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Low Energy Solutions to the Helmholtz Equation on a Torus

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Bachelor's Project Mathematics

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Abstract

This Bachelor Project focuses on the low energy solution to the Helmholtz Equation on a torus, in particular it focuses on its relation to the solution to the same equation on the surface of a sphere. The first two chapters provide background information on the field of scattering and control theory, the Helmholtz Equation (as well as the Maxwell Equations) and toroidal coordinates. Then a review of topics from partial differential equations is carried out, with information concerning harmonics functions, the maximum principle, Poisson's Formula, and the Sommerfeld Radiation Condition. Finally, the solution on the torus is carried out by means of an approximation through the Laplacian Equation for the specific case of a compactly supported solution, and the results show that the first power term of the solution is non-zero and analogous to that of the solution to the spherical case.

Contents

Chapter 1

Introduction & Background

This Bachelor Project will focus on carrying out computations to analytically solve the Maxwell and Helmholtz equations on the surface of a torus. In order to achieve this, we want to calculate the expansions in terms of powers of λ for the solutions by imposing Dirichlet boundary conditions in toroidal coordinates. Then, through a change of basis, we want to transform the terms into spherical harmonics, which will result in the first order terms λ^0 and λ^1 having form $1/r$ and $1/r^2$. These give a solution to the low energy problems in spherical coordinates when known results from scattering theory are applied, which relates to the way the waves describing the solutions scatter [18, 16, 17]. In turn, this plays a role in solving the control theory problem of forcing the solutions to 0 in a finite amount of time through the implementation of an external control.

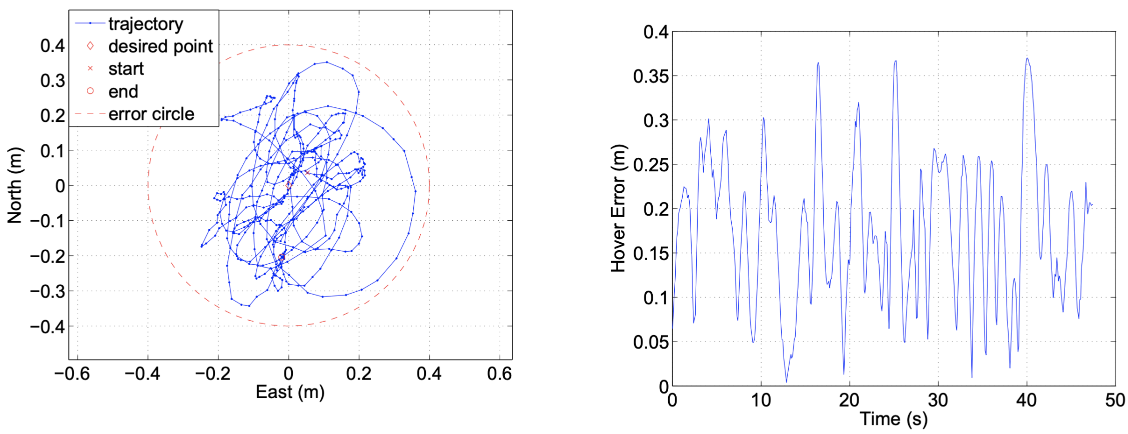
We would like to carry out this computation for both cases: the Helmholtz equation and the Maxwell equations. The desired conclusion using the known literature on scattering theory has been shown to be that in the solution to the Helmholtz equation, the nontrivial genus/topology of the torus does not matter for low energy solutions, meaning that both in the spherical and toroidal case the terms will play the same role. What previous authors claim is that this is not the case, and the main objective of the thesis project will be to show they are missing information [7]. On the other hand, the desired result for the solution to the Maxwell equations would be that these terms have no influence in the spherical case, whereas they are substantial in the toroidal case [18, 16, 17]. Unfortunately, the goals for the Maxwell case require technical computations that are beyond the scope of this thesis, so only the relevant computations for the Helmholtz case will be carried out.

The rest of this chapter will provide a very first introduction to the fields of study concerned with this project, as well as the equations in question and the toroidal coordinate system. Next, in the second chapter, the reader will find a review of methods and results of partial differential equations, including Legendre functions, the Maximum Principle, and more. Then finally the computations will be carried out in the third chapter.

1.1 Scattering and Control Theory

The fields of study concerned with the topics and computations that will be carried out in this Bachelor Project are scattering theory and control theory, both of which study physical phenomena that are governed and described by partial differential equations. In particular, scattering theory studies the phenomenon which is obtained when the trajectory of a wave or of a particle is interrupted and diverted due to its collision with other objects or particles. This is known as scattering, and is a fundamental phenomenon required to explain and understand how things such as light, sound, and radiation behave through time in real-life scenarios. On the other hand, control theory studies how a particular physical system can be influenced to behave in a desired way through the use of an external control, which influences the system accordingly depending on its current state and the desired outcome.

Example 1.1.1. Drones and other flying motorized vehicles are usually equipped with systems that collect a lot of information about the current state of the vehicle (e.g. velocity in each direction, torque, tilt angle). Given the knowledge of the physical laws that describe motion, normally given by differential and partial differential equations, these inputs can be fed through algorithms and calculations to produce an output in the form of a set of actions to be carried out by the motors [6]. For example, this can be used in order to stabilise the vehicle, whether it is in motion or attempting to remain in the same position. This is very clearly an example of control theory, since it involves differential equations and a predetermined desired solution. In particular, Figure 1.1 shows the performance of a drone attempting to remain still while hovering in the air. The figure shows that through the methods of control theory it is possible to keep the vehicle relatively stable, since its distance from the original stationary point never goes over 40 centimetres, which is comparable to human assisted results [6].



(a) (b)

Figure 1.1: [6]

Example 1.1.2. Another interesting and lesser known real-life application of control theory concerns earthquakes. Earthquakes are the result of a burst of energy due to the movement of the tectonic plates, which leads to the release of seismic waves that make the surface of the Earth shake. Since these seismic waves are described by partial differential equations wtih multiple variables and inputs, in particular by the diffusion and wave equation, control theory can be applied to reduce the intensity or risk that they pose [14]. Some researchers have theorised that fluid injections into the Earth's surface, as seen in Figure 1.2, could act as the required external control, so advancements in this field could potentially lead to the prevention of earthquakes [5].

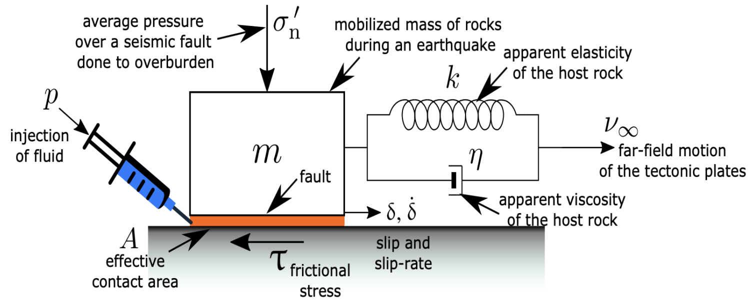


Figure 1.2: [5]

1.2 Helmholtz Equation

The equation is named after Hermann von Helmholtz, a German applied mathematician. It is a partial differential equation useful for a wide variety of applications, most importantly the solutions for other partial differential equations such as the heat equation, the wave equation and the Schrödinger equations. The equation is as follows: [12]

$$-\Delta u = \lambda^2 u,$$

where Δ is the Laplacian operator defined as

$$\Delta u(x, y, z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

The Helmholtz equation arises in the solutions to multiple partial differential equations involving a time variable, for example the heat equation

$$\frac{\partial u}{\partial t} = \gamma \Delta u$$

(where in this case the Laplacian operator only concerns the spatial coordinates) undergoes a first separation of variables through the ansatz

$$u = e^{-\lambda t} v(\mathbf{x}),$$

where $v(\mathbf{x})$ needs to satisfy the Helmholtz Equation. A similar derivation of the Helmholtz Equation can be found in the solutions to a wide variety of differential equations. Most notably the heat, wave, and Schrödinger equations which are of great importance when studying most physical phenomena (e.g. light, fluid dynamics) [12]. This makes solutions to the Helmholtz Equation necessary in the study of most physical phenomena, like the ones mentioned in Example 1.1.1 and in Example 1.1.2.

