university of groningen
faculty of science and engineering
mathematics and applied mathematics

Extremal Rational Elliptic Surfaces with Additive Fibers

Bachelor's Project Mathematics
July 2022
Student: M.A. van Straaten
First supervisor: Prof. dr. C. Salgado
Second assessor: Prof. dr. J. Top

Abstract

We provide explicit constructions of extremal rational elliptic surfaces with at least one additive fiber over an algebraically closed field. We also provide blow-downs to its different minimal models, namely $\mathbb{P}^2$, $\mathbb{F}_0$ and $\mathbb{F}_2$. Where found we exhibit pencils of cubics where blowing-up at the base points results in the particular extremal rational elliptic surface.

Contents

Introduction

An elliptic curve E is a nonsingular curve of genus 1 over a field k, together with a point on the curve with coordinates in k. It can be proven that an elliptic curve can be written as

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad \text{where } a_i \in k.$$

The topic of elliptic curves has been of interest for mathematical research for decades, with the proof of Fermat's Last Theorem as its most famous use. One of their most prominent properties is that their rational points form a finitely generated abelian group whenever k is a number field. This has structural implications for elliptic curves themselves as well as surfaces containing them. In the preliminaries this will be illustrated with the introduction of elliptic surfaces.

An elliptic surface is a surface with an elliptic fibration over a curve. Each point on the curve corresponds to a fiber of the surface. Moreover, almost every fiber on the surface is an elliptic curve. The finite set of fibers that are not elliptic are the singular fibers. All singular fibers were discovered and classified by Kunihiko Kodaira [12].

One of the main tools in this thesis is the blow-up. Not like the chemical explosion process, but more in a mathematical inflating kind of sense. In short, the blow-up of a point is an isomorphism everywhere, except at the point. The blow-up replaces the point by a line. By blowing up the base points of a 1-dimensional family of cubic curves, a rational elliptic surfaces can be constructed. These surfaces are extremal if the rank of the group of rational points is zero. Reversing the blow-up process turns lines into points, and can be used to go from rational elliptic surfaces to $\mathbb{P}^2$, or other types of surfaces.

The main goal of this thesis is to explicitly construct the blow-down of extremal rational elliptic surfaces with at least one additive fiber over an algebraically closed field to the minimal surfaces $\mathbb{P}^2$, $\mathbb{F}_0$ and $\mathbb{F}_2$. The construction of extremal rational elliptic surfaces from a pencil of cubics in $\mathbb{P}^2$ can be found in literature (see for instance [13],[2]), but, to the writer's knowledge, their explicit blow-down cannot. Moreover, the blow-downs to the other surfaces are not described in literature, to the extend of the writer's knowledge.

The minimal model of elliptic surfaces provides a tool to study elliptic surfaces. For instance, strict transforms of curves and singular fibers can be described in an explicit way in the minimal model. Therefore, the construction and study of these minimal models is proven to be relevant to the field of Algebraic Geometry.

The idea for this thesis came from my supervisor Prof. Dr. Cecília Salgado. She taught the Master's course *Caput Algebra and Geometry on Algebraic Curves*, of which I attended some lectures. These sessions inspired my interest in this topic, and she came up with the idea to study extremal rational elliptic surfaces with additive fibers. Anna de Bruijn has done a similar project, studying the surfaces with multiplicative fibers. Her thesis is complementary to this study, and is highly recommended to read.

This thesis is organized as follows. The first chapter will cover introductory topics in algebraic geometry. Elliptic surfaces and in particular extremal rational elliptic surfaces are introduced in the second chapter. The third chapter focuses on the construction of the minimal surface $\mathbb{P}^2$ of extremal rational elliptic surfaces with at least one additive fiber. The constructions of the same elliptic surfaces to a minimal surface called the Hirzebruch surface $\mathbb{F}_n$ are discussed in Chapter 4.

1 General Preliminaries

The object of study in this thesis are extremal rational elliptic surfaces with at least one additive fiber. To aid the understanding of this topic, this chapter introduces relevant basic topics in algebraic geometry. Excellent references regarding algebraic geometry are written by Hartshorne [11], Gathmann [8] and Shafarevich [21]. For a more thorough study of the concepts treated in this chapter, these references are highly recommended. A more advanced reader is advised to skip this preliminary chapter.

1.1 Affine and Projective Space

During this thesis projective and affine spaces and their respective varieties come up frequently. To ensure that all concepts are clear to the reader, they are clarified in a compact manner below. Whenever k is mentioned, it is always considered to be a field. Moreover, $\bar{k}$ denotes a fixed algebraic closure of the field k.

For $n \in \mathbb{N}$, $\mathbb{A}_k^n$ denotes the *affine n-space* over a field k. An element of $\mathbb{A}_k^n$ is of the form $(a_1, ..., a_n)$, with $a_i \in k$. The subscript k in $\mathbb{A}_k^n$ denotes the dependence of the affine space on the field k and is omitted when the field is clear from the context. The affine space will be the workplace for the set of solutions of polynomials.

Definition 1.1. *For a subset $F \subset k[X_1, ..., X_n]$ of polynomials, we define the* zero locus *of F as*

$$V(F) := \{P \in \mathbb{A}^n \mid f(P) = 0,\ \forall f \in F\}.$$

A subset $V(F)$ of $\mathbb{A}^n$ of this form is called an affine algebraic set.

An algebraic set is *reducible* if it can be expressed as $V(F) = V(G) \cup V(H)$ for some proper algebraic subsets $V(G), V(H) \subset V(F)$. Otherwise, it is *irreducible*. An irreducible affine algebraic set $V(F)$ is called an *affine variety*.

From affine spaces projective spaces can be constructed. In projective spaces some useful properties hold, for instance with regard to intersections between curves. This will be discussed in *Bézout's theorem* in section 1.2.1.

Definition 1.2. *The* projective space $\mathbb{P}^n$ *over a field k is the set of 1-dimensional linear subspaces through* $(0, ..., 0) \in \mathbb{A}^{n+1}$.

$\mathbb{P}^n$ consists of elements of the form $(x_1, ..., x_{n+1})$ with $x_i \in k$, subject to the following equivalence relation:

$$(x_1, ..., x_{n+1}) \sim (\lambda x_1, ..., \lambda x_{n+1}), \quad \text{where } \lambda \in k^*.$$

A point in projective space $[x_1 : ... : x_{n+1}]$ denotes the equivalence class of $(x_1, ..., x_{n+1})$, and the individual $x_0, ..., x_n$ are called the *homogeneous coordinates* for the corresponding point in $\mathbb{P}^n$. The projective space $\mathbb{P}^n$ may be identified with the equivalence class of points in $\mathbb{A}^{n+1} \setminus \{0, ..., 0\}$.

This thesis will also encounter *projective product spaces* $\mathbb{P}^n \times ... \times \mathbb{P}^m$. A product space $\mathbb{P}^n \times \mathbb{P}^m$ can be defined as the quotient of $\{\mathbb{A}^{n+1} \setminus 0\} \times \{\mathbb{A}^{m+1} \setminus 0\}$ by the product of the multiplicative group k^*.

Definition 1.3. *A polynomial $F \in k[X_1, ..., X_n]$ of the form*

$$F = \sum_{i_1,...,i_n \in \mathbb{N}} a_{i_1,...,i_n} X_1^{i_1} \cdot ... \cdot X_n^{i_n}$$

is homogeneous of degree d *if all its terms have the same degree $d = i_1 + ... + i_n$.*

Polynomials in affine space $\mathbb{A}^n$ can be transformed into homogeneous polynomials in $\mathbb{P}^n$ and vice versa. These processes are called *homogenization* and *dehomogenization* respecively. A non-homogeneous polynomial $f(x_1, ..., x_n) \in k[X]$ of degree d can be homogenized to $f(x_0, ..., x_n)$ by introducing the variable x_0 in the polynomial as follows:

$$f(x_0, x_1, ..., x_n) = x_0^d f\left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right).$$

Replacing a homogeneous polynomial $f(x_0, ..., x_n) \in k[X]$ by $f(y_1, ..., y_{i-1}, 1, y_{i+1}, ..., y_n)$ is called dehomogenization with respect to x_i.

By replacing polynomials by homogeneous polynomials, projective algebraic sets can be defined.

Definition 1.4. *To a homogeneous ideal $F \subset \bar{k}[X_1, ..., X_n]$ generated by homogeneous polynomials, we associate a subset of $\mathbb{P}^{n-1}$ by the rule*

$$V(F) := \{P \in \mathbb{P}^{n-1} \mid f(P) = 0, \textit{ for all homogeneous } f \in F\}.$$

A subset $V(F)$ of $\mathbb{P}^{n-1}$ of this form is called a projective algebraic set.

A projective algebraic set $V(F)$ is irreducible if it cannot be expressed as $V(F) = V(G) \cup V(H)$ for some proper projective algebraic subsets $V(G), V(H) \subset V(F)$. An irreducible projective algebraic set $V(F)$ is called a *projective variety.*

Definition 1.5. *If V is a projective algebraic set, the* (homogeneous) ideal *of V, denoted $I(V)$, is the ideal of $\bar{k}[X]$ generated by*

$$I(V) := \{f \in \bar{k}[X] \mid f \textit{ is homogeneous and } f(P) = 0 \textit{ for all } P \in V\}.$$

A projective algebraic set $V(F)$ is a projective variety if and only if $I(V)$ is a prime ideal in $\bar{k}[X]$.

Definition 1.6. *A subset Y of a variety X is a* subvariety *of X if Y is itself a variety.*

Definition 1.7. *The* dimension *of a variety V, denoted $dim(V)$, is the integer n such that the largest nesting of distinct proper subvarieties of V consists of n subvarieties:*

$$V_0 \subset V_1 \subset ... \subset V_{n-1} \subset V.$$

The codimension *of a closed subvariety W of V is equal to $codim(W) = dim(V) - dim(W)$.*

1.1.1 Maps between Varieties

There are different types of maps between varieties. The maps are given coordinate wise by rational functions in a function field. Chapter I.3 in Silverman [23] is recommended for a more elaborate discussion of algebraic maps between projective varieties.

Definition 1.8. *Let V be a variety defined over a field k. The* coordinate ring *of V is*

$$k[V] = \frac{k[X]}{I(V)}.$$

The field of fractions of $k[V]$ is called the function field *of V, denoted $k(V)$.*

Considering that $I(\mathbb{P}^n) = \{f \in \bar{k}[X] \mid f \text{ is homogeneous and } f(P) = 0, \text{ for all } P \in \mathbb{P}^n\} = \{0\}$, the function field of $\mathbb{P}^n$ is $\bar{k}(X_0, ..., X_n)$. For a projective variety $V \subset \mathbb{P}^n$ its function field $k(V)$ is defined by $\frac{f(X_0,...,X_n)}{g(X_0,...,X_n)}$, with $g(X_0, ..., X_n) \notin I(V)$ and $f, g \in \bar{k}[X_0, ..., X_n]$ homogeneous polynomials of the same degree.

Definition 1.9. *Let $V_1, V_2 \subset \mathbb{P}^n$ be two projective varieties. A* rational map *$\phi : V_1 \to V_2$ is a map of the form*

$$f_i : V_1 \to V_2, \qquad \phi = [f_0, ..., f_n]$$

where the rational functions $f_i \in \bar{k}(V_1)$ have the property that for every $P \in V_1$ at which all functions f_i are defined,

$$\phi(P) = [f_0(P), ..., f_n(P)] \in V_2.$$

Remark. A rational map ϕ is not necessarily defined at every point of V_1. In case that a rational function f_i is not defined at $P \in V_1$, the map ϕ might still be defined at P. This is done by replacing f_i by gf_i for an appropriate $g \in \bar{k}(V_1)$.

Definition 1.10. *A rational map $\phi = [f_0, ..., f_n] : V_1 \to V_2$ is* regular *or* defined *at a point $P \in V_1$ if there is a function $g \in \bar{k}(V_1)$ such that*

1. *each gf_i is regular at P.*

2. *there is some j for which $(gf_j)(P) \neq 0$.*

If such a function g exists, we set $\phi(P) = [(gf_1)(P), ..., (gf_n)(P)]$.

A rational map $\phi : X \to Y$ is *birational* if there exists a rational map $\psi : Y \to X$ such that $\phi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \phi = \mathrm{id}_X$ as rational maps. In this case the varieties X, Y are said to be birational.

Example 1.1. *Take the polynomials G: $z^3 = 0$ and F: $zy^2 = x^3$ in $k(\mathbb{P}^2)$. Define the map $\phi = [F, G]$, $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ by $(x : y : z) \mapsto (F(x, y, z)\colon G(x, y, z))$. At the points $p \in V(F) \cap V(G)$, $\phi(p) = (0 : 0) \notin \mathbb{P}^1$. However, F, G are defined at p. Since $gF(p) = g(zy^2 - x^3)(p) = 0$ and $gG(p) = g(z^3)(p) = 0$ for all $g \in \bar{k}(\mathbb{P}^2)$, ϕ is not regular at the points $p \in V(F) \cap V(G)$. These points are in the* indeterminacy locus *of the rational map ϕ.*

Definition 1.11. *A rational map $\phi : V_1 \to V_2$ between two varieties V_1, V_2 that is regular at every point $P \in V_1$ is called a* morphism.

A morphism $\phi : V_1 \to V_2$ is called an *isomorphism* if there exists a morphism $\psi : V_2 \to V_1$ such that $\phi \circ \psi = \mathrm{id}_{V_2}$ and $\psi \circ \phi = \mathrm{id}_{V_1}$. In this case the varieties V_1, V_2 are said to be isomorphic.

1.2 Linear System of Curves

Using (homogeneous) polynomials, curves in affine and projective space can be defined. As this thesis considers curves in $\mathbb{P}^2$, curves will be defined over this space.

1.2.1 Algebraic Curves

The aim of this subsection is to define algebraic curves and introduce two consequential theorems regarding their points of intersection.

Definition 1.12. *An* affine algebraic curve *is an affine variety of dimension 1. Similarly, a* projective algebraic curve *is a projective variety of dimension 1.*

By definition 1.12, the maps from Section 1.1.1 can also be applied to curves. An affine algebraic plane curve is the zero locus of an irreducible polynomial in two variables. An projective algebraic plane curve is the zero locus of an irreducible homogeneous polynomial in three variables. The *degree* of a plane curve is the degree of the polynomial by which it is defined.

In projective spaces, there are extra points on the line at infinity. For $\mathbb{P}^2$ these extra points will be discussed in Section 1.5.2. Due to these extra points, a general statement can be made regarding the number of intersections (counting multiplicity) of two curves. This is the topic of Bézout's theorem:

Theorem 1.1. ***Bézout***
Let k be algebraically closed. Let X and Y be projective plane curves, with X nonsingular and not contained in Y. Then the sum of the multiplicities of intersection of X and Y at all points of $X \cap Y$ equals the product of the degrees of X and Y .

Proof. The proof can be found in [21]. ■

Our curves of interest are cubics, and another useful result about their common intersection points can be made.

Theorem 1.2. ***Cayley Bacharach***
Given eight points $P_1, ..., P_8$ in the plane, no four colinear and no seven lying on a conic, there is a uniquely determined point P_9 (possibly an infinitely near point) such that every cubic through $P_1, ..., P_8$ also passes through P_9. This is still true if P_2 is infinitely near P_1, and P_8 is infinitely near any one of $P_1, ..., P_7$.

Proof. The proof can be found in Chapter V in [11]. ■

1.2.2 Linear Systems

A projective curve of degree d is the zero locus of an ideal generated by homogeneous polynomials of degree d. A curve in $\mathbb{P}^2$ can be identified with a polynomial whose zero locus defines the curve. As this thesis considers curves in $\mathbb{P}^2$, the term curve will henceforth also be used to describe a polynomial.

A homogeneous polynomial of a certain degree $d \geq 1$ in $\mathbb{P}^2$ can be systematically defined by numbering all existing monomials $M_1, ..., M_N$ of degree d. Every monomial is of the form

$M_i = X^{p_i}Y^{q_i}Z^{r_i}$, where $p_i + q_i + r_i = d$. The number of monomials of degree d is equal to $N = \frac{1}{2}(d+1)(d+2)$ [7]. Using the numbered monomials, any homogeneous polynomial in $\mathbb{P}^2$ can be defined.

Definition 1.13. *A homogeneous polynomial of degree d in $\mathbb{P}^2$ is of the form:*

$$F = \sum_{i \in \{1,2,\ldots,N\}} a_i M_i,$$

where $M_1, \ldots, M_N$ are the monomials of degree d and $a_i \in k$.

Giving a curve of degree d is the same thing as choosing $a_1, \ldots, a_N \in k$, not all zero, except that $(a_1, \ldots, a_N)$ and $(\lambda a_1, \ldots, \lambda a_N)$ define the same curve.

Using this convention of notation for curves, curves of degree d can be associated to a projective space. Namely, curves of degree d form a projective space of dimension $d(d+3)/2$ [7]. In other words, each curve of degree d corresponds to a unique point in $\mathbb{P}^{d(d+3)/2}$ and vice versa. Therefore, curves can be considered as points in $d(d+3)/2-$dimensional projective space.

Example 1.2. *A projective cubic curve $a_1Y^3 + a_2Y^2Z + a_3YZ^2 + a_4Y^2X + a_5YX^2 + a_6X^3 + a_7X^2Z + a_8XZ^2 + a_9Z^3 + a_{10}XYZ = 0$ corresponds to a point $(a_1, \ldots, a_{10}) \in \mathbb{P}^9$.*

When putting conditions on the set of all curves of degree d, the curves that satisfy the conditions form a subset of $\mathbb{P}^{d(d+3)/2}$. If the subset is a linear subvariety [7], it is called a *linear system of plane curves.*

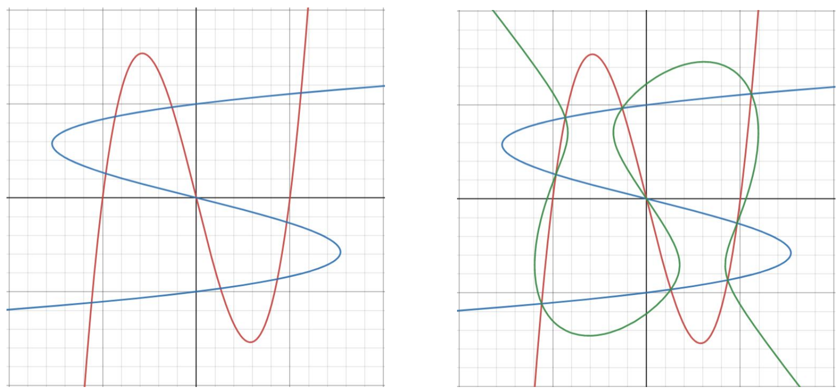
One of the conditions all curves of degree d could be subject to is passing through a given set of points. In fact, imposing all curves to pass through a certain point decreases the dimension of the corresponding projective space $\mathbb{P}^{d(d+3)/2}$ by one.

