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Extremal rational elliptic surfaces with semi-stable fibers

Bachelor's Project Mathematics

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Abstract

This thesis contains geometric constructions of semi-stable extremal rational elliptic surfaces over the rational numbers and a discussion of their torsion structure via examples. Moreover, it includes blow-downs of the surfaces to their minimal models in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. The preliminaries cover topics in algebraic geometry and number theory such as blowing up at a point, the Mordell-Weil theorem and bad reduction on fibers of a rational elliptic surface.

Contents

1 Introduction

Elliptic curves are an important topic in number theory as they are famous for their use in the proof of Fermat's last theorem by Andrew Wiles. We find applications for elliptic curves in elliptic curve cryptography (ECC) and integer factorization. The of study elliptic surfaces is an expansion of the theory on elliptic curves and helps us understand the latter in the context of algebraic geometry. An elliptic curve $(E, \mathcal{O})$ consist of a complete curve E of genus one together with a point $\mathcal{O}$. When an elliptic curve E is defined over a field k with characteristic other than 2 or 3 we write its equation in Weierstrass normal form

$$E : y^2 = x^3 + Ax + B$$

so that $4A^3 - 27B^2 \neq 0$. A rational elliptic surface $\pi : S \to \mathbb{P}^1$ can be defined in a similar manner, the elliptic surface S is determined by a Weierstrass equation with coefficients $A(t), B(t)$ in the function field $k(t)$.

$$y^2 = x^3 + A(t)x + B(t). \tag{1}$$

This defines an elliptic surface, a family of curves of which all but finitely many members are elliptic curves over k. The family members are given by fibers $\pi^{-1}(t_0)$ of the surface, where we take $t_0 \in \mathbb{P}^1(k)$. Singular fibers are the curves in the family that have discriminant equal to zero. On the elliptic surface S the singular fibers are either reducible or irreducible. We study the reducible fibers that are moreover semi-stable: they are the fibers that appear as nodal curves on the surface, see Figure 1. Together with the elliptic surface we define the generic fiber ε_η, an elliptic curve over the function field $k(t)$ above the generic point η given by Equation (1). This description allow us to study rational elliptic surfaces in a two-folded way, namely as an algebraic surfaces which contains a 1-dimensional family of elliptic curves, or as an elliptic curve over the function field $k(t)$.

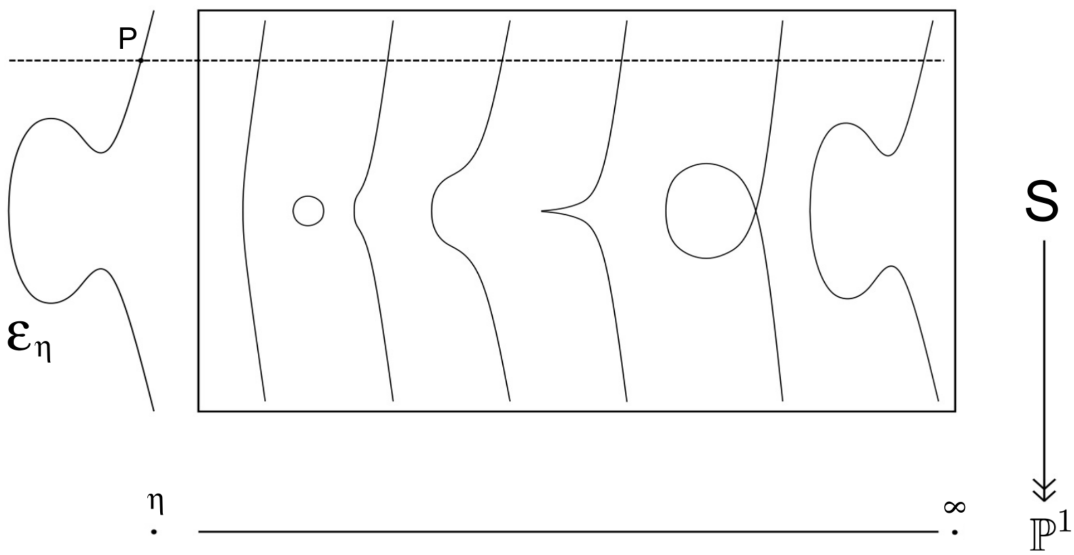


Figure 1: Weierstrass model of an elliptic surface S over $\mathbb{P}^1$ with generic fiber ε_η above generic point η. The second curve in the surface from the right to left is a nodal curve.

The preliminaries of this thesis include the basic definitions on varieties and morphisms in Chapter 2. In the next chapter we define Weil divisors and discuss the self-intersection of a divisor on a surface. In Chapter 4 we discuss linear systems of curves and Bézout's theorem. For our construction of semi-stable fibers, we need theory on blowing up points on curves, which is given in Chapter 5. We discuss the basic theory of elliptic curves in Chapter 6 together with the Mordell-Weil theorem for a number field k. Naturally, the next chapter introduces elliptic surfaces. We focus on the study of extremal rational elliptic surfaces in Chapter 8. Finally, in Chapter 9 we give our constructions of rational elliptic surfaces via blow-ups of points in the plane.

The main goal of this bachelor thesis is to give explicit constructions of examples of extremal rational elliptic surfaces with semi-stable fibers over $\mathbb{Q}$ or a quadratic extension and work out the blow-ups of base points that give rise to the configurations of semi-stable reducible fibers in the Néron model of the surface. A similar construction can be found in the bachelor thesis of Marit van Straaten, where explicit constructions of extremal rational elliptic surfaces with non-reduced fibers are given.

2 Preliminaries on algebraic varieties

The sources that were used for Chapter 2 are [8], [9], [18] and [22].

2.1 Algebraic varieties

2.1.1 Affine varieties

Consider any field k. **Affine n-space** $\mathbb{A}^n := k^n$ consists of n-tuples $(x_1, x_2, \ldots, x_n)$ with coordinates in k. We define polynomials $f \in k[x_1, \ldots, x_n]$ modulo units, i.e. for two polynomials f, g on $\mathbb{A}^n$ we have $f \sim g$ if and only if $g = \lambda \cdot f$ for $\lambda \in k^*$ a unit. Affine 2-space, also known as the affine plane, is the ambient space for **plane curves**: zero sets of non-constant polynomials in $k[x, y]$.

Definition 2.1 (Affine variety)**.** Let $T \subset k[x_1, \ldots, x_n]$ be a set of polynomials. The zero set

$$V_a(T) = \{P \in \mathbb{A}^n : f(P) = 0 \text{ for all } f \in T\},$$

is called a zero locus of T. A subset in $\mathbb{A}^n$ of this form is called an **affine variety**[1].

Example 2.2. A polynomial $f \in k[x, y]$ has zero set, or plane curve $V_a(f) = \{P = (x, y) \in \mathbb{A}^2 : f(P) = 0\}$ associated with it. The set of points in the plane curve depend on the choice of the field k. For example, $V_a(x^2 + y^2 - 1)$ over the field $\mathbb{R}$ has zero set

$$V_a(x^2 + y^2 - 1) := \left\{ \left(\tilde{x}, \sqrt{1 - \tilde{x}^2} \right) \in \mathbb{A}^2 : \tilde{x} \in [-1, 1] \right\}$$

over $\mathbb{A}^2$. Note that points $(\tilde{x}, \sqrt{1 - \tilde{x}^2})$ with $|\tilde{x}| > 1$ or $\mathrm{Im}(\tilde{x}) \neq 0$ are only in the zero set if we define $\mathbb{A}^2$ over $\mathbb{C}$, an algebraic extension of $\mathbb{R}$ and the classic example of an algebraically closed field. ♙

Remark 2.3. For two polynomials $f, g \in k[x_1, \ldots, x_n]$ we have

$$V_a(f) \cup V_a(g) = V_a(fg) \text{ and } V_a(f) \cap V_a(g) = V_a(f, g).$$

Definition 2.4 (The ideal of a set $S \subset \mathbb{A}^n$)**.** Let $S \subset \mathbb{A}^n$, then

$$I_a(S) = \{f \in k[x_1, \ldots, x_n] : f(P) = 0 \text{ for all } P \in S\}$$

is called the ideal of S.

2.1.2 Projective varieties

For k a field, we define **projective n-space** $\mathbb{P}^n$ as the set of all 1-dimensional linear subspaces of k^{n+1}. We write

$$\mathbb{P}^n = (k^{n+1} \setminus \{0\}) / \sim .$$

Two non-zero points $(x_1, \ldots x_{n+1})$ and $(y_1, \ldots, y_{n+1}) \in k^{n+1}$ in the same 1-dimensional linear subspace are equivalent in the following sense: if there exists a scalar $\lambda \in k^*$ such that $x_i = \lambda y_i$ for all i we write $(x_1, \ldots x_{n+1}) \sim (y_1, \ldots, y_{n+1})$. The equivalence class of a point $(x_1, \ldots x_{n+1})$ in $\mathbb{P}^n$ is commonly denoted by $[x_1 : \cdots : x_{n+1}]$.

Polynomials in projective n-space need to satisfy an extra constraint because they are generally not well-defined functions in $\mathbb{P}^n$. For example if $n = 2$ the polynomial $F = x^3 - y^2 + z$ maps $[1 : -1 : 0]$ to 0 and it maps $[-1 : 1 : 0]$ to -2. However, $[1 : -1 : 0] = [-1 : 1 : 0]$ and thus it is ill-defined as a function on $\mathbb{P}^2$. However, if F satisfies

$$F([\lambda x, \lambda y, \lambda z]) = F(\lambda [x, y, z]) = \lambda^d F([x, y, z]). \tag{2}$$

for some $d \in \mathbb{N}$, the zero locus is well-defined on $\mathbb{P}^2$, i.e. $F(x_0, y_0, z_0) = 0$ if and only if $F(\lambda x_0, \lambda y_0, \lambda z_0) = 0$. A polynomial $F \in k[x, y, z]$ satisfying Equation (2) is called **homogeneous of degree** d. A projective variety is defined just like an affine variety, only the set $T \subset k[x_1, \ldots, x_{n+1}]$ contains only homogeneous polynomials and the zero locus $V_p(T) \subset \mathbb{P}^n$.

[1]Depending on the literature, subsets of this form are known as affine algebraic sets and not affine algebraic varieties, see Chapter 1.1 in Hartshorne [9].

Definition 2.5 (Projective variety)**.** Let $T \subset k[x_1, \ldots, x_{n+1}]$ be a set of homogeneous polynomials. The zero set

$$V_p(T) = \{P \in \mathbb{P}^n : F(P) = 0 \text{ for all } F \in T\},$$

is called a zero locus of T. A subset of this form in $\mathbb{P}^n$ is a **projective variety**.

Remark 2.6. For two homogeneous polynomials $F, G \in k[x, y, z]$ we have

$$V_p(F) \cup V_p(G) = V_p(FG) \text{ and } V_p(F) \cap V_p(G) = V_p(F, G).$$

Definition 2.7 (The ideal of a set $S \subset \mathbb{P}^n$)**.** Let $S \subset \mathbb{P}^n$, then

$$I_p(S) = \{F \in k[x_1, \ldots, x_{n+1}] : F \text{ homogeneous and } F(P) = 0 \text{ for all } P \in S\}$$

is called the ideal of S.

2.1.3 The affine and infinite part of projective space

There exists a geometric interpretation of **projective n-space** given by the embedding of affine n-space. Consider the map $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ defined by $(x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_n : 1]$. The image of this mapping is the subset $U_{n+1} = \{[x_1 : \ldots : x_{n+1}] : x_{n+1} \neq 0\}$ of $\mathbb{P}^n$. The remaining points are of the form $[x_1 : \ldots : x_n : 0]$ and the bijective mapping

$$\mathbb{P}^n \setminus U_{n+1} \to \mathbb{P}^{n-1}, \quad [x_1 : \ldots : x_n : 0] \mapsto [x_1 : \ldots : x_n]$$

allows us to write it as $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$. We call $\mathbb{A}^n$ the **affine part** and $\mathbb{P}^{n-1}$ the **infinite part** of $\mathbb{P}^n$.

2.1.4 Homogenization and dehomogenization of polynomials

Consider a polynomial $f \in k[x_1, \ldots, x_n]$ of degree d given by

$$f = \sum_{i_1 + \cdots + i_n \leq d} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

The **homogenization** f^h of f in e.g. the $(n+1)$-th coordinate is then given by

$$f^h = \sum_{i_1 + \cdots + i_n \leq d} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \cdot x_{n+1}^{d - (i_1 + \cdots + i_n)},$$

such that f^h is a homogeneous polynomial in $k[x_1, \ldots, x_{n+1}]$. The reverse action of dehomogenization consists of taking $x_{n+1} = 1$. For a homogeneous polynomial $F \in k[x_1, \ldots, x_{n+1}]$, the affine set of points and the set of points at infinity for each coordinate x_i are given by $V_p(F) \cap \mathbb{A}^2 = V_a(F(x_i = 1))$ and $V_p(F(x_i = 0))$ respectively.

2.1.5 The local ring

Definition 2.8 (Local rings of $\mathbb{A}^2$)**.** Let $P \in \mathbb{A}^2$ be a point and k a fixed ground field.

(a) The local ring of $\mathbb{A}^2$ at P is defined as the set of rational functions given by

$$\mathscr{O}_P := \mathscr{O}_P(\mathbb{A}^2) := \left\{ \frac{f}{g} : f, g \in k[x, y] \text{ with } g(P) \neq 0 \right\} \quad \subset k(x, y).$$

(b) The local ring admits a well-defined ring homomorphism

$$\mathscr{O}_P \to k, \frac{f}{g} \mapsto \frac{f(P)}{g(P)}$$

which we will call the evaluation map. Its kernel will be denoted by

$$I_P := I_P(\mathbb{A}^2) := \left\{ \frac{f}{g} : f, g \in k[x, y] \text{ with } f(P) = 0 \text{ and } g(P) \neq 0 \right\} \quad \subset \mathscr{O}_P.$$

This definition translates well to a definition of the local ring $\mathscr{O}_P = \mathscr{O}_P(\mathbb{P}^2)$ of projective space $\mathbb{P}^2$, except we need to make a few restrictions and change the maps appropriately. First of all we need $f, g \in k[x, y]$ to be homogeneous and of the same degree, because f/g has to be well-defined and thus we require that

$$\frac{f(\lambda x, \lambda y, \lambda z)}{g(\lambda x, \lambda y, \lambda z)} = \frac{\lambda^d f(x, y, z)}{\lambda^d g(x, y, z)} = \frac{f(x, y, z)}{g(x, y, z)}.$$

Similarly to (b) in Definition 2.8 we define the evaluation map $\mathscr{O}_P \to k$ for projective space with kernel denoted $I_P(\mathbb{P}^2)$. For any point $P = [x_0 : y_0 : z_0]$ in $\mathbb{P}^2$ there exists an isomorphism $\mathscr{O}_P(\mathbb{A}^2) \to \mathscr{O}_P(\mathbb{P}^2)$ given by homogenization in a non-zero coordinate of P.

2.2 Intersection multiplicity and multiplicity of points inside varieties

2.2.1 Intersection multiplicity

Definition 2.9 (Intersection multiplicity of curves in $\mathbb{A}^2$)**.** For a point $P \in \mathbb{A}^2$ and two polynomials f and g in $k[x, y]$ we define the **intersection multiplicity** of the curves defined by f and g at their point of intersection P to be

$$\mu_P(f, g) := \dim \mathscr{O}_P/\langle f, g\rangle \quad \in \mathbb{N} \cup \{\infty\},$$

where dim denotes the dimension as a vector space over k.

We would like to define intersection multiplicity for curve in the projective plane $\mathbb{P}^2$. Take homogeneous polynomials $F_1, \ldots, F_n$ in $\mathscr{O}_p$, that generate the ideal

$$\langle F_1, \ldots, F_n\rangle := \left\{\frac{a_1}{b_1}F_1 + \cdots + \frac{a_n}{b_n}F_n : a_i = 0 \text{ or } a_i, b_i \text{ homogeneous with } \deg(a_i F_i) = \deg b_i \text{ for all } i\right\}$$

in $\mathscr{O}_P(\mathbb{P}^2)$. The isomorphism $\mathscr{O}_P(\mathbb{A}^2) \to \mathscr{O}_P(\mathbb{P}^2)$ given by homogenization in a non-zero coordinate of P, takes the ideal $\langle f, g\rangle$ to the ideal $\langle f^h, g^h\rangle$. Therefore, the intersection multiplicity is preserved under homogenization of curves. This allows us to define intersection multiplicity for projective plane curves analogously to the definition for affine plane curves. For a point $P \in \mathbb{P}^2$ and two homogeneous polynomials F and G in $k[x, y, z]$ we define the intersection multiplicity of the curves defined by F and G at their point of intersection P to be

$$\mu_P(F, G) := \dim \mathscr{O}_P/\langle F, G\rangle \quad \in \mathbb{N} \cup \{\infty\},$$

where dim denotes the dimension as a vector space over k.

Curves defined by polynomials f, g may intersect in more than one point. We denote the the total intersection multiplicity, i.e. the sum of the intersection multiplicities of the curves at each point in their intersection, by

$$\sum_{P \in f \cap g} \mu_P(f, g) = \mu_{f \cap g}(f, g).$$

Example 2.10. Consider the lines defined by $y - x$ and $2y - 5x + 3$ in $\mathbb{Q}[x, y]$, they intersect at the point $P = (1, 1)$. Notice that the ideal $\langle y - x, 2y - 5x + 3\rangle\mathscr{O}_{(1,1)} = \langle x - 1, y - 1\rangle\mathscr{O}_{(1,1)}$ generates the kernel of the evaluation map I_P. By the First Isomorphism Theorem on the ring homomorphism in Definition 2.8 (b) we find that $\mu_P(y - x, 2y - 5x + 3) = 1$. ♙

2.2.2 Multiplicity of points

Definition 2.11 (Multiplicity of points in $\mathbb{A}^n$)**.** Let $f \in k[x_1, \ldots, x_n]$ be a polynomial and let f_m denote the homogeneous part of f of degree $m \in \mathbb{N}$. The smallest m for which f_m is zero is called the multiplicity $m_O(f)$ of f at the origin $O := (0, \ldots, 0)$. For any point $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$ we compute $m_P(f)$ by first making a linear coordinate transformation $x_i \to x_i - a_i$ which translates P into the origin $(0, \ldots, 0)$. The multiplicity is preserved under coordinate transformations.

We define multiplicity of a point $P = [a_1 : \ldots : a_{n+1}]$ in projective n-space $\mathbb{P}^n$ by making use of the isomorphism $\mathscr{O}_{\mathbb{A}^n,P} \cong \mathscr{O}_{\mathbb{P}^n,P}$ given by (de)homogenization in a non-zero coordinate of P. If an affine or projective variety $X = V(F)$ contains the point P we call the point smooth in X if $m_P(F) = 1$ and singular otherwise.

Definition 2.12 (Nonsingular affine variety)**.** A variety is **nonsingular** or smooth if all its points are smooth.

2.3 Zariski topology and irreducibility of varieties

Assume k to be algebraically closed unless stated otherwise.

Proposition 2.13. *The union of two varieties is a variety. The intersection of any family of varieties is a variety. The empty set and the ambient space (i.e. $\mathbb{A}^n$ or $\mathbb{P}^n$) are varieties.*

Proof. For the first two statement we refer back to Remarks 2.3 and 2.6. We can take an arbitrary set T of (homogeneous) polynomials in $k[x_1, \ldots, x_n]$ to generate a variety. $X = V(T)$. The variety X can be expressed as an arbitrary intersection of varieties, each generated by a polynomial in T. Notice that the empty set is a variety generated by a constant polynomial in k^*, e.g. 1. The ambient space (i.e. $\mathbb{A}^n$ or $\mathbb{P}^n$) is a variety generated by the zero polynomial 0.

Definition 2.14 (Zariski topology)**.** We define the **Zariski topology** on $\mathbb{A}^n$ and $\mathbb{P}^n$ by taking the open subsets to be the complements of varieties.

To see why this is a topology, we refer to Proposition 2.13. Namely, the intersection of two open sets is open and the union of any family of open sets is open. Furthermore, the empty set and the whole space are both open. Then the closed subsets are the varieties with respect to this topology. Varieties in $\mathbb{A}^n$ or $\mathbb{P}^n$ are topological spaces with respect to the Zariski subspace topology. An open subset in $\mathbb{A}^n$ or $\mathbb{P}^n$ is called a **quasi-affine variety** or a **quasi-projective variety** respectively.

Definition 2.15 (Irreducibility of varieties)**.** A nonempty subset Y of a topological space X with respect to the Zariski topology is **irreducible** if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, both closed in Y. The empty set is not considered irreducible.

Since we are mainly focused on the properties of curves, we use the following proposition more often. Let $I(X)$ denote either the affine or projective ideal of a variety X, defined as in Definitions 2.4 and 2.7.

Proposition 2.16. *A variety X is irreducible if and only if $I(X)$ is a prime ideal.*

Proof. A proof can be found in [9, p. 4], Corollary I.4.

Remark 2.17. For a polynomial $f \in k[x_1, \ldots, x_n]$ irreducible in $k[x_1, \ldots, x_n]$ a unique factorization domain, we find that f^n for any $n \in \mathbb{N}$ generates a prime ideal $I(X)$ where $X = V(f^n)$. Then X is an irreducible variety.

Remark 2.18. Note that affine n-space $\mathbb{A}^n$ has ideal $I_a(\mathbb{A}^n) = \langle 0 \rangle$ which is prime in the domain $k[x_1, \ldots, x_n]$. By Proposition 2.16 this implies $\mathbb{A}^n$ is irreducible as a variety. Analogously, $\mathbb{P}^n$ is irreducible as a variety.

Using the Zariski topology we define the dimension of a variety.

Definition 2.19 (Dimension of a variety)**.** If X is a topological space, we define the **dimension** of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $X_0 \subset X_1 \subset \ldots \subset X_n$ of distinct irreducible closed subsets of X. We define the dimension of a variety to be its dimension as a topological space.

Example 2.20. Consider the (irreducible) variety $\mathbb{P}^2$ as a topological space with respect to the Zariski topology. Take the coordinates $[t : u]$ in $\mathbb{P}^1$. We recognise that the point $P = [1 : 0 : 0]$ is the zero set of the polynomials y and z homogeneous in the polynomial ring $k[x, y, z]$. One way to see P is irreducible is by the fact that $k[x, y, z]/\langle y, z \rangle$ is isomorphic with the (principal ideal) domain $k[x]$. This implies $\langle y, z \rangle$ is a prime ideal in the ring $k[x, y, z]$. The point P lies on the projective line $L = [t : u : 0]$, the zero set of the irreducible homogeneous polynomial z. The line L lies in the ambient space $\mathbb{P}^2$. We find the topological space $\mathbb{P}^2$ is at least of dimension 2, the line L at least of dimension 1 and the point P at least of dimension 0.

