the rule (11). So we are left with A4. This is a generalization of Theorem 2.9. Since the shuffle product satisfies the distributive law (see Definition 3.7), it is enough to show $Z(\omega_1 ⧢ \omega_2) = Z(\omega_1)Z(\omega_2)$ for words $\omega_1, \omega_2 \in \mathfrak{H}^0_{⧢}$, which are not linear combinations of words. By the integral representation from Theorem 2.5, we can write

$$Z(\omega_i) = \int_{\Delta^{u_i}} I_i,$$

where we use I_i to denote the integrand of the product of all the differential forms $\eta_{\varepsilon_i}(t_i)$ and u_i is the weight of theb word ω_i. Then we have

$$\begin{aligned} Z(\omega_1)Z(\omega_2) &= \int_{\Delta^{u_1}} I_1 \int_{\Delta^{u_2}} I_2 \\ &= \int_{\Delta^{u_1} \times \Delta^{u_2}} I_1 I_2 \\ &= \sum_{\sigma \in \mathcal{S}(u_1,u_2)} \int_{\widetilde{\sigma}(\Delta^{u_1+u_2})} I_1 I_2, \end{aligned}$$

where we use the notation from Section 2.2 and equation (6). On the other hand, we can use Lemma 3.9 for the shuffle product. We see that in the shuffle product $\omega_1 ⧢ \omega_2$ we permute the x's and y's with all possible permutations of type (u_1, u_2). By Remark 4 this means we permute the 0's and 1's in λ. By Remark 1 this can be written as the integral over $\widetilde{\sigma}(\Delta^{u_1+u_2})$. Taking all permutations, we can reason that

$$\begin{aligned} Z(\omega_1 ⧢ \omega_2) &= \sum_{\sigma \in \mathcal{S}(u_1,u_2)} \int_{\widetilde{\sigma}(\Delta^{u_1+u_2})} I_1 I_2 \\ &= Z(\omega_1)Z(\omega_2). \end{aligned}$$

□

3.4 Finite Double Shuffle Relation

With these definitions and theorems in hand, we can finally generalize the finite double shuffle relation. From now on, if we write $\mathfrak{H}^1$ or $\mathfrak{H}^0$, the stuffle and shuffle product are defined on them. [Gil and Fresán, 2017] proves in Theorem 1.160 the following result.

Theorem 3.11 (Finite Double Shuffle Relation). *For any words $\omega, \nu \in \mathfrak{H}^0$ we have*

$$Z(\omega * \nu) = Z(\omega ⧢ \nu).$$

Proof. By combining Theorems 3.6 and 3.10 we have

$$Z(\omega * \nu) = Z(\omega)Z(\nu) = Z(\omega ⧢ \nu).$$

□

Let us check if we indeed recover the finite double shuffle relation we saw in equation (9). Recall that $z_n = x^{n-1}y$. We take the words $\omega = \nu = xy$, hence $\omega = \nu = z_2$. Then

$$\begin{aligned} \omega * \nu &= z_2 * z_2 \\ &= z_2(\mathbf{1} * z_2) + z_2(z_2 * \mathbf{1}) + z_4(\mathbf{1} * \mathbf{1}) \\ &= z_2 z_2 + z_2 z_2 + z_4 \\ &= 2 z_2 z_2 + z_4, \end{aligned}$$

and

$$\begin{aligned}
\omega \shuffle \nu &= xy \shuffle xy \\
&= x(y \shuffle xy) + x(xy \shuffle y) \\
&= x\Big(y(\mathbf{1} \shuffle xy) + x(y \shuffle y)\Big) + x\Big(x(y \shuffle y) + y(xy \shuffle \mathbf{1})\Big) \\
&= 2xyxy + 2x^2(y \shuffle y) \\
&= 2xyxy + 2x^2\Big(y(\mathbf{1} \shuffle y) + y(y \shuffle \mathbf{1})\Big) \\
&= 2xyxy + 4x^2y^2 \\
&= 2z_2z_2 + 4z_3z_1.
\end{aligned}$$

Then using Theorem 3.11 we obtain

$$\begin{aligned}
Z(\omega * \nu) &= Z(\omega \shuffle \nu), \\
Z(2z_2z_2 + z_4) &= Z(2z_2z_2 + 4z_3z_1), \\
2Z(z_2z_2) + Z(z_4) &= 2Z(z_2z_2) + 4Z(z_3z_1), \\
Z(z_4) &= 4Z(z_3z_1), \\
\zeta(4) &= 4\zeta(3,1).
\end{aligned}$$

Hence we indeed recover the finite double shuffle relation we saw in equation (9).

Example 6. Let us explore another example in order to see that this finite double shuffle relation can derive many linear relations for MZV. Take $\omega = z_2z_1 = xy^2$ and $\nu = z_2 = xy$. Then it can be shown by expanding the stuffle and shuffle product, similar as has been done above, that

$$\omega * \nu = z_2z_1z_2 + 2z_2z_2z_1 + z_2z_3 + z_4z_1,$$

$$\omega \shuffle \nu = z_2z_1z_2 + 6z_3z_1z_1 + 3z_2z_2z_1.$$

Then using Theorem 3.11 we obtain

$$\begin{aligned}
Z(\omega * \nu) &= Z(\omega \shuffle \nu), \\
Z(z_2z_1z_2 + 2z_2z_2z_1 + z_2z_3 + z_4z_1) &= Z(z_2z_1z_2 + 6z_3z_1z_1 + 3z_2z_2z_1), \\
Z(z_2z_1z_2) + 2Z(z_2z_2z_1) + Z(z_2z_3) + Z(z_4z_1) &= Z(z_2z_1z_2) + 6Z(z_3z_1z_1) + 3Z(z_2z_2z_1), \\
\zeta(2,1,2) + 2\zeta(2,2,1) + \zeta(2,3) + \zeta(4,1) &= \zeta(2,1,2) + 6\zeta(3,1,1) + 3\zeta(2,2,1), \\
\zeta(2,3) + \zeta(4,1) &= 6\zeta(3,1,1) + \zeta(2,2,1).
\end{aligned}$$

■

The question might be asked why it is called the *finite* double shuffle relation. The reason for this is, that while it can derive many linear combinations for MZV, it can not derive all of them. Take for example the relation

$$\zeta(3) = \zeta(2,1),$$

which was proved by Euler and it can also be derived from Corollary 2.7. Trying to obtain this from the finite double shuffle relation, one may argue to take $\omega = z_2$ and $\nu = z_1$. But this is not allowed since we need $\omega, \nu \in \mathfrak{H}^0$. But we have $\nu \in \mathfrak{H}^1 \backslash \mathfrak{H}^0$. So one could say that we need to enlarge the domain of the function $Z : \mathfrak{H}^0 \to \mathbb{R}$ to $\mathfrak{H}^1$. This can be done by the regularization of the function Z.

The theory behind this regularization needs a bit more background, which we do not discuss in this paper. This is explained in more detail in Section 1.7 of [Gil and Fresán, 2017] and another good reference is Chapter 2 of [Ihara et al., 2006]. The regularization of the finite double shuffle relation is called the regularized double shuffle relation and its importance can be seen by the following conjecture.

Conjecture 3.12. *All linear relations over $\mathbb{Q}$ for MZV are implied by the relation*

$$Z_R(\omega \sqcup\!\sqcup\, \nu - \omega * \nu) = 0, \qquad \omega \in \mathfrak{H}^0, \nu \in \mathfrak{H}^1,$$

where $Z_R : \mathfrak{H}^1 \to \mathbb{R}$ is the extension of $Z : \mathfrak{H}^0 \to \mathbb{R}$.

Some evidence for this conjecture can be found in [Li and Qin, 2016], where they derive some other relations from the regularized double shuffle relation. As we said, we do not study this conjecture in this paper. Luckily, it is conjectured that more relations imply all other relations for MZV. One of them is the integral-series identity.

4 Integral-Series Identity

In this section, we explore the integral-series identity, which is derived by Masanobu Kaneko and Shuji Yamamoto [Kaneko and Yamamoto, 2018]. Since this relation is quite new, there are not many other sources. Therefore this section is based on [Kaneko and Yamamoto, 2018].

4.1 Circled Stuffle Product

We defined the stuffle and shuffle product for the sub-algebra $\mathfrak{H}^1$. We can define another product on $\mathfrak{H}^1$.

Definition 4.1. *The circled stuffle product on $\mathfrak{H}^1$, denoted by $\circledast$, is the unique distributive operator that is defined by the following axioms*

C1. $\mathbf{1} \circledast \omega = \omega \circledast \mathbf{1} = \omega, \qquad \omega \in \mathfrak{H}^1,$

C2. $z_k\omega \circledast z_l\nu = z_{k+l}(\omega * \nu), \qquad \omega, \nu \in \mathfrak{H}^1, k, l \geq 1,$

where $*$ is the stuffle product on $\mathfrak{H}^1$.

Remark 5. For two words $\omega, \nu \in \mathfrak{H}^1$ we have that $\omega \circledast \nu \in \mathfrak{H}^0$. Namely, if $\omega = z_{s_1} \dots z_{s_k}$ and $\nu = z_{r_1} \dots z_{r_h}$ then $s_1, r_1 \geq 1$ so $s_1 + r_1 \geq 2$. Therefore

$$\omega \circledast \nu = z_{s_1} \dots z_{s_k} \circledast z_{r_1} \dots z_{r_h} = z_{s_1+r_1}(z_{s_2} \dots z_{s_k} * z_{r_2} \dots z_{r_h}) \in \mathfrak{H}^0$$

by construction of $\mathfrak{H}^0$. ♦

Furthermore, for a positive multi-index $\boldsymbol{s} = (s_1, \dots, s_k)$ let $\boldsymbol{s}^\star$ denote the formal sum of the multi-indexes of the form

$$(s_1 \circ s_2 \circ \dots \circ s_k),$$

where we either place ′,′ or ′+′ on the spot of ′∘′. So on every spot we have two options and we have $k-1$ spots. Hence we have 2^{k-1} elements in the formal sum. As an example we have

$$(3, 1, 2)^\star = (3, 1, 2) + (4, 2) + (3, 3) + (6).$$

In terms of words, using map (10), this is

$$(z_3z_1z_2)^\star = z_3z_1z_2 + z_4z_2 + z_3z_3 + z_6.$$

Example 7. With this $\star$ notation of a multi-index, we can write the formal sum of all positive multi-indexes of weight s in an abstract form. Namely

$$(\underbrace{1,\ldots,1}_{s\text{ times}})^\star = \sum_{k=1}^{s} \sum_{\substack{s_1+\ldots+s_k=s,\\ s_i\geq 1}} (s_1,\ldots,s_k),$$

where the right hand side is the formal sum of all multi-indexes of weight s, because it takes all lengths k and all possibilities in each length such that the weight is equal to s. To see this equality, we notice that we have s amount of 1's in the left hand side, so every element of the formal sum is of weight s. Moreover, any multi-index $(s_1,\ldots,s_j)$ for $1\leq j\leq k$ in the right hand side can be formed from $(\underbrace{1,\ldots,1}_{s\text{ times}})^\star$ by taking the first s_1 amount of 1's together, then we have place a $','$ and then we take s_2 amount of 1's together, and so on. So the equality follows, since we can construct any multi-index of weight s with length between 1 and k. Hence there exists precisely 2^{s-1} positive multi-indexes of weight s since there are that many multi-indexes in the formal sum of $(\underbrace{1,\ldots,1}_{s\text{ times}})^\star$.