Proposition 1.3. *Let $P \in \mathbb{P}^2$ be a fixed point. The set of curves of degree d containing the point P forms a hyperplane in $\mathbb{P}^{d(d+3)/2}$.*

Proof. The proof can be found in Chapter 5.2 in [7]. ∎

A family of cubics that contains 8 points such that no four are colinear and no seven lay on a conic is a 1-dimensional linear system of curves, also called a *pencil of cubics*. Any curve in this family corresponds to a point in $\mathbb{P}^1$. By the Cayley Bacharach theorem, each curve in the family will also contain the same ninth point. The 9 common points of the family are called the *base points* of the pencil.

For two projective cubic curves F, G, the family of cubic curves sharing their nine common points is the pencil $\{tF(x:y:z) + uG(x:y:z) = 0 \mid (t:u) \in \mathbb{P}^1,\ (x:y:z) \in \mathbb{P}^2\}$. The pencil is called a *linear system of plane curves*. It is linear as $(t:u) \in \mathbb{P}^1$ are linear terms. Figure 1.a shows an example of two cubics F, G and their nine points of intersection. In Figure 1.b another curve in the pencil is shown that also intersect the same nine points.



(a) The cubics $F : y = 4x(x-1)(x+1)$ and $G : x = 4y(y-1)(y+1)$.

(b) A member of the pencil given by $2F + G = 0$.

Figure 1: A pencil of cubics.

1.3 Blow-up of a Point

For two projective cubic curves F, G the rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, defined by $(x : y : z) \mapsto (F(x,y,z) \colon G(x,y,z))$ is not defined at the points of intersection of F and G, see Example 1.1. Ideally, the map ϕ would be defined everywhere, making it a morphism. In order to achieve this, a procedure called a *blow-up* can be used.

A blow-up is a birational map. Its main uses are to resolve singularities and the indeterminacy locus of a rational map, by replacing points with a line. In the upcoming sections, the base points of pencils of cubics will be the points that are blown-up. This section will discuss how blowing up a point works in respectively affine and projective space. Mainly pages 28 and 29 of [11] and Chapter 4 of [21] are referenced.

Firstly, let us take a look at the affine space $\mathbb{A}^n$. The blow-up of the point $P = (0, ..., 0) \in \mathbb{A}^n$ will be constructed. It is enough to describe the procedure for a specific point, as any point can be translated to it. Blowing up of $\mathbb{A}^n$ at P takes $\mathbb{A}^n$ to a closed subset $S \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$, defined by the equations $\{x_i y_j = x_j y_i \mid i, j = 1, ..., n\}$, with $x_1, ..., x_n$ the affine coordinates of $\mathbb{A}^n$, and $y_1, ..., y_n$ the projective coordinates of $\mathbb{P}^{n-1}$.

$$\begin{tikzcd}
\mathrm{S} \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \\
 & \mathbb{A}^n
\end{tikzcd}$$

There exists a natural morphism $\pi : S \to \mathbb{A}^n$ by restriction to the first factor. This map allows for some useful properties:

1. π gives an isomorphism between $S \setminus \pi^{-1}(P)$ and $\mathbb{A}^n \setminus \{P\}$. This can be seen by letting $Q = (a_1, ..., a_n) \in \mathbb{A}^n \setminus \{P\}$, where at least one $a_i \neq 0$. Let $Q \times (y_1, ..., y_n) \in \pi^{-1}(Q) \subset S$, for some $(y_1, ..., y_n) \in \mathbb{P}^{n-1}$. In S we are subject to the condition $a_i y_j = a_j y_i$. This means that $y_j = \frac{a_j}{a_i} y_i$, and hence $(y_1, ..., y_n)$ is uniquely defined by Q. Moreover, we can set $y_j = a_j$, meaning we can take $(y_1, ..., y_n) = (a_1, ..., a_n)$. Thus $\pi^{-1}(Q)$ consists of a single point $(a_1, ..., a_n) \times (a_1, ..., a_n)$.

   There is an inverse morphism to π between $\mathbb{A}^n \setminus \{P\}$ and $S \setminus \pi^{-1}(P)$ given by $\Psi(a_1, ..., a_n) = (a_1, ..., a_n) \times (a_1, ..., a_n)$, showing the isomorphism between $\mathbb{A}^n \setminus \{P\}$ and $S \setminus \pi^{-1}(P)$.

2. $\pi^{-1}(P) \cong \mathbb{P}^{n-1}$. By definition, $\pi^{-1}(P)$ consists of all points $P \times (y_1, ..., y_n)$, with $(y_1, ..., y_n) \in \mathbb{P}^{n-1}$. Since $P = (0, ..., 0)$, the condition $x_i y_j = x_j y_i$ is satisfied for all $(y_1, ..., y_n) \in \mathbb{P}^{n-1}$.

3. Points of $\pi^{-1}(P)$ correspond to the lines through P in $\mathbb{A}^n$. For the explicit 1-1 correspondence see page 28 in [11].

When we wish to blow-up a point P of a closed subvariety $C \subset \mathbb{A}^n$, we simply define the blow up as $S = (\pi^{-1}(C \setminus \{P\}))$, with $\pi : S \to \mathbb{A}^n$ the blow-up as defined above.

This thesis mainly considers blow-ups in $\mathbb{P}^2$. The blow-up of a point in $\mathbb{P}^2$ is defined as follows:

Definition 1.14. *Blow-up of a point in* $\mathbb{P}^2$
Let S be a surface and $p \in S$. Then there exist a surface $\tilde{S}$ and a morphism $\pi : \tilde{S} \to S$, which are unique up to isomorphism, such that

1. *the restriction of π to $\pi^{-1}(S \setminus \{P\})$ is an isomorphism onto $S \setminus \{P\}$;*

2. *$\pi^{-1}(P) = E$, say, is isomorphic to $\mathbb{P}^1$.*

π is the blow-up of S at p, and E is called the exceptional divisor *or exceptional curve of the blow-up. The self-intersection of an exceptional divisor is $E^2 = -1$. The intersection number of divisors will be discussed in Section 1.4.*

Projective blow-ups have a similar construction to affine blow-ups. Take two projective spaces $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$, with homogeneous coordinates $[x_0 : ... : x_n]$ and $[y_1 : ... : y_n]$ respectively. A point in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ is denoted $P = [x : y] = [x_0 : ... : x_n : y_1 : ... : y_n]$. Similar to the affine case, the blow up at $P = [1 : 0 : ... : 0] \in \mathbb{P}^n$ is defined by the map $\pi : \Pi \to \mathbb{P}^n$, where $\Pi \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ is the closed subvariety defined by $x_i y_j = x_j y_i$, where $i, j = 1, ..., n$. Again the map π allows for useful properties:

1. π gives an isomorphism between $\mathbb{P}^n \setminus \{P\}$ and $\Pi \setminus (\{P\} \times \mathbb{P}^{n-1})$, given by the inverse $\pi^{-1} : \mathbb{P}^n \setminus \{P\} \to \Pi \setminus (\{P\} \times \mathbb{P}^{n-1})$, where $[x_0 : ... : x_n] \mapsto ((x_0 : ... : x_n), (x_1 : ... : x_n))$.

2. $\pi^{-1}(P) \cong P \times \mathbb{P}^{n-1}$ and since $P = [1 : 0 : ... : 0]$, the condition $x_i y_j = x_j y_i$ is satisfied for all $(y_1, ..., y_n) \in \mathbb{P}^{n-1}$, so $\pi^{-1}(P) \cong \mathbb{P}^{n-1}$.

3. Again, points of $\pi^{-1}(P)$ correspond to lines through the point $P \in \mathbb{P}^n$. For an elaborate construction of this correspondence, see page 114 of [21].

Let $\pi : \hat{S} \to S$ be the blow-up of a point p, and consider an irreducible curve C on S that passes through p with multiplicity[1] m. The closure of $\pi^{-1}(C \setminus \{P\})$ in $\hat{S}$ is an irreducible curve $\hat{C}$ on $\hat{S}$, which we call the *strict transform* of C [3].

Proposition 1.4. *Let $\pi : \hat{S} \to S$ be the blow-up of a point p, and consider an irreducible curve C on S that passes through p with multiplicity m. The* proper transform *of C is*

$$\pi^*(C) = \hat{C} + mE,$$

where $\hat{C}$ is the strict transform of C.

Proof. The proof can be found in [3]. ■

Example 1.3. *Consider two cubics $F, G \in \mathbb{P}^2$, intersecting in 9 distinct points. In order to resolve the indeterminacy locus of the rational map ϕ mentioned at the beginning of this section, we blow up at the points of intersection $P_1, ..., P_9$. When blowing up at a point P_1, F and G are separated at P_1 and the point is replaced by the exceptional divisor E_1 (see Figure 2). The proper transform of the cubic F is $\pi^*(F) = \tilde{F} + E_1$, where $\tilde{F}$ is the strict transform of F. After repeating the blow-up process for the other 8 points $P_2, ..., P_9$ the two curves have no more points of intersection, meaning that the indeterminacy locus of ϕ has been resolved.*

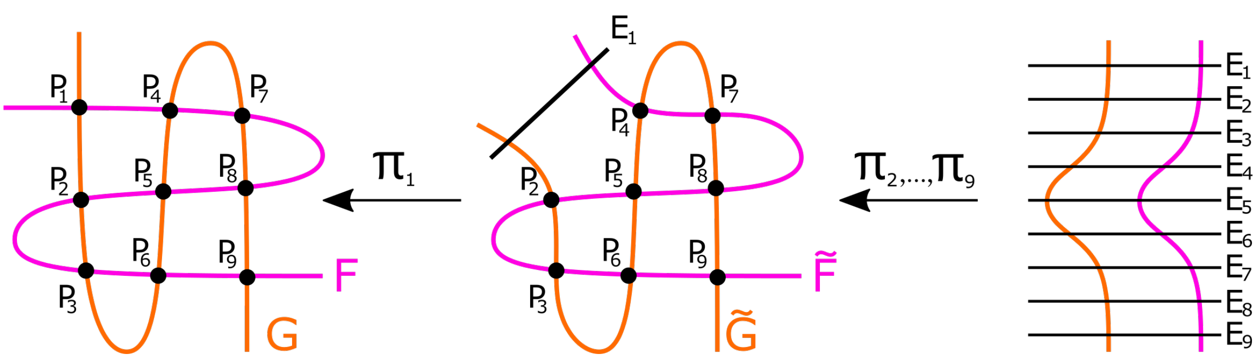


Figure 2: The blow-up of at the intersection points of two cubics.

Example 1.4. *In order to illustrate a blow-up at the base points of a pencil of cubics, let $S\colon y^2z = x^3 + tz^3$ be a pencil of cubics, $G\colon zy^2 = x^3$ and $F\colon z^3 = 0$. The map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, defined by $(x : y : z) \mapsto (F(x,y,z)\colon G(x,y,z))$ is not defined at the basepoints of S. The point $P = [1 : 0 : 0]$ is such a point. In order to resolve the indeterminacy locus of ϕ, this example will blow-up at P.*

In order to illustrate the blow-up, we dehomogenise to $y^2 = x^3 + t$. At $t = 0$, the pencil has a singular point at $P = (0,0)$. To resolve this singularity, we blow up at P. The result is visible in Figure 3. The blow-up is defined by the equations $y^2 = x^3$ and $xu = yv$, for $(v : u) \in \mathbb{P}^1$.

[1] Intersection multiplicity is discussed in Chapter 2 of [9].

To see what the strict transform $\hat{S}$ of S looks like, we look along different points of E. We have $E \cong \mathbb{P}^1$ by Definition 1.14. For $(v : u) \in \mathbb{P}^1$, it always holds that either $v \neq 0$ and/or $u \neq 0$. Let us consider the case where $v \neq 0$. Then we can set $u = \frac{u}{v}$ and $v = 1$. The equations describing the blow-up become $y^2 = x^3$ and $xu = y$. Substituting for y results in $x^2u^2 = x^3$. This gives two irreducible components, the first one being E described by $x = 0$ and $y = 0$, where u can be chosen arbitrarily.

The second irreducible component is the strict transform $\hat{S}$, given by $x = u^2$ and $y = u^3$. $\hat{S}$ meets the exceptional curve in $u = 0$. Consequently, the strict transform tangentially intersects the exceptional curve at $u = 0$, see Figure 3.

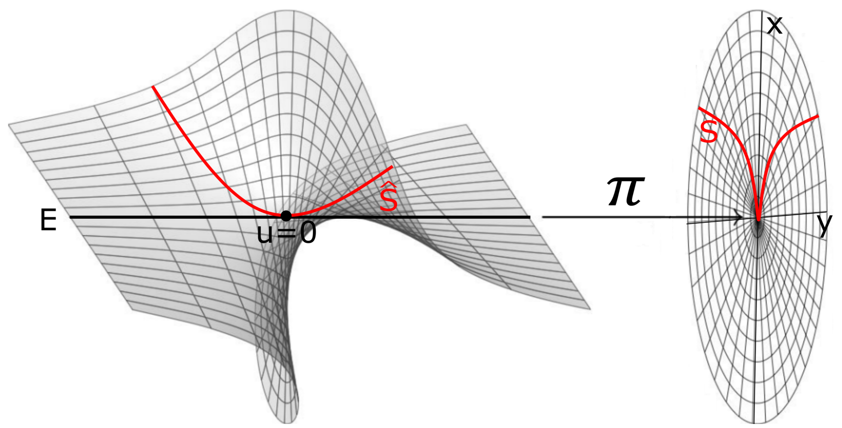


Figure 3: Blowing up[2] $y^2 = x^3 + t$ at $t = 0$ and $v \neq 0$.

As previously mentioned, a blow-up is a birational map, with as inverse a *blow-down*. A blow-down takes a $(-1)-$curve isomorphic to $\mathbb{P}^1$ and replaces it with a point, according to the reverse process of a blow-up.

1.4 Divisors

In this section divisors are studied. More precisely, a divisor is a formal sum of subvarieties of codimension 1 in a variety. Furthermore, this section includes a proof that the self intersection of the exceptional divisor E of a blow-up is -1.

Definition 1.15. *Let X be an irreducible variety. A collection of distinct irreducible closed*[3] *subvarieties $C_1, ..., C_r$ of codimension 1 in X with assigned integer multiplicities $k_1, ..., k_r$ will be called a divisor on X. A divisor is written*

$$D = k_1C_1 + ... + k_rC_r.$$

The support *of a divisor $D = k_1C_1 + ... + k_rC_r$ with all $k_1 \neq 0$ is the variety $C_1 \cup ... \cup C_r$, denoted Supp(D). A divisor $D = C_i$ is called a* prime divisor*.*

[3]The algebraic subvarieties are the closed sets the Zariski topology. For a more detailed discussion of the Zariski topology see [15].

The addition of two divisors $D = k_1C_1 + ... + k_rC_r$, $D' = k'_1C_1 + ... + k'_rC_r$, provided we allow the coefficients k_i to take the value zero, is defined as

$$D + D' = (k_1 + k'_1)C_1 + ... + (k_r + k'_r)C_r.$$

The set of divisors on X form a group under this operation, denoted $Div(X)$, generated by the prime divisors.

Example 1.5. *Let X be a curve. A divisor D in $Div(X)$ is a finite formal sum of points in X, e.g. $D = P_1 + 3P_2$, or $D = 2P_3 - 4P_5$. In case X is a surface, a divisor D in $Div(X)$ is a finite formal sum of curves over X, e.g. $D = C_1$, or $D = 3E - B$.*

For a nonzero rational function $f \in k(X)$, its divisor is defined as:

$$div(f) = \sum_{C_i} v_{C_i}(f) \cdot C_i$$

Where $v_{C_i}(f)$ corresponds to the order of the zero/pole of f along prime divisor C_i. Divisors of this form are called *principal divisors* [3].

Result 1.1. *[21] Let $f, l \in k(X)$. There exist $g, h \in k[X]$ such that $f = \frac{g}{h}$. Two properties of principal divisors are*

i. $div(f \cdot l) = div(f) + div(l)$

ii. $div(f) = div(g) - div(h)$

The principal divisors form a subgroup of the group of all divisors, denoted $PDiv(X)$. Using this subgroup, the *Picard group* can be defined.

Definition 1.16. *The* Picard group *is the quotient group given by $Pic(X) = Div(X)/PDiv(X)$. The equivalence between $D_1, D_2 \in Pic(X)$ is defined by:*

$$D_1 \sim D_2 \quad \text{if } D_1 = D_2 + div(f) \quad \text{for some } f \in k(X).$$

In case $D_1 \sim D_2$, D_1 and D_2 are said to be linearly equivalent.

The following theorem defines the intersection product between two divisors.

Theorem 1.5. *Intersection Number*
There is a unique pairing $Div(X) \times Div(X) \to \mathbb{Z}$, denoted by $C \cdot D$ for any two divisors C, D, such that

i. if C and D are nonsingular curves meeting transversally, then $C \cdot D = \#(C \cap D)$.

ii. it is symmetric: $C \cdot D = D \cdot C$

iii. it is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$

iv. it depends only on the linear equivalence classes: if $C \sim C'$, then $C \cdot D = C' \cdot D$ for all $D \in Div(X)$.

Proof. The proof of this theorem can be found in Chapter V in [11]. ■

In what follows a proposition will be stated that relates the genus of a curve with its self-intersection and its intersection with the canonical divisor of the variety. For a definition of the canonical divisor, see [11].

Proposition 1.6. *Adjunction Formula*
If C is a nonsingular curve of genus $g(C)$ on the surface X, and K is the canonical divisor on X, then

$$2g(C) - 2 = C \cdot (C + K)$$

Proof. The proof of the Adjunction formula can be found in Chapter V in [11]. ■

As the intersection product only depends on linear equivalence classes of the divisors, the self-intersection of divisors can also be computed by calculating the intersection product of the divisor with a divisor in its equivalence class.

Example 1.6. *In this example the self-intersection of a line $L \in \mathbb{P}^2$ will be determined. Calculating $(L \cdot L) = L^2$ is not trivial, as one cannot simply count the number of intersections with multiplicity. Fortunately the adjunction formula in Proposition 1.6 can be used. The canonical divisor K in $\mathbb{P}^2$ is $-3H$, where H is any line in $\mathbb{P}^2$ (page 361 in [11]). Note since a line has degree $d = 1$, its genus according to the genus degree formula in Section 1.5.1 is $g(L) = 0$. Therefore*

$$\begin{aligned} 2g(L) - 2 &= L \cdot (L - 3H) \\ -2 &= L^2 - 3(L \cdot H) \end{aligned}$$

Since two lines with no common components intersect in one point by Bézout's theorem, we have that $(L \cdot H) = 1$.

$$\begin{aligned} -2 &= L^2 - 3 \\ L^2 &= 1 \end{aligned}$$

Therefore, all lines in $\mathbb{P}^2$ have self intersection 1.