In general we find by Proposition 1.7 in [9, p. 6] that the dimensions of affine and projective n-space $\mathbb{A}^n$ and $\mathbb{P}^n$ are equal to n. Thus in Example 2.20 the dimension of $\mathbb{P}^2$ is 2, the line has dimension 1 and the point dimension 0.

Definition 2.21 (Codimension of a subvariety)**.** Let X be a subvariety of a variety Y. We define the **codimension** of X in Y as

$$\operatorname{codim}(X) := \dim(Y) - \dim(X).$$

Remark 2.22. A variety of dimension 2 is called an algebraic surface and a variety of dimension 1 inside an algebraic surface is called an algebraic curve. We say that a curve has codimension 1.

2.4 Morphisms and rational maps

We discuss maps between irreducible varieties for k an algebraically closed field.

Definition 2.23 (Affine coordinate ring)**.** Let X be an irreducible affine variety over $k[x_1, \dots, x_n]$ and $I_a(X)$ the affine ideal of X as a subset of $\mathbb{A}^n$. We call

$$k[X] := k[x_1, \dots, x_n]/I_a(X)$$

the **affine coordinate ring** of X.

Since X is assumed to be irreducible, meaning $I_a(X)$ is a prime ideal, we know $k[X]$ must be an integral domain. Therefore, we can define its quotient field.

Definition 2.24 (Affine function field)**.** Let X be an irreducible affine variety.

(a) The quotient field

$$k(X) := \text{ Quot } k[X] = \left\{ \frac{f}{g} : f, g \in k[X] \text{ with } g \neq 0 \right\}$$

of the coordinate ring is called the **affine function field** of X.

(b) A rational function $\phi \in k(X)$ is called regular at a point $P \in X$ if it can be written as $\phi = f/g$ with $f, g \in k[X]$ and $g(P) \neq 0$. The regular functions at P form a subring of $k(X)$ which we denote by

$$\mathscr{O}_P(X) := \left\{ \frac{f}{g} : f, g \in k[X] \text{ with } g(P) \neq 0 \right\} \quad \subset k(X)$$

and which is called the local ring of X at P.

(c) There is a well-defined evaluation map

$$\mathscr{O}_P(X) \to k, \frac{f}{g} \mapsto \frac{f(P)}{g(P)}$$

which we will simply write as $\phi \mapsto \phi(P)$ for $\phi \in \mathscr{O}_P(X)$, and whose kernel is

$$I_P(X) := \left\{ \frac{f}{g} : f, g \in k[X] \text{ with } f(P) = 0 \text{ and } g(P) \neq 0 \right\}.$$

For a projective variety X we analogously define the **homogeneous coordinate ring** $k[X] := k[x_1, \dots, x_{n+1}]/I_p(X)$ with function field $k(X)$. Here we restrict our choice of $f, g \in k[x_1, \dots, x_{n+1}]$ to homogeneous polynomials of the same degree. Note that if f is an irreducible curve with $X = V(f)$ we sometimes denote $k(f) := k(X)$.

Definition 2.25 (Rational map)**.** Let X and $Y \subset \mathbb{P}^n$ be irreducible projective varieties. A **rational map** $\varphi : X \to Y$ given by

$$\varphi = [\phi_1 : \dots : \phi_{n+1}],$$

where $\phi_1, \dots, \phi_{n+1} \in k(X)$ such that for every point $P \in X$ at which $\phi_1, \dots, \phi_{n+1}$ are all defined[2],

$$\varphi(P) = [\phi_1(P) : \dots : \phi_{n+1}(P)] \in Y.$$

Note that each ϕ_i is defined at all but finitely many points. Therefore, a rational map $\varphi = [\phi_1 : \dots : \phi_{n+1}]$ between varieties is defined at all but finitely many points.

Definition 2.26 (Birational map)**.** A **birational** map $\varphi : X \to Y$ is a rational map which admits an inverse, namely a rational map $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$ as rational maps. If there is a birational map from X to Y, we say that X and Y are birationally equivalent, or simply birational.

Definition 2.27 (Morphism)**.** A rational map

$$\varphi = [\phi_1 : \dots : \phi_{n+1}] : X \longrightarrow Y$$

is regular (or defined) at $P \in X$ if there exists a function $G \in k(X)$ such that

[2] This includes that there exists an i so that $\phi_i(P) \neq 0$.

(a) each $G\phi_i$ is regular at P;

(b) there is some i for which $(G\phi_i)(P) \neq 0$.

If such a G exists, then we set
$$\varphi(P) = [(G\phi_1)(P) : \cdots : (G\phi_{n+1})(P)] .$$
A rational map regular at every point P in X is called a **morphism**.

Two irreducible varieties X, Y are called **isomorphic** if there exists a morphism $\varphi : X \to Y$ which admits an inverse morphism $\psi : Y \to X$ with $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$.

3 Preliminaries on Weil divisors

The definitions and results in Chapter 3 are based on [9] and [18].

3.1 Weil divisors

Definition 3.1 (Weil divisors)**.** Let X be an irreducible variety. A finite collection of irreducible closed subvarieties $C_1, \ldots, C_r$ of codimension 1 in X with assigned integer multiplicities $k_1, \ldots, k_r$ defines a **divisor** on X. A divisor is a formal sum

$$D = k_1 C_1 + \cdots + k_r C_r. \tag{3}$$

If all $k_i = 0$, we write $D = 0$. If all $k_i \geq 0$ and some $k_i > 0$ then we write $D > 0$; in this case D is said to be **effective**. An irreducible codimension 1 subvariety C_i taken with multiplicity 1 is called a **prime divisor**. If all the $k_i \neq 0$ in Equation (3) then the variety $C_1 \cup \cdots \cup C_r$ is called the **support** of D and denoted by Supp D.

Definition 3.2 (Degree of a divisor)**.** Let X be an irreducible variety and let $D = k_1 C_1 + \cdots + k_r C_r$ be a divisor on X. The degree of D is given by

$$\deg D := \sum_{i=1}^{r} k_i.$$

The divisor of a rational function $\phi = f/g$ is given by

$$\operatorname{div}(\phi) := \sum \mu_{\phi \cap C}(\phi, C) C$$

where we sum over all the irreducible closed subvarieties C. Since there are only finitely many such C with $\mu_{\phi \cap C}(\phi, C) \neq 0$, $\operatorname{div}(\phi)$ is well-defined, see [18, p. 149]. A divisor of the form $\operatorname{div}(\varphi)$ is called a **principal** divisor.

Divisors form a group under addition denoted $\operatorname{Div} X$. Moreover, there exists a subgroup denoted $\operatorname{Div}^0 X$ consisting of all $D \in \operatorname{Div} X$ such that $\deg D = 0$.

Definition 3.3 (Principal divisor and divisor class group)**.** Let X be an irreducible variety.

(a) A divisor on X is called principal if it is the divisor of a non-zero rational function. The set of all principal divisors will be denoted by

$$\operatorname{Prin} X := \{\operatorname{div}(\phi) : \phi \in k(X)^*\}.$$

It is a subgroup of $\operatorname{Div} X$, and also of $\operatorname{Div}^0 X$.

(b) The quotient group $\operatorname{Cl} X := \operatorname{Div} X / \operatorname{Prin} X$ is called the **divisor class group** on X. Two divisors D_1 and D_2 defining the same element in $\operatorname{Cl} X$, i.e. with $D_1 - D_2 = \operatorname{div} \phi$ for a rational function $\phi \in k(X)^*$, are said to be linearly equivalent, written $D_1 \sim D_2$. Restricting to divisors of degree 0, we also set

$$\operatorname{Cl}^0 X := \operatorname{Div}^0 X / \operatorname{Prin} X,$$

which is a subgroup of $\operatorname{Cl} X$.

Definition 3.4 (Néron-Severi group)**.** Let X be a nonsingular projective variety of dimension at least 2 defined over an algebraically closed field k, let $\operatorname{Cl} X$ be the divisor class group of X and let $\operatorname{Cl}^0 X$ be the subgroup of divisors with degree zero. The quotient group $\operatorname{Cl} X / \operatorname{Cl}^0 X$ is called the **Néron–Severi group** of X and is denoted by $\operatorname{NS}(X)$.

3.2 Divisors over a surface

In this section we let S be an irreducible projective variety of dimension 2 over k algebraically closed, also known as a projective surface.

Divisors over a surface S over an algebraically closed field k are finite formal sums of curves. Namely, the irreducible subvarieties C_i of codimension 1 are smooth algebraic curves when they are defined over $\mathbb{P}^2$ since they are of dimension 1 as topological spaces with respect to the Zariski topology. Note that all curves are divisors but the opposite does not hold since a divisor is a formal sum of curves with the possibly negative coefficients. However, if a divisor is effective, i.e. a formal sum of curves with non-negative coefficients, it is a curve. We define intersection multiplicity of effective divisors over a surface as the formal sum of the intersection multiplicity of the curves in the sum, see Definition 2.9. By using the notation $-D$ for the inverse of a divisor D we can generalize the notion of intersection multiplicity for all divisors in $\operatorname{Div} S$. In order to compute with intersection multiplicity of divisors we state the following theorem from Hartshorne [9, p. 357].

Theorem 3.5. *There is a unique pairing* $\operatorname{Div} S \times \operatorname{Div} S \to \mathbb{Z}$, $(D_1, D_2) \mapsto D_1.D_2$ *for any two divisors* D_1, D_2*, such that for* D_3 *a third divisor*

(1) if D_1 *and* D_2 *are nonsingular curves meeting transversely, then* $D_1.D_2 = \#(D_1 \cap D_2)$*, the number of points of* $D_1 \cap D_2$*,*

(2) it is symmetric: $D_1.D_2 = D_2.D_1$*,*

(3) it is additive: $(D_1 + D_2).D_3 = D_1.D_3 + D_2.D_3$*, and*

(4) it depends only on the linear equivalence classes: if $D_1 \sim D_2$ *then* $D_1.D_3 = D_2.D_3$*.*

Proof. A proof can be found in [9, p. 358], Theorem 1.1. ◻

Due to property (4) in above theorem we often define a pairing $\operatorname{Cl} S \times \operatorname{Cl} S \to \mathbb{Z}$ instead. That is, the pairing respects the linear equivalence of divisors.

Remark 3.6. If D is a divisor on a surface S, we define its **self-intersection** $D^2 := D.D$. The self-intersection of a divisor D on S can not be calculated directly, but rather by using the linear equivalence of divisors in the divisor class group $\operatorname{Cl} S$, see property (4) Theorem 3.5. That is, if $D' \sim D$ for some $D' \in \operatorname{Div} S$, then $D^2 = D'.D$.

Let X and Y be irreducible varieties on S. Consider a morphism $\varphi : X \to Y$ such that $\varphi(X)$ is dense[3] in Y and a divisor $D = k_1 C_1 + \cdots + k_r C_r$ in $\operatorname{Cl} Y$. The map $\varphi^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ defined by

$$\varphi^* D = \sum_{i=1}^{r} k_i \cdot \varphi^{-1}(C_i)$$

is a group homomorphism: the mapping of divisors is linear over $\mathbb{Z}$ and for rational functions f on Y we have $\varphi^* \operatorname{div}(f) = \operatorname{div}(f \circ \varphi)$ ensuring it is well-defined. We call φ^* the **pullback of divisors**, see [18, p. 152].

Definition 3.7 (Degree of a morphism)**.** If $\varphi : X \to Y$ is a finite[4] dominant morphism of irreducible varieties X and Y, we define the **degree** of φ to be the degree of the field extension $[k(X) : k(Y)]$.

From this definition we derive that the degree of a map between surfaces X, Y is the number of points in the pre-image of almost all points in Y, except for finitely many point called **ramification points**. For X, Y irreducible varieties and φ a morphism of degree d we have

$$(\varphi^* D_1.\varphi^* D_2) = d \cdot (D_1.D_2) \tag{4}$$

for any divisors D_1, D_2 in $\operatorname{Cl} Y$.

[3] That is, φ is a **dominant** morphism, see Hartshorne page 23 ex. 3.17.

[4] See Hartshorne page 84 for the definition of a finite morphism.

4 Linear systems of curves

The definitions and results in Chapter 4 can be found in [7] and [8] in more detail.

4.1 Linear systems

We work in the projective plane $\mathbb{P}^2$.

Let F be a homogeneous polynomial in $k[x, y, z]$ of the form

$$F = \sum_{i+j \leq d} a_{i,j} x^i y^j z^{d-(i+j)},$$

where $a_{i,j} \in k$ not all zero. Using an inductive argument we know that F consists of at most $(d+1)(d+2)/2$ terms of degree d. Let $N := (d+1)(d+2)/2$ and re-assign indices $1, \ldots, N$ to the coefficients $a_{i,j}$ in lexicographical order. Notice that F is determined by our choice of $a_1, \ldots, a_N$ up to units: choosing $\lambda a_1, \ldots, \lambda a_N$ for some $\lambda \in k^*$ also gives F. We obtain a one-to-one correspondence between homogeneous polynomials F of degree d and projective points $[a_1 : a_2 : \ldots : a_N]$ in projective $N-1$ space, where $N - 1 = (d+1)(d+2)/2 - 1 = d(d+3)/2$.

Consider the family of polynomials F of degree d that give rise to plane curves containing a point P. As we discussed each polynomial of degree d relates directly to a point in $\mathbb{P}^{d(d+3)/2}$. We argue that the family gives rise to an irreducible variety in $\mathbb{P}^{d(d+3)/2}$, because it generates a linear dependence between the coefficients $a_1, \ldots, a_N$. As a result we can speak about the dimension of the family of polynomials as the dimension of the irreducible variety of points in $\mathbb{P}^{d(d+3)/2}$ with respect to the Zariski topology. The restriction to a point in $\mathbb{P}^2$ reduces the dimension of the coefficients $a_1, \ldots, a_N$ of polynomials in the family by one. Therefore, the family gives rise to a hyperplane in $\mathbb{P}^{d(d+3)/2}$. The dimension of the family of polynomials is then equal to $d(d+3) - 1$.

We generalize this construction to include more points in the curves that are generated. Let $P_1, \ldots, P_n$ be distinct points in $\mathbb{P}^2, r_1, \ldots, r_n$ nonnegative integers. We set

$$\mathcal{C}\left(d; r_1 P_1, \ldots, r_n P_n\right) = \left\{ \text{ homogeneous polynomials } F \text{ of degree } d \mid m_{P_i}(F) \geq r_i, 1 \leq i \leq n \right\}. \tag{5}$$

Theorem 4.1. *Let $\mathcal{C}$ be a family of homogeneous polynomials F given by* (5)*. We have*

(1) $\mathcal{C}\left(d; r_1 P_1, \ldots, r_n P_n\right)$ is a subvariety of $\mathbb{P}^{d(d+3)/2}$ of dimension

$$\dim \mathcal{C}\left(d; r_1 P_1, \ldots, r_n P_n\right) \geq \frac{d(d+3)}{2} - \sum \frac{r_i\left(r_i+1\right)}{2},$$

(2) if $d \geq \left(\sum r_i\right) - 1$ then equality holds.

Proof. For the proof see Theorem 1 in [7, p. 56].

4.2 Bézout's theorem

For homogeneous polynomials F, G over an algebraically closed field there exists a direct correspondence between the degrees of both homogeneous polynomials and the number of points in which the curves they define intersect, so-called **base points**. The number of base points is independent of the curves and is solely determined by the degrees.

Theorem 4.2 (Bézout's theorem)**.** *Let F and G be homogeneous polynomials without common factor over a field k not necessarily algebraically closed. Then*

$$\sum_{P \in F \cap G} \mu_P(F, G) \leq \deg F \cdot \deg G.$$

Moreover, equality holds if k is algebraically closed.

Proof. A proof can be found in [8, p. 31], Corollary 4.6. ✍

Theorem 4.3 (Cayley-Bacharach). *Given 8 points $P_1, \ldots, P_8$ in the plane, no 4 collinear, and no 7 lying on a conic, there is a uniquely determined point P_9 such that every cubic through $P_1, \ldots, P_8$ also passes through P_9.*

Proof. See the proof of Corollary 4.5 in [9, p. 400]. ✍

Example 4.4. Consider two homogeneous polynomials $F(x, y, z)$ and $G(x, y, z)$ of degree 3 without common irreducible factor over an algebraically closed field k. By Bézout's theorem, see Theorem 4.2, the intersection of the curves defined by F, G contains exactly 9 base points. The base points $P_1, \ldots, P_9$ may not all be distinct. Note that no 4 base points are collinear and no 7 are on a conic, which can be seen directly from Bézout's theorem. We know from Theorem 4.3 that the family of curves generated by homogeneous polynomials of degree 3 which contains 8 base points also contains the 9th base point. We derive from Theorem 4.1 that the dimension of the family with $d = 3$ through all 9 points must be at least $d(d+3)/2 - 8 = 9 - 8 = 1$ if all base points are distinct, i.e. $r_i = 1$. We construct a one-dimensional family of plane cubic curves by taking $\mathbb{P}^1 = [t : u]$ and $tF(x, y, z) + uG(x, y, z) = 0$. We call a 1-dimensional family of subvarieties a pencil. Therefore, the construction is called the **pencil of cubics**. ♙

5 Blow-up of a point

In Chapter 5 is inspired by constructions and results for the blow-up at a point as they are given in [7, p. 86] and [9, p. 28].

5.1 Introduction

Let F and G be two homogeneous polynomials in $k[x, y, z]$ of degree 3 without common factor. We know from Bézout's theorem (see Theorem 4.2) that the cubic curves they define meet in exactly 9 points if k is algebraically closed. Consider the map $\mathbb{P}^2 \to \mathbb{P}^1$ defined by sending $[x : y : z] \dashrightarrow [F(x, y, z) : G(x, y, z)]$. It is a rational map but not a morphism: the mapping is not defined at the 9 base points of the curves of F and G. In order to resolve the 9 indeterminacy points we create a set called the **blow-up**

$$S = \{\ [x : y : z] \times [t : u]\ :\ uF(x, y, z) - tG(x, y, z) = 0\ \} \subset \mathbb{P}^2 \times \mathbb{P}^1.$$

Using the commutative diagram underneath with $\varepsilon : S \to \mathbb{P}^2$ the blow-up map (we define it further on) and π a surjective map on S, projection of the second factor outside the pre-image of ε of the base points. We aim to create a morphism for points outside of the intersection sending $[x : y : z] \to [F(x, y, z) : G(x, y, z)]$ and separately sending the 9 base points to exactly 9 lines.

$$\begin{tikzcd}
 & S \arrow[dl, "\varepsilon"'] \arrow[dr, "\pi"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "{[x:y:z]\to[F(x,y,z):G(x,y,z)]}"'] & & \mathbb{P}^1
\end{tikzcd}$$

In the next example we describe for a specific curve the process of blowing up a point on the curve. For now we avoid the technicalities of the construction, they are discussed further on this chapter.

Example 5.1. Consider the projective curve $C : y^2z = x^3 + x^2z$ which has one singular point $[0 : 0 : 1]$. We know from Definition 2.11 that the singular point has multiplicity 2. We resolve it by replacing the point with a projective line E. This projective line is isomorphic with $\mathbb{P}^1 = [t : u]$. We let the variable t function as a parameter for $\mathbb{P}^1$ by taking $t \in k$ if $u \neq 0$ and $t = \infty$ if $u = 0$. Blowing up maps the curve C into a union $\widetilde{C} \cup E$. The curve $\widetilde{C}$ is induced by the transformation of all nonsingular points on C. It intersects E at the points $t = \pm 1$, since the curve C has two different tangent lines at $[0 : 0 : 1]$ with slopes 1 and -1. The parameter t corresponds bijectively to the slope of the lines through the singular point.

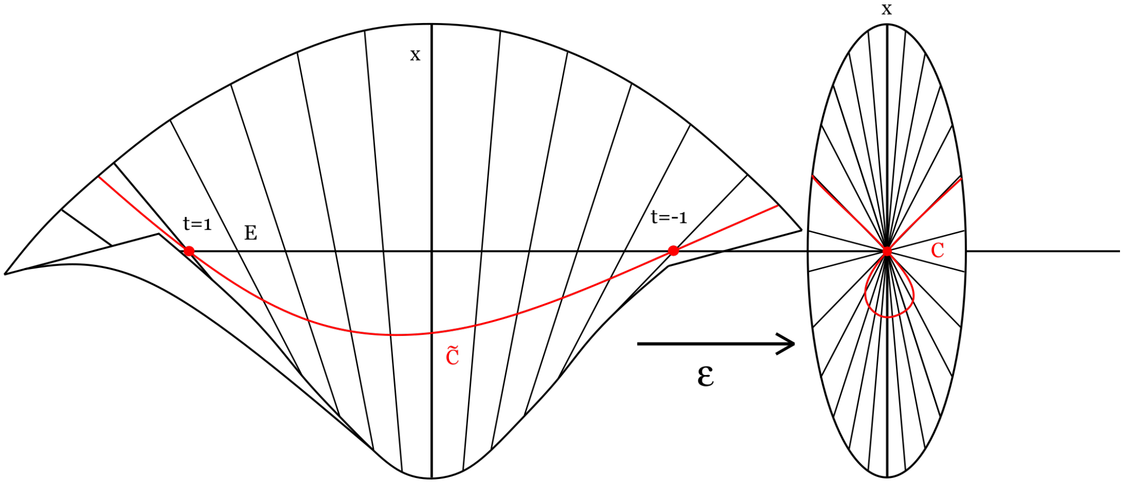


Figure 2: Blow-up of $(0, 0)$ on the curve $C : y^2 = x^3 + x^2$ on the affine plane $z = 1$.