In case of admissible multi-index, we notice the first spot should be a $'+'$, otherwise it might be non-admissible. So we take

$$(2,\ \underbrace{1,\ldots,1}_{s-2\text{ times}}\ )^\star.$$

So have $s-2$ spots, and have two options for every spot. Hence we have 2^{s-2} elements in the formal sum. So there exists 2^{s-2} admissible multi-indexes of weight s. ■

Lemma 4.2. *For $\omega\in\mathfrak{H}^0$ written by $\omega = z_{s_1}\ldots z_{s_k}$, we have*

$$Z(\omega^\star) = \sum_{n_1\geq\ldots\geq n_k\geq 1} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}}.$$

Proof. The main observation is that we can decompose the summation on the right hand side in the cases in which $n_i = n_{i+1}$ and $n_i > n_{i+1}$ for each $2\leq i\leq k-1$. In the case $n_i = n_{i+1}$ we get one variable and can add the powers of n_i and n_{i+1}. Therefore this case is equivalent to the case in which we place a $'+'$ on the spot between s_i and s_{i+1} (the powers) in $\omega^\star$. In the case $n_i > n_{i+1}$ we still have two variables and do not add the powers of n_i and n_{i+1}. Therefore this case is equivalent to the case in which we place a $','$ on the spot between s_i and s_{i+1} (the powers) in $\omega^\star$. Therefore this equation follows. □

Example 8. In the next sections we compute products of the form $\omega \circledast \nu^\star$ for $\omega,\nu\in\mathfrak{H}^1$. Therefore we give a small example here. Take $\omega = z_3z_1$ and $\nu = z_1z_2$. Then

$$\nu^\star = (z_1z_2)^* = z_1z_2 + z_3.$$

Then

$$\begin{aligned}\omega\circledast\nu^\star &= z_3z_1 \circledast (z_1z_2+z_3)\\ &= z_3z_1\circledast z_1z_2 + z_3z_1\circledast z_3\\ &= z_4(z_1 * z_2) + z_6(z_1 * \mathbf{1})\\ &= z_4\left[z_1z_2+z_2z_1+z_3\right] + z_6z_1\\ &= z_4z_1z_2 + z_4z_2z_1 + z_4z_3 + z_6z_1.\end{aligned}$$

■

4.2 2-Posets and Hasse Diagrams

Definition 4.3. *A partially ordered set (or poset) is a set $\mathcal{A}$ and a relation $\preceq$ such that for all $a, b, c \in \mathcal{A}$*

- *$a \preceq a$ (Reflexive),*
- *$a \preceq b$ and $b \preceq c$, then $a \preceq c$ (Transitive),*
- *$a \preceq b$ and $b \preceq a$, then $a = b$ (Anti-Symmetry).*

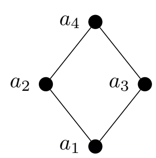
We denote a poset by the pair $(\mathcal{A}, \preceq)$. We say that $a, b \in \mathcal{A}$ are comparable if $a \preceq b$ or $a \succeq b$, otherwise a, b are called incomparable. We write $a < b$ if $a \preceq b$ and $a \neq b$. If $\mathcal{A}$ is a finite set we speak of a finite poset. In addition if $\mathcal{A} = \{a_1, \ldots, a_n\}$, then we say $\mathcal{A}$ is totally ordered if $a_1 < \ldots < a_n$.

Some examples of posets are $(\mathbb{R}, \leq)$ and $(\mathcal{P}(\mathbb{R}), \subseteq)$ where $\mathcal{P}$ stands for the power set.

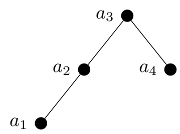
If $(\mathcal{A}, \preceq)$ is a finite poset then we can visualize the poset with a so-called Hasse diagram. In this diagram elements are denoted by dots, comparable elements are linked with a line and the difference in height of two linked dots determines how they are related by the relation $\preceq$. For example, if $a_1 < a_2$ then the dot of a_1 is lower than the dot of a_2.

Example 9. Let $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$. Then we get the following Hasse diagram for the different posets:

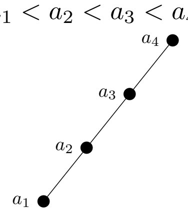
$a_1 < a_2 < a_4$ and $a_1 < a_3 < a_4$



$a_1 < a_2 < a_3 > a_4$



$a_1 < a_2 < a_3 < a_4$

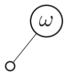


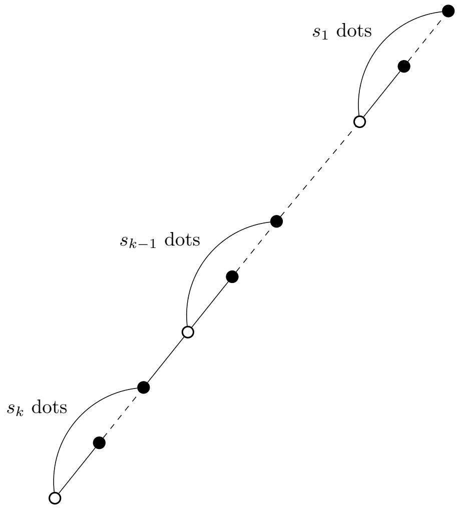
■

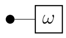
From now on we leave out the labeling of the dots. We move on with the convention that the dot to the left is a_1, the dot second from the left is a_2, and so on.

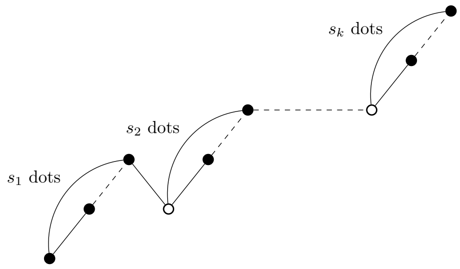
Definition 4.4. *A 2-poset is a finite poset $(\mathcal{A}, \preceq)$ with a map δ from $\mathcal{A}$ to $\{x, y\}$, called the label map of $\mathcal{A}$. We denote a 2-poset by the pair $(\mathcal{A}, \delta)$.*

We introduce this poset terminology so that we can use this for MZV. Assume we have a 2-poset $(\mathcal{A}, \delta)$. For a word $\omega = z_{s_1} \ldots z_{s_k}$ corresponding to the positive multi-index $\boldsymbol{s} = (s_1, \ldots, s_k)$ we create some notation.

We write  for the totally ordered Hasse diagram:

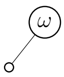


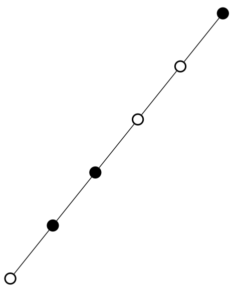
Furthermore, we write  for the Hasse diagram;

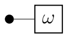


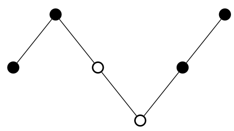
where the elements denoted by an empty dot are mapped to y under δ and the elements denoted by solid dots are mapped to x under δ.

Remark 6. By Remark 3 this means that the total number of dots determines the weight of the multi-index and the number of empty dots determines the length of the multi-index. ⧫

Example 10. Take multi-index $(2, 1, 3)$, equivalently word $\omega = z_2 z_1 z_3$. Then the Hasse diagram denoted by  is given by:



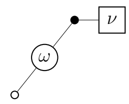
The Hasse diagram denoted by  is given by:



■

Let us define two functions. For $\omega, \nu \in \mathfrak{H}^1$ we define

$$\mu(\omega, \nu) = W\left(\quad \right),$$



where the function W on a 2-poset (or Hasse diagram) is defined by the following 2 rules

- For a totally ordered 2-poset $(\mathcal{A}, \delta)$, with $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $a_1 < \ldots < a_n$ we have

$$W(\mathcal{A}) = \delta(a_n) \ldots \delta(a_1). \tag{14}$$

  Remember that the elements denoted by an empty dot are mapped to y and the elements denoted by solid dots are mapped to x.

- Let $(\mathcal{A}, \delta)$ be a 2-poset. For two incomparable elements $a, b \in \mathcal{A}$, we define

$$W(\mathcal{A}) = W(\mathcal{A}_a^b) + W(\mathcal{A}_b^a), \tag{15}$$

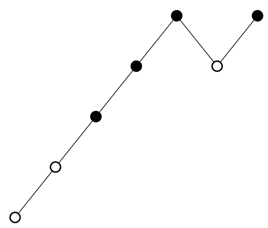
  where $\mathcal{A}_a^b$ (resp. $\mathcal{A}_b^a$) is the 2-poset by adjoining the rule $a < b$ (resp. $b < a$). This is equivalent to asking what are the possibilities to totally order $\mathcal{A}$ with the given relations. Thereafter, you sum over all totally orders with the given rule for W for totally ordered 2-posets. So for example if we have the rules $a_1 < a_2$ and $a_2 > a_3 < a_4$, then we can obtain the following totally orders

  1. $a_3 < a_1 < a_2 < a_4$,
  2. $a_1 < a_3 < a_2 < a_4$,
  3. $a_1 < a_3 < a_4 < a_2$,
  4. $a_3 < a_1 < a_4 < a_2$,
  5. $a_3 < a_4 < a_1 < a_2$.

Let us explore a more difficult example.