The solution to this equation can be obtained through further separation of variables, which results in multiple ordinary differential equations which can be solved in terms of eigen-solutions. These can be uniquely determined by applying the given boundary conditions. Depending on the domain, a change of coordinates might need to be applied in order for the equation to be solved. For example, the Helmholtz equation in polar coordinates is as follows:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda^2 u = 0,$$

whereas in spherical coordinates it is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\cos\varphi}{r^2 \sin\varphi}\frac{\partial u}{\partial \varphi} + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} + \lambda^2 u = 0,$$

with u defined respectively either as $u(r, \theta)$ or as $u(r, \varphi, \theta)$. When solved for these cases, the solutions will be in terms of additional functions such as Bessel functions or spherical harmonics.

Example 1.2.1. We take the general homogeneous Helmholtz problem

$$\Delta u + \lambda^2 u = 0,$$

in spherical coordinates. Then by forcing the variables in the solution to be separable we get the simpler form

$$u(r, \theta, \varphi) = v(r)p(\theta)q(\varphi),$$

where each function v, p, and q are single valued functions. Then, by plugging in this form of u into the Helmholtz equation and by manipulating the resulting equation, it is possible to separate completely each variable, which yields three different ODE's. In particular, for the rotationally symmetric case which does not depend on the azimuth angle φ, the final eigensolutions will be of form

$$u_n = r^{-1/2}[AJ_{n+1/2}(\lambda r) + BH^{(2)}_{n+1/2}(\lambda r)][CP_n(\cos\theta) + DQ_n(\cos\theta)],$$

where A, B, C and D are uniquely determined through the boundary conditions, and where the functions J_i are the Bessel function of the first kind and the functions P_i and Q_i are respectively the Legendre polynomials of the first kind and Legendre functions of the second kind [8]. A brief explanation and introduction to these function will be given at the beginning of the following chapter. Since these are the polynomials and functions making up spherical, cylindrical as well as toroidal harmonics, they cannot be overlooked, as they will be present in most solutions that take place on these surfaces.

1.3 Maxwell Equations

The Maxwell Equations describe the behaviour of magnetic and electrical fields and their interactions through a system of coupled equations. The equations are governed by the electric vector field $\mathbf{E}(t, \mathbf{x})$ and the magnetic vector field $\mathbf{B}(t, \mathbf{x})$, but they also include current $\mathbf{J}$ and charges ρ as well as the permeability constant in free space μ_0 and the permittivity constant in free space ϵ_0. The equations are as follows [4]:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \end{aligned}$$

where $\nabla \cdot \mathbf{v}$ is the divergence of a vector field defined as:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

and $\nabla \times \mathbf{v}$ is the curl of a vector field defined as:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

These two operators are commonly found in differential equations that involve vector fields, and they have a very tangible physical interpretation that clearly explains their importance. The divergence of a vector field can be seen as a measure of whether a given point of the field is acting as a source or as a sink for the field, whereas the curl associates every point with a vector that indicates a measure for the direction and intensity of some rotational behaviour of the field at a given point (see Figure 1.3 and Figure 1.4 for 2-dimensional simplifications).

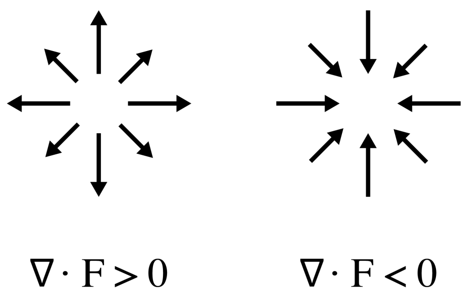


Figure 1.3

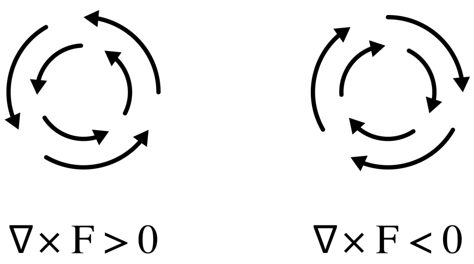


Figure 1.4

In the context of this thesis, the Maxwell equations that will be discussed are the ones that would be observed in empty space, which means no currents and charges are included in the equations. This results in a more simple and symmetric system of equations [4]:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

These can then be rearranged by applying the properties of curl and divergence, resulting in two standard uncoupled wave equations:

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \Delta \mathbf{B}$$
$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \Delta \mathbf{E},$$

where if we plug in the permeability constant μ_0 and the permittivity constant ϵ_0 we get the standard form of the wave equation with the wave speed c equal to the speed of light:

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \, \Delta \mathbf{B}$$
$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \, \Delta \mathbf{E}.$$

In order to solve these equations in spaces other than $\mathbb{R}^1$, we make use of the trick described above to transform a heat equation into a Helmholtz equation. The same can be done here for the wave equation by introducing an ansatz. Without loss of generality we take the ansatz of the first equation concerning the magnetic field:

$$B_k(t, \mathbf{x}) = \cos(\omega_n t) \, v(\mathbf{x}), \qquad \tilde{B_k}(t, \mathbf{x}) = \sin(\omega_n t) \, v(\mathbf{x}),$$

Which once again leads to the Helmholtz equation.

1.4 Toroidal Coordinates

This project focuses on solutions to partial differential equations in toroidal coordinates with particolar boundary conditions on the surface of a torus. A torus is a geometrical surface which is obtained by rotating a circle around a point outside of it, as can be seen in Figure 1.5. In order to describe tori in a more efficient manner, an appropriate coordinate system is necessary, which is the toroidal coordinate system. The toroidal coordinate system is the three-dimensional coordinate system used whenever tori are present, since it allows toroidal surfaces to be easily expressed by fixing one of the coordinates, similarly to how in spherical coordinates a sphere is defined simply by fixing the radius of it. In particular, toroidal coordinates are an extension of bipolar coordinates obtained by applying a rotation about

