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A QCD Investigation of the Proton Mass

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Abstract

In quantum chromodynamics (QCD), the proton mass can be derived from the matrix elements of the energy-momentum tensor (EMT). A natural question that arises is whether the proton mass can be decomposed, e.g. into a quark and a gluon part. Multiple decompositions have been synthesized, depending on different criteria like gauge invariance, Lorentz invariance and locality. As of yet there is no one universal decomposition of the proton mass. In this thesis we compare the existing decompositions and their relation to each other. This thesis contains a review of QCD in general, the EMT and how it is related to the proton mass. Special attention is paid to the renormalization of the EMT for which operator mixing must be involved. The different decompositions of the proton mass and the role of the trace anomaly are examined. We compute numerical results for the mass decompositions and present and discuss graphs of the running of these decompositions.

Contents

1 Introduction

Nearly all the mass of known matter is contained in protons and neutrons - the particles that make up the nuclei of atoms. And yet we know surprisingly little about the distribution of mass inside particles like the proton itself. When the proton was first discovered, scientists thought it to be an elementary particle, just like the electron. Later, as more and more particles (particularly hadrons) were discovered, scientists started noticing patterns among the masses and charges of the hadrons. This eventually led to the belief the proton consists of three smaller particles called quarks: two up- and one down-quark. They are held together by massless gluons in a ball of radius ~ 1 fm. The total mass of the proton (~ 938 MeV) can however not be explained by the sum of the masses of the three quarks (~ 9 MeV). The rest of the mass must be contained in the kinetic energy of the quarks and gluons and the confinement energy of the quarks. A simple uncertainty principle argument for instance shows that the confinement energy of the quarks must be ~ 300 MeV [1]. The modern picture of a proton is a sea of quarks and gluons described by QCD with three valence quarks: two up- and one down-quark. A natural question that arises in this picture is: how much quark and how much gluon is a proton? This thesis tries to give some insights into this question.

In this context, the QCD energy-momentum tensor (EMT) plays a key role. Its matrix elements relate to properties like the mass and momentum of a proton, but also its spin, angular momentum and even pressure and shear distributions [2]. In this thesis we focus on the mass of the proton and how it can be decomposed into contributions from the masses and energies of the quarks and gluons. We investigate several decompositions from literature [3, 4, 5]. All of these decompositions are derived from the EMT via either the Hamiltonian (which is related to the energy-component of the EMT) or the EMT trace. From the phenomenological point of view, we compute the numerical results of the mass decompositions and discuss their scale dependence. For the numerical inputs of the necessary matrix elements we use the parton momentum fractions and the scalar charge of the proton.

The thesis is organized as follows: in chapter 2, we review QCD and how it is based upon the gauge symmetry $SU(3)$. We derive the (classical) QCD Lagrangian, equations of motion and the EMT. In chapter 3, we then review quantization and renormalization of QCD. We discuss the scale dependence of the theory and calculate the QCD beta function and quark mass anomalous dimension. In chapter 4, we discuss operator renormalization and operator mixing. We then apply this to the renormalization of the operators that form the EMT. We also derive the EMT trace anomaly from the dilatation current. Lastly, in chapter 5 we discuss how the mass of the proton is derived from the EMT, either through its trace or the Hamiltonian. We then investigate four mass decompositions: one 2-term decomposition by Tanaka [5], a 2-term decomposition by Lorcé [3], a 4-term decomposition by Ji [4] and an improvement on Ji's decomposition by Rodini et al. [2]. We use numerical inputs from [2] to compute the numerical values of the mass decompositions and produce graphs of the running of these decompositions as function of scale. Based on these graphs we discuss the large- and small-scale behaviour of the mass decompositions.

2 An introduction to QCD

A good grasp of the theory of QCD is needed to fully understand the content of this thesis. Therefore this chapter explains the basics of QCD as a field theory and how it obtained from the $SU(3)$ gauge symmetry of the color charge in quarks. Section 2.1 treats the Lie group $SU(3)$ and its representations for quarks and gluons. Section 2.2 uses this theory to build the QCD Lagrangian. The theory in these sections is mostly adapted from [6]. Readers who are familiar with QCD can skip these sections. In section 2.3 the energy-momentum tensor is discussed and the QCD EMT is derived.

2.1 The Lie group $SU(3)$

According to QCD, quarks possess a a set of three related charges called color charges. Therefore they are represented mathematically by a three-component vector of spinor fields $\psi_i(x)$ with $i = 1, 2, 3$. The theory of QCD is invariant under local rotations of this vector, meaning only colorless particles can exist in nature. These rotations are unitary operators and can therefore be form the gauge symmetry group $U(3)$. The pure phase transformations form a $U(1)$ subgroup and can be described with a QED-like theory. These transformations are not of interest to us and therefore we remove this subgroup to form the simple Lie group $SU(3)$. We next discuss some general properties of this group.

The Lie group $SU(3)$ is defined as the 8-dimensional group of all 3×3 unitary matrices g with determinant 1. Any element of $SU(3)$ can be written as the exponential of a Hermitian matrix. If we consider matrices infinitesimally close to the unit matrix[1], we can write:

$$g(\alpha) = \mathbf{1} + i\alpha^a T^a + \mathcal{O}(\alpha^2), \tag{1}$$

where $a = 1, \ldots, 8$ and repeated indices are summed over. The anti-hermitian matrices T^a are called the generators of the group. A property of Lie groups is that the commutator of two generators is again a linear combination of generators:

$$[T^a, T^b] = if^{abc}T^c. \tag{2}$$

The f^{abc} determine the structure of the group and are therefore called the structure constants of $SU(3)$. They are completely anti-symmetric and obey the Jacobi identity.

To describe QCD mathematically, we must use different matrix representations of $SU(3)$ for the different parts of the theory. For a d-dimensional representation r, we replace T^a with $d \times d$ matrices t^a_r in the equations above. The matrices are normalized based on the traces of their products. Specifically, we can choose a basis for the generators T^a such that, for any irreducible representation:

$$\mathrm{Tr}\left[t^a_r t^b_r\right] = C(r)\delta^{ab}. \tag{3}$$

[1] As $SU(3)$ is a continuously generated group, any element can be reached by the repeated action of these infinitesimal elements.

where $C(r)$ is a constant for every representation. Furthermore, the operator T^2 is an invariant of $SU(3)$. Therefore T^2 takes a constant value on each irreducible representation:

$$t_r^a t_r^a = C_2(r) \cdot \mathbf{1}. \tag{4}$$

Here $C_2(r)$ is a constant, called the quadratic Casimir operator, for every representation.

We introduce two important representations of $SU(3)$. The quark fields transform under the fundamental representation N of $SU(3)$. The fundamental representation acts on three-vectors and its elements are simply the matrices that define $SU(3)$: 3×3, unitary matrices with a determinant of 1. The quark fields then transforms under an infinitesimal element of $SU(3)$ as:

$$\psi_i \longrightarrow \left[e^{i\alpha^a t_N^a}\right]^{ij} \psi_i \approx \left[\delta^{ij} + i\alpha^a (t_N^a)^{ij}\right] \psi_j, \tag{5}$$

where $a = 1, \ldots, 8$ and $t_N^a = \frac{1}{2}\lambda^a$, λ^a being the Gell-Mann matrices. With this choice of t_N^a, we have $C(N) = \frac{1}{2}$ and $C_2(N) = \frac{4}{3}$.

As we will see below, QCD requires the introduction of another type of field called gluon fields transforming under the adjoint representation G of $SU(3)$ (plus a gauge transformation that will be explained later). The representation matrices of the adjoint representation are given by the structure constants: $(t_G^b)^{ac} = if^{abc}$. Similar to the fundamental representation, we can view the action of $SU(3)$ as a vector rotation, this time acting on an eight-component vector A_a. We get the transformation:

$$A_a \longrightarrow \left[e^{i\alpha^b t_G^b}\right]^{ac} A_c \approx \left[\delta^{ac} + i\alpha^b f^{abc}\right] A_c. \tag{6}$$

The constants of the adjoint representation equal $C(g) = C_2(G) = 3$. In the rest of this thesis, we will use the notation $T^a = t_N^a$ for the generators of $SU(3)$ in the fundamental representation and the structure constants for the adjoint representation. As we are using structure constants, it is useful to write equations (3) and (4) in terms of structure constants for the adjoint representation. We have:

$$\mathrm{Tr}\left[t_G^a t_G^b\right] = f^{acd} f^{bcd} = C(G)\delta^{ab}, \tag{7}$$

and:

$$(t_G^c)^{ad}(t_G^c)^{db} = f^{acd} f^{bcd} = C_2(G)\delta^{ab}, \tag{8}$$

which proves that the two constants are indeed equal to one another.

2.2 The Lagrangian formulation of QCD

The fundamental quantity of quantum field theories is the action S. The principle of least action states that if a system evolves between times t_1 and t_2, it will follow the "path" in configuration space for which S is an extremum. The action can be expressed in terms of a functional, called the Lagrangian L, as:

$$S = \int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i), \tag{9}$$

where $\mathcal{L}$ is the Lagrangian density and the ϕ_i represent all the fields in the theory. The principle of least action then leads to a set of equations involving the Lagrangian, called the relativistic Euler-Lagrange equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = \frac{\partial \mathcal{L}}{\partial \phi_i}. \tag{10}$$

Finally, solving these equations in terms of the ϕ_i results in the equations of motion for the system.

As quarks are fermionic particles, we start with the Dirac Lagrangian for the quark fields:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{11}$$

where $\bar{\psi}_i = (\psi_i)^\dagger \gamma^0$ and $\gamma^\mu,\ \mu = 0,1,2,3$ are the gamma matrices. A sum over quark flavors and colors is understood. This Lagrangian is already invariant under a global $SU(3)$ transformation as it only depends on the combination $\bar{\psi}\psi$:

$$\bar{\psi}\psi \longrightarrow \bar{\psi}\big[e^{-i\alpha^a T^a}\big]\big[e^{i\alpha^a T^a}\big]\psi = \bar{\psi}\psi. \tag{12}$$

However, we stipulate that the Lagrangian, like the theory, must be invariant under an gauge $SU(3)$ transformation. This differs from a global $SU(3)$ transformation in that it allows an independent symmetry transformation at every point in spacetime. Therefore the kinetic term in (11) is no longer invariant; the gauge transformation shifts the Lagrangian by $-\bar{\psi}\partial_\mu \alpha^a(x) T^a \psi$. To obtain a gauge invariant Lagrangian, we replace ∂_μ by a covariant derivative. A covariant derivative D_μ is defined such that (for infinitesimal transformations):

$$D_\mu \psi(x) \longrightarrow \big[\mathbf{1} + i\alpha^a(x) T^a\big] D_\mu \psi(x), \tag{13}$$

i.e. $D_\mu \psi(x)$ has the same transformation law as $\psi(x)$. This means that the transformation of the covariant derivative has to counteract the extra term. Therefore we introduce a eight new fields A^a_μ which appear in the covariant derivative:

$$D_\mu = \partial_\mu - igA^a_\mu T^a, \tag{14}$$

where have arbitrarily extracted a constant g. To ensure equation (13), we must have the following transformation for $A^a_\mu T^a$:

$$A^a_\mu T^a \longrightarrow A^a_\mu T^a + i\big[\alpha^a(x) T^a, A^b_\mu T^b\big] + \frac{1}{g}\partial_\mu \alpha^a(x), \tag{15}$$

or:

$$A^a_\mu \longrightarrow \big(\delta^{ac} - f^{abc}\alpha^b(x)\big) A^c_\mu + \frac{1}{g}\partial_\mu \alpha^a(x), \tag{16}$$

where we recognize as a transformation under the adjoint representation of $SU(3)$ plus a gauge transformation. The physical interpretation of A^a_μ are the eight gluon fields. The constant g is the coupling constant of the strong interaction.

Next we need to add a kinetic term for the gluon fields to the Lagrangian. The transformation law of the covariant derivative implies that:

$$[D_\mu, D_\nu]\psi \longrightarrow \big[\mathbf{1} + i\alpha^a(x)T^a\big][D_\mu, D_\nu]\psi. \tag{17}$$

Simultaneously, the commutator of two covariant derivatives is not a differential operator but merely a matrix operator:

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}T^a, \tag{18}$$

with:

$$F^a_{\mu\nu}T^a = \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a - ig\big[A^a_\mu T^a, A^b_\nu T^b\big]. \tag{19}$$

The tensor $F^a_{\mu\nu}$ is called the gluon field tensor, similar to the electromagnetic field tensor of QED. It equals:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{20}$$

Although the gluon field tensor is not sensitive to the gauge term of the A^a_μ transformation, it is not gauge invariant. Nor should it be, as there are eight field strengths, each associated with a given direction of rotation in the abstract $SU(3)$ space. Similar how the transformation (13) requires the transformation (16), so does equation (17) require the following transformation:

$$F^a_{\mu\nu}T^a \longrightarrow F^a_{\mu\nu}T^a + i\big[\alpha^a(x)T^a, F^b_{\mu\nu}T^b\big]. \tag{21}$$

Many gauge invariant combinations can be made from the gluon field tensor, for instance $-\frac{1}{2}\operatorname{Tr}\left[(F^a_{\mu\nu}T^a)^2\right] = -\frac{1}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta}$. Using this term, we obtain the QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta}. \tag{22}$$

It contains massive quarks interacting with massless gluons. The gluons, unlike the photons of QED, also interact among each other due to the A^a_μ cubic and quartic terms in the Lagrangian. The whole Lagrangian depends on two parameters: the coupling constant g and the fermion mass m. From the Lagrangian we also find the equations of motion:

$$i\not{\partial}\psi = m\psi - g\not{A}^a T^a\psi, \tag{23}$$

$$-i\partial_\mu\bar{\psi}\gamma^\mu = m\bar{\psi} - \bar{\psi}g\not{A}^a T^a, \tag{24}$$

$$\partial^\mu F^a_{\mu\nu} = gf^{abc}F^b_{\mu\nu}A^{c\mu} - g\bar{\psi}\gamma_\nu T^a\psi. \tag{25}$$

where equations (23) and (24) hold for all quark flavors and colors.

2.3 The Energy-momentum Tensor

The energy-momentum tensor, denoted $T^{\mu\nu}(x)$, describes the energy- and momentum density and flux of a system. It is the Noether current associated with spacetime translation invariance of the Lagrangian. This means that it is conserved ($\partial_\mu T^{\mu\nu} = 0$) and connected to the energy and momentum of a system via $P^\mu = \int d^3x T^{0\mu}(x)$). The EMT plays a crucial role in understanding global properties like mass of particles, due to its relation to energy and momentum.

Since we'll be using the EMT to compute the proton mass, we better derive the EMT for QCD. As a Noether current, the EMT is given by:

$$T_C^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i - g^{\mu\nu} \mathcal{L}. \tag{26}$$

A problem with this canonical formula for the EMT is that it does not satisfy gauge invariance nor symmetry in its indices μ and ν. Gauge invariance is desirable for the EMT, as our physical world does not depend on what gauge we choose. Symmetry in its indices μ and ν is desirable if we ever want to unify QFT and general relativity, as there the EMT is symmetric. We can derive a new EMT from equation (26) such that these properties are met, without changing the meaning of the EMT in field theory. First of all, we can use the Belinfante-Rosenfeld procedure to symmetrize the EMT [2]. To enforce gauge invariance, we use the fact that the Lagrangian is only defined up to a derivative. This means there is some arbitrariness in the EMT. Specifically, the charges P^ν and conservation law are unchanged if we make the transformation:

$$\tilde{T}^{\mu\nu} = T_C^{\mu\nu} + \partial_\rho \Theta^{[\mu\rho]\nu}, \tag{27}$$

where $\Theta^{[\mu\rho]\nu}$ is local in the fields and anti-symmetric in the indices μ and ρ [7].

For $\mathcal{L}_{\text{QCD}}$, the EMT is:

$$T_C^{\mu\nu} = i\bar{\psi}\gamma^\mu \partial^\nu \psi - (F^a)^{\mu\rho}(\partial^\nu A^a_\rho) - g^{\mu\nu}\left[-\frac{1}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta} + \bar{\psi}(i\not{D} - m)\psi\right]. \tag{28}$$

The first two terms of $T_C^{\mu\nu}$ are not gauge invariant. To rectify this, we make the transformation of equation (27) with $\Theta^{\mu\rho,\nu} = (F^a)^{\mu\rho}(A^a)^\nu$. We get:

$$\tilde{T}^{\mu\nu} = i\bar{\psi}\gamma^\mu D^\nu \psi - (F^a)^{\mu\rho}(F^a)^\nu{}_\rho - g^{\mu\nu}\left[-\frac{1}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta} + \bar{\psi}(i\not{D} - m)\psi\right]. \tag{29}$$

Next we use the Belinfante-Rosenfeld procedure to make the EMT symmetric. For the case of QCD, this is equivalent to ignoring the anti-symmetric part. Lastly we will use the equations of motion to simplify the EMT, as we will be using the EMT on physical states only. Therefore the EMT that we will use throughout the rest of this thesis is:

$$T^{\mu\nu} = i\bar{\psi}\gamma^{\{\mu} D^{\nu\}}\psi - (F^a)^{\mu\rho}(F^a)^\nu{}_\rho + \frac{g^{\mu\nu}}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta}, \tag{30}$$

where $\gamma^{\{\mu} D^{\nu\}} = \frac{1}{2}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)$.

In this chapter we have set up a classical field theory for the strong interaction. We have modelled it as a manifestation of a $SU(3)$ gauge symmetry in the abstract space formed by three quark color charges. We have seen that gauge fields needed to be added to keep the theory $SU(3)$ gauge-invariant. These fields are the gluon fields and they act as the force carriers of the strong interaction. We have set up the Lagrangian of the theory and derived the equations of motion. Also, we have discussed the EMT and its role in mass computations. We've ended this chapter with a derivation of the EMT for QCD. In the next chapter we will quantize the theory, fully obtaining QCD. We will see that the Lagrangian is central to this quantization. Also, we will run into some divergences, which we will deal with using renormalization.

3 Into the quantum realm

Everything so far was based on a classical theory. Of course we need to quantize the theory, as the universe is quantum in nature at the small scales. This quantization will introduce UV-divergences in the theory, a sign that something is amiss with our theory at very high energies. As we do not have a better theory on hand, we will fix these divergences using a process called renormalization. We also treat the scale dependency of the quantized theory.

3.1 Quantization

Central to quantization is a function called the correlation function, or Green's function,

$$G^{(n)}(x_1, \ldots, x_n) = \langle \Omega | \, T\{O_1(x_1) \ldots O_n(x_n)\} \, |\Omega\rangle \, . \tag{31}$$

We will also often use the Fourier transformed Green's functions:

$$G^{(n)}(k_1, \ldots, k_n) = \int d^4x_1 \ldots d^4x_n \; e^{i \sum_{i=1}^{n} k_i x_i} G^{(n)}(x_1, \ldots, x_n). \tag{32}$$

Green's functions are somewhat abstract quantities which describe a process in which the vacuum (denoted by $|\Omega\rangle$) becomes exited to form particles and operators (the O_i) at certain points in spacetime (the x_i) and eventually decaying back down to the vacuum. The square of a correlation function is related to the probability that this process occurs.

Throughout this work, we will use the path integral formulation to quantize the theory. The path integral formulation describes the evolution of quantum states, e.g. from $|\phi_a\rangle$ to $\langle\phi_b|$. It identifies a measure for every possible way or "path" that this evolution can occur:

$$O_1(x_1) \ldots O_n(x_n) e^{\frac{i}{\hbar} \int_{-T}^{+T} d^4x \;\, \mathcal{L}(\phi_i, \partial^\mu \phi_i)}.$$

In the exponential we recognize the action, which ensures the principle of least action: around the true classical path the action varies vary slowly and the phases add up; far away from the true classical path the action varies rapidly and the measures cancel each other out. Therefore paths around the true classical path are much more likely to occur. In the classical limit $\hbar \to 0$, only the classical path survives and we regain classical Lagrangian mechanics. According to the principle of superposition the true quantum path is then given by the sum of the measures of all paths. The continuous limit of this sum is the functional integral over all fields of the theory:

$$G^{(n)}(x_1, \ldots, x_n) = \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi_i O_1(x_1) \ldots O_n(x_n) e^{\frac{i}{\hbar} \int_{-T}^{+T} d^4x \;\, \mathcal{L}(\phi_i, \partial^\mu \phi_i)}}{\int \mathcal{D}\phi_i e^{\frac{i}{\hbar} \int_{-T}^{+T} d^4x \;\, \mathcal{L}(\phi_i, \partial^\mu \phi_i)}}. \tag{33}$$

The denominator here is a normalization factor that rid us of any phase and overlap factors that do not contribute to the physics. The limit $T \to \infty(1 - i\epsilon)$

has been chosen to project out the vacuum state from $|\phi_a\rangle$ and $\langle\phi_b|$. From now on we set $\hbar = 1$.

The number of possible paths for a process is immense: quarks can interact with each other, quark-antiquark pairs can be created or annihilated, gluons can be emitted or absorbed etc. A general analytical solution for path integrals in the interacting theory is not known in more than 2 spacetime dimensions [6]. Therefore we turn to a perturbative solution. We expand the exponentials in equation (33) as a series in g:

$$e^{i\int d^4x\mathcal{L}_{\text{QCD}}} = e^{i\int d^4x\mathcal{L}_0}\left[1 + i\int d^4y\ \mathcal{L}_{\text{int}} + \dots\right], \tag{34}$$

where $\mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi + (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$ is the free theory Lagrangian and $\mathcal{L}_{\text{int}} = \mathcal{L}_{\text{QCD}} - \mathcal{L}_0$ contains all the interactions (and is proportional to g). When inserted into a correlation function, the n'th term of this expansion corresponds to paths with n interactions. So depending on the needed precision of our answer, we just need to compute all paths up to n interactions. Even more, we only need to consider connected paths. Pieces that are disconnected from all external points (i.e. vacuum fluctuations) exponentiate to a constant that is present in both the numerator and the denominator of the correlation function and therefore cancels out.

The easiest way to find all connected paths is by making use of Feynman diagrams. Feynman diagrams are pictorial representations of amplitudes of a single path. They are connected to the mathematical expressions by a set of Feynman rules. Connected paths translate to 1-particle irreducible (1PI) diagrams, which are diagrams that cannot be split in two by cutting one propagator. For a full review on Feynman diagrams, see for instance [6, 8].