Example 11. Take $\omega = z_3 z_1$ and $\nu = z_1 z_2$. Then

$$\mu(\omega, \nu) = \mu(z_3 z_1, z_1 z_2) = W\left(\quad \right).$$



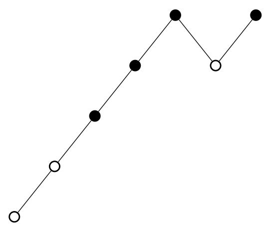
We see that some elements in this Hasse diagram are incomparable. So we have to use rule (15) of the function W. We have the relations $a_1 < a_2 < a_3 < a_4 < a_5$ and $a_5 > a_6 < a_7$. So how many totally orders are possible?

$$
\begin{array}{ll}
a_6 < a_1 < a_2 < a_3 < a_4 < a_5 < a_7, & a_6 < a_1 < a_2 < a_3 < a_7 < a_4 < a_5, \\
a_1 < a_6 < a_2 < a_3 < a_4 < a_5 < a_7, & a_1 < a_6 < a_2 < a_3 < a_7 < a_4 < a_5, \\
a_1 < a_2 < a_6 < a_3 < a_4 < a_5 < a_7, & a_1 < a_2 < a_6 < a_3 < a_7 < a_4 < a_5, \\
a_1 < a_2 < a_3 < a_6 < a_4 < a_5 < a_7, & a_1 < a_2 < a_3 < a_6 < a_7 < a_4 < a_5, \\
a_1 < a_2 < a_3 < a_4 < a_6 < a_5 < a_7, & a_6 < a_1 < a_2 < a_7 < a_3 < a_4 < a_5, \\
a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7, & a_1 < a_6 < a_2 < a_7 < a_3 < a_4 < a_5, \\
a_6 < a_1 < a_2 < a_3 < a_4 < a_7 < a_5, & a_1 < a_2 < a_6 < a_7 < a_3 < a_4 < a_5, \\
a_1 < a_6 < a_2 < a_3 < a_4 < a_7 < a_5, & a_6 < a_1 < a_7 < a_2 < a_3 < a_4 < a_5, \\
a_1 < a_2 < a_6 < a_3 < a_4 < a_7 < a_5, & a_1 < a_6 < a_7 < a_2 < a_3 < a_4 < a_5, \\
a_1 < a_2 < a_3 < a_6 < a_4 < a_7 < a_5, & a_6 < a_7 < a_1 < a_2 < a_3 < a_4 < a_5, \\
a_1 < a_2 < a_3 < a_4 < a_6 < a_7 < a_5. &
\end{array}
$$

So we get in total 21 totally ordered 2-posets. Taking W of these totally ordered 2-posets, we can use rule (14). So for example take the first totally ordered 2-poset $\mathcal{A} = \{a_6 < a_1 < a_2 < a_3 < a_4 < a_5 < a_7\}$, then

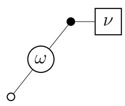
$$W(\mathcal{A}) = \delta(a_7)\delta(a_5)\delta(a_4)\delta(a_3)\delta(a_2)\delta(a_1)\delta(a_6) = xxxxyyy = z_5 z_1 z_1,$$

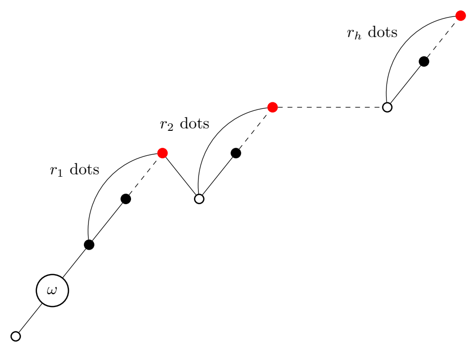
where we looked whether a_i is denoted by a solid or an empty dot in the Hasse diagram. Doing this for all of them and sum them, we find

$W\Bigg($  $\Bigg) = 12 z_5 z_1 z_1 + 5 z_4 z_2 z_1 + 2 z_3 z_3 z_1 + z_2 z_4 z_1 + z_4 z_1 z_2.$

■

Remark 7. For two words $\omega, \nu \in \mathfrak{H}^1$ we have that $\mu(\omega, \nu) \in \mathfrak{H}^0$. Namely, if $\omega = z_{s_1} \dots z_{s_k}$ and $\nu = z_{r_1} \dots z_{r_h}$ then

$\mu(\omega, \nu) = W\Big($ 

$\Big) = W\Bigg($ 

$\Bigg).$

By construction of this function W, we calculate all possible totally orders of the given Hasse diagram. In all totally orders, we must have one of the red dots as the highest dot.

Since this dot is solid, it is mapped under δ to x. Therefore any word in the formal sum, when calculating this function W, begins with the letter x. Therefore all words start with a word z_{k_1} with $k_1 \geq 2$. Hence $\mu(\omega,\nu) \in \mathfrak{H}^0$ by construction of $\mathfrak{H}^0$. ♦

4.3 Integral-Series Identity

With Remark 5 and 7, we can see that for $\omega,\nu \in \mathfrak{H}^1$ we have $\omega \circledast \nu^\star, \mu(\omega,\nu) \in \mathfrak{H}^0$. Therefore we can use the function $Z : \mathfrak{H}^0 \to \mathbb{R}$ from Section 3.1. We can obtain a representation for $Z(\omega \circledast \nu^\star)$ and $Z(\mu(\omega,\nu))$. Eventually, we can show that they are actually equal. First we prove the representation for $Z(\omega \circledast \nu^\star)$. [Kaneko and Yamamoto, 2018] gives it in equation 2.4. The proof is self-written to explain this equation.

Lemma 4.5. *Write the non-empty words $\omega,\nu \in \mathfrak{H}^1$ as $\omega = z_{s_1}\dots z_{s_k}$ and $\nu = z_{r_1}\dots z_{r_h}$. Then*

$$Z(\omega \circledast \nu^\star) = \sum_{1\leq n_k<\dots<n_1=m_1\geq\dots\geq m_h\geq 1} \frac{1}{n_1^{s_1}\dots n_k^{s_k} m_1^{r_1}\dots m_h^{r_h}}.$$

Proof. We notice that we can rewrite the right hand side as

$$\begin{aligned}
\sum_{1\leq n_k<\dots<n_1=m_1\geq\dots\geq m_h\geq 1} \frac{1}{n_1^{s_1}\dots n_k^{s_k} m_1^{r_1}\dots m_h^{r_h}} &= \sum_{1\leq n_k<\dots<n\geq\dots\geq m_h\geq 1} \frac{1}{n_2^{s_2}\dots n_k^{s_k} m_2^{r_2}\dots m_h^{r_h}} \frac{1}{n^{s_1+r_1}} \\
&= \sum_{n\geq 1} \left[\sum_{1\leq n_k<\dots<n\geq\dots\geq m_h\geq 1} \frac{1}{n_2^{s_2}\dots n_k^{s_k} m_2^{r_2}\dots m_h^{r_h}} \right] \frac{1}{n^{s_1+r_1}} \\
&= \sum_{n\geq 1} \left[\sum_{1\leq n_k<\dots<n_2<n} \frac{1}{n_2^{s_2}\dots n_k^{s_k}} \right] \left[\sum_{n\geq m_2\geq\dots\geq m_h\geq 1} \frac{1}{m_2^{r_2}\dots m_h^{r_h}} \right] \frac{1}{n^{s_1+r_1}}.
\end{aligned}$$

We can decompose the summation

$$\sum_{n\geq m_2\geq\dots\geq m_h\geq 1} \frac{1}{m_2^{r_2}\dots m_h^{r_h}}.$$

We do this by taking the cases in which we take the first amount of m_i's equal to n up to some j, i.e $n = m_i$ for $2 \leq i \leq j \leq h$. In that case we can rewrite the sum as

$$\begin{aligned}
\sum_{n\geq m_2\geq\dots\geq m_h\geq 1} \frac{1}{m_2^{r_2}\dots m_h^{r_h}} &= \sum_{j=2}^{h} \left(\sum_{n>m_{j+1}\geq\dots\geq m_h\geq 1} \frac{1}{n^{r_2+\dots+r_j} m_{j+1}^{r_{j+1}}\dots m_h^{r_h}} \right) \\
&= \sum_{j=2}^{h} \left(\sum_{n>m_{j+1}\geq\dots\geq m_h\geq 1} \frac{1}{m_{j+1}^{r_{j+1}}\dots m_h^{r_h}} \right) \frac{1}{n^{r_2+\dots+r_j}}.
\end{aligned}$$

We write $\nu = z_{r_1+\dots+r_j}\nu_j$ with $\nu_j = z_{r_{j+1}}\dots z_{r_h}$.

Then by translating Lemma 4.2 to the function ϕ_n, which is defined in section 3.2, we can write this as

$$
\begin{aligned}
\sum_{n \geq m_2 \geq \ldots \geq m_h \geq 1} \frac{1}{m_2^{r_2} \ldots m_h^{r_h}} &= \sum_{j=2}^{h} \left(\sum_{n > m_{j+1} \geq \ldots \geq m_h \geq 1} \frac{1}{m_{j+1}^{r_{j+1}} \ldots m_h^{r_h}} \right) \frac{1}{n^{r_2+\ldots+r_j}} \\
&= \sum_{j=2}^{h} \phi_{n-1}(\nu_j^\star) \frac{1}{n^{r_2+\ldots+r_j}}.
\end{aligned}
$$

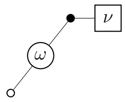
Then writing $\omega = z_{s_1}\omega'$ with $\omega' = z_{s_2} \ldots z_{s_k}$, we get in the first computations that

$$
\begin{aligned}
&\sum_{1 \leq n_k < \ldots < n_1 = m_1 \geq \ldots \geq m_h \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k} m_1^{r_1} \ldots m_h^{r_h}} \\
&= \sum_{n \geq 1} \left[\sum_{1 \leq n_k < \ldots < n_2 < n} \frac{1}{n_2^{s_2} \ldots n_k^{s_k}} \right] \left[\sum_{n \geq m_2 \geq \ldots \geq m_h \geq 1} \frac{1}{m_2^{r_2} \ldots m_h^{r_h}} \right] \frac{1}{n^{s_1+r_1}} \\
&= \sum_{n \geq 1} \left[\phi_{n-1}(\omega') \right] \left[\sum_{j=2}^{h} \phi_{n-1}(\nu_j^\star) \frac{1}{n^{r_2+\ldots+r_j}} \right] \frac{1}{n^{s_1+r_1}} \\
&= \sum_{j=2}^{h} \left(\sum_{n \geq 1} \left[\phi_{n-1}(\omega') \right] \left[\phi_{n-1}(\nu_j^\star) \right] \frac{1}{n^{s_1+r_1+\ldots+r_j}} \right) \\
&= \sum_{j=2}^{h} \left(\sum_{n \geq 1} \left[\phi_{n-1}(\omega' * \nu_j^\star) \right] \frac{1}{n^{s_1+r_1+\ldots+r_j}} \right) \\
&= \sum_{j=2}^{h} \left(\lim_{p \to \infty} \sum_{p \geq n \geq 1} \left[\phi_{n-1}(\omega' * \nu_j^\star) \right] \frac{1}{n^{s_1+r_1+\ldots+r_j}} \right) \\
&= \sum_{j=2}^{h} \left(\lim_{p \to \infty} \phi_p \left(z_{s_1+r_1+\ldots+r_j} (\omega' * \nu_j^\star) \right) \right) \\
&= \lim_{p \to \infty} \phi_p \left(z_{s_1} \omega' \circledast \sum_{j=2}^{h} z_{r_1+\ldots+r_j} \nu_j^\star \right) \\
&= \lim_{p \to \infty} \phi_p \left(\omega \circledast \nu^\star \right) \\
&= Z(\omega \circledast \nu^\star),
\end{aligned}
$$

where in fourth equality we used Lemma 3.5, in sixth equality Lemma 3.4 and seventh equality we used linearity of ϕ_p and the circled stuffle product. The limit exists since $\omega \circledast \nu^\star \in \mathfrak{H}^0$ for $\omega, \nu \in \mathfrak{H}^1$ by Remark 5.