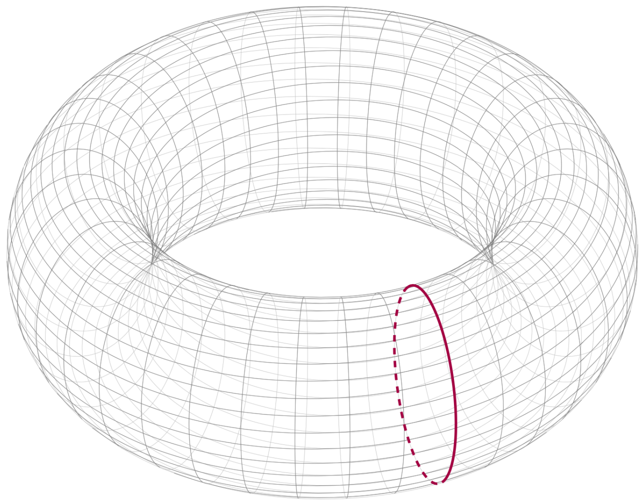


Figure 1.5

the y-axis. Bipolar coordinates are the two-dimensional coordinate system described by

$$\alpha \geq 0,$$
$$-\pi < \beta \leq \pi,$$

and a constant a [9]. This constant describes the location of two foci on the x-axis. Given these two points, the Apollonian Circles are constructed, as seen in Figure 1.6. The Apollonian circles consist of two infinite sets of circles which satisfy a particular condition with respect to the foci. The first set (shown in blue in the figure) includes infinite circles with centres on the x-axis, where each circle is represented by a value of α. On the other hand, the second set (shown in red in the figure) includes infinite circle that pass through both foci with centres on the y-axis. Each circle in this set is described by a value of β. Therefore, given an arbitrary point $p = (x_p, y_p)$ in two-dimensional euclidean space, its representation in bipolar coordinates can be given by identifying the α_p value of the circle from the first set that passes through the point, and similarly with β_p for the second set of circles. Then the point will be described by (α_p, β_p). If this two-dimensional system is rotated alongside the y-axis, we obtain a three-dimensional coordinate system that we call toroidal coordinates. Similarly to the bipolar case, these coordinates are described by (α, β, φ), accompanied by a ring of radius a on the xy-plane. The coordinates have the following ranges:

$$\alpha \geq 0,$$
$$-\pi < \beta \leq \pi,$$
$$0 \leq \varphi < 2\pi.$$

The azimuth angle φ defines a plane perpendicular to the xy-plane, then we can take any such plane to be an instance of the bipolar coordinate system, where α and β are obtained as described above through the construction of the Apollonian Circles. Alternatively, we can see that any coordinate taken alone defines a surface in the space: φ defines a vertical plane, α defines a torus that is cut in half by the xy-plane (equivalent to rotating one of the blue circles in Figure 1.6, and β defines a sphere centered on the z-axis. Then the intersection of

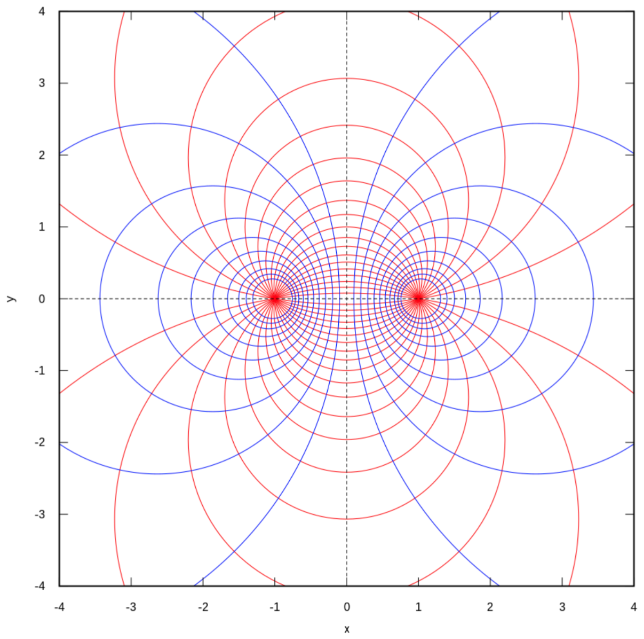


Figure 1.6: [11]

these three surfaces, which will have coordinates (α, β, φ), will represent a unique point in 3-dimensional space. Toroidal coordinates can be defined in terms of euclidean coordinates as follows:

$$x = a\frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\cos\varphi,$$
$$y = a\frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\sin\varphi,$$
$$z = a\frac{\sin\beta}{\cosh\alpha - \cos\beta}.$$

Through these identities it is then possible to transform toroidal coordinates into spherical polar coordinates (r, θ, φ) as follows:

$$r = a\frac{\sqrt{\sinh^2\alpha + \sin^2\theta}}{\cosh\alpha - \cos\theta},$$
$$\theta = \arccos\frac{\sin\varphi}{\sqrt{\sinh^2\alpha + \sin^2\theta}},$$

with the azimuth angle φ being equal across both coordinates systems [10].

If a change of variable to toroidal coordinates is applied to Laplace's equation, we obtain the form

$$\frac{\partial}{\partial\alpha}\left(\frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\frac{\partial u}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\frac{\partial u}{\partial\beta}\right) + \\ + \frac{1}{(\cosh\alpha - \cos\beta)\sinh\alpha}\frac{\partial^2 u}{\partial\varphi^2} = 0. \quad (1.1)$$

This is a particularly hard instance of Laplace's equation though, since it is not possible to separate variables as is. Therefore the substitution

$$v = \frac{1}{\sqrt{2\cosh\alpha - 2\cos\beta}}\,u \tag{1.2}$$

is used to transform (1.1) into the simpler form

$$\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} + \coth\alpha\frac{\partial v}{\partial\alpha} + \frac{1}{4}v + \frac{1}{\sinh^2\alpha}\frac{\partial^2 v}{\partial\varphi^2} = 0, \tag{1.3}$$

which allows for finding the solution through separation of variables as follows:

$$v = A(\alpha)B(\beta)C(\varphi). \tag{1.4}$$

Example 1.4.1. We take the case of the Laplacian problem on the exterior/interior of a torus, particularly in the case of rotational symmetry, meaning that the solution will be independent of φ. This means that

$$C(\varphi) = 1$$

everywhere, and we will be able to combine (1.3) and (1.4) as follows:

$$\frac{1}{A}\frac{\partial^2 A}{\partial\alpha^2} + \frac{1}{B}\frac{\partial^2 B}{\partial\beta^2} + \frac{\coth\alpha}{A}\frac{\partial A}{\partial\alpha} + \frac{1}{4} = 0, \tag{1.5}$$

from which we can obtain the two ordinary differential equations that make up the final solution:

$$\frac{\partial^2 B}{\partial\beta^2} + \lambda^2 B = 0, \tag{1.6}$$

$$\frac{\partial^2 A}{\partial\alpha^2} + \coth\alpha\frac{\partial A}{\partial\alpha} + A\left(\frac{1}{4} - \lambda^2\right) = 0. \tag{1.7}$$

We know from the separation of variables on wave equations that the solution to (1.6) will be of form

$$B_\lambda = C\cos\lambda\beta + D\cos\lambda\beta,$$

whereas the solutions to (1.7) are given by

$$A_\lambda = EP_{\lambda-1/2}(\cosh\alpha) + FQ_{\lambda-1/2}(\cosh\alpha), \tag{1.8}$$

where $P_{n-1/2}(\cosh\alpha)$ and $Q_{n-1/2}(\cosh\alpha)$ are so called "toroidal functions", once again expressed in terms of Legendre functions. More information on these can be found in the next chapter.