5.2 Blow-up of a point

We provide a general construction of a blow-up of a point on the surface $\mathbb{P}^2$. Consider the point $O = [0 : 0 : 1]$. The blow-up of the point O is given by the projective variety

$$S = \{ty - ux = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1.$$

First let us define the morphism $\varepsilon : S \to \mathbb{P}^2$ for points $P = [a_1 : a_2 : a_3]$ in $\mathbb{P}^2 \setminus \{O\}$. In S this gives $ta_2 = ua_1$. Since $P \neq O$, meaning either $a_1 \neq 0$ or $a_2 \neq 0$, this defines a unique point $[t : u]$ in $\mathbb{P}^1$. We set $t = a_1$ and $u = a_2$. It follows that $P \times [a_1 : a_2] \in S$. The blow-up map ε at points distinct from O is defined by

$$\varepsilon([x : y : z] \times [x : y]) = [x : y : z].$$

It is the composition of the injection of S in $\mathbb{P}^2 \times \mathbb{P}^1$ and projection onto the first factor shown in the commutative diagram underneath. The inverse morphism $\psi : \mathbb{P}^2 \setminus \{O\} \to S$ is given by $\psi(P) = [a_1 : a_2 : a_3] \times [a_1 : a_2]$. Therefore, $\mathbb{P}^2 \setminus \{O\}$ is isomorphic with $S \setminus \varepsilon^{-1}(O)$ and $\varepsilon^{-1}(P)$ uniquely defines a point in S.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\varepsilon"] & \mathbb{P}^2 \times \mathbb{P}^1 \arrow[d] \\ & \mathbb{P}^2 \end{tikzcd}$$

Next let us define ε for the point O. By filling out the first two coordinates of O in the equation $ty - ux = 0$ we observe that $[t : u]$ can be chosen completely freely in S. Thus we obtain an isomorphism between $\varepsilon^{-1}(O)$ and $\mathbb{P}^1$. We have

$$\varepsilon^{-1}(O) = \{O\} \times \mathbb{P}^1 \cong \mathbb{P}^1.$$

Let us fix the point $P = [a_1 : a_2 : a_3]$ in $\mathbb{P}^2 \setminus \{O\}$. We aim to show that the points on the projective line $\varepsilon^{-1}(O)$ correspond to lines through the point O. A line through O and P is uniquely defined by the equation $a_2x - a_1y = 0$. Since either $a_1 \neq 0$ or $a_2 \neq 0$, the line L is given by the parametrization $[a_1t : a_2t : u]$ where we take $[t : u] \in \mathbb{P}^1$.

Then consider the line $L' = \varepsilon^{-1}(L \setminus \{O\})$ in $S \setminus \varepsilon^{-1}(O)$. It is given by

$$[a_1t : a_2t : u] \times [a_1 : a_2]$$

where $[t : u] \in \mathbb{P}^1$ and $t \neq 0$. Allowing for $t = 0$ gives us the closure $\overline{L'}$ of L' with respect to the projective line $\varepsilon^{-1}(O)$. The line $\overline{L'}$ meets the line $\varepsilon^{-1}(O)$ in the point $[a_1 : a_2]$ in $\mathbb{P}^1$. We conclude that a point $P \in \mathbb{P}^2$ defines a unique line which in its turn defines a unique point on the exceptional curve E with coordinates $[a_1 : a_2]$. Thus the lines trough O correspond bijectively to points on E.

In order to blow up at a point $Q = [b_1 : b_2 : b_3]$ different from O we choose a non-zero coordinate and make a linear coordinate change to send the other coordinates to zero. Normalizing with respect to the non-zero coordinate allows one to use the construction for O with minor adaptations.

Definition 5.2. The blow-up of a surface $\widetilde{S}$ in a smooth point P consists of a surface S together with a morphism $\varepsilon : S \to \widetilde{S}$ such that

(a) $\varepsilon^{-1}(\{P\}) = E$ is a smooth rational curve and

(b) ε gives an isomorphism when restricted to the open subsets $S \backslash \varepsilon^{-1}(\{P\})$ and $\widetilde{S} \backslash \{P\}$.

The curve E is called the exceptional curve or **exceptional divisor**.

Example 5.3 (A specific pencil of cubics)**.** Consider the projective polynomials $F = x^3 - 8xz^2 - yz^2$ and $G = y^3 - 8yz^2 - xz^2$. Since F, G are both of degree 3, we know from Bézout's theorem, see Theorem 4.2, that there exists at least one point $P \in \mathbb{CP}^2$ in the intersection of the curves they define. Moroever, Bézout's theorem also tells us that the number of points counting multiplicities must be equal to $\deg F \cdot \deg G = 9$ since $\mathbb{C}$ is algebraically closed. The 9 base points $P_1, \ldots, P_9$ in this example are pairwise distinct and real, see Equation (6). Since they are pairwise distinct, each of them has multiplicity equal to 1.

The blow-up of the points $P_1, \ldots, P_9$ is given by

$$S = \{uF(x, y, z) - tG(x, y, z) = 0\}\,.$$

We define the morphism $\varepsilon : S \to \mathbb{P}^2$ by $\varepsilon\left([x : y : z] \times [F(x, y, z) : G(x, y, z)]\right) = [x : y : z]$ for all points outside of the intersection. The base points P_i are sent to exceptional curves E_i respectively.

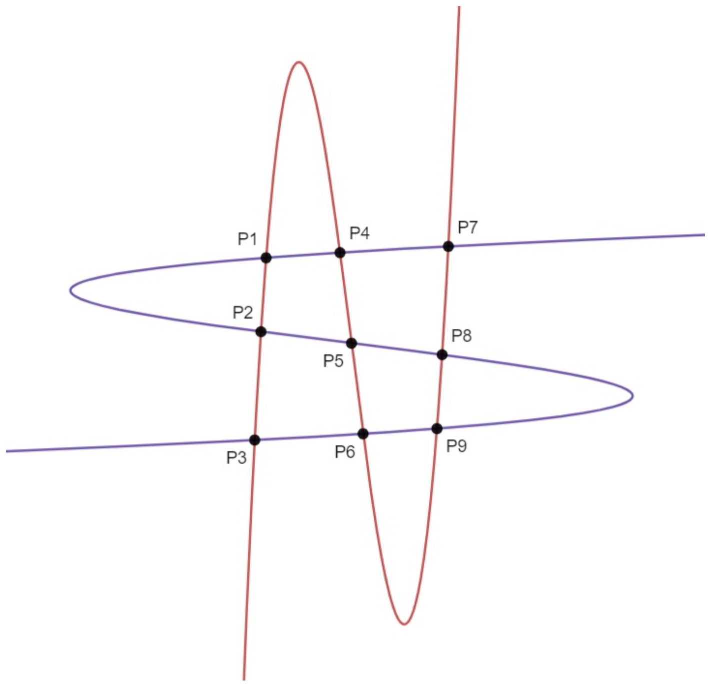


Figure 3: Base points of the dehomogenized cubic curves $x^3 - 8x - y = 0$ and $y^3 - 8y - x = 0$ in $\mathbb{A}^2$

$$\begin{aligned}
P_1 &= [3 : 3 : 1]\\
P_2 &= \left[\sqrt{4+\sqrt{15}} : \left(\sqrt{4+\sqrt{15}}\right)^3 - 8\sqrt{4+\sqrt{15}} : 1\right]\\
P_3 &= \left[\sqrt{7} : -\sqrt{7} : 1\right]\\
P_4 &= \left[\sqrt{4-\sqrt{15}} : \left(\sqrt{4-\sqrt{15}}\right)^3 - 8\sqrt{4-\sqrt{15}} : 1\right]\\
P_5 &= [0 : 0 : 1]\\
P_6 &= \left[-\sqrt{4-\sqrt{15}} : \left(-\sqrt{4-\sqrt{15}}\right)^3 - 8\sqrt{4-\sqrt{15}} : 1\right]\\
P_7 &= \left[-\sqrt{7} : \sqrt{7} : 1\right]\\
P_8 &= \left[-\sqrt{4+\sqrt{15}} : \left(-\sqrt{4+\sqrt{15}}\right)^3 - 8\sqrt{4+\sqrt{15}} : 1\right]\\
P_9 &= [-3 : -3 : 1]
\end{aligned} \tag{6}$$

Note that all nine points have non-zero z-coordinate. We blow up at the points where $F(x,y,z) = G(x,y,z) = 0$. Each P_i will be replaced with an E_i. The lines through each point P_i parameterize each exceptional curve E_i, which is isomorphic with $[t : u]$. By setting $u = 1$ we can use t as a parameter. The curves F, G intersect each E_i at the point with parameter value t equal to the slope of their tangents at P_i. Let us calculate the slope of the tangent lines by using implicit differentiation.

$$\frac{\partial F}{\partial x} = 3x^2 - 8z^2 \qquad \frac{\partial F}{\partial y} = -z^2$$
$$\frac{\partial G}{\partial x} = -z^2 \qquad \frac{\partial G}{\partial y} = 3y^2 - 8z^2$$

We find that the slope of the tangent lines at each P_i are given by

$$-\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y}(P_i) \quad \text{and} \quad -\frac{\partial G}{\partial x}\Big/\frac{\partial G}{\partial y}(P_i)$$

for F and G respectively. In this specific example we find that if F has tangent line with slope λ_i at the point P_i, then G has tangent line with slope $1/\lambda_i$ at the same point. As a result of blowing up the cubic curves are completely separated by 9 exceptional curves, see Figure 4. ♙

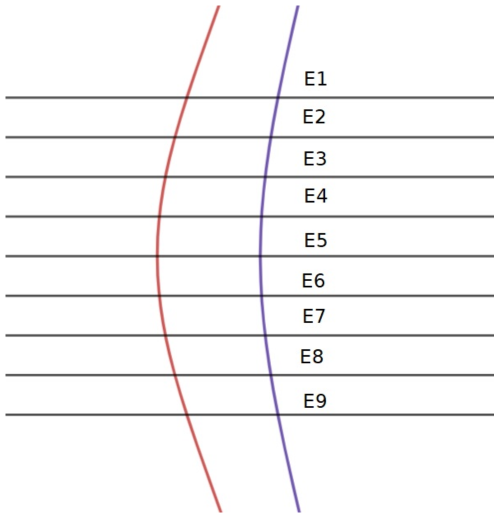


Figure 4: Blow-up of the pencil of cubics $F = x^3 - 8x - y$ and $G = y^3 - 8y - x$ in $\mathbb{A}^2$

Remark 5.4. From [18, p. 150] (see also for the proof) we know that the divisor class group $\mathrm{Cl}(\mathbb{P}^2)$ is isomorphic with $\mathbb{Z}$. Blowing up gives rise to a new divisor class in Cl. As a result the blow-up of $\mathbb{P}^2$ in the 9 base points of a pencil of cubics S has divisor class $\mathrm{Cl}\, S$ isomorphic with $\mathbb{Z}^{10}$.

5.3 The exceptional divisor

By blowing up a point we create a new line E isomorphic with $\mathbb{P}^1$ called the exceptional divisor. The curve E is a divisor over a surface $\widetilde{S}$. In this section we aim to prove E has self-intersection (-1), see Theorem 5.7. We use the following lemma's.

Lemma 5.5. *For any divisor D on a nonsingular variety X, and any finite number of points $P_1, \ldots, P_m \in X$, there exists a divisor D' with $D' \sim D$ such that $P_i \notin \mathrm{Supp}(D')$ for $i = 1, \ldots, m$.*

Lemma 5.6 (The proper transform)**.** *Let C be an effective divisor on $\mathbb{P}^2$, let P be a point of multiplicity m on C, and let $\varepsilon : S \to \mathbb{P}^2$ be the blow-up map with center P. Then $\varepsilon^* : \mathrm{Cl}(\mathbb{P}^2) \to \mathrm{Cl}(S)$ and*

$$\widetilde{C} = \varepsilon^* C - mE.$$

*Moreover $\widetilde{C}$ is called the **strict transform** and $\varepsilon^* C$ the **proper transform**.*

Theorem 5.7. *Let E be the exceptional divisor of a blow-up map of a point P on $\mathbb{P}^2$. It has self-intersection (-1), i.e.*

$$E^2 = (-1).$$

Proof. Consider a curve $C \subset \mathbb{P}^2$ such that $P \in C$ with multiplicity $m_P(C) = 1$. We blow-up at the point P using the map $\varepsilon : S \to \mathbb{P}^2$. Take another curve C' such that $C \sim C'$ in the divisor class group of $\mathbb{P}^2$, and moreover $P \notin C'$. We know such a curve C' exists by Lemma 5.5. Since ε as we defined it for blow-ups is a morphism, we find that $\varepsilon^* C := \widetilde{C} + E$ is a divisor. Rewriting gives $\widetilde{C} = \varepsilon^* C - E$, the strict transform. We know that ε is a map of degree 1 since $\mathbb{P}^2 \setminus \varepsilon^{-1}(P)$ is isomorphic with $S \setminus \{P\}$. Therefore, by Equation (4) we have $(\varepsilon^* C)^2 = C^2$. We compute

$$1 \overset{(1)}{=} \widetilde{C} \cdot E \overset{(2)}{=} (\varepsilon^* C - E) \cdot E \overset{(3)}{=} (\varepsilon^* C \cdot E) - E^2 \overset{(4)}{=} -E^2$$

which gives $E^2 = -1$. Here equality (1) holds since P has multiplicity 1 in the intersection between $\widetilde{C}$ and E. Equality (2) holds by Lemma 5.6 and equality (3) is an expansion of the brackets using the additive property from Theorem 3.5. For equality (4) recall C and C' are equivalent in the divisor class group of $\mathbb{P}^2$, and

$$\widetilde{C} = \varepsilon^* C - E \quad \text{and} \quad \widetilde{C'} =: \varepsilon^* C' - E = \varepsilon^* C'$$

where the last equality holds since $P \notin C'$. The equivalence $C \sim C'$ implies there exists a rational function $\phi \in k(x, y, z)$ such that $C = C' + \operatorname{div}(\phi)$. Therefore,

$$\varepsilon^* C = \varepsilon^* C' + \varepsilon^* \operatorname{div}(\phi) = \varepsilon^* C' + \operatorname{div}(\phi \circ \varepsilon).$$

We find that $\varepsilon^* C \sim \varepsilon^* C'$. However, C' does not contain P and thus $\varepsilon^* C'$ and E do not intersect, i.e. $\varepsilon^* C'.E = 0$. ♘

Corollary 5.8. *Using the notation from earlier,*

$$\widetilde{C}^2 = C^2 - m^2$$

whenever we blow up at a point P with multiplicity $m_P(C) = m$.

Proof. We have from Lemma 5.6 that

$$\begin{aligned}
\widetilde{C}^2 &= (\varepsilon^*(C) - mE)^2 \\
&= (\varepsilon^*(C))^2 - 2m\varepsilon^*(C) \cdot E + m^2 E^2 \\
&= C^2 - m^2.
\end{aligned}$$

Note that $\varepsilon^*(C) \cdot E = 0$ by the same argument we made in the proof of Proposition 5.7. Finally we derive $(\varepsilon^*(C))^2 = C^2$ from Equation (4). Together with the fact that $E^2 = -1$ this completes the argument. ♘

Example 5.9. Consider the projective curve $C : zy^2 = x^3$. It has a singularity of order 2 at $P := [0 : 0 : 1]$ that we resolve using the blow-up. Figure 5 displays the transformations in the plane $z = 1$, for now we assume $z = 1$. In the first blow-up $xu = ty$ we take $t = 1$ such that u parameterizes $\mathbb{P}^1$. This gives $y^2 = x^3$ and $y = ux$. Substituting $y = ux$ we find that

$$\begin{aligned}
u^2 x^2 &= x^3 \\
x^2(u^2 - x) &= 0.
\end{aligned}$$

Then $x = y = 0$ and u arbitrary give an exceptional curve E_1, $u^2 = x$ and $y = ux$ give the strict transform $\widetilde{C}_1$. The new curve intersects the exceptional curve at $u = 0$ with intersection multiplicity 2. The strict transform is locally smooth around the point $u = 0$. Since $\widetilde{C}_1$ and E_1 are tangent at $u = 0$, after blowing up the strict transform $\widetilde{C}_2$ and E_1 intersect the new exceptional curve E_2 in one point. In the third and last blow-up E_1, E_2 and the strict transform $\widetilde{C}_3$ are separated and the intersection is resolved.

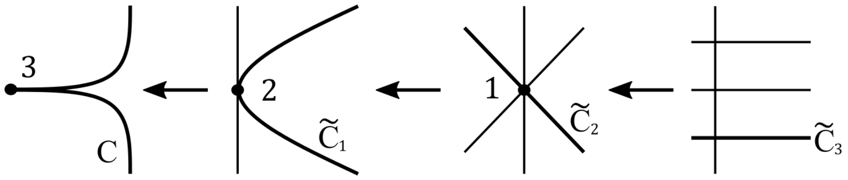


Figure 5: Blow-ups of the curve $C : zy^2 = x^3$ at $[0 : 0 : 1]$ drawn in the plane $z = 1$.

♙

Remark 5.10. As we discussed in Remark 5.4, $\mathrm{Cl}(\mathbb{P}^2)$ is isomorphic with $\mathbb{Z}$. As a result any two lines L, L' are in the same coset, meaning they are linearly equivalent denoted by $L \sim L'$. By Bézout's theorem, see Theorem 4.2, they intersect in exactly one point. More generally, any two curves in $\mathrm{Cl}(\mathbb{P}^2)$ generated by polynomials of the same degree are linearly equivalent. Then by Remark 3.6 we have $L^2 = 1$. Similarly we have for a smooth conic D that $D^2 = 4$ and for a smooth cubic C we have $C^2 = 9$ in $\mathbb{P}^2$, see [18, p. 150].

6 Preliminaries on elliptic curves

The definitions and results in Chapter 6 can be found in more detail in [21] and [22].

6.1 Elliptic curves

6.1.1 Introduction

An **elliptic curve** over a field k given by $(E, \mathcal{O})$ is a nonsingular cubic polynomial E in two variables together with a point $\mathcal{O}$. The genus of a nonsingular polynomial of degree d in two variables is given by $(d-1)(d-2)/2$. Hence an equivalent definition of elliptic curve is that E is a nonsingular curve of genus one together with a point [18, p. 207].

Definition 6.1. An elliptic curve is a pair $(E, \mathcal{O})$, where E is a nonsingular curve of genus one and $\mathcal{O} \in E$. We often leave the point $\mathcal{O}$ implicit in our notation.

6.1.2 The Weierstrass equation

Assume k to be a perfect field[5]. Every elliptic curve has associated to it a **Weierstrass equation** $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Namely can write any cubic polynomial in two variables to this form by making a change of variables. On the condition that char $(\bar{k}) \neq 2, 3$ we can further simplify the equation by substitution of y with $\frac{1}{2}(y - a_1x - a_3)$. This yields the **extended Weierstrass form**

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

where $b_2 = a_1^2 + 4a_2$, $b_4 = 2a_4 + a_1a_3$ and $b_6 = a_3^2 + 4a_6$. Making yet another substitution given by

$$(x, y) \longmapsto \left(\frac{x - 3b_2}{36}, \frac{y}{108}\right)$$

eliminates the x^2-term and gives us an equation of the form $y^2 = x^3 - 27c_4x - 54c_6$. We have $c_4 = b_2^2 - 24b_4$ and $c_6 = -b_2^3 + 36b_2b_4 - 216b_6$. We baptize it the **Weierstrass normal form** [21, p. 42].

6.1.3 The discriminant

Consider a curve in Weierstrass normal form, $E : y^2 = x^3 + Ax + B$ with $A = -27c_4$ and $B = -54c_6$, the **discriminant** Δ is given by $\Delta = -16\left(4A^3 + 27B^2\right)$. It determines whether a curve given by a Weierstrass equation is singular. If it is non-singular, the curve is an elliptic curve.

Proposition 6.2. *A curve given by a Weierstrass equation satisfies:*

1. *It is nonsingular if and only if* $\Delta \neq 0$.
2. *It has a node if and only if* $\Delta = 0$ *and* $c_4 \neq 0$.
3. *It has a cusp if and only if* $\Delta = 0$ *and* $c_4 = 0$.

In cases 2. and 3. there is only the one singular point.

Proof. A proof can be found in [21, p. 45], Proposition 1.4. ⌂

[5] Every algebraic extension of a perfect field is separable (e.g. $\mathbb{Q}$).

6.1.4 Elliptic curves in the projective plane

Note that an elliptic curve $(E, \mathcal{O})$ is a projective curve: it has affine part E and infinite part $\mathcal{O}$ in projective 2-space $\mathbb{P}^2$. We obtain the homogeneous equation for the projective curve E^h by homogenizing the Weierstrass equation of E with a third coordinate z. For an elliptic curve E in Weierstrass normal form $y^2 = x^3 + Ax + B$ this gives

$$E^h : zy^2 = x^3 + Axz^2 + Bz^3.$$

We obtain the point at infinity by filling out $z = 0$ in E^h, this gives $x^3 = 0$, a point of inflection. Thus in the case of curves given by a Weierstrass equation we take $\mathcal{O} = [0 : 1 : 0]$.