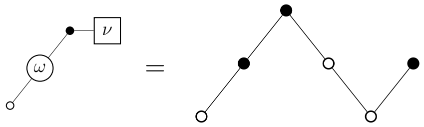
□

For $\omega, \nu \in \mathfrak{H}^1$, we set $\mathcal{A}(\omega, \nu)$ to denote the rules of the 2-poset of the Hasse diagram



. (16)

Example 12. If we take $\omega = z_2$ and $\nu = z_1 z_1 z_2$ we have the Hasse diagram



.

So we denote the rules $a_1 < a_2 < a_3 > a_4 > a_5 < a_6$ by $\mathcal{A}(\omega, \nu)$. ■

Note that we have seen that the number of dots is equal to the weight of the multi-index. So we have $u := wt(\boldsymbol{s}) + wt(\boldsymbol{r})$ number of dots in the Hasse diagram (16), where $\boldsymbol{s}$ is the multi-index corresponding to ω and $\boldsymbol{r}$ the multi-index corresponding to ν. If we say that for $\mathcal{A}(\omega, \nu)$ the a_i's satisfy $0 < a_i < 1$ for each $1 \leq i \leq u$, we can say that $\mathcal{A}(\omega, \nu)$ is the subset of $\mathbb{R}^u$ containing all $(a_1, \ldots, a_u)$ satisfying the rules of $\mathcal{A}(\omega, \nu)$. Then an integral over $\mathcal{A}(\omega, \nu)$ is equivalent to the sum of all integrals over all possible totally orders of $\mathcal{A}(\omega, \nu)$. So let $\mathcal{A}_i$ for $i \in I$, where I is some index set, denote a totally order of $\mathcal{A}(\omega, \nu)$, then

$$\int_{\mathcal{A}(\omega,\nu)} g(a_1, \ldots, a_u) da_1 \ldots da_u = \sum_{i \in I} \int_{\mathcal{A}_i} g(a_1, \ldots, a_u) da_1 \ldots da_u, \tag{17}$$

where $g : \mathbb{R}^u \to \mathbb{R}$ is some function. The following result has been self-written to improve the proof of the integral-series identity, which we see after this proof.

Lemma 4.6. *Write the non-empty words $\omega, \nu \in \mathfrak{H}^1$ as $\omega = z_{s_1} \ldots z_{s_k}$ and $\nu = z_{r_1} \ldots z_{r_h}$. Denote $u := wt(s_1, \ldots, s_k) + wt(r_1, \ldots, r_h)$. Then*

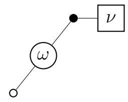
$$Z(\mu(\omega, \nu)) = \int_{\mathcal{A}(\omega,\nu)} \eta(a_1) \ldots \eta(a_u),$$

where

$$\eta(a) = \begin{cases} \frac{da}{1-a}, & a \text{ is denoted by an empty dot} \\ \frac{da}{a}, & a \text{ is denoted by a solid dot} \end{cases}.$$

Proof. We have

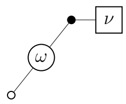
$$Z(\mu(\omega, \nu)) = Z\left(W\left(\right.\right.$$



$$\left.\left.\right)\right).$$

We have to totally order this Hasse diagram and sum over all possibilities. Therefore we have

$$Z(\mu(\omega, \nu)) = Z\left(W\left(\right.\right.$$



$$\left.\left.\right)\right)$$

$$= Z\left(\sum_{i \in I} W(\mathcal{A}_i)\right).$$

We know by the definition of the function W that $W(\mathcal{A}_i)$ is a word in $\mathfrak{H}^0$. Therefore we write it in terms of the basis of $\mathfrak{H}^0$, i.e. $W(\mathcal{A}_i) = z_{i_1} \dots z_{i_j}$ with $1 \leq j \leq u$, where j depends on the totally order given by $\mathcal{A}_i$.
Then

$$
\begin{aligned}
Z(\mu(\omega,\nu)) &= Z\left(\sum_{i\in I} W(\mathcal{A}_i)\right) \\
&= \sum_{i\in I} Z\left(W(\mathcal{A}_i)\right) \\
&= \sum_{i\in I} Z(z_{i_1}\dots z_{i_j}) \\
&= \sum_{i\in I} \zeta(i_1,\dots,i_j) \\
&= \sum_{i\in I} \lambda(\underbrace{0,\dots,0}_{i_1-1 \text{ times}},1,\underbrace{0,\dots,0}_{i_2-1 \text{ times}},1,\dots,\underbrace{0,\dots,0}_{i_j-1 \text{ times}},1).
\end{aligned}
$$

We can write out this λ in terms of an integral. We have

$$\lambda(\varepsilon_1,\dots,\varepsilon_k) = \int_{\Delta^k} \eta_{\varepsilon_1}(t_1)\dots\eta_{\varepsilon_k}(t_k),$$

where

$$\eta_\varepsilon(t) = \begin{cases} \frac{dt}{1-t}, & \varepsilon = 1 \\ \frac{dt}{t}, & \varepsilon = 0 \end{cases}.$$

But we replace t_i by a_i. In that case we have two changes in the integral denoted by λ.

- We have

$$
\begin{aligned}
\eta_\varepsilon(a) &= \begin{cases} \frac{da}{1-a}, & \varepsilon = 1 \\ \frac{da}{a}, & \varepsilon = 0 \end{cases} \\
&= \begin{cases} \frac{da}{1-a}, & a \text{ is mapped to } y \text{ under } \delta \\ \frac{da}{a}, & a \text{ is mapped to } x \text{ under } \delta \end{cases} \\
&= \begin{cases} \frac{da}{1-a}, & a \text{ is denoted by an empty dot} \\ \frac{da}{a}, & a \text{ is denoted by a solid dot} \end{cases} \\
&= \eta(a)
\end{aligned}
$$

  Because $\varepsilon = 0$ if we have a x and $\varepsilon = 1$ if we have a y, see Remark 4.

- We have that Δ^u changes to the totally ordered 2-poset $\mathcal{A}_i$. Let us clarify this. By construction of the function W the elements $a_i \in \mathcal{A}_i$ are mapped to x and y. So the order of the a_i's determine the order of x and y. By Remark 4 the order of the 0's and 1's in λ is determined by x and y. Therefore, the order of the a_i's determine the order of the 0's and 1's in λ. By Remark 1 we know that the 0's and 1's in λ determine the order of the t_i's in Δ^k. So the a_i's must determine the order of the t_i's. Therefore, by replacing t_i by a_i, Δ^u becomes the totally order 2-poset $\mathcal{A}_i$.

Then by these two points we have

$$\begin{aligned}\lambda(\underbrace{0,\dots,0}_{i_1-1\text{ times}},1,\underbrace{0,\dots,0}_{i_2-1\text{ times}},1,\dots,\underbrace{0,\dots,0}_{i_j-1\text{ times}},1) &= \int_{\Delta^k}\eta_0(t_1)\dots\eta_1(t_u)\\ &= \int_{\mathcal{A}_i}\eta(a_1)\dots\eta(a_u).\end{aligned}$$

So we get

$$\begin{aligned}Z(\mu(\omega,\nu)) &= \sum_{i\in I}\lambda(\underbrace{0,\dots,0}_{i_1-1\text{ times}},1,\underbrace{0,\dots,0}_{i_2-1\text{ times}},1,\dots,\underbrace{0,\dots,0}_{i_j-1\text{ times}},1)\\ &= \sum_{i\in I}\int_{\mathcal{A}_i}\eta(a_1)\dots\eta(a_u)\\ &= \int_{\mathcal{A}(\omega,\nu)}\eta(a_1)\dots\eta(a_u),\end{aligned}$$

where in the last equality we used rule (17). □

We are ready to state the main theorem of this section. [Kaneko and Yamamoto, 2018] proves it by proof by example in Theorem 4.1, we follow this, but take a different example.

Theorem 4.7 (Integral-Series Identity)**.** *For non-empty words* $\omega,\nu\in\mathfrak{H}^1$ *we have*

$$Z(\omega\circledast\nu^\star) = Z(\mu(\omega,\nu)). \tag{18}$$

Proof. By Lemmas 4.5 and 4.6 equation (18) is equivalent to showing

$$\sum_{1\le n_k<\dots<n_1=m_1\ge\dots\ge m_h\ge 1}\frac{1}{n_1^{s_1}\dots n_k^{s_k}m_1^{r_1}\dots m_h^{r_h}} = \int_{\mathcal{A}(\omega,\nu)}\eta(a_1)\dots\eta(a_u).$$

This can be done by computing the integral from the inside to the outside and using geometric series. We have done something similar in the proof of Theorem 2.5. Since the proof works exactly the same for this theorem, only the boundaries are different, we work out a specific example to show the computations.