By plugging these results into equation (1.4) and applying the inverse substitution used to obtain (1.2), we get the eigensolutions [8]

$$u_\lambda = \sqrt{2\cosh\alpha - 2\cos\beta}\,[C\cos\lambda\beta + D\cos\lambda\beta]\cdot \\ \cdot [EP_{\lambda-1/2}(\cosh\alpha) + FQ_{\lambda-1/2}(\cosh\alpha)], \quad (1.9)$$

where coefficients C, D, E, F depend on the initial and boundary conditions. In particular, E and F depend on whether the equation is being solved for the interior or exterior of a torus (with reasonable boundedness conditions). This is because of the behaviour of the toroidal functions for values of α close to 0 (points at a large distance from the origin) or for large values of α (points very close to the radial ring). This results in $E = 0$ for any eigensolution to the interior problem, whereas we obtain $F = 0$ in the case of the exterior problem.

When this problem is encountered with Dirichlet boundary conditions and continuity conditions at $\beta = -\pi$ and $\beta = \pi$, the eigenvalues λ will always be integers. This allows us to write the final solution as an infinite sum of all the possible eigensolution through superposition:

$$u(\alpha,\beta) = \sqrt{2\cosh\alpha - 2\cos\beta}\sum_{n=0}^{\infty}[C\cos n\beta + D\cos n\beta]\cdot \\ \cdot [EP_{n-1/2}(\cosh\alpha) + FQ_{n-1/2}(\cosh\alpha)]. \quad (1.10)$$

Chapter 2

Review

The problem with toroidal coordinates, is that it is not possible to construct solutions to the Helmholtz equation using a coordinate system built around the torus. This is because the literature on the topic claims that the toroidal coordinate system does not allow for the separation of variables to be carried out successfully for this equation. For this reason, we will shift our attention to the Laplace's equation since we are focusing on the low energy solutions to the equations at hand. This is because in the case of small values for λ, Laplace's equation $-\Delta u = 0$ with particular boundary conditions can be seen as a perturbation of the Helmholtz equation $-\Delta u = \lambda^2 u$ with the same boundary conditions. Therefore, results for the former can be used as approximations for the latter. As a consequence, this section will be dedicated to properties and results obtained by studying the Laplacian operator, since it can provide insight for the objective at hand.

2.1 Harmonics

Harmonics are a large set of families of functions which describe important parts of the solutions to various differential equations, especially of the partial kind. These families of equations have infinite elements that depend on one or more parameters, and each family is obtained according to the problem discussed and its boundary conditions (and shape).

Bessel Functions

Bessel functions arise by solving the second order differential equation known as Bessel's differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0,$$

where the arbitrary complex number α is the order of the resulting Bessel function. Depending on the singularity that the solution obtains, the solution will be described differently. If a solution is not singular at the origin, it is a Bessel function of the first kind, commonly referred to as $J_\alpha(x)$. This equation is commonly found when solving Helmholtz and Laplace's equations in either cylindrical or spherical coordinates. In particular, one obtains $\alpha = n$ in the cylindrical case (Figure 2.1), and $\alpha = n + 1/2$ in the spherical case (Figure 2.2), with $n \in \mathbb{N}$.

A way to generalize a description for Bessel functions of the first kind is through infinite sums. This sum in particular is a series expansion around the origin [1]:

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

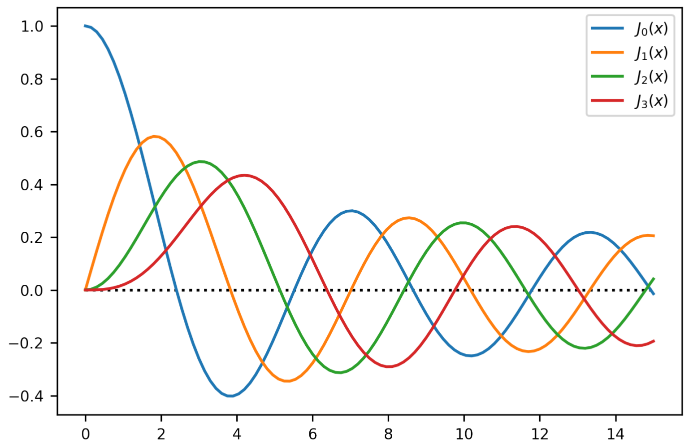


Figure 2.1

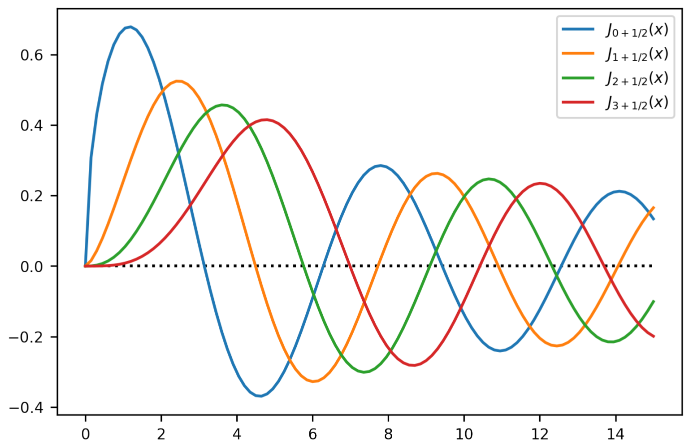


Figure 2.2

The solutions that do have a singularity at the origin are called Bessel functions of the second kind, and they are referred to by $Y_\alpha(x)$ (Figure Figure 2.3). In the case that α is non-integer, then the following relation holds [3]:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha x) - J_{-\alpha}(x)}{\sin(\alpha x)}.$$

When instead α is an integer, the expression for Y_α is either given by a more complicated series, or it can be obtained by taking the limit of the previous expression as α tends to the integer value. Lastly, the complex solutions to the Bessel equations are given by the Bessel functions of the third kind, commonly known as Hankel functions of the first and second

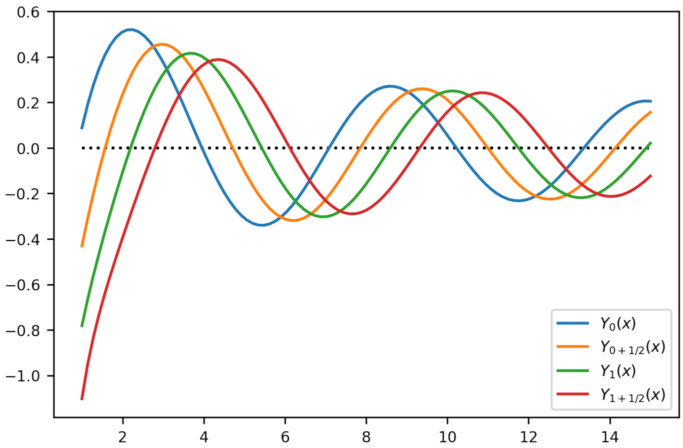


Figure 2.3

kind. These are a linear combination of the Bessel functions of first and second kind with respect to the complex basis $\{1, i\}$. The distinction between the first and second kind of Hankel functions is as follows [1]:

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x),$$
$$H_\alpha^{(2)}(x) = J_\alpha(x) - iY_\alpha(x).$$

Legendre Functions

Similarly to Bessel functions, Legendre functions are the solutions to the Legendre differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

Once again, we call the solution that attains no singularities a Legendre function of the first kind, denoted by $P_n(x)$. In the case that n is an integer, then this solution will be a polynomial of order n (Figure Figure 2.4). The most compact way to explicitly express these polynomials is given by [1]