In order to debate the self intersection of exceptional divisors, the pull-back map of a divisor needs to be discussed.

Proposition 1.7. *Let $\phi : X \to Y$ be a regular map of non-singular irreducible varieties, and D a divisor on Y. In particular, if $\phi(X)$ is dense in Y then the pullback ϕ^* of any divisor $D \in Div(Y)$ defines a homomorphism*

$$\phi^* : Div(Y) \to Div(X).$$

The pullback of a principal divisor $f \in K(X)$ is

$$\phi^*(div(f)) = div(\phi^* f). \tag{1}$$

As a result, ϕ^* also maps the principal divisors of Y to the principal divisors of X. Therefore ϕ^* defines a homomorphism $\phi^* : Pic(Y) \to Pic(X)$.

Proof. The proof can be found in [21]. ■

Proposition 1.8. *Let S, S' be surfaces, $\phi : S \to S'$ a generically finite morphism of degree*[4] *d, D and D' divisors on S. Then $\phi^*D \cdot \phi^*D = d(D \cdot D')$.*

Proof. The proof can be found in [3]. ■

Finally, the self intersection of the exceptional divisor E can be determined.

Proposition 1.9. *The exceptional divisor E as in Definition 1.14 has self-intersection*

$$(E \cdot E) = E^2 = -1.$$

Proof. Let $\pi : S \to X$ be the blow up of a point P on a nonsingular variety X. By Theorem 3.1 in Chapter 3.1.3 in [21] we can take curves $c, c' \in X$, such that we have a point $P \in c$ (with multiplicity m), but $P \notin c'$, while $c \sim c'$. Hence we can say $c' = c + div(f)$ for some $f \in K(X)$. As a result,

$$\begin{aligned} \pi^*c' &= \pi^*c + \pi^*(div(f)) \\ &\overset{\text{eq. (1)}}{=} \pi^*c + div(\pi^*f) \\ &\implies \pi^*c' \sim \pi^*c \end{aligned}$$

Since π is an isomorphism outside of P, and $E = \pi^{-1}(P)$, we have $E \cdot \pi^*(c') = 0$. Since $\pi^*c' \sim \pi^*c$, by Definition 1.5.ii we have

$$E \cdot \pi^*(c) = 0. \tag{2}$$

Moreover, we have that $\pi^*c = \tilde{c} + mE$ (see Proposition 1.4), and $E \cdot \tilde{c} = m$. Then

$$\begin{aligned} \tilde{c} &= \pi^*(c) - mE \\ E \cdot \tilde{c} &= E \cdot (\pi^*c - mE) \\ m &= E \cdot \pi^*c - mE^2 \\ m &\overset{\text{eq. (2)}}{=} -mE^2 \\ \implies E^2 &= -1 \end{aligned}$$

■

1.5 Elliptic Curves

An elliptic curve is a smooth projective curve of genus 1, defined over a field k together with a $k-$rational point. This is a point whose coordinates are elements of k. This section will define elliptic curves, and discuss the group structure of the $k-$rational points.

[4] Meaning that the pre-image at almost all points in S' consists of d points, except for the points on a Zariski closed set.

1.5.1 Definition of Elliptic Curves

This thesis studies cubic curves in two variables over a field. Cubic curves with a point over k are elliptic curves.

Definition 1.17. *An* elliptic curve *E over a field k is a nonsingular cubic curve, together with a k-rational point. A k-rational point is a point on the curve where $(x, y) \in k^2$.*

Proposition 1.10. *An elliptic curve can be given by*

$$E: \; y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad \text{where } a_i \in k.$$

Proof. The proof can be found in Chapter III.3 of [23]. ■

An elliptic curve over a field of $Char(k) \neq 2$ can be written as

$$E: \; y^2 = x^3 + ax^2 + bx + c, \qquad \text{where } a, b, c \in k.$$

Moreover, in the case that $Char(k) \neq 2, 3$, any elliptic curve over a field k can be written in the shorter *Weierstrass normal form*: $y^2 = x^3 - 27c_4x - 54c_6$. How this can be achieved is shown in Chapter 3.1 in [23]. It uses two substitutions, respectively

$$y \mapsto \frac{1}{2}(y - a_1x - a_3), \qquad (x, y) \mapsto \left(\frac{x - 3b_2}{36}, \frac{y}{108}\right),$$

where $b_2 = 4a_2 + a_1^2$, $b_4 = 2a_4 + a_3a_1$, $b_6 = 4a_6 + a_3^2$, $c_4 = b_2^2 - 24b_4$, $c_6 = -b_3^2 + 36b_2b_4 - 216b_6$. From these substitutions it is visible that the Weierstrass normal form can only be attained in fields of characteristic unequal to 2 or 3, as in fields of characteristic 2 or 3 x and y would be substituted by 0.

In order to see all intersection points of elliptic curves this thesis considers elliptic curves in the projective space $\mathbb{P}^2$. Therefore, the affine coordinates must be transformed to homogeneous coordinates. To this end, the following substitution is used:

$$x = \frac{X}{Z}, \qquad y = \frac{Y}{Z}.$$

This gives the general expression for a projective elliptic curve:

$$ZY^2 + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3, \qquad \text{where } a_i \in k. \tag{3}$$

Definition 1.3 shows that projective elliptic curves are defined by homogeneous polynomials of degree 3 in three variables. The fact that every elliptic curve has genus 1 is a consequence of the genus-degree formula.

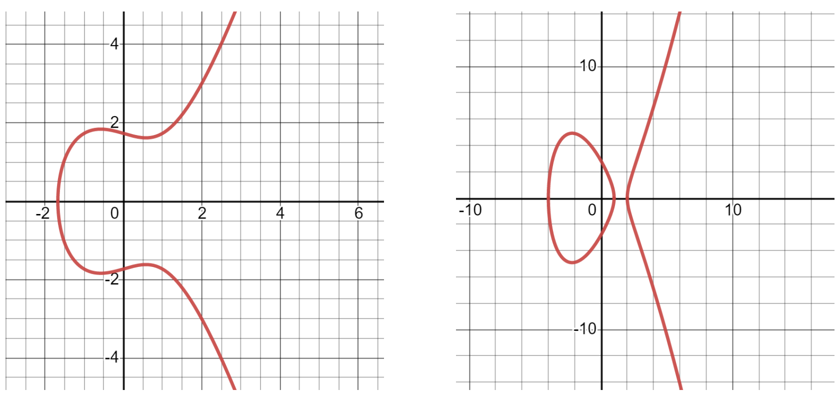
Proposition 1.11. *Genus-Degree Formula*
The genus $g(C)$ of a smooth irreducible projective plane curve C of degree d is equal to $g(C) = \frac{1}{2}(d-2)(d-1)$.

Proof. See Hartshorne's book Algebraic Geometry [11]. ■

In Figure 4 two examples of elliptic curves over $\mathbb{Q}$ are shown. Elliptic curves over $\mathbb{Q}(t)$ will be relevant for this thesis. An elliptic curve over $\mathbb{Q}(t)$ is of the form

$$E: \ y^2 = x^3 + A(t)x + B(t),$$

where $A(t), B(t) \in \mathbb{Q}(t)$.These type of elliptic curves resurface in Section 2.1 on elliptic surfaces.



(a) $y^2 = x^3 - x + 3$ (b) $y^2 = (x+4)(x-1)(x-2)$

Figure 4: Two elliptic curves over $\mathbb{Q}$ with corresponding polynomial.

1.5.2 Group Law on Elliptic Curves

This section defines an action on the set of points on an elliptic curve. It is shown that these points form an abelian group under this action together with the unit element $\mathcal{O}$.

Using the coordinate transformation in Section 1.5.1, the Weierstrass normal form in projective space becomes

$$Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in k.$$

A point on the curve is given by $[X : Y : Z] \in \mathbb{P}^2$. Now the unit element $\mathcal{O}$ mentioned at the beginning of this section can be defined. This element is called the *point at infinity* of $\mathbb{P}^2$, because it is not in the dehomogenized affine model when taking $Z = 1$. In $\mathbb{P}^2$ the points at infinity are the lines through (0,0,0) and the plane where $Z = 0$. Substituting $Z = 0$ in the general expression of a plane curve in equation 3 gives $X^3 = 0$, hence the point at infinity $\mathcal{O}$ is $[0 : 1 : 0]$. A more thorough discussion of the point at infinity in $\mathbb{P}^n$ can be found in Chapter 4 of [7].

The point $\mathcal{O}$ can be thought of as the point where all vertical lines $X = a$ meet. Moreover, $\mathcal{O}$ is counted as a rational point. Addition on elliptic curves can be geometrically defined by setting $\mathcal{O}$ as the origin.

By Bézout's theorem, there are always three intersections between a line and an elliptic curve in $\mathbb{P}^2$, counting multiplicity. As shown in Figure 4.a, by taking the line defined by P and Q for any two points P and Q on our curve, one can find a third intersection point which is denoted $P * Q$. The case for $Q = P$ is shown in Figure 4.b. There, drawing the tangent line to the curve at the point P gives the third intersection $P * P$. This procedure defines the group law on the points of an elliptic curve.

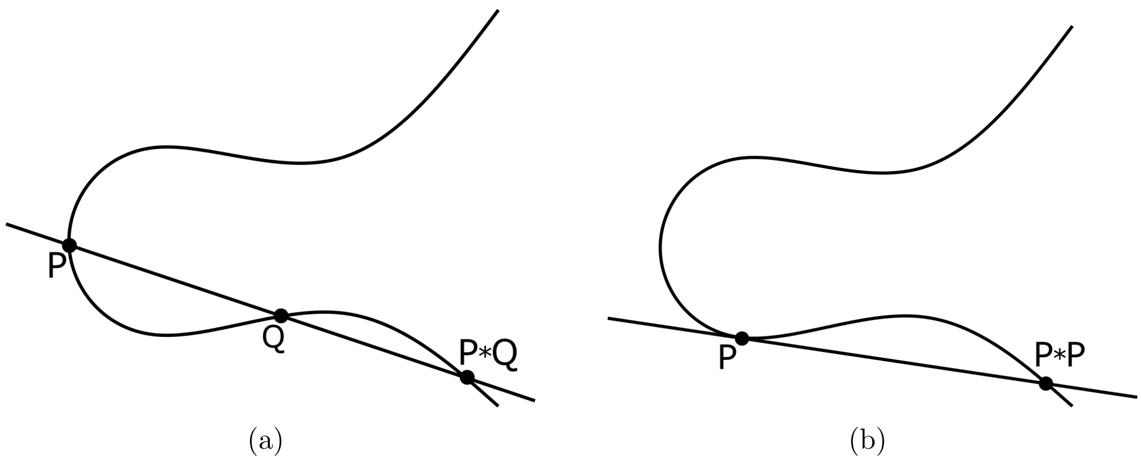


Figure 5: Intersection of the line through points on an elliptic curve

Definition 1.18. *The addition of two points P and Q on an elliptic curve is defined as $P + Q = \mathcal{O} * (P * Q)$.*

An exemplary addition can be found in Figure 6. The lines $X = a$ meet in $\mathcal{O}$ hence the line intersecting $\mathcal{O}$ and $P * Q$ is a vertical line through $P * Q$. In case that the third intersection point of the line through P and Q is $\mathcal{O}$ we have $P + Q = \mathcal{O}$.

From $P * Q = Q * P$, it follows that the addition of points on a curve is commutative:

$$P + Q = \mathcal{O} * (P * Q) = \mathcal{O} * (Q * P) = Q + P.$$

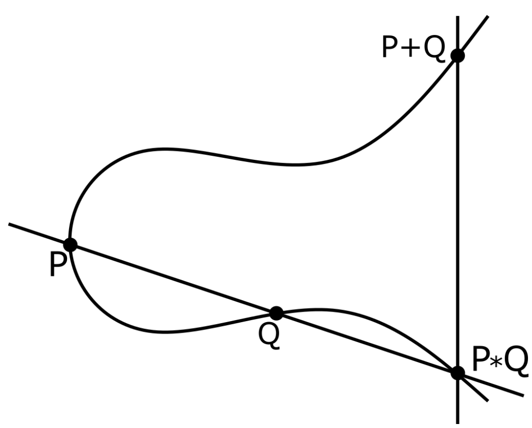


Figure 6: The addition of two points P, Q on an elliptic curve

Addition on elliptic curves turns the points of an elliptic curve into a commutative group, as shown in Chapter III of [23].

Example 1.7. *To illustrate how the addition of points on a curve precisely works, consider the elliptic curve* $E : y^2 = x^3 + 3x + 4$*. Two points on this curve are* $P = (-1, 0)$ *and* $Q = (0, 2)$*. In order to calculate* $P + Q$*, we first determine* $P * Q$*. The line through* P, Q *is* $y = \frac{2-0}{0-(-1)}x + b = 2x + b$*. Filling in one of the points gives* $y = 2x + 2$*. The intersection between this line and* E *is equal to* $P * Q$*. It turns out that* $P * Q = (5, 12)$*. Now, we can calculate* $P + Q = (P * Q) * \mathcal{O}$ *by finding the second intersection of the elliptic curve and* $X = 5$*. By the symmetry of* E*, we find that* $P + Q = (5, -12)$*.*

Definition 1.19. *A point* P *on an elliptic curve* E *over* k *is a* torsion point of order n *if* n *is the smallest positive integer for which* $nP = \mathcal{O}$*. If no such* n *exists,* P *is not a torsion point.*

In fact, the rational points of an elliptic curve defined over a number field are a finitely generated abelian group.

Theorem 1.12. *Mordell-Weil*
Let k *be a number field. The set* $E(k)$ *consisting of* k*-rational points of an elliptic curve* E *over* k *is a finitely generated abelian group, the* Mordell-Weil group*. In particular, it can be written as*

$$E(k) \cong Tor(E(k)) \oplus \mathbb{Z}^r,$$

where r *is the rank of* $E(k)$ *and* $Tor(E(K))$ *is the torsion group of* $E(k)$*, consisting of all* k*-rational points of finite order.*

Proof. The proof of the Mordell-Weil theorem is given by Silverman in Chapter 8 of his book Arithmetic of Elliptic Curves [23]. It uses a weak statement of the theorem, which states that for $m \geq 2$ the quotient group $E(k)/mE(k)$ is a finite group. Moreover, the *height* of a point $P = [x_0, ..., x_n]$ in projective space is used. This notion is expanded to points on elliptic curves. Using these tools, Silverman proves the Mordell-Weil theorem. ■

1.6 Good and Bad reduction

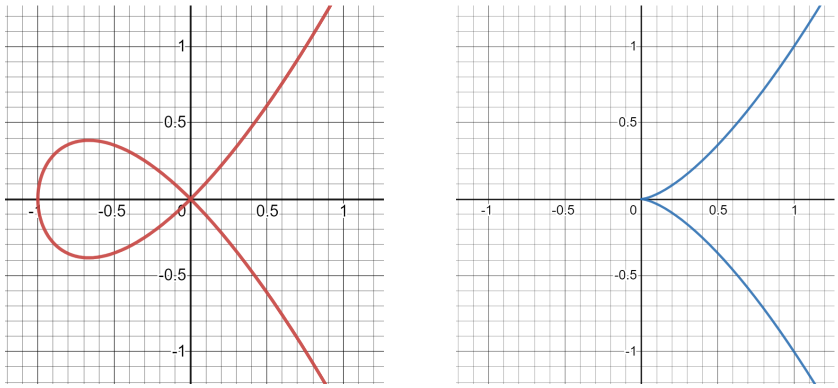
As seen in Section 1.5.1, any elliptic curve can be put into the Weierstrass normal form $y^2 = x^3 + Ax + B$, for some $A, B \in k$, $Char(k) \neq 2, 3$. The discriminant Δ of an elliptic curve E in Weierstrass normal form is

$$\Delta(E) = -16(4A^3 + 27B^2).$$

Definition 1.20. *A curve* E *is* non-singular *if and only if the discriminant* $\Delta(E)$ *is non-zero. Any curve that has a singular point is a singular curve.*

Definition 1.21. *A point* $P = (x_0, x_1, x_2) \in \mathbb{P}^2$ *on a curve defined by the homogeneous polynomial* $F \in k[X, Y, Z]$ *is* singular *if all partial derivatives vanish at that point, e.g.*

$$\frac{dF}{dX}(P) = 0, \qquad \frac{dF}{dY}(P) = 0, \qquad \frac{dF}{dZ}(P) = 0.$$



(a) $y^2 = x^3 + x^2$ (b) $y^2 = x^3$

Figure 7: Two singular curves with a node and a cusp, respectively.

A singular point is a *node* if the tangent lines to the curve at P have different slopes, while it is a *cusp* if the slopes are equal. An example of a node and cusp are given in Figure 7.

Based on its singularity after reducing A and B, a non-singular plane curve either has good or bad reduction at a particular local parameter. In order to thoroughly discuss reduction, the notion of valuation and discrete valuation rings over a field will be defined, according to the definitions by Altman and Kleiman in Chapter 23 of [1].

Definition 1.22. *Let k be a field. A* discrete valuation *of k is a surjective function $v : k^\times \to \mathbb{Z}$ such that, for every $x, y \in k^\times$*

1. $v(x \cdot y) = v(x) + v(y)$
2. $v(x + y) \geq min\{v(x), v(y)\}$ *if* $x \neq -y$

As per convention, $v(0) = \infty$.

In order to define the discrete valuation with respect to a local parameter t, notions regarding local rings will be defined according to [9].

Definition 1.23. *Let F be an irreducible non-singular affine curve over an algebraically closed field k. The coordinate ring of F is defined as*

$$A(F) := \frac{k[x, y]}{\langle F \rangle}.$$

Definition 1.24. *Let F be an affine curve. The local ring of F at a point $P \in F$ is denoted by*

$$\mathcal{O}_{F,P} := \left\{ \frac{f}{g} \mid f, g \in A(F) \text{ with } g(P) \neq 0 \right\}.$$

In particular, $\mathcal{O}_{F,P}$ is a discrete valuation ring.