Propagators

To find the basic propagators we work in the free theory ($g = 0$), in which we have no interactions. The quark propagator is then given by the following two-point Green's function:

$$\langle\Omega|\,T\{\psi(x_1)\bar{\psi}(x_2)\}\,|\Omega\rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \psi(x_1)\bar{\psi}(x_2)e^{i\int d^4x\left[\bar{\psi}(i\partial\!\!\!/-m)\psi\right]}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{i\int d^4x\left[\bar{\psi}(i\partial\!\!\!/-m)\psi\right]}}. \tag{35}$$

where we have left the limits on the time integrals implicit; they are the same as equation (33). The gluon contributions in the numerator and denominator have canceled out. As quarks are fermions and obey anticommutation relations, we have used Grassmann fields for the quark fields ψ and $\bar{\psi}$. More information on Grassmann fields can be found in for instance [6]. Both the denominator and numerator in equation (35) are Gaussian integrals over Grassmann variables. The denominator of this equation is $\det(i\partial\!\!\!/ - m)$. The numerator is this same determinant multiplied by $S_F(x_1 - x_2)$, the functional inverse of the operator $-i(i\partial\!\!\!/ - m)$:

$$-i(i\partial\!\!\!/ - m)S_F(x_1 - x_2) = \delta^{(4)}(x_1 - x_2).$$

Performing a Fourier transform, we find:

$$\langle\Omega|\, T\{\psi(x_1)\bar{\psi}(x_2)\}\,|\Omega\rangle = S_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x_1-x_2)}}{\not{k} - m + i\epsilon}. \tag{36}$$

The gluon propagator is given by another two-point Green's function:

$$\langle\Omega|\, T\{A^a_\mu(x_1)A^b_\nu(x_2)\}\,|\Omega\rangle = \frac{\int \mathcal{D}A\; A^a_\mu(x_1)A^b_\nu(x_2)e^{i\int d^4x\left[-\frac{1}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta}\right]}}{\int \mathcal{D}A\; e^{i\int d^4x\left[-\frac{1}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta}\right]}}, \tag{37}$$

where $\mathcal{D}A \equiv \prod_{a=1}^{8}\prod_{i=0}^{3}\mathcal{D}A^a_i$ now consists of regular complex fields. This time the quark contributions have canceled out. The integrals in equation (37) can be rewritten as Gaussian integrals by partial integration of the exponent[2]:

$$\int d^4x\left[-\frac{1}{4}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta}\right] = \frac{1}{2}\int d^4x A^a_\mu(x)\delta^{ab}\big(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\big)A^b_\nu(x). \tag{38}$$

When trying to evaluate the Gaussian integrals we run into trouble. Because gluons are massless they have only two polarization states. Therefore the operator $g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu$ is singular and has no inverse. In fact, the functional integrals in equation (37) are badly divergent. The reason for this is gauge invariance: the Lagrangian is invariant under a gauge transformation and hence we are integrating over a continuous infinity of physically equivalent field configurations. A solution to this problem was found by Faddeev and Popov [9]. They introduced a delta function picking out one field configuration of each set of gauge equivalent configurations. They then showed that this delta function can be expressed as an addition to the Lagrangian:

$$\mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghost}} = \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c, \tag{39}$$

where $D^{ac}_\mu = \partial_\mu\delta^{ac} + gf^{abc}A^b_\mu$ is the covariant derivative in the adjoint representation. The gauge fixing term $\mathcal{L}_{\text{g.f.}}$ contains a new parameter ξ, called the gauge-fixing parameter. As ξ is not physical (it breaks the gauge invariance of the Lagrangian), it should not appear in any physical results. Indeed the value of the correlation function of any gauge-invariant operator is independent of ξ [6]. The ghosts c^a are anticommuting fields belonging to the adjoint representation. Just like the gauge fixing parameter they are not physical and can't appear as external lines of Feynman diagrams. They interact with the gluons and serve as negative degrees of freedom that cancel the effects of the unphysical timelike and longitudinal polarization states of the gluons that were added by the gauge fixing term. The ghost propagator equals:

$$\langle\Omega|\, c^a(x_1)\bar{c}^b(x_2)\,|\Omega\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\delta^{ab}e^{-ik(x_1-x_2)}, \tag{40}$$

[2]Remember that we are working in the free theory ($g = 0$) here. Therefore we do not have any gluon-gluon interactions.

which can be derived much like the other propagators. The ghost and gauge fixing terms in the Lagrangian do affect the energy-momentum tensor. However, as they are not physical, they vanish in any physical state. As we are concerned in the proton mass (which is a physical state), we ignore these contributions and use equation (30) as our formula for the EMT.

With the extra terms $\mathcal{L}_{\text{g.f.}}$ and $\mathcal{L}_{\text{ghost}}$, equation (38) changes to:

$$\int d^4x\left[-\frac{1}{4}F^2+\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right]=\int d^4x A^a_\mu(x)\delta^{ab}\big(g^{\mu\nu}\partial^2-(1-\frac{1}{\xi})\partial^\mu\partial^\nu\big)A^b_\nu(x),$$

The operator $\delta^{ab}(g^{\mu\nu}\partial^2-(1-\frac{1}{\xi})\partial^\mu\partial^\nu)$ is indeed invertible and we denote this inverse by $D_F^{\mu\nu,ab}(x_1-x_2)$. Evaluating the Gaussian integrals and performing a Fourier transformation, we find the photon propagator:

$$\begin{aligned}\langle\Omega|\,T\{A^\mu(x_1)A^\nu(x_2)\}\,|\Omega\rangle &= D_F^{\mu\nu,ab}(x_1-x_2)\\ &=\int\frac{d^4k}{(2\pi)^4}\frac{-ie^{-ik(x_1-x_2)}}{k^2+i\epsilon}\left(g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right),\end{aligned}\tag{41}$$

Often a specific value of ξ is chosen in computations. Two convenient choices are the Landau gauge ($\xi=0$) and the Feynman gauge ($\xi=1$). In these gauges the gluon propagator takes on an especially simple form. We will be working in the Feynman gauge in our computations.

Interactions

All interactions are treated as perturbations to the free theory and therefore appear of the correlation function on the same footing as external fields. This makes computing their Feynman rules particularly simple. As an example we treat the basic quark-quark-gluon interaction $g\bar{\psi}\gamma^\nu A^b_\nu T^b\psi$. We start with the three-point Green's function:

$$\langle\Omega|\,T\{\psi(x_1)\bar{\psi}(x_2)A^a_\mu(x_3)\}\,|\Omega\rangle\,.$$

At zeroth order of the perturbation (i.e. the free theory) this correlation function is 0, as it contains an odd integral over A^μ. At first order, we have:

$$\int\mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi\left[ig\int d^4y\;\bar{\psi}(y)\gamma^\nu A^b_\nu(y)T^b\psi(y)\psi(x_1)\bar{\psi}(x_2)A^a_\mu(x_3)e^{i\int d^4x\mathcal{L}_0}\right]$$

Performing the path integrals, we get:

$$ig\int d^4y S_F(x_1-y)\gamma_\nu T^b S_F(y-x_2)D_F^{\mu\nu,ab}(x_3-y).$$

This expression still includes the propagation of the fields. To obtain the interaction itself we multiply by the inverse of the propagators and get:

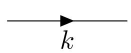
$$ig\int d^4y\gamma_\nu\delta^{(4)}(y-x_1)\delta^{(4)}(x_2-y)\delta^{\mu\nu}\delta^{ab}\delta^{(4)}(x_3-y).$$

Performing a Fourier transform and evaluating everything in momentum space we arrive at:

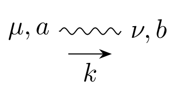
$$ig\gamma^{\mu}T^{a}\delta^{(4)}(k_1 + k_2 + k_3) \tag{42}$$

where k_i represents the momentum of the particle coming from x_i. Note that this result looks very similar to the original Lagrangian term; this is a benefit of pulling the interactions out of the exponent. The other interactions can be derived in a similar fashion, although one must be careful with the different ways the gluons can connect to the 3-gluon or 4-gluon vertex.

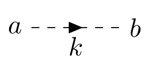
In conclusion we can describe the quantized theory by means of abstract quantities called Green's functions, whose square relate to the probability that the process they describe occurs. Green's functions can be calculated perturbatively to the desired accuracy by computing the Feynman diagrams that are related to said Green's function. The Feynman rules that are needed for computing the Feynman diagrams are:

1. 

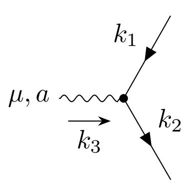
$= \dfrac{i}{\not{k} - m}$ (quark propagator),

2. 

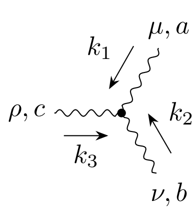
$= \dfrac{ig^{\mu\nu}\delta^{ab}}{k^2}$ (gluon propagator),

3. 

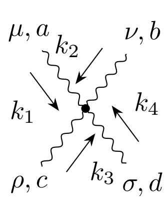
$= \dfrac{i\delta^{ab}}{k^2}$ (ghost propagator),

4. 

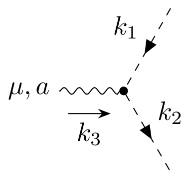
$= -ie\gamma^{\mu}T^{a}$,

5. 

$$= gf^{abc}\big[g^{\mu\nu}(k_1 - k_2)^{\rho} + g^{\nu\rho}(k_2 - k_3)^{\mu} + g^{\mu\rho}(k_3 - k_1)^{\nu}\big]$$

6. 

$$= -ig^2\big[f^{abx}f^{cdx}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{acx}f^{bdx}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{adx}f^{bcx}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big]$$

7. 

$= -gf^{abc}k_2^{\mu},$

8. Impose momentum conservation at each vertex,

9. Integrate over each undetermined (free) momentum: $\int \frac{d^4k}{(2\pi)^4}$,

10. Add a minus sign for all fermion and ghost loops

11. Divide by the symmetry factor of the diagram

where we have suppressed the $+i\epsilon$ terms in the propagators. This concludes the quantization of QCD using the path integral formulation. There is however still one problem with the theory: many Feynman diagrams contain divergent integrals over free momenta. This problem can be resolved by renormalizing the theory, which we will do in the next section.

3.2 Renormalization

Anyone who has ever worked on QCD knows that the world of quantum field theory is plagued with divergences. Mathematically, UV-divergences show up as the $k \to \infty$ limit of the momentum integrals over free momenta in the Feynman diagrams. This indicates that quantum field theories are not perfectly valid theories, but only work up to a certain momentum scale. The physical meaning of these divergences is however not so clear. Due to the strong similarities between QFT and statistical mechanics, many believe that QFT is merely an emergent theory. Just like statistical mechanics describes the emergent properties of lots of atoms together, QFT might describe the emergent properties of many incredibly tiny objects. Some even say that it is the quantization of spacetime itself (which should be around the Planck scale or 10^{-35} m) that breaks quantum field theory. Whatever the true physical reasoning is behind the divergences, we need to get rid of them to be able to use the theory. We do this by renormalizing the theory.

General principles of renormalization

The process of renormalization starts with introducing a regulator in the theory. The most obvious choice is a momentum cutoff: we lower the upper limit of the momentum integrals from ∞ to Λ. Another choice, called dimensional regularization, is to change the number of dimensions of the theory from 4 to d. For a sufficient small d the integrals are convergent. With this regulator we obtain expressions for the divergences as limits of $\Lambda \to \infty$ (or $d \to 4$). Next, we change our theory to use fields and parameters that are are actually divergent themselves: their divergences "absorb" the divergences from the ultraviolet behaviour of the Feynman diagrams. This absorption of infinitely many UV-divergences by finitely many fields and parameters is not possible for every quantum field theory. The theories for which this is possible are called renormalizable. It is proven [6] that theories in which the coupling constants are dimensionless or have a positive mass dimension are renormalizable. Lucky for us, this is the case with QCD: the coupling constant g is dimensionless and the quark mass m obviously has a positive mass dimension.

Denoting the divergent or 'bare' fields and parameters with a subscript B, we start from the Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}_B(i\not{D} - m_B)\psi_B - \frac{1}{4}(F_B^a)^{\mu\nu}(F_B^a)_{\mu\nu} + \bar{c}_B^a(-\partial^\mu D_\mu^{ac})c_B^c. \tag{43}$$

with $(F_B^a)^{\mu\nu} = \partial^\mu A_B^{\nu,a} - \partial^\nu A_B^{\mu,a} + g_B f^{abc} A_B^{\mu,b} A_B^{\nu,c}$, $D^\mu = \partial^\mu - ig_B A_B^{\mu,a} T^a$ and $D^{\mu,ac} = \partial^\mu \delta^{ac} + g_B f^{abc} A_B^{\mu,b}$. We ignore the gauge fixing term here for simplicity (this can be done by setting $\xi = \infty$). We express the divergences in the bare parameters and fields by so called renormalization constants. What is left we call renormalized fields and parameters, denoted with a subscript R:

$$A_B^{\mu,a} = \sqrt{Z_3} A_R^{\mu,a}, \qquad \psi_B = \sqrt{Z_2}\psi_R, \qquad c_R^a = \sqrt{Z_c} c_R^a,$$
$$m_B = Z_m m_R, \qquad g_B = Z_g g_R.$$

The renormalized parameters g_R and m_R then refer to the physical quark mass and coupling constant, Green's functions built from renormalized fields and parameters are finite and agree with experiment. As the coupling constant will be used very often, we drop the subscript R from here on and use simply g for the renormalized coupling constant.

After renormalization we are working with an effective theory: it is (probably) not the true way how nature works, but we can nonetheless describe nature with the theory. An aspect of effective theories is that they are momentum scale dependent. If we look at nature with low energies and long wavelengths, then we observe it coarsely. Higher energies and shorter wavelengths can reveal a finer theory with possibly different characteristics. Therefore the effective theory must change as function of scale. In renormalized field theories this manifests itself as a momentum scale dependence of the parameters. Every renormalization scheme introduces the scale dependence in one way or another. For the momentum cutoff this is the cutoff scale Λ. For dimensional regularization the scale dependence is not so clear. By changing the number of dimensions the coupling constant becomes dimensionful. This is remedied by introducing a momentum scale and rescaling the renormalized coupling constant: $g \to \mu^{\frac{\epsilon}{2}} g$ with $\epsilon = 4 - d$.

The precise formulation of the renormalization constants depends on the choice of renormalization scheme. We can however deduce a general structure from the free theory. In the free theory ($g = 0$) there obviously are no loop diagrams and therefore no divergences. So in the limit $g \to 0$ all renormalization constants must converge to unity. This means we can express all renormalization constants as: $Z = 1 + \mathcal{O}(g)$, which allows us to rewrite the Lagrangian in terms of renormalized fields and parameters as follows:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{ren}} + \mathcal{L}_{c.t.}, \tag{44}$$

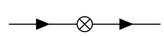
where $\mathcal{L}_{\text{ren}}$ is the Lagrangian of equation (43) but with renormalized instead of bare fields and parameters and $\mathcal{L}_{c.t.}$ is given by:

$$\begin{aligned}\mathcal{L}_{c.t.} = -\frac{1}{4}(\partial_\mu (A_R)^a_\nu - \partial_\nu (A_R)^a_\mu)^2 \delta_3 - \frac{1}{2} g f^{abc} (\partial_\mu (A_R)^a_\nu - \partial_\nu (A_R)^a_\mu)(A_R)^{\mu,b}(A_R)^{\nu,c}\delta_1^{3g} \\ - \frac{1}{4} g^2 f^{abc} f^{ade} (A_R)^b_\mu (A_R)^c_\nu (A_R)^{\mu,d} (A_R)^{\nu,e} \delta_1^{4g} + \bar{\psi}_R (i\not{\partial}\delta_2 - m\delta_m)\psi_R \\ + ig\bar{\psi}_R \not{A}_R \psi_R \delta_1 - \bar{c}^a_R \partial^2 c^a_R \delta^c_2 - g f^{abc} \bar{c}^a_R \partial^\mu (A_R)^b_\mu c^c_R \delta^c_1. \end{aligned} \tag{45}$$

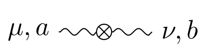
These extra terms are called the counterterms. The δ_i are related to the renormalization constants by:

$$\begin{gathered} \delta_3 = Z_3 - 1, \qquad \delta_2 = Z_2 - 1, \qquad \delta_m = Z_m Z_2 - 1, \qquad \delta_2^c = Z_c - 1, \\ \delta_1^{3g} = Z_g (Z_3)^{3/2} - 1, \qquad \delta_1^{4g} = (Z_g)^2 (Z_3)^2 - 1, \\ \delta_1 = Z_g Z_2 (Z_3)^{1/2} - 1, \qquad \delta_1^c = Z_g Z_c (Z_3)^{1/2} - 1. \end{gathered}$$

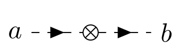
For the computation of the renormalization constants it is easier to treat the counterterms as extra vertex terms with their own Feynman rule and notation. The Feynman rules of the counterterms are given below[3]. Demanding that Green's functions including counterterms are finite allows us to compute the values of the δ_i and therefore the renormalization constants.



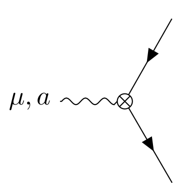
$= i(\not{p}\delta_2 - m\delta_m),$



$= -i(g^{\mu\nu}k^2 - k^\mu k^\nu)\delta^{ab}\delta_3,$



$= -ip^2 \delta^{ab} \delta_2^c,$



$= ig\gamma^\mu T^a \delta_1.$

[3]Only the counterterms needed to complete the renormalization of QCD to one-loop order have been given.

1-loop structure of QCD

We choose to calculate the renormalization constants in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, combined with dimensional regularization. In dimensional regularization the UV-divergences appear as poles in $\frac{1}{\epsilon}$, often together with a term $\ln 4\pi e^{-\gamma_E}$. The $\overline{\text{MS}}$ scheme then absorbs the combination $\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]$ in the renormalization constants. Note that when working with dimensional regularization that special care must be taken of terms of $\frac{\epsilon}{\epsilon}$! Terms proportional to ϵ can still provide finite contributions to equations if they later get multiplied by $\frac{1}{\epsilon}$. In this subsection we calculate the renormalization constants, using Feynman diagrams and the δ_i. Not all δ_i have to be calculated, as there are eight counterterms and only five renormalization constants. We choose to focus on $\delta_{1,2,3,m}$ and δ_2^c. All the needed Feynman diagrams are computed in appendix A.1.

Figure 2: All Feynman diagrams for $\langle\Omega|\, \psi_R(x_1)\bar{\psi}_R(x_2)\, |\Omega\rangle$ up to one-loop order, including counterterms.

The δ_2 and δ_m can be computed from the quark self-energy, i.e. the Green's function $\langle\Omega|\, \psi_R(x_1)\bar{\psi}_R(x_2)\, |\Omega\rangle$. Up to 1-loop order, there is one divergent diagram and one counterterm diagram, depicted in figure 2; their sum must be finite. Therefore the δ_2 and δ_m must equal (up to one-loop order):

$$\delta_2 = -C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{46}$$

$$\delta_m = -4C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{47}$$

Here $\alpha_s = \frac{g^2}{4\pi}$ is the QCD equivalent of the fine structure constant.

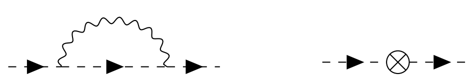


Figure 3: All Feynman diagrams for $\langle\Omega|\, c_R^a(x_1)\bar{c}_R^b(x_2)\, |\Omega\rangle$ up to one-loop order, including counterterms.

For δ_2^c we consider the ghost self-energy $\langle\Omega|\, c_R^a(x_1)\bar{c}_R^b(x_2)\, |\Omega\rangle$, which is very similar to the electron self-energy. We again have one divergent diagram and one counterterm diagram up to 1-loop order, depicted in figure 3. If we insist their divergences cancel, δ_2^c must be given by (up to one-loop order):

$$\delta_2^c = -\frac{1}{2}C_2(G)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{48}$$

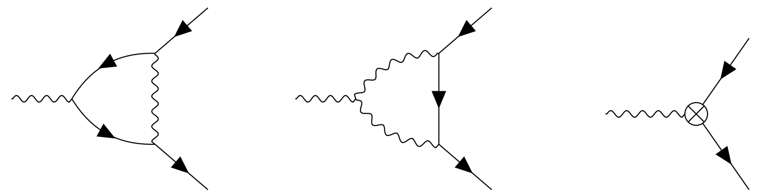


Figure 4: All Feynman diagrams for $\langle\Omega|\,\psi_R(x_1)A_R^{\mu,a}(x_2)\bar{\psi}_R(x_3)\,|\Omega\rangle$ up to one-loop order, including counterterms.

To compute δ_1 we examine the effective quark-quark-gluon vertex, given by $\langle\Omega|\,\psi_R(x_1)A_R^{\mu,a}(x_2)\bar{\psi}_R(x_3)\,|\Omega\rangle$. We now have two divergent diagrams and one counterterm diagram up to 1-loop order, depicted in figure 4. As their divergences must cancel out, δ_1 has to equal:

$$\delta_1 = -\big(C_2(N) + C_2(G)\big)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{49}$$

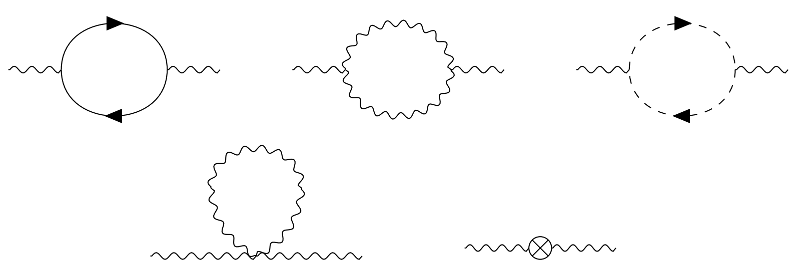


Figure 5: All Feynman diagrams for $\langle\Omega|\,A_R^{\mu,a}(x_1)A_R^{\nu,b}(x_2)\,|\Omega\rangle$ up to one-loop order, including counterterms.