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

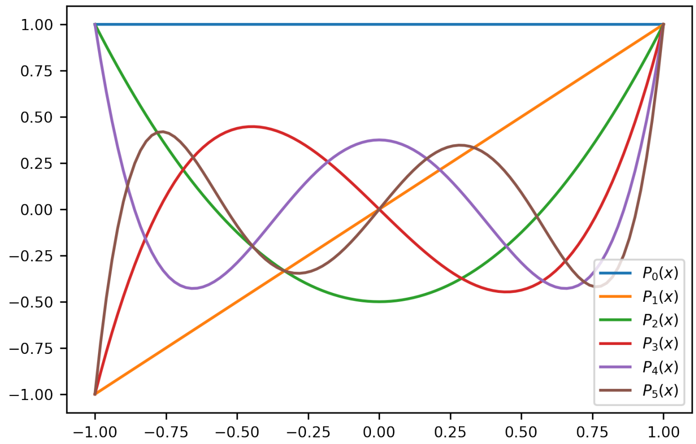


Figure 2.4

Additionally, the Legendre polynomials obey a recurrence relation which makes it possible to iteratively compute the Legendre polynomial of arbitrary order n. This relations is

$$P_{n+1}(x) = \frac{(2n+1)xP_n(x) - nP_{n-1}(x)}{n+1}. \tag{2.1}$$

The other possible solution will always have two singularities, namely at $x = -1$ and at $x = 1$. Solutions of this kind are called the Legendre functions of the second kind, and they're indicated by $Q_n(x)$ (Figure Figure 2.5). Unlike the first kind solutions, these do not simplify to polynomials when n is an integer. Nevertheless, the recurrence relation (2.1) described for the first kind solution is maintained for these as well, so the Legendre functions of the second kind for integer orders can all be obtained from the first two orders [1]

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$
$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1.$$

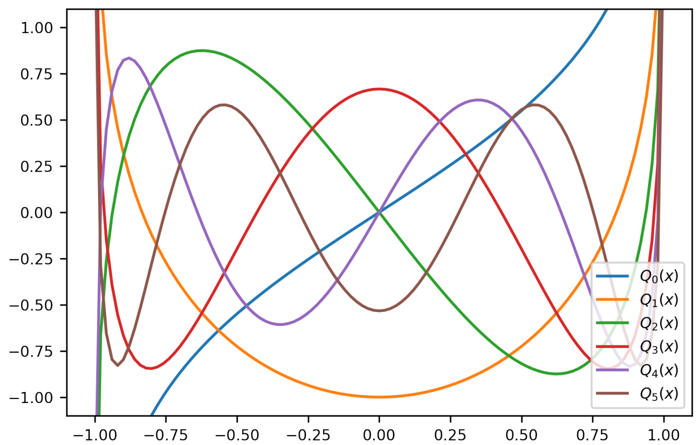


Figure 2.5

Associated Legendre Functions

Once the Legendre functions are established as solutions to the Legendre differential equation, we introduce the associated Legendre differential equation, which resembles the first very closely. In fact, this equation can be seen as a generalization of the first which adds a component m. When m is set to be equal to 0, we obtain the Legendre equations once again. This generalized formula ultimately increases the complexity of the solutions, as well as their usefulness.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0.$$

Similarly to how we defined the Legendre functions, we define n as the degree, and similarly to Bessel's equation we call m the order. For these functions, we mainly consider the case where both n and m are integers. Once again, the solution with no singularities is a polynomial, and when both the order and the degree are positive we have a formula to determine the associated Legendre polynomial based off of the corresponding Legendre polynomial [1]

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}(P_n(x)). \tag{2.2}$$

Additionally, we have the solutions with singularities (again at $|x| = 1$ like in the normal Legendre equation) which are obtained very similarly by

$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}(Q_n(x)). \tag{2.3}$$

Spherical and Toroidal Harmonics

By spherical harmonics we refer to the solutions to the angular portion of Laplace's equation in spherical coordinates. The solution to this only depends on the two angular coordinates, and is obtained through separation of variables. We denote spherical harmonics of degree n and order m by Y_n^m, not to be confused with Bessel's equations of the second kind, and they are as follows [2]:

$$Y_n^m(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)e^{im\varphi},$$

where the first term within the square root is somewhat of an arbitrary choice, chosen such that the normalization will allow for some convenient equalities. We can then see the term $P_n^m(\cos\theta)$, which is a reparameterization of the associated Legendre polynomials. Therefore formula (2.2) can be modified as

$$P_n^m(\cos\theta) = (-\sin)^m\frac{d^m}{d(\cos\theta)^m)}(P_n(\cos\theta)).$$

Similarly, by toroidal harmonics we refer to the components of the solutions to Laplace's equation in toroidal coordinates that include the coordinate α (the one that when kept fixed represents a torus). As we've previously seen in Example 1.4.1, these consist of the functions

$$P_{n-1/2}(\cosh\alpha) \quad \text{and} \quad Q_{n-1/2}(\cosh\alpha),$$

with half-integer degree $n - 1/2$. But in order to maintain the highest degree of generality we take the corresponding associated Legendre functions to be the actual toroidal harmonics [10]:

$$P^m_{n-1/2}(\cosh\alpha) \quad \text{and} \quad Q^m_{n-1/2}(\cosh\alpha).$$

Similarly to spherical harmonics, we can apply a reparameterization of formula (2.2) and (2.3) to obtain an explicit expression for toroidal harmonics.

2.2 The Maximum Principle

Definition 2.1.1. Let $\Omega \subset \mathbb{R}^n$ be an open, connected domain. A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is said to be:

- subharmonic if $-\Delta u \leq 0$ in Ω
- superharmonic if $-\Delta u \geq 0$ in Ω
- harmonic if $-\Delta u = 0$ in Ω

Theorem 2.1.2 (Weak Maximum Principle). Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded, connected domain and that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic in Ω. Then,

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$$

Proof. First consider the special case where $-\Delta u < 0$ in Ω. We show that u cannot have a maximum in Ω. Suppose $x_0 \in \Omega$ is a maximum of u. Then by differential calculus

$$\begin{aligned} \partial_{x_i} u(x_0) &= 0, \\ \partial^2_{x_i} u(x_0) &\leq 0, \qquad i = 1, .., n \\ \implies -\Delta u(x_0) &\geq 0, \end{aligned}$$

and this contradicts the fact that $-\Delta u < 0$ in Ω. Hence u attains its maximum on $\overline{\Omega}$ on $\partial\Omega$, so that

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x).$$

Now consider the general case $-\Delta u \leq 0$ in Ω. Define

$$v_\epsilon(x) = u(x) + \epsilon\, e^{x_i}$$

so that

$$-\Delta v_\epsilon(x) = \underbrace{\underbrace{-\,\Delta u(x)}_{\leq 0}\ \underbrace{-\,\epsilon\, e^{x_i}}_{<0}}_{<0},$$

by the first part we have that

$$\max_{x\in\overline{\Omega}} v_\epsilon(x) = \max_{x\in\partial\Omega} v_\epsilon(x).$$

Taking the limit $\epsilon \to 0$, so that $v_\epsilon(x) \to u(x)$ uniformly on $\overline{\Omega}$, we have that