6.1.5 Examples of curves corresponding to Weierstrass equations

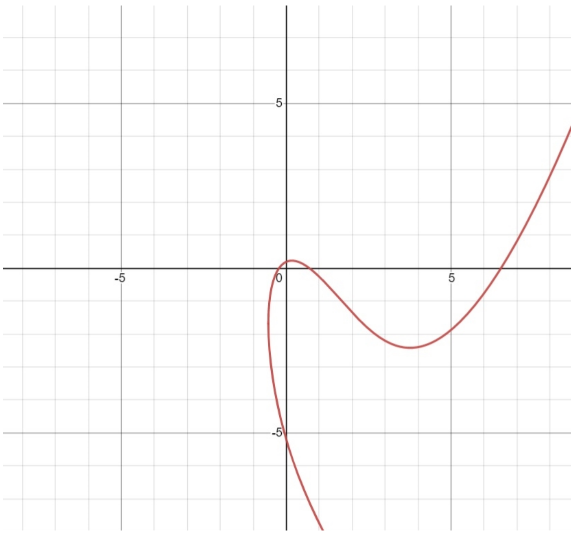


Figure 6: $C_1 : y^2 + 3xy + 5 = x^3 - 7x^2 + 3x$

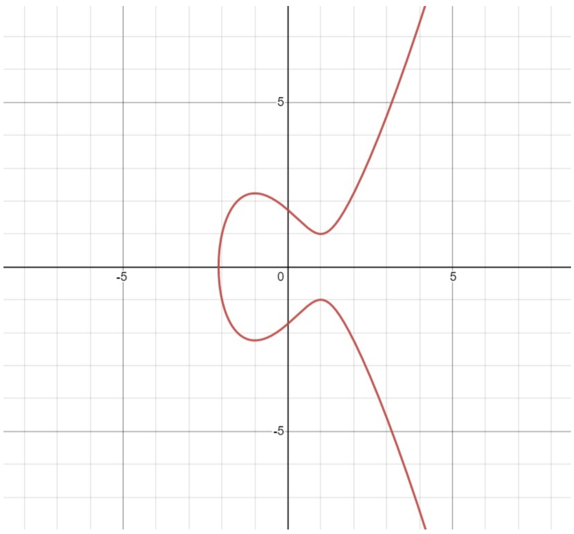


Figure 7: $C_2 : y^2 = x^3 - 3x + 3$

Example 6.3. The Weierstrass equation $C_1 : y^2 + 3xy + 5 = x^3 - 7x^2 + 3x$ gives us the elliptic curve in Figure 6. ♙

Example 6.4. Consider $C_2 : y^2 = x^3 - 3x + 3$, an elliptic curve in Weierstrass normal form. One can see in Figure 7 that the graph is symmetric in the x-axis, which can also be seen from its equation. ♙

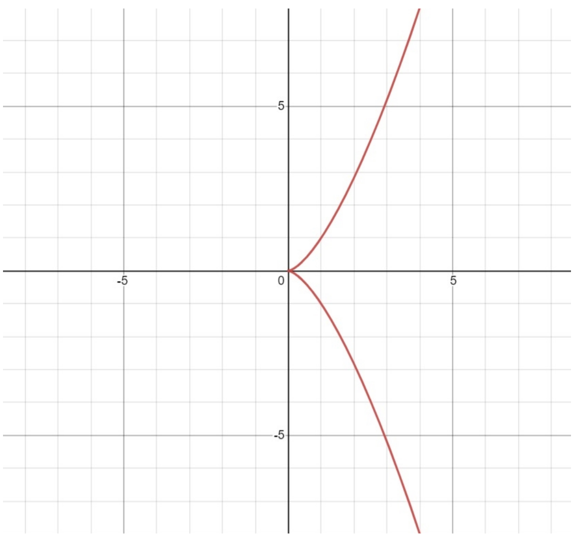


Figure 8: $C_3 : y^2 = x^3$

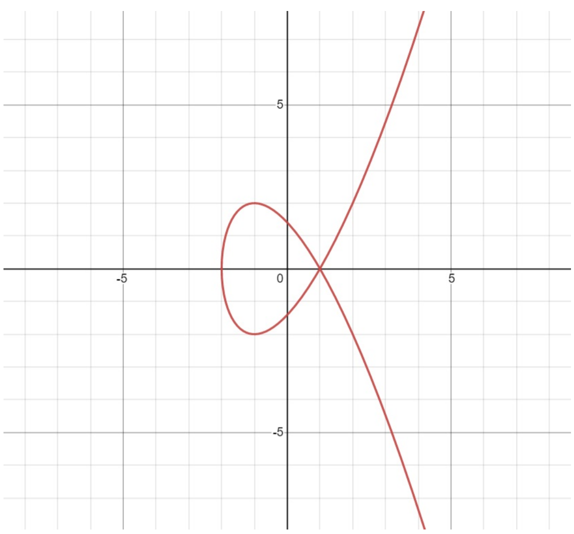


Figure 9: $C_4 : y^2 = x^3 - 3x + 2$

Example 6.5. Consider the curve $C_3 : y^2 = x^3$ in Figure 8. The curve has a **cusp** at the origin $(0, 0)$, a singularity. The origin has two tangent lines, both with slope $y = 0$. The curve is singular and therefore not elliptic. ♙

Example 6.6. Consider the curve $C_4 : y^2 = x^3 - 3x + 2$ in Figure 9. The curve has a **node** at $(1, 0)$, a singularity. Note that the point $(1, 0)$ where the curve intersects itself has two distinct tangent lines. The curve is singular and therefore not elliptic. ♙

6.2 Group law

6.2.1 Introduction

Let E be an elliptic curve over the field k defined by the Weierstrass equation $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Its set of k-rational points is given by

$$E(k) = \left\{(x, y) \in k^2 : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\right\} \cup \{\mathcal{O}\}.$$

We claim $E(k)$is group with group law $\oplus$, we write $(E(k), \oplus, \mathcal{O})$. We define the group law $\oplus$ as follows (see [21, p. 51]).

Definition 6.7. Let P, Q be two distinct points on $E(k)$. There exists a unique line L through P and Q. By Bézout's theorem, see Theorem 4.2, it intersects the curve E in precisely one more point[6], which we call $R \in E(k)$. Then draw another line L' through the points R and $\mathcal{O}$, so that L' intersects with $E(k)$ in a point which we will call $P \oplus Q$.

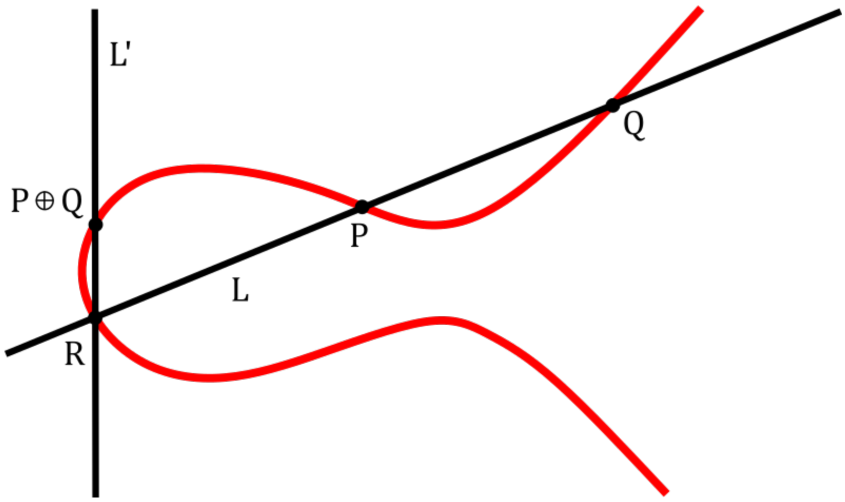


Figure 10: Group law for two distinct points $P, Q \in E(k)$ and the elliptic curve E drawn in red.

Consider a point $P \in E(k)$ and the tangent line with E at the point P. By Bézout's theorem, it intersects with E in a third point, say $Q \in E(k)$. Next we take another line L' through Q and $\mathcal{O}$ and define their third intersection as the point $P \oplus P = 2P$.

Together with the group law $\oplus$ it can be proven that $(E(k), \oplus, \mathcal{O})$ is an abelian group. For the proof see Proposition 2.2 in [21, p. 51].

Definition 6.8 (Group structure $E(k)$)**.** For all $P, Q, R \in E(k)$ we see that $(E(k), \oplus, \mathcal{O})$ defines a group and $\mathcal{O} \in E(k)$ and $\oplus$ is a map defined by $(P, Q) \mapsto P \oplus Q$, satisfying

1. (associativity) $P \oplus (Q \oplus R) = (P \oplus Q) \oplus R$,
2. (unit element) $P \oplus \mathcal{O} = P = \mathcal{O} \oplus P$,

[6] This point R is again in $E(k)$ because E, L are a cubic and a line intersecting in P, Q both in $E(k)$, therefore the third base point R must also be in $E(k)$.

3. (inverses) $P \oplus -P = \mathcal{O} = -P \oplus P$,
4. (commutativity) $P \oplus Q = Q \oplus P$.

In finitely many cases, adding an element in $E(k)$ to itself, gives us its inverse. An inflection point $P \in E(k)$ in the group $E(k)$ of an elliptic curve E is a point such that $2P = -P$. We know from the Corollary in [7, p. 59] that an elliptic curve has nine inflection points. We can find them by studying the **Hessian** H, the determinant of the matrix of second partial derivatives of a polynomial.

Theorem 6.9. *Let k be field with characteristic 0 and let F be a homogeneous polynomial of degree n, and assume F contains no lines. We have*

(1) $P \in H \cap F$ if and only if P is either an inflection or a multiple point of F,

(2) $\mu_P(H, F) = 1$ if and only if P is an ordinary inflection point.

Proof. The outline of the proof can be found on page 59 in [7, p. 59]. ✍

Example 6.10. The elliptic curve $E : y^2 = x^3 + 1$ over the rationals has nine inflection points. Three inflection points are P its additive inverse $-P$ and the point $\mathcal{O}$. The former two points are of order 3, i.e. $\pm 3P = \mathcal{O}$. We find $\pm P$ using Theorem 6.9. Consider the implicit function $F(x, y) = y^2 - x^3 - 1 = 0$, and compute its Hessian. The matrix has entries

$$\frac{\partial^2 F}{\partial x^2} = -6x \qquad \frac{\partial^2 F}{\partial x \partial y} = 0$$

$$\frac{\partial^2 F}{\partial y \partial x} = 0 \qquad \frac{\partial^2 F}{\partial y^2} = 2$$

and thus the Hessian is given by $H(x, y) = -12x$. A point Q is an inflection point or a multiple point of F if $Q \in H \cap F$, see Theorem 6.9. However, the curve E is nonsingular and thus Q must be an inflection point. Taking $H(x, y) = 0$ implies $\pm P$ have x-coordinate equal to 0. Since $P \in F$ the points $\pm P$ have y-coordinate ± 1 respectively. Then $\pm P = (0, \pm 1)$.

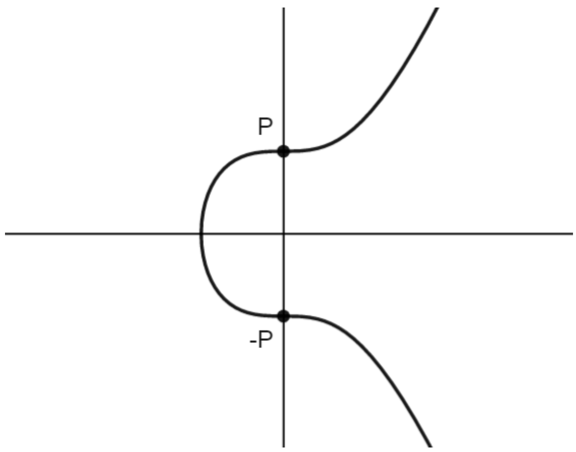


Figure 11: $E : y^2 = x^3 + 1$ and two non-trivial inflection points $\pm P$.

We check this result as follows. Let us compute $2P$ first. For the tangent at the point P we implicitly differentiate E and obtain

$$\left.\frac{\partial y}{\partial x}\right|_P = \frac{3 \cdot 0}{2 \cdot 1} = 0.$$

This gives us the slope of the line $L' : y = 1$ through P. We find all the intersection points of L' with E by taking $y = 1$ in E which yields

$$1 = x^3 + 1.$$

We obtain the unique solution $P = (0, 1)$ and by Bézout it has multiplicity 3. Therefore, P is an inflection point. The line L' through P and $\mathcal{O}$ is given by $x = 0$ and intersects with E at the point $-P$. Then $2P = -P$, i.e. $3P = \mathcal{O}$. ♙

6.3 Mordell-Weil Theorem

In the previous section we saw that the set $E(k)$ is an abelian group. In 1901 Henri Poincaré posed the question if it is moroever a finitely generated group if k is an arbitrary number field (i.e. finite extension of $\mathbb{Q}$). Louis Mordell showed in 1922 that this is indeed the case for $k = \mathbb{Q}$. In 1928, André Weil proved the result for arbitrary number fields. The theorem they proved is called the Mordell-Weil theorem. Let $E(k)_{\text{tor}}$ be the set of points in $E(k)$ with finite order.

Theorem 6.11 (Mordell-Weil)**.** *Let E be an elliptic curve over a number field k. The group $E(k)$ is of the form*

$$E(k) \cong E(k)_{\text{tor}} \oplus \mathbb{Z}^r$$

*and the **rank** r a non-negative integer*

The proof in 'Arithmetic of elliptic curves' by Joseph Silverman uses a **height-function** h which maps $E(k)$ to $[0, \infty)$, see [21, p. 239]. The height-function is used as an upper bound for a set of points in $E(k)$. That is, in the proof of the theorem a lemma is used which states: for every non-negative number M, the set

$$\{P = (x, y) \in E(k) : h(P) \leq M\}$$

is finite. In the proof yet another result called the Descent theorem is used which gives inequality relations on the height of points, see Theorem 3.1 in [21, p. 218]. By manipulating the inequalities and using a set of generators together with torsion points in $E(k)$ one arrives at the result.

In the case of number fields k the number of possible torsion groups is bounded by the degree of the extension over the rationals, a fact which was proved by Merel in 1996 in the paper 'Bornes pour la torsion des courbes elliptiques sur les corps de nombre'. In particular there are only finitely many options for the structure of the subgroup of torsion points $E(\mathbb{Q})_{\text{tor}}$. This result is known as Mazur's theorem, named after Barry Mazur, an American mathematician. It appeared for the first time in 1978 in the paper 'Modular curves and the Eisenstein ideal', see Theorem 8 in [12, p. 35]. The proof is far too difficult, but the theorem itself is relatively easy to understand.

Theorem 6.12 (Mazur)**.** *Let $E/\mathbb{Q}$ be an elliptic curve. Then the torsion subgroup $E_{tor}(\mathbb{Q})$ of $E(\mathbb{Q})$ is isomorphic to one of the following fifteen groups:*

$$\begin{aligned} &\mathbb{Z}/N\mathbb{Z} \qquad \textit{with } 1 \leq N \leq 10 \textit{ or } N = 12, \\ &\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z} \qquad \textit{with } 1 \leq N \leq 4. \end{aligned}$$

Further, each of these groups occurs as $E_{tor}(\mathbb{Q})$ for some elliptic curve $E/\mathbb{Q}$.

6.4 Bad reduction

6.4.1 Introduction

Curves in Weierstrass form are singular if their discriminant is zero and nonsingular otherwise. In the case curve E over a field k is nonsingular we classify the curve using the notion of valuation v.

Definition 6.13 (Discrete valuation)**.** Let k be a field. We define a discrete valuation of a field k to be a surjective function $v : k^* \to \mathbb{Z}$ such that, for every $x, y \in k^*$,

(1) $v(x \cdot y) = v(x) + v(y)$

(2) $v(x + y) \geq \min\{v(x), v(y)\}$ if $x \neq -y$.

Then $v(1) = 0$ and $v\left(x^{-1}\right) = -v(x)$. As a convention, we define $v(0) := \infty$.

We normalise the local parameter π with respect to a valuation V so that $v(\pi) = 1$. A valuation is in that regard always defined with respect to a local parameter π and its dependence is denoted by a subscript v_π. Let us define $R = \{x \in k : v(x) \geq 0\}$, the discrete valuation ring (DVR) in k, see [1, p. 138]. From Definition 6.13 we derive that π is necessarily irreducible. Moreover, any element $x \in k$ has a unique factorization

$$x = n/d \cdot \pi^m,$$

where n, d are in their lowest terms and m an integer. Then $v(x) = m$.

6.4.2 Minimal Weierstrass equation

Definition 6.14. Let E/k be an elliptic curve. A Weierstrass equation for E is called a minimal (Weierstrass) equation for E at v if $v(\Delta)$ is minimized subject to the condition that $a_1, a_2, a_3, a_4, a_6 \in R$. This minimal value of $v(\Delta)$ is called the valuation of the minimal discriminant of E at v.

We discuss a method to minimize a Weierstrass equation such that it is minimal according to Definition 6.14. If $a_i \in R$ but $v(\Delta) \geq 12$ with respect to any local parameter, we make a substitution $(x, y) \mapsto \left(u^{-2}x, u^{-3}y\right)$ where we take u divisible by a sufficiently large power of π such that $0 \leq v(\Delta) < 12$. If, on the other side, $v(\Delta) < 12$ but not all $a_i \in R$, we make the reverse substitution with appropriate u such that $0 \leq v(\Delta) < 12$. Important to note is that every elliptic curve E/k has a minimal Weierstrass equation and this equation is unique up to a change of coordinates [21, p. 186]. Therefore we assume every general Weierstrass equation to be minimal unless stated otherwise.

6.4.3 Good and bad reduction

Define $\mathcal{M} = \{x \in k : v(x) > 0\}$, the maximal ideal of R and π so that $\mathcal{M} = \pi R$. Let us define **reduction modulo** π by sending the coefficients of the Weierstrass equation E of an elliptic curve to their representations modulo π in $R/\pi R$. We obtain the elliptic curve

$$\widetilde{E} : y^2 + \tilde{a}_1 xy + \tilde{a}_3 y = x^3 + \tilde{a}_2 x^2 + \tilde{a}_4 x + \tilde{a}_6$$

which is called the reduction of E/k modulo π. We moreover define $\widetilde{E}_{\mathrm{ns}}(k)$ as the set of nonsingular points: these are the points with non-zero partial derivatives.

Definition 6.15. Let E/k be an elliptic curve, and let $\widetilde{E}$ be the reduction modulo $\mathcal{M}$ of a minimal Weierstrass equation for E.

(a) E has good (or stable) reduction if $\widetilde{E}$ is nonsingular.

(b) E has multiplicative (or semi-stable) reduction if $\widetilde{E}$ has a node.

(c) E has additive (or unstable) reduction if $\widetilde{E}$ has a cusp.

In cases (b) and (c) we say that E has bad reduction. If E has multiplicative reduction, then the reduction is said to be **split** if the slopes of the tangent lines at the node are in k, and otherwise it is said to be **nonsplit**.

Let $K = R/\mathcal{M}$. The next proposition provides us with a method to deduce the type of reduction from the Weierstrass equation of a curve.

Proposition 6.16. *Let E/k be an elliptic curve given by a minimal Weierstrass equation*

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

Let Δ be the discriminant of this equation, and let c_4 be the usual expression involving $a_1, \ldots, a_6$ as described above.

(a) E has good reduction if and only if $v(\Delta) = 0$, i.e., $\Delta \in R^*$. In this case $\tilde{E}/K$ is an elliptic curve.

(b) E has multiplicative reduction if and only if $v(\Delta) > 0$ and $v(c_4) = 0$, i.e., $\Delta \in \mathcal{M}$ and $c_4 \in R^*$. In this case $\widetilde{E}_{\mathrm{ns}}$ is the multiplicative group,

$$\widetilde{E}_{\mathrm{ns}}\left(\overline{K}\right) \cong \overline{K}^*$$

(c) E has additive reduction if and only if $v(\Delta) > 0$ and $v(c_4) > 0$, i.e., $\Delta, c_4 \in \mathcal{M}$. In this case $\widetilde{E}_{\mathrm{ns}}$ is the additive group,

$$\widetilde{E}_{\mathrm{ns}}\left(\overline{K}\right) \cong \overline{K}^+.$$

Proof. A proof can be found in [21, p. 196], Proposition 5.1. ♘

Example 6.17. In this example we determine the reduction of the elliptic curve $E : y^2 = x^3 + 2x^2 + 3x + 6$ modulo 7. In this specific case we take $k = \mathbb{Q}$. Then

$$R = \{x \in \mathbb{Q} : v_7(x) \geq 0\} =: \mathbb{Z}_7.$$

In order to check whether it has good or bad reduction we ensure the equation is minimal. Indeed all coefficients of E are in $\mathbb{Z}_7$. Before we compute the determinant we bring E to Weierstrass normal form first. Using the earlier formulated recipe, we find that E has Weierstrass normal form

$$y^2 = x^3 + 2160x + 214272.$$

Then E has discriminant equal to $\Delta = -20479168217088 = -2^{18} \cdot 3^{13} \cdot 7^2$ and so $v_7(\Delta) = 2$. We find that E is indeed minimal with respect to v_7. Now we can check whether the Weierstrass normal form of E has good or bad reduction. Since $c_4 = -1/3$ and thus $v_7(c_4) = 0$, we find that it has bad multiplicative reduction modulo 7. Reducing the Weierstrass normal form of E modulo 7 gives

$$y^2 = x^3 - \bar{3}x + \bar{2}$$

over $K = \mathbb{Z}_7/7\mathbb{Z}_7$. From Figure 9 we conclude that the curve is nodal. ♙

7 Elliptic surfaces

7.1 Introduction

Assume k to be a number field unless we define geometric properties, in that case take k algebraically closed.

We study 1-dimensional families of elliptic curves in a geometric context. An elliptic curve is a variety of dimension 1 as a topological space. Therefore, a 1-dimensional family of elliptic curves gives rise to a variety of dimension 2 called an elliptic surface. What follows is based on [14] and [17] where the details can be found.

Definition 7.1 (Elliptic surface)**.** Let C a smooth projective curve over k. An **elliptic surface** S over C is a smooth projective surface S with an elliptic fibration over C, i.e. a surjective morphism

$$\pi : \quad S \to C,$$

such that

(a) almost all[7] fibers are smooth elliptic curves,

(b) no fiber contains an exceptional curve with self-intersection -1.

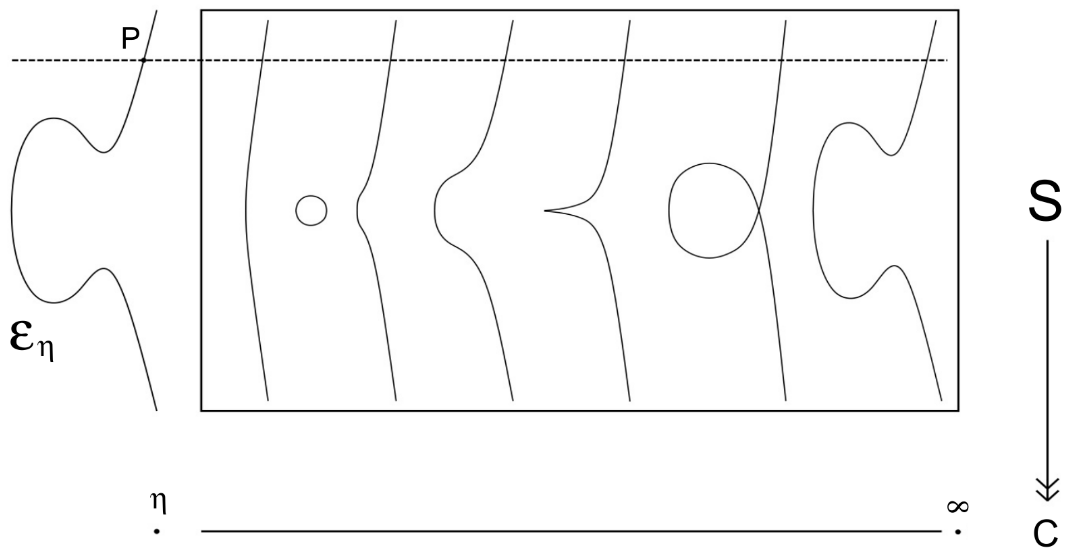


Figure 12: Elliptic surface S over C with generic fiber ε_η above generic point η.

If we blow up a point in a fiber this gives rise to a (-1)-curve contained within the fiber. The resulting surface is no longer elliptic due to condition (b) in Definition 7.1. Thus, there are no fibers on an elliptic surface containing exceptional curves with self-intersection (-1). However, if we forget about the structure of the elliptic fibration $\pi : S \to C$ we can expand the definition of an elliptic surface to surfaces over a smooth projective curve C where we allow for (-1)-curves in the fibers of the surface. The model of the surface S which has no fibers that contain (-1)-curves and which respects the elliptic fibration is called the **relatively minimal model**.

Definition 7.2 (Relatively minimal model of an elliptic surface)**.** An elliptic surface $\pi : S \to C$ is relatively minimal if S is smooth and there are no (-1)-curves in the fibers of π.

Suppose S is an elliptic surface which does not necessarily respect the structure of the fibration, i.e. is not necessarily relatively minimal. By blowing down all (-1)-curves that occur within its fibers, we reach a relatively minimal model of the surface. In the algebraic closure over which the surface is defined, a relatively minimal model is unique up to a birational mapping.

[7]Except for finitely many.

Proposition 7.3. *Given an elliptic surface $\pi : S \to C$, there is a unique smooth relatively minimal elliptic surface $\pi_1 : S_1 \to C$ birational to $\pi : S \to C$ as elliptic surfaces over C.*

Proof. A proof can be found in [13, p. 17], Corollary II.1.3. ♘

There are two conventions that we will adopt in this text:

(1) every elliptic surface has a section.

Definition 7.4 (Section of an elliptic surface)**.** A **section** of an elliptic surface $\pi : S \to C$ is a morphism $f : C \to S$ such that $\pi \circ f = \mathrm{id}_C$.

(2) every elliptic surface S has a singular fibre. In particular, S is not isomorphic to a product $\Gamma \times C$ where Γ denotes a nonsingular (elliptic) curve.

Due to condition (a) in Definition 7.1 and the adopted convention that each elliptic surface S has a singular fiber, we know there exist finitely many points $P \in C$ such that $\pi^{-1}(P)$ defines a singular curve.

7.2 The generic fiber of an elliptic surface

The generic point η of the curve C is dense in C with respect to the Zariski topology. The generic point has all the properties that are true for almost all points in C. For instance, we know all but finitely many fibers in the surface are elliptic curves. Therefore, the fiber above the generic point is an elliptic curve over the function field of the base curve C, denoted by $k(C)$. It is called the **generic fiber**. More on the generic point can be found in [24]. Let ε_η be the generic fiber above the generic point η of C as in Figure 12. The set of $k(C)$-rational points $\varepsilon_\eta(k(C))$ is an abelian group under addition. The $k(C)$-rational points correspond directly to the group of sections on S, called the **Mordell-Weil group** $\mathrm{MW}(S)$ of the surface S.

Proposition 7.5. *Let $\pi : S \to C$ be an elliptic surface defined over k. Let ε_η be the generic fiber of the surface and $\mathrm{MW}(S)$ the group of sections of the surface. Then there is a group isomorphism between the $k(C)$-rational points on ε_η and the group of sections $\mathrm{MW}(S)$ of the surface.*

Proof. A proof can be found in [20, p. 210], Proposition 3.10. ♘

For the elliptic surfaces we consider in this thesis, the Mordell-Weil theorem holds more generally for sections on an elliptic surface, i.e. points on the elliptic curve ε_η over the function field $k(C)$. We assume that the Mordell-Weil group $\mathrm{MW}(S)$ of the surface S is finitely generated. Since $\varepsilon_\eta(k(C))$ and $\mathrm{MW}(S)$ are isomorphic this implies $\varepsilon_\eta(k(C))$ is finitely generated. Consequently, we have the following definition.

Definition 7.6 (Generic rank of an elliptic surface)**.** The **generic rank of an elliptic surface** is the rank of the generic fiber given by

$$\mathrm{rank}\,(\varepsilon_\eta(k(C))) = \mathrm{rank}\,(\varepsilon_\eta(k(C))_{\mathrm{tor}} \oplus \mathbb{Z}^r) = r.$$

8 Extremal rational elliptic surfaces

8.1 Rational elliptic surface

Let us introduce rational elliptic surfaces by means of an example. The definitions and results stated in this section can be found in [13] and [17].

Example 8.1. Let t parameterize $\mathbb{P}^1 = [t : u]$. Consider the pencil of cubics given by

$$F + tG = 0,$$

over an algebraically closed field k where F, G are homogeneous polynomials of degree 3 without common irreducible component. The cubic curves defined by F and G meet in 9 base points. Note that the map sending $[x : y : z] \mapsto [F(x, y, z) : G(x, y, z)]$ is not defined at the base points. In order to resolve the indeterminacy we create the blow-up

$$S = \{F(x, y, z) + tG(x, y, z) = 0\}.$$

The set is a rational elliptic surface since it is birational with $\mathbb{P}^2$ by construction via the blow-up map ε.

$$\begin{tikzcd} & S \arrow[dl, "\varepsilon"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "{[x:y:z] \to [F(x,y,z):G(x,y,z)]}"'] & & \mathbb{P}^1 \end{tikzcd}$$

In Example 8.1 we state that the nine-fold blow up of a pencil of cubics over an algebraically closed field is a rational elliptic surface (RES). We find that over an algebraically closed field the converse statement is true.

Theorem 8.2. *Let $\pi : \mathrm{S} \to \mathbb{P}^1$ be an elliptic surface with section over k algebraically closed. Then the rational elliptic surface S is isomorphic with the 9-fold blow-up of the plane $\mathbb{P}^2$ at the base points of a pencil of generically smooth cubic curves which induces the fibration π.*

Proof. A proof can be found in [13, p. 37], Lemma IV.1.2.

Over an algebraically closed field k any rational elliptic surface is the blow-up of a pencil of cubics. As a result a RES is a surface above the nonsingular curve $\mathbb{P}^1$. Moreover, every RES is in one-to-one correspondence with its generic fiber ε_η. The latter is an elliptic curve over the function field $k\left(\mathbb{P}^1\right)$. We take t as a parameter of $\mathbb{P}^1$ so that $k(\mathbb{P}^1) \cong k(t)$. Then ε_η admits a Weierstrass equation

$$\varepsilon_\eta : y^2 = x^3 + A(t)x + B(t),$$

where $A(t), B(t) \in k(t)$. Using the valuation of primes $(t - t_0)$ in $k[t]$ we can then minimize the Weierstrass equation with respect to the discriminant $\Delta(\varepsilon_\eta(t))$. In order for minimality of the Weierstrass equation to hold we need that $v(c_4) < 4$ or $v(c_6) < 6$ for all $t \in \mathbb{P}^1$. After minimizing we choose a positive integer n such that $\deg(a_i) \leq n \cdot i$ as a polynomial. Using a second variable s we homogenize the a_i so that $a_i(t, s)$ are homogeneous polynomials in two variables t, s of degree $n \cdot i$. Then the discriminant $\Delta(\varepsilon_\eta(t, s))$ is of degree $12n$. The integer n is equal to the arithmetic genus $\chi(S)$ of an elliptic surface S, i.e. $n = \chi(S)$.

Definition 8.3 (Arithmetic genus)**.** Let $(X, \mathscr{O}_X)$ be a projective scheme of dimension r over a field $k = \bar{k}$. We define the arithmetic genus $\chi(X)$ by

$$\chi(X) = (-1)^r \left(\chi_e\left(\mathscr{O}_X\right) - 1\right),$$

where χ_e is the Euler characteristic.

An elliptic surface is an example of a projective scheme of dimension $r = 2$. For more context, see Exercise III.5.3 in [9, p. 230]. We know from [17, p. 39] that a rational elliptic surface S has arithmetic genus equal to $\chi(S) = 1$.

Proposition 8.4. *For any $P \in \varepsilon_\eta(k(t))$ on the generic fiber, we have $P^2 = -\chi(S)$ as a section.*

Proof. A proof can be found in [17, p. 27], Corollary 6.9. For the adjunction formula mentioned in the proof, see [9, p. 361] Proposition 1.5.

Therefore, in the case of rational elliptic surfaces with section, all sections $P \in \varepsilon_\eta(k(t))$ have self-intersection -1. The final blow-up of the base points in the pencil of cubics gives rise to sections on the surface.

8.2 Bad reduction of rational elliptic surfaces over the function field $\mathbb{Q}(t)$.

Consider the rational elliptic surface S over the curve $\mathbb{P}^1$ defined by $\varepsilon_\eta : y^2 = x^3 + A(t)x + B(t)$ with coefficients $A(t), B(t) \in \mathbb{Q}[t]$ so that $\Delta(\varepsilon_\eta) \neq 0$. Our goal is to determine for which $t_0 \in \mathbb{P}^1(\mathbb{Q})$ this elliptic curve[8] has good or bad reduction, i.e. for which value t_0 of the parameter t the surface has singular fibers. Note that the ideals $(t-t_0)\mathbb{Q}[t]$ where $t_0 \in \mathbb{Q}$ are maximal in $\mathbb{Q}[t]$ because $\mathbb{Q}[t]$ is a principle ideal domain and $(t-t_0)$ an irreducible element in $\mathbb{Q}[t]$. The valuation v_{t-t_0} in $\mathbb{Q}(t)$

$$v_{t-t_0}\left(n/d \cdot (t-t_0)^m\right) = m, \tag{7}$$

where n and d are coprime integers, gives rise to $\mathbb{Q}[t]$ a discrete valuation ring with local parameter $t - t_0$. Minimizing the discriminant Δ of the equation with respect to this valuation gives us the unique minimal Weierstrass equation over $\mathbb{Q}(t)$. To determine for which t_0 the elliptic curve ε_η has good or bad reduction we reduce the coefficients of its Weierstrass equation modulo $(t-t_0)$. If $m \geq 1$ at $t = t_0$ after minimizing the discriminant, it has bad reduction at the fiber $\pi^{-1}(t_0) =: E_{t_0}$. We often say ε_η has bad reduction at $t - t_0$ instead, i.e. we take $t - t_0 = 0$.

Example 8.5. Take the surface S defined by $\varepsilon_\eta : y^2 = x^3 + (t-1)x + t^2 - 1$ over $\mathbb{P}^1$. By computing the discriminant we find that the Weierstrass equation is minimal and minimized with respect to any valuation v_{t-t_0} for some $t_0 \in \mathbb{Q}$. Namely,

$$\Delta\left(\varepsilon_\eta\right) = -16\left(4(t-1)^3 + 27(t+1)^2(t-1)^2\right).$$

In the cases $t_0 = \pm 1$ we have $v_{t-t_0}(\Delta) < 12$. Since $v_{t-1}(\Delta) > 0$ and $v_{t-1}(c_4) = v_{t-1}(48(t-1)) = 1$ the elliptic curve ε_η has bad reduction at $t = 1$. Therefore, the surface S must have a singular fiber at $t = 1$. Reducing modulo $(t-1)$ yields $E_1 : y^2 = x^3$. Note that for any other $(t - t_0)$ the fiber E_{t_0} has good reduction since $v_{t-t_0}(\Delta) = 0$.

Let us compute the reduction at $t = \infty$. We make the coordinate change $t = 1/s$ and send $s^3 y \mapsto y$ and $s^2 x \mapsto x$. This yields the Weierstrass equation corresponding to the fiber in the surface S at $t = \infty$ given by

$$E_\infty : y^2 = x^3 + (s^3 - s^4)x + s^4 - s^6.$$

We only look at the reduction for s. Since $\Delta(E_\infty) = -16(4(s^3 - s^4)^3 + 27(s^4 - s^6)^2)s^3$ we find that E_∞ has bad reduction at s.

[8] We consider ε_η as an elliptic curve over the function field $\mathbb{Q}(t)$.

8.3 Kodaira fiber types

The singular fibers on complex surfaces were first classified by Kunihiko Kodaira in the years 1960 – 63, see [10]. In 1975 John Tate developed an algorithm for determining the singular fibers over a perfect field, see [23]. We discuss the classification of singular fibers using a part of Tate's algorithm for RES. Figure 13 provides a table with all the possible fiber types. Singular fibers are either reducible or irreducible. The irreducible fibers of an elliptic surface that are nodal or cuspidal[9] rational curves with self-intersection 0. All other types are reducible fibers, i.e. they have multiple irreducible components, and each component in a reducible fiber is a smooth rational curve with self-intersection −2, see [17, p. 11].

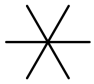
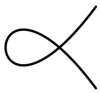
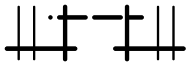
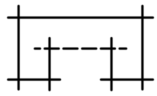
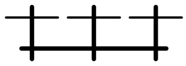
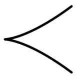
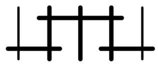
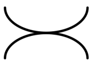
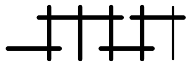
Kodaira type	Number of components	Fiber	Kodaira type	Number of components	Fiber
I_0	1		IV	3	
I_1	1		I_0^*	5	
I_2	2		I_n^*	n+5	
I_n	n		IV^*	7	
II	1		III^*	8	
III	2		II^*	9	

Figure 13: Kodaira fiber types and the number of irreducible components of each type. The lines in thick print have multiplicity higher than one. The exact multiplicity of these components can be found in [17, p. 13].

In this bachelor project we focus explicitly on the **semi-stable** fibers, a subset of the singular fibers that have Kodaira type I_n where $n \geq 1$. To determine the semi-stable fibers of a rational elliptic surface we follow part of Tate's algorithm. Consider a RES $\pi : S \to \mathbb{P}^1$ with a singular fiber at[10] t. We bring its equation to Weierstrass normal form

$$y^2 = x^3 + c_4(t)x + c_6(t)$$

where $c_4(t), c_6(t) \in k[t]$. Recall the determinant of the Weierstrass equation in normal form given by

$$1728\Delta = c_4(t)^3 - c_6(t)^2 \tag{8}$$

If the fiber is singular and in Weierstrass normal form, a singularity exists at $(x_0, 0)$. We know from Proposition 6.2 that only one such singular point exists meaning the y-coordinate is necessarily zero, since Weierstrass normal form admits a symmetry in the y-coordinate of points in the curve. From here we make an important case distinction. Let v_t be a valuation with respect to the local parameter t, i.e. let $t_0 = 0$.

Let $y^2 = x^3 + c_4(t)x + c_6(t)$ define an elliptic surface over $\mathbb{P}^1$. Suppose there is a singular fiber at $t - t_0$. If $v_{t-t_0}(c_4) = 0$, i.e. $t - t_0 \nmid c_4$, the singular fiber is called **multiplicative** or semi-stable, and otherwise **additive**.

[9] The Kodaira types of these fibers are I_1 and II, as can be seen from Figure 13

[10] In case the singular fiber is situated at $(t - t_0)$ or $1/t$ we first make the substitution $t \mapsto t - t_0$ or $t \mapsto 1/t$ respectively.

In extended Weierstrass form, the equation corresponding to a surface with semi-stable fibers always describes a nodal curve. Suppose $v_t(\Delta) = n$. We blow-up base points of the pencil to resolve the singularity that exists at t and consequently create $n-1$ exceptional curves. As can be seen from the final blow-up, the singular fiber at t is of Kodaira type I_n. Let us consider the case $n > 1$. Although the fiber itself is defined over the field k as a whole, its irreducible components might not be. Let $y^2 = ax^2$ with $a^2 \in k$ denote the slope(s) of the tangent lines of the Weierstrass equation at $(0,0)$. Let u be the parameter of $\mathbb{P}^1 = [t : u]$. The first blow-up resolves the node and the exceptional curve intersects the strict transform at $u = \pm\sqrt{a}$. If $n > 1$ we continue blowing up to a second exceptional curve. The exceptional curves are conjugate over $k(\sqrt{a})/k$ if and only if $\sqrt{a} \notin k$. If $\sqrt{a} \in k$ we refer to it as split reduction and otherwise as nonsplit reduction.

The final blow-up of the pencil gives rise to the **Maximal Disjoint Configuration** of a singular fiber (of type) I_n. If $n > 1$, it contains exactly n irreducible components of multiplicity $m = 1$ and self-intersection -2. That is, the components are irreducible over the lowest extension of k where they are defined separately and no more singular point exists within this model.

Example 8.6. Let $k = \mathbb{Q}$. We determine the Kodaira fiber types in the elliptic surface $E_t : y^2 = x^3 + (t-1)^2x^2 + t^2x + t^3$ over $\mathbb{P}^1$. We find that the normal form of above Weierstrass equation[11] is already minimal and minimized with respect to any valuation v_{t-t_0} for some $t_0 \in \mathbb{Q}$. Namely, for the extended Weierstrass equation we have

$$\Delta(E_t) = -16t^3\left(4t^6 - 25t^5 + 46t^4 - 19t^3 + 46t^2 - 25t + 4\right) \tag{9}$$

where all factors are irreducible in $\mathbb{Q}$. In the case $t_0 = 0$ we find $v_t(\Delta) < 12$. Since $v_t(\Delta) = 3 > 0$ and

$$v_t(c_4) = v_t(-432t^4 + 1728t^3 - 1296t^2 + 1728t - 432) = 0$$

we find that E_t has bad multiplicative reduction at $t = 0$. Our computation $v_t(\Delta) = 3$ determines the singular fiber at $t = 0$ is of Kodaira type I_3. The irreducible factor of order 6 in the determinant $\Delta(E_t)$, see Equation (9), admits 6 distinct roots in an extension of $\mathbb{Q}$. This gives exactly 6 fibers of type I_1 in the extension. Lastly, we compute the reduction at $t = \infty$. We make the coordinate change $t = 1/s$ and send $s^3y \mapsto y$ and $s^2x \mapsto x$. This gives a Weierstrass over $\mathbb{Q}(s)$.

$$E_s : y^2 = x^3 + (1-s)^2x^2 + s^2x + s^3$$

We only look at the reduction for $s = 1/t = 0$. Notice that the equation for the fiber at $1/t = 0$ is completely analogous to the equation for the fiber at $t = 0$. Then the singular fiber at $1/t = 0$ is multiplicative and of Kodaira type I_3. ♁

8.4 Extremal rational elliptic surface

8.4.1 Introduction

Let us start by defining the set $R = \{t \in \mathbb{P}^1 : \pi^{-1}(t) \text{ is singular and reducible}\}$ and let m_t denote the number of irreducible components of $\pi^{-1}(t)$ over k. The rank of the Néron-Severi group $\mathrm{NS}(S)$ for an elliptic surface S is bounded from above[12], see [17, p. 42]. In case the surface S is rational, the rank attains this bound and is called **maximal**.

Definition 8.7. A rational elliptic surface $\pi : S \to \mathbb{P}^1$ with section is called **extremal** if $\operatorname{rank} \mathrm{NS}(S) = 2 + \sum_{t \in R}(m_t - 1)$ is maximal and $\varepsilon_\eta(k(t))$ is finite[13].

Consider an extremal rational elliptic surface S with generic fiber ε_η. Since the group of $k(t)$-rational points of the generic fiber is finite, the one-to-one correspondence with the sections of the surface implies the number of sections of the surface must be finite. We know from Proposition 8.4 that sections on an elliptic surface have self-intersection -1 and correspond to exceptional curves in the final blow-up of a pencil of cubics. The total blow-up of the pencil gives rise to a finite number of components naturally. Thus only a finite number of components can have self-intersection less than zero. This line of reasoning leads us to the following proposition.

Proposition 8.8. *Let S be a rational elliptic surface. Then the following are equivalent:*

(a) S is extremal.

[11] Given by $y^2 = x^3 + (-432t^4 + 1728t^3 - 1296t^2 + 1728t - 432)x + (345t^6 - 20736t^5 + 36288t^4 + 8640t^3 + 36288t^2 - 20736t + 3456)$.
[12] The Picard number $\rho(S)$ is equal to the rank of the Néron-Severi group.
[13] That is, $\operatorname{rank}(\varepsilon_\eta(k(t))) = 0$.

(b) The number of representations of S as a blow-up of $\mathbb{P}^2$ is finite.

(c) The number of smooth rational curves C with $C^2 < 0$ is finite.

(d) The number of reduced irreducible curves C with $C^2 < 0$ is finite

Proof. A proof can be found in [13, p. 75], Proposition VIII.1.2.

8.4.2 The Shioda-Tate formula

In this thesis we are interested in constructions of extremal rational elliptic surfaces with semi-stable singular fibers.

Any elliptic surface $\pi : S \to C$ adhere to the Shioda-Tate formula which relates the rank of the Néron-Severi group $\mathrm{NS}(S)$ and the rank of the Mordell-Weil group $\mathrm{MW}(S)$ with the number of components m_t in each singular fiber $E_t \in R$ of the surface. The aim is to find conditions on (the number of components of) fibers in extremal RES using the Shioda-Tate formula. Since the surface of interest is extremal we know that the rank of $\mathrm{MW}(S)$ is zero. Thus, it remains to determine the rank of $\mathrm{NS}(S)$.

For any elliptic surface we know from [17, p. 23], Theorem 6.2 that the Néron-Severi group is finitely generated and torsion-free. This implies $\mathrm{NS}(S)$ is isomorphic with $\mathbb{Z}^n$ for a surface S and n a positive integer. Recall from Theorem 8.2 and Remark 5.4 that given a RES $\pi : S \to \mathbb{P}^1$ over an algebraically closed field, its divisor class group $\mathrm{Cl}(S)$ is isomorphic with $\mathbb{Z}^{10}$. In the case of extremal RES the rank of the Néron-Severi group $\mathrm{NS}(S)$ and the divisor class group $\mathrm{Cl}(S)$ coincide. Therefore, $\mathrm{NS}(S)$ has rank equal to 10.

Theorem 8.9 (Shioda-Tate formula)**.** *Let S be an elliptic surface with section. Then*

$$\operatorname{rank} \mathrm{NS}(S) = 2 + \sum_{t \in R} (m_t - 1) + \operatorname{rank} \mathrm{MW}(S).$$

Proof. A proof can be found in [19, p. 216], Proposition 2.3.

Let $\pi : S \to \mathbb{P}^1$ be an extremal RES. Then $\operatorname{rank} \mathrm{NS}(S) = 10$ and $\operatorname{rank} \mathrm{MW}(S) = 0$ which implies $8 = \sum_{t \in R} (m_t - 1)$. Unfortunately it is not possible to further determine the number of components in each singular fiber, because we have no information on the number of fibers except for the fact that there is a finite number of them. By using theory beyond the scope of this thesis it can be proven that there are at least 4 fibers for extremal RES with only semi-stable fibers.

Theorem 8.10. *Let $\pi : S \to \mathbb{P}^1$ be a non-trivial semi-stable fibration. Then π admits at least 4 singular fibers.*

Proof. The proof uses the result of Theorem 8.9 and can be found in [4, p.103], where it is a proof for the theorem on page 100.

Therefore, in the case of extremal RES with exclusively semi-stable fibers, at least 4 such fibers are on the surface. Since every RES is the blow-up of a pencil in nine base points if $k = \bar{k}$, the semi-stable fibers are of Kodaira type I_n with $1 \leq n \leq 9$ an integer. This can also be seen from the equality $8 = \sum_{t \in R} (m_t - 1)$ and the fact that the number of irreducible components m_t of a fiber of type I_n is equal to n. The constraints that we need in order to determine exactly which configurations can occur are not complete, but in order to complete them we go beyond the scope of this bachelor project. For more information on the constraints, see [17, p. 43]. In total 6 configurations of extremal RES with semi-stable fibers exist that all admit exactly 4 singular fibers, they are given by

$I_9, 3I_1$	$2I_4, 2I_2$
$4I_3$	I_6, I_3, I_2, I_1
$I_8, I_2, 2I_1$	$2I_5, 2I_1.$

The extremal RES with singular fibers of above Kodaira types are unique up to up to isomorphism or coordinate change and have a Mordell-Weil group that is pre-determined.

Theorem 8.11. *For any possible configuration of semi-stable fibers over an algebraically closed field k, there exists a unique extremal rational elliptic surface with pre-determined Mordell-Weil group.*

Proof. A proof can be found in [14, p. 550], Theorem 5.4. ♘

The full torsion structure of the Mordell-Weil group for extremal RES generated by a pencil of cubics defined over the rationals in general is well-known. A list with configurations and the corresponding torsion of the Mordell-Weil group can be found in [16], see 'The list' on pages $7 - 14$. We give the full torsion of the Mordell-Weil group for semi-stable extremal RES, over a finite extension over the rationals where all the sections of the RES are defined, see Table 1. That is, although the pencil of cubics is defined over the rationals, the sections in the Mordell-Weil group are generally not defined over the rationals. In the next chapter we discuss examples of semi-stable extremal RES and their torsion over the rationals.

Fibration	Torsion
$I_9, 3I_1$	$\mathbb{Z}/3\mathbb{Z}$
$I_8, I_2, 2I_1$	$\mathbb{Z}/4\mathbb{Z}$
I_6, I_3, I_2, I_1	$\mathbb{Z}/6\mathbb{Z}$
$2I_5, 2I_1$	$\mathbb{Z}/5\mathbb{Z}$
$2I_4, 2I_2$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$4I_3$	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

Table 1: All extremal RES with semi-stable fibers only and their full torsion.

9 Explicit examples of semi-stable extremal rational elliptic surfaces

9.1 Introduction

In this section we work out examples of extremal rational elliptic surfaces with semi-stable fibers from Beauville's 'The stable families of elliptic curves over $\mathbb{P}^1$ with four singular fibers' [5] and some of our own examples. We work out the blow-up of the examples to a Maximal Disjoint Configuration (MDC), such that all components are irreducible over $\mathbb{Q}$ and MDC defined over $\mathbb{Q}$ as a whole.

The blow-up of the fibers is shown separately each time, but in reality all fibers are connected via the sections that are shown in red in the final blow-up. In every step we blow-up at most once in every base point. The black lines denote the original irreducible components and the blue lines the components obtained through the blow-up map. Red lines denote the curves with self-intersection -1, all other components have self-intersection -2. The fibers as a whole have self-intersection 0.

We choose to work out the reducible fibers only since they provide a more interesting case. All irreducible singular fibers are of type I_1, nodal curves with base points outside the singular point and of order 1. The self-intersection of a cubic is 9 and thus the nine times blow up provides us a curve with self-intersection 0, see Corollary 5.8. For the same reason nonsingular fibers in a RES have self-intersection 0.

9.2 Pencil of cubics with fibers $I_9 3I_1$

The example corresponding to the fibers $I_9 3I_1$ given in [5] is given by the pencil

$$S_1 : x^2y + y^2z + z^2x + txyz = 0$$

where t parameterizes $\mathbb{P}^1$. The base points of this pencil are given by $p_0 = [0 : 1 : 0]$, $p_1 = [1 : 0 : 0]$ and $p_2 = [0 : 0 : 1]$ all with multiplicity 3. Since these are all rational points, the blow-ups of the pencil are defined over $\mathbb{Q}$ as a whole[14]. The surface S_1 has bad places at $1/t$, $t - 1$ and $t^2 - 3t + 9$. Only one of the fibers is reducible i.e. has Kodaira type I_n with $n > 1$. The reducible fiber is the blow-up at $t = \infty$. The final blow up or MDC contains 3 sections defined over $\mathbb{Q}$, see Figure 14. This corresponds to full torsion of the Mordell-Weil group $\mathrm{MW}(S_1)$ over $\mathbb{Q}$, which is $\mathbb{Z}/3\mathbb{Z}$, see 1.

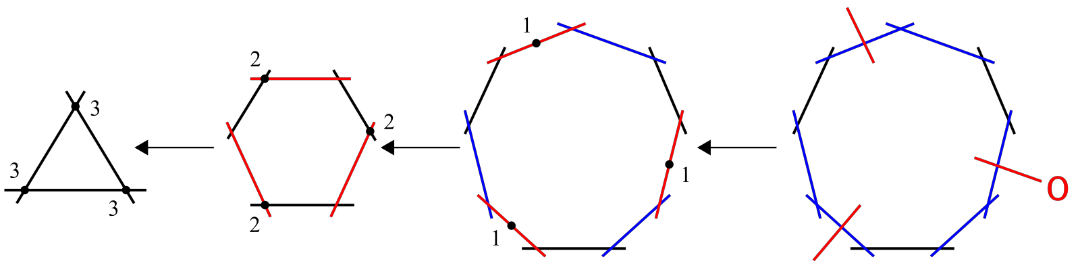


Figure 14: The fiber I_9 at $1/t$ on S_1 and drawn over $\mathbb{Q}$. We blow up from left to right, the rightmost picture gives us the MDC of the pencil.

9.3 Pencil of cubics with fibers $2I_4 2I_2$

The example corresponding to the fibers $2I_4 2I_2$ given in [5] is given by the pencil

$$S_2 : x(x^2 + z^2 + 2yz) + t(x + y)(x - y)z = 0.$$

The base points of this pencil are given by $p_0 = [0 : 1 : 0]$ with multiplicity 3 and $p_1 = [1 : -1 : 1]$, $p_2 = [-1 : -1 : 1]$, $p_3 = [0 : 0 : 1]$ all with multiplicity 2. Therefore all blow-ups of the pencil are defined over $\mathbb{Q}$ as a whole. The surface S_2 has singular fibers at $1/t$, t, $t - 1$ and $t + 1$ that are of Kodaira fiber type I_4, I_4, I_2 and I_2 respectively. All of the

[14] A line is uniquely defined by two points. If the points are both rational, the line is defined over the rationals.

fibers are reducible and the I_2 fibers are very similar, therefore we only draw one of them. The MDC contains 4 sections defined over $\mathbb{Q}$, see Figure 15. However, this does not correspond to full torsion: from Table 1 we learn that $\mathrm{MW}(S_2)$ has torsion group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and thus if the pencil of cubics given in [5] has full torsion Mordell-Weil group over the rationals, exactly 8 sections exist. From the group structure of the sections, see [11, p. 4], we derive that the example has full torsion Mordell-Weil group over the rationals. Hence, exactly four sections are 'hiding' somewhere in the surface.

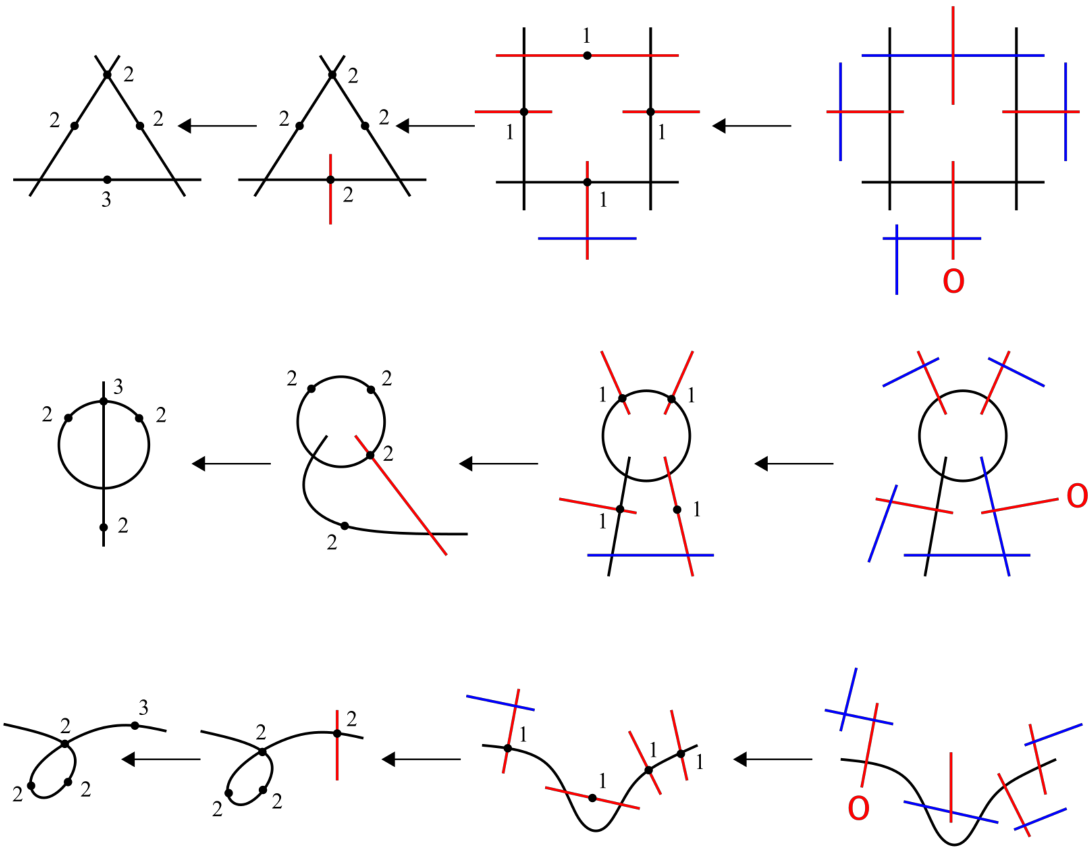


Figure 15: The fibers I_4, I_4 and I_2 at $1/t$, t and $t+1$ or $t-1$ respectively, on S_2 and drawn over $\mathbb{Q}$.

The first three 'hidden' section are all lines, each of them defined by two points in the set $\{p_0, p_1, p_2\}$. The fourth section is a conic passing through p_0, p_1, p_2 with multiplicity 1 and through p_3 with multiplicity 2. The equations of the lines and conic are necessarily irreducible and given by

$$y + z = 0, \qquad x + z = 0, \qquad x - z = 0, \qquad x^2 + yz = 0.$$

For the drawn blow-ups of the hidden sections, see Appendix A.

9.4 Pencil of cubics with fibers $4I_3$

The example in [5] is a pencil of cubics known as the Hesse pencil, see [3]. The equation for this pencil is given by

$$S_3 : x^3 + y^3 + z^3 + txyz = 0.$$

There are nine base points of order 1 on this pencil. The base points of this pencil are given by

$$\begin{aligned} p_0 &= [0, 1, -1] & p_1 &= [0, 1, -\zeta] & p_2 &= \left[0, 1, -\zeta^2\right] \\ p_3 &= [1, 0, -1] & p_4 &= \left[1, 0, -\zeta^2\right] & p_5 &= [1, 0, -\zeta] \\ p_6 &= [1, -1, 0] & p_7 &= [1, -\zeta, 0] & p_8 &= \left[1, -\zeta^2, 0\right] \end{aligned} \tag{10}$$

where $\zeta = (-1+\sqrt{-3})/2$. The base points are not defined over $\mathbb{Q}$, but rather over a quadratic extension[15] $\mathbb{Q}(\sqrt{-3})$. The surface S_3 has singular fibers at $1/t$, $t+3$, $t+3\zeta$ and $t+3\zeta^2$ all corresponding to Kodaira fiber types I_3. In total this gives 4 triangles and 12 edges. The edges are given by the following equations.

$$\begin{aligned}
E_\infty: &\quad xyz = 0\\
E_{-3}: &\quad (x+y+z)\left(x+\zeta y+\zeta^2 z\right)\left(x+\zeta^2 y+\zeta z\right) = 0\\
E_{-3\zeta}: &\quad (x+\zeta y+z)\left(x+\zeta^2 y+\zeta^2 z\right)(x+y+\zeta z) = 0\\
E_{-3\zeta^2}: &\quad \left(x+\zeta^2 y+z\right)(x+\zeta y+\zeta z)\left(x+y+\zeta^2 z\right) = 0.
\end{aligned}$$

Note that the non-trivial element of the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$, which is given by $\sigma(\sqrt{-3}) = -\sqrt{-3}$, permutes two of the edges in E_{-3} and it permutes all components of the triangles $E_{-3\zeta}$, $E_{-3\zeta^2}$[16]. All intersections are preserved under the permutations. In the triangles, every edge contains exactly three base points, the distribution of these points for every fiber is as follows. For E_∞ each row in Equation (10) corresponds to the base points on an edge, for E_{-3} each column, for $E_{-3\zeta}$ each diagonal pointing north-east and for $E_{-3\zeta^2}$ each diagonal pointing north-west.

The MDC of the pencil of cubics (see Figure 16) contains 9 sections and this corresponds well with the full torsion of $\mathrm{MW}(S_3)$ over $\mathbb{Q}(\sqrt{-3})$, which is given by $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. The torsion of the Mordell-Weil group over the rationals is $\mathbb{Z}/3\mathbb{Z}$, and these are the sections defined over $\mathbb{Q}$ corresponding to the points p_0, p_3, p_6.

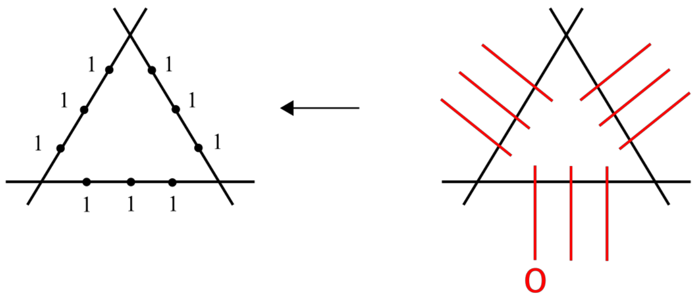


Figure 16: The fiber I_3 at $1/t$, $t+3$, $t+3\zeta$ or $t+3\zeta^2$ on S_3 and drawn over $\mathbb{Q}(\sqrt{-3})$.

The Hesse pencil is the only example in [5] where the base points, and for that matter the sections, are not defined over $\mathbb{Q}$. This raises the question if there exists an example of a pencil with all base points defined over $\mathbb{Q}$ that gives rise to a $4I_3$ configuration. We argue there is no such pencil using the following theorem.

Theorem 9.1 (Specialization theorem)**.** *Let $\pi: S \to \mathbb{P}^1$ be a non-isotrivial[17] rational elliptic surface defined over $\mathbb{Q}$. Then for all but finitely many $t \in \mathbb{P}^1(\mathbb{Q})$, the fiber E_t is a non-singular elliptic curve, and there is a specialization homomorphism*

$$\sigma_t : \varepsilon_\eta\left(\mathbb{Q}(t)\right) \to E_t(\mathbb{Q})$$

mapping the group of sections defined over $\mathbb{Q}$ to the group of rational points on the fiber. The specialization map σ_t is injective for all but finitely many $t \in \mathbb{P}^1(\mathbb{Q})$.

Proof. For the proof, see [15].

From Mazur's theorem, see Theorem 6.12, we argue that there exist no elliptic curves $E/\mathbb{Q}$ where $E(\mathbb{Q})$ is of rank zero and has full torsion group $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ over the rationals. We argue by contraposition using Theorem 9.1. We view generic torsion of an elliptic fiber as the torsion of the $\mathbb{Q}(t)$-rational points of the generic fiber as an elliptic curve over $\mathbb{Q}(t)$. Note that σ_t is a group homomorphism with respect to the abelian structure of torsion points. Thus, by injectivity of the map if no elliptic curves with rank zero and $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$-torsion exist over $\mathbb{Q}$ in the image of σ_t, we can not find an extremal RES with generic torsion $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ over the function field $\mathbb{Q}(t)$.

The relation between the generic fiber and nonsingular fibers on the surface in Theorem 9.1 provides us with a relation between the rank of the generic fiber and the rank of rational points of the nonsingular fibers as elliptic curves over $\mathbb{Q}$.

[15] Eisenstein's criterion shows x^2+3 is irreducible over $\mathbb{Q}$, $\pm\sqrt{-3}$ are its roots which give rise to the quadratic field extension $\mathbb{Q}\left(\sqrt{-3}\right)$.

[16] Using Equation (11) and Figure 20 in the next chapter, we denote the permutations by $C_1^{-3} \leftrightarrow C_2^{-3}$ and $C_i^{-3\zeta} \leftrightarrow C_i^{-3\zeta^2}$.

[17] Not isomorphic to a constant curve over a finite extension of $\mathbb{Q}$.

Due to the injectivity of σ_t for all but finitely many t, we find that the rank of nonsingular fibers $E_t(\mathbb{Q})$ as elliptic curves over the rationals, is greater or equal than the generic rank r of the surface. In other words, the generic rank is bounded from above by the rank of nonsingular fibers in the surface. Therefore, whenever an elliptic curve is of rank 0 and appears as a nonsingular fiber in a RES, the surface must be extremal.

Example 9.2. We use 'The L-functions and modular forms database' with URL http://www.lmfdb.org to give us an elliptic curve over the rationals with rank 0. This yields the cubic curve in $\mathbb{P}^2$ defined by $F(x,y,z) = y^2z + yz^2 - x^3 - x^2z + 769xz^2 - 8470z^3$ [18]. Using the construction in [2, p. 2] we construct a Hesse pencil $F + tH = 0$ where $H(F)$ denotes the Hessian of F. From the matrix

$$\begin{aligned}
&\frac{\partial^2 F}{\partial x^2} = -6x - 2z && \frac{\partial^2 F}{\partial x \partial y} = 0 && \frac{\partial^2 F}{\partial x \partial z} = -2x + 1538 \\
&\frac{\partial^2 F}{\partial y \partial x} = 0 && \frac{\partial^2 F}{\partial y^2} = 2z && \frac{\partial^2 F}{\partial y \partial z} = 2y + 2z \\
&\frac{\partial^2 F}{\partial z \partial x} = -2x + 1538 && \frac{\partial^2 F}{\partial z \partial y} = 2y + 2z && \frac{\partial^2 F}{\partial z^2} = 2y + 1538x - 50820z
\end{aligned}$$

we compute the Hessian, which is given by

$$H(F) = -18464x^2z + 24xy^2 + 24xyz + 616016xz^2 + 8y^2z + 8yz^2 - 4527600z^3.$$

The pencil of cubics $S : F + tH = 0$ generates a semi-stable extremal RES with configuration $4I_3$. The point $[0 : 1 : 0]$ is the only base point defined over the rationals. As a result the zero section $\mathcal{O}$ is the only section defined over $\mathbb{Q}$, meaning the torsion group $\mathrm{MW}(S)$ over the rationals is trivial. ♗

9.5 Pencil of cubics with fibers $I_6 I_3 I_2 I_1$

The example corresponding to the fibers $I_6 I_3 I_2 I_1$ given in [5] is given by the pencil

$$S_4 : (x + y)(x + z)(y + z) + txyz = 0.$$

The base points of this pencil are $p_0 = [0 : 1 : 0]$, $p_1 = [0 : 0 : 1]$ and $p_2 = [1 : 0 : 0]$ with order 2 and $p_3 = [0 : -1 : 1]$, $p_4 = [-1 : 1 : 0]$ and $p_5 = [-1 : 0 : 1]$ with order 1. The base points are all defined over $\mathbb{Q}$ and thus the MDC of the pencil must defined over the rationals. The surface S_4 has singular fibers at $1/t$, t, $t - 1$ and $t + 8$ that are of Kodaira fiber type I_6, I_3, I_2 and I_1 respectively. The MDC contains 6 sections over the rationals, see Figure 17, which corresponds to full torsion of the Mordell-Weil group $\mathrm{MW}(S_4)$ over $\mathbb{Q}$, given by $\mathbb{Z}/6\mathbb{Z}$.

[18] This is $19.a1$ in the data base.