For δ_3 we look at the vacuum polarization $\langle\Omega|\,A_R^{\mu,a}(x_1)A_R^{\nu,b}(x_2)\,|\Omega\rangle$. The divergent and counterterm diagrams up to 1-loop order are depicted in figure 5. The quark loop diagram has to be multiplied by the number of fermions n_f. The other three loop diagrams make little sense separately, but combined yield a divergence of:

$$\begin{aligned} iC_2(G)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\int_0^1 dx\ \Delta^{\frac{\epsilon}{2}}\Big[&(2 - 2\epsilon + \frac{1}{2}\epsilon^2)g^{\mu\nu}\Gamma\Big(\frac{\epsilon}{2} - 1\Big)\Delta \\ &+ \frac{1}{2}\big([(2x-1)^2(2-\epsilon) - 2(x+1)(2-x) - 2x(x-1)]p^\mu p^\nu \\ &\qquad + [(2-x)^2 + (1+x)^2 - 2(3-\epsilon)(x-1)^2]p^2 g^{\mu\nu}\big)\Gamma\Big(\frac{\epsilon}{2}\Big)\Big]. \end{aligned}$$

We recognize that $(2 - 2\epsilon + \frac{1}{2}\epsilon^2) = -(2-\epsilon)(\frac{\epsilon}{2} - 1)$, which means that now the two gamma functions contribute to the same divergence (because of the relation

$x\Gamma(x) = \Gamma(x+1)$). Using $\Delta = x(x-1)p^2$, we find:

$$iC_2(G)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\int_0^1 dx\ \Delta^{\frac{\epsilon}{2}}\frac{1}{2}\big([(2x-1)^2(2-\epsilon)-2(x+1)(2-x)-2x(x-1)]p^\mu p^\nu$$
$$+ [(2-x)^2 + (1+x)^2 - 2(3-\epsilon)(x-1)^2 - 2(2-\epsilon)x(x-1)]p^2 g^{\mu\nu}\big)\Gamma\Big(\frac{\epsilon}{2}\Big).$$

Next we tackle the $p^\mu p^\nu$ coefficient: this simply equals $(2-\epsilon)(2x-1)^2 - 4$. As the total of the three diagrams must be proportional to $(p^2 g^{\mu\nu} - p^\mu p^\nu)$, the p^2 coefficient must also equal $(2-\epsilon)(2x-1)^2 - 4$. So we get:

$$iC_2(G)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\Gamma\Big(\frac{\epsilon}{2}\Big)\int_0^1 dx\ \Delta^{\frac{\epsilon}{2}}\frac{1}{2}[(2-\epsilon)(2x-1)^2 - 4](p^2 g^{\mu\nu} - p^\mu p^\nu).$$

The divergent part of this diagram is therefore:

$$-\frac{5}{3}C_2(G)\frac{\alpha_s}{4\pi}(-i[p^2 g^{\mu\nu} - p^\mu p^\nu]\delta^{ab})\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right].$$

Finally, the divergences of all four diagrams must be cancelled by the counterterm diagram. Therefore we get:

$$\delta_3 = \big(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(N)\big)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{50}$$

Now that we have computed the needed counterterms, we can calculate the renormalization constants. The results are:

$$Z_2 = 1 - C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{51}$$
$$Z_3 = 1 + \big(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(N)\big)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{52}$$
$$Z_m = 1 - 3C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{53}$$
$$Z_g = 1 + \big(\frac{2}{3}n_f C(N) - \frac{11}{6}C_2(G)\big)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{54}$$
$$Z_c = 1 - \frac{1}{2}C_2(G)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{55}$$

This concludes the renormalization of QCD. It is instructive to check how the cancellation of divergences works when using renormalization constants for the quark propagator. We know that the bare propagator is (up to on-loop order) the sum of two diagrams: the basic propagator and the quark self energy diagram, computed in appendix A.1. Together, we have:

$$\langle\Omega|\,\psi_B(x)\bar{\psi}_B(y)\,|\Omega\rangle = \frac{i}{\not{p} - m_B} + \frac{i}{\not{p} - m_B}C_2(N)\frac{\alpha_s}{4\pi}\frac{2}{\epsilon}(i[\not{p} - 4m_B])\frac{i}{\not{p} - m_B}$$

plus some additional finite terms. This time we have not ignored the external legs. This is because the external legs depend on the bare mass, while the renormalized propagator should only depend on renormalized quantities. To remove the dependence on bare quantities, we use $m_B = Z_m m_R$ and write:

$$\begin{aligned}\frac{i}{\not{p} - m_B} &= \frac{i}{\not{p} - m_R + (Z_m - 1)m_R}, \\ &= \sum_{n=0}^{\infty} \frac{i}{\not{p} - m_R}\left(\frac{(Z_m - 1)m_R}{\not{p} - m}\right)^n.\end{aligned}$$

The bare propagator then equals:

$$\langle\Omega|\, \psi_B(x)\bar{\psi}_B(y)\, |\Omega\rangle = \frac{i}{\not{p} - m_R} + \frac{i}{\not{p} - m_R} C_2(N)\frac{\alpha_s}{4\pi}\frac{2}{\epsilon}(i[\not{p} - m_R])\frac{i}{\not{p} - m_R},$$

where we've used equation (53) to express everything as a series in α_s. As the one-loop correction already is of order α_s, we only keep the leading order term of the external legs in that term. The renormalized propagator differs from the bare propagator by a factor Z_2^{-1}, coming from the quark fields. Multiplying the basic propagator by this factor, we find an additional term of $\mathcal{O}(\alpha_s)$:

$$\frac{iZ_2^{-1}}{\not{p} - m_R} = \frac{i}{\not{p} - m_R} + \frac{i}{\not{p} - m_R}[-i(Z_2^{-1} - 1)(\not{p} - m_R)]\frac{i}{\not{p} - m_R}$$

If we use equation (51) to again express everything as a series in α_s, we note that this additional term precisely cancels the divergence in the bare propagator:

$$\langle\Omega|\, \psi_R(x)\bar{\psi}_R(y)\, |\Omega\rangle = \frac{i}{\not{p} - m_R}. \tag{56}$$

This concludes the proof. Even though this confirms that the renormalized quark propagator is finite, it is good to note that equation (56) is not an exact equation. Additional finite contributions to the renormalized propagator have been suppressed in this derivation. The precise nature of these additional terms depends on the renormalization scheme and the scale at which we probe the theory.

3.3 Beta function and mass anomalous dimension

We have established before that QCD is an effective theory and therefore dependent on the scale at which we probe the theory. This scale dependence manifests itself in the momentum scale dependence, or 'running', of the parameters of the theory. If we fully want to understand QCD, we need to investigate the running of the parameters. To do so, we consider the following differential equations:

$$\frac{\partial g}{\partial \ln \mu} = \beta(g), \tag{57}$$

$$\frac{\partial m_R}{\partial \ln \mu} = m_R \gamma_m(g). \tag{58}$$

The two functions on the right hand side are called the QCD beta function and quark mass anomalous dimension respectively. They are the backbone of many renormalization calculations and even appear in for instance quantum anomalies.

We first focus on the beta function. We use the fact that the bare parameter g_B is scale independent and write:

$$0 = \frac{\partial g_B}{\partial \ln \mu} = \frac{\epsilon}{2}\mu^{\frac{\epsilon}{2}} Z_g g + \mu^{\frac{\epsilon}{2}} \frac{\partial Z_g}{\partial \ln \mu} g + \mu^{\frac{\epsilon}{2}} Z_g \beta(g).$$

All terms have a factor $\mu^{\frac{\epsilon}{2}}$ in common, which we divide out. Also, Z_g is only dependent on μ through g^2. If we define $\beta_0 \equiv \frac{11}{3}C_2(G) - \frac{4}{3}n_f C(N)$, we get $Z_g = 1 - \frac{\beta_0}{2}\frac{\alpha_s}{4\pi}\Big[\frac{\epsilon}{2} + \ln 4\pi e^{-\gamma_E}\Big]$ and the equation above becomes:

$$\Big(1 - \frac{3}{2}\beta_0 \frac{\alpha_s}{4\pi}\Big[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\Big]\Big)\beta(g) + \Big(1 - \frac{1}{2}\beta_0 \frac{\alpha_s}{4\pi}\Big[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\Big]\Big)\frac{\epsilon}{2} g = 0,$$

from which we deduce that up to 1-loop order the beta function is given by:

$$\beta(g) = -g\Big(\frac{\epsilon}{2} + \beta_0 \frac{\alpha_s}{4\pi} + \dots\Big). \tag{59}$$

The equation above only holds when using dimensional regularization in $4 - \epsilon$ dimensions. When we take the limit $\epsilon \to 0$, we obtain the well-known expression of the beta function, that is independent of the renormalization scheme (up to this order) [6]:

$$\beta(g) = -\beta_0 \frac{g^3}{(4\pi)^2} + \dots. \tag{60}$$

Depending on the value of β_0, we have two possible behaviours of the coupling constant g. If there are many quark flavors ($n_f > \frac{33}{2}$), the beta function is positive. In this case g would become small at large distances and large at small distances, meaning charges at large distances do not affect each other. However, in our universe there are only 6 known quark flavors. Therefore the beta function is negative and g increases at large distances, while it tends to zero as the distance decreases. This behaviour is called 'asymptotic freedom'. Quarks and gluons at very small distances act like free particles, hardly noticing each other. In the large-distance limit however, the coupling constant increases to infinity. In fact, g might reach infinity even at a finitely large distance[4]. It is interesting to calculate an approximate formula for α_s. If we assume that $\alpha_s \ll 1$, we have:

$$\frac{\partial \alpha_s}{\partial \ln \mu} = -\beta_0 \frac{\alpha_s^2}{2\pi}.$$

This is a simple differential equation which can be solved by standard methods. The result is:

[4] We do not know this for sure, as perturbation theory fails when g becomes large.

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 - \alpha_s(\mu_0)\beta_0 \frac{\log \mu/\mu_0}{2\pi}}, \tag{61}$$

with initial conditions $\alpha_s(\mu_0)$ at $\mu = \mu_0$. This confirms the story above, that $|\alpha_s|$ decreases with increasing energy and vice versa. Of course equation (61) is only valid when α_s is weak, as we have ignored higher order terms. To gain an estimate for the lower bound at which α_s is weak, note that equation (61) contains a pole at $\mu = \mu_0 e^{\frac{2\pi}{\beta_0 \alpha_s(\mu_0)}} \equiv \Lambda$. This defines the scale at which α_s becomes strong and perturbation theory becomes useless for our calculations. Using $\alpha_s(2\text{ GeV}) = 0.269$ (from [10]) and assuming 4 active quark flavours, we find a lower bound of $\Lambda = 0.121$ GeV. This is in agreement with experimental measurements, which yielded a value of $\Lambda \approx 0.2$ GeV [6]. QCD perturbation theory is valid when μ is larger than these values, say above $\mu = 1$ GeV. Finally, we can rewrite equation (61) in a simpler form using Λ:

$$\alpha_s(\mu) = \frac{2\pi}{-\beta_0 \ln(\mu/\Lambda)}. \tag{62}$$

This is the most simple display of the running of the coupling constant.

The quark mass anomalous dimension can be found via a similar way as the beta function. We use the fact that m_B is scale independent and write:

$$0 = \frac{\partial m_B}{\partial \ln \mu} = \frac{\partial Z_m}{\partial \ln \mu} m_R + Z_m m_R \gamma_m(g).$$

We divide by m_R and again note that Z_m is only dependent on μ through g^2. Therefore we get:

$$\begin{aligned} &-6C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]\left(\frac{\epsilon}{2} + \beta_0\frac{\alpha_s}{4\pi} + \dots\right) \\ &\qquad + \left(1 - 3C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]\right)\gamma_m(g) = 0. \end{aligned}$$

Note that we have used equation (59) for $\beta(g)$ instead of equation (60) as we are still working in $d = 4 - \epsilon$ dimensions. From here we conclude that the quark mass anomalous dimension up to 1-loop order in 4 dimensions equals:

$$\gamma_m(g) = -6C_2(N)\frac{\alpha_s}{4\pi} + \dots. \tag{63}$$

4 Operator renormalization

Composite operators are products of fields evaluated at the same point in spacetime. Examples are the quark number current $\bar{\psi}\gamma^\mu\psi(x)$ and the energy momentum tensor $T^{\mu\nu}(x)$. Calculations involving composite operators must evaluate multiple fields infinitely close to each other. As we have seen, QFT is not fit to handle these small distances and UV-divergences arise in the calculation of Green's functions. These divergences generally can't be removed by the field renormalizations alone. Instead we introduce a separate renormalization of the composite operators, much like the field renormalization. Effectively, we treat the composite operator as a field of its own, although related to the basic fields of the theory. Sections 4.1 and 4.2 are based on [6] and deal with the basics of operator renormalization and mixing. Sections 4.3 deals with the EMT trace anomaly and is an extension of chapter 19.5 of [6]. Finally, section 4.4 treats the renormalization of the EMT.

4.1 The basics

Let us consider a general composite operators $\mathcal{O}(x)$, constructed from the fields $\phi_1, \ldots, \phi_n$ (ϕ can be any type of field). We define a finite, renormalized operator $\mathcal{O}_R$ which is a rescaled version of the operator $\mathcal{O}_B$, built from bare fields[5]:

$$\mathcal{O}_R = Z_\mathcal{O}\mathcal{O}_B. \tag{64}$$

The renormalized operator is renormalization scheme dependent, as is the case with all renormalized quantities. One must however be careful to use the same regularization as for the field renormalizations. One can use a different renormalization scheme from the field renormalization [2]. So far throughout this thesis we have used dimensional regularization with the $\overline{\text{MS}}$ scheme. We will be using the same combination for composite operator renormalization. The renormalized composite operators are also scale dependent and therefore have an anomalous dimension. Its value is computed as follows[6]:

$$\gamma_\mathcal{O}(\mu) = \frac{1}{Z_\mathcal{O}}\frac{\partial Z_\mathcal{O}}{\partial \ln \mu}. \tag{65}$$

The value of the renormalization constant is calculated by using Green's functions. We define the renormalized Green's function[7]:

$$G_R^{(n)}(p_1, \ldots, p_n) = \langle\Omega|\, \mathcal{O}_R(0)\phi_{1,R}(p_1)\ldots\phi_{n,R}(p_n)\,|\Omega\rangle\,. \tag{66}$$

The zero-momentum insertion of the operator is chosen deliberately, as the renormalization constants are momentum-independent and this is the easiest

[5]Note that $Z_\mathcal{O}$ is defined opposite of how we defined renormalization constants so far. We have done this to be more in agreement with other papers on renormalization of the EMT.

[6]The opposite sign stems again from the fact that we've defined $Z_\mathcal{O}$ opposite of the theory so far.

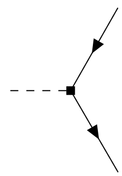
[7]We define all operators to have their vacuum expectation value removed. For instance, whenever we write $\mathcal{O} = \bar{\psi}\psi$, we are actually saying: $\mathcal{O} = \bar{\psi}\psi - \langle\Omega|\,\bar{\psi}\psi\,|\Omega\rangle$.

choice. We relate $G_R^{(n)}$ to a bare Green's function by:

$$G_R^{(n)}(p_1, \ldots, p_n) = \left(\prod_{i=1}^{n} Z_{\phi_i}^{-\frac{1}{2}} \right) Z_{\mathcal{O}} \langle \Omega | \, \mathcal{O}_B(0) \phi_{1,B}(p_1) \ldots \phi_{n,B}(p_n) \, | \Omega \rangle \, . \quad (67)$$

The bare Green's function can be computed by evaluating the corresponding Feynman diagrams. As the renormalized Green's function must be finite at all orders, we then can calculate $Z_{\mathcal{O}}$ to the desired order.

As an example, we consider the composite operator $[\bar{\psi}\psi](x)$. It interacts with quarks via:

$[\bar{\psi}\psi]$  $= 1.$

where the square dot with dashed line indicates an interaction with a composite operator. Up to 1-loop order, there are four Feynman diagrams associated with the Green's function $\langle \Omega | \, [\bar{\psi}\psi](0)\psi(p_1)\bar{\psi}(p_2) \, | \Omega \rangle$, depicted in figure 6. Of these,

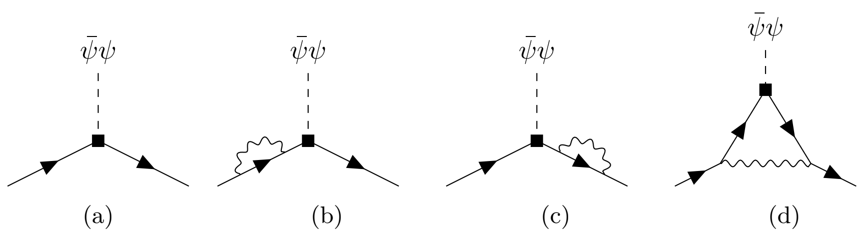


Figure 6: All Feynman diagrams for $\langle \Omega | \, [\bar{\psi}\psi](0)\psi(p_1)\bar{\psi}(p_2) \, | \Omega \rangle$ up to 1-loop order.

diagram (d) (and of course diagram (a)) is 1PI and is computed in appendix A.2. Diagrams (b) and (c) are not 1PI, but can be viewed as quantum corrections to the external fields. In fact, this holds at all orders: the full Green's function can be written as:

$$\langle \Omega | \, [\bar{\psi}\psi](0)\psi(p_1)\bar{\psi}(p_2) \, | \Omega \rangle = (Z_2)^2 \Gamma_{\bar{\psi}\psi}, \quad (68)$$

where $\Gamma_{\bar{\psi}\psi}$ is the sum of all 1PI diagrams. Up to 1-loop order, we then have:

$$\langle \Omega | \, [\bar{\psi}_B\psi_B](0)\psi_B(p_1)\bar{\psi}_B(p_2) \, | \Omega \rangle = 1 + 2\delta_2 + 4C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right] + \mathcal{O}(\alpha_s^2),$$

where we have subtracted external legs. Here we see that only renormalizing the fields would not have been enough: divergences of the type of diagram (d) would still be present. Therefore we define the renormalized operator $[\bar{\psi}\psi]_R(x)$

similar to equation (64). We find that the following combination must be finite at all orders:

$$\frac{Z_{\bar{\psi}\psi}}{Z_2}\left(1+2\delta_2+4C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]+\mathcal{O}(\alpha_s^2)\right),$$

from where we conclude that:

$$Z_{\bar{\psi}\psi}=1-3C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]+\mathcal{O}(\alpha_s^2), \tag{69}$$

and therefore:

$$\gamma_{\bar{\psi}\psi}=6C_2(N)\frac{\alpha_s}{4\pi}+\mathcal{O}(\alpha_s^2). \tag{70}$$

You might notice that $\gamma_{\bar{\psi}\psi}$ is equal to $-\gamma_m$. This is no coincidence. We will see later that the composite operator $m\bar{\psi}\psi$ is RG-invariant (renormalization group invariant). Therefore we obtain the relation:

$$0=\frac{\partial m_R(\bar{\psi}\psi)_R}{\partial \log\mu}=\gamma_m m_R(\bar{\psi}\psi)_R+m_R\frac{\partial Z_{\bar{\psi}\psi}}{\partial\log\mu}(\bar{\psi}\psi)_B=(\gamma_m+\gamma_{\bar{\psi}\psi})m_R(\bar{\psi}\psi)_R,$$

proving that $\gamma_{\bar{\psi}\psi}=-\gamma_m$ to all orders.

4.2 Operator mixing

Often when working with composite operators, there exist multiple, linearly independent operators which possess the same quantum numbers, symmetries and other characteristics. These operators form a linear space on which renormalization acts as a linear operator. Therefore the renormalization constant needs to be generalized to a matrix and the operators mix under renormalization. If we consider a linearly independent set $\mathcal{O}^i$, $i=1,\ldots,N$, of operators with the same characteristics, the relation between renormalized and bare renormalization is:

$$\mathcal{O}_R^i=Z_{\mathcal{O}}^{ij}\mathcal{O}_B^j \tag{71}$$

where the $\mathcal{O}_B^j$ are built from bare fields. In turn the anomalous dimension function also must be generalized to a matrix:

$$\gamma_{\mathcal{O}}^{ij}=\mu\frac{\partial Z_{\mathcal{O}}^{ik}}{\partial\mu}[Z_{\mathcal{O}}^{-1}]^{kj}. \tag{72}$$

The value of the renormalization matrix is again calculated by using Green's functions. Each $\mathcal{O}^i$ is constructed from fields $\phi_1^i,\ldots,\phi_{n_i}^i$ (where again ϕ can be any type of field). Therefore we define renormalized Green's functions similar to equation (66) for each operator O^i. However, because all the composite operators have the same quantum numbers etc, we can also consider the Green's function of $\mathcal{O}^i$ with the fields that make up $\mathcal{O}^j$ where $i\neq j$. So we get the Green's functions:

$$G_{i,j,R}^{(n_j)}=\langle\Omega|\,\mathcal{O}_R^i(0)\phi_{1,R}^j(p_1)\ldots\phi_{n_j,R}^j(p_{n_j})\,|\Omega\rangle\,. \tag{73}$$

where $i, j = 1, \dots, N$. Once again we write everything in terms of bare operators and fields. This time we get a sum of N bare Green's functions on the right hand side:

$$G^{(n_j)}_{i,j,R} = \left(\prod_{k=1}^{n_j} (Z_{\phi^j_k})^{-\frac{1}{2}} \right) \sum_{l=1}^{N} Z^{il} \langle\Omega| \mathcal{O}^l_B(0) \phi^j_{1,B}(p_1) \dots \phi^j_{n_j,B}(p_{n_j}) |\Omega\rangle . \quad (74)$$

The relation between the renormalized and bare Green's functions, similar to equation (67), yield a system of N^2 equations. This system can then be solved for the N^2 entries of the renormalization matrix.