There is a well-defined evaluation map from the local ring to the field k, given by

$$\mathcal{O}_{F,P} \to k, \qquad \frac{f}{g} \mapsto \frac{f(P)}{g(P)}.$$

The kernel of this evaluation map is

$$I_{F,P} := \left\{ \frac{f}{g} \mid f, g \in A(F) \text{ with } f(P) = 0 \text{ and } g(P) \neq 0 \right\}.$$

Definition 1.25. *The ideal $I_{F,P}$ can be written as $I_{F,P} = \langle t \rangle$ for some $t \in \mathcal{O}_{F,P}$. We call t a* local parameter *for F at P.*

Given a local parameter t for F at P, any $x \in k^{\times}$ has a unique factorization related to the local parameter t, given by $x = ut^n$, where $u \in k$ satisfies $v(u) = 0$. Furthermore it is assumed that the valuation is normalized, i.e. $v(t) = 1$. Hence, $v(x) = v(ut^n) = v(u) + v(t^n) = v(t^n) := n$.

Example 1.8. *In $\mathbb{Q}$, we can define valuation with respect to a prime in Z. For a prime p and $\alpha \in \mathbb{Q}$ satisfying $p \nmid \alpha$, we have $v_p(\alpha p^n) = n$. For instance with respect to the prime 3, the valuation of the integer 9 gives $v_3(9) = v_3(3) + v_3(3) = 2$. Note that this corresponds with $9 = 3^2$.*

Moreover, a discrete valuation gives rise to the discrete valuation ring $R := \{x \in k \mid v(x) \geq 0\}$, with maximal ideal $\mathcal{M} := \{x \in k \mid v(x) > 0\}$. The residue field is defined as $k_v = R \setminus \mathcal{M}$. For a local parameter t with valuation v, any element of $\mathcal{M}$ is of the form ut^n, $n \geq 1$ and $u \in k$. Therefore, $\mathcal{M} = \langle t \rangle$.

Definition 1.26. ***Reduction***
Let E be an elliptic curve over k, and let $\tilde{E}$ be the reduction modulo $\mathcal{M}$ of a minimal Weierstrass equation for E.

1.) E has good reduction *if $\tilde{E}$ is non-singular.*

2.) E has multiplicative reduction *if $\tilde{E}$ has a node (see Figure 7.a).*

3.) E has additive reduction *if $\tilde{E}$ has a cusp (see Figure 7.b).*

In cases 2.) and 3.) E is said to have bad reduction.

Multiplicative reduction is *split* if the slopes of the tangent lines at the node are in k, otherwise the reduction is non-split.

Example 1.9. *Take the elliptic curve $E: \; y^2 = x^3 + t^2x$ defined over the field $\mathbb{Q}(t)$. The local parameters are of the form $t - \alpha$, with $\alpha \in \mathbb{Q}$. Given the local parameter t, the reduced elliptic curve is $\tilde{E}: \; y^2 = x^3$. $\tilde{E}$ is a singular curve, with a cusp at the singular point (see Figure 7.b). Hence, E has additive reduction modulo t.*

Remark. Let E be an elliptic curve over a field k defined by $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, where $a_i \in R$ for all $i = 1, ..., 6$. The corresponding Weierstrass form $E : y^2 = x^3 - 27c_4x - 54c_6$, with $c_4, c_6 \in k$, is minimal if one of the following hold:

- $v(\Delta) < 12$
- $v(c_4) < 4$
- $v(c_6) < 6$

For an arbitrary field k, an algorithm to determine whether a Weierstrass form is minimal is given in [23].

Proposition 1.13. *Let E be an elliptic curve over a field k, given by a minimal Weierstrass equation*

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let c_4 be such that in Weierstrass form we have $E : y^2 = x^3 - 27c_4x - 54c_6$.

1.) E has good reduction modulo $\mathcal{M}$ if and only if $v(\Delta) = 0$, i.e. $\Delta \in R^*$. In this case $\tilde{E}$ is an elliptic curve over k_v.

2.) E has multiplicative reduction modulo $\mathcal{M}$ if and only if $v(\Delta) > 0$ and $v(c_4) = 0$, i.e. $\Delta \in \mathcal{M}$ and $c_4 \in R^*$. The set of nonsingular points of $\tilde{E}(k_v)$ is isomorphic to the multiplicative group $\bar{k_v}^*$.

3.) E has additive reduction modulo $\mathcal{M}$ if and only if $v(\Delta) > 0$ and $v(c_4) > 0$, i.e. $\Delta, c_4 \in \mathcal{M}$. The set of nonsingular points of $\tilde{E}(k_v)$ is isomorphic to the additive group $\bar{k_v}^+$.

Proof. The proof can be found in [23]. ■

Moreover, there are only finitely many primes where E/k has bad reduction, since it only has bad reduction at the local parameters dividing the discriminant Δ.

Example 1.10. *The elliptic curve $E : \ y^2 = x^3 - 2x + 1$ defined over $\mathbb{Q}$ has discriminant $\Delta = 80 = 2^4 * 5$. Hence, E has bad reduction modulo the local parameters 2 and 5. E has $v_5(c_4) = v_5(\frac{2}{27}) = 0$ and $v_2(c_4) = v_2(\frac{2}{27}) = 1$. Therefore, by Proposition 1.13 we find that E has multiplicative reduction modulo 5 and additive reduction modulo 2.*

2 Elliptic Surfaces

As mentioned in the introduction on blow-ups, the rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, given by $(x : y : z) \mapsto (F(x,y,z) : G(x,y,z))$ for two projective cubic curves F, G can be defined everywhere after blowing-up the points in the base points of the pencil defined by F and G. In particular, *rational elliptic surfaces* can be constructed by blowing up these base points.

2.1 Definition of Elliptic Surfaces

Definition 2.1. *An* elliptic surface *S over a smooth projective curve C over k is a smooth projective surface S with an elliptic fibration over C, i.e. a surjective morphism $f : S \to C$, such that*

1. *All but finitely many fibers are smooth curves of genus 1*
2. *No fiber contains an exceptional curve of the first kind*

An exceptional curve of the first kind is a smooth rational curve of self-intersection -1 (also called a (-1)-curve). A fiber is defined as $f^{-1}(t)$ for a point $t \in C$.

The property in Definition 2.1.2 can be assumed, since in the case that a surface $f : S \to C$ does contain a fiber with a $(-1)-$curve, the $(-1)-$curve can be blown down. This results in a surface birational to the original surface.

In this thesis the assumption is made that every elliptic fibration has a singular fiber. In particular, an elliptic surface is not isomorphic to $\varepsilon \times C$ where ε is an elliptic curve, so the fibration is not constant. In addition, it is assumed that every elliptic surface has a *section* over S.

Definition 2.2. *A* section *of an elliptic surface $f : S \to C$ is a morphism $\pi : C \to S$ such that $f \circ \pi = id_C$.*

Definition 2.3. *The* generic curve *ε of an elliptic surface $f : S \to C$ is an elliptic curve over the function field $k(C)$. We denote by ε the fiber corresponding to the* generic point *η, meaning $\varepsilon = f^{-1}(\eta)$. The generic point is defined as the point whose closure in the Zariski topology is the whole of C.*

Every section of a fibration corresponds to a $k(C)$-rational point P on the generic fiber, defined by the intersection $\pi(C) \cap \varepsilon$. As a matter of fact, there is a correspondence between $k(C)-$rational points of the generic fiber and sections.

Proposition 2.1. *Let $f : S \to C$ be an elliptic surface defined over k. Let $\varepsilon/k(C)$ be the generic fiber of the surface and denote the group of sections of the surface by $\pi(C/k)$. Then there is a group isomorphism between the $k(C)-$rational points on $\varepsilon/k(C)$, denoted $\varepsilon(k(C))$, and the sections of the surface:*

$$\varepsilon(k(C)) \cong \pi(C/k).$$

Proof. The proof can be found on page 211 of [22]. ∎

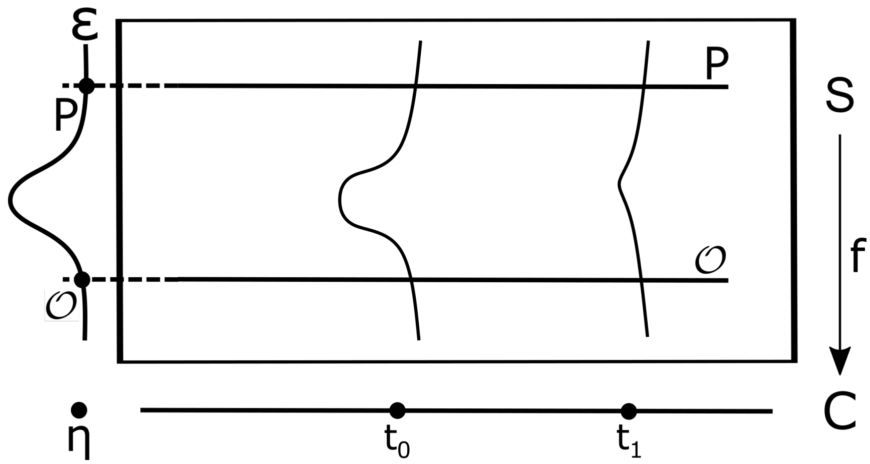


Figure 8: The points on the generic fiber ε corresponding to two sections $P, \mathcal{O}$ of a surface S.

For a schematic drawing of fibers, sections and the generic fiber on an elliptic surface, see Figure 8. The $k(C)-$rational points on the generic curve ε form the Mordell-Weil group of ε, by Theorem 1.12. Therefore, one of the points is assigned as the unit element $\mathcal{O}$ of the Mordell-Weil group, and because of the isomorphism from Proposition 2.1 the corresponding section is also denoted by $\mathcal{O}$.

2.1.1 Rational Elliptic Surfaces

Rational elliptic surfaces are defined by being birational to $\mathbb{P}^2$. Two surfaces S, S' are birational if there exists a composition of birational maps between them.

Definition 2.4. *A surface S is said to be rational if it is birational to $\mathbb{P}^2$.*

As blow-ups are rational maps, blow-ups also define whether surfaces are birational. Hence, if there exists a map $f : S \to S'$ between two surfaces S, S' given by the composition of blow-ups and blow-downs, S and S' are birational.

Theorem 2.2. *Let k be an algebraically closed field. Let $\pi : S \to \mathbb{P}^1$ be a* rational elliptic surface*. Then S is the 9-fold blow-up of the plane $\mathbb{P}^2$ at the base points of a pencil of generically smooth cubics curves which induces the fibration π.*

Proof. The proof can be found on page 37 in [16]. ■

In order to show the construction of a rational elliptic surface from a pencil of cubics, let the smooth cubic curves $F, G \in \mathbb{P}^2$ have no factor in common. By Bézout, F and G have nine points of intersection. The map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $(x_0 : y_0 : z_0) \mapsto (F(x_0, y_0, z_0) : G(x_0, y_0, z_0))$ is not defined at these base points. Blowing up these nine points results in a surface $S \cong \{(x : y : z) \times (t : u) \mid tF + uG = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1$. Projection onto the second factor $\pi : S \to \mathbb{P}^1$ makes S an elliptic surface. Moreover, it is a rational elliptic surface since S is birational to $\mathbb{P}^2$. The diagram corresponding to this construction can be found in Figure 9.

$$\begin{tikzcd} & S \arrow[dl, "\substack{\text{Blow up of}\\ \text{points in}\\ F\cap G}"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi"] & & \mathbb{P}^1 \end{tikzcd}$$

Figure 9: The blow up of a pencil of cubics in $\mathbb{P}^2$ gives a rational elliptic surface S.

For a rational elliptic surface, Miranda and Persson [17] state that the Picard number is $\rho(S) = 10$. This number is due to the blow-ups at the nine base points of a pencil of cubics.

The construction of a rational elliptic surface can also be reversed by blowing down the sections. This gives a minimal model for the rational elliptic surface. This will be the topic of section 4.1, where the blow-down of rational elliptic surfaces to Hirzebruch surfaces are found.

Definition 2.5. *Over an algebraically closed field $\bar{k}$, any rational elliptic surface whose fibers do not contain $(-1)-$curves is called* relatively minimal*. Over a non-algebraically closed field k, a rational elliptic surface is called* minimal *if there are no curves that can be blown down over k.*

2.1.2 Reduction and Elliptic Surfaces

For $u \neq 0$, a pencil of cubics $\{tF + uG = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1$ can be written as $\{tF + G = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1$, with $(t : 1) = (\frac{t}{u} : \frac{u}{u}) \in \mathbb{P}^1$. If $u = 0$, the point $(t : u)$ corresponds to the point $t = \infty$. From here onward, $(t : 1) \in \mathbb{P}^1$ is denoted as t, making the notation for a pencil $\{tF+G = 0\}$.

Given a point P on the curve $\mathbb{P}^1$, a rational elliptic surface $\pi : S \to \mathbb{P}^1$ can have bad reduction at local parameters $t \in \mathcal{O}_{\mathbb{P}^1,P}$. If the elliptic surface is the blow-up of a pencil of cubics, the blow-up will give singular fibers at the corresponding places of bad reduction. All singular fibers are of the form I_n for $n > 0$, I_n^* for $n \geq 0$, II, III, IV, II^*, III^* or IV^*. They were classified by Kodaira [12] and can be found in Figure 10. When the reduction is multiplicative, the corresponding fiber is of type I_n. In the case of additive reduction the fiber type has an asterisk. Irreducible components make up a singular fiber.

Definition 2.6. *A fiber is* reducible *if it has more than one irreducible components. A fiber is* non-reduced *if is contains an irreducible component with multiplicity > 1.*

Proposition 2.3. *An* irreducible component *of a reducible fiber is a smooth rational curve with self intersection -2.*

Proof. The proof can be found on page 7 in [16]. ∎

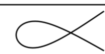
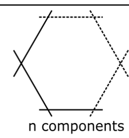
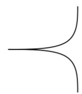
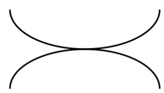
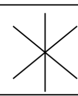
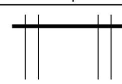
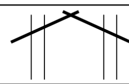
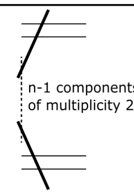
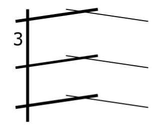
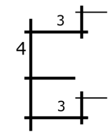
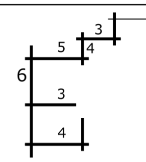
Reduction type	Number of components	Fiber
I_0	1	Any smooth fiber
I_1	1	
I_n	n	
II	1	
III	2	
IV	3	
I_0^*	5	
I_1^*	6	
I_n^*	n+5	
IV^*	7	
III^*	8	
II^*	9	

Figure 10: All types of fibers. Multiple components have a thick line, where if no multiplicity is indicated signifies multiplicity 2.

Using Tate's algorithm [25], the singular fiber belonging to a local parameter t can be determined. This algorithm gives the types of fibers a surface has based on certain properties of the extended Weierstrass form $y^2 = x^3 + a_2x^2 + a_4x + a_6$, in a given characteristic unequal to 2 or 3.[5] The discriminant of the extended Weierstrass is of the form

$$\Delta = -27a_6^2 + 18a_2a_4a_6 + a_2^2a_4^2 - 4a_2^3a_6 - 4a_4^3. \tag{4}$$

Proposition 1.13 indicates that we encounter a singular fiber at t whenever $v(\Delta) > 0$. The valuation of a_4 and a_6 specify the type of singular fiber at t, according to Table 1.

Example 2.1. *Let S be the pencil $S : \{tF + G = 0\}$, where $G : x^3 - zy^2 = 0,\ F : z^3 = 0$. In affine space this is equal to the pencil $y^2 = x^3 + t$ (see Example 1.4), therefore $a_2 = 0,\ a_4 = 0,\ a_6 = t$. The discriminant in equation 4 becomes $\Delta = -27t^2$. Hence, there is a singular fiber at $t = 0$. As $a_4 = 0, a_6 = t$, $v(a_4) = \infty$ and $v(a_6) = 1$, this indicates a fiber of type II at $t = 0$ according to Table 1.*

[5]The cases where the characteristic is equal to 2 or 3 can be found in Section 4.5 in [20].

Fiber type	$v(a_4)$	$v(a_6)$
I_0	$\begin{cases} 0 & \geq 0 \\ \geq 0 & 0 \end{cases}$	
I_n with $(n > 0)$	0	0
II	≥ 1	1
III	1	≥ 2
IV	≥ 2	2
I_0^*	$\begin{cases} 2 & \geq 3 \\ \geq 2 & 3 \end{cases}$	
I_n^* with $(n > 0)$	2	3
IV^*	≥ 3	4
III^*	3	≥ 5
II^*	≥ 4	5

Table 1: Types of fiber based on valuation of Weierstrass coefficients [20]. No sign implies an equality, a greater or equal sign implies that a greater or equal value to what is stated is sufficient.

Moreover, the parameter at infinity has to be considered. Letting $t = \infty$*, we can set* $t = \frac{1}{s}$*, resulting in the local parameter* $s = 0$*. In affine space, the pencil becomes* $y^2 = x^2 + \frac{1}{s}$*. Rewriting gives*

$$s^6 y^2 = s^6 x^3 + s^5.$$

Let $\mathrm{y} = s^3 y,\ \ \mathrm{x} = s^2 x$*, then*

$$\mathrm{y}^2 = \mathrm{x}^3 + s^5.$$

Hence, we get the discriminant $\Delta = -27s^{10}$*. We find that* $a_4 = 0, a_6 = s^5$*,* $v(a_4) = \infty$ *and* $v(a_6) = 5$*. Table 1 indicates that we hence have a type II* fiber at* $t = \infty$*. Therefore, the pencil of cubics* $S : \{tF + G = 0\}$ *gives rise to the elliptic surface with the fibers II*,II. The corresponding elliptic surface can be seen in Figure 11. At all other parameters* $t \neq 0, \infty$*, the surface consists of nonsingular fibers.*

2.1.3 Extremal Rational Elliptic Surfaces

The sections of an elliptic surface form the Mordell-Weil group of the generic fiber of the surface by Proposition 2.1. The rank of the Mordell-Weil group determines whether an elliptic surface is extremal.

Definition 2.7. *An elliptic fibration* $f : S \to C$ *is* extremal *if the rank of the Mordell-Weil group is zero.*

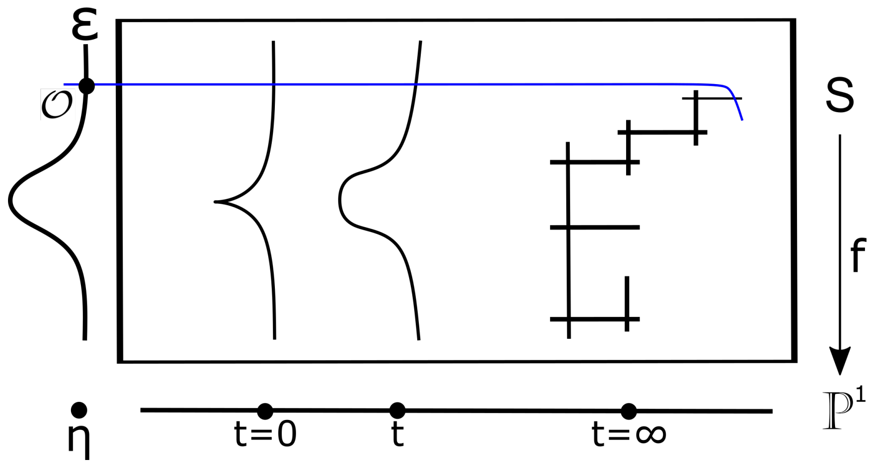


Figure 11: The rational elliptic surface $f : S \to C$ with singular fibers II^*, II at respectively $t = \infty$ and $t = 0$.