Take $\omega = z_2$ and $\nu = z_1z_1z_2$. Then in Example 12 we see that the dots a_1,a_4,a_5 are empty dots and a_2,a_3,a_6 are solid dots. Hence

$$\eta(a_i) = \begin{cases}\frac{da_i}{1-a_i}, & \text{for } i=1,4,5\\ \frac{da_i}{a_1}, & \text{for } i=2,3,6\end{cases}.$$

In the computation of the integral, we use geometric series. In other words, we use the rules that for $0<a<1$ we have

$$\sum_{n\ge 1}a^{n-1} = \frac{1}{1-a},\qquad \sum_{m\ge n\ge 1}a^{n-1} = \frac{1-a^m}{1-a}.$$

So for the integral computation we get

$$
\begin{aligned}
Z(\mu(\omega,\nu)) &= \int_{\mathcal{A}(\omega,\nu)} \eta(a_1)\eta(a_2)\eta(a_3)\eta(a_4)\eta(a_5)\eta(a_6) \\
&= \int_{0<a_1<a_2<a_3>a_4>a_5<a_6<1} \frac{da_1}{1-a_1}\frac{da_2}{a_2}\frac{da_3}{a_3}\frac{da_4}{1-a_4}\frac{da_5}{1-a_5}\frac{da_6}{a_6} \\
&= \int_0^1\int_0^{a_6}\int_{a_5}^1\int_{a_4}^1\int_0^{a_3}\int_0^{a_2} \frac{da_1}{1-a_1}\frac{da_2}{a_2}\frac{da_3}{a_3}\frac{da_4}{1-a_4}\frac{da_5}{1-a_5}\frac{da_6}{a_6} \\
&= \int_0^1\int_0^{a_6}\int_{a_5}^1\int_{a_4}^1\int_0^{a_3}\left(\int_0^{a_2}\sum_{n\geq 1} a_1^{n-1}da_1\right)\frac{da_2}{a_2}\frac{da_3}{a_3}\frac{da_4}{1-a_4}\frac{da_5}{1-a_5}\frac{da_6}{a_6} \\
&= \int_0^1\int_0^{a_6}\int_{a_5}^1\int_{a_4}^1\left(\int_0^{a_3}\sum_{n\geq 1}\frac{a_2^{n-1}}{n}da_2\right)\frac{da_3}{a_3}\frac{da_4}{1-a_4}\frac{da_5}{1-a_5}\frac{da_6}{a_6} \\
&= \int_0^1\int_0^{a_6}\int_{a_5}^1\left(\int_{a_4}^1\sum_{n\geq 1}\frac{a_3^{n-1}}{n^2}da_3\right)\frac{da_4}{1-a_4}\frac{da_5}{1-a_5}\frac{da_6}{a_6} \\
&= \int_0^1\int_0^{a_6}\left(\int_{a_5}^1\sum_{n\geq 1}\frac{1}{n^3}\left(\frac{1-a_4^n}{1-a_4}\right)da_4\right)\frac{da_5}{1-a_5}\frac{da_6}{a_6} \\
&= \int_0^1\int_0^{a_6}\left(\int_{a_5}^1\sum_{n\geq 1}\frac{1}{n^3}\left(\sum_{n\geq m\geq 1} a_4^{m-1}\right)da_4\right)\frac{da_5}{1-a_5}\frac{da_6}{a_6} \\
&= \int_0^1\left(\int_0^{a_6}\sum_{n\geq 1}\frac{1}{n^3}\sum_{n\geq m\geq 1}\frac{1}{m}\left(\frac{1-a_5^m}{1-a_5}\right)da_5\right)\frac{da_6}{a_6} \\
&= \int_0^1\left(\int_0^{a_6}\sum_{n\geq 1}\frac{1}{n^3}\sum_{n\geq m\geq 1}\frac{1}{m}\left(\sum_{m\geq l\geq 1} a_5^{l-1}\right)da_5\right)\frac{da_6}{a_6} \\
&= \int_0^1\sum_{n\geq 1}\frac{1}{n^3}\sum_{n\geq m\geq 1}\frac{1}{m}\sum_{m\geq l\geq 1}\frac{a_6^{l-1}}{l}da_6 \\
&= \sum_{n\geq 1}\frac{1}{n^3}\sum_{n\geq m\geq 1}\frac{1}{m}\sum_{m\geq l\geq 1}\frac{1}{l^2} \\
&= \sum_{1\leq n\geq m\geq l\geq 1}\frac{1}{n^3 m l^2} \\
&= \sum_{1\leq n_1=m_1\geq m_2\geq m_3\geq 1}\frac{1}{n_1^2 m_1 m_2 m_3^2} \\
&= Z(\omega \circledast \nu^\star).
\end{aligned}
$$

Hence we showed equation 18 for a special case, the general case follows by computing the integrals in exactly the same way. □

Therefore we see why the identity is called the integral-series identity since the left hand side of equation (18) can be expressed as a series and the right hand side as an integral.

Example 13. For $\omega = z_3 z_1$ and $\nu = z_1 z_2$ we found in examples 8 and 11 that

$$\omega \circledast \nu^\star = z_4 z_1 z_2 + z_4 z_2 z_1 + z_4 z_3 + z_6 z_1,$$

$$\mu(\omega, \nu) = 12 z_5 z_1 z_1 + 5 z_4 z_2 z_1 + 2 z_3 z_3 z_1 + z_2 z_4 z_1 + z_4 z_1 z_2.$$

Hence by the integral-series identity we have

$$\begin{aligned} Z(\omega \circledast \nu^\star) &= Z(\mu(\omega, \nu)), \\ Z(z_4 z_1 z_2 + z_4 z_2 z_1 + z_4 z_3 + z_6 z_1) &= Z(12 z_5 z_1 z_1 + 5 z_4 z_2 z_1 + 2 z_3 z_3 z_1 + z_2 z_4 z_1 + z_4 z_1 z_2), \\ \zeta(4,1,2) + \zeta(4,2,1) + \zeta(4,3) + \zeta(6,1) &= 12\zeta(5,1,1) + 5\zeta(4,2,1) + 2\zeta(3,3,1) + \zeta(2,4,1) + \zeta(4,1,2), \\ \zeta(4,3) + \zeta(6,1) &= 12\zeta(5,1,1) + 4\zeta(4,2,1) + 2\zeta(3,3,1) + \zeta(2,4,1). \end{aligned}$$

■

4.4 Relations derived from Integral-Series Identity

We explored the integral-series identity, because likewise as for the regularized double shuffle relation, we have the following conjecture.

Conjecture 4.8. *All linear relations over $\mathbb{Q}$ for MZV are implied by the integral-series identity.*

This means that we should be able to derive any relation by filling in some ω and ν. We give some evidence for this conjecture in this section. Firstly we derive Hoffman's relation, which was proven by Theorem 5.1 in [Hoffman, 1992]. The proof is self-written, only the values that we take are confirmed to work by Remark 7.3 in [Kaneko and Yamamoto, 2018].

Theorem 4.9 (Hoffman's Relation)**.** *Let $\boldsymbol{s} = (s_1, \dots, s_k)$ be an admissible multi-index, then*

$$\sum_{i=1}^{k} \zeta(s_1, \dots, s_{i-1}, s_i + 1, s_{i+1}, \dots, s_k) = \sum_{\substack{i=1, \\ s_i \geq 2}}^{k} \sum_{j=0}^{s_i - 2} \zeta(s_1, \dots, s_{i-1}, s_i - j, j+1, s_{i+1}, \dots, s_k). \quad (19)$$

Proof. Take $\omega = z_{s_1 - 1} z_{s_2} \dots z_{s_k}$ and $\nu = z_1 z_1$ in Theorem 4.7. Firstly,

$$\begin{aligned} \omega \circledast \nu^\star &= z_{s_1-1} z_{s_2} \dots z_{s_k} \circledast (z_1 z_1)^\star \\ &= z_{s_1-1} z_{s_2} \dots z_{s_k} \circledast (z_1 z_1 + z_2) \\ &= z_{s_1-1} z_{s_2} \dots z_{s_k} \circledast z_1 z_1 + z_{s_1-1} z_{s_2} \dots z_{s_k} \circledast z_2 \\ &= z_{s_1}(z_{s_2} \dots z_{s_k} * z_1) + z_{s_1+1}(z_{s_2} \dots z_{s_k} * \mathbf{1}) \\ &= z_{s_1}(z_{s_2} \dots z_{s_k} * z_1) + z_{s_1+1} z_{s_2} \dots z_{s_k}. \end{aligned}$$

Then computing the stuffle product we get

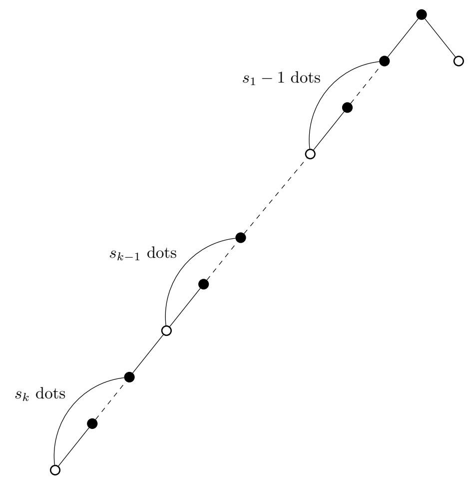
$$
\begin{aligned}
z_{s_2}\dots z_{s_k} * z_1 &= z_1 z_{s_2}\dots z_{s_k} + z_{s_2}(z_{s_3}\dots z_{s_k} * z_1) + z_{s_2+1} z_{s_3}\dots z_{s_k} \\
&= z_1 z_{s_2}\dots z_{s_k} + z_{s_2} z_1 z_{s_3}\dots z_{s_k} + z_{s_2} z_{s_3}(z_{s_4}\dots z_{s_k} * z_1) \\
&\quad + z_{s_2} z_{s_3+1} z_{s_4}\dots z_{s_k} + z_{s_2+1} z_{s_3}\dots z_{s_k} \\
&\vdots \\
&= \left(\sum_{i=2}^{k} z_{s_2}\dots z_{s_{i-1}} z_1 z_{s_i}\dots z_{s_k}\right) + z_{s_2}\dots z_{s_k} z_1 \\
&\quad + \sum_{j=2}^{k} z_{s_2}\dots z_{s_{j-1}} z_{s_j+1} z_{s_{j+1}}\dots z_{s_k}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\omega \circledast \nu^{\star} &= z_{s_1}(z_{s_2}\dots z_{s_k} * z_1) + z_{s_1+1} z_{s_2}\dots z_{s_k} \\
&= z_{s_1}\left[\left(\sum_{i=2}^{k} z_{s_2}\dots z_{s_{i-1}} z_1 z_{s_i}\dots z_{s_k}\right) + z_{s_2}\dots z_{s_k} z_1 + \sum_{j=2}^{k} z_{s_2}\dots z_{s_{j-1}} z_{s_j+1} z_{s_{j+1}}\dots z_{s_k}\right] \\
&\quad + z_{s_1+1} z_{s_2}\dots z_{s_k} \\
&= \left(\sum_{i=2}^{k} z_{s_1} z_{s_2}\dots z_{s_{i-1}} z_1 z_{s_i}\dots z_{s_k} + z_{s_1} z_{s_2}\dots z_{s_k} z_1\right) \\
&\quad + \left(\sum_{j=2}^{k} z_{s_1} z_{s_2}\dots z_{s_{j-1}} z_{s_j+1} z_{s_{j+1}}\dots z_{s_k} + z_{s_1+1} z_{s_2}\dots z_{s_k}\right) \\
&= \sum_{i=1}^{k} z_{s_1}\dots z_{s_i} z_1 z_{s_{i+1}}\dots z_{s_k} + \sum_{j=1}^{k} z_{s_1}\dots z_{s_{j-1}} z_{s_j+1} z_{s_{j+1}}\dots z_{s_k},
\end{aligned}
$$

where the single terms could be included in the sum by changing the starting point to $i = 1$ and changing the order of z_1 in the first sum and the starting point to $j = 1$ in the second sum. On the other hand

$$\mu(\omega, \nu) = W\left(\quad\right),$$



where we have $s_1 - 1$ dots since this equals the subscript of the first z in ω. We have to see which possibilities we have to total order this Hasse diagram. We can do this by placing the most right empty dot between any other 2 dots in the left array. Working this out we see that we get the general formula

$$\mu(\omega,\nu) = 2\sum_{i=1}^{k} z_{s_1}\dots z_{s_i} z_1 z_{s_{i+1}}\dots z_{s_k} + \sum_{\substack{i=1,\\ s_i\geq 3}}^{k}\sum_{j=1}^{s_i-2} z_{s_1}\dots z_{s_{i-1}} z_{s_i-j} z_{j+1} z_{s_{i+1}}\dots z_{s_k}.$$

The first summation appears when we place the empty dot next to an empty dot. In this case we get z_1. We have a coefficient of 2 in the first sum, because if you swap two non-empty dots, nothing changes. The double summation appears when you place between two solid dots. In that case, the empty dot we place splits the z_{s_i} in z_{s_i-j} and z_{j+1}. Furthermore, the double sum is only for $s_i \geq 3$, since for $s_i \in \{1,2\}$ it is included in the first sum. Then we have

$$\begin{aligned}
\mu(\omega,\nu) - \omega \circledast \nu^\star &= \left(2\sum_{i=1}^{k} z_{s_1}\dots z_{s_i} z_1 z_{s_{i+1}}\dots z_{s_k} + \sum_{\substack{i=1,\\ s_i\geq 3}}^{k}\sum_{j=1}^{s_i-2} z_{s_1}\dots z_{s_{i-1}} z_{s_i-j} z_{j+1} z_{s_{i+1}}\dots z_{s_k}\right)\\
&\quad - \left(\sum_{i=1}^{k} z_{s_1}\dots z_{s_i} z_1 z_{s_{i+1}}\dots z_{s_k} + \sum_{j=1}^{k} z_{s_1}\dots z_{s_{j-1}} z_{s_j+1} z_{s_{j+1}}\dots z_{s_k}\right)\\
&= \sum_{i=1}^{k} z_{s_1}\dots z_{s_i} z_1 z_{s_{i+1}}\dots z_{s_k} + \sum_{\substack{i=1,\\ s_i\geq 3}}^{k}\sum_{j=1}^{s_i-2} z_{s_1}\dots z_{s_{i-1}} z_{s_i-j} z_{j+1} z_{s_{i+1}}\dots z_{s_k}\\
&\quad - \sum_{j=1}^{k} z_{s_1}\dots z_{s_{j-1}} z_{s_j+1} z_{s_{j+1}}\dots z_{s_k}\\
&= \sum_{\substack{i=1,\\ s_i\geq 2}}^{k}\sum_{j=0}^{s_i-2} z_{s_1}\dots z_{s_{i-1}} z_{s_i-j} z_{j+1} z_{s_{i+1}}\dots z_{s_k} - \sum_{j=1}^{k} z_{s_1}\dots z_{s_{j-1}} z_{s_j+1} z_{s_{j+1}}\dots z_{s_k},
\end{aligned}$$

where in the last equality we could add the first sum to the second sum, by which the starting point becomes $j=0$ and we can take $s_i \geq 2$. By the integral-series identity we have

$$Z(\mu(\omega,\nu) - \omega \circledast \nu^\star) = 0.$$

So the expression we derived is under Z also equal to zero, in that way we obtain exactly equation (19). □

The second relation we derive is the restricted sum formula. [Kaneko and Yamamoto, 2018] proves it in Proposition 7.1. We follow the proof but explain some reasoning in more detail.

Theorem 4.10 (Restricted Sum Formula). *For positive integers s, k, p such that $s \geq k + p$ we have*

$$\sum_{\substack{s_1+\dots+s_p=k+p-1,\\ s_j\geq 1}} \zeta(s_1+s-k-p+1, s_2,\dots,s_p) = \sum_{\substack{s_1+\dots+s_k=s-p,\\ s_j\geq 1}} \zeta(s_1+1, s_2,\dots,s_k, \underbrace{1,\dots,1}_{p-1\ times}).$$

Proof. Let $\mathcal{X}(s,k,p)$ denote the left hand side and $\mathcal{Y}(s,k,p)$ the right hand side. So we want to prove $\mathcal{X}(s,k,p) = \mathcal{Y}(s,k,p)$ for all integers s,k,p such that $s \geq k+p$. Let $m := s-k-p+1$ and take $\omega = z_m \underbrace{z_1 \ldots z_1}_{p-1 \text{ times}}$ and $\nu = \underbrace{z_1 \ldots z_1}_{k \text{ times}}$ in Theorem 4.7. Firstly, from Example 7 we have

$$(\underbrace{z_1 \ldots z_1}_{k \textbf{ times}})^\star = \sum_{i=1}^{k} \sum_{\substack{s_1+\ldots+s_i=k, \\ s_j \geq 1}} z_{s_1} \ldots z_{s_i}$$
$$= \sum_{i=0}^{k-1} \sum_{\substack{s_1+\ldots+s_{i+1}=k, \\ s_j \geq 1}} z_{s_1} \ldots z_{s_{i+1}},$$

where we started the summation at $i=0$ because it is useful for the rest of the proof. So then

$$\omega \circledast \nu^\star = z_m \underbrace{z_1 \ldots z_1}_{p-1 \text{ times}} \circledast (\underbrace{z_1 \ldots z_1}_{k \textbf{ times}})^\star$$
$$= z_m \underbrace{z_1 \ldots z_1}_{p-1 \text{ times}} \circledast \left(\sum_{i=0}^{k-1} \sum_{\substack{s_1+\ldots+s_{i+1}=k, \\ s_j \geq 1}} z_{s_1} \ldots z_{s_{i+1}} \right)$$
$$= \sum_{i=0}^{k-1} \sum_{\substack{s_1+\ldots+s_{i+1}=k, \\ s_j \geq 1}} z_{s_1+m} \Big(\underbrace{z_1 \ldots z_1}_{p-1 \text{ times}} * z_{s_2} \ldots z_{s_{i+1}} \Big).$$

Then without computing this stuffle product explicit, we can reason that any word in this formal sum must satisfy 3 things:

- The first z must have a subscript greater than m, because the first subscript is $s_1 + m$ with $s_1 \geq 1$.
- The weight of the word is equal to s. Because summing over all subscripts of the z's we have $\underbrace{s_1 + \ldots + s_{i+1}}_{=k} + m + p - 1 = s$.
- The length of the word is equal to h where $p \leq h \leq k+p-1$. This follows from the fact that from $z_{s_1} \ldots z_{s_{i+1}}$ we can have at most length k and at least 1, since $0 \leq i \leq k-1$. Furthermore, from $\underbrace{z_1 \ldots z_1}_{p-1 \text{ times}}$ we have always length $p-1$. Therefore together, under the stuffle product, we have length $p \leq h \leq k+p-1$.

We count how many times a word that satisfies these three conditions appears in the formal sum. This is equivalent to asking how many times the word can be constructed from $\omega \circledast \nu^\star$. Write such a word as $z_{r_1} \ldots z_{r_h}$. Since r_1 is uniquely determined by s_1 in $m + s_1$, we only have to look how $z_{r_2} \ldots z_{r_h}$ is constructed from the stuffle product

$$\underbrace{z_1 \ldots z_1}_{p-1 \text{ times}} * z_{s_2} \ldots z_{s_{i+1}}.$$

If we recall the construction of the stuffle product

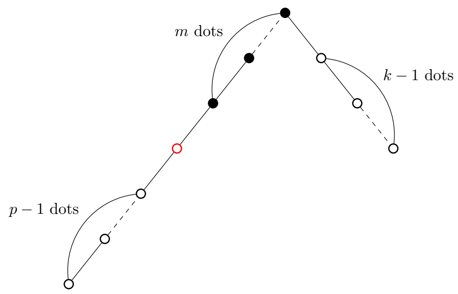
$$z_k\omega * z_l\nu = z_k(\omega * z_l\nu) + z_l(z_k\omega * \nu) + z_{k+l}(\omega * \nu), \qquad \omega, \nu \in \mathfrak{H}^1, k, l \geq 1,$$

we can reason that the numbers r_l for $2 \leq l \leq h$ in $z_{r_2} \dots z_{r_h}$ are constructed by the numbers $s_2, \dots, s_{i+1}$ and $p-1$ amount of 1. We are allowed to create any r_l by a single number of by adding 1 with a s_j for $1 \leq j \leq i+1$. Since every r_l is at least one, we can always place the 1's. After that, the r_l's are uniquely determined by the s_i's. Meaning, we can look at the possibilities of the position of the z_1's in $z_{r_2} \dots z_{r_h}$, because after these possibilities $z_{r_2} \dots z_{r_h}$ is uniquely determined by the values $s_1, \dots, s_{i+1}$. Since we have $p-1$ amount of z_1's and $j-1$ positions for them in $z_{r_2} \dots z_{r_h}$, we get $\binom{j-1}{p-1}$ choices. So this is how many times a word, that satisfies the three conditions, appears in the formal sum. So under Z we can reason, with these three conditions and the counting argument, that

$$\begin{aligned}
Z(\omega \circledast \nu^\star) &= \sum_{j=p}^{k+p-1} \binom{j-1}{p-1} \sum_{\substack{s_1+\dots+s_j=p+q-1,\\ s_l \geq 1}} \zeta(s_1+m, s_2, \dots, s_j) \\
&= \sum_{j=p}^{k+p-1} \binom{j-1}{p-1} \sum_{\substack{s_1+\dots+s_j=p+q-1,\\ s_l \geq 1}} \zeta(s_1+s-k-p+1, s_2, \dots, s_j) \\
&= \sum_{j=p}^{k+p-1} \binom{j-1}{p-1} \mathcal{X}(s, k+p-j, j) \\
&= \sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \mathcal{X}(s, k-i, p+i),
\end{aligned}$$

where we used the change of variables $j = p+i$ in the last equation. On the other hand

$$\mu(\omega, \nu) = W\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right).$$

To compute this, we must find all possible totally orders of this Hasse diagram. Therefore we can place the right array of empty dots along the left array. To be more precise, first we place i amount of empty dots from the right array of $k-1$ empty dots among the $p-1$ empty dots in the left array (below the red empty dot). If we count how many possibilities we have for this, we notice that we have in total $p-1+i$ empty dots, and have to place i empty dots among them. So we have $\binom{p-1+i}{i} = \binom{p-1+i}{p-1}$ choices. Secondly, we are left with $k-1-i$ empty dots from the right array. We have to place them among the m solid dots (above the red empty dot). This forms all admissible words of length $k-i$ (because we have $k-1-i$ empty dots, plus the red empty dot) and of weight $s-p-i+1$ (because we have in total $m+k-i = s-p-i+1$ dots). See also Remark 6.

Hence with this all we can reason that by summing over i from 0 to $k-1$ we have

$$\mu(\omega,\nu) = \sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \sum_{\substack{s_1+\ldots+s_{k-i}=s-p-i+1,\\ s_j\geq 1, s_1\geq 2}} z_{s_1} z_{s_2} \ldots z_{s_{k-i}} \underbrace{z_1 \ldots z_1}_{p+i-1 \text{ times}}$$
$$= \sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \sum_{\substack{s_1+\ldots+s_{k-i}=s-p-i,\\ s_j\geq 1}} z_{s_1+1} z_{s_2} \ldots z_{s_{k-i}} \underbrace{z_1 \ldots z_1}_{p+i-1 \text{ times}},$$

where the $p+i-1$ amount of z_1's come from the $p+i-1$ empty dots at the end of the array. Furthermore, in the last equation we could remove $+1$ from the weight and add it to the first entry. Then we have

$$Z(\mu(\omega,\nu)) = Z\left(\sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \sum_{\substack{s_1+\ldots+s_{k-i}=s-p-i,\\ s_j\geq 1}} z_{s_1+1} z_{s_2} \ldots z_{s_{k-i}} \underbrace{z_1 \ldots z_1}_{p+i-1 \text{ times}} \right)$$
$$= \sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \sum_{\substack{s_1+\ldots+s_{k-i}=s-p-i,\\ s_j\geq 1}} \zeta(s_1+1, s_2 \ldots, s_{k-i}, \underbrace{1,\ldots,1}_{p+i-1 \text{ times}})$$
$$= \sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \mathcal{Y}(s, k-i, p+i).$$

So then by the integral-series identity we have

$$Z(\omega \circledast \nu^\star) = Z(\mu(\omega,\nu)),$$
$$\sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \mathcal{X}(s, k-i, p+i) = \sum_{i=0}^{k-1} \binom{p+i-1}{p-1} \mathcal{Y}(s, k-i, p+i). \tag{20}$$

Remember that we had to show that $\mathcal{X}(s,k,p) = \mathcal{Y}(s,k,p)$. By using equation (20), we show this by induction on k. Since k is a positive integer, we start the base-case on $k=1$. So we need

$$\mathcal{X}(s,1,p) = \mathcal{Y}(s,1,p),$$

this follows immediate by choosing $k=1$ in equation (20). Hence the equation holds for $k=1$. Now by the induction hypothesis assume that the equation $\mathcal{X}(s,k',p) = \mathcal{Y}(s,k',p)$ holds for all s,k',p with $k' < k$. We need to show that it holds for k, i.e. $\mathcal{X}(s,k,p) = \mathcal{Y}(s,k,p)$. By the induction hypothesis we have that $\mathcal{X}(s,k-i,p+i) = \mathcal{Y}(s,k-i,p+i)$ for $1 \leq i \leq k-1$. Therefore in equation (20) everything cancels and we are left with the case when $i=0$. Hence $\binom{p-1}{p-1}\mathcal{X}(s,k,p) = \binom{p-1}{p-1}\mathcal{Y}(s,k,p)$. Hence by induction we proved that $\mathcal{X}(s,k,p) = \mathcal{Y}(s,k,p)$ for all integers s,k,p such that $s \geq k+p$. □

An immediate consequence of the restricted sum formula is the sum formula, [Kaneko and Yamamoto, 2018] proves it in Corollary 7.2.

Corollary 4.11 (Sum Formula). *Let s,k be positive integers such that $s \geq k+1$ then*

$$\zeta(s) = \sum_{\substack{s_1+\ldots+s_k=s,\\ s_j\geq, s_1\geq 2}} \zeta(s_1,\ldots,s_k).$$

Proof. By taking $p = 1$ in the restricted sum formula we obtain the sum formula. Namely

$$\sum_{s_1=k} \zeta(s_1 + s - k) = \sum_{\substack{s_1+\ldots+s_k=s-1,\\ s_j\geq 1}} \zeta(s_1+1, s_2, \ldots, s_k,)$$

$$\zeta(s) = \sum_{\substack{s_1+\ldots+s_k=s,\\ s_j\geq 1, s_1\geq 2}} \zeta(s_1, s_2, \ldots, s_k,),$$

where we could remove the $+1$ from the first entry and add it to the total weight. □

In order words, the sum formula describes that $\zeta(s)$ is equal to all MZV with admissible multi-indexes of weight s and a fixed length k. We derive one more relation. [Gil and Fresán, 2017] proves in Corollary 1.57 the Euler sum formula. We derive it in a different way.

Corollary 4.12 (Euler's Sum Formula)**.** *If $s \geq 3$, then*

$$\zeta(s) = \sum_{j=1}^{s-2} \zeta(s-j, j).$$

Proof. To see this, we take the length to be $k = 2$ in the sum formula and see that for $s \geq 3$ we have

$$\begin{aligned}
\zeta(s) &= \sum_{\substack{s_1+s_2=s,\\ s_1\geq 2, s_2\geq 1}} \zeta(s_1, s_2) \\
&= \sum_{\substack{s_1=s-s_2,\\ s_1\geq 2, s_2\geq 1}} \zeta(s_1, s_2) \\
&= \sum_{s-s_2\geq 2, s_2\geq 1} \zeta(s-s_2, s_2) \\
&= \sum_{1\leq j\leq s-2} \zeta(s-j, j).
\end{aligned}$$

We could also have taken $\boldsymbol{s} = s$ in Hoffman's relation. □

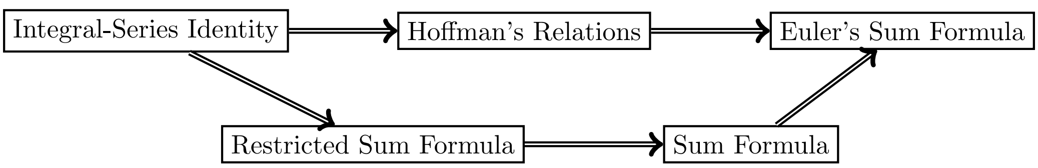
Example 14. By the Euler's sum formula we can see that we have

$$\zeta(4) = \zeta(3,1) + \zeta(2,2),$$

$$\zeta(5) = \zeta(4,1) + \zeta(3,2) + \zeta(2,3).$$

■

So in this section we have seen that:



Which gives some evidence for Conjecture 4.8. Other important relations for MZV can be found in the poster of Henrik Bachmann, see [Bachmann, 2018]. In this poster relations that can be derived from each other are linked with an arrow. Furthermore, all relations that are conjectured to imply all other relations are marked in red.

5 Zagier's and Hoffman's Conjecture

Throughout this paper, we have seen many relations for MZV: Duality theorem, finite double shuffle relation, integral-series identity, Hoffman's relation, restricted sum formula, sum formula and Euler's sum formula. But why are they useful and what can we achieve with them? To see this, it is interesting to explore the spaces spanned by MZV. This section is based on Section 1.2 and 1.4 of [Gil and Fresán, 2017].

Definition 5.1. *The $\mathbb{Q}$-subvector space of $\mathbb{R}$ generated by all MZV is denoted by $\mathcal{Z}$, i.e.*

$$\mathcal{Z} := \langle \zeta(\boldsymbol{s}) | \boldsymbol{s} \textit{ admissible multi-index} \rangle_{\mathbb{Q}}.$$

Moreover, $\mathcal{Z}_s$ denotes the $\mathbb{Q}$-subvector space generated by all MZV of weight s, i.e.

$$\mathcal{Z}_s := \langle \zeta(\boldsymbol{s}) | \boldsymbol{s} \textit{ admissible multi-index and } wt(\boldsymbol{s}) = s \rangle_{\mathbb{Q}}.$$

In particular $\mathcal{Z}_0 = \mathbb{Q}$ and $\mathcal{Z}_1 = \{0\}$.

Note that $\mathcal{Z}_s$ contains exactly 2^{s-2} generators. This follows from Example 7. So we have that 2^{s-2} MZV span the space $\mathcal{Z}_s$. Do the generators also form a basis for $\mathcal{Z}_s$ and would we have $\dim_{\mathbb{Q}} \mathcal{Z}_s = 2^{s-2}$? Therefore the generators need to be linearly independent. Now we see the importance of exploring linear relations for MZV. Because of all these linear relations for MZV we have seen, we can say that they are not linearly independent at all. Therefore the dimension is also smaller than 2^{s-2}. But what is the basis and dimension for $\mathcal{Z}_s$ then? This is not solved yet. But we can give the nowadays conjectures.

Conjecture 5.2 (Zagier's Conjecture)**.** *Define the recursion formula*

$$d_s = d_{s-2} + d_{s-3}, \qquad s \geq 3,$$

with initial values $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$. Then the equality $\dim_{\mathbb{Q}} \mathcal{Z}_s = d_s$ holds.

Conjecture 5.3 (Hoffman's Conjecture)**.** *Let $\boldsymbol{s} = (s_1, \ldots, s_k)$. The set*

$$\{\zeta(\boldsymbol{s}) : s_i \in \{2,3\}, 1 \leq i \leq k \textit{ and } wt(\boldsymbol{s}) = s\}$$

is a basis for $\mathcal{Z}_s$.

Progress toward Hoffman's conjecture was made by Francis Brown in 2012. [Brown, 2012] proves in Theorem 1.1 that the Hoffman Conjecture for so-called motivic MZV holds. Without explaining and discussing the proof of Brown's theorem in this paper. We can use this to our advantage. [Gil and Fresán, 2017] explains in Corollary 5.94 how this can be translated to normal MZV.

Corollary 5.4. *Every MZV can be written as a linear combination of MZV with entries $2, 3$.*

Remark 8. In terms of the Hoffman Conjecture this means that the set

$$\{\zeta(\boldsymbol{s}) : s_i \in \{2,3\}, 1 \leq i \leq k \text{ and } wt(\boldsymbol{s}) = s\}$$

spans the space $\mathcal{Z}_s$, where $\boldsymbol{s} = (s_1, \ldots, s_k)$. Hoffman's conjecture states that they are linearly independent as well. ♦

The following result is an observation made in this paper.

Theorem 5.5. *Hoffman's conjecture holds if and only if Zagier's conjecture holds.*

Proof. $'\implies'$ Assume that Hoffman's conjecture holds. Since $\mathcal{Z}_0 = \mathbb{Q}$, $\mathcal{Z}_1 = \{0\}$ and $\mathcal{Z}_2 = \langle\zeta(2)\rangle_{\mathbb{Q}}$ we have $\dim_{\mathbb{Q}} \mathcal{Z}_0 = 1$, $\dim_{\mathbb{Q}} \mathcal{Z}_1 = 0$ and $\dim_{\mathbb{Q}} \mathcal{Z}_2 = 1$. Therefore it coincides with the initial values $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$ from Zagier's conjecture. By Hoffman's conjecture we are given a basis for $\mathcal{Z}_s$. Then we can compute the dimension of $\mathcal{Z}_s$ by counting the number of basis elements. In other words, count the possible MZV of weight s with entries 2 and 3. We can separate this in the cases that MZV end on a 2 and end on a 3. But then we reduce 2 and 3 in weight and count the number of basis elements of the spaces $\mathcal{Z}_{s-2}$ and $\mathcal{Z}_{s-3}$. In other words, we look at the dimension for $\mathcal{Z}_{s-2}$ and $\mathcal{Z}_{s-3}$. In mathematical reasoning this looks like

$$\begin{aligned}
\dim_{\mathbb{Q}} \mathcal{Z}_s &= \#\{\zeta(s_1,\dots,s_k) : s_i \in \{2,3\}, 1 \le i \le k \text{ and } wt(s_1,\dots,s_k) = s\}\\
&= \#\{\zeta(s_1,\dots,s_{k-1},2) : s_i \in \{2,3\}, 1 \le i \le k-1 \text{ and } wt(s_1,\dots,s_{k-1},2) = s\}\\
&\quad + \#\{\zeta(s_1,\dots,s_{k-1},3) : s_i \in \{2,3\}, 1 \le i \le k-1 \text{ and } wt(s_1,\dots,s_{k-1},3) = s\}\\
&= \#\{\zeta(s_1,\dots,s_{k-1}) : s_i \in \{2,3\}, 1 \le i \le k-1 \text{ and } wt(s_1,\dots,s_{k-1}) = s-2\}\\
&\quad + \#\{\zeta(s_1,\dots,s_{k-1}) : s_i \in \{2,3\}, 1 \le i \le k-1 \text{ and } wt(s_1,\dots,s_{k-1}) = s-3\}\\
&= \dim_{\mathbb{Q}} \mathcal{Z}_{s-2} + \dim_{\mathbb{Q}} \mathcal{Z}_{s-3}.
\end{aligned}$$

Hence setting $d_s := \dim_{\mathbb{Q}} \mathcal{Z}_s$, we indeed obtain the recursion formula from Zagier's conjecture.

$'\impliedby'$ Assume Zagier's conjecture holds. In that case, we know the dimension of $\mathcal{Z}_s$ for all $s \ge 0$. We need to show that

$$\{\zeta(\boldsymbol{s}) : s_i \in \{2,3\}, 1 \le i \le k \text{ and } wt(\boldsymbol{s}) = s\}$$

is the basis of $\mathcal{Z}_s$. We know by the above computation that the cardinality of this set is equal to the dimension. By Corollary 5.4 we have that the set spans $\mathcal{Z}_s$. Now we only need to show they are linearly independent. Say for the matter of contradiction that they are linearly dependent. In that case, we can remove some MZV from the set, such that they are linearly independent. Then this set would still span the space. In that case, we have a basis. But that would mean that we have a basis with fewer elements than the set initially had. Therefore the dimension has to be reduced, which is a contradiction with the assumption that Zagier's conjecture holds. Hence we prove that they are linearly independent, i.e. form a basis. □

We try to make Corollary 5.4 explicit for a couple of weights.

Weight is 3
By Corollary 5.4 we should have

$$\mathcal{Z}_3 = \langle\zeta(3)\rangle_{\mathbb{Q}}.$$

But we have by definition

$$\mathcal{Z}_3 = \langle\zeta(3), \zeta(2,1)\rangle_{\mathbb{Q}}.$$

This means we should be able to express $\zeta(2,1)$ in terms of $\zeta(3)$. By Corollary 2.7, we have exactly $\zeta(3) = \zeta(2,1)$.

Weight is 4
By Corollary 5.4 we should have

$$\mathcal{Z}_4 = \langle\zeta(2,2)\rangle_{\mathbb{Q}}. \tag{21}$$

But we have by definition

$$\mathcal{Z}_4 = \langle\zeta(4), \zeta(2,2), \zeta(3,1), \zeta(2,1,1)\rangle_{\mathbb{Q}}.$$

This means we should be able to express $\zeta(4), \zeta(3,1), \zeta(2,1,1)$ in terms of $\zeta(2,2)$. We have

$$\begin{cases} \zeta(4) = \zeta(2,1,1), & \text{Corollary 2.7,} \\ \zeta(4) = 4\zeta(3,1), & \text{Equation (9),} \\ \zeta(4) = \zeta(3,1) + \zeta(2,2), & \text{Example 14.} \end{cases}$$

Trying to express every MZV of weight 4 in terms of $\zeta(2,2)$, we obtain

$$\zeta(2,2) = \frac{3}{4}\zeta(4) = \frac{3}{4}\zeta(2,1,1) = 3\zeta(3,1).$$

Therefore we have indeed that equality (21) holds.

Weight is 5
By Corollary 5.4 we should have

$$\mathcal{Z}_5 = \langle \zeta(3,2), \zeta(2,3) \rangle_{\mathbb{Q}}. \tag{22}$$

But we have by definition

$$\mathcal{Z}_5 = \langle \zeta(5), \zeta(4,1), \zeta(3,2), \zeta(2,3), \zeta(3,1,1), \zeta(2,2,1), \zeta(2,1,2), \zeta(2,1,1,1) \rangle_{\mathbb{Q}}.$$

This means we should be able to express $\zeta(5), \zeta(4,1), \zeta(3,1,1), \zeta(2,2,1), \zeta(2,1,2), \zeta(2,1,1,1)$ in terms of $\zeta(3,2), z(2,3)$. We have

$$\begin{cases} \zeta(5) = \zeta(2,1,1,1), & \text{Corollary 2.7,} \\ \zeta(4,1) = \zeta(3,1,1), & \text{Example 2,} \\ \zeta(3,2) = \zeta(2,2,1), & \text{Example 2,} \\ \zeta(2,3) = \zeta(2,1,2), & \text{Example 2,} \\ \zeta(2,3) + \zeta(4,1) = 6\zeta(3,1,1) + \zeta(2,2,1), & \text{Example 6,} \\ \zeta(5) = \zeta(4,1) + \zeta(3,2) + \zeta(2,3), & \text{Example 14.} \end{cases}$$

Trying to express every MZV of weight 5 in terms of $\zeta(3,2), \zeta(2,3)$, we obtain

$$\begin{pmatrix} \zeta(5) \\ \zeta(4,1) \\ \zeta(3,1,1) \\ \zeta(2,2,1) \\ \zeta(2,1,2) \\ \zeta(2,1,1,1) \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ -\frac{1}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{1}{5} \\ 1 & 0 \\ 0 & 1 \\ \frac{4}{5} & \frac{6}{5} \end{pmatrix} \begin{pmatrix} \zeta(3,2) \\ \zeta(2,3) \end{pmatrix}.$$

Therefore we have indeed that equality (22) holds.

In weights 3 and 4 we prove that $\zeta(3)$ and $\zeta(2,2)$ span $\mathcal{Z}_3$ and $\mathcal{Z}_4$, respectively. Since a single nonzero element is also linearly independent, we can say that it forms a basis for $\mathcal{Z}_3$ and $\mathcal{Z}_4$ as well. Hence Hoffman's conjecture holds in these weights. However, we can not say that Hoffman's conjecture holds in weight 5. We know that $\zeta(3,2), \zeta(2,3)$ span $\mathcal{Z}_5$ as we just showed. But they might be linearly dependent, i.e. there exists a $r \in \mathbb{Q}$ such that $\zeta(3,2) = r\zeta(2,3)$. In that case, it does not form a basis, but only one of them forms a basis for Z_5, which means that both Zagier's conjecture and Hoffman's conjecture would fail. Moreover, there does not exist a single s for which we know that $\dim_{\mathbb{Q}} \mathcal{Z}_s \geq 1$. There might always exist linear combinations that we do not know, which can lower the dimension.

There has also been a progression toward Zagier's conjecture. [Terasoma, 2001] proves in Theorem 1.2 that the dimension of $\mathcal{Z}_s$ is bounded from above by d_s, which is given by the recursion formula in Zagier's conjecture, i.e.

$$\dim_{\mathbb{Q}} \mathcal{Z}_s \leq d_s.$$

We can also see this from the fact that the set

$$\{\zeta(\boldsymbol{s}) : s_i \in \{2, 3\}, 1 \leq i \leq k \text{ and } wt(\boldsymbol{s}) = s\}$$

span the space $\mathcal{Z}_s$ (see Corollary 5.4). So the dimension is at least lower or equal to the number of elements in this set. In Theorem 5.5 we have seen that this number is equal to d_s. Hence we notice this upper bound. Proving that d_s is also a lower bound, would prove Zagier's conjecture.

Conclusion

But where has this all been good for? Let us summarize what we did in this paper. In the introduction, we questioned the values of the Riemann zeta function at the odd integers. We do not know much about them, but we conjecture them to be algebraically independent over the rationals, see the Transcendence conjecture. It could help to look at algebraic relations of zeta values, for example multiplication of zeta values. MZV arose naturally by doing this. This was the motivation to explore MZV and their relations. After seeing so many linear relations, it was time to look into the spaces spanned by MZV. Zagier's and Hoffman's conjecture gives the dimension and the basis for these spaces. [Gil and Fresán, 2017] shows in Corollary 5.106 how important this all was in terms of the zeta values at odd integers. It states that Zagier's conjecture, equivalent to Hoffman's conjecture (see Theorem 5.5), implies the Transcendence conjecture. So now we have at least a way of proving the Transcendence conjecture. So to conclude this paper, we see how important it can be by questioning new problems and investigating these new problems. Because eventually, they can answer the main question.

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