Let us consider an example. We define the linearly independent operators:

$$\mathcal{O}^{\mu\nu}_g = -(F^{\mu\rho})_a (F^{\nu}{}_{\rho})_a, \qquad \mathcal{O}^{\mu\nu}_f = \bar{\psi}(i\gamma^{\{\mu} D^{\nu\}})\psi, \quad (75)$$

both with the trace in Lorentz space subtracted. Within QCD, these form a basis for all the gauge invariant Dirac scalar, Lorentz tensor, colorless and traceless operators that are independent under the equations of motion and symmetric under the exchange of μ and ν. Therefore they only mix among themselves and the mixing matrix looks like:

$$\begin{bmatrix} \mathcal{O}^{\mu\nu}_{g,R} \\ \mathcal{O}^{\mu\nu}_{f,R} \end{bmatrix} = \begin{bmatrix} Z_{gg} & Z_{gf} \\ Z_{fg} & Z_{ff} \end{bmatrix} \begin{bmatrix} \mathcal{O}^{\mu\nu}_{g,B} \\ \mathcal{O}^{\mu\nu}_{f,B} \end{bmatrix} . \quad (76)$$

To calculate the renormalization constants, we consider the induced relations between bare and renormalized Green's functions:

$$\begin{aligned}
\langle\Omega| \mathcal{O}^{\mu\nu}_{g,R} \psi_R \bar{\psi}_R |\Omega\rangle &= \frac{Z_{gg}}{Z_2} \langle\Omega| \mathcal{O}^{\mu\nu}_{g,B} \psi_B \bar{\psi}_B |\Omega\rangle + \frac{Z_{gf}}{Z_2} \langle\Omega| \mathcal{O}^{\mu\nu}_{f,B} \psi_B \bar{\psi}_B |\Omega\rangle , \\
\langle\Omega| \mathcal{O}^{\mu\nu}_{g,R} A^{\sigma,a}_R A^{\tau,b}_R |\Omega\rangle &= \frac{Z_{gg}}{Z_3} \langle\Omega| \mathcal{O}^{\mu\nu}_{g,B} A^{\sigma,a}_B A^{\tau,b}_B |\Omega\rangle + \frac{Z_{gf}}{Z_3} \langle\Omega| \mathcal{O}^{\mu\nu}_{f,B} A^{\sigma,a}_B A^{\tau,b}_B |\Omega\rangle , \\
\langle\Omega| \mathcal{O}^{\mu\nu}_{f,R} \psi_R \bar{\psi}_R |\Omega\rangle &= \frac{Z_{fg}}{Z_2} \langle\Omega| \mathcal{O}^{\mu\nu}_{g,B} \psi_B \bar{\psi}_B |\Omega\rangle + \frac{Z_{ff}}{Z_2} \langle\Omega| \mathcal{O}^{\mu\nu}_{f,B} \psi_B \bar{\psi}_B |\Omega\rangle , \\
\langle\Omega| \mathcal{O}^{\mu\nu}_{f,R} A^{\sigma,a}_R A^{\tau,b}_R |\Omega\rangle &= \frac{Z_{fg}}{Z_3} \langle\Omega| \mathcal{O}^{\mu\nu}_{g,B} A^{\sigma,a}_B A^{\tau,b}_B |\Omega\rangle + \frac{Z_{ff}}{Z_3} \langle\Omega| \mathcal{O}^{\mu\nu}_{f,B} A^{\sigma,a}_B A^{\tau,b}_B |\Omega\rangle ,
\end{aligned}$$

The renormalized Green's functions are finite to all orders. As there are no renormalizations at zero-loop order, the bare and renormalized Green's function agree at this order. The bare Green's functions can be calculated from the definition of the correlation function using Wick contractions. The results are:

$$\begin{aligned}
&\langle\Omega| \mathcal{O}^{\mu\nu}_{g,R} A^{\sigma,a}_R A^{\tau,b}_R |\Omega\rangle = -2\delta^{ab}\big(p^{\{\mu}p^{\nu\}} g^{\sigma\tau} + p^2 g^{\sigma\{\mu} g^{\nu\}\tau} - p^\sigma p^{\{\mu} g^{\nu\}\tau} - p^\tau p^{\{\mu} g^{\nu\}\sigma}\big), \\
&\langle\Omega| \mathcal{O}^{\mu\nu}_{f,R} \psi_R \bar{\psi}_R |\Omega\rangle = \gamma^{\{\mu} p^{\nu\}}, \\
&\langle\Omega| \mathcal{O}^{\mu\nu}_{f,R} A^{\sigma,a}_R A^{\tau,b}_R |\Omega\rangle = \langle\Omega| \mathcal{O}^{\mu\nu}_{g,R} \psi_R \bar{\psi}_R |\Omega\rangle = 0,
\end{aligned}$$

all with the trace in $\mu-\nu$ space subtracted. All other orders of the bare Green's functions can be calculated by means of Feynman diagrams. Both sides of the

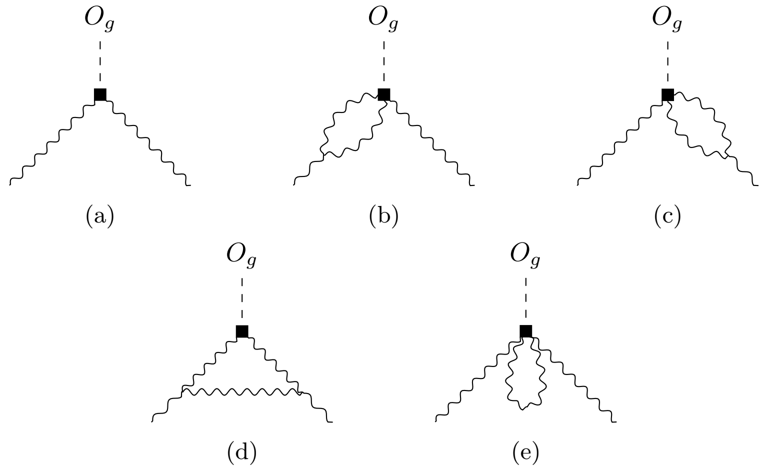


Figure 7: All 1PI diagrams for $\langle\Omega|\, \mathcal{O}^{\mu\nu}_{g,0} A^{\alpha,a}_0 A^{\beta,b}_0 \,|\Omega\rangle$ up to 1-loop order.

equations above must have the same overall structure. Therefore it is sufficient to compare one term of the Green's functions to calculate the renormalization constants. We choose to focus on the terms $-2\delta^{ab}p^{\{\mu}p^{\nu\}}g^{\sigma\tau}$ and $\gamma^{\{\mu}p^{\nu\}}$ for interactions with gluons and quarks respectively.

For $\langle\Omega|\, \mathcal{O}^{\mu\nu}_{g,B} A^{\alpha,a}_B A^{\beta,b}_B \,|\Omega\rangle$, there are five 1PI diagrams up to one loop order, depicted in figure 1, and eight diagrams for the external gluon field renormalization. The external field corrections yield a divergence of $2Z_3$ (multiplied by the basic interaction), where Z_3 is given in equation (52). Of the diagrams in figure 7, diagram (e) contains no terms with p^μ and therefore does not contribute to $-2\delta^{ab}p^{\{\mu}p^{\nu\}}g^{\sigma\tau}$. Diagram (b) is equal to diagram (c). The divergences of all the diagrams are calculated in appendix A.2. All together, we get:

$$\langle\Omega|\, \mathcal{O}^{\mu\nu}_{g,B} A^{\alpha,a}_B A^{\beta,b}_B \,|\Omega\rangle = \left(1 + 2\delta_3 - \frac{5}{3}C_2(G)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right] + \mathcal{O}(\alpha_s^2)\right) \\ [-2\delta^{ab}p^{\{\mu}p^{\nu\}}g^{\sigma\tau} + \ldots]. \quad (77)$$

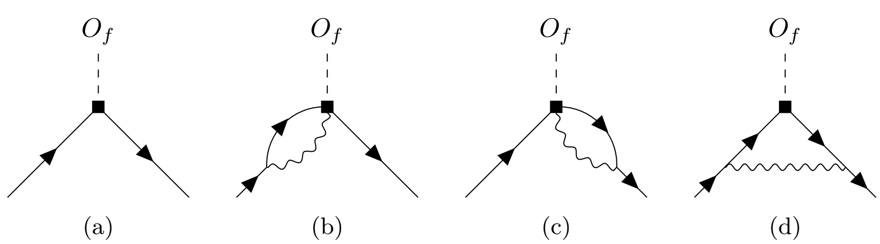


Figure 8: All 1PI diagrams for $\langle\Omega|\, \mathcal{O}^{\mu\nu}_{f,0} \psi_0 \bar{\psi}_0 \,|\Omega\rangle$ up to 1-loop order.

Similarly, for $\langle\Omega|\,\mathcal{O}^{\mu\nu}_{f,B}\psi_B\bar{\psi}_B\,|\Omega\rangle$ there are four 1PI diagrams up to 1-loop order, depicted in figure 8, and two diagrams for the external quark field renormalization. Just as in the last section, yield the external field corrections a divergence of $2\delta_2$. Of the 1PI diagrams, diagram (b) is equal to diagram (c). The divergences of all the diagrams are calculated in appendix A.2. All together, we get:

$$\langle\Omega|\,\mathcal{O}^{\mu\nu}_{g,B}\psi_B\bar{\psi}_B\,|\Omega\rangle = \left(1+2\delta_2-\frac{5}{3}C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]+\mathcal{O}(\alpha_s^2)\right)[\gamma^{\{\mu}p^{\nu\}}-(\text{trace})]. \tag{78}$$

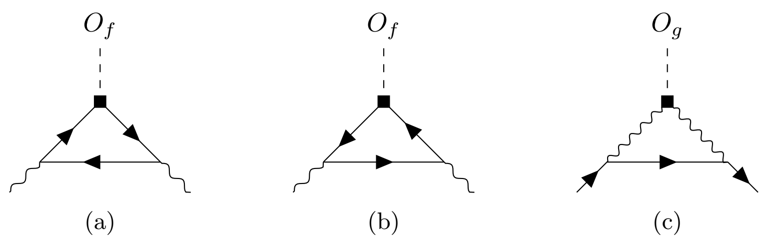


Figure 9: All diagrams for $\langle\Omega|\,\mathcal{O}^{\mu\nu}_{f,0}A_0^{\alpha,a}A_0^{\beta,b}\,|\Omega\rangle$ (diagrams (a) and (b)) and $\langle\Omega|\,\mathcal{O}^{\mu\nu}_{g,0}\psi_0\bar{\psi}_0\,|\Omega\rangle$ (diagram (c)) up to 1-loop order.

For $\langle\Omega|\,\mathcal{O}^{\mu\nu}_{f,B}A_B^{\alpha,a}A_B^{\beta,b}\,|\Omega\rangle$, there are a total of $2n_f$ diagrams up to 1-loop order, diagrams (a) and (b) of figure 9 for each type of fermion. Their contribution is identical. For $\langle\Omega|\,\mathcal{O}^{\mu\nu}_{g,B}\psi_B\bar{\psi}_B\,|\Omega\rangle$ there is one diagram up to 1-loop order, diagram (c) of figure 9. There are no external field corrections at this order. The divergences of the diagrams are calculated in appendix A.2. The results are:

$$\langle\Omega|\,\mathcal{O}^{\mu\nu}_{f,B}A_B^{\alpha,a}A_B^{\beta,b}\,|\Omega\rangle = \left(\frac{4}{3}n_f C(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]+\mathcal{O}(\alpha_s^2)\right)[-2\delta^{ab}p^{\{\mu}p^{\nu\}}g^{\sigma\tau}+\ldots]. \tag{79}$$

$$\langle\Omega|\,\mathcal{O}^{\mu\nu}_{g,B}\psi_B\bar{\psi}_B\,|\Omega\rangle = \left(\frac{8}{3}C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]+\mathcal{O}(\alpha_s^2)\right)[\gamma^{\{\mu}p^{\nu\}}-(\text{trace})]. \tag{80}$$

We are now able to solve for the renormalization constants order by order. At leading order, we have:

$$Z_{gg} = Z_{ff} = 1,$$
$$Z_{fg} = Z_{gf} = 0.$$

At next-to-leading order, the following combinations must be finite:

$$\frac{8}{3}C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]+Z_{gf},$$
$$Z_{gg}-\delta_3+2\delta_3-\frac{5}{3}C_2(G)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right],$$
$$Z_{ff}-\delta_2+2\delta_2-\frac{5}{3}C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right],$$
$$Z_{fg}+\frac{4}{3}n_f C(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right].$$

All together, we conclude:

$$Z_{gg}=1+\frac{4}{3}n_f C(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right], \tag{81}$$
$$Z_{gf}=-\frac{8}{3}C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right], \tag{82}$$
$$Z_{fg}=-\frac{4}{3}n_f C(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right], \tag{83}$$
$$Z_{ff}=1+\frac{8}{3}C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]. \tag{84}$$

From here the anomalous dimension and therefore the scaling of the operators $\mathcal{O}_f$ and $\mathcal{O}_g$ could be calculated. This anomalous dimension is also a matrix, meaning the scaling depends on both operators. One could say that the operators "talk" to each other. We will later see that the components of mass rules also "talk" to each other.

4.3 Computation of the trace of the EMT

Before we tackle the renormalization of the EMT, we will compute the EMT trace. This is a well-known result in the literature that is important to cover here as well. The EMT is related to the divergence of the dilatation current, which is the Noether current of scale invariance. Therefore the EMT can be viewed as a measure of the scale invariance of a theory. In classical QCD, we have $T^{\mu}_{\ \mu}=m\bar{\psi}\psi$. The meaning of this is clear: the masses of the fermions break the scale invariance of the theory. In a massless theory we would have $T^{\mu}_{\ \mu}=0$ and the dilatation current would indeed be a conserved current. Quantization breaks scale invariance by breaking the energy spectrum into tiny discrete steps (quanta). Therefore quantization should add a non-zero term to the trace of the EMT that vanishes when we take the classical limit. This term is called the quantum trace anomaly. In this section we show that a quantum trace anomaly indeed arises when we quantize QCD.

We define a scale transformation as a transformation of spacetime $x^\mu \longrightarrow \lambda^{-1}x^\mu$, $\lambda>0$. Given a field theory $\mathcal{L}(\phi_i,\partial^\mu\phi_i)$ in d dimensions, the fields ϕ_i are transformed as: $\phi_i(x) \longrightarrow \lambda^{D_i}\phi_i(\lambda x)$, where D_i is the mass dimension of the field ϕ_i. For an infinitesimal scaling $\lambda = 1+\epsilon$, the change in ϕ_i takes the form:

$$\delta\phi_i(x) = \epsilon[D_i + x^\mu\partial_\mu]\phi_i(x).$$

Here we have ignored the terms of order $\mathcal{O}(\epsilon^2)$. For a theory with no mass terms or dimensionful couplings, the Lagrangian transforms similar to a field:

$$\delta\mathcal{L} = \epsilon[d + x^\mu\partial_\mu]\mathcal{L} = \epsilon\partial_\mu(x^\mu\mathcal{L}). \tag{85}$$

As this is a 4-divergence, the action is invariant under this transformation and the theory possesses a conserved current, called the dilatation current:

$$D^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(D_i + x^\nu\partial_\nu)\phi_i - x^\mu\mathcal{L}. \tag{86}$$

satisfying $\partial_\mu D^\mu = 0$. If the Lagrangian does have a scale dependency (e.g. through quantization), then equation (85) changes to $\delta\mathcal{L} = \epsilon\big[\partial_\mu(x^\mu\mathcal{L}) + \Delta\big]$, where Δ is not a 4-divergence. This means the dilatation current is not conserved but instead satisfies: $\partial_\mu D^\mu = \Delta$. We interpret this as Δ being the measure of scale dependency of the theory.

The dilatation current is related to the energy-momentum tensor. We recognize the sum of the second and third term of equation (86) as $x_\nu T_C^{\mu\nu}$. It was shown by Callan, Coleman and Jackiw [11] that it is possible to define a new energy-momentum tensor $T^{\mu\nu}$ such that $D^\mu = x_\nu T^{\mu\nu}$. Therefore the divergence of the dilatation current is equal to the trace of the energy-momentum tensor. We won't repeat the proof here, but instead show that the QCD energy-momentum tensor defined in equation (30) satisfies: $\partial_\mu D^\mu = T^\mu_{\ \mu}$. In QCD, the sum in equation (86) reaches over the fields A^a_μ, ψ^i and $\bar\psi^i$. We have:

$$\sum_i D_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\phi_i = F^{\nu\mu}A_\nu + \frac{3}{2}i\bar\psi\gamma^\mu\psi. \tag{87}$$

The second term on the rhs is a conserved current of QCD and therefore its divergence equals zero. Inserting the rest into the divergence of equation (86), we find:

$$\partial_\mu D^\mu = \partial_\mu((F^a)^{\mu\nu}A^a_\nu) + (T_C)^\mu_{\ \mu} = T^\mu_{\ \mu}. \tag{88}$$

where $T^{\mu\nu}$ coincides with equation (30). Note that this implies that $T^\mu_{\ \mu} = \Delta$ and therefore the EMT trace is also a measure of scale dependency of the theory.

Next let us calculate the QCD dilatation current. To this end we consider the renormalized QCD Lagrangian. Apart from fields and derivatives, the renormalized QCD Lagrangian contains the dimensionful parameters g and m. They transform under a scale transformation as:

$$\delta g_R(\mu) = \epsilon\mu\frac{\partial g_R}{\partial\mu} = \epsilon\beta(g),$$
$$\delta m_R(\mu) = -\epsilon\mu\frac{\partial m_R}{\partial\mu} = -\epsilon m_R\gamma_m(g).$$

The calculation simplifies by rescaling the gauge fields $gA_\mu^a \longrightarrow A_\mu^a$ such that the parameter g is removed from the covariant derivative and only appears in front of the F^2 term. The total Lagrangian then transforms under a scale transformation as:

$$\begin{aligned}\delta\mathcal{L} = \epsilon(-\frac{2\beta}{g} + d + x^\mu\partial_\mu)\frac{-(F^2)_R}{4g^2} + \epsilon(d + x^\mu\partial_\mu)(\bar{\psi}\not{D}\psi)_R \\ - \epsilon(-\gamma_\mu + d - 1 + x^\mu\partial_\mu)(m\bar{\psi}\psi)_R, \\ = \epsilon(d + x^\mu\partial_\mu)\mathcal{L} + \frac{\epsilon\beta}{2g^3}(F^2)_R + \epsilon(\gamma_\mu + 1)(m\bar{\psi}\psi)_R.\end{aligned}$$

From here we extract Δ and conclude that:

$$(T_R)^\mu{}_\mu = \Delta = \frac{\beta}{2g}(F^2)_R + (1 + \gamma_\mu)(m\bar{\psi}\psi)_R, \tag{89}$$

where we have undone the rescaling of the gauge fields. The EMT trace anomaly is given by $\frac{\beta}{2g}(F^2)_R + \gamma_m(m\bar{\psi}\psi)_R$. If we take the classical limit, i.e. $\beta = \gamma_m = 0$, we indeed re-obtain the classical EMT trace.

4.4 Renormalization of the EMT

In this section we use the theory from the previous three sections to renormalize the energy-momentum tensor of QCD. We define the operators:

$$\begin{aligned}\mathcal{O}_1 &= -(F^a)^{\mu\rho}(F^a)^\nu{}_\rho, & \mathcal{O}_2 &= g^{\mu\nu}(F^a)^{\alpha\beta}(F^a)_{\alpha\beta}, \\ \mathcal{O}_3 &= i\bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi, & \mathcal{O}_4 &= g^{\mu\nu}m\bar{\psi}\psi,\end{aligned} \tag{90}$$

which form a basis for all operators with the same symmetries and quantum numbers as the EMT. We have omitted the Lorentz indices on $\mathcal{O}_i$ for ease of notation. Using the operators of equation (90), we write:

$$T^{\mu\nu} = \mathcal{O}_1 + \frac{\mathcal{O}_2}{4} + \mathcal{O}_3. \tag{91}$$

While the total EMT is independent under renormalization [2], the operators that make up $T^{\mu\nu}$ do need to be separately renormalized and mix under renormalization:

$$\begin{bmatrix}\mathcal{O}_{1,R} \\ \mathcal{O}_{2,R} \\ \mathcal{O}_{3,R} \\ \mathcal{O}_{4,R}\end{bmatrix} = \begin{bmatrix} Z_T & Z_M & Z_L & Z_S \\ 0 & Z_F & 0 & Z_C \\ Z_Q & Z_B & Z_\psi & Z_K \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathcal{O}_{1,B} \\ \mathcal{O}_{2,B} \\ \mathcal{O}_{3,B} \\ \mathcal{O}_{4,B}\end{bmatrix} \tag{92}$$

We give a couple remarks regarding the structure of the renormalization matrix. First of all, the linear space of EMT operators can be split into two subspaces: the Lorentz trace- and traceless operators. The trace operators, like $\mathcal{O}_2$ and $\mathcal{O}_4$, are essentially Lorentz scalars multiplied by the metric tensor. Therefore the Lorentz trace- and traceless operators transform under different representations of the Lorentz group [4]. This in turn means that they do not mix under renormalization. One can see this in the renormalization matrix, as the operators $\mathcal{O}_{1,3}$ do not appear in the mixing of $\mathcal{O}_{2,4}$. Secondly, the operator $\mathcal{O}_4$ vanishes in the limit $m \to 0$. Again, this is an extra characteristic which is why $\mathcal{O}_4$ only mixes with itself. In fact, $\mathcal{O}_4$ does not need renormalization at all. This was shown to one-loop order in section 4.1. To show this is true at all orders, note that $m_B\bar{\psi}_B\psi_B$ appears in the renormalized Lagrangian. Therefore:

$$
\begin{aligned}
m_B \frac{\partial}{\partial m_B} & \langle\Omega|\, \phi_1(x_1)\ldots\phi_n(x_n)\,|\Omega\rangle\,, \\
&= \langle\Omega| \int d^4x\; m_B\bar{\psi}_B(x)\psi_B(x)\phi_1(x_1)\ldots\phi_n(x_n)\,|\Omega\rangle \\
&\qquad\qquad - \langle\Omega| \int d^4x\; m_B\bar{\psi}_B(x)\psi_B(x)\,|\Omega\rangle\,\langle\Omega|\,\phi_1(x_1)\ldots\phi_n(x_n)\,|\Omega\rangle\,, \\
&= \langle\Omega| \int d^4x \Big(m_B\bar{\psi}_B(x)\psi_B(x) - \langle\Omega|\, m_B\bar{\psi}_B(x)\psi_B(x)\,|\Omega\rangle \Big)\phi_1(x_1)\ldots\phi_n(x_n)\,|\Omega\rangle\,.
\end{aligned}
$$

must be a finite quantity ($\langle\Omega|\,\phi_1(x_1)\ldots\phi_n(x_n)\,|\Omega\rangle$ is finite in the renormalized theory). We recognize the operator $m_B\bar{\psi}_B(x)\psi_B(x) - \langle\Omega|\, m_B\bar{\psi}_B(x)\psi_B(x)\,|\Omega\rangle$ as the operator $[m_B\bar{\psi}_B\psi_B](x)$ with the vacuum expectation value subtracted. So a zero momentum insertion of the operator $[m_B\bar{\psi}_B\psi_B](x)$ in any Green's function has to be finite. And since renormalization is independent of the momentum inserted, we conclude that $m\bar{\psi}\psi$ is a finite operator and needs no renormalization.

As stated before, the trace- and traceless parts of the linear space of EMT operators do not mix under renormalization. Therefore we can compute the renormalization of the trace- and traceless operators separately. Since any operator can be split into a trace part and a traceless part, we then can build up the EMT renormalization matrix from these results. For the renormalization of the trace operators, we look at the EMT trace. The bare trace in d dimensions is given by:

$$
(T_B)^{\mu}{}_{\mu} = \frac{d-4}{4}(F^{\alpha\beta}F_{\alpha\beta})_B + (m\bar{\psi}\psi)_B \tag{93}
$$

where we have used the equations of motion to obtain $m\bar{\psi}\psi$. Classically, the first term drops out in the limit $d \to 4$. In the renormalized theory however, the bare operators contain divergences that, multiplied by $\frac{4-d}{4}$, yield additional finite contributions to the trace of the EMT. The renormalized trace is given by equation (89). Expressing it in bare operators, we get:

$$
(\Theta_R)^{\mu}{}_{\mu} = \frac{\beta Z_F}{2g}(F^2)_B + (1 + \gamma_m + \frac{\beta Z_C}{2g})(m\bar{\psi}\psi)_B. \tag{94}
$$

We find Z_F and Z_C by comparing equations (93) and (94):

$$
Z_F = -\frac{g\epsilon}{2\beta} = 1 + \left(\frac{4}{3}n_f C(N) - \frac{11}{3}C_2(G)\right)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{95}
$$

$$
Z_C = -\frac{2g\gamma_m}{\beta} = 12 C_2(N)\frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{96}
$$

This completes the renormalization of the space of trace operators, which is fully spanned by $\mathcal{O}_2$ and $\mathcal{O}_4$. The space of traceless operators is spanned by the operators $\mathcal{O}_g$ and $\mathcal{O}_f$ from equation (75).

Now that we know the renormalization of the two subspaces, we split the renormalization equations of $\mathcal{O}_1$ and $\mathcal{O}_3$ into their trace and traceless parts. Defining $\hat{\mathcal{O}}_i^{\mu\nu} = \frac{1}{d} g^{\mu\nu}(g_{\alpha\beta}\mathcal{O}_i^{\alpha\beta})$ and $\tilde{\mathcal{O}}_i = \mathcal{O}_i - \hat{\mathcal{O}}_i$, we get:

$$\tilde{\mathcal{O}}_{1,R} = Z_T \tilde{\mathcal{O}}_{1,B} + Z_L \tilde{\mathcal{O}}_{3,B}, \tag{97}$$

$$\tilde{\mathcal{O}}_{3,R} = Z_Q \tilde{\mathcal{O}}_{1,B} + Z_\psi \tilde{\mathcal{O}}_{3,B}, \tag{98}$$

and:

$$\hat{\mathcal{O}}_{1,R} = Z_T \hat{\mathcal{O}}_{1,B} + Z_M \mathcal{O}_{2,B} + Z_L \hat{\mathcal{O}}_{3,B} + Z_S \mathcal{O}_{4,B}, \tag{99}$$

$$\hat{\mathcal{O}}_{3,R} = Z_Q \hat{\mathcal{O}}_{1,B} + Z_B \mathcal{O}_{2,B} + Z_\psi \hat{\mathcal{O}}_{3,B} + Z_K \mathcal{O}_{4,B}, \tag{100}$$

The renormalization constants $Z_{T,L,Q,\psi}$ are given by $Z_{gg,gf,fg,ff}$ respectively. To find the remaining renormalization constants, we work out the traces of $\mathcal{O}_i$. For the bare operators, we find:

$$\hat{\mathcal{O}}_{1,B} = \frac{1}{d} g^{\mu\nu}(-F^{\alpha\beta}F_{\alpha\beta})_B = -\frac{1}{d}\mathcal{O}_{2,B}, \tag{101}$$

$$\hat{\mathcal{O}}_{3,B} = \frac{1}{d} g^{\mu\nu}(\bar{\psi} i \not{D} \psi)_B = \frac{1}{d}\mathcal{O}_{4,B}. \tag{102}$$

The traces of the renormalized operators are not so simple, because renormalization and the trace operation do not commute [12]. We parameterize the possible anomalous terms by four unknown constants $x, y, x', y' = \mathcal{O}(\alpha_s)$:

$$\hat{\mathcal{O}}_{1,R} = -\frac{1+x'}{d}\mathcal{O}_{2,R} - \frac{y'}{d}\mathcal{O}_{4,R}, \tag{103}$$

$$\hat{\mathcal{O}}_{3,R} = \frac{x}{d}\mathcal{O}_{2,R} + \frac{1+y}{d}\mathcal{O}_{4,R}. \tag{104}$$

Actually, we can reduce the number of unknown constants by 2 by looking at the sum $\hat{\mathcal{O}}_{1,R} + \hat{\mathcal{O}}_{3,R}$. On the one hand, this simply equals:

$$\hat{\mathcal{O}}_{1,R} + \hat{\mathcal{O}}_{3,R} = g^{\mu\nu}\left[\frac{1+(y-y')}{d}(m\bar{\psi}\psi)_R + \frac{(x-x')-1}{d}(F^{\alpha\beta}F_{\alpha\beta})_R\right]. \tag{105}$$

On the other hand is the sum related to the full renormalized EMT by:

$$\hat{\mathcal{O}}_{1,R} + \hat{\mathcal{O}}_{3,R} = \frac{1}{d} g^{\mu\nu} g_{\alpha\beta}\left[T_R^{\alpha\beta} - \frac{1}{4}\mathcal{O}_{2,R}^{\alpha\beta}\right]. \tag{106}$$

The operator $\mathcal{O}_2$ is already a trace (multiplied by the metric tensor) and can therefore be contracted with the metric tensor to form the EMT trace. The full EMT is scale independent, so $(T_R)^\mu{}_\mu = (T^\mu{}_\mu)_R$. Using equation (89) to write the EMT trace in terms of the renormalized operators, we get:

$$\hat{\mathcal{O}}_{1,R} + \hat{\mathcal{O}}_{3,R} = g^{\mu\nu}\left[\frac{1+\gamma_m}{d}(m\bar{\psi}\psi)_R + \frac{\frac{\beta}{2g} - \frac{4-\epsilon}{4}}{d}(F^{\alpha\beta}F_{\alpha\beta})_R\right]. \tag{107}$$

Comparing equations (105) and (107), we find two relations between the four unknown constants x, y, x', y'. Therefore we can rewrite the traces of operators $\mathcal{O}_1$ and $\mathcal{O}_3$ as:

$$\hat{\mathcal{O}}_{1,R} = -\frac{1 - \frac{\beta}{2g} + x}{d}\mathcal{O}_{2,R} - \frac{y - \gamma_m}{d}\mathcal{O}_{4,R}, \tag{108}$$

$$\hat{\mathcal{O}}_{3,R} = \frac{x}{d}\mathcal{O}_{2,R} + \frac{1+y}{d}\mathcal{O}_{4,R}. \tag{109}$$

The exact values of the constants x, y depend on the chosen renormalization scheme. In [2], the x, y are computed for several different schemes. In this thesis we use the $\overline{\text{MS}}$ scheme.

With these expressions for the bare and renormalized trace, we are able to rewrite equations (99) and (100) in terms of the full operators. Next we use equation (92) to express everything in bare operators. We then collect terms containing the same operator. As both sides of the equation have to be equal for all values of $\mathcal{O}_{i,B}$, the following relations must be true:

$$Z_M = \frac{Z_T}{d} - \frac{Z_F}{d}\left(1 - \frac{\beta}{2g} + x\right), \tag{110}$$

$$Z_S = -\frac{Z_L}{d} - \frac{Z_C}{d}\left(1 - \frac{\beta}{2g} + x\right) - \frac{y - \gamma_m}{d}, \tag{111}$$

$$Z_B = \frac{Z_Q}{d} + \frac{x}{d}Z_F, \tag{112}$$

$$Z_K = -\frac{Z_\psi}{d} + \frac{x}{d}Z_C + \frac{1+y}{d}. \tag{113}$$

This solves all entries of the renormalization matrix in terms of the two constants x and y. Before computing x and y in the $\overline{\text{MS}}$ scheme, we first compute them in the easier MS scheme. In this scheme the renormalization constants only contain poles in ϵ. There, all renormalization constants Z_X, with $X = T, M, L, S, F, C, Q, B, \psi, K$, take the following form:

$$Z_X\Big|_{\text{MS}} = (0,1)_X + \alpha_s\frac{2a_{1,X}}{\epsilon} + \mathcal{O}(\alpha_s^2), \tag{114}$$

where $(0,1)_X = 1$ for $X = T, F, \psi$ and zero otherwise. We can find x and y by taking the Laurent series of equations (112) and (113) around $\epsilon = 0$ and collecting the ϵ^0 terms:

$$\frac{1}{8}\Big[(\alpha_s a_{1,Q} + \dots) + x(2 + \alpha_s a_{1,F} + \dots)\Big] = 0,$$
$$\frac{1}{8}\Big[-(2 + \alpha_s a_{1,\psi} + \dots) + x(\alpha_s a_{1,C} + \dots) + 2(1+y)\Big] = 0.$$

Solving these equations by order in α_s yields:

$$x = -\alpha_s\frac{a_{1,Q}}{2}, \tag{115}$$

$$y = \alpha_s\frac{a_{1,\psi}}{2}. \tag{116}$$

We next relate these equations to the $\overline{\text{MS}}$ scheme. There, the formula for the renormalization constants is:

$$Z_X\Big|_{\overline{\text{MS}}} = (0,1)_X + \alpha_s \frac{2\bar{a}_{1,X}}{\epsilon} S_\epsilon + \mathcal{O}(\alpha_s^2), \tag{117}$$

where $S_\epsilon = (4\pi e^{-\gamma_E})^{\frac{\epsilon}{2}} = 1 + \frac{\epsilon}{2}\ln 4\pi e^{-\gamma_E} + \ldots$. Because of this factor S_ϵ, using the same trick as for the MS scheme does not work: the resulting equations will still be dependent on coefficients of Z_B and Z_K. However, as the difference between the MS and $\overline{\text{MS}}$ schemes is finite, the divergent parts of equations (114) and (117) should agree with each other. This implies that $a_{1,X} = \bar{a}_{1,X}$ (this relation is more complicated for higher order coefficients, but this order is sufficient for our purposes). So we conclude that:

$$x = \frac{2}{3} n_f C(N) \frac{\alpha_s}{4\pi}, \tag{118}$$

$$y = \frac{4}{3} C_2(N) \frac{\alpha_s}{4\pi}. \tag{119}$$

and finally:

$$Z_M = \frac{11}{12} C_2(G) \frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{120}$$

$$Z_S = -\frac{7}{3} C_2(N) \frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{121}$$

$$Z_B = -\frac{1}{3} n_f C(N) \frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right], \tag{122}$$

$$Z_K = -\frac{2}{3} C_2(N) \frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{123}$$

This completes the renormalization of the EMT. We can summarize the results as follows:

$$Z = \mathbf{1} + \frac{\alpha_s}{4\pi}\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right] Z_2, \tag{124}$$

where:

$$Z_2 = \begin{bmatrix} \frac{4}{3} n_f C(N) & \frac{11}{12} C_2(G) & -\frac{8}{3} C_2(N) & -\frac{7}{3} C_2(N) \\ 0 & \frac{4}{3} n_f C(N) - \frac{11}{3} C_2(G) & 0 & 12 C_2(N) \\ -\frac{4}{3} n_f C(N) & -\frac{1}{3} n_f C(N) & \frac{8}{3} C_2(N) & -\frac{2}{3} C_2(N) \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{125}$$

As always, the anomalous dimension matrix is key to calculating the scale behaviour of the EMT components. The anomalous dimension matrix is defined by:

$$\gamma^{ij} = \frac{\partial Z^{ik}}{\partial \ln \mu} [Z^{-1}]^{kj}. \tag{126}$$

We have:

$$\frac{\partial Z}{\partial \ln \mu} = \frac{2g}{(4\pi)^2}\beta(g)\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right] Z_2 + \mathcal{O}(\alpha_s^2),$$
$$= -\frac{2\alpha_s}{4\pi} Z_2 + \mathcal{O}(\alpha_s^2).$$

As this part is already of next-to-leading order, we only need to compute the leading order of Z^{-1}, which is equal to unity. So we conclude:

$$\gamma_Z = -\frac{\alpha_s}{2\pi}\begin{bmatrix} \frac{4}{3}n_f C(N) & \frac{11}{12}C_2(G) & -\frac{8}{3}C_2(N) & -\frac{7}{3}C_2(N) \\ 0 & \frac{4}{3}n_f C(N) - \frac{11}{3}C_2(G) & 0 & 12C_2(N) \\ -\frac{4}{3}n_f C(N) & -\frac{1}{3}n_f C(N) & \frac{8}{3}C_2(N) & -\frac{2}{3}C_2(N) \\ 0 & 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\alpha_s^2). \tag{127}$$

Here we again see that $(m\bar{\psi}\psi)_R$ is scale-independent and that $(F^{\alpha\beta}F_{\alpha\beta})_R$ only mixes with $(m\bar{\psi}\psi)_R$.

Now that we have found the anomalous dimension matrix, we compute the values of $\mathcal{O}_{i,R}$ up to 1-loop order as a function of energy scale. We start from the differential equation:

$$\frac{\partial \mathcal{O}_{i,R}}{\partial \ln \mu} = \gamma_Z^{ij}(\mu)\mathcal{O}_{j,R},$$
$$= -\frac{\alpha_s}{2\pi} Z_2^{ij}\mathcal{O}_{j,R}.$$

with the initial value $\mathcal{O}_{i,R}(\mu_0)$ at some energy μ_0. We insert the solution of $\alpha_s(\mu)$ to obtain:

$$\frac{\partial}{\partial \ln \mu}\mathcal{O}_{i,R} = \frac{Z_2^{ij}}{\beta_0 \ln \frac{\mu}{\Lambda}}\mathcal{O}_{j,R}. \tag{128}$$

As the matrix part of this differential equation is not dependent on μ, the solution is simply a matrix exponential:

$$\mathcal{O}_{i,R}(\mu) = e^{\int_{\ln \mu_0}^{\ln \mu} \frac{Z_2^{ij}}{\beta_0 \ln \frac{\mu}{\Lambda}} d\ln \mu}\mathcal{O}_{j,R}(\mu_0). \tag{129}$$

The integral in the exponent is of the form $\int_a^b \frac{1}{x}dx$ and therefore has a solution of the form $\ln(b/a)$:

$$\int_{\ln \mu_0}^{\ln \mu} \frac{Z_2^{ij}}{\beta_0 \ln \frac{\mu}{\Lambda}} d\ln \mu = \frac{Z_2^{ij}}{\beta_0}\ln\left|\frac{\ln \frac{\mu}{\Lambda}}{\ln \frac{\mu_0}{\Lambda}}\right|,$$

We use equation (62) to dispose of Λ in favour of α_s and arrive at:

$$\mathcal{O}_{i,R}(\mu) = e^{-\frac{Z_2^{ij}}{\beta_0}\ln\left|\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right|}\mathcal{O}_{j,R}(\mu_0). \tag{130}$$

This result is valid up to next-to-leading order. Due to the mixing very little can be said about the running behaviour purely based on this expression. Therefore we do not discuss the scaling of the operators, but use the same technique for solving the matrix differential equation to solve for the running of the mass decompositions in the next chapter.

5 Investigating the proton mass

Now that we have treated the energy-momentum tensor of QCD, we focus our attention on the main goal of this thesis: investigating the proton mass. For most purposes the proton is said to consist of three quarks, two up and one down. However, the sum of the masses of these quarks ($\sim$ 9 MeV) is nowhere near the mass of the proton ($\sim$ 938 MeV). Therefore the 3-quark picture is replaced by a sea of quarks and gluons that contains three more quarks then anti-quarks. Pictorially, one can say that the three quarks exchange gluons, which in turn briefly split into a quark-antiquark pair, which exchange gluons again, etc. This picture raises the question of how much of the proton mass can be attributed to the quarks and how much to the gluons? Can we make a further decomposition, splitting the quark contribution into for instance a mass and a kinetic term? In this chapter we try to give an answer to these questions by examining the relation between the energy-momentum tensor and the proton mass. We give two formula's for obtaining the mass from the EMT and use these to discuss several decompositions of the proton mass. We also investigate the scale dependence of these mass sum rules, which originates in the just examined scale dependence of the EMT operators.

5.1 Deriving the proton mass

There are many ways one can calculate the mass of a proton, but we focus on deriving the proton mass from the energy-momentum tensor. Consider a proton state, $|P\rangle$, with four-momentum $P^\mu = (E, \mathbf{P})$ and mass $M = E^2 - \mathbf{P}^2$. The state is normalized according to $\langle P'|P\rangle = \frac{E}{M}(2\pi)^3\delta^{(3)}(\mathbf{P'} - \mathbf{P})$. Due to Lorentz symmetry, the forward matrix element of the EMT in $|P\rangle$ must have the following general shape:

$$\langle P| T^{\mu\nu}(x) |P\rangle = A(0)P^\mu P^\nu + \bar{C}(0)g^{\mu\nu}M^2, \tag{131}$$

where $A(\Delta)$ and $\bar{C}(\Delta)$ are scalars depending on the transferred momentum $\Delta = (P' - P)^2$. The Hamiltonian of the system, satisfying $H|P\rangle = E|P\rangle$, is related to the EMT by $H = \int d^3x\, T^{00}$. Therefore we have the relation:

$$\langle P| H |P\rangle = \frac{E^2}{M}(2\pi)^3\delta^{(3)}(\mathbf{0}). \tag{132}$$

Combining equations (131) and (132), we find that $\bar{C}(0)$ must be zero and we get the formula:

$$\langle P| T^{\mu\nu}(0) |P\rangle = \frac{P^\mu P^\nu}{M}. \tag{133}$$

Even though an x-dependence was written inside the forward matrix elements above, the forward matrix elements themselves are independent of x. To show this, we define the translation operator $\hat{T}(x) = e^{i\hat{P}^\mu x_\mu}$, where $\hat{P}^\mu$ is the momentum operator satisfying $\hat{P}^\mu |P\rangle = P^\mu |P\rangle$. Starting from a generic forward matrix element $\langle P| O(x) |P\rangle$, we use $\hat{T}(x)$ to translate $O(x)$ to the point

$x = 0$:

$$\begin{aligned} \langle P| O(x) |P\rangle &= \langle P| e^{i\hat{P}^\mu x_\mu} O(0) e^{-i\hat{P}^\mu x_\mu} |P\rangle , \\ &= \langle P| e^{iP^\mu x_\mu} O(0) e^{-iP^\mu x_\mu} |P\rangle , \\ &= \langle P| O(0) |P\rangle . \end{aligned}$$

Indeed the final result is independent of x (this does not hold for a general matrix element, only forward matrix elements). To avoid confusion about the x-dependence, we shall use $\langle P| O(0) |P\rangle$ to indicate forward matrix elements of x-dependent operators.

A formula for the mass of the proton is then obtained by taking the trace of equation (133) [4]:

$$M = \langle P| T^\mu_{\ \mu}(0) |P\rangle \tag{134}$$

An advantage of this expression is that it is Poincaré invariant. Indeed it relies on the identity $P^2 = M^2$, which defines mass as the square root of one of the invariant Casimir operators of the Poincaré group [3]. However, the formula only relies on the trace of the mass at a single point in spacetime. This is problematic for a proton, which is not a point particle. Also, the precise formula depends on the normalization of the proton, a feature that is not desirable for a physical interpretation.

To avoid these issues, Lorcé [3] suggests using a different formula for the proton mass. This formula uses the normalized expectation value of some spatially extended operator:

$$\langle O\rangle \equiv \frac{\langle P| \int d^3x \ O(x) |P\rangle}{\langle P|P\rangle}. \tag{135}$$

The normalized expectation value of $T^{00}(x)$ is then equal to the normalized forward matrix element of the Hamiltonian. From equation (132) we see that if we equate this in the rest frame of the proton, we get:

$$\langle T^{00}\rangle\big|_{\mathbf{P}=0} = M, \tag{136}$$

Although this formula solves the issues surrounding formula (134), we have lost frame independence: the formula only holds in the rest frame of the proton. Because the forward matrix element is independent of x, the integral over x simply evaluates to the space-time volume $V = (2\pi)^3\delta^{(3)}(0)$. This factor cancels the delta function divergence of the norm $\langle P|P\rangle$. We then find:

$$M = \langle P| T^{00}(0) |P\rangle \big|_{\mathbf{P}=0} \tag{137}$$

Both equations (134) and (136) (or equivalently equation (137)) have been used in the literature and to calculate the proton mass. In the rest of chis chapter we review several mass decompositions from the literature and will therefore also use both equations.

5.2 The mass decompositions

Both formula's for the proton mass are linearly related to (parts of) the EMT. This allows us to examine proton mass decompositions through decompositions of the EMT. The most common decomposition of the classical (symmetric) EMT of equation (30) into a quark and gluon part is:

$$T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}, \tag{138}$$

with:

$$T_q^{\mu\nu} = i\bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi, \tag{139}$$

$$T_g^{\mu\nu} = -(F^a)^{\mu\rho}(F^a)^{\nu}{}_{\rho} + \frac{g^{\mu\nu}}{4}(F^a)^{\alpha\beta}(F_a)_{\alpha\beta}. \tag{140}$$

where in equation (139) a sum over quark flavors and colors and in equation (140) a sum over gluon colors is understood. The quark part also contains a gluonic component, due to the covariant derivative. The gluonic part of the EMT contains both the kinetic energy of the gluons and the gluon-gluon interactions. While the matrix element of the full EMT was constricted by the Hamiltonian of the system, the matrix elements of $T_q^{\mu\nu}$ and $T_g^{\mu\nu}$ suffer no such constriction. Therefore we introduce 4 form factors $A_i(0)$ and $\bar{C}_i(0)$ $(i = q, g)$ such that, similar to equation (131):

$$\langle P| T_i^{\mu\nu}(0) |P\rangle = A_i(0)\frac{P^\mu P^\nu}{M} + \bar{C}_i(0)Mg^{\mu\nu}. \tag{141}$$

The conservation of the total EMT imposes two constraints on these form factors:

$$A_q(0) + A_g(0) = 1, \qquad \bar{C}_q(0) + \bar{C}_g(0) = 0. \tag{142}$$

Therefore any mass rule based on this decomposition (without further decomposition), has at most two independent terms.

After renormalization equation (138) holds for both the bare and renormalized operators. The bare operators however are not necessarily finite: they can contain divergences that cancel when we sum them to form the total EMT. This translates to the proton mass: if we base our mass decompositions on the bare operators, we might end up with divergences in the mass terms, even though their sum is finite. Therefore all proton mass decompositions described below are based on (and expressed in) renormalized operators.

Tanaka's mass sum rule

The first mass rule we treat is a two-term decomposition by Tanaka et al. [5]. It is based on the trace of the decomposition depicted in equation (138). The traces of the operators $i\bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi$ and $-(F^a)^{\mu\rho}(F^a)^{\nu}{}_{\rho}$ have been treated in section 4.4. Using the calculated values of x and y, we find:

$$(T_{q,R})^{\mu}{}_{\mu} = \left(1 + \frac{4}{3}C_2(N)\frac{\alpha_s}{4\pi}\right)(m\bar{\psi}\psi)_R + \frac{2}{3}n_f C(N)\frac{\alpha_s}{4\pi}(F^{\alpha\beta}F_{\alpha\beta})_R, \tag{143}$$

$$(T_{g,R})^{\mu}{}_{\mu} = \frac{14}{3}C_2(N)\frac{\alpha_s}{4\pi}(m\bar{\psi}\psi)_R - \frac{11}{6}C_2(G)\frac{\alpha_s}{4\pi}(F^{\alpha\beta}F_{\alpha\beta})_R. \tag{144}$$

Their sum indeed reproduces the total QCD trace. In the background field method the 1-loop corrections to the gluon kinetic term in the effective Lagrangian can be separated into a quark-loop ($\propto \frac{4}{3}n_f C(N)$) and a gluon-loop ($\propto -\frac{11}{3}C_2(G)$) contribution [6]. This separation coincides with the decomposition of the $\langle (F^{\alpha\beta}F_{\alpha\beta})_R\rangle$ term in equations (143) and (144). A similar correspondence could be invoked on the quark mass term. Because of this simple correspondence, [5] states that this decomposition is well-defined and of physical origin. The mass sum rule is given by:

$$M = \bar{M}_q + \bar{M}_g, \tag{145}$$

with:

$$\bar{M}_q = \langle P|\,(T_{q,R})^{\mu}{}_{\mu}(0)\,|P\rangle\,, \qquad \bar{M}_g = \langle P|\,(T_{g,R})^{\mu}{}_{\mu}(0)\,|P\rangle\,. \tag{146}$$

The quark and gluon contributions in this decomposition are related to the form factors via:

$$\bar{M}_i = M(A_i + 4\bar{C}_i) \tag{147}$$

Lorcé's 2-term mass sum rule

A second two-term sum rule, proposed by Lorcé [3], also uses the decomposition of equation (138), but focuses on the T_i^{00} to find the mass (equation (137)). To give more insight in this sum rule, we rewrite the T_i^{00} as follows:

$$T_{q,R}^{00} = (m\bar{\psi}\psi)_R + (i\psi^{\dagger}\mathbf{D}\cdot\alpha\psi)_R, \tag{148}$$

$$T_{g,R}^{00} = \frac{1}{2}(E^2 + B^2)_R. \tag{149}$$

where $\alpha^i = \gamma^0\gamma^i$. In the first term on the r.h.s. of equation (148) we recognize the quark mass term. The second term in that equation is usually referred to as the kinetic plus potential energy of the quarks. The operator in equation (149) represents the total energy stored in the gluon fields. The sum rule is then given by:

$$M = \tilde{M}_q + \tilde{M}_g, \tag{150}$$

with:

$$\tilde{M}_q = \langle P|\,T_{q,R}^{00}(0)\,|P\rangle\,\big|_{\mathbf{P}=0}, \qquad \tilde{M}_g = \langle P|\,T_{g,R}^{00}(0)\,|P\rangle\,\big|_{\mathbf{P}=0}. \tag{151}$$

In terms of the EMT form factors we have:

$$\tilde{M}_i = M(A_i + \bar{C}_i), \tag{152}$$

proving that $\bar{M}_i \neq \tilde{M}_i$ (unless $\bar{C}_i = 0$).

Lorcé also shows an argument why his sum rule is in his eyes the best sum rule. He poses that a proton can be seen as made up from two effective coupled fluids, one consisting of quarks and one consisting of gluons. In this picture, the EMT form factors can be related to the partial internal energy U_i and the partial pressure-volume work W_i:

$$U_i = (A_i + \bar{C}_i)M, \qquad W_i = -\bar{C}_i M. \tag{153}$$

The internal energies represent the sum of all the potential and kinetic energies. Therefore the total mass of a system is given by the sum of all internal energies in the rest frame:

$$M = \sum_i U_i. \tag{154}$$

The pressure-volume work represents the pressures and stresses of the fluids on each other and the outside world. In a stable system, there is no interaction with the outside world and the total pressure-volume work should vanish:

$$\sum_i W_i = 0. \tag{155}$$

These statements are in agreement with the conditions posed on the EMT form factors. Of course any mass rule should only be concerned with the internal energies; the pressure-volume work that one fluid exerts on another should be irrelevant for the mass. His own sum rule does obey this principle: $\tilde{M}_i = U_i$. Tanaka's mass components however include a pressure part: $\bar{M}_i = U_i - 3W_i$. Therefore (in the effective coupled two-fluid picture) Tanaka's sum rule is not physical and the fact that the $\sum_i \bar{M}_i = M$ is the result of the total pressure-volume work vanishing.

In the effective coupled two-fluid picture it is also non-sensible to compose a mass sum rule consisting of more than two components, as this would inadvertently lead to a pressure contribution. A sum rule based on more than two fluids is however also impossible, as quarks (or gluons) would then be part of two fluids simultaneously. Lorcé does offer a 4-part decomposition, but he refrains from giving a physical interpretation to this 4-term decomposition and says it can at best be seen as some sort of virial decomposition. Of course, all of this is dependent on whether the effective coupled multi-fluid picture can be used to describe a proton.

Ji's mass sum rule

We now turn to the four-term sum rule proposed by Ji [4], focused on the Hamiltonian density $T^{00}(0)$. In this sum rule the EMT is decomposed into a traceless part and trace part according to:

$$T^{\mu\nu} = \bar{T}^{\mu\nu} + \hat{T}^{\mu\nu}, \tag{156}$$

where $\hat{T}^{\mu\nu} = \frac{1}{d} g^{\mu\nu} T^{\alpha}_{\ \alpha}$. As the traceless and trace parts do not mix under renormalization, both are finite and scale-independent composite operators and the separation is completely physical. This implies:

$$M = \langle P| \bar{T}^{00}(0) |P\rangle \big|_{\mathbf{P}=0} + \langle P| \hat{T}^{00}(0) |P\rangle \big|_{\mathbf{P}=0}. \tag{157}$$

From equation (134) one can see that the second term on the right hand side, i.e. the trace part, contributes exactly $\frac{1}{4}M$. It is tempting therefore to conclude that the trace of the EMT contributes 25% of the proton mass. This would however be wrong. The second term on the right hand side is actually only a quarter of the trace, as can be seen in the definition of $\hat{T}^{\mu\nu}$. The traceless part of equation (157) can however be expressed in terms of the trace part [2]. Recall that, due to equation (131), in the rest frame the forward matrix element of the EMT is zero except for T^{00}. In particular, in the rest frame the EMT has no off-diagonal elements and we have the relation:

$$\langle P| \bar{T}^{00}(0) |P\rangle \big|_{\mathbf{P}=0} = \frac{3}{4} \langle P| T^{\mu}_{\ \mu}(0) |P\rangle \big|_{\mathbf{P}=0} \tag{158}$$

So again we find that the proton mass is fully determined by the forward matrix elements of the EMT trace.

Ji then further decomposes both the trace and traceless parts. The traceless part is decomposed into a quark and gluon part:

$$T^{\mu\nu}_{q,R} = \big(\bar{\psi} i D^{\{\mu} \gamma^{\nu\}} \psi - (\text{trace})\big)_R, \qquad T^{\mu\nu}_{g,R} = -\big((F^a)^{\mu\alpha} (F^a)^{\nu}_{\ \alpha} - (\text{trace})\big)_R. \tag{159}$$

The trace in $T^{\mu\nu}_{q,R}$ is given by $\big(\frac{g^{\mu\nu}}{d} m\bar{\psi}\psi\big)_R$. As $m\bar{\psi}\psi$ is finite, we can ignore the ϵ terms arising from the factor $\frac{1}{d}$ and write:

$$T^{\mu\nu}_{q,R} = (\bar{\psi} i D^{\{\mu} \gamma^{\nu\}} \psi)_R - \frac{g^{\mu\nu}}{4} (m\bar{\psi}\psi)_R. \tag{160}$$

For $T^{\mu\nu}_{g,R}$, we write the trace as follows:

$$\big(\frac{g^{\mu\nu}}{d} (F^a)^{\alpha\beta} (F^a)_{\alpha\beta}\big)_R = \big(\big[\frac{1}{4} - \frac{1}{4}\frac{4-d}{d}\big] g^{\mu\nu} (F^a)^{\alpha\beta} (F^a)_{\alpha\beta}\big)_R. \tag{161}$$

The former term then combines with the $F^{\mu\alpha} F^{\nu}_{\ \alpha}$ to form the total gluon contribution to the EMT, while in the latter term we recognize the trace anomaly. Therefore the total gluon part equals:

$$\begin{aligned} T^{\mu\nu}_{q,R} = -\big((F^a)^{\mu\alpha} (F^a)^{\nu}_{\ \alpha} - \frac{g^{\mu\nu}}{4} (F^a)^{\alpha\beta} (F^a)_{\alpha\beta}\big)_R \\ - \frac{\beta}{8g} \big((F^a)^{\alpha\beta} (F^a)_{\alpha\beta}\big)_R - \frac{\gamma_m}{4} (m\bar{\psi}\psi)_R. \end{aligned} \tag{162}$$

The trace of the EMT, as given in equation (89), is decomposed into a quark mass contribution and a gluon field tensor contribution:

$$T^{\mu\nu}_{m,R} = \frac{g^{\mu\nu}}{4} (1 + \gamma_m) (m\bar{\psi}\psi)_R, \qquad T^{\mu\nu}_{a,R} = \frac{g^{\mu\nu}}{4} \frac{\beta}{2g} (F^{\alpha\beta} F_{\alpha\beta})_R \tag{163}$$

Finally, Ji's mass sum rule is based upon T^{00}, equation (137). Using a similar rewriting as for equations (148) and (149), we find:

$$M'_q = \langle P| (\psi^\dagger i\mathbf{D}\cdot\alpha\psi)_R |P\rangle \big|_{\mathbf{P}=0} + \langle P| \frac{3}{4}(m\bar{\psi}\psi)_R |P\rangle \big|_{\mathbf{P}=0}, \tag{164}$$

$$M'_m = \langle P| \frac{1+\gamma_m}{4}(m\bar{\psi}\psi)_R |P\rangle \big|_{\mathbf{P}=0}, \tag{165}$$

$$M_g = \langle P| \frac{1}{2}(\mathbf{B}^2+\mathbf{E}^2)_R |P\rangle \big|_{\mathbf{P}=0} - \langle P| \frac{\beta}{4g}(\mathbf{B}^2-\mathbf{E}^2) |P\rangle \big|_{\mathbf{P}=0} - \frac{\gamma_m}{4}(m\bar{\psi}\psi)_R |P\rangle \big|_{\mathbf{P}=0}, \tag{166}$$

$$M_a = \langle P| \frac{\beta}{4g}(\mathbf{B}^2-\mathbf{E}^2)_R |P\rangle \big|_{\mathbf{P}=0}. \tag{167}$$

Ji then assigns a physical meaning to the gluon pieces: M_g represents the gluon kinetic and potential energy as in the classical case, i.e. without the trace anomaly; M_a represents the gluon part of the trace anomaly. Originally, Ji defined $\frac{1}{2}(\mathbf{B}^2+\mathbf{E}^2)_R$ as the traceless gluon kinetic and potential energy [13], opposed to our full gluon EMT definition which also contains the trace anomaly. Therefore we have to subtract the trace anomaly separately in equation (166) to agree with Ji's mass rule. To obtain suitable linear combinations of renormalized operators for the quark part of the decomposition, Ji then reorganizes M_q and M_m:

$$M_q = \langle P| (\psi^\dagger i\mathbf{D}\cdot\alpha\psi)_R |P\rangle \big|_{\mathbf{P}=0}, \tag{168}$$

$$M_m = \langle P| (1+\frac{\gamma_m}{4})(m\bar{\psi}\psi)_R |P\rangle \big|_{\mathbf{P}=0}. \tag{169}$$

We can now easily assign a physical meaning to the quark pieces: M_q represents the quark kinetic plus potential energy and M_m represents the quark mass term. They uphold the relations $M'_m + M_a = \frac{1}{4}M$ and $M'_q + M_g = \frac{3}{4}M$, as these combinations yield the trace- and traceless parts of the EMT. Therefore even though Ji's mass rule has four components, only two of these are independent. This is in agreement with the text below equation (142).

Rodini's mass sum rule

Rodini et al. [2] proposes a three-term mass rule using a similar method as Ji. Their decomposition differs from Ji's method in the order in which they renormalize the EMT and take out the trace. The non-commutation of the renormalization and the trace operation results in a consequent additional scheme dependence. They do not apply the separation (157) to the full EMT, but to the individual quark and gluon parts:

$$T^{\mu\nu}_{i,R} = \bar{T}^{\mu\nu}_{i,R} + \hat{T}^{\mu\nu}_{i,R}. \tag{170}$$

Using the relations for the traces of the renormalized quark and gluon parts (equations (143) and (144)), we find:

$$\bar{T}^{00}_{q,R} = (m\bar{\psi}\psi)_R + (i\psi^\dagger \mathbf{D}\cdot\alpha\psi)_R - \frac{1+y}{4}(m\bar{\psi}\psi)_R - \frac{x}{4}(F^{\alpha\beta}F_{\alpha\beta})_R, \tag{171}$$

$$\hat{T}^{00}_{q,R} = \frac{1+y}{4}(m\bar{\psi}\psi)_R + \frac{x}{4}(F^{\alpha\beta}F_{\alpha\beta})_R, \tag{172}$$

$$\bar{T}^{00}_{g,R} = \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R - \frac{\gamma_m - y}{4}(m\bar{\psi}\psi)_R - \frac{1}{4}\big(\frac{\beta}{2g} - x\big)(F^{\alpha\beta}F_{\alpha\beta})_R, \tag{173}$$

$$\hat{T}^{00}_{g,R} = \frac{\gamma_m - y}{4}(m\bar{\psi}\psi)_R - \frac{1}{4}\big(\frac{\beta}{2g} - x\big)(F^{\alpha\beta}F_{\alpha\beta})_R. \tag{174}$$

We see here that Ji's method of decomposing the EMT is equal to setting $x = y = 0$, also called the D2 scheme. In this scheme the entire trace anomaly originates from the gluon sector. Rodini's mass rule is then given by:

$$M = \hat{M}'_q + \hat{M}'_m + \hat{M}'_g + \hat{M}'_a, \tag{175}$$

with:

$$\hat{M}'_q = \langle P|\, \bar{T}^{00}_{q,R}(0)\, |P\rangle\, \big|_{\mathbf{P}=0}, \qquad \hat{M}'_m = \langle P|\, \hat{T}^{00}_{q,R}(0)\, |P\rangle\, \big|_{\mathbf{P}=0}, \tag{176}$$

$$\hat{M}'_g = \langle P|\, \bar{T}^{00}_{g,R}(0)\, |P\rangle\, \big|_{\mathbf{P}=0}, \qquad \hat{M}'_a = \langle P|\, \hat{T}^{00}_{g,R}(0)\, |P\rangle\, \big|_{\mathbf{P}=0}. \tag{177}$$

Just like Ji, Rodini et al. want to obtain suitable linear combinations of renormalized operators. To this end, they consider the following linear combination:

$$\begin{aligned}
\hat{M}_q &= \hat{M}'_q + c_{qm}\hat{M}'_m + c_{qa}\hat{M}'_q,\\
\hat{M}_m &= (1 - c_{qm})\hat{M}'_m + c_{ma}\hat{M}'_a,\\
\hat{M}_g &= \hat{M}'_g + c_{ga}\hat{M}'_a,\\
\hat{M}_a &= (1 - c_{qa} - c_{ma} - c_{ga})\hat{M}'_a,
\end{aligned}$$

with:

$$\begin{aligned}
c_{qm} &= \frac{(-3+y)\frac{\beta}{2g} + x(3 - \gamma_m)}{(1+y)\frac{\beta}{2g} - x(1+\gamma_m)},\\
c_{qa} &= -c_{ma} = \frac{4x}{(1+y)\frac{\beta}{2g} - x(1+\gamma_m)},\\
c_{ga} &= 1.
\end{aligned}$$

This leads to the very simple result:

$$\hat{M}_q = \langle P|\, (\psi^\dagger i\mathbf{D}\cdot\alpha\psi)_R\, |P\rangle\, \big|_{\mathbf{P}=0}, \tag{178}$$

$$\hat{M}_m = \langle P|\, (m\bar{\psi}\psi)_R\, |P\rangle\, \big|_{\mathbf{P}=0}, \tag{179}$$

$$\hat{M}_g = \langle P|\, \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R\, |P\rangle\, \big|_{\mathbf{P}=0}, \tag{180}$$

$$\hat{M}_a = 0. \tag{181}$$

The connection with the original quark- and gluon- decomposition of equation (138) is now lost, as the quark part ($\hat{M}_q$ and $\hat{M}_m$) has absorbed part of the gluon part of the decomposition. This sum rule can however be seen as a finer version of Lorcé's decomposition, splitting the quark part into a quark mass term and a quark kinetic and potential energy term.

The mass rules of both Ji and Rodini et al. are based on a decomposition of the full 00-component of the EMT. In dimensional regularization, the entire trace anomaly is derived from the bare gluon EMT (as is shown by equation (93)). As dimensional regularization only affects the spatial part of spacetime, T^{00} is somewhat special as it is left untouched by dimensional regularization. The full trace anomaly is then contained in the spatial part of the trace and should not appear in any mass rule based on T^{00}. Rodini et al. base their mass rule on this premise, setting $\hat{M}_a = 0$. This line of reasoning is however not true in general. Other regularization schemes, like the momentum cutoff, do affect the time-dimension and therefore do induce (part of) the trace anomaly in T^{00} [13]. Therefore the question whether the trace anomaly yields truly a separate contribution to the proton mass remains.

5.3 Numerical results

Now that we have examined the mass decompositions from a operator point of view, let us compute their numerical values. In the last section it was stated that mass decompositions based on the EMT decomposition (138) have at most two independent terms. This implies that only two independent numerical inputs are needed to fix all the terms of the different mass decompositions. It is very hard to extract information of the matrix elements of the EMT directly from experiment, due to the weakness of the gravitational force [3]. All that we need though are the form factors $A_{q,g}(0)$ and $\bar{C}_{q,g}(0)$, or equivalently the parameters a (the average fraction of hadron momentum carried by the quarks) and b (related to the proton scalar charge $\langle P|\,\bar{\psi}\psi\,|P\rangle$) [4]. We follow [2] and define a parameter b as:

$$Mb = (1+\gamma_m)\,\langle P|\,(m\bar{\psi}\psi)_R\,|P\rangle\,\big|_{\mathbf{P}=0}. \tag{182}$$

Using the trace of the EMT, we then find for the scalar gluon operator:

$$M(1-b) = \frac{\beta}{2g}\,\langle P|\,(F^{\alpha\beta}F_{\alpha\beta})_R\,|P\rangle\,\big|_{\mathbf{P}=0}. \tag{183}$$

There are multiple methods for fixing the parameter b, two of which are explained in [2]. We take a rough weighted average of the results of these two methods at the scale $\mu_0 = 2$ GeV, yielding $b = 0.2$.

For the second numerical input we define the parton momentum fractions a_i in the proton as:

$$\frac{3}{4}Ma_q = \langle P|\,\bar{T}^{00}_{q,R}\,|P\rangle\,\big|_{\mathbf{P}=0}, \qquad \frac{3}{4}Ma_g = \langle P|\,\bar{T}^{00}_{g,R}\,|P\rangle\,\big|_{\mathbf{P}=0}, \tag{184}$$

where a_q is a shorthand notation for the sum of the momentum fractions of all active quark flavours. The a_i satisfy the sum rule $a_q + a_g = 1$, similar to the

i	$\bar{M}_i$	$\tilde{M}_i$	M_i	$\hat{M}_i$
q	-0.074	0.394	0.234	0.234
m	-	-	0.167	0.160
g	1.002	0.544	0.350	0.544
a	-	-	0.188	-

Table 1: The numerical results of the different mass decompositions at $\mu = 2$ GeV. All results are in units of GeV and for $\mathcal{O}(\alpha_s)$ accuracy.

form factors $A_i(0)$. Therefore they yield one independent numerical input. We take a_i from [2] at the scale $\mu_0 = 2$ GeV and four active quark flavours, yielding $a_q = 0.586$ (and $a_g = 0.414$). Of course the calculations of this thesis are only of one-loop order and not of two-loop order, but this will make little numerical difference. We write the values of the forward matrix elements used in the mass decompositions as functions of the numerical inputs:

$$\langle P| \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R |P\rangle \big|_{\mathbf{P}=0} = \frac{3}{4}Ma_g + \frac{\gamma_m - y}{4(1+\gamma_m)}Mb + \Big(\frac{1}{4} - \frac{gx}{2\beta}\Big)M(1-b), \tag{185}$$

$$\langle P| (F^{\alpha\beta}F_{\alpha\beta})_R |P\rangle \big|_{\mathbf{P}=0} = 2\langle P| (\mathbf{B}^2 - \mathbf{E}^2)_R |P\rangle \big|_{\mathbf{P}=0} = \frac{2gM(1-b)}{\beta}, \tag{186}$$

$$\langle P| (\psi^\dagger i\mathbf{D}\cdot\alpha\psi)_R |P\rangle \big|_{\mathbf{P}=0} = \frac{3}{4}Ma_q + \frac{y-3}{4(1+\gamma_m)}Mb + \frac{gx}{2\beta}M(1-b), \tag{187}$$

$$\langle P| (m\bar{\psi}\psi)_R |P\rangle \big|_{\mathbf{P}=0} = \frac{Mb}{1+\gamma_m}. \tag{188}$$

Lastly, the value of the coupling constant at $\mu = 2$ GeV is given by $\alpha_s = 0.269$ (one loop order) [2].

In table 1 we present the numerical results for the several sum rules at $\mu = 2$ GeV. All decompositions agree that most of the proton mass at $\mu = 2$ GeV comes from the gluons. The quark mass term seems to contribute 0.16 GeV to the proton mass. As the masses of the three valence quark add up to about 0.009 GeV, this validates the quark-gluon sea picture with more quarks than just the three valence quarks. In the MIT bag model, three massless quarks confined to a spherical cavity of radius 1 fm have a total kinetic energy of 0.6 to 0.7 GeV [4]. Assuming the quark kinetic energy of a proton is somewhat similar to this, the quark-gluon interactions contribute -0.3 to -0.4 GeV in the models of Rodini et al. and Ji. This negative contribution can be interpreted as a binding energy of the quarks and gluons. Lorcé's decomposition does not make a distinction between the quark mass term and other quark contributions, but is still in rough agreement . A surprising contrast to this is Tanaka's decomposition, based on the EMT trace. There the quarks actually have a very small, possibly even slightly negative total contribution to the proton mass [8]. Given that the quark

[8]Because we performed our analysis only at 1-loop order, the uncertainties are too large ($\alpha_s^2 \sim 0.07$) to state with certainty that the quark contribution is negative.

mass and kinetic contributions to $\bar{M}_q$ are definitely positive, this would imply that the quark-gluon interactions yield contribution to the proton mass roughly equal to the negative sum of the mass and kinetic terms. This is a much larger contribution than the -0.3 to -0.4 GeV. Of course whether we can make the distinction between kinetic energy and quark-gluon interaction is questionable: any decomposition that separates the two is no longer gauge invariant. Lastly, we note that in Ji's mass rule the different contributions are fairly close to each other, making the gluon dominance the weakest in that decomposition.

Of course the results of table 1 are but a snapshot of the proton mass at one particular scale. To obtain a full picture of the proton mass decompositions, we must also investigate the running of the mass decompositions. To do so, we first calculate the running of the forward matrix elements of equations (185) through (188):

$$\frac{\partial}{\partial \log \mu} \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R = \frac{\partial}{\partial \log \mu} \mathcal{O}^{00}_{1,R} + \frac{1}{4} \frac{\partial}{\partial \log \mu} \mathcal{O}^{00}_{2,R}, \tag{189}$$

$$\frac{\partial}{\partial \log \mu} (\mathbf{B}^2 - \mathbf{E}^2)_R = \frac{\partial}{\partial \log \mu} \mathcal{O}^{00}_{2,R}, \tag{190}$$

$$\frac{\partial}{\partial \log \mu} (\psi^\dagger i\mathbf{D} \cdot \alpha \psi)_R = \frac{\partial}{\partial \log \mu} \mathcal{O}^{00}_{3,R}, \tag{191}$$

$$\frac{\partial}{\partial \log \mu} (m\bar{\psi}\psi)_R = 0. \tag{192}$$

The renormalization of 00-components of the EMT operators is identical to the renormalization of the full operators. Therefore we obtain the differential equation:

$$\frac{\partial}{\partial \log \mu} \begin{bmatrix} \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R \\ (\mathbf{B}^2 - \mathbf{E}^2)_R \\ (\psi^\dagger i\mathbf{D} \cdot \alpha\psi)_R \\ (m\bar{\psi}\psi)_R \end{bmatrix} = -\frac{\alpha_s}{2\pi} A \begin{bmatrix} \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R \\ (\mathbf{B}^2 - \mathbf{E}^2)_R \\ (\psi^\dagger i\mathbf{D} \cdot \alpha\psi)_R \\ (m\bar{\psi}\psi)_R \end{bmatrix} \tag{193}$$

with:

$$A = \begin{bmatrix} \frac{4}{3}n_f C(N) & \frac{1}{3}n_f C(N) & -\frac{8}{3}C_2(N) & \frac{2}{3}C_2(N) \\ 0 & \frac{4}{3}n_f C(N) - \frac{11}{3}C_2(G) & 0 & 12C_2(N) \\ -\frac{4}{3}n_f C(N) & -\frac{1}{3}n_f C(N) & \frac{8}{3}C_2(N) & -\frac{2}{3}C_2(N) \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{194}$$

This differential equation has the same structure as the differential equation at the end of section 4.4 and therefore has a similar solution:

$$\begin{bmatrix} \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R(\mu) \\ (\mathbf{B}^2 - \mathbf{E}^2)_R(\mu) \\ (\psi^\dagger i\mathbf{D} \cdot \alpha\psi)_R(\mu) \\ (m\bar{\psi}\psi)_R(\mu) \end{bmatrix} = e^{-\frac{A}{\beta_0} \log \left| \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right|} \begin{bmatrix} \frac{1}{2}(\mathbf{B}^2 + \mathbf{E}^2)_R(\mu_0) \\ (\mathbf{B}^2 - \mathbf{E}^2)_R(\mu_0) \\ (\psi^\dagger i\mathbf{D} \cdot \alpha\psi)_R(\mu_0) \\ (m\bar{\psi}\psi)_R(\mu_0) \end{bmatrix} \tag{195}$$

From here we can calculate the running of the different mass decompositions. Figure 10 shows the result for the four decompositions treated in the last section. We see that our conclusions at $\mu = 2$ GeV do not relate to other scales due scale dependency and the mass operators mixing with each other. All decompositions agree that at small distances (high energy) the gluon energy contributes the most to the proton mass and the quark energy contribution the least. An extended momentum scale domain even shows that the quark energy contribution is negative in the high-energy limit for all decompositions. This supports the hypothesis that at a high energy scale the proton consists mainly of gluons. Obviously a dominance of gluons in the proton increases the gluon energy contribution to the proton mass. Also, with more gluons, we have more quark-gluon interaction per quark. As the quark-gluon interaction to the proton mass is negative, this results in a lower quark energy at a higher momentum scale.

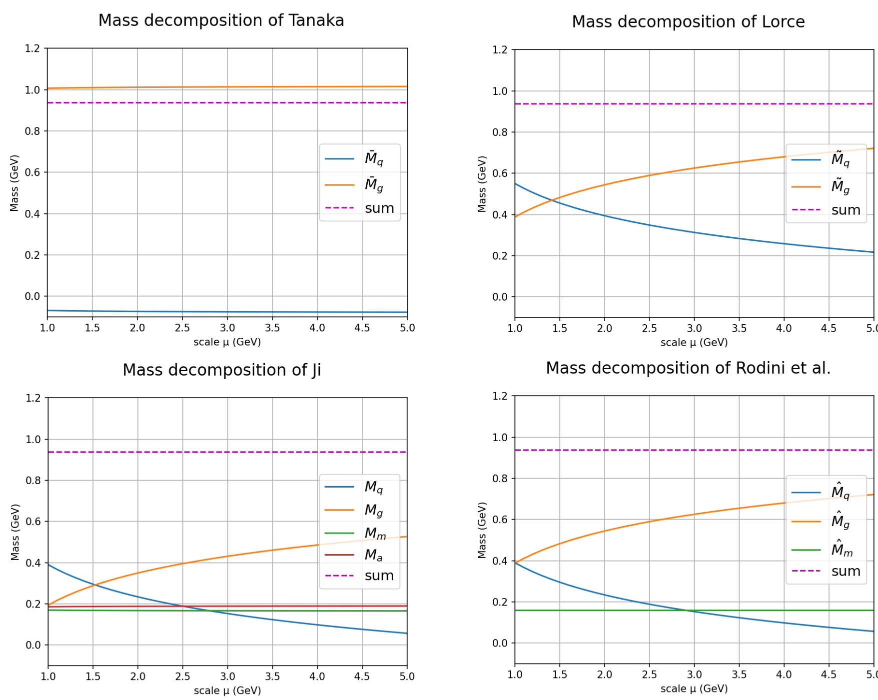


Figure 10: Scale dependence of the different mass decompositions. All mass components are graphed together with their sum for the scale range $\mu = 1 - 5$ GeV. In this region we have 4 active quark flavours, which we treat as if they have identical masses.

At the lower end of the momentum scale domain, the decompositions do not agree as much with each other. Specifically, the decomposition of Tanaka is very weakly scale dependent, which would imply that the gluon contribution dominates the proton mass at all scales. The other three decompositions show that the quark and gluon contributions would become roughly equal at a low momentum scale. Of course we only know this for sure above 1 GeV, as below that scale the higher-order effects become important and we cannot use perturbation theory anymore. Moreover, there is also a physical reason why we can't say much about momentum scales smaller than 1 GeV. The Compton wavelength associated with a particle of 1 GeV is 1fm, roughly the size of a proton. Therefore, probing the proton at a momentum scale < 1 GeV means we are averaging over an area larger than the proton. Lastly, we note that the quark mass term is RG-invariant, as we have shown before.

In conclusion, the decompositions based on the trace (Tanaka) and the Hamiltonian (Lorcé, Ji and Rodini et al.) yield very different numerical results for the quark- and gluon contributions to the proton mass. They seem to agree more with each other in the high-momentum scale region, but we cannot provide a physical reasoning for this phenomenon. The high-momentum scale behaviour can be explained by assuming that the proton consists mainly of gluons at high scale. The low-momentum scale behaviour of the Hamiltonian based decompositions show a roughly equal contributions of the quarks and gluons to the proton mass.

6 Conclusions

This thesis deals with the mass of the proton and how it can be decomposed, in the theory of QCD. We have reviewed QCD as a gauge theory with symmetry group $SU(3)$, its Lagrangian and energy-momentum tensor, a key component for investigating the mass of the proton. We have quantized and renormalized QCD, paying special attention to the scale dependency of the theory. We have seen that QCD is an asymptotically free theory, meaning it behaves as a free theory at very small distances / high momenta. On the other hand, QCD seems to be unworkable below an energy of $\Lambda \approx 0.2$ GeV, due to a diverging coupling constant. For perturbative purposes, a lower momentum limit of ~ 1 GeV is advised, unless one has access to many-loop calculations.

Next, we have reviewed operator renormalization, which is needed on top of the field renormalization when working with composite operators. We have seen that in the case of multiple operators with the same quantum numbers, these operators mix under renormalization. Next, we have renormalized the operators that form the EMT. Even though the EMT as a whole is RG-independent, mixing does take place between the operators. We have shown that the trace of the EMT is a measure for the scale-invariance of a theory. The scale dependence that is introduced by quantizing the theory leads to the well-known EMT trace anomaly, which we have derived.

In chapter 5 we have derived the proton mass from the EMT via forward matrix elements. We have treated two different derivations, one focusing on the EMT trace and the other on the Hamiltonian (related to the 00-component of the EMT). Even though the trace method was Lorentz invariant, just like mass itself, it was local and dependent on the renormalization of the proton state. Both these properties are not desirable for a proton mass formula. The Hamiltonian method did not have these issues, but required the proton to be at rest, singling out one frame of reference. We then looked at four mass decompositions. The two-term decomposition by Tanaka, based on the trace of the EMT, separated the mass into a quark- and gluon part, mimicking the quark- and gluon loop contributions to the effective gluon kinetic term in the effective Lagrangian of the background field method. Lorcé based his two-term decomposition of the Hamiltonian and reasoned an effective two-fluid picture, in which only the internal energy of the quark- or gluon fluid can truly contribute to the mass. The pressure effects of the fluids cancel each other out and should not be accounted towards any mass decomposition. Ji's decomposition based on the Hamiltonian acknowledges four terms: a quark mass contribution, a quark kinetic and potential energy contribution, a gluon energy contribution and a contribution from the trace anomaly. Lastly, a three-term contribution by Rodini et al. reinterprets the trace anomaly as part of the gluon kinetic and potential energy, claiming that there is no true separate trace anomaly contribution to the proton mass. This is true when using dimensional regularization and the mass formula based on T^00, but not for other regulators like the momentum cutoff nor for the mass formula based on the trace of the EMT.

Finally, we have used numerical inputs for the matrix elements that make up the decompositions and computed the numerical values of all decompositions. We have also computed the running of all the decompositions graphically. All decompositions agreed that at high energies the gluon energy was the main contribution to the proton mass. At the low energy limit however, there was controversy between the decompositions. This controversy seems to originate in the different formula's to calculate the proton mass. The decomposition by Tanaka showed that the quark contribution was very small and even slightly negative for all energies. The other decompositions agreed that the quark (energy) contribution became more dominant at low energies. The quark mass contribution, present in Ji's and Rodini's decompositions, was responsible for about 0.16 GeV. It would be interesting to see what numerical values these same decompositions would be for a pion, which is known to be the lightest hadron and acts as a nearly massless goldstone boson. It is expected that the quark mass term plays a much more dominant role in pion decompositions.

Acknowledgements

I would like to thank the Rijksuniversiteit Groningen for allowing me to perform this research and to work in their offices. I also would like to thank prof. dr. Lorcé, on giving some useful insights regarding the mass sum rules by Rodini et al. and Ji. A special thanks goes out to prof. dr. Boer, my supervisor and mentor for this project. Our weekly conversations and your feedback have always been pleasant and inspiring, giving me new energy to continue my research. I also would like to thank my girlfriend Anne Lambrechts, for being there during the fun and the rough times and always listening to me when I started rambling on about physics.

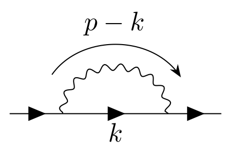
A Calculation of Feynman Diagrams

In this appendix the divergent parts of 1-loop Feynman diagrams needed in this thesis are calculated. The diagrams are all computed using dimensional regularization and the external legs are ignored. The general structure is as follows:

1. Change the number of dimensions from 4 to d.
2. Combine the denominators using Feynman parameters.
3. Shift the integration variable such that the denominator takes the form $[l^2 - \Delta]^n$ for some $n \in \mathbb{N}$.
4. Simplify the numerator, use the substitution $l^\mu l^\nu \to \frac{1}{d} g^{\mu\nu} l^2$ to express everything as much as possible in l^2 and throw away all terms without UV-divergences (terms of l^x with $x < 2n - 4$ or terms odd in l).
5. Use the identities of appendix A.4 of [6] to perform the integrals.

Finally, the result can be interpreted in $d = 4 - \epsilon$ dimensions, yielding the divergences as poles in $\frac{1}{\epsilon}$.

A.1 QCD diagrams



The diagram above equals the following integral:

$$\int \frac{d^d k}{(2\pi)^d} (ig\gamma^\mu T^a) \frac{i(\not{k} + m)}{k^2 - m^2} (ig\gamma_\mu T^a) \frac{-i}{(p-k)^2}.$$

By introducing a Feynman parameter and shifting the integration variable to $l = k - xp$, this integral simplifies to:

$$-g^2 C_2(N) \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{N}{[l^2 - \Delta]^2},$$

where:

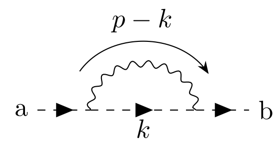
$$\begin{aligned} N &= \gamma^\mu (\not{l} + x\not{p} + m)\gamma_\mu, \\ &\sim (2-d)x\not{p} + dm, \end{aligned}$$

Performing the momentum integral, the full diagram equals (setting $d = 4 - \epsilon$):

$$-iC_2(N) \frac{\alpha_s}{4\pi} (4\pi)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx \, \left((\epsilon - 2)x\not{p} + (4-\epsilon)m\right)\Delta^{-\frac{\epsilon}{2}}.$$

Finally using the identities $\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$ and $x^\epsilon = 1 + \epsilon \ln x + \mathcal{O}(\epsilon^2)$, the divergent part of the diagram equals:

$$C_2(N)\frac{\alpha_s}{4\pi}(i[\not{p} - 4m])\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{196}$$



The diagram above equals the following integral:

$$\int \frac{d^dk}{(2\pi)^d}(-gf^{xyb}p^\mu)\frac{i}{k^2}(-gf^{ayx}k_\mu)\frac{-i}{(p-k)^2}.$$

By introducing a Feynman parameter and shifting the integration variable to $l = k - xp$, this integral simplifies to:

$$-g^2C_2(G)\delta^{ab}\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{N}{[l^2-\Delta]^2},$$

where:

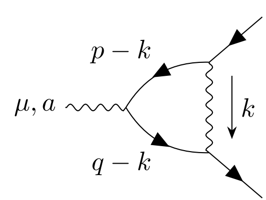
$$\begin{aligned} N &= p^\mu(l+xp)_\mu, \\ &\sim xp^2, \end{aligned}$$

Performing the momentum integral, the full diagram equals (setting $d = 4 - \epsilon$):

$$-iC_2(G)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\ xp^2\Delta^{-\frac{\epsilon}{2}}.$$

Finally, the divergent part of the diagram equals:

$$\frac{1}{2}C_2(G)\frac{\alpha_s}{4\pi}(-ip^2\delta^{ab})\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{197}$$



The diagram above equals the following integral:

$$\int \frac{d^4k}{(2\pi)^4}(ig\gamma^\alpha T^x)\frac{i(\not{q} - \not{k} + m)}{(q-k)^2 - m^2}(ig\gamma^\mu T^a)\frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2}(ig\gamma_\alpha T^x)\frac{-i}{k^2}.$$

By introducing Feynman parameters and shifting the integration variable to $l = k - xp - yq$, this integral simplifies to:

$$g^3[C_2(N) - \frac{1}{2}C_2(G)]T^a \int_0^1 dx \int_0^x dy \int \frac{d^d l}{(2\pi)^d} \frac{2N^\mu}{[l^2 - \Delta]^3},$$

where:

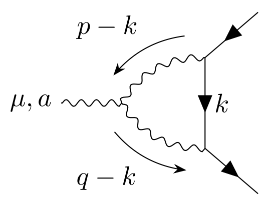
$$\begin{aligned} N^\mu &= \gamma^\alpha([1-y]\not{q} - x\not{p} - \not{l} + m)\gamma^\mu(-y\not{q} + [1-x]\not{p} - \not{l} + m)\gamma_\alpha, \\ &\sim \gamma^\alpha(-\not{l})\gamma^\mu(-\not{l})\gamma_\alpha, \\ &\sim l^2 \frac{(2-d)^2}{d}\gamma^\mu. \end{aligned}$$

Performing the momentum integral, the full diagram equals (setting $d = 4 - \epsilon$):

$$i[C_2(N) - \frac{1}{2}C_2(G)]\frac{\alpha_s}{4\pi} g\gamma^\mu T^a \frac{(\epsilon-2)^2}{2}(4\pi)^{\frac{\epsilon}{2}}\Gamma\Big(\frac{\epsilon}{2}\Big)\int_0^1 dx \int_0^x dy \ \Delta^{-\frac{\epsilon}{2}}.$$

Finally, the divergent part of the diagram equals:

$$[C_2(N) - \frac{1}{2}C_2(G)]\frac{\alpha_s}{4\pi}(ig\gamma^\mu T^a)\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right] \tag{198}$$



The diagram above equals the following integral:

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4}(ig\gamma^\alpha T^x)\frac{i(\not{k}+m)}{k^2 - m^2}(ig\gamma^\beta T^y)\frac{-i}{(p-k)^2}\frac{-i}{(q-k)^2} \\ \times gf^{axy}\big[g^{\mu\alpha}(2q - p - k)^\beta + g^{\alpha\beta}(2k - p - q)^\mu + g^{\beta\mu}(2p - q - k)^\alpha\big]. \end{aligned}$$

By introducing Feynman parameters and shifting the integration variable to $l = k - xp - yq$, this integral simplifies to:

$$-\frac{1}{2}g^3 C_2(G)T^a \int_0^1 dx \int_0^x dy \int \frac{d^d l}{(2\pi)^d}\frac{2N^\mu}{[l^2 - \Delta]^3},$$

where:

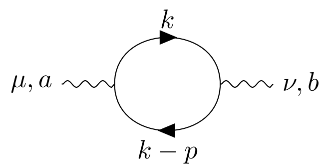
$$\begin{aligned} N^\mu = \gamma^\alpha(\not{l} + x\not{p} + y\not{y} + m)\gamma^\beta\big[&g^{\mu\alpha}([2-y]q - [1+x]p - l)^\beta \\ &+ g^{\alpha\beta}(2l - [1-2x]p - [1-2y]q)^\mu + g^{\beta\mu}([2-x]p - [1+y]q - l)^\alpha\big], \\ \sim \gamma^\alpha \not{l}\gamma^\beta\big[&g^{\mu\alpha}(-l)^\beta + g^{\alpha\beta}(2l)^\mu + g^{\beta\mu}(-l)^\alpha\big], \\ \sim -(4 - &\frac{4}{d})l^2\gamma^\mu. \end{aligned}$$

Performing the momentum integral, the full diagram equals (setting $d = 4 - \epsilon$):

$$iC_2(G)\frac{\alpha_s}{4\pi} g\gamma^\mu T^a (3-\epsilon)(4\pi)^{\frac{\epsilon}{2}} \Gamma\Big(\frac{\epsilon}{2}\Big) \int_0^1 dx \int_0^x dy\ \Delta^{-\frac{\epsilon}{2}}.$$

Finally, the divergent part of the diagram equals:

$$\frac{3}{2} C_2(G)\frac{\alpha_s}{4\pi}(ig\gamma^\mu T^a)\Big[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\Big]. \tag{199}$$



The diagram above equals:

$$(-1)\int \frac{d^d k}{(2\pi)^d}\, \mathrm{Tr}\Big\{(ig\gamma^\mu T^a)\frac{i(\not k - \not p + m)}{(k-p)^2 - m^2}(ig\gamma^\nu T^b)\frac{i(\not k + m)}{k^2 - m^2}\Big\}.$$

By introducing a Feynman parameter and shifting the integration variable to $l = k - xp$, this integral simplifies to:

$$-g^2 C(N)\delta^{ab}\int_0^1 dx \int \frac{d^d l}{(2\pi)^d}\frac{N^{\mu\nu}}{[l^2 - \Delta]^2},$$

where:

$$\Delta = x(x-1)p^2 + m^2,$$

and:

$$\begin{aligned} N^{\mu\nu} &= \mathrm{Tr}\{\gamma^\mu(\not l - [1-x]\not p + m)\gamma^\nu(\not l + x\not p + m)\}, \\ &\sim \Big(\frac{8}{d} - 4\Big) g^{\mu\nu} l^2 - x(x-1)(4g^{\mu\nu}p^2 - 8p^\mu p^\nu) + 4m^2 g^{\mu\nu}. \end{aligned}$$

Performing the momentum integral, the full diagram equals (setting $d = 4 - \epsilon$):

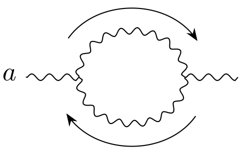
$$\begin{aligned} iC(N)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}} \int_0^1 dx\ \Delta^{\frac{\epsilon}{2}} \Big[& 4g^{\mu\nu}\Big(\frac{\epsilon}{2} - 1\Big)\Gamma\Big(\frac{\epsilon}{2} - 1\Big)\Delta \\ & + [x(x-1)(4g^{\mu\nu}p^2 - 8p^\mu p^\nu) + 4m^2 g^{\mu\nu}]\Gamma\Big(\frac{\epsilon}{2}\Big)\Big]. \end{aligned}$$

This expression seems to contain two different divergences. However, using the relation $x\Gamma(x) = \Gamma(x+1)$ we see that they actually both gamma functions contribute to the same divergence of $\Gamma\big(\frac{\epsilon}{2}\big)$. Writing out Δ, we then combine the terms and find:

$$iC(N)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\Gamma\Big(\frac{\epsilon}{2}\Big)\int_0^1 dx\ \Delta^{\frac{\epsilon}{2}} 8x(x-1)(g^{\mu\nu}p^2 - p^\mu p^\nu).$$

Finally, the divergent part of the diagram equals:

$$\frac{4}{3}C(N)\frac{\alpha_s}{4\pi}(-i[g^{\mu\nu}p^2]\delta^{ab} - p^\mu p^\nu)\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]. \tag{200}$$



The diagram above equals:

$$\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} g f^{axy}[g^{\mu\alpha}(2p-k)^\beta + g^{\alpha\beta}(2k-p)^\mu + g\beta\mu(-p-k)^\alpha]\frac{-i}{k^2}$$
$$\times g f^{byx}[g^{\nu\beta}(-p-k)^\alpha + g^{\beta\alpha}(2k-p)^\nu + g^{\alpha\nu}(2p-k)^\beta]\frac{-i}{(k-p)^2}.$$

The factor $\frac{1}{2}$ is a symmetry factor. By introducing a Feynman parameter and shifting the integration variable to $l = k - xp$, this integral simplifies to:

$$g^2 C_2(G)\delta^{ab}\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{N^{\mu\nu}}{[l^2-\Delta]^2},$$

where:

$$\Delta = x(x-1)p^2,$$

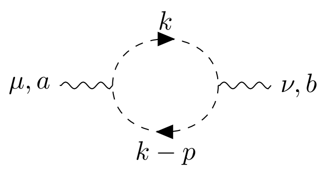
and:

$$\begin{aligned} N^{\mu\nu} &= \frac{1}{2}[g^{\mu\alpha}([2-x]p-l)^\beta + g^{\alpha\beta}(2l+[2x-1]p)^\mu + g^{\beta\mu}(-[1+x]p-l)^\alpha] \\ &\quad \times [g^{\nu\beta}(-[1+x]p-l)^\alpha + g^{\beta\alpha}(2l+[2x-1]p)^\nu + g^{\alpha\nu}([2-x]p-l)^\beta], \\ &\sim \frac{3}{d}(d-1)g^{\mu\nu}l^2 + \frac{1}{2}\big[(2x-1)^2(d-2) - 2(x+1)(2-x)\big]p^\mu p^\nu \\ &\quad + \big[(2-x)^2 + (1+x)^2\big]p^2 g^{\mu\nu}. \end{aligned}$$

Performing the momentum integral, the full diagram equals (setting $d = 4 - \epsilon$):

$$\begin{aligned} iC_2(G)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\int_0^1 dx\,\Delta^{\frac{\epsilon}{2}}\Big[&-\frac{3}{2}(3-\epsilon)g^{\mu\nu}\Gamma\Big(\frac{\epsilon}{2}-1\Big)\Delta + \frac{1}{2}\big([(2x-1)^2(2-\epsilon) \\ &- 2(x+1)(2-x)]p^\mu p^\nu + [(2-x)^2 + (1+x)^2]p^2 g^{\mu\nu}\big)\Gamma\Big(\frac{\epsilon}{2}\Big)\Big]. \end{aligned} \tag{201}$$

At this point we cannot proceed any further, as the two gamma functions yield different divergences. Therefore we continue with the two diagrams that are intimately linked with this diagram: the ghost loop correction and the gluon tadpole correction to the vacuum polarization.



The diagram above equals:

$$(-1)\int \frac{d^dk}{(2\pi)^d}(-gf^{xay}k^\mu)\frac{i}{(k-p)^2}(-gf^{ybx}(k-p)^\nu)\frac{i}{k^2}.$$

By introducing a Feynman parameter and shifting the integration variable to $l = k - xp$, this integral simplifies to:

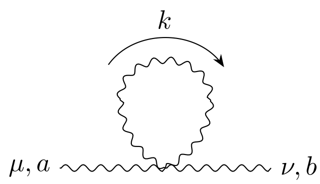
$$-g^2C_2(G)\delta^{ab}\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{N^{\mu\nu}}{[l^2-\Delta]^2},$$

where Δ is equal to the gluon loop diagram and:

$$\begin{aligned} N^{\mu\nu} &= (l+xp)^\mu(l+[x-1]p)^\nu, \\ &\sim \frac{1}{d}g^{\mu\nu}l^2 + x(x-1)p^\mu p^\nu. \end{aligned}$$

Performing the momentum integral, the full diagram equals (setting $d = 4-\epsilon$):

$$iC_2(G)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\int_0^1 dx\ \Delta^{\frac{\epsilon}{2}}\left[\frac{1}{2}g^{\mu\nu}\Gamma\Big(\frac{\epsilon}{2}-1\Big)\Delta - x(x-1)p^\mu p^\nu\Gamma\Big(\frac{\epsilon}{2}\Big)\right]. \quad (202)$$



The diagram above equals:

$$\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}[-2ig^2f^{axy}f^{bxy}(d-1)g^{\mu\nu}]\frac{-i}{k^2}.$$

Due to its intimate link with the gluon loop correction and the ghost loop correction to the vacuum polarization, we force it to look like the other diagram by multiplying it by $1 = \frac{(k-p)^2}{(k-p)^2}$. Next we introduce a Feynman parameter and shift the integration variable to $l = k - xp$. The integral simplifies to:

$$-g^2C_2(G)\delta^{ab}g^{\mu\nu}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{N}{[l^2-\Delta]^2},$$

where Δ is equal to the gluon loop diagram and:

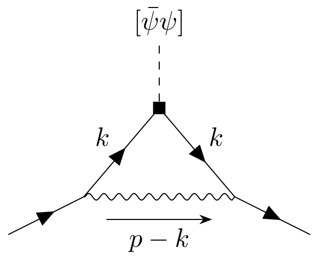
$$N = (d-1)(l + [x-1]p)^2,$$
$$\sim (d-1)l^2 + (d-1)(x-1)^2p^2.$$

Performing the momentum integral, the full diagram equals (setting $d = 4 - \epsilon$):

$$iC_2(G)\delta^{ab}\frac{\alpha_s}{4\pi}(4\pi)^{\frac{\epsilon}{2}}\int_0^1 dx\ \Delta^{\frac{\epsilon}{2}}\Big[(6 - \frac{7}{2}\epsilon + \frac{1}{2}\epsilon^2)\Gamma\Big(\frac{\epsilon}{2} - 1\Big) - (3-\epsilon)(x-1)^2p^2\Gamma\Big(\frac{\epsilon}{2}\Big)\Big]. \quad (203)$$

A.2 Operator diagrams

Interactions with $\bar{\psi}\psi$



Expressed as an integral, this diagram equals:

$$\int \frac{d^dk}{(2\pi)^d}(ig\gamma^\mu T^a)\frac{i(\not{k}+m)}{k^2-m^2}(1)\frac{i(\not{k}+m)}{k^2-m^2}(ig\gamma_\mu T^a)\frac{-i}{(p-k)^2}$$

We combine the denominators by introducing a Feynman parameter and shift the integration value to $l = k - xp$. The result is:

$$-ig^2C_2(N)\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{2(1-x)N}{[l^2-\Delta]^3}, \quad (204)$$

with:

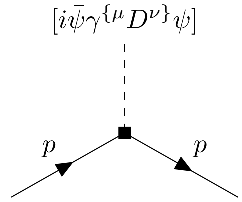
$$N = \gamma^\alpha(\not{l} + x\not{p} + m)(\not{l} + x\not{p} + m)\gamma_\alpha.$$

Only the term with two factors of l results in a divergence. Therefore we ignore the rest and write $N = dl^2$. The full Feynman diagram therefore contains the term:

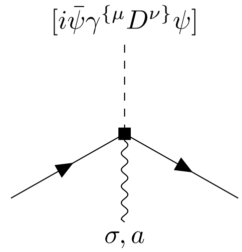
$$-ig^2C_2(N)\int_0^1 dx\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-\frac{d}{2})}{2}\Delta^{d/2-2}2(1-x)d,$$

which contains the divergence:

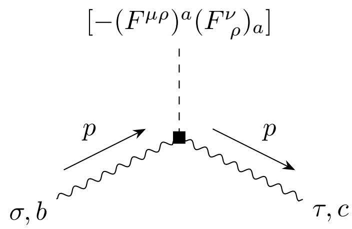
$$4C_2(N)\frac{\alpha_s}{4\pi}\Big[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\Big]. \quad (205)$$



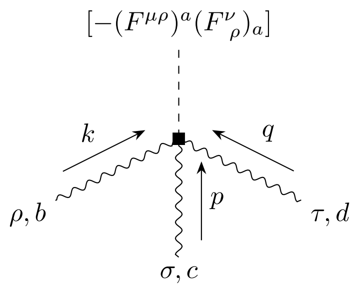
$$= \gamma^{\{\mu}p^{\nu\}},$$



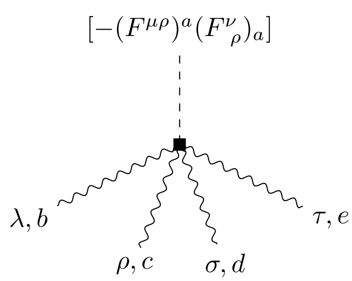
$$= -g\gamma^{\{\mu}g^{\nu\}\sigma}T^a,$$



$$= -2\delta^{bc}\Big[p^{\{\mu}p^{\nu\}}g^{\sigma\tau} + p^2 g^{\sigma\{\mu}g^{\nu\}\tau} - p^{\sigma}p^{\{\mu}g^{\nu\}\tau} - p^{\tau}p^{\{\mu}g^{\nu\}\sigma}\Big],$$



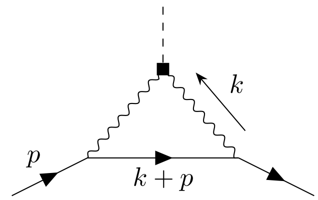
$$= 2if^{bcd}\Big[g^{\rho\sigma}g^{\tau\{\mu}(p-k)^{\nu\}} + g^{\sigma\tau}g^{\rho\{\mu}(q-p)^{\nu\}} + g^{\tau\rho}g^{\sigma\{\mu}(k-q)^{\nu\}} + g^{\rho\{\mu}g^{\nu\}\sigma}(p-k)^{\tau} + g^{\sigma\{\mu}g^{\nu\}\tau}(q-k)^{\rho} + g^{\tau\{\mu}g^{\nu\}\rho}(k-q)^{\sigma}\Big],$$



$$= 2g^2\Big[f^{xbc}f^{xde}\big(g^{\lambda\{\mu}[g^{\nu\}\sigma}g^{\rho\tau} - g^{\nu\}\tau}g^{\rho\sigma}] - g^{\rho\{\mu}[g^{\nu\}\sigma}g^{\lambda\tau} - g^{\nu\}\tau}g^{\rho\sigma}]\big) + f^{xbd}f^{xce}\big(g^{\lambda\{\mu}[g^{\nu\}\rho}g^{\sigma\tau} - g^{\nu\}\tau}g^{\rho\sigma}] - g^{\sigma\{\mu}[g^{\nu\}\rho}g^{\lambda\tau} - g^{\nu\}\tau}g^{\lambda\rho}]\big) + f^{xbe}f^{xcd}\big(g^{\lambda\{\mu}[g^{\nu\}\rho}g^{\sigma\tau} - g^{\nu\}\sigma}g^{\rho\tau}] - g^{\tau\{\mu}[g^{\nu\}\rho}g^{\lambda\sigma} - g^{\nu\}\sigma}g^{\lambda\rho}]\big)\Big].$$

Figure 11: Interactions of the traceless EMT-like operators.

Interactions with $-(F^{\mu\rho})^a(F^{\nu}{}_{\rho})_a$ and $i\bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi$.



The Feynman diagram above equals:

$$\int \frac{d^dk}{(2\pi)^d}(ig\gamma^\alpha T^a)\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}(ig\gamma^\beta T^b)\left(\frac{-i}{k^2}\right)^2(-2\delta^{ab})\Big[k^{\{\mu}k^{\nu\}}g_{\alpha\beta} + k^2 g_\alpha{}^{\{\mu}g^{\nu\}}_{\ \beta} - k_\alpha k^{\{\mu}g^{\nu\}}_{\ \beta} - k_\beta k^{\{\mu}g^{\nu\}}_{\ \alpha}\Big] - (\text{trace}). \quad (206)$$

We combine the denominators by introducing a Feynman parameter. The result is:

$$-ig^2C_2(N)\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{2(1-x)N^{\mu\nu}}{[l^2-\Delta]^3}, \quad (207)$$

with:

$$l = k + xp,$$
$$N^{\mu\nu} = 2\gamma^\alpha(\not{l}+(1-x)\not{p}+m)\gamma^\beta\Big[(l-xp)^{\{\mu}(l-xp)^{\nu\}}g_{\alpha\beta} + (l-xp)^2 g_\alpha{}^{\{\mu}g^{\nu\}}_{\ \beta} - (l-xp)_\alpha(l-xp)^{\{\mu}g^{\nu\}}_{\ \beta} - (l-xp)_\beta(l-xp)^{\{\mu}g^{\nu\}}_{\ \alpha}\Big] - (\text{trace}),$$

For our case, we are only interested in traceless, divergent terms that contain the structure $\gamma^{\{\mu}p^{\nu\}} = \gamma^{\{\nu}p^{\mu\}}$. Ignoring all other terms and replacing all $l^\alpha l^\beta \to \frac{g^{\alpha\beta}}{d}l^2$, $N^{\mu\nu}$ simplifies to:

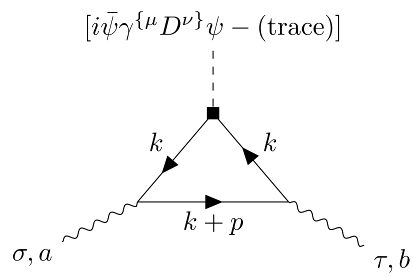
$$N^{\mu\nu} = 2(2-\frac{4}{d})(x+1)l^2\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big].$$

The full Feynman diagram therefore contains the term:

$$-ig^2C_2(N)\int_0^1 dx \frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-\frac{d}{2})}{2}\Delta^{d/2-2}4(2-\frac{4}{d})(1-x^2)\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big],$$

which contains the divergence:

$$\frac{8}{3}C_2(N)\Big(\frac{g}{4\pi}\Big)^2\Big[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\Big]\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big]. \quad (208)$$



The Feynman diagram above equals:

$$(-1)\int \frac{d^dk}{(2\pi)^d}\,\mathrm{Tr}\left[(ig\gamma^\sigma T^a)\frac{i(\not{k}+m)}{k^2-m^2}\gamma^{\{\mu}k^{\nu\}}\frac{i(\not{k}+m)}{k^2-m^2}(ig\gamma^\tau T^b)\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right] - \text{(trace)}. \tag{209}$$

We combine the denominators by introducing a Feynman parameter. The result is:

$$-ig^2C(N)\delta^{ab}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{2(1-x)N^{\mu\nu\sigma\tau}}{[l^2-\Delta]^3}, \tag{210}$$

with:

$$\begin{aligned} l &= k + xp, \\ N^{\mu\nu\sigma\tau} &= \mathrm{Tr}\left[\gamma^\sigma(\not{l}-x\not{p}+m)\gamma^{\{\mu}(l-xp)^{\nu\}}(\not{l}-x\not{p}+m)\right. \\ &\qquad \left.\gamma^\tau(\not{l}+(1-x)\not{p}+m)\right] - \text{(trace)}. \end{aligned}$$

For our case, we are only interested in traceless, divergent terms that contain the structure $p^{\{\mu}p^{\nu\}}g^{\sigma\tau}$. Ignoring all other terms, the denominator simplifies to:

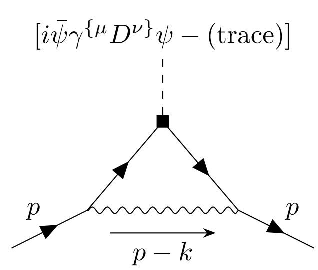
$$N^{\mu\nu\sigma\tau} = 2\Big[(\frac{4}{d}-2)x - x^2\Big]l^2p^{\{\mu}p^{\nu\}}g^{\sigma\tau} - \text{(trace)}.$$

The full Feynman diagram therefore contains the term:

$$-ig^2C(N)\delta^{ab}\int_0^1 dx\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-\frac{d}{2})}{2}\Delta^{d/2-2}4(1-x)\Big[(\frac{4}{d}-2)x-x^2\Big]\big[p^{\{\mu}p^{\nu\}}g^{\sigma\tau}-\text{(trace)}\big],$$

which contains the divergence:

$$\frac{2}{3}C(N)\Big(\frac{g}{4\pi}\Big)^2\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]\big[(-2\delta^{ab})p^{\{\mu}p^{\nu\}}g^{\sigma\tau} - \text{(trace)}\big]. \tag{211}$$



The Feynman diagram above equals:

$$\int \frac{d^dk}{(2\pi)^d}(ig\gamma^\alpha T^a)\frac{i(\not{k}+m)}{k^2-m^2}\gamma^{\{\mu}k^{\nu\}}\frac{i(\not{k}+m)}{k^2-m^2}(ig\gamma_\alpha T^a)\frac{-i}{(p-k)^2}.$$

We combine the denominators by introducing a Feynman parameter. The result is:

$$-ig^2C_2(N)\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{2(1-x)N^{\mu\nu}}{[l^2-\Delta]^3}, \tag{212}$$

with:

$$l = k - xp,$$
$$N^{\mu\nu} = \gamma^\alpha(\not{l}+x\not{p}+m)\gamma^{\{\mu}(l+xp)^{\nu\}}(\not{l}+x\not{p}+m)\gamma_\alpha.$$

For our case, we are only interested in traceless, divergent terms that contain the structure $\gamma^{\{\mu}p^{\nu\}} = \gamma^{\{\nu}p^{\mu\}}$. Ignoring all other terms, $N^{\mu\nu}$ simplifies to:

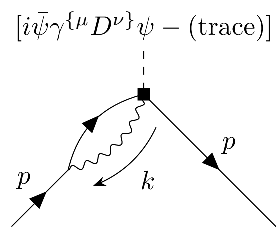
$$N^{\mu\nu} = x\frac{(d-2)^2}{d}l^2\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big].$$

The full Feynman diagram therefore contains the term:

$$-ig^2C_2(N)\int_0^1 dx\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-\frac{d}{2})}{2}\Delta^{d/2-2}2(1-x)x\frac{(d-2)^2}{d}\big[\gamma^{\{\mu}p^{\nu\}}-(\text{trace})\big].$$

which contains the divergence:

$$\frac{1}{3}C_2(N)\Big(\frac{g}{4\pi}\Big)^2\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big]. \tag{213}$$



The Feynman diagram above equals:

$$\int\frac{d^dk}{(2\pi)^d}(ig\gamma^\alpha T^a)\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}(-g\gamma^{\{\mu}g^{\nu\}}{}_\alpha T^a)\frac{-i}{k^2} - (\text{trace}).$$

We combine the denominators by introducing a Feynman parameter. The result is:

$$ig^2C_2(N)\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{N^{\mu\nu}}{[l^2-\Delta]^2}, \tag{214}$$

with:

$$l = k + xp,$$
$$N^{\mu\nu} = \gamma^{\{\mu}(\not{l} + (1-x)\not{p} + m)\gamma^{\nu\}} - (\text{trace}).$$

For our case, we are only interested in traceless, divergent terms that contain the structure $\gamma^{\{\mu}p^{\nu\}} = \gamma^{\{\nu}p^{\mu\}}$. Ignoring all other terms, the numerator simplifies to:

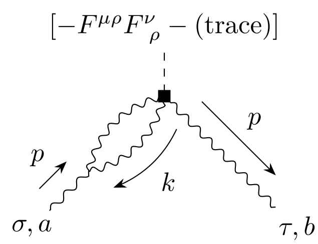
$$N^{\mu\nu} = 2(1-x)\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big].$$

The full Feynman diagram therefore contains the term:

$$ig^2 C_2(N)\int_0^1 dx \frac{i}{(4\pi)^{d/2}}\Gamma(2-\frac{d}{2})\Delta^{d/2-2}2(1-x)\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big]$$

which contains the divergence:

$$-C_2(N)\Big(\frac{g}{4\pi}\Big)^2\Bigg[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\Bigg]\big[\gamma^{\{\mu}p^{\nu\}} - (\text{trace})\big]. \tag{215}$$



The Feynman diagram above equals:

$$\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} g f^{axy}[g^{\sigma\alpha}(p-k)^\beta + g^{\alpha\beta}(2k+p)^\sigma + g^{\beta\sigma}(-2p-k)^\alpha]\frac{-i}{(p+k)^2}2igf^{bxy}$$
$$\times\Big[g^{\tau\alpha}g^{\beta\{\mu}(p-k)^{\nu\}} + g^{\alpha\beta}g^{\tau\{\mu}(2k+p)^{\nu\}}$$
$$+ g^{\beta\tau}g^{\alpha\{\mu}(-2p-k)^{\nu\}} + g^{\tau\{\mu}g^{\nu\}\alpha}(p-k)^\beta$$
$$+ g^{\alpha\{\mu}g^{\nu\}\beta}(2k+p)^\tau + g^{\beta\{\mu}g^{\nu\}\tau}(-2p-k)^\alpha\Big]\frac{-i}{k^2} - (\text{trace}).$$

We combine the denominators by introducing a Feynman parameter. The result is:

$$-ig^2C_2(G)\delta^{ab}\int_0^1 dx\int \frac{d^dl}{(2\pi)^d}\frac{N^{\mu\nu\sigma\tau}}{[l^2-\Delta]^2}, \tag{216}$$

with:

$$l = k + xp,$$
$$N^{\mu\nu\sigma\tau} = \text{a humongous expression}$$

For our case, we are only interested in traceless, divergent terms that contain the structure $p^{\{\mu}p^{\nu\}}g^{\sigma\tau}$. Ignoring all other terms, the numerator simplifies to:

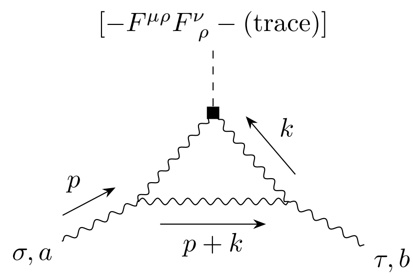
$$N^{\mu\nu\sigma\tau} = \big((1+x)^2 + (2-x)^2\big)\big[p^{\{\mu}p^{\nu\}}g^{\sigma\tau} - (\text{trace})\big].$$

The full Feynman diagram therefore contains the term:

$$-ig^2 C_2(G)\delta^{ab}\int_0^1 dx \frac{i}{(4\pi)^{d/2}}\Gamma(2-\frac{d}{2})\Delta^{d/2-2} \\ \times \big((1+x)^2 + (2-x)^2\big)\big[p^{\{\mu}p^{\nu\}}g^{\sigma\tau} - (\text{trace})\big],$$

which contains the divergence:

$$-\frac{14}{6}C_2(G)\Big(\frac{g}{4\pi}\Big)^2\left[\frac{2}{\epsilon} + \ln 4\pi e^{-\gamma_E}\right]\big[(-2\delta^{ab})p^{\{\mu}p^{\nu\}}g^{\sigma\tau} - (\text{trace})\big]. \tag{217}$$



The Feynman diagram above equals:

$$\int \frac{d^dk}{(2\pi)^d} g f^{axy}[g^{\sigma\alpha}(2p+k)^\beta + g^{\alpha\beta}(-p-2k)^\sigma + g^{\beta\sigma}(k-p)^\alpha]\frac{-i}{k^2} \\ \times (-2\delta^{yz})[k^{\{\mu}k^{\nu\}}g^{\rho\beta} + k^2 g^{\rho\{\mu}g^{\nu\}\beta} - k^\rho k^{\{\mu}g^{\nu\}\beta} - k^\beta k^{\{\mu}g^{\nu\}\rho}]\frac{-i}{k^2} \\ \times g f^{bzx}[g^{\tau\rho}(-p+k)^\alpha + g^{\rho\alpha}(-2k-p)^\tau + g^{\alpha\tau}(2p+k)^\rho]\frac{-i}{(p+k)^2} - (\text{trace}).$$

We combine the denominators by introducing a Feynman parameter. The result is:

$$2ig^2 C_2(G)\delta^{ab}\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{2(1-x)N^{\mu\nu\sigma\tau}}{[l^2-\Delta]^3}, \tag{218}$$

with:

$$l = k + xp, \\ N^{\mu\nu\sigma\tau} = \text{a humongous expression}$$

For our case, we are only interested in traceless, divergent terms that contain the structure $p^{\{\mu}p^{\nu\}}g^{\sigma\tau}$. Ignoring all other terms, the denominator simplifies to:

$$N^{\mu\nu\sigma\tau} = (\frac{-4x^2}{d} + 5x^2 + \frac{4x}{d} + 4 - \frac{8}{d})l^2\big[p^{\{\mu}p^{\nu\}}g^{\sigma\tau} - (\text{trace})\big].$$

The full Feynman diagram therefore contains the term:

$$2ig^2C_2(G)\delta^{ab}\int_0^1 dx \frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-\frac{d}{2})}{2}\Delta^{d/2-2}2(1-x)$$
$$\times\Big(\frac{-4x^2}{d}+5x^2+\frac{4x}{d}+4-\frac{8}{d}\Big)\big[p^{\{\mu}p^{\nu\}}g^{\sigma\tau}-(\text{trace})\big].$$

which contains the divergence:

$$3C_2(G)\Big(\frac{g}{4\pi}\Big)^2\left[\frac{2}{\epsilon}+\ln 4\pi e^{-\gamma_E}\right]\big[(-2\delta^{ab})p^{\{\mu}p^{\nu\}}g^{\sigma\tau}-(\text{trace})\big]. \tag{219}$$

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