This thesis focuses on extremal rational elliptic surfaces. By the Mordell-Weil theorem 1.12, the group of $k(\mathbb{P}^1)-$rational points on the generic curve ε of an extremal rational elliptic surface is equal to

$$Tor(\varepsilon(k(\mathbb{P}^1))) \oplus \mathbb{Z}^r = Tor(\varepsilon(k(\mathbb{P}^1))) \oplus \mathbb{Z}^0 = Tor(\varepsilon(k(\mathbb{P}^1))).$$

Hence, the Mordell-Weil group of an extremal rational elliptic surface is the torsion group. Furthermore, a rational elliptic surface being extremal has consequences for the number of curves with negative self intersection [17].

Proposition 2.4. *Let X be a rational elliptic surface. Then the following are equivalent:*

i. X is extremal

ii. The number of representations as a blow-up of $\mathbb{P}^2$ is finite

iii. The number of rational curves C with $C^2 < 0$ is finite

iv. The number of reduced curves C with $C^2 < 0$ is finite

Proof. The proof can be found on page 75 in [16]. ■

As there are only finitely many curves with negative self-intersection by Proposition 2.4, the number of blow-downs possible on an extremal rational elliptic surface is in fact finite.

In Table 2 all extremal rational elliptic surfaces and their corresponding torsion group can be found. As follows from Section 2.1.2, the first 10 surfaces contain additive fibers. The torsion column indicates the group structure of the set of torsion points of the generic curve, and hence of the set of sections of the surface.

Remark. As extremal rational elliptic surfaces are a type of rational elliptic surface, an extremal rational elliptic surface over an algebraically closed field $\bar{k}$ can be constructed by blowing up the base points of a pencil of cubics in $\mathbb{P}^2$.

Fibration	Rank MW	Torsion
II^*, II	0	$\{0\}$
$II^*, 2I_2$	0	$\{0\}$
$I_4^*, 2I_1$	0	$\mathbb{Z}/2\mathbb{Z}$
III^*, III	0	$\mathbb{Z}/2\mathbb{Z}$
III^*, I_2, I_1	0	$\mathbb{Z}/2\mathbb{Z}$
IV^*, IV	0	$\mathbb{Z}/3\mathbb{Z}$
IV^*, I_3, I_1	0	$\mathbb{Z}/3\mathbb{Z}$
$I_2^*, 2I_2$	0	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
I_1^*, I_4, I_1	0	$\mathbb{Z}/4\mathbb{Z}$
$2I_0^*$	0	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$I_9, 3I_1$	0	$\mathbb{Z}/3\mathbb{Z}$
$I_8, I_2, 2I_1$	0	$\mathbb{Z}/4\mathbb{Z}$
I_6, I_3, I_2, I_1	0	$\mathbb{Z}/6\mathbb{Z}$
$2I_5, 2I_1$	0	$\mathbb{Z}/5\mathbb{Z}$
$2I_4, 2I_2$	0	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$4I_3$	0	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

Table 2: All extremal rational elliptic surfaces and their torsion [18].

Considering theorem 2.2, an extremal rational elliptic surface can always be blown-down to $\mathbb{P}^2$ if the field is algebraically closed. The statement does not necessarily hold over more general fields. Some blow-downs to $\mathbb{P}^2$ are not defined over general fields, as the following example illustrates. The preliminaries on Galois actions on sections of fibers can be found in [14].

Example 2.2. *Take the pencil of cubics* $S: \ \{(t^2-t+1)y^2 - x^3 - x = 0\}$*, whose blow-up of base points gives the surface with the* $2I_0^*$ *fibers. The roots of* x^3+x *correspond to the sections. As the roots are* $x = 0, \pm i$*, there are four sections* $(0,0,1), (i,0,1), (-i,0,1)$ *and* $\mathcal{O}$*. The sections containing* $\pm i$ *are not defined over* $\mathbb{Q}$*, and hence cannot be blown down over* $\mathbb{Q}$ *individually. Therefore, the extremal rational elliptic surface constructed from* S *has torsion group* $\mathbb{Z}/2\mathbb{Z}$ *over* $\mathbb{Q}$*. As a result, the minimal surface for this elliptic surface is not* $\mathbb{P}^2$*, but is birational to a minimal Châtelet surface (Figure 12) [6].*

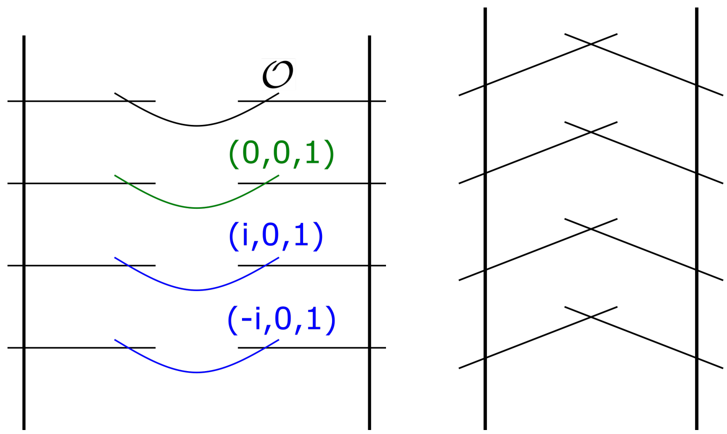
A well-known formula regarding elliptic surfaces is the *Shioda-Tate formula.* The formula relates the number of components of singular fibers to the rank of the elliptic surface.

Proposition 2.5. ***Shioda-Tate formula***
The Picard number[6] ρ *of an elliptic surface* $f: S \to C$ *is given by*

$$\rho(S) = rk(MW) + 2 + \sum_{v \in \Sigma} (m_v - 1),$$

where Σ *is the finite set of points* $P_1, ..., P_n$ *on* C *such that* $f^{-1}(P_i)$ *is a singular fiber. The number of irreducible components of the singular fiber is denoted by* m_v*.*

[6]The Picard number ρ is the rank of the Néron-Severi group of an elliptic surface. More details can be found in [16].



(a) The extremal rational elliptic surface defined by $S:\ \{(t^2-t+1)y^2-x^3-x=0\}$

(b) The minimal Châtelet surface.

Figure 12: An extremal rational elliptic surface whose minimal model contains $(-1)-$curves.

Proof. The proof can be found in [24] ■

As mentioned in Section 2.1.1, the Picard number of a rational elliptic surface S is $\rho(S) = 10$. From this it follows that $\sum_{v\in\Sigma}(m_v - 1) = 8$ for extremal rational elliptic surfaces. As a result, a singular fiber on an extremal rational elliptic surface can have at most 9 irreducible components.

3 Blow-downs to $\mathbb{P}^2$

This section studies the construction of the minimal model $\mathbb{P}^2$ from extremal rational elliptic surfaces with at least one additive fiber. By blowing down $(-1)-$curves in the surface in a certain order, $\mathbb{P}^2$ will be formed. Hence, the curves in the minimal model will have a self intersection that agrees with Bézout's theorem. The sum of the intersection multiplicities between these curves in the minimal model will always be nine. This is due to the fact that each pencil in $\mathbb{P}^2$ whose blow-up defines the surface has nine base points, meaning there will be nine blow-downs.

In order to properly understand the change in self intersection after a blow-down, the following proposition will be proven.

Proposition 3.1. *Take a surface S with a curve l, with self intersection $l^2 = n$. A blow-up $\pi : S \to C$ of a point P on l results in the self intersection of the strict transform l' of l to be $l'^2 = n - 1$.*

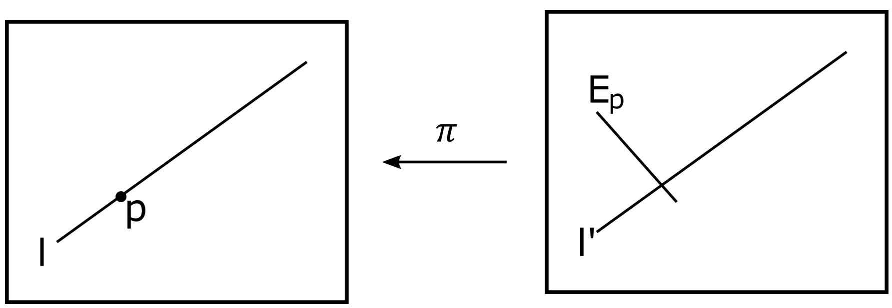


Figure 13: The blow-up of a point P on a line l.

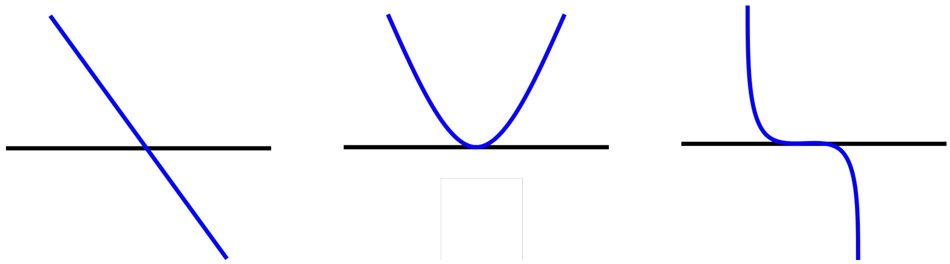
Proof. Let l have self intersection $l^2 = n$. As $\pi^*(l) = l' + E_p$,

$$\begin{aligned}(\pi^*(l))^2 &= (l' + E_p)^2 \\ &= l'^2 + 2l'E_p + E_p^2 \\ &= l'^2 + 1\end{aligned}$$

As a blow-up is a degree 1 map, by Proposition 1.8 we have $\pi^*(l)^2 = l^2$. Therefore, the self intersection of l' is $n - 1$. ∎

Moreover, the type of intersection between curves corresponds to the intersection multiplicity. If two curves intersect *transversally* (Figure 14.a), then the intersection multiplicity is 1. If two curves intersect *tangentially*, (Figure 14.b), then the intersection multiplicity is ≥ 2. If two curves intersect in a *point of inflection* (Figure 14.c), then the intersection multiplicity is ≥ 3 [9].

What follows are the blow-downs of extremal rational elliptic surfaces with at least one additive fiber to $\mathbb{P}^2$ over the algebraically closed field $\bar{\mathbb{Q}}$. For all figures the green numbers represent the self-intersection of the corresponding curve, the blue numbers represent the



(a) (b) (c)

Figure 14: Intersection multiplicity by way of intersecting.

intersection multiplicity. Bold lines have multiplicity 2 or higher, in the case the line multiplicity is higher than 2 this is indicated. As the singular points of II and I_1 are not one of the base points of the pencil, their blow-down does not give any additional information regarding the base points of the pencil. Therefore, the singular fibers II and I_1 are not drawn in the blow-downs. The pencils whose blow up generates the particular elliptic surface were found either in a paper by Kurumadani [13], or by trial and error using the magma code in Appendix 5.

3.1 $2I_0^*$

The blow down of the surface with $2I_0^*$ fibers is visible in Figure 15. After contracting of the sections, there are eight exceptional curves that can be blown-down. However, keeping in mind that in $\mathbb{P}^2$ lines have self-intersection 1 (Example 1.6) we must blow down such that one of the vertical components has self intersection 1. The last blow-down is the contraction of the other vertical component, resulting in two cubics.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with $2I_0^*$ is given by

$$S : \{y^2z - x^3 - xz^2 + ty^2z = 0\} \text{ [13]}.$$

Its Weierstrass form is

$$y^2 = x^3 + t^2x.$$

The bad places of the pencil are at $t = -1$ and $t = \infty$, both corresponding to a I_0^* fiber.

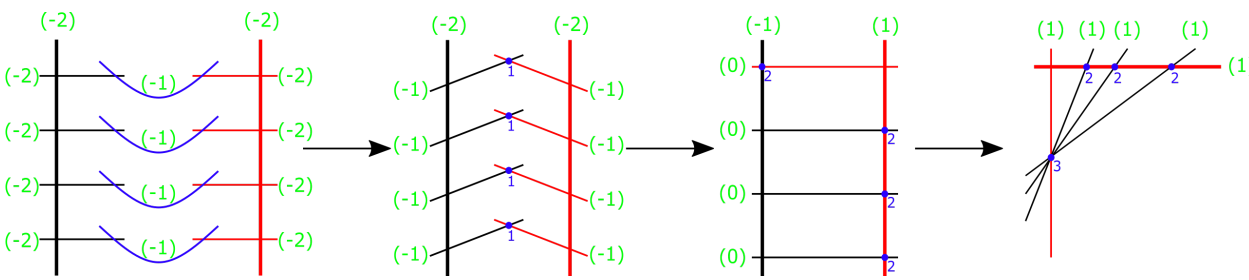


Figure 15: Blow down of $2I_0^*$.

3.2 I_1^*, I_4, I_1

The blow down of the surface with I_1^*, I_4, I_1 fibers is visible in Figure 16. The first four blow-downs concern the sections. For the next series of blow-downs, one of the components of the I_4 fiber, and three components of the I_1^* fiber are contracted. Then the last blow-down that results in all lines having self intersection (1) is done on the last remaining $(-1)-$curve.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with I_1^*, I_4, I_1 is given by

$$S : \{t(z^2 + xy)(x + y) + x^2(x - y - 2z) = 0\} \text{ [13]}.$$

Its Weierstrass form is

$$y^2 = x^3 + (-6912t^4 + 27648t^3 - 6912t^2)x + (221184t^6 - 1327104t^5 + 1658880t^4 + 221184t^3).$$

The bad places of the pencil are at $t = 0$, $t = 4$ and $t = \infty$, corresponding to the I_1^*, I_1 and I_4 fibers, respectively.

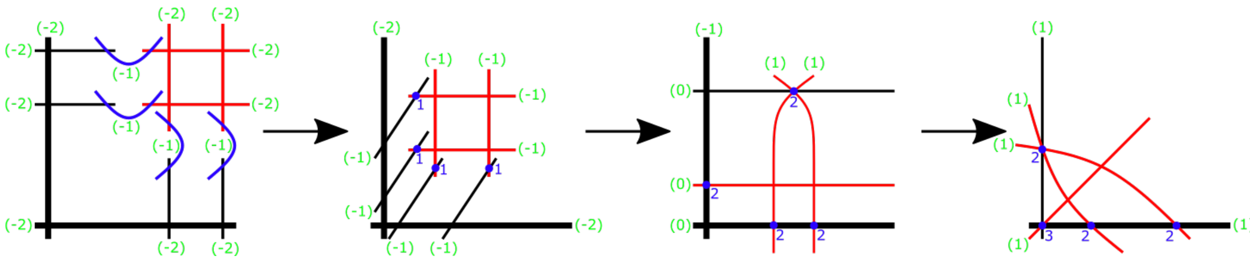


Figure 16: Blow down of I_1^*, I_4, I_1.

3.3 IV^*, IV

The blow down of the surface with IV^*, IV fibers is visible in Figure 17. The first three blow-downs are done on the sections, and afterwards the three newly created (-1)−curves on the IV^* fiber are blown down. This again creates three new (-1)−curves on the former IV^* fiber. After these are blown down, the minimal surface $\mathbb{P}^2$ has been constructed.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with IV^*, IV fibers is given by

$$S : \{tz^3 + x^3 - y^3 = 0\}.$$

Its Weierstrass form is

$$y^2 = x^3 - \frac{314928}{t^4}.$$

The bad places of the pencil are at $t = 0$ and $t = \infty$, corresponding to the IV^* and IV fibers, respectively.

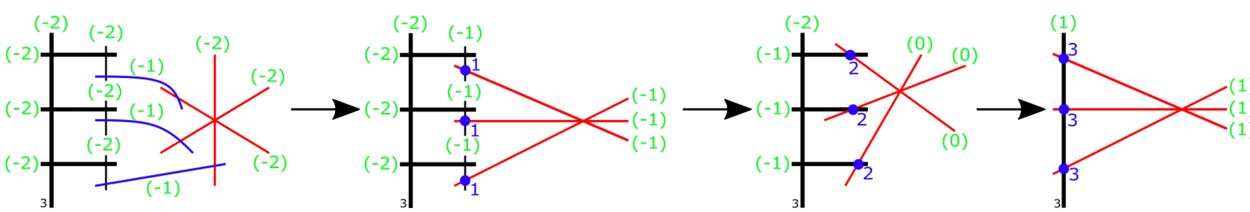


Figure 17: Blow down of IV^*, IV.

3.4 IV^*, I_3, I_1

The blow down of the surface with IV^*, I_3, I_1 fibers is visible in Figure 18. The blow down to the minimal surface follows the same path as the blow down of IV^*, IV.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with IV^*, I_3, I_1 fibers is given by

$$S : \{y^2 z = z(x - tz)^2 + x^3\} \text{ [13]}.$$

Its Weierstrass form is

$$y^2 = x^3 + (-2592t - 432)x + (46656t^2 + 31104t + 3456).$$

The bad places of the pencil are at $t = 0$, $t = \frac{-4}{27}$ and $t = \infty$, corresponding to the I_3, I_1 and IV^* fibers, respectively.[7]

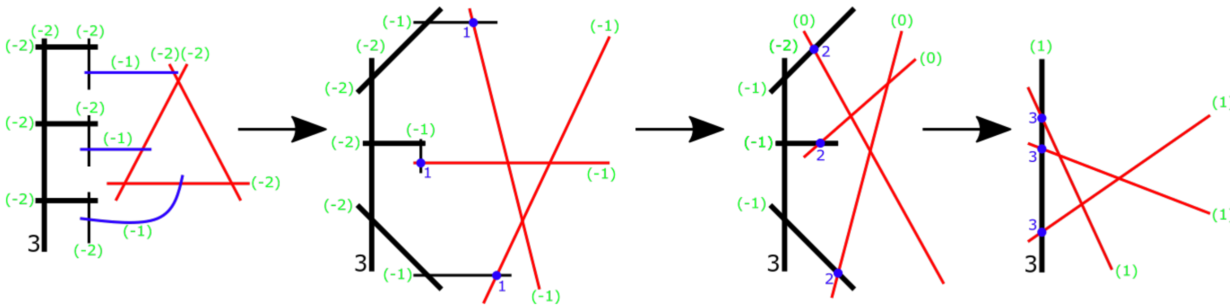


Figure 18: Blow down of IV^*, I_3, I_1.

3.5 $I_4^*, 2I_1$

The blow down of the surface with $I_4^*, 2I_1$ fibers is visible in Figure 19. In order to blow-down to the minimal surface $\mathbb{P}^2$, the two sections are contracted first. The next six blow-downs are applied on the $(-1)-$curves newly created by the previous blow-down. Then, there is a surface with four $(-1)-$curves. One $(-1)-$curve only intersecting another $(-1)-$curve is blown down, followed by the blow down of the curve intersecting both a $(-1)-$curve and $(-2)-$curve. Any other combination of blow-downs does not result in the minimal surface $\mathbb{P}^2$. After the last remaining $(-1)-$curve has been blown-down, the minimal surface $\mathbb{P}^2$ is found.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with $I_4^*, 2I_1$ is given by

$$S : \{tz(x+z)^2 + y^2z - x^3 - x^2z = 0\} \text{ [13]}.$$

Its Weierstrass form is

$$y^2 = x^3 + (-432t^2 - 1728t - 432)x + (-3456t^3 - 20736t^2 - 25920t + 3456).$$

The bad places of the pencil are at $t = 0,\ t = -4$ and $t = \infty$, corresponding to the two I_1 and I_4^* fibers, respectively.

[7]For a study of the elliptic curves $y^2 = x^3 + A(x - t)$ describing this surface, see the Master's thesis of Monique van Beek [4].

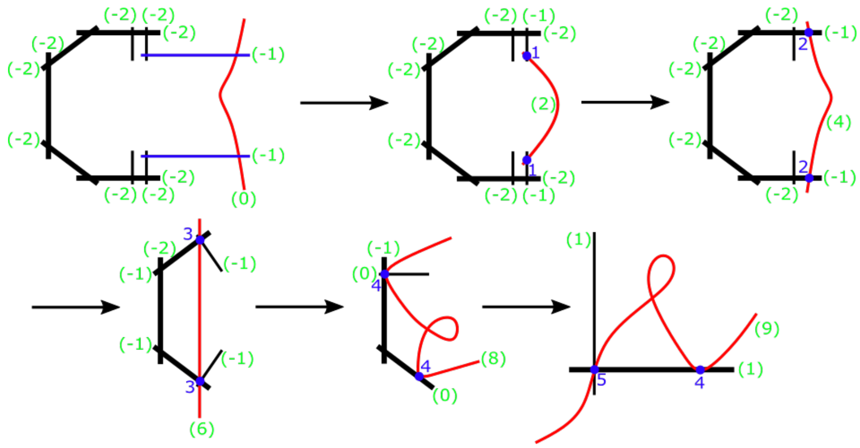


Figure 19: Blow down of $I_4^*, 2I_1$.

3.6 II^*, II and $II^*, 2I_1$

The blow down of the surfaces with II^*, II and $II^*, 2I_1$ fibers is visible in Figure 20. There is no choice in which $(-1)-$curve to blow down for the first seven blow-downs. The first blow down is of the section, followed by the blow-down of the $(-1)-$curve generated by the previous blow-down. For the eighth blow-down there are two choices of exceptional curve to blow down. However, only blowing down the $(-1)-$curve intersecting both the other $(-1)-$curve and the $(-2)-$curve results in the minimal surface $\mathbb{P}^2$.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with II^*, II is given by

$$S : \{tz^3 + x^2 - zy^2 = 0\}.$$

Its Weierstrass form is

$$y^2 = x^3 + t.$$

The bad places of the pencil are at $t = 0$ and $t = \infty$, corresponding to the two II and II^* fibers, respectively.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with $II^*, 2I_1$ is given by

$$S : \{tz^3 + x^3 + x^2z - zy^2 = 0\}.$$

Its Weierstrass form is

$$y^2 = x^3 - 432 * x + (46656 * t + 3456).$$

The bad places of the pencil are at $t = 0$, $t = \frac{-4}{27}$ and $t = \infty$, corresponding to the two I_1 and II^* fibers, respectively.

As previously mentioned, the fibers II and I_1 are not drawn in the blow down. This is the reason that figure 20 suffices to show the blow-down for both surfaces.

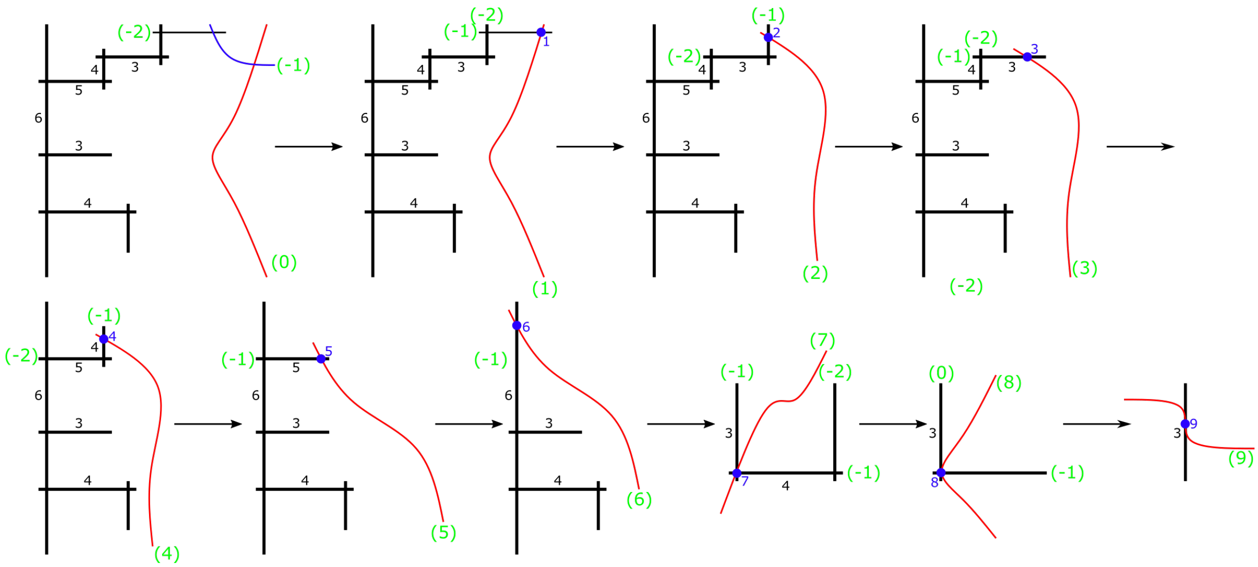


Figure 20: Blow down of II^*, II and $II^*, 2I_1$.

3.7 III^*, III

The blow down of the surface with III^*, III fibers is visible in Figure 21. Firstly, the sections are blown down. This is followed by four blow downs on the former III^* fiber. Then, one of the two available $(-1)-$curves is blown-down. The component this curve intersects is blown down next, and after blowing down the remaining $(-1)-$curve the minimal model $\mathbb{P}^2$ has been constructed.

The minimal model contains a line of multiplicity 3, and two curves with self intersection 1 and 4. As lines are $(1)-$curves and conics are $(4)-$curves, these curves are drawn as a line and conic.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with III^*, III is given by

$$S : \{tz^3 + x^2y - zy^2 = 0\}.$$

Its Weierstrass form is

$$y^2 = x^3 + tx.$$

The bad places of the pencil are at $t = 0$ and $t = \infty$, corresponding to the two III and III^* fibers, respectively.

3.8 III^*, I_2, I_1

The blow down of the surface with III^*, I_2, I_1 fibers is visible in Figure 22. The order of blowing down is the same as for III^*, III.

As for III^*, I_2, I_1, the minimal model contains a line with multiplicity 3, and two curves with self intersection 1 and 4. As lines are $(1)-$curves and conics are $(4)-$curves, these curves are drawn as a line and conic. The only difference in the final minimal model between III^*, III and III^*, I_2, I_1 is that for the minimal model of III^*, III the conic and line are tangent, while for III^*, I_2, I_1 the line intersects the conic twice transversally.

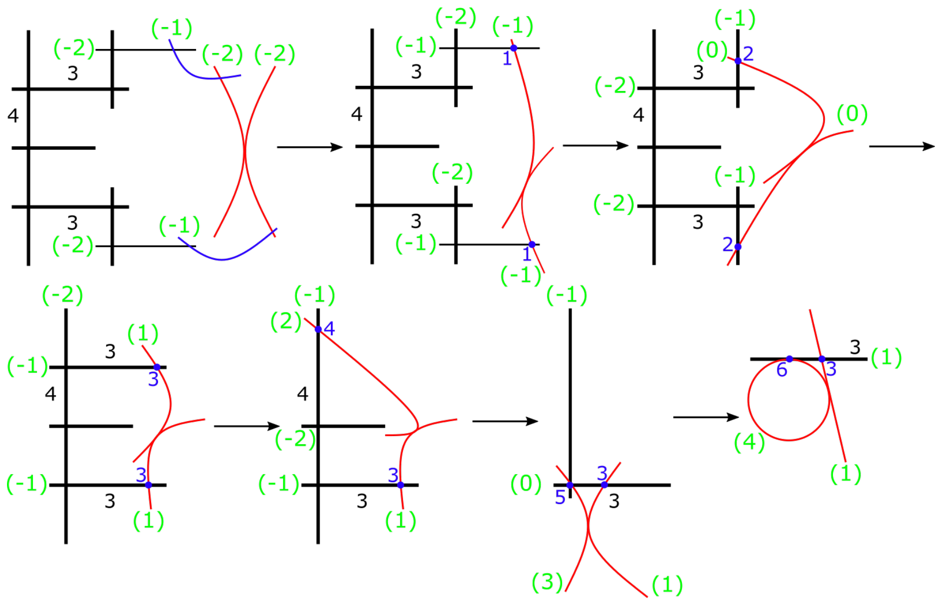


Figure 21: Blow down of III^*, III.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with III^*, I_2, I_1 is given by

$$S : \{tz^3 + x^2y - zy^2 + zxy = 0\}.$$

Its Weierstrass form is

$$y^2 = x^3 + (1296t - 432)x + (-15552t + 3456).$$

The bad places of the pencil are at $t = 0,\ t = \frac{-1}{64}$ and $t = \infty$, corresponding to the two $I_2,\ I_1$ and III^* fibers, respectively.

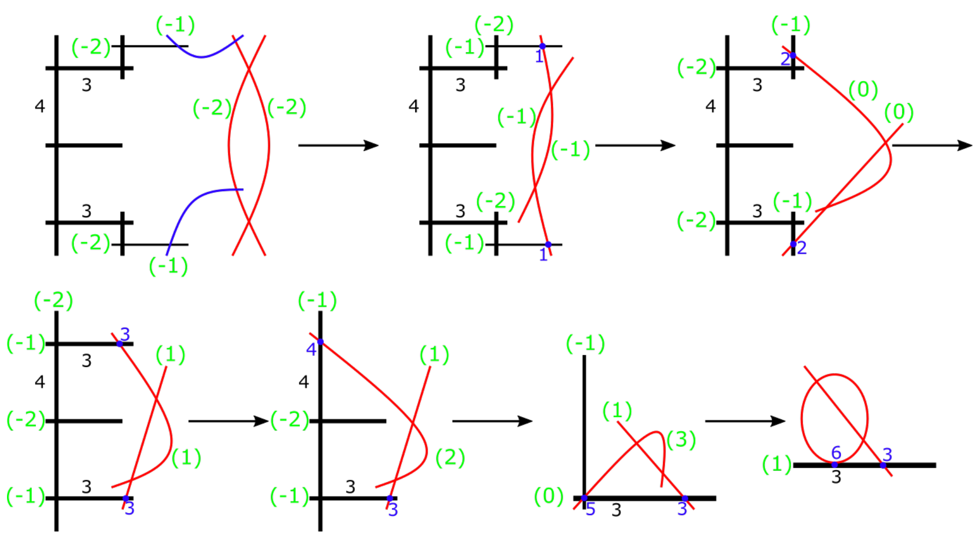


Figure 22: Blow down of III^*, I_2, I_1.

3.9 $I_2^*, 2I_2$

The blow down of the surface with $I_2^*, 2I_2$ fibers is visible in Figure 23. As this is a more cluttered elliptic surface, the colour scheme in Figure 23 is changed compared to the preceding blow-downs. Each of the I_2 fibers and the self intersection of their components are given in red or orange. Moreover, to keep track of the intersection points for each component, the intersection points and their multiplicities have been given different colours. The blow-down itself is not that complicated. After blowing down the sections, three of the four newly formed $(-1)-$curves are blown down. The only way to get to $\mathbb{P}^2$ is by blowing down the $(-1)-$curve that intersects the $(-2)-$curve. For the final blow-down there is only one $(-1)-$curve that can be blown down, resulting in the surface $\mathbb{P}^2$.

A pencil whose blow-up at its base points gives an extremal rational elliptic surface with $I_2^*, 2I_2$ is given by

$$S : \{t(zx^2 + xz^2 - y^2z) + x^3 + zx^2 = 0\}.$$

Its Weierstrass form is

$$y^2 = x^3 + (-432t^4 + 432t^3 - 432t^2)x + (3456t^6 - 5184t^5 - 5184t^4 + 3456t^3).$$

The bad places of the pencil are at $t = 0,\ t = 1$ and $t = \infty$, corresponding to the two $I_2^*,\ I_2$ and I_2 fibers, respectively.

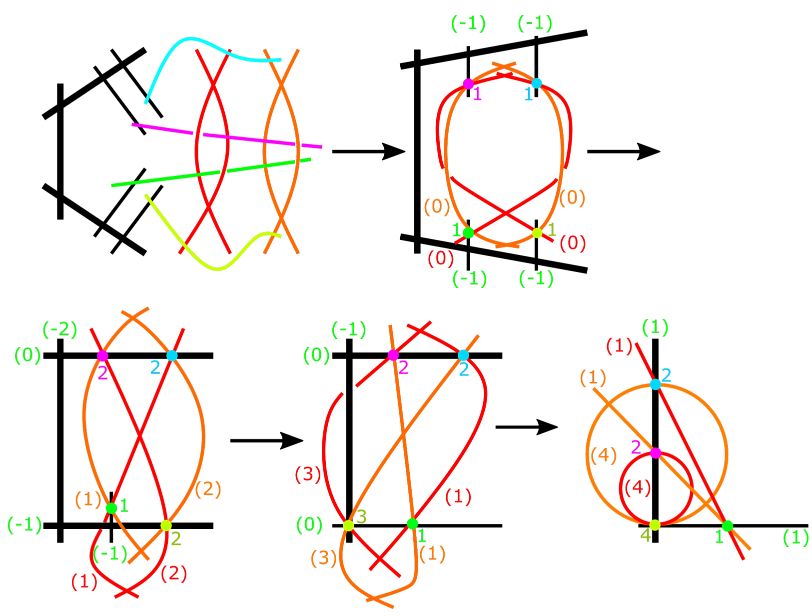


Figure 23: Blow down of $I_2^*, 2I_2$.

4 Hirzebruch Surfaces

Besides $\mathbb{P}^2$, extremal rational elliptic surfaces can also be blown down to the *Hirzebruch surfaces* $\mathbb{F}_n$.

Theorem 4.1. *Let k be an algebraically closed field. Any minimal model of a rational function field is isomorphic to $\mathbb{P}^2$ or to one of the Hirzebruch surfaces $\mathbb{F}_n$, $n \neq 1$.*

Proof. The proof by Hartshorne can be found in [10]. ■

This theorem forms the basis for this section. The goal is to find out which order of blow-downs results in which minimal model.

The fact that an arbitrary (-1)-curve can be blown down follows from Castelnuovo's criterion [19], stated in Theorem 4.2.

Theorem 4.2. ***Castelnuovo's Criterion***
If Y is a curve on a surface X, with $Y \cong \mathbb{P}^1$ and $Y^2 = -1$, then there exists a morphism $f : X \to X_0$ to a (nonsingular projective) surface X_0, and a point $P \in X_0$, such that X is isomorphic via f to the blow-up of X_0 at P, and Y is the exceptional curve.

Proof. The proof can be found in Chapter V of Hartshorne [11]. ■

As a result, any exceptional curve of the first kind can be blown down. This gives various options for minimal models which are not $\mathbb{P}^2$, as in the previous chapter. By Theorem 4.1, a different order of blowing down may result in a minimal surface $\mathbb{F}_n$, the *Hirzebruch surface.*

Definition 4.1. *[8] The* Hirzebruch surface $\mathbb{F}_n$ *is a projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$.*

This definition is quite technical, and uses a lot a new concepts. Hence, this thesis will focus on the two relevant Hirzebruch spaces $\mathbb{F}_0, \mathbb{F}_2$ and their construction. These are the only relevant Hirzebruch surfaces due to the fact that there are no curves of self intersection $(-n)$, $n > 2$ in a rational elliptic surface, as shown in Proposition 4.3.

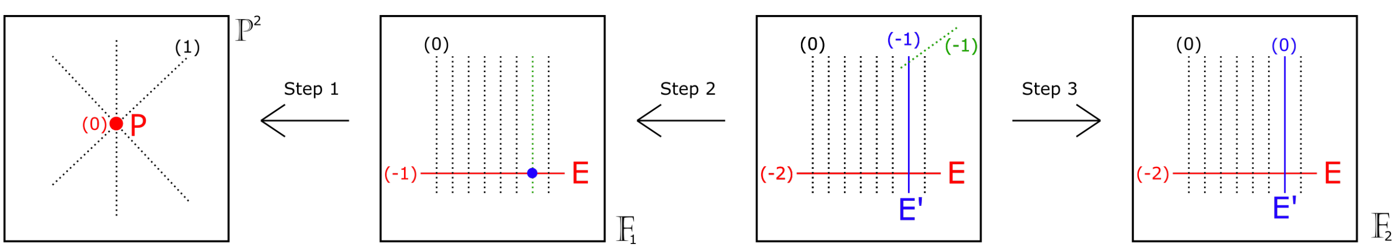


Figure 24: Construction of the Hirzebruch surface $\mathbb{F}_2$

The Hirzebruch surface $\mathbb{F}_n$ can be constructed from $\mathbb{P}^2$ using blow-ups and blow-downs. Starting with a point P in $\mathbb{P}^2$ and considering all lines intersecting P, $\mathbb{F}_1$ is created by blowing up at the point P, as visible in step 1 of Figure 24. This results in the exceptional curve E of self intersection (-1), which is intersected by lines of self intersection (0) at every point. These lines of self intersection (0) correspond to the lines through P, in line with the definition of a blow-up.

Moreover, $\mathbb{F}_2$ is constructed by selecting a point on the exceptional curve E, and doing another blow-up at the selected point. The result is visible after step 2 in Figure 24. Afterwards, blowing down one of the $(-1)-$curves results in a curve with self intersection (-2), intersected at every point by a curve of self intersection (0). This is the Hirzebruch surface $\mathbb{F}_2$. This is step 3 in Figure 24. Repeating steps 2 and 3 another $n-2$ times results in a curve of self intersection $(-n)$, intersected at every point by a curve of self intersection (0), and is hence the Hirzebruch surface $\mathbb{F}_n$.

The construction of $\mathbb{F}_0$ (Figure 25) differs slightly from the construction of the other Hirzebruch surfaces. Starting again with a point P_1 and the lines intersecting it in $\mathbb{P}^2$, a second point P_2 on one of the intersecting lines is picked. Both P_1 and P_2 are blown up, of which the result is visible after step 1 of Figure 25. Subsequently, the $(-1)-$curve connecting the two other $(-1)-$curves is blown down, resulting in a ruled surface with only lines of self intersection (0).

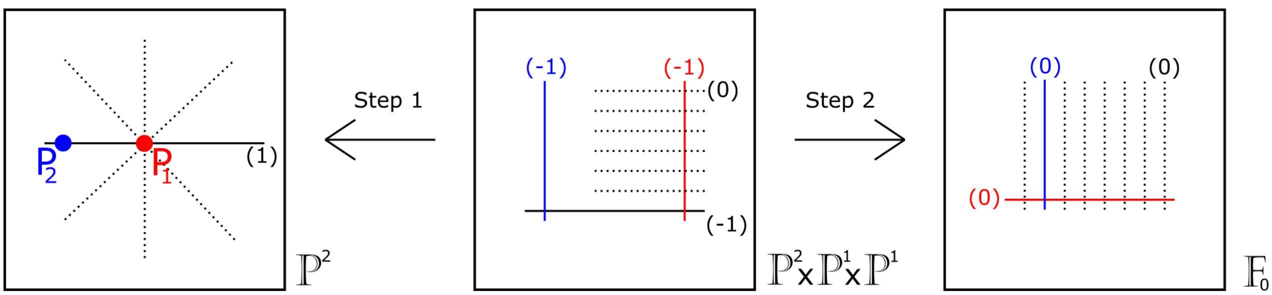


Figure 25: The construction of the Hirzebruch space $\mathbb{F}_0$

From the above constructions it follows that $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up at a point. Moreover, there is the isomorphism $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ [8].

As previously mentioned, the only Hirzebruch surfaces of interest for minimal models of extremal rational elliptic surfaces are $\mathbb{F}_0$ and $\mathbb{F}_2$. This follows from the following proposition:

Proposition 4.3. *On a rational elliptic surface, the only curves with negative self intersection have self intersection -1 or -2.*

Proof. The anti-canonical divisor $-K_X$ of a rational elliptic surface X is linearly equivalent to a fiber F [16]. As the general fiber of an elliptic surface is smooth, only finitely many fibers in X are not smooth. We consider a smooth fiber F. Let C be a curve in X. Then $C \cdot F \geq 0$. Therefore, $C \cdot K_X \leq 0$.

Take a curve C on X, with negative self intersection $-n$, $n \in \mathbb{Z}^+$. By the adjunction formula in Proposition 1.6:

$$2g(C) - 2 = C^2 + C \cdot K$$
$$2g(C) - 2 = \underbrace{-n + C \cdot K}_{<0}$$

The genus degree formula $g(C) = \frac{1}{2}(d-1)(d-2)$ implies $g(C) \geq 0$. Therefore, in order for the left hand side to be less than zero $g(C) = 0$ and

$$-2 = -n + C \cdot K$$

And as $C \cdot K \leq 0$, it must be that $n \leq 2$, hence C can only have negative self intersection (-1) or (-2). ∎

Consequently, there are no $(-n)-$curves for $n > 2$. Since blowing down can only increase a curve's self-intersection, a Hirzebruch surface $\mathbb{F}_n$ with $n > 2$ can never be constructed from blowing down rational elliptic surfaces, as the lowest self intersection of a component on a rational elliptic surface is (-2).

Beauville provides some more technical properties of Hirzebruch spaces in Proposition IV.1 in [3].

Proposition 4.4.

i. $Pic(\mathbb{F}_n) = \mathbb{Z}h \oplus \mathbb{Z}f$, *with* $f^2 = 0$, $f \cdot h = 1$, $h^2 = n$.

ii. if $n > 0$, *then there is a unique irreducible curve* B *on* $\mathbb{F}_n$ *with negative self-intersection. If* b *denotes the class of* B *in* $Pic(\mathbb{F}_n)$, *then* $b = h - nf$ *and satisfies* $b^2 = -n$.

iii. $\mathbb{F}_n$ *and* $\mathbb{F}_m$ *are not isomorphic unless* $n = m$. $\mathbb{F}_n$ *is minimal except if* $n = 1$.

Proof. The proof can be found in [3]. ∎

In order to find pencils that correspond to the minimal model $\mathbb{F}_n$, the *bidegree* for curves in each Hirzebruch surface needs to be determined [3].

Definition 4.2. *let* $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ *have coordinates* $(X, Y) \times (Z, W)$. *A curve* C *in* $\mathbb{P}^1 \times \mathbb{P}^1$ *is* bihomogeneous, *meaning that it has degree* m *in* (X, Y) *and degree* n *in* (Z, W). C *is said to have* bidegree (m, n) *and* $C \sim mh + nf$.

Example 4.1. *The curve* $C : y^3zw + x^2yw^2 + xy^2z^2 = 0$ *is bihomogeneous of bidegree* $(3, 2)$.

Proposition 4.5. *For Hirzebruch surface* $\mathbb{F}_n$, *the canonical divisor* $K_{\mathbb{F}_n}$ *is*

$$K_{\mathbb{F}_n} = -(n+2)f - 2h. \tag{5}$$

Proof. The proof can be found in [5]. ∎

Proposition 4.6. *Genus 1 curves in the Hirzebruch surface* $\mathbb{F}_0$ *have bidegree* $(2, 2)$.

Proof. To calculate the bidegree of genus 1 curves in $\mathbb{F}_0$, take a curve $C = mh + nf$ of bidegree (m, n). By equation 5, we have $K_{\mathbb{F}_0} = -2f - 2h$. By the adjunction formula in Proposition 1.6:

$$\begin{aligned}
2g(C) - 2 =& C \cdot (C + K_{\mathbb{F}_0}) \\
0 =& (mh + nf)^2 + (mh + nf)(-2f - 2h) \\
=& m^2h^2 + 2mnfh + n^2f^2 - 2mhf \\
& - 2mh^2 - 2nf^2 - 2nhf
\end{aligned}$$

As $f^2 = 0,\ fh = 1,\ h^2 = 0,$

$$\begin{aligned} 0 &= 2mn - 2m - 2n \\ &= mn - m - n \end{aligned}$$

This implies that $mn = m + n$, hence C is of bidegree $(2, 2)$. Moreover, this means that all curves of genus 1 in $\mathbb{F}_0$ have bidegree $(2, 2)$ [11]. ■

Proposition 4.7. *Genus 1 curves in the Hirzebruch surface $\mathbb{F}_2$ have bidegree $(2, 4)$.*

Proof. The canonical divisor in $\mathbb{F}_2$ is $K_{\mathbb{F}_2} = -2h-4f$, by equation 5. For a curve $C \sim mh+nf$ of bidegree (m, n), Proposition 1.6 gives

$$\begin{aligned} 2g(C) - 2 &= C \cdot (C + K_{\mathbb{F}_2}) \\ 0 &= (mh + nf)^2 + (mh + nf)(-2h - 4f) \\ 0 &= -2m^2 + 2mn - 2n \end{aligned}$$

The equation has integer solution $(m, n) = (2, 4)$. Hence, the bidegree of any curve of genus 1 in $\mathbb{F}_2$ is $(2, 4)$. ■

Knowing the bidegree of divisors in $\mathbb{F}_0$ and $\mathbb{F}_2$ allows for finding the curves whose blow-up at the base points results in the rational elliptic surface. In the next section, some pencils are given, whose blow-up results in the particular extremal rational elliptic surface. These were found using the Magma code in Appendix 5.

4.1 Blow-downs to Hirzebruch Surfaces

4.1.1 $2I_0^*$

The blow-down of $2I_0^*$ to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 26. The first four blow downs are the same as in Section 3.1. However, to construct a $(-2)-$curve intersected by only $(0)-$curves, the four $(-1)-$curves intersecting one $(-2)-$curve must be blown down. This is indicated by the green blow down, which results in minimal surface $\mathbb{F}_2$. On the other hand, to get to $\mathbb{F}_0$ each $(-2)-$curve must have two $(-1)-$curves that get blown down. This contraction is shown by the orange blow down.

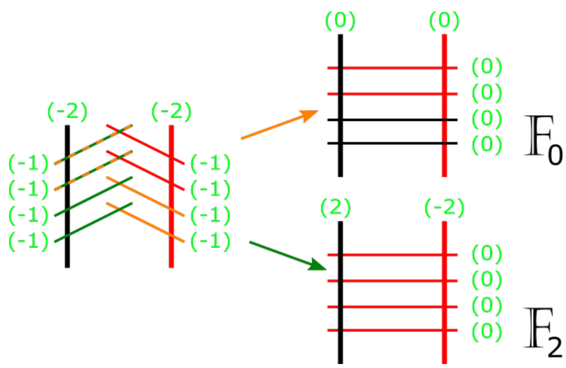


Figure 26: Blow down from $2I_0^*$ to Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$.

4.1.2 I_1^*, I_4, I_1

The blow-down of I_1^*, I_4, I_1 to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 27. Again, the same four curves as in 3.2 are blown down first. Then, two $(-1)-$curves intersecting the same $(-2)-$curve are blown down. To get to the Hirzeburch surface $\mathbb{F}_0$, the other two $(-1)-$curves intersecting the $(-2)-$curve are blown down, as indicated by the green components. The minimal model $\mathbb{F}_2$ is constructed by blowing down the orange $(-1)-$curves.

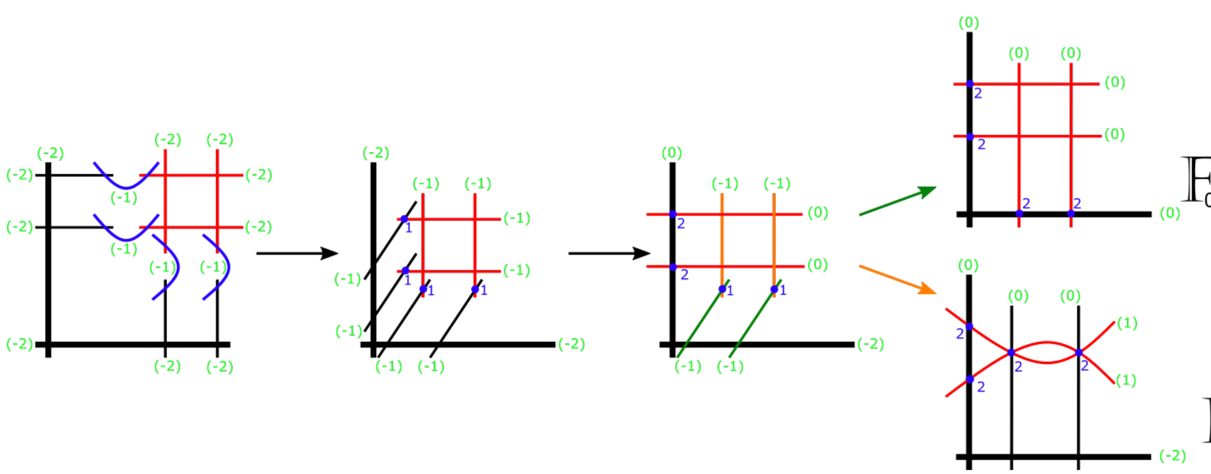


Figure 27: Blow down from I_1^*, I_4, I_1 to Hirzebruch surface $\mathbb{F}_2$.

4.1.3 IV^*, IV

The blow-down of IV^*, IV to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 28. After blowing down the sections, there is an immediate distinction between the blow downs to $\mathbb{F}_0$ and $\mathbb{F}_2$. To reach $\mathbb{F}_0$, the three green $(-1)-$curves on the former IV^* fiber are blown down. Subsequently, one of the $(-1)-$curves is blown down, followed by the $(-1)-$curve intersecting the other two $(-1)-$curves. This results in the minimal model $\mathbb{F}_0$. To find the minimal model $\mathbb{F}_2$, one of the components of the former IV fiber is blown down. This is followed by the blow-down of the two $(-1)-$curves on the former IV^* fiber. Lastly, the two remaining $(-1)-$curves are contracted, resulting in $\mathbb{F}_2$.

A pencil in $\mathbb{F}_0$ such that the blow up at its the base points results in the rational elliptic surface with the IV^*, IV fibers is given by

$$x^2zw + t(xyz^2 + y^2w^2) = 0,$$

where $(x, y) \times (w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{F}_0$.

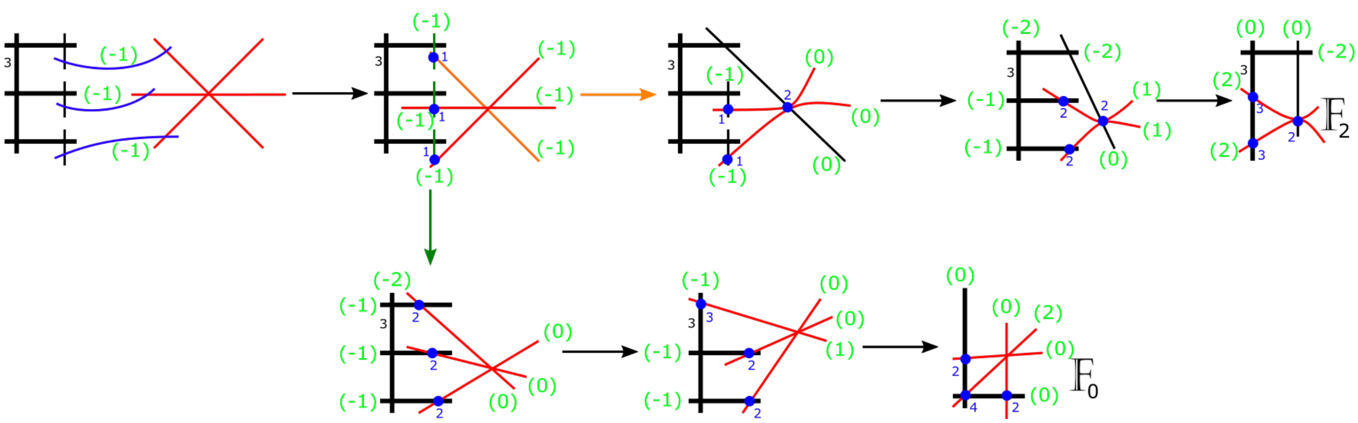


Figure 28: Blow down from IV^*, IV to Hirzebruch surface $\mathbb{F}_2$.

4.1.4 IV^*, I_3, I_1

The blow-down of IV^*, I_3, I_1 to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 29. This blow down follows the same procedure as the blow-down from IV^*, IV to $\mathbb{F}_0$ and $\mathbb{F}_2$.

4.1.5 $I_4^*, 2I_1$

The blow-down of $I_4^*, 2I_1$ to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 30. The order of the first six blow downs is the same as in Section 3.5. However, to get to the minimal surface $\mathbb{F}_2$ the orange coloured $(-1)-$curves are blown down. To get to $\mathbb{F}_0$ the green $(-1)-$curves are blown down.

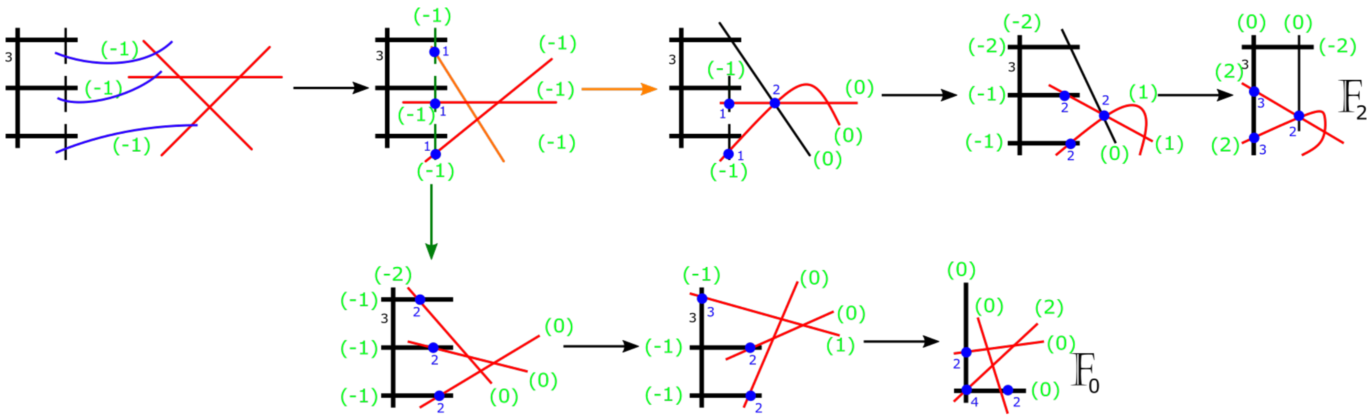


Figure 29: Blow down from IV^*, I_3, I_1 to Hirzebruch surface $\mathbb{F}_2$.

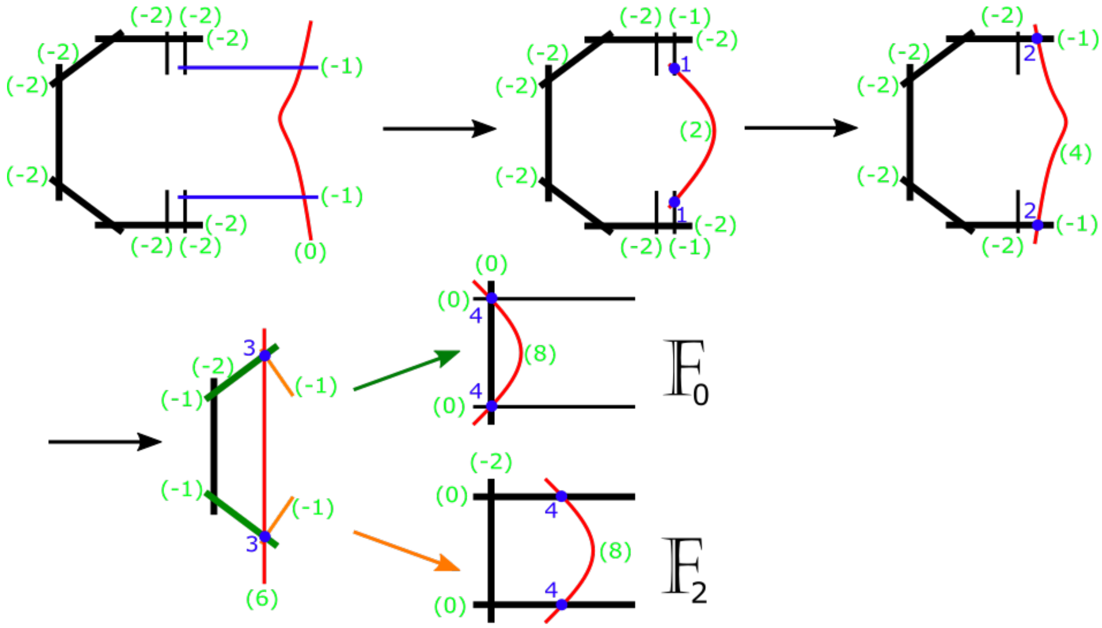


Figure 30: Blow down from $I_4^*, 2I_1$ to Hirzebruch surfaces $\mathbb{F}_2$ and $\mathbb{F}_0$.

4.1.6 II^*, II and $II^*, 2I_1$

The blow-down of II^*, II and $II^*, 2I_1$ to $\mathbb{F}_2$ can be found in Figure 31. The first seven blow-downs coincide with the case for the minimal model $\mathbb{P}^2$, in Section 3.6. However, the eighth blow down concerns the $(-1)-$curve with line multiplicity 3. The result is the minimal model $\mathbb{F}_2$.

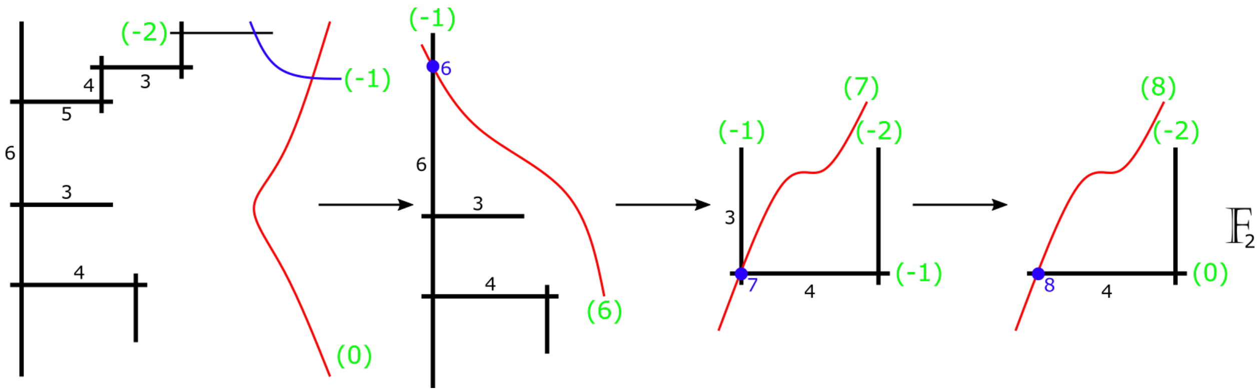


Figure 31: Blow down from II^*, II and $II^*, 2I_1$ to Hirzebruch surface $\mathbb{F}_2$.

4.1.7 III^*, III

The blow-down of III^*, III to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 32. After contracting the sections, the blow-downs already differ from the one in Section 3.7. To get to the minimal model $\mathbb{F}_2$, the orange components are contracted consecutively. After contracting the two remaining $(-1)-$curves, the minimal model $\mathbb{F}_2$ has been constructed. To blow down to $\mathbb{F}_0$ instead the green $(-1)-$curve is contracted. The figure shows the successive blow-downs to reach the minimal model in $\mathbb{F}_0$ in green.

A pencil in $\mathbb{F}_0$ such that the blow up at its the base points results in the rational elliptic surface with the III^*, III fibers is given by

$$z^2xy + y^2w^2 + t(x^2w^2) = 0,$$

where $(x, y) \times (w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{F}_0$.

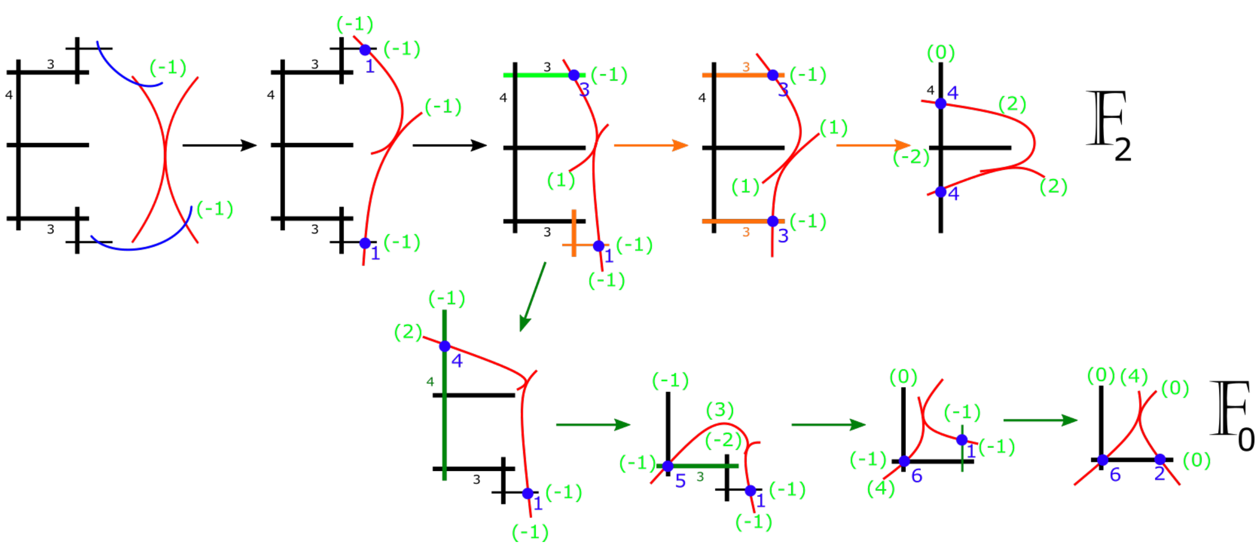


Figure 32: Blow down from III^*, III Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$.

4.1.8 III^*, I_2, I_1

The blow-down of III^*, I_2, I_1 to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 33. The blow down of III^*, I_2, I_1 to $\mathbb{F}_0$ and $\mathbb{F}_2$ follows the same procedure as the blow down of III^*, III.

A pencil in $\mathbb{F}_0$ such that the blow up at its the base points results in the rational elliptic surface with the III^*, I_2, I_1 fibers is given by

$$x^2w^2 + t(x^2zw - x^2z^2 + y^2zw) = 0,$$

where $(x, y) \times (w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{F}_0$.

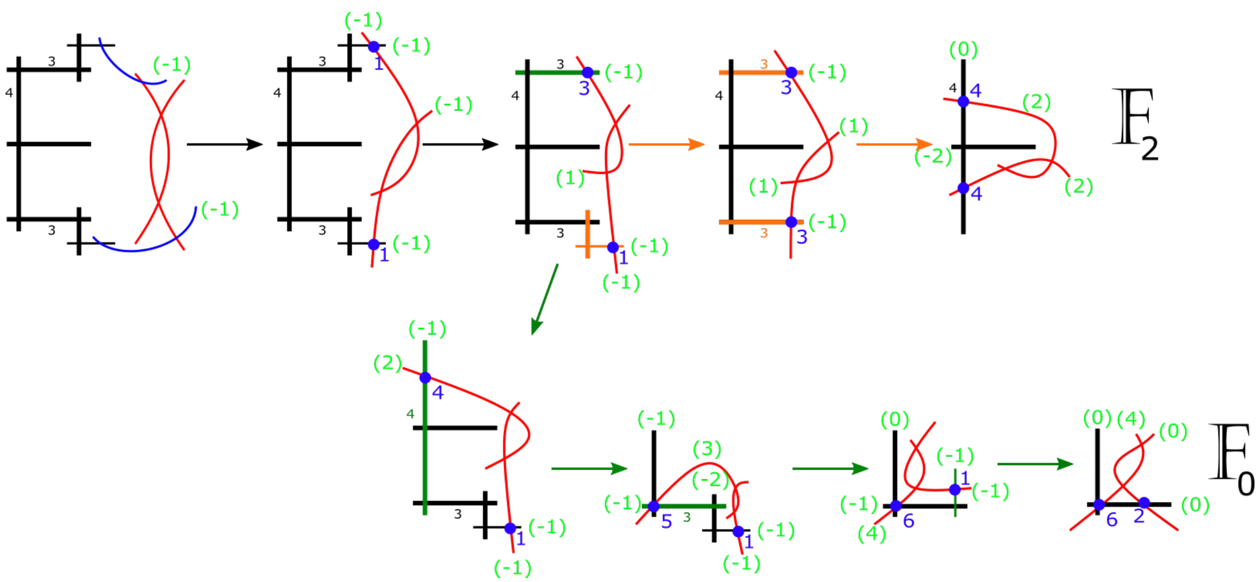


Figure 33: Blow down from III^*, I_2, I_1 to Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$.

4.1.9 $I_2^*, 2I_2$

The blow-down of $I_2^*, 2I_2$ to $\mathbb{F}_0$ and $\mathbb{F}_2$ can be found in Figure 34. First, the four sections are blown down. Then one $(-1)-$curve on each $(-2)-$curve intersecting a $(-1)-$curve is blown down. This results in a model that can either become the minimal model $\mathbb{F}_2$ by blowing down the green components, or the minimal model $\mathbb{F}_0$ by blowing down the blue components.

A pencil in $\mathbb{F}_0$ such that the blow up at its base points results in the rational elliptic surface with the $I_2^*, 2I_2$ fibers is given by

$$w^2y^2 - zwy^2 + t(x^2zw - x^2z^2 + y^2zw) = 0,$$

where $(x, y) \times (w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{F}_0$.

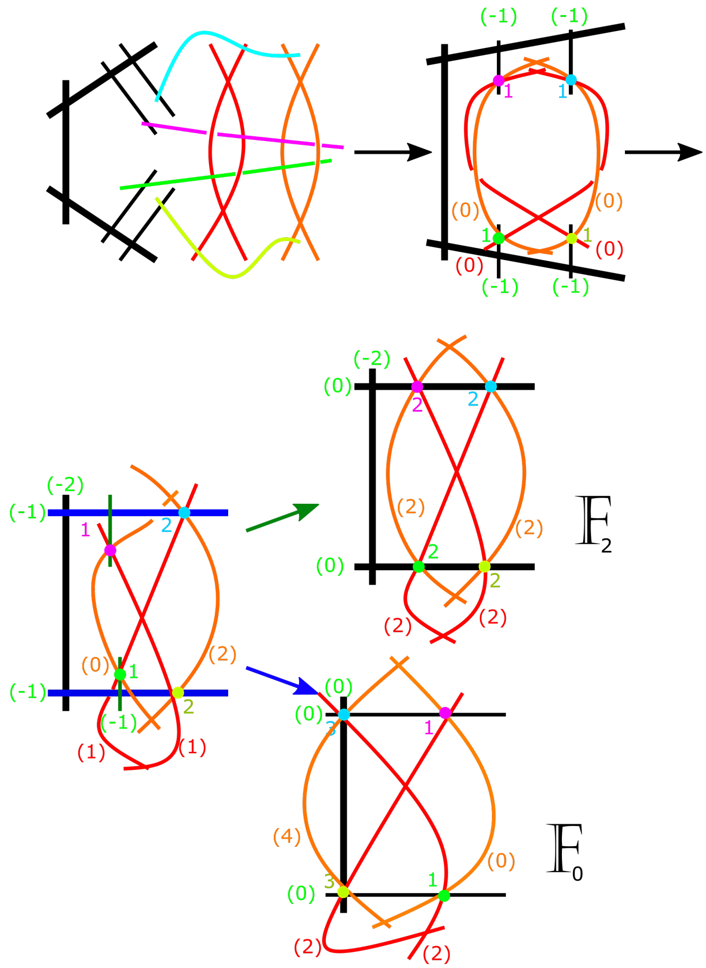


Figure 34: Blow down from $I_2^*, 2I_2*$ to Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$.

5 Discussion

This thesis has explicitly discussed blow-downs of extremal rational elliptic surfaces with at least one additive fiber to the surfaces $\mathbb{P}^2$, $\mathbb{F}_0$ and $\mathbb{F}_2$ over an algebraically closed field. A few pencils in $\mathbb{F}_0$ whose blow-up at the base points results in an extremal rational elliptic surface have been found explicitly. However, not for every extremal rational elliptic surface an explicit pencil in $\mathbb{F}_0$ and $\mathbb{F}_2$ has been found. In a perfect world, for each Hirzebruch surface a reference for a pencil would have been found, preferably in the same style as [13].

As discussed in Section 2.1, a non-algebraically closed field could result in problems when blowing down. There is some literature about the Galois theory on elliptic surfaces that is connected to this topic, for instance [14]. However, the writer has not found an extensive reference. For the writer of this thesis personally, a new aim would be to fully understand how Galois permutations on components of singular fibers work, and how this influences the possible blow-downs for elliptic surfaces over non-algebraically closed fields. For instance, why can certain extremal rational elliptic surface always be blown down to $\mathbb{P}^2$ over k, while for others $\bar{k}$ is a necessary condition. The writer's search engine skills have not yet found a reference on it, and hence it would seem like a suitable next project.

References

[1] Altman, Allen and Kleiman, Steven. *A term of commutative algebra*. Worldwide Center of Mathematics, 2013.

[2] Beauville, Arnaud. "Les familles stables de courbes elliptiques sur P1 admettant quatre fibres singulieres". In: *CR Acad. Sci. Paris Sér. I Math* 294.19 (1982), pp. 657–660.

[3] Beauville, Arnaud. *Complex algebraic surfaces*. 34. Cambridge University Press, 1996.

[4] Beek, Monique van. "On elliptic curves of the form $y^2 = x^3 + a(x-b)^2$". Master's thesis, University of Groningen, 2010. http://fse.studenttheses.ub.rug.nl/id/eprint/9259, 2010.

[5] Brodie, Callum R et al. "Index formulae for line bundle cohomology on complex surfaces". In: *Fortschritte der Physik* 68.2 (2020), p. 1900086.

[6] Cantoral-Farfán, Victoria et al. "Fields of definition of elliptic fibrations on covers of certain extremal rational elliptic surfaces". In: *Women in Numbers Europe III*. Springer, 2021, pp. 171–205.

[7] Fulton, William. *Algebraic Curves*. Addison Wesley Publishing Company, 1969.

[8] Gathmann, Andreas. "Algebraic geometry". In: *Notes for a class taught at the University of Kaiserslauten (2002/2003) available at http://www. mathematik. unikl. de/˜ gathmann/class/alggeom-2002/main. pdf* (2002).

[9] Gathmann, Andreas. "Plane Algebraic Curves". In: *Class Notes TU Kaiserslautern* (2018).

[10] Hartshorne, Robin. "Curves with high self-intersection on algebraic surfaces". In: *Publications Mathématiques de l'IHÉS* 36 (1969), pp. 111–125.

[11] Hartshorne, Robin. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 2013.

[12] Kodaira, Kunihiko. "On the structure of compact complex analytic surfaces, II". In: *American Journal of Mathematics* 88.3 (1966), pp. 682–721.

[13] Kurumadani, Y. "Pencils of cubic curves and rational elliptic surfaces". In: *RIMS preprints RIMS-1800, Japan* (2014).

[14] MANE, TARIKA. "GALOIS ACTIONS ON TORSIONS OF ELLIPTIC CURVES". In: (2020).

[15] McCasland, RL, Moore, ME, and Smith, PF. "An introduction to Zariski spaces over Zariski topologies". In: *The Rocky Mountain journal of mathematics* (1998), pp. 1357–1369.

[16] Miranda, Rick. "The basic theory of elliptic surfaces". In: *Dottorato di Ricerca in Matematica, ETS Editrice* (1989).

[17] Miranda, Rick and Persson, Ulf. "On extremal rational elliptic surfaces". In: *Mathematische Zeitschrift* 193.4 (1986), pp. 537–558.

[18] Persson, Ulf. "Configurations of Kodaira fibers on rational elliptic surfaces". In: *Mathematische Zeitschrift* 205.1 (1990), pp. 1–47.

[19] Reid, Miles. “Chapters on algebraic surfaces”. In: *arXiv preprint alg-geom/9602006* (1996).

[20] Schütt, Matthias and Shioda, Tetsuji. “Elliptic surfaces”. In: *Algebraic geometry in East Asia—Seoul 2008*. Vol. 60. Mathematical Society of Japan, 2010, pp. 51–161.

[21] Shafarevich, Igor Rostislavovich and Reid, Miles. *Basic algebraic geometry*. Vol. 2. Springer, 1994.

[22] Silverman, Joseph H. *Advanced topics in the arithmetic of elliptic curves*. Vol. 151. Springer Science & Business Media, 1994.

[23] Silverman, Joseph H. *The arithmetic of elliptic curves*. Vol. 106. Springer, 2009.

[24] Tate, John. “On the conjectures of Birch and Swinnerton-Dyer and a geometric analog”. In: *Séminaire Bourbaki* 9.306 (1965), pp. 415–440.

[25] Tate, John. “Algorithm for determining the type of a singular fiber in an elliptic pencil”. In: *Modular functions of one variable IV*. Springer, 1975, pp. 33–52.

Appendices

This thesis contains pencils whose blow-up gives a certain type of rational elliptic surface. These pencils were tested using Magma, an algebraic computational software. The following code was used to confirm the pencils in $\mathbb{P}^2$ and find their Weierstrass Form.

```
P<x,y,z> := ProjectiveSpace(Rationals(),2);
k<t> := FunctionField(Rationals());
P2<x,y,z> := ProjectiveSpace(k,2);
C := Curve(P2,t*z*(x^2+x*z-y^2)+x^3+z*x^2);
pt := C![0,1,0];
E,toE := EllipticCurve(C,pt);
KodairaSymbols(E);
BadPlaces(E);
WeierstrassModel(E);
```

The following code was used to confirm the pencils in $\mathbb{F}_0$.

```
P<x,y,z> := ProjectiveSpace(Rationals(),2);
k<t> := FunctionField(Rationals());
P2<x,y,w,z> := ProductProjectiveSpace(k,[1,1]);
C := Curve(P2, w^2*x^2+t*(x^2*z*w-x^2*z^2+y^2*z*w));
pt := C![0,1,1,0];
E,toE := EllipticCurve(C,pt);
KodairaSymbols(E);
BadPlaces(E);
```