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$$

□

Corollary 2.1.3 (Weak Minimum Principle). Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded, connected domain and that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is superharmonic in Ω. Then,

$$\min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$$

Proof. Apply the weak maximum principle to $-u$. □

Corollary 2.1.4 (Comparison Principle). Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded, connected domain and that $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta u_1 &= f_1 \text{ in } \Omega, \qquad & -\Delta u_2 &= f_2 \text{ in } \Omega, \\ u_1 &= g_1 \text{ in } \partial\Omega, \qquad & u_2 &= g_2 \text{ in } \partial\Omega, \end{aligned}$$

where

$$\begin{aligned} f_1(x) &\le f_2(x), \quad x \in \Omega, \\ g_1(x) &\le g_2(x), \quad x \in \partial\Omega. \end{aligned}$$

Then,

$$u_1(x) \le u_2(x), \quad x \in \overline{\Omega}$$

Proof. The function $w = u_1 - u_2$ satisfies

$$\begin{aligned} -\Delta w &= f_1 - f_2 \le 0 \quad \text{in } \Omega, \\ w &= g_1 - g_2 \le 0 \quad \text{on } \partial\Omega. \end{aligned}$$

So by the weak maximum principle,

$$\begin{aligned} \max_{x\in\overline{\Omega}} w(x) &= \max_{x\in\partial\Omega} w(x) \le 0 \\ &\implies w(x) \le 0 \quad \text{on } \overline{\Omega} \\ &\implies u_1(x) \le u_2(x) \quad \text{on } \overline{\Omega}. \end{aligned}$$

□

Remark. The previous result explains the use of the term "subharmonic". Suppose $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta u_1 &\le 0 \text{ in } \Omega, \qquad & -\Delta u_2 &= 0 \text{ in } \Omega, \\ u_1 &= g \text{ on } \partial\Omega, \qquad & u_2 &= g \text{ on } \partial\Omega. \end{aligned}$$

Then, by the Comparison Principle $u_1(x) \le u_2(x)$ on $\overline{\Omega}$. In other words, a subharmonic function is always less than or wqual to the harmonic function with the same boundary values.

Corollary 2.1.5 (Uniqueness). Tbe Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded, connected domain, has at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Proof. Suppose there are two solutions u_1, u_2. Then $w = u_1 - u_2$ satisfies

$$\begin{aligned} -\Delta w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

By the weak maximum principle we obtain

$$\max_{x \in \overline{\Omega}} w(x) = \max_{x \in \partial\Omega} w(x),$$

and similarly by the weak minimum principle

$$\min_{x \in \overline{\Omega}} w(x) = \min_{x \in \partial\Omega} w(x).$$

Hence it follows that

$$\begin{aligned} w(x) &= 0, \quad x \in \overline{\Omega}, \\ \implies u_1(x) &= u_2(x), \quad x \in \overline{\Omega}. \end{aligned}$$

□

Remark. As shown in the previous proof, we have that when $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in an open, bounded, connected domain and zero on the boundary of the domain, u will be zero over the entire closure of the domain.

Theorem 2.1.6 (Strong Maximum Principle). Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded, connected domain and that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic in Ω.

Then either:

- u does not achieve its maximum value on $\overline{\Omega}$ at a point of Ω,
- u is constant on $\overline{\Omega}$.

The proof for this theorem is not given here since it is long and cumbersome.

2.3 Poisson's Formula

Through formal arguments based upon Green's functions, the solution to

$$\begin{aligned} \Delta u &= 0, \qquad x \in B_a(0) \subset \mathbb{R}^2, \\ u &= g, \qquad x \in \partial B_a(0) \end{aligned}$$

is given by the formula

$$u(x) = \frac{a^2 - |x|^2}{2\pi a} \int_{|y|=a} \frac{g(y)}{|x-y|^2} dy.$$

Similarly, the solution to the same problem in a 3-dimensional ball is

$$u(x) = \frac{a^2 - |x|^2}{4\pi a} \int_{|y|=a} \frac{g(y)}{|x-y|^3} dy.$$

This result can be generalized for an n-dimensional ball. In fact, if we take an n-dimensional ball instead of one in $\mathbb{R}^2$ or $\mathbb{R}^3$, we find that the solution is given by Poisson's Formula:

$$u(x) = \frac{a^2 - |x|^2}{n w_n a} \int_{|y|=a} \frac{g(y)}{|x-y|^n} dy,$$

where w_n is the volume of the unit ball in $\mathbb{R}^n$.

We now have this famous representation formula for the ball in the case that the behaviour on the boundary is radially and axially symmetric (independent of both coordinates θ and φ). Another well known representation formula is the one for the similar case, but considered on the exterior of the ball rather than the interior.

2.4 Sommerfeld Radiation Condition

Since scattering and control theory deal with physical phenomena rather than just theoretical equations, we want to make sure that we restrict the solutions that we obtain to ones that are physically possible. For this reason we introduce the Sommerfeld Radiation Condition. This condition was introduced by Arnold Sommerfeld in 1912, and it requires that a given solution satisfies the condition

$$\lim_{r\to\infty} \left(\frac{\partial u}{\partial r} - i\lambda^2 u \right) = 0. \tag{2.4}$$

What this condition basically implies, in simple terms, is that the solutions are "out coming", meaning that they generate outwards rather than generating at infinity and vanishing at the origin [12]. If a solution did generate at infinity and vanished at the origin, we would define that "unphysical", since such a behaviour is not observed in any physical phenomena. By observing the form of this condition and focusing on the easier cases, it is important to realise that if the solution is compactly supported (meaning that it vanishes when sufficiently distant from the origin) then it will also satisfy this condition, since for sufficiently large r the entire expression inside of the limit will equal 0 [13].

Chapter 3

Laplace's Case

At this point, we would like to solve the exterior problem for the torus. Note that if we give the torus boundary conditions equal to 0, and we assume that the solution is 0 in the exterior of a large ball, then the overall solution will be the trivial 0 solution. This follows by the combination of the weak maximum and minimum principles, as was shown above. Regardless, this is not necessarily the case for Maxwell's equation, therefore we focus on the Helmholtz problem. For the reasons specified previously at the beginning of Chapter 2, we will analyze the Laplacian problem as an approximation of the Helmholtz case. In particular, we will analyze the problem with the setting detailed by Shushkevich, namely that of a torus enclosed by a spherical shell in free three-dimensional space [15]. We consider a torus centred at the origin described by toroidal coordinates with minor radius r and major radius R. We let the radial constant be $c = \sqrt{R^2 - r^2}$, then the the torus T is described as follows:

$$T = \{\alpha = \alpha_0, \ \ 0 \le \beta \le 2\pi, \ \ 0 \le \varphi \le 2\pi\},$$

where

$$\alpha_0 = \ln\left(\frac{R}{r} + \sqrt{\left(\frac{R}{r}\right)^2 - 1}\right).$$

We then introduce the spherical shell S as the top part of a sphere centered at the origin with radius d such that $T \subset B_d(0)$. A visualization of the problem at hand can be seen in Figure 3.1. In particular we focus on the cases where d is very large. The shell is described in spherical coordinates as follows:

$$S = \{r = d, \ \ 0 \le \theta \le \theta_0, \ \ 0 \le \varphi \le 2\pi\},$$

The Laplacian problem is then represented by solving a set of two Laplace's equations, one on the exterior of the sphere describing the shell, and one on the interior of it and the exterior of the torus:

$$\begin{aligned} \Delta U_1 &= 0 \qquad \text{in } W_1 = E_3/B_d(0), \\ \Delta U_2 &= 0 \qquad \text{in } W_2 = E_3/(W_1 \cup T). \end{aligned}$$

We introduce boundary conditions in order to obtain a unique set of solutions. Firstly, we

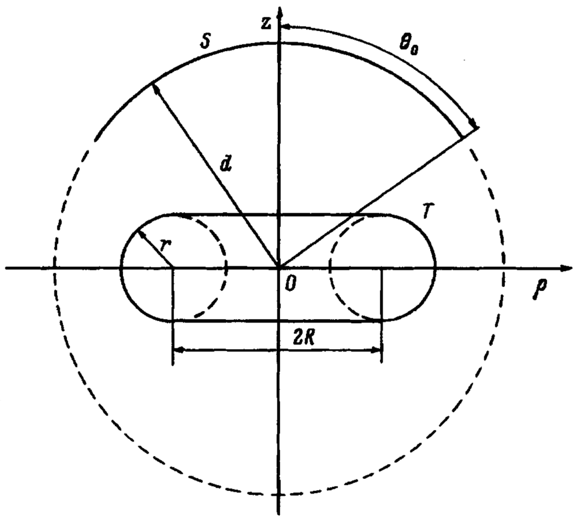


Figure 3.1: [15]

demand that U_1 vanishes at infinity. Then we put in place Dirichlet boundary conditions

$$U_2 = V_s \qquad \text{on } S, \tag{3.1}$$
$$U_2 = V_t \qquad \text{on } T. \tag{3.2}$$

Additionally, in order to ensure continuity between the two different solutions, we have that

$$U_1 = U_2 \qquad \text{on } \partial B_d(0), \tag{3.3}$$
$$\frac{\partial U_1}{\partial r} = \frac{\partial U_2}{\partial r} \qquad \text{on } \partial B_d(0)/S, \tag{3.4}$$

where $\partial B_d(0)/S$ is the open part of the spherical shell.

3.1 General solutions

The general solution for unspecified boundary conditions is as follows. The solutions depend on multiple unknown coefficients b_n, a_n, M_n and N_n. These are then determined by specifying the Dirichlet boundary conditions (3.1) and (3.2).

$$U_1(r,\theta) = \sum_{n=0}^{\infty} b_n \left(\frac{d}{r}\right)^{n+1} P_n(\cos\theta) \qquad \text{in } W_1, \tag{3.5}$$
$$U_2 = U_2^{(1)}(r,\theta) + U_2^{(2)}(\alpha,\beta) \qquad \text{in } W_2, \tag{3.6}$$

where

$$U_2^{(1)}(r,\theta) = \sum_{n=0}^{\infty} a_n \left(\frac{r}{d}\right)^{n} P_n(\cos\theta). \tag{3.7}$$

$$U_2^{(2)}(\alpha,\beta) = \sqrt{2(\cosh\alpha - \cos\beta)} \sum_{n=0}^{\infty} (M_n \cos n\beta + N_n \sin n\beta) \frac{P_{n-\frac{1}{2}}(\cosh\alpha)}{P_{n-\frac{1}{2}}(\cosh\alpha_0)}. \tag{3.8}$$

Here we have made the assumption that since the solution approaches 0 as we go to infinity, the torus will be compactly supported, meaning the solution will be equal to 0 everywhere outside of a large ball. We do this because since we already have that the solution vanishes at infinity, assuming that the solution will be compactly supported is a great approximation. The toroidal component $U_2^{(2)}$ can be rewritten in complex form by replacing the sum contained in it by

$$\sum_{n=-\infty}^{\infty} x_n \frac{P_{n-\frac{1}{2}}(\cosh\alpha)}{P_{n-\frac{1}{2}}(\cosh\alpha_0)} e^{in\beta},$$

where $x_n = \frac{1}{2}(M_n \mp iN_n)$ such that $\overline{x_n} = x_{-n}$.

Since the solution U_2 is composed of a function in spherical coordinates and a function in toroidal coordinates, we introduce different variable changes to turn U_2 into a function of one coordinate system. These changes are as follows.

$$U_2^{(1)}(\alpha,\beta) = \frac{\sqrt{2(\cosh\alpha - \cos\beta)}}{2\pi} \cdot \sum_{n=-\infty}^{\infty} \left(\sum_{p=0}^{\infty} \left(\frac{c}{d}\right)^p a_p D_p^n \right) Q_{n-\frac{1}{2}}(\cosh\alpha)\, e^{in\beta}, \tag{3.9}$$

$$U_2^{(2)}(r,\theta) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{c}{r}\right)^{n+1} \left(\sum_{s=-\infty}^{\infty} \frac{x_s D_n^s}{P_{s-\frac{1}{2}}(\cosh\alpha_0)} \right) P_n(\cos\theta). \tag{3.10}$$

If we let

$$f_n = (-1)^n \left(\sum_{s=-\infty}^{\infty} \frac{x_s D_n^s}{P_{s-\frac{1}{2}}(\cosh\alpha_0)} \right), \tag{3.11}$$

we get that (3.7) and (3.10) can be combined to be

$$U_2(r,\theta) = \sum_{n=0}^{\infty} \left(a_n \left(\frac{r}{d}\right)^n + f_n \left(\frac{c}{r}\right)^{n+1} \right) P_n(\cos\theta), \tag{3.12}$$

which together with condition (3.3) and the orthogonality of the Legendre polynomials gives

$$b_n = a_n + f_n \left(\frac{c}{d}\right)^{n+1} \tag{3.13}$$

3.2 $V_s = 0$

Now that we have given expressions for the general solutions to this problem, we focus on the particular case that is of interest to us. Namely we look at the case where we set V_s from condition (3.1) to be equal to 0. As a consequence of (3.3) we get that the solution on the spherical shell must be equal to 0 for both equations U_i. Therefore, in order to satisfy (3.3) we have that

$$U_2 = 0 \qquad \text{on } \partial B_d(0),$$

which results in $b_n = 0$ for $n \in \{0, 1, 2, \dots\}$. Therefore from (3.5) we get that the solution

U_1 is zero on its entire domain W_1. What this means is that the union of U_1 and U_2 on the entire space will be compactly supported, as it will be zero outside of the sphere. Then, as described at the end of the previous chapter, any of the solutions obtained from this particular case will satisfy the Sommerfeld Radiation Condition, meaning that the results will be able to be used as a representation of natural phenomena. With this result for the coefficients b_n, from (3.13) we get that

$$a_n = -f_n \left(\frac{c}{d}\right)^{n+1}, \tag{3.14}$$

which gives

$$U_2(r,\theta) = \sum_{n=0}^{\infty} a_n \left(\left(\frac{r}{d}\right)^n - \left(\frac{d}{r}\right)^{n+1}\right) P_n(\cos\theta). \tag{3.15}$$

An explicit form for the coefficients a_n can then be determined through the evaluation of the following integral:

$$a_n = \int_0^{\theta_0} \phi(x) \cos\left(\left(n+\frac{1}{2}\right) x\right) dx, \tag{3.16}$$

where in the case that the radius d of the shell is much larger than the radial constant c, $\phi(x)$ can be approximated by

$$\phi(x) = \phi_0(x) + \mu\phi_1(x) + \mu^2\phi_2(x) + \mu^3\phi_3(x) + \dots, \tag{3.17}$$

where $\mu = c/d$, and the functions $\phi_i(x)$ up to $i = 3$ are given by Shushkevich through more complicated computations. Since we are in the situation where the value of d is much larger than that of c, we decide to arbitrarely ignore values of i larger than 2 for the sake of simplicity, since increasing powers of μ will quickly go to zero. Through direct computation we obtain:

$$\begin{aligned} a_0 \approx \mu \frac{V_t T_0}{\pi^2}(\theta_0 + \sin\theta_0) &+ \\ + \mu^2 \left[\frac{2\,\alpha_{00}\, V_t\, T_0\, R_{00}}{2\pi^3}(\theta_0 + \sin\theta_0) + \frac{V_t\, T_1}{\pi^2} \sin\theta_0(\cos\theta_0 + 1)\right]&. \end{aligned} \tag{3.18}$$

The different coefficients introduced above follow from a series of calculations, and their values depend on the choice of the radii for the torus as well as on the constant V_t. In the case that $V_t > 0$, all these coefficients are positive, meaning that a_0 will be non-zero. What this implies is that the eigen-solution at $n = 0$

$$U_{2,\, n=0}(r,\theta) = a_0 \left(1 - \frac{d}{r}\right)$$

depends on a factor of $1/r$. This is meaningful since we have that the first order term of an eigensolution to Laplace's problem in spherical coordinates is

$$u_0(r) = \tilde{a}\,\frac{1}{r},$$

which is very similar to the result obtained in the toroidal case just observed.

Chapter 4

Conclusions

What can be concluded from the results obtained in the previous chapter is that in the case of the Helmholtz equation, there are terms which are of the form 1/r (analogous to those on the surface of a sphere) that arise for zero boundary conditions on domains which are enclosed by arbitrarily large balls that entirely support the solutions. Importantly, these terms can be proven to be non-zero. These have been shown to be present when there are non-zero boundary conditions on the torus. Regardless, the opposite case where there is no charge on the surface of the torus rather than on the sphere (zero boundary conditions on the torus and non-zero boundary conditions on the sphere) has not been carried out in this project, therefore no conclusions can be made about it. This is one of the many possible variations of this research that could be investigated, and since these variations are many, I will discuss those separately in the following section.

4.1 Further Research

As I have briefly mentioned in the introduction to this paper, the original intention for this Bachelor Project was to cover this topic for both the Helmholtz and Maxwell equations, but due to lack of time and a much higher complexity for the latter case, this paper only addresses the former. Therefore, there is a chance for further research and computations to be carried out for the Maxwell equations, which might also give interesting results.

As I mentioned at the end of the previous section, variations in the boundary conditions for the Helmholtz case can lead to a difference in results. Additionally, for the case that we have considered, we have only achieved a conclusion for the very first order term, but it might be possible to achieve comparable results and more insightful analogies with the spherical case if higher order terms are computed. In the future, a progression of this project would aim to show that these terms of the form $1/r$ affect the rates at which waves disperse. In particular, if there are no such terms, the waves would be observed to disperse faster when the $\lambda = 0$ expansion is used to approximate solutions of the Helmholtz equation. Lastly, in order to easily satisfy the Sommerfeld Radiation Condition, we limited our scope to solutions that assume compact support in order compute these terms. Nevertheless, by studying the Sommerfeld Radiation Condition more carefully there must be ways to compute the solutions when the exterior is genuinely an infinite space and we do not assume compact support.

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Appendix

The following is the Python code used to obtain the figures displayed in section 2.1.

```
# Packages

import numpy as np
import matplotlib.pyplot as plt
import scipy.special as ss


# Bessel Functions of the First Kind

ioi=101
#A = list(map(float, input().split()))  # for personalized inputs
#B = list(map(int, input().split()))    # for personalized inputs
A = np.array([0, 1, 2, 3])
x = np.linspace(0,15,ioi)
plt.figure(dpi=300)
plt.plot(x,np.zeros(ioi),'k:',markersize=2)
for i in range(len(A)):
  a = A[i]
  plt.plot(x,ss.jv(a,x),label='$J_{%3.0f}(x)$'%A[i])
plt.legend(loc='upper right', fontsize=8)
plt.xticks(fontsize=8)
plt.yticks(fontsize=8)

A = np.array([0.5, 1.5, 2.5, 3.5])
B = np.array([0, 1, 2, 3])
x = np.linspace(0,15,ioi)
plt.figure(dpi=300)
plt.plot(x,np.zeros(ioi),'k:',markersize=2)
for i in range(len(A)):
  a = A[i]
  plt.plot(x,ss.jv(a,x),label='$J_{%3.0f + 1/2}(x)$'%B[i])
plt.legend(loc='upper right', fontsize=8)
plt.xticks(fontsize=8)
plt.yticks(fontsize=8)

# Bessel Functions of the First Kind
```

```
#A = list(map(float, input().split()))  # for personalized inputs
A = np.array([0, 0.5, 1, 1.5])
x = np.linspace(1,15,ioi)
plt.figure(dpi=300)
plt.plot(x,np.zeros(ioi),'k:',markersize=2)
for i in range(len(A)):
  a = A[i]
  if (a - 0.5) % 1 == 0:
    atemp = int(a - 0.5)
    plt.plot(x,ss.yv(a,x),label='$Y_{%3.0f + 1/2}(x)$'%atemp)
  else:
    plt.plot(x,ss.yv(a,x),label='$Y_{%3.0f}(x)$'%A[i])
plt.legend(loc='lower1 right', fontsize=8)
plt.xticks(fontsize=8)
plt.yticks(fontsize=8)


# Legendre Polynomials of the First Kind

#N = list(map(int, input().split())) # for personalized inputs
N = np.array([0, 1, 2, 3, 4, 5])
x = np.linspace(-1,1,ioi)
plt.figure(dpi=300)
for i in range(len(N)):
  n = N[i]
  plt.plot(x,ss.eval_legendre(n,x),label='$P_{%3.0f}(x)$'%N[i])
plt.legend(loc='lower right', fontsize=8)
plt.xticks(fontsize=8)
plt.yticks(fontsize=8)


# Legendre Functions of the Second Kind

def relation(Ppre,P,n):
  P_post = lambda x: ((2*n+1)*x*P(x)-n*Ppre(x))/(n+1)
  return P_post

def Qn(N):
  Q0 = lambda x: (np.log((1+x)/(1-x)))/2
  Q1 = lambda x: (np.log((1+x)/(1-x)))*x/2 - 1
  if N == 0:
    Pnew = Q0
  if N == 1:
    Pnew = Q1
  else:
    Pold=Q0
```

```
    Pcurr=Q1
    for i in range(1,N):
      Pnew = relation(Pold,Pcurr,i)
      Pold = Pcurr
      Pcurr = Pnew
  return Pnew

#N = list(map(int, input().split()))  # for personalized inputs
N = np.array([0, 1, 2, 3, 4, 5])
x = np.linspace(-0.999,0.999,ioi)
plt.figure(dpi=300)
for i in range(len(N)):
  n = N[i]
  plt.plot(x,Qn(n)(x),label='$Q_{%3.0f}(x)$'%N[i])
plt.legend(loc='lower right', fontsize=8)
plt.xticks(fontsize=8)
plt.yticks(fontsize=8)
plt.ylim([-1.1, 1.1])
```