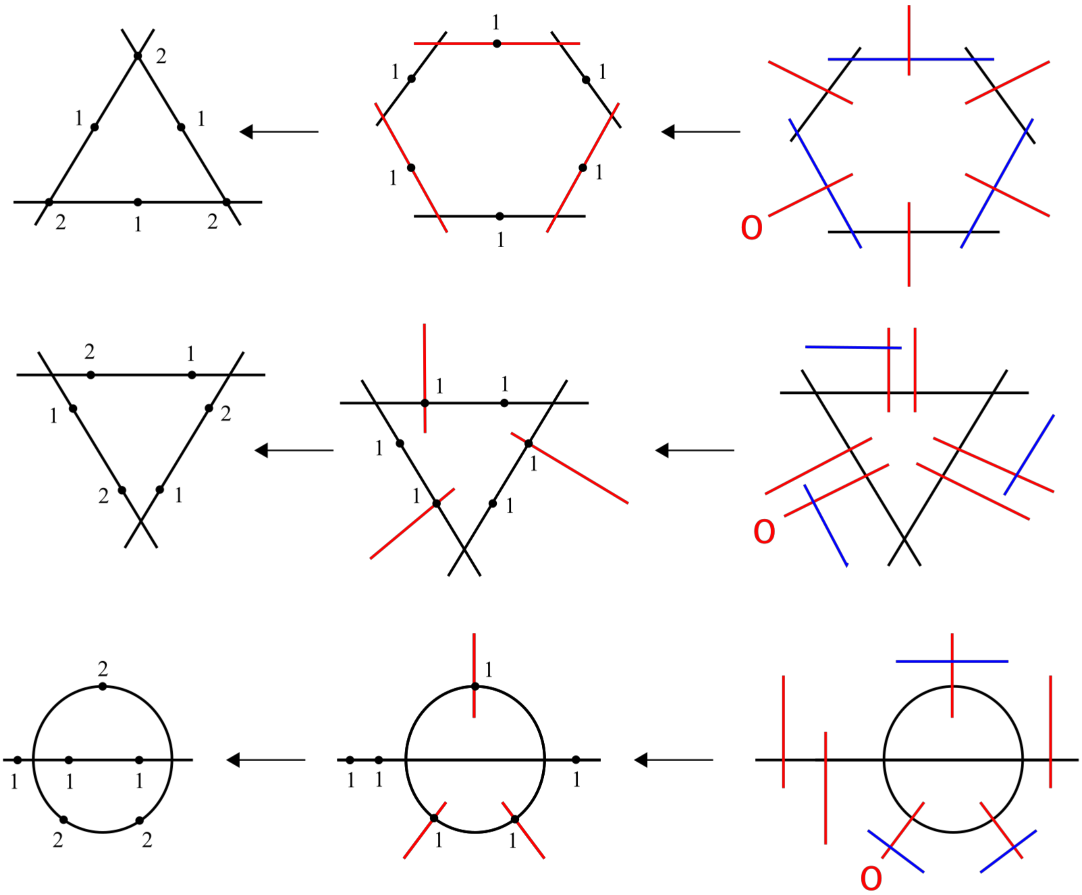


Figure 17: The fibers I_6, I_3 and I_2 at $1/t$, t and $t-1$ respectively, on S_4 and drawn over $\mathbb{Q}$.

9.6 Pencil of cubics with fibers $I_8I_22I_1$

The example corresponding to the fibers $I_8I_22I_1$ given in [5] is given by the pencil

$$S_5 : (x+y)(xy-z^2)+txyz = 0.$$

The base points of this pencil are $p_0 = [0:1:0]$, $p_1 = [1:0:0]$ with order 3, $p_2 = [0:0:1]$ with order 2 and $p_3 = [-1:1:0]$ with order 1. The base points are all defined over $\mathbb{Q}$ and so are the blow-ups of the pencil. The surface S_5 has singular fibers at $1/t$, t and t^2+16 that are of Kodaira fiber type I_8, I_2, and $2I_1$ respectively. We only draw the first two types. The MDC contains 4 sections over $\mathbb{Q}$ (see Figure 18) which corresponds to full torsion of the Mordell-Weil group $\mathrm{MW}(S_4)$ over the rationals, given by $\mathbb{Z}/4\mathbb{Z}$.

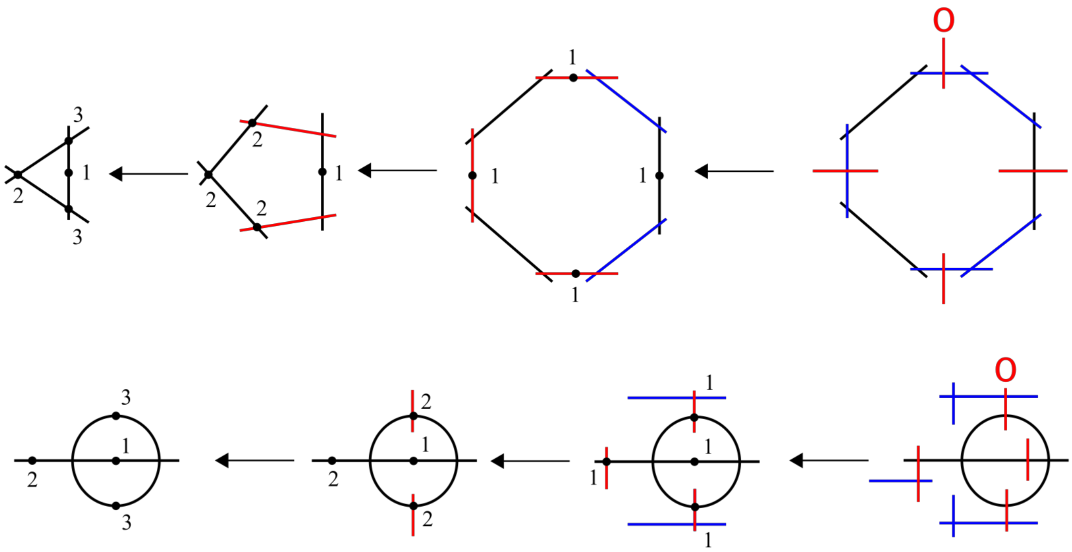


Figure 18: The fibers I_8 and I_2 at $1/t$ and t respectively, on S_5 and drawn over $\mathbb{Q}$.

9.7 Pencil of cubics with fibers $2I_5 2I_1$

The example corresponding to the fibers $2I_5 2I_1$ given in [5] is given by the pencil

$$S_6 : x(x-z)(y-z) + t(x-y)yz = 0.$$

The base points of this pencil are $p_0 = [1:0:1]$ with order 1 and $p_1 = [0:1:0]$, $p_2 = [1:0:0]$, $p_3 = [0:0:1]$ $p_4 = [1:1:1]$ with order 2. The base points are all defined over $\mathbb{Q}$ and therefore the blow-up of the pencil is defined over the rationals. The surface S_6 has singular fibers at $1/t$, t and $t^2 - 11t - 1$ that are of Kodaira fiber type I_5, I_5, and $2I_1$ respectively. The MDC contains 5 sections, see Figure 19 over $\mathbb{Q}$ which corresponds to full torsion of the Mordell-Weil group $\mathrm{MW}(S_6)$ over the rationals given by $\mathbb{Z}/5\mathbb{Z}$.

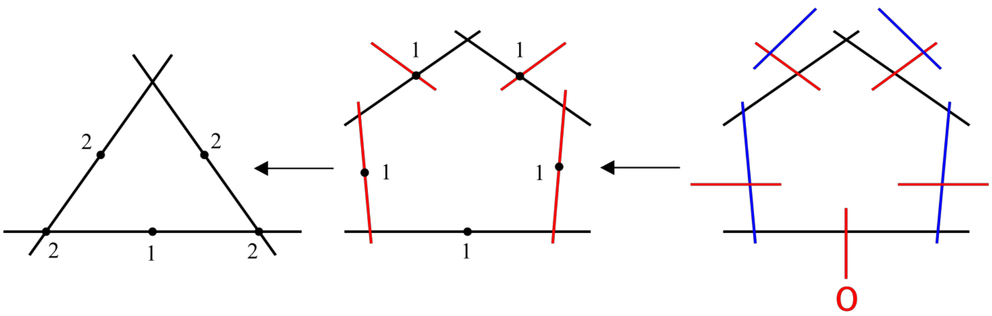


Figure 19: The fiber I_5 at $1/t$ or t on S_6 and drawn over $\mathbb{Q}$.

We give an example of a pencil of cubics that defines an extremal RES with configuration $2I_5 2I_2$ and trivial torsion over $\mathbb{Q}$.

Example 9.3. Consider the pencil of cubics defined over $\mathbb{Q}$ and given by

$$S : x(y^2 + x^2 - 2xz + z^2) + tz(y^2 + x^2 + 2xz + z^2).$$

The base points of this pencil are $p_0 = [0:1:0]$ with order 1 and $p_1 = [0:1:i]$, $p_2 = [0:-1:i]$, $p_3 = [i:1:0]$ $p_4 = [i:-1:0]$ with order 2, defined over the quadratic extension $\mathbb{Q}(i)$. The surface S has singular fibers at $1/t$, t and $t^2 - 11t - 1$ that are of Kodaira fiber type I_5, I_5, and $2I_1$ respectively. Of all sections only the zero section $\mathcal{O}$ is defined over the rationals meaning the torsion of the Mordell-Weil group $\mathrm{MW}(S)$ is trivial over the rationals. The non-trivial element of the Galois group $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ given by $\sigma(i) = -i$ permutes four of the components in each 5-gon. Using Equation (11) and Figure 20 in the next chapter, we denote the permutations for each 5-gon by $C_1 \leftrightarrow C_4$ and $C_2 \leftrightarrow C_3$. ♙

10 Description of the minimal models of semi-stable extremal rational elliptic surfaces

10.1 Introduction

Every rational elliptic surface over a field k is isomorphic to the blow-up in nine base points of a pencil of cubics if k is algebraically closed. In this chapter we discuss the reverse process of blowing up base points in pencils that are defined over the rationals. We know from Castelnuovo's criterion, see Theorem 10.1, that (-1)-curves isomorphic with $\mathbb{P}^1$ on a rational elliptic surface are exceptional curves of blow-ups in points on the surface. Therefore, such (-1)-curves can always be contracted.

Theorem 10.1 (Castelnuovo)**.** *If Y is a curve on a surface S, with $Y \cong \mathbb{P}^1$ and $Y^2 = -1$, then there exists a morphism $\varphi : S \to \widetilde{S}$ to a (nonsingular projective) surface $\widetilde{S}$, and a point $P \in \widetilde{S}$, such that S is isomorphic via φ to the blow-up of $\widetilde{S}$ with center P, and Y is the exceptional curve.*

Proof. See Theorem 5.7 in [9, p. 414] for the proof. ◻

Reversing the process of blowing up base points in a pencil is done by blowing down all possible (-1)-curves. We reach the minimal model of an elliptic surface with respect the field $\mathbb{Q}$, if no blow-down of (-1)-curves can be made over the field. However, the rationals are not algebraically closed and the conjugacies that exist between components restricts the number of ways we can blow down. If two (-1)-curves are conjugate in a surface, they can only be blow down if they do not intersect, since blowing down only one of the curves decreases the self-intersection of the other. We can only blow down (-1)-curves and so this scenario causes the whole model to not be defined over $\mathbb{Q}$. If two (-1)-curves are conjugate and do not intersect, they are blown down simultaneously for the same reason. This can be done since they do not intersect.

In this chapter we look at minimal models of pencils in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. The minimal model of a surface is useful to us since more is known about projective 2 and 1×1-space than about the surface the MDC of the pencil is defined over. In the case of extremal rational elliptic surfaces, Proposition 8.8 tells us there are a finite number of possibilities to blow down an MDC to the minimal model of an extremal RES. In this section we work out the contractions given on pages $9-10$ in the paper 'Fields of definition of elliptic fibrations on covers of certain extremal rational elliptic surfaces' by Victoria Cantoral-Farfán et al [6]. It is our aim to explain and construct the proof of the following proposition with the help of figures.

Proposition 10.2. *Let S be a semi-stable extremal rational elliptic surface defined over k and m the order of the Mordell-Weil group. Then the following hold.*

(i) If m is odd and S has a unique reducible fiber then S admits a contraction over k to $\mathbb{P}^1 \times \mathbb{P}^1$.

(ii) If m is odd and S has at least two reducible fibers then S admits a contraction over k to $\mathbb{P}^2$.

(iii) If m is even then S admits a contraction over k to $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. See proof of Proposition 4.7 in [6, p. 8-10]. ◻

In the contractions we consider the irreducible components. Separately the components are not necessarily defined over the rationals but rather over a finite extension. Recall components are conjugate when the reduction of the nodal curve in Weierstrass form is non-split. As a result the blow-down of (-1)-curves needs to satisfy the following restrictions. Galois actions preserve intersection of components and when we blow-down a component not defined over $\mathbb{Q}$, we also need to blow-down its conjugate component. As a rule we choose $\mathcal{O}$ so that it is defined over the smallest field extension of $\mathbb{Q}$. We label the irreducible components of a MDC fiber of type I_n in clockwise direction $C_0, C_1, \dots, C_{n-1}$ such that

$$(\mathcal{O}.C_0) = 1 \qquad C_i^2 = -2 \qquad (C_i.C_j) = 1 \text{ if and only if } |i-j| = 1. \tag{11}$$

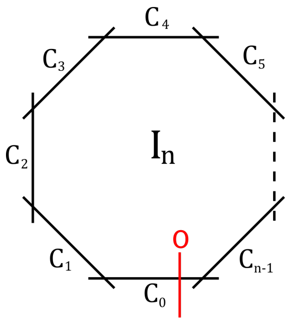


Figure 20: Labelling irreducible components of an MDC of type I_n.

For each semi-stable surface we show the contraction on sections of the bad fiber I_n such that n is the highest integer occurring in the Kodaira types of the surface. The first blow-down always concerns the sections since they are the only curves with self-intersection -1 in the MDC. Note that contracting often affects the components in other bad fibers on the surface and we take this into account in our text. In order to keep the figures comprehensible we link colors to the self-intersection of curves.

Blue lines have self-intersection -2, red lines -1, green lines 0 and black lines 1. Moreover, dashed red lines are exceptional curves that we blow-down each transformation.

10.2 Blow-down for the configuration $I_9 3I_1$

First we contract all sections simultaneously, see Figure 21. This can be done since all sections are disjoint. The only (-1)-curves this yields are the images of C_0, C_3 and C_6. We point out that the images of C_3, C_6 may be conjugate. In the second blow-down we contract all three (-1)-curves. In the third blow-down there are two possibilities depending on where the components are defined separately. If they are all defined over the rationals we contract the images of C_1, C_4 and C_7. We obtain a bad fiber of three curves with self-intersection 1 in the minimal model, this gives $\mathbb{P}^2$. However, the images of C_i and C_{9-i} can be conjugate and so we propose blowing down the images of C_2, C_7 instead. We reach four curves with self intersection 0, this gives a pencil in $\mathbb{P}^1 \times \mathbb{P}^1$, see Figure 21. Since $I_9 3I_1$ has torsion equal to $\mathbb{Z}/3\mathbb{Z}$ and a unique reducible fiber, this is in accordance with Proposition 10.2.

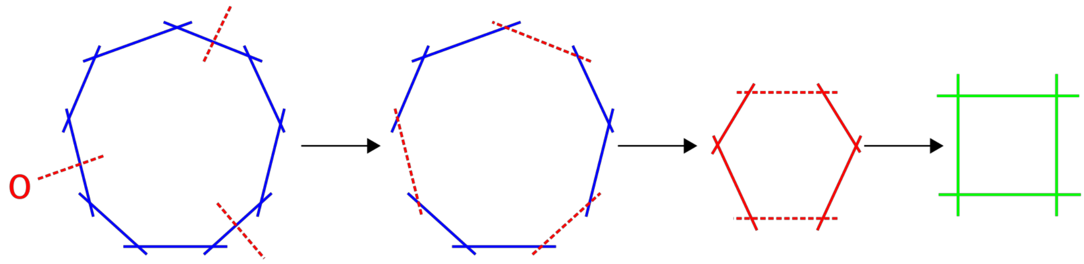


Figure 21: The contractions on the fiber of type I_9.

The other bad places $3I_1$ are nodal curves with self-intersection 0. After contracting the sections and all other images of components that obtain self-intersection -1, the images of the nodal curves have self-intersection 8. This does not make sense for an irreducible cubic in $\mathbb{P}^2$, but this is possible for the space $\mathbb{P}^1 \times \mathbb{P}^1$.

10.3 Blow-down for the configuration $2I_4 2I_2$

We contract all eight sections simultaneously in the fiber I_4, this give us a pencil in $\mathbb{P}^1 \times \mathbb{P}^1$. Since the fibers of type I_4 both intersect all sections once, the contractions are analogous. Since $2I_4 2I_2$ has torsion equal to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, this is in line with Proposition 10.2. For the bad fiber I_2 we find that contracting all sections gives two curves intersecting in two points, both with self-intersection 2. No other contractions can be made.

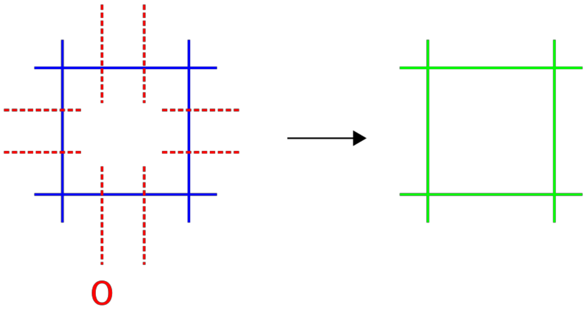


Figure 22: The contractions on a fiber of type I_4.

10.4 Blow-down for the configuration $4I_3$

Contracting all sections gives a $\mathbb{P}^2$ in all bad fibers, namely three lines with self-intersection 1. This is in accordance with Proposition 10.2.

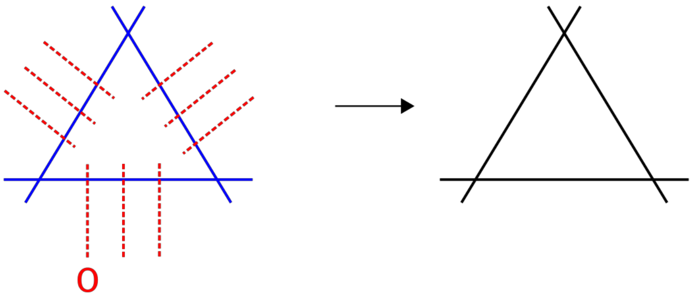


Figure 23: The contractions on a fiber of type I_3.

10.5 Blow-down for the configuration $I_6I_3I_2I_1$

First we contract all sections in the bad fiber I_6. The images of all C_i obtain self-intersection -1. Again, if no conjugate components exist we can contract the images of C_0, C_2 and C_4 to a pencil in $\mathbb{P}^2$ over the rationals, see Figure 24. However, if at least one component is not defined over the rationals separately, we contract the images of C_0 and C_3. Namely, the possible conjugacies that can occur are between C_i and C_{6-i}. We obtain a pencil in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $I_6I_3I_2I_1$ has torsion equal to $\mathbb{Z}/6\mathbb{Z}$, this is in accordance with Proposition 10.2.

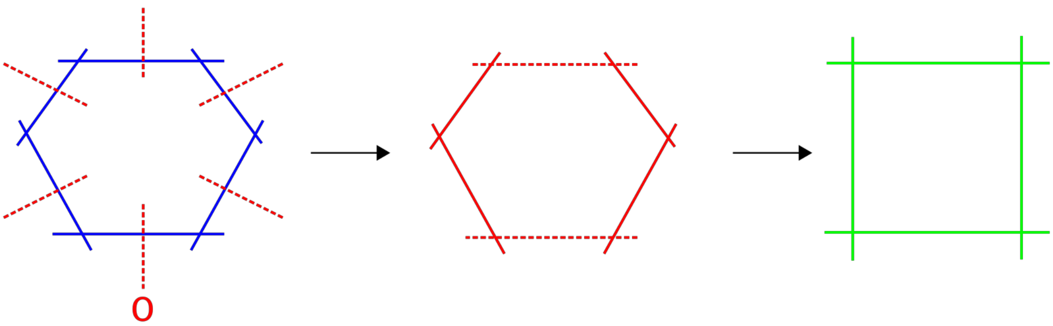


Figure 24: The contractions on the fiber of type I_6.

To verify the contractions for all reducible bad fibers we use the MDC in Figure 25. First of all we look at the contractions on the bad fiber of type I_3. Contracting yields one curve with self-intersection 0 and two with self-intersection 1. For the bad fiber I_2 this yields two components with self-intersection 2 and for I_1 it yields a curve with self-intersection 8. Note that the minimal models of these contractions exist in $\mathbb{P}^1 \times \mathbb{P}^1$ but not in $\mathbb{P}^2$.

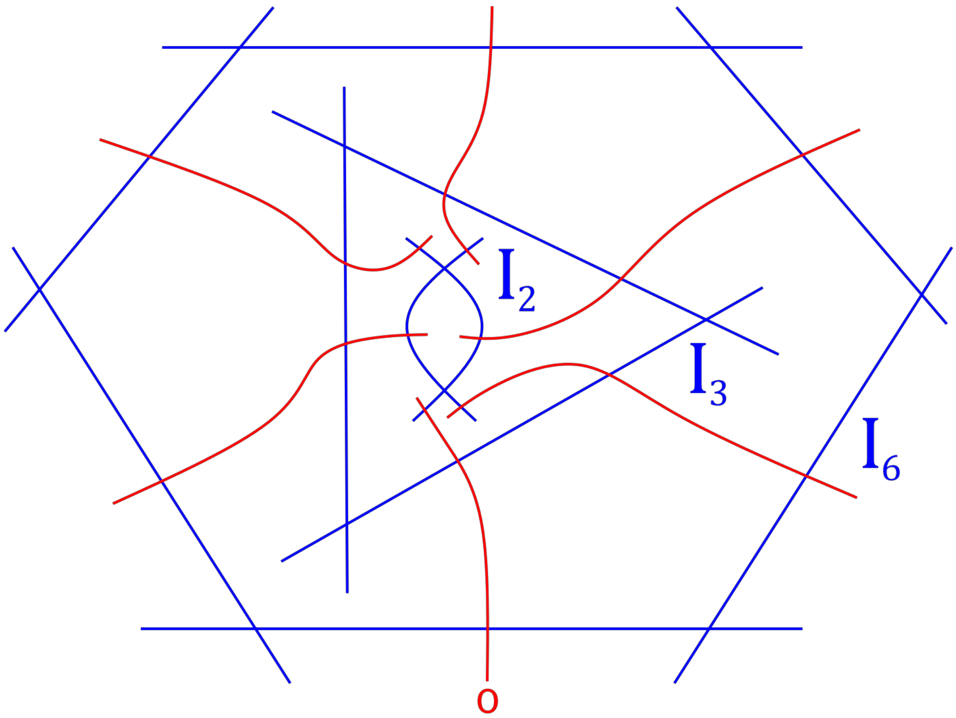


Figure 25: The complete MDC intersection pattern of the sections with the reducible fibers I_6, I_3 and I_2.

10.6 Blow-down for the configuration $I_8I_22I_1$

Again we contract all sections. As a result the images of C_0, C_2, C_4 and C_6 have self-intersection -1 after the first blow-down. If no conjugate components exist we can contract the images of C_1, C_3, C_6 (or equivalently the images of C_2, C_5, C_7) to a pencil in $\mathbb{P}^2$. In case we do not know, possible conjugacies that can occur are between C_i and C_{8-i}. We contract C_2 and C_6 which are possibly conjugate. This gives four lines with self-intersection 0 intersecting in four points in $\mathbb{P}^1 \times \mathbb{P}^1$, see Figure 26. Since $I_8I_22I_1$ has torsion equal to $\mathbb{Z}/4\mathbb{Z}$, this is in accordance with Proposition 10.2.

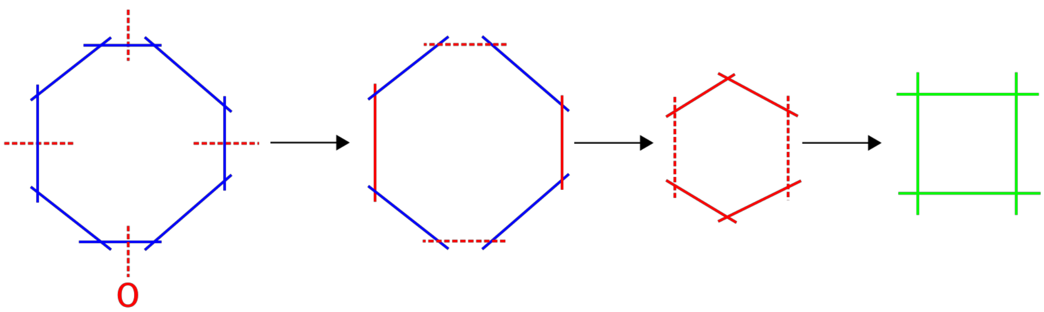


Figure 26: The contractions on the fiber of type I_8.

To verify the contractions for all reducible bad fibers we use the MDC in Figure 27. In the bad fiber I_2 the contractions yield two curves intersecting in two places both with self-intersection 2. For the $2I_1$ this yields two curves both with self-intersection 8.

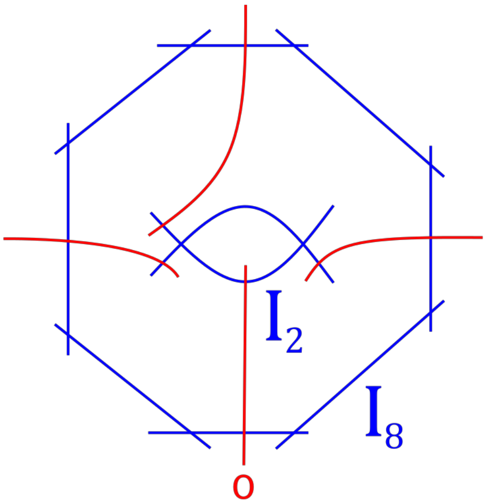


Figure 27: The complete MDC intersection pattern of the sections with the reducible fibers I_8 and I_2.

10.7 Blow-down for the configuration $2I_5 2I_1$

First we contract all sections. The images of C_i after the first blow-down have self-intersection -1 as a result. The possible conjugacies that can occur are between C_i and C_{5-i}. Next we contract the components C_1, C_4 (or equivalently the components C_2, C_3) in both I_5. We claim that this yields three lines with self-intersection 1 intersecting in three points. This gives a pencil in $\mathbb{P}^2$. Note that this is not straightforward from Figure 28. Therefore, we add a figure with the drawing of both I_5 and the sections between them, see Figure 29. Since $2I_5 2I_1$ has torsion equal to $\mathbb{Z}/5\mathbb{Z}$ and two reducible fibers, this is in accordance with Proposition 10.2.

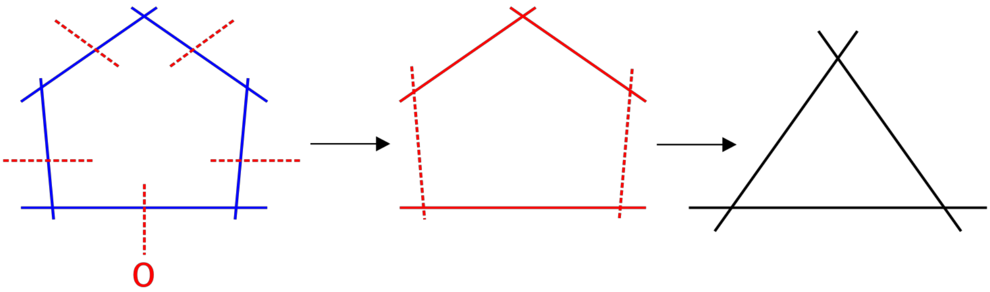


Figure 28: The contractions on a fiber of type I_5.

We draw two bad fibers simultaneously in MDC, both with components in blue and sections in red. Note that the components do not intersect outside their fiber, only the sections do. The bad fibers of type I_1 have self-intersection 0. After contracting nine times (five sections and four components, two in each I_5) have self-intersection 9, satisfying Remark 5.10.

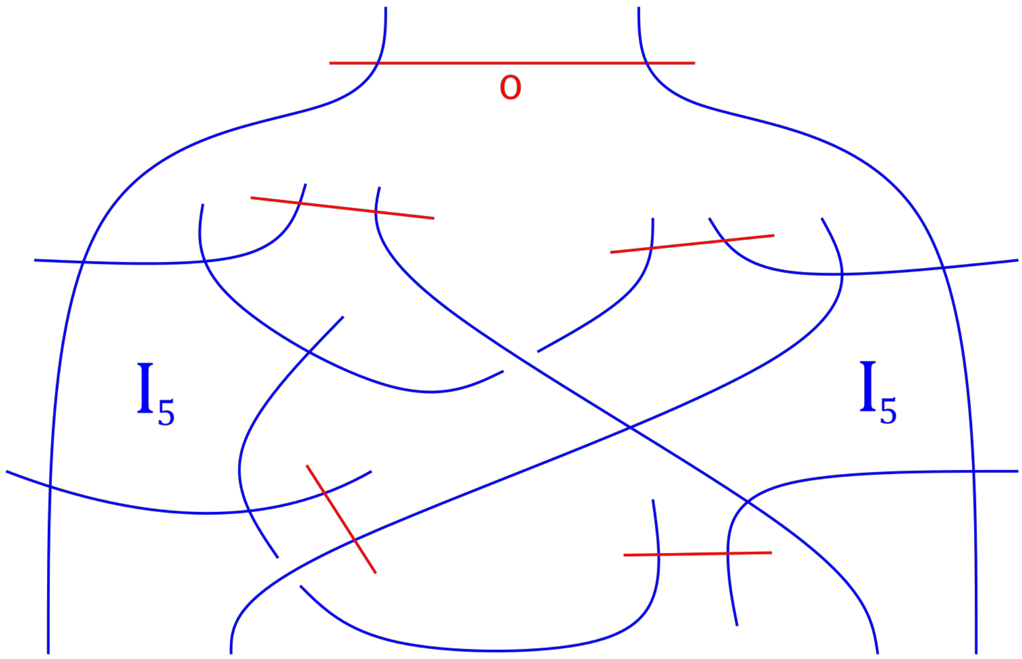


Figure 29: The complete MDC intersection pattern of the sections with the reducible fibers I_6, I_3 and I_2.

Remark 10.3. The converse implications in Proposition 10.2 are not always true. As we discussed in the cases $I_9 3I_1$, $I_8 I_2 2I_1$ and $I_6 I_3 I_2 I_1$, if all reduction is split we can find contractions to both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

11 Discussion

In Chapter 9 of this thesis one finds the geometric constructions of semi-stable extremal rational elliptic curves in [5] via examples. In this chapter we generate the Maximal Disjoint Configuration (MDC) of a pencil in $\mathbb{P}^2$ over the rationals, by blowing up the base points of a specific pencil of cubics. Studying specific pencils of cubics is of a general importance, since two distinct pencils in $\mathbb{P}^2$ with the same configuration of Kodaira fiber types have an isomorphic MDC. As a result, the minimal models of the pencils have the same number of base points including multiplicity in $\mathbb{P}^2$. The figures therefore describe the blow-up in the base points of any pencil of cubics with the same configuration of semi-stable fibers.

In the same chapter we moreover discuss the torsion of the Mordell-Weil group in each example. We find by Mazur's Theorem, see Theorem 6.12, that the only surface where there exists no example with full torsion defined over $\mathbb{Q}$, is $4I_3$. For this configuration we use the Specialization Theorem, see Theorem 9.1, and the vast amount of knowledge that exists on elliptic curves, to generate another example of a pencil with trivial torsion over the rationals, see Example 9.2. In Chapter 10 one can find the blow-downs of all semi-stable extremal rational elliptic curves in MDC to a minimal model of the surface in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In the chapter we work out the constructions given in the proof of Proposition 4.7 in [6], see Proposition 10.2, with the help of figures.

Suggestions for further research include studying and developing a computation method to rewrite an elliptic curve with Weierstrass normal form $y^2 = x^3 + A(t)x + B(t)$ to a pencil of cubics $F + tG = 0$. The method of reduction for Weierstrass normal form is well-known and this would enable us to construct rational elliptic surfaces with specific reduction and torsion over the rationals. Moreover, the structure of the pencil of cubics allows us to study the conjugacy of components in case the reduction is nonsplit. We aim to construct examples where the Mordell-Weil group of semi-stable extremal rational elliptic surfaces is defined over a number field.

A Blow-ups of hidden sections $2I_4 2I_2$

Underneath you find for each 'hidden' section in the pencil given by $S_2 : x(x^2+z^2+2yz)+t(x+y)(x-y)z = 0$ the blow-ups of each singular fiber. The pencil has fibers of Kodaira type $2I_4 2I_2$. Depending on the section we draw the fibers of type I_2 at $t \pm 1$ only once or twice. In case they are drawn only once for a section, the way the sections intersect the components of the fiber are analogous for both I_2 fibers.

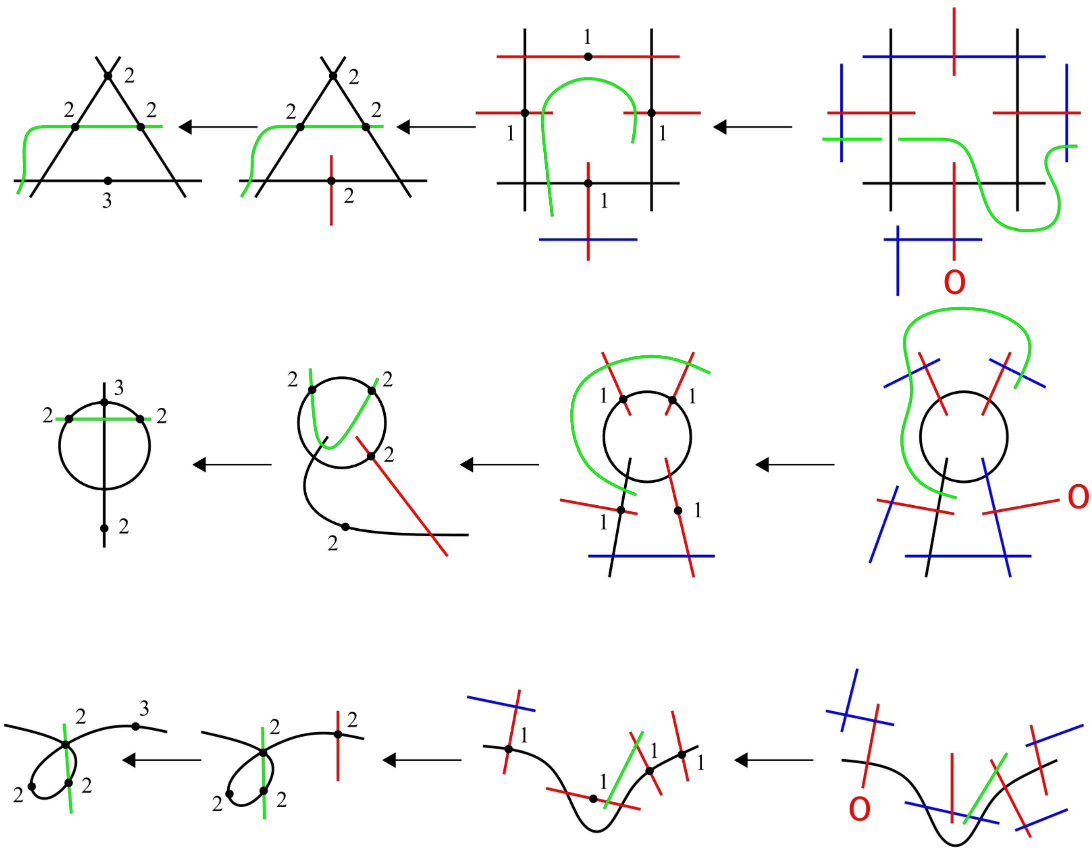


Figure 30: The section $y+z=0$ drawn in green.

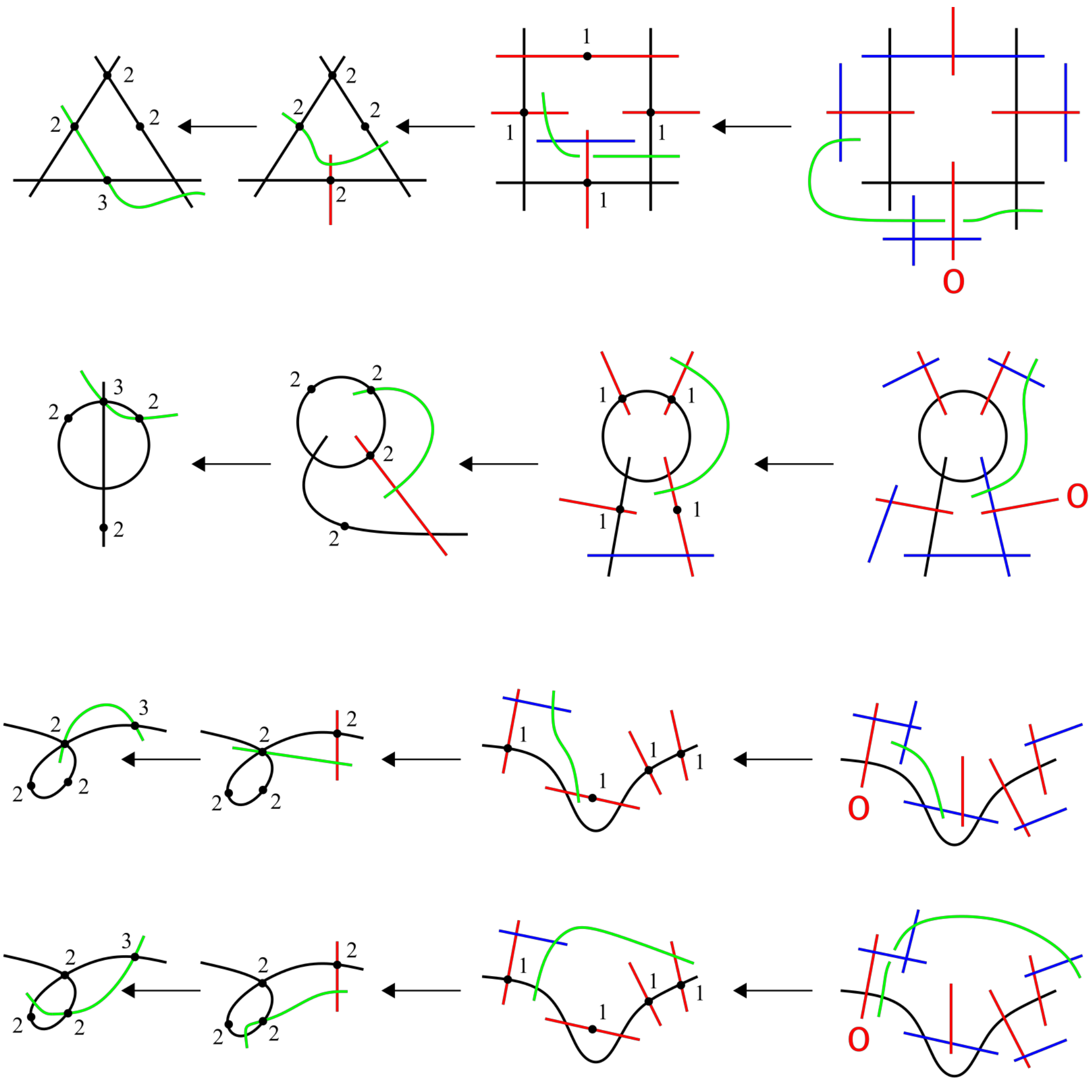


Figure 31: The section $x + z = 0$ drawn in green.

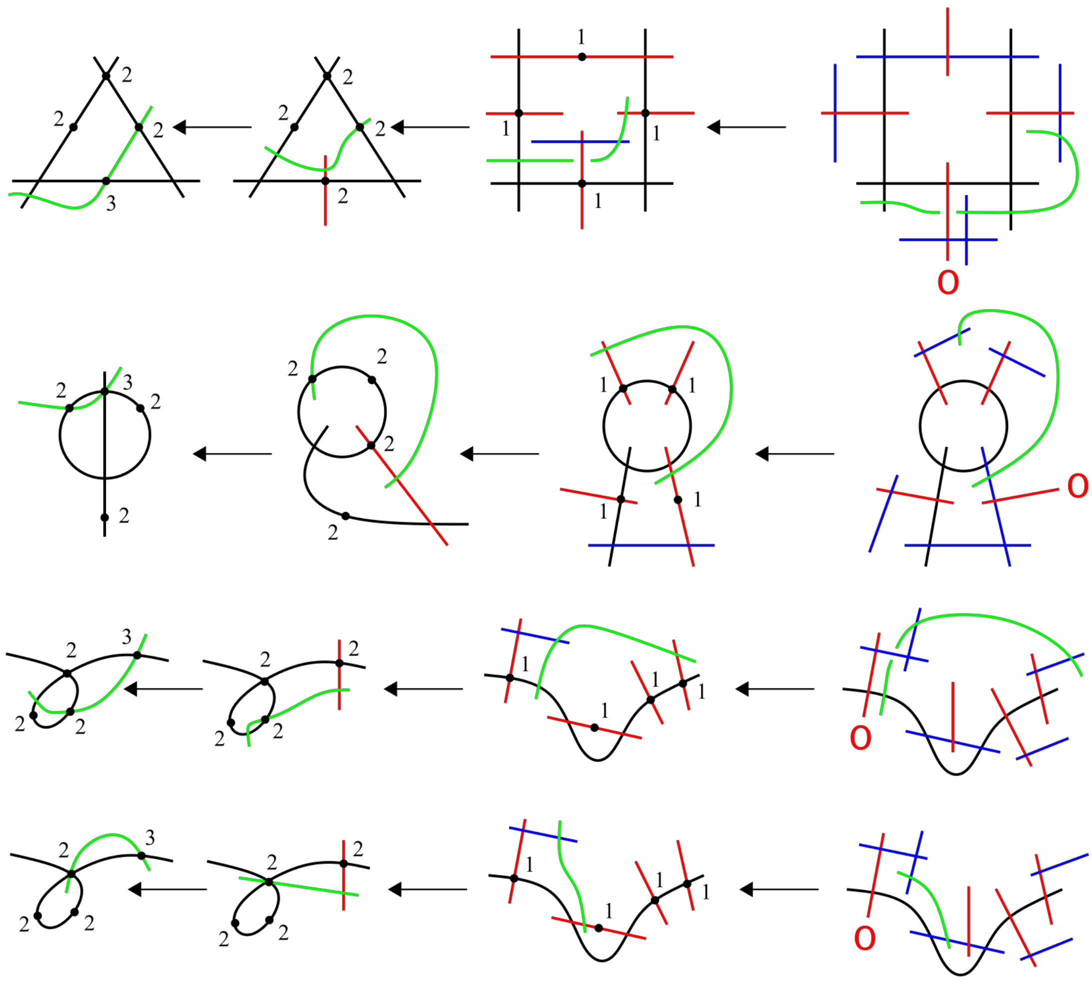


Figure 32: The section $x - z = 0$ drawn in green.

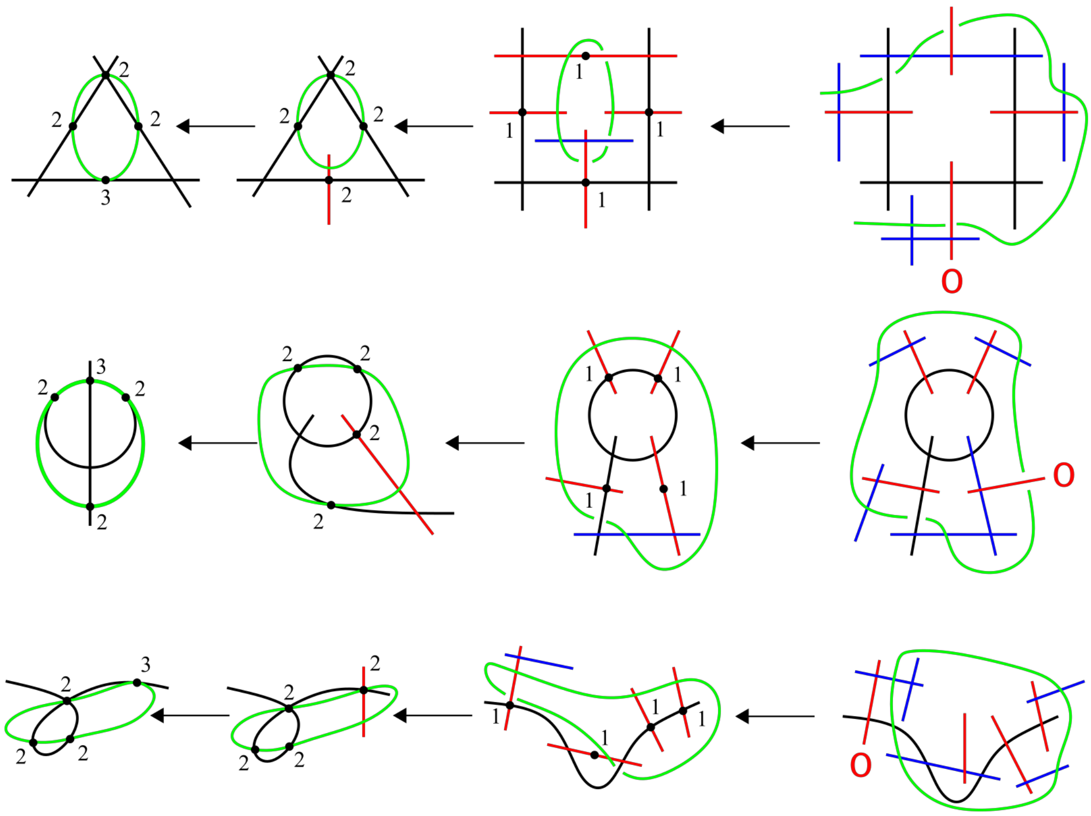


Figure 33: The section $x^2 + yz = 0$ drawn in green.

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