university of groningen

faculty of science and engineering

The Occurrences of Sliding Solutions in the Hegselmann-Krause Model

Leonardo Bertamini

university of groningen

faculty of science and engineering

University of Groningen

The Occurences of Sliding Solutions in the Hegselmann-Krause Model

Bachelor's Thesis

Under the supervision of
Prof. Stefan Trenn
and
Prof. Henk van Waarde

Leonardo Bertamini (s3922928)

February 8, 2023

Contents

Page

Abstract **4**

1 Introduction **5**

2 Background Literature **6**
2.1 Opinion Dynamics 6
2.2 The Hegselmann-Krause Model 7
2.3 Sliding Solutions 8

3 Preliminaries **9**

4 Methodology **12**
4.1 Research Questions 12
4.2 Analysis Outline 12

5 The Model **14**
5.1 Discrete Time Model 14
5.2 Example 15
5.3 Continuous Time Model 16
5.4 Dimension Reduction 18
5.5 State Space 19
5.6 Example 20

6 Results **22**
6.1 Sliding Solutions 22
6.6 Super Symmetric Model 25
6.6.1 Preliminaries 25
6.6.2 Boundary Testing 26
6.6.3 Zonal proof 29
6.7 Asymmetric Model 32
6.7.1 Basic Zones 32
6.7.2 Section 1 32
6.7.3 Section 2 34
6.7.4 Section 3 36

7 Discussion **37**
7.1 Summary of Main Contributions 37
7.2 Limitations 37
7.3 Future Work 38

8 Conclusion **39**

Bibliography **40**

Abstract

This project investigated the Hegselmann-Krause model in opinion dynamics. This paper followed a previous paper that investigated the stability of the model and its convergence. The occurrences of sliding solutions due to discontinuities in the model was investigated. This means that the vector field varies discontinuously as solutions approach a surface. At this point, solutions can display many different behaviours, but this paper focused on the case where the solution slides along the switching surface. To do this the model will be viewed as a discontinuous piecewise linear model. This paper is significant as it improved the understanding of an important model in the field of opinion dynamics. Each zone was investigated and extended to the different boundary cases. Patterns were investigated to make more general proofs. This project proved no sliding solutions in the super symmetric case. The asymmetric case proved to be complex but no sliding solutions could be constructed in three dimensions. However, no evidence was found that they could not exist in higher dimensions.

1 Introduction

"Opinions are the medium between knowledge and ignorance" is a quote from Greek Philosopher Plato over 2000 years ago [1]. He described opinions as a human striving away from ignorance and towards knowledge, but still falling short of a binary truth or falsehood. In a world where all questions have more nuanced answers than yes or no, opinions now hold more importance than ever before.

The dictionary definition of an opinion is not that different. "A view that is formed, not necessarily based on fact or knowledge" [2]. This specification, that it is not formed around fact, seems to make it weaker, however, this is precisely why opinions can shift and evolve over time. Opinions are a crucial element to human identity. Thus their formulation and subsequent evolution over time is a formative part of who one is at current and who one will become in the future. They can be a window into one's mental state, or an expression of one's inner philosophy. Opinions, much like people, can be influenced by many things. The primary element that influences and alters one's views are the opinions of people adjacent to them (i.e. parents/guardians). However, it is far more complicated than this. Many children have different opinions to their parents maybe consciously or unconsciously. This is due to a whole spectrum of interactions where one voices their opinions and listens to others and this can affect their stance.

The field of opinion dynamics is the practice of simulating the evolution of opinions over time due to interactions with agents of similar views [3]. In consequence, many models have been designed to simulate the interactions of agents with similar views and their convergence properties.

The field of modelling agent interactions based on encounters has changed drastically with the growth of social media. Now opinions are interacting at higher rates and with more effect than in the past. Social media is constantly being used or misused to shift public opinion or to garner favour in politics and thus the field of opinion dynamics is more important than ever at this current point in time.

In this paper, the focus will remain on one specific model, namely, the Hegselmann-Krause model [4]. This paper is a progression of a previous Bachelor's thesis which investigated the convergence properties of said model [5]. That paper ignored the possibilities of sliding solutions which are a specific occurrence due to model discontinuities. This paper will look to improve the understanding of this model and whether these problems arise.

2 Background Literature

2.1 Opinion Dynamics

Some of the early papers relating to the origins of opinion dynamics are papers describing crowd mentality such as Mackey (1841) [6]. These were observations made on how behaviour shifts or becomes less rational when one is part of a group or a crowd. Other famous studies have been found showing similar behaviours such as Zimbardo's Stanford prison experiment [7] or Milgram's experiments [8]. Evidence was shown that people may behave differently to conform to what they perceive as an intellectual or an authoritative figure. These behaviours are at the core of how opinions shift in everything from sports to politics. This fascination with conformity stems back to psychologist Solomon Asch [9]. Asch found that a person in a classroom told to answer an easy question will be more likely to answer incorrectly if they see everyone else in the class answer incorrectly first. Allen and Levine [10] later found that if one other dissident is placed in the classroom then the absence of group unanimity lowers conformity.

The field of opinion dynamics has grown rapidly in importance in the last 20 years with the emergence of social media. Opinion dynamics works by simulating networks and the connected agents. So this might be one's group of acquaintances with a few external influences. However, the internet has made this method infinitely more complex. Where once opinion interactions were happening with limited agents at the pub with friends or in one direction over the radio, now opinions are interacting at higher rates than the past. Twitter and Facebook are prime examples of a constant interaction between various opinions from random agents and this constantly evolves your own stance. This has been used effectively to shift the sentiment of different policies or political movements. Social media has played a significant role in the Trump Campaign, the Brexit Referendum and Russia's invasion of Ukraine [11]. There have also been social media campaigns on recent policy discussions such as the COVID-19 vaccine [12].

Another important factor to assess is how perspectives shift due to new relationships with new people. For example, people being more open about their sexuality or gender means more people have interactions where they can be open about their values. Subsequently, this shifts the general population. This phenomenon is called shifting the Overton window, where the said window is the range of acceptable policies to the public [13]. Certain agents positing extreme or new positions may shift what is acceptable for the general public. An example of this is the following paper which followed groups in Turkey and how friends can shift group perspectives and reduce prejudice [14]. People are now able to tell stories they were not comfortable before and this means opinion dynamics is only growing in importance in terms of shifting norms and prejudices. This shift is something that has to be accounted for in the models.

How opinions evolve is intrinsically connected to an agent's personality. Being a leader or a follower, being confident or unsure, being loud or quiet. Thus there are a variety of papers investigating how personalities influence different opinion shifts and how to simulate this in a mathematical context. This is addressed later in the paper. Chen, Glass and McCartney investigated reputation, stubbornness, appeal and extremeness in the following paper, as indicators for how effective they would be at shifting group opinion [15]. Results show that successful opinion leaders should have low stubbornness, lower extremeness and high appeal as expected. Another paper, [16] investigated similar personality traits but in the context of groups, so stubborn groups were found to be moderating instead of polarizing for example. This literature is important to understand the kind of things that opinion dynamics models have to simulate.

2.2 The Hegselmann-Krause Model

Opinion dynamics has been a long standing branch of psychology. One of the first models of opinion dynamics can be traced back to French (1956) [17] who used his model to simulate something he referred to as social power. Another early example was Katz and Lazarsfeld (1955) [18] who used the model throughout the 40s to investigate how 'opinion leaders' shift conformity. Since then a variety of other models were introduced such as De Groot (1974) [19] and Lehrer (1975) [6]. De Groot's model is still in use today, here it was used in the context of assessing reactions to COVID policies [20]. All of these models were, however, linear. The first non-linear model was Krause (1997) [21] and was popularised in 2002 [22] and called the Hegselmann-Krause model. This was a collaboration between philosopher Rainer Hegselmann and mathematician Ulrich Krause. Hegselmann was aware that the idea of a non-linear model being an improvement predated him and dated back as early as Abelson (1964) [4]. The non-linearity allowed better simulations of real human behaviour, which is often irrational or unintuitive. The irrationality of human decision making was shown extensively by the Nobel Prize winner Daniel Kahneman, who showed precisely that human behaviour is often illogical [23].

The HK model has also seen many adaptations and improvements to address different problems. This is part of why it has remained such a significant model in opinion dynamics over time. It has both a discrete time and a continuous time form. It can also be applied to an infinite number of agents. However, this project will focus on the 3 agent case. This paper discusses how the HK model being heterogeneous allowed for it to better represent behaviour as it accounts for more individual traits that determine where opinions shift. In this paper, we will also compare the homogeneous model to the heterogeneous case. Hou (2021), [24] improves upon the HK model by adding scenarios where agents may express stances against their true values which complicates the model significantly.

The Hegselmann-Krause extends further than mathematics. It has been shown to be useful recently in the medical industry in fields as pressing as cancer research [15], specifically to predict chemotherapeutic resistance. Furthermore, it has been used in more obvious applications such as business and finance [25], where modelling opinions is an important aspect. It has also been used to track sentiment analysis of huge internet outrages such as that of the Suez Canal blockage of 2021 [26]. The model was able to assess the sentiment of social media posts regarding the news story and thus track public discourse as the ordeal progressed. As was previously mentioned, the application of opinion dynamics models to places such as the internet is more important than ever before. This shows more than ever why the Hegselmann-Krause model is an important model to investigate.

2.3 Sliding Solutions

The theory on sliding solutions that will be focused on in this paper is from Filippov's book 'Differential Equations with Discontinuous Righthand Sides' [27]. This book details, what are aptly named, 'Filippov solutions', how they are defined and how to deal with them. However, this book is not specific enough to this case and thus only the general theory is applied.

To spot sliding solutions, the concept of Lie derivatives is used as is explained by Da Silva et al. [28]. This concept uses the gradient transpose of the boundary to show that the vector fields on either side have opposite directionality. It gives useful conditions to spot sliding solutions and assess when they might appear within the Hegselmann-Krause model. This paper is used in tandem with the following [29] which also details Filippov solutions in an accessible manner and the ways to approach them.

3 Preliminaries

Definition 3.1. *A symmetric matrix is a square matrix whose transpose is equal to itself. For matrix M, this would mean $M^T = M$. A matrix which is not symmetric is referred to as asymmetric.*

Definition 3.2. *A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous at a point x = a if $\lim_{x \to a} f(x) = f(a)$. The function is continuous on an interval if it is continuous at each point in the interval.*

Definition 3.3. *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on the interval $[a,b]$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever a finite collection of mutually disjoint sub-intervals of $[a,b]$, given by $\{[x_i, y_i] : i = 1,...,n\}$, satisfies $\sum_{i=1}^{n} |y_i - x_i| < \delta$, then the following holds: $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$.*

Definition 3.4. *A piecewise linear function is a function which is comprised of some amount of linear segments defined over an equal number of intervals. Note that a piecewise linear function may have jump discontinuities. Similarly, a piecewise continuous function is one comprised of continuous segments, and a piecewise differentiable function is comprised of differentiable segments.*

Definition 3.5. *A point $x \in \mathbb{D} \subset \mathbb{R}^n$ is an interior point if there exists an open ball with centre x such that the ball is completely contained in $\mathbb{D}$. The interior of a set $\mathbb{D}$ is the union of all its open subsets, or equivalently, the set of all interior points of $\mathbb{D}$. The boundary of $\mathbb{D}$ is the set of points x such that every neighbourhood of x contains at least one point that is in $\mathbb{D}$ and one point that is not in $\mathbb{D}$.*

Definition 3.6. *A subset V of the vector space $\mathbb{R}^n$ is called convex if the line segment between any two points in V is also in V.*

Definition 3.7. *A conical combination of vectors $x_1,...,x_n$ is a vector of the form $\theta_1 x_1 + \theta_2 x_2 + ... + \theta_n x_n$ where θ_i are non-negative real numbers. Given λ points $\{v_1,...,v_\lambda\}$ where $v_l \in \mathbb{R}^n, \lambda \in \mathbb{N}$, a conical hull, denoted cone$\{v_l\}_{l=1}^{\lambda}$, is the set of points $v \in \mathbb{R}^n$ such that $v = \sum_{l=1}^{\lambda} \theta_l v_l$.*

Definition 3.8. *A convex hull, denoted conv$\{v_l\}_{l=1}^{\lambda}$, is the conical hull in which $\sum_{l=1}^{\lambda} \theta_l = 1$, for $\{v_l\}_{l=1}^{\lambda}$ and θ_i. Equivalently, the convex hull of a given finite set S is the intersection of all the convex sets which contain S.*

Definition 3.9. *Using the same notation as the definition above, and considering θ_l as 'weights,' a convex polyhedron is the convex hull containing all the weighted averages of a finite set of points. The face of a polyhedron P refers to a planar region which is a subset of P, and at least one dimension lower. Then, a face with dimension 0 is equivalent to a vertex and a face with dimension 1 is an edge.*

Definition 3.10. *A polyhedral partition of a set $X \subseteq \mathbb{R}^n$ is a group of polyhedra $\{X_s\}_{s=1}^{S}$*

for $S \in \mathbb{N}$, so that $X = \cup_{s=1}^{S} X_s$, under the conditions that the interior int(X_s) is non-empty, int(X_s) $\cap$ int(X_r) $= \varnothing$ for $s \neq r$, and the intersection of two polyhedra is either empty or a common face.

Definition 3.11. ***(Euler's Explicit Method)***
For an initial value problem $\dot{x} = f(t,x)$ with $f(t_0) = x_0, t_0 = 0$, we choose a step size h (the smaller h is the usually better results), then generate x_{k+1} using the formula $x_{k+1} = x_k + f(t_k, x_k)h$.

Definition 3.12. *A graph, denoted G = (V,E), is comprised of*

1. *a set of n vertices V = {1, 2, ..., n}*

2. *a set of edges E, where $E \subseteq V \times V$.*

Within a graph, the neighbouring vertices (or adjacent vertices) to a given vertex are those which are directly attached via an edge as follows.

Definition 3.13. *The neighbour set of a vertex i is expressed as*

$$N_i = \{j \in V \mid (j,i) \in E\}.$$

A path from vertex i to vertex j is the sequence of adjacent vertices which start at i and terminate at j. If a path can be found between i and j, we say these vertices are connected. The graph is called undirected if $(i, j) \in E$ implies $(j,i) \in E$, and directed otherwise. Direction is shown via an arrow, as opposed to a line. Simple graphs are undirected graphs which have no loops (i.e. no edges which start and ends at the same vertex) and have a maximum of one edge between any two vertices.

Definition 3.14. *The degree of a vertex v_i is equal to the number of edges which meet with that vertex, denoted by deg(v_i), in the other words this is the cardinality of N_i.*

Definition 3.15. *The Degree matrix of a simple graph G = (V,E), is the $n \times n$ matrix defined as*

$$D_{ij} = \begin{cases} deg(v_i) & i = j \\ 0 & otherwise. \end{cases}$$

Definition 3.16. *The adjacency matrix of a simple graph G = (V,E), is the $n \times n$ matrix A in which the element A_{ij} has the value 1 if the vertices v_i and v_j are adjacent, and $A_{ij} = 0$ if they are not. The adjacency matrix of a simple graph has 0 on the diagonal.*

Definition 3.17. *The Laplacian matrix of a simple graph G above, is the $n \times n$ matrix given by L = D - A, where D is the degree matrix and A is the adjacency matrix. Equivalently, this is given element-wise:*

$$L_{ij}(G) = \begin{cases} deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.18. *For any interval I = [a,b], in the set $\mathbb{R}$ let $l(I) = b - a$ denote its length. For any subset $S \subseteq \mathbb{R}$ the Lebesgue outer measure $\lambda^*(S)$ is defined as an infimum $\lambda^*(S) = inf\{\sum_{i=1}^{\infty} l(I_K)\}$ where I_k is a sequence of open intervals with $S \subset \bigcup_{i=1}^{\infty} I_k$. Having measure zero refers to having a Lebesgue Measure as the null set.*

Definition 3.19. *An equilibrium of a system is a solution which does not change with time. The differential equation $\dot{x} = f(x)$ has an equilibrium solution or point at $x(t) = x_0 \in \mathbb{R}^n$ if $f(x_0) = 0$.*

Definition 3.20. *A hyperplane is a subspace with 1 dimension less than its ambient space. In $\mathbb{R}^3$ a hyperplane would be two dimensional.*

Note: In the following definitions we will be referring to a piecewise linear system of the form:

$$\dot{x} = A_s x + b_s,$$

where $A_s \in \mathbb{R}^{n \times n}, b_s \in \mathbb{R}^n, x \in X_s$ and $s \in \Sigma$. We have that $\{X_s\}_{s=1}^S$ is the polyhedral partition of $\mathbb{R}^n$ and $S \in \mathbb{N}$ is finite size of the partition. Let $\Sigma = \{1, 2...S\}$. The index set of current modes at $x \in \mathbb{R}^n$ is defined as $\Sigma^x = \{s \in \Sigma \mid x \in X_s\}$. The system can now be rewritten as:

$$\dot{x} = \{A_s x + b_s \mid s \in \Sigma^x\}.$$

Definition 3.21. *The map $\xi : [t_0, T) \to \mathbb{R}$ where $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, is referred to as classical solution of the system if and only if:*

1. *ξ is differentiable on $[t_0, T)$,*

2. *and for all $t \in [t_0, T)$, $\dot{\xi}(t) \in \{A_s \xi(t) \mid s \in \Sigma^{\xi(t)}\}$.*

Definition 3.22. ***(Caratheodory Solution)***

The map $\xi : [t_0, T) \to \mathbb{R}$ where $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, is referred to as Caratheodory solution of the system if:

1. *ξ is absolutely continuous on $[t_0, T)$,*

2. *and for almost all $t \in [t_0, T)$, $\dot{\xi}(t) \in \{A_s \xi(t) \mid s \in \Sigma^{\xi(t)}\}$.*

Lemma 1. $\sum_{i=0}^{N} \binom{N}{i} = 2^N$.

4 Methodology

This paper is a progression from a previous paper investigating the stability of the HK model [5]. The previous paper assumed no sliding solutions were present to prove asymptotic stability and in this paper this assumption will be assessed. There is minimal previous research into the sliding solutions of the HK Model and that is why this paper holds a value for both mathematics and opinion dynamics. It is important to improve our knowledge of an important model and understand which assumptions are reasonable to make when investigating its stability. Thus to approach this problem, the following research questions are proposed.

4.1 Research Questions

This paper examines the following questions:

1. Do sliding solutions exist for the 3 agent HK Model?

2. If they do, what conditions are necessary for sliding solutions to appear?

3. Can the findings from this paper be extended to the HK Model with more agents?

Only 3 agents will be assessed as this is the easiest way to depict the state space with the facilities at hand, however, an extension into higher dimensions may be explored. It is common in mathematics to approach the base case to draw out information that can be implemented in higher dimensions.

4.2 Analysis Outline

To answer these questions this paper will be broken up into smaller sections. The paper will start by detailing the Hegselmann-Krause Model and how it functions.

Firstly, the discrete time model will be explored. This is the simpler and more intuitive version of the model and it helps understand the more complex continuous time model. Examples will be shown to help visualise what the model is simulating.

Following this, the continuous time model will be introduced. To do this, the influence functions will be explained along with the boundary levels and what they simulate in the real world. Then the piecewise linear model will be introduced. Next this model will be reduced by a dimension by centring the final agent to have opinion '0' and measuring the other agent's distance with respect to said agent. This allows us to depict the model in a 2 dimensional visualization called the state space. This state space is of particular importance to the study of sliding solutions.

The paper will then start investigating the occurrences of sliding solutions in said state space. To start, a sliding solution will be defined and differentiated from a Caratheodory solution, and how to spot them will be addressed.

This paper will start its investigation looking precisely at the super symmetric model where the two boundary matrices are symmetrical and the same as each other. The approach will initially be testing the individual boundary points but will become an investigation of the general zonal behaviour due to their assigned matrix. If a sliding solution is found, the conditions leading to its existence will be assessed, and whether this is a problem for the proof of asymptotic stability in the previous paper.

Finally, this paper will investigate the asymmetrical case and try to extend the findings into higher dimensions. Again the boundary points will be investigated along with the vector's orientation towards a [1,1] vector dividing the state space in half. Due to symmetry certain assumptions can be made. This is because certain agents will behave in similar ways to others. A more general argument will be made about the possibilities of sliding solutions from a real world standpoint. Then project limitations and future work will be discussed, in the context of improving our understanding of this important model in opinion dynamics.

5 The Model

As was previously discussed, the HK model is the collaboration between a philosopher, Hegselmann, and a mathematician, Krause. It was significant as it was the first non-linear opinion dynamics model.

Different models use different systems to simulate the evolution of opinions. Some models give opinions discrete values one through ten for example. Other models use a continuous space where an opinion can take any value within the closed set $[0,1]$.

The unique property of the Hegselmann-Krause model is the property of egocentric trimming. This means that agents are only influenced by agents with an opinion close enough to their own. An agent within this range is referred to as a 'Neighbour' and then the model can be detailed in graph form. The ability to put this model in graph form is precisely how this paper looks to investigate sliding solutions.

In this next section, the model will be introduced in 2 sections, the discrete time version and the continuous time version.

5.1 Discrete Time Model

The simpler version of the model is to use discrete time states. This is to say that the model is computed only at specific time values $[t, t+1\ldots]$.

The first thing to address when introducing an opinion dynamics model is how to express the agents and their interactions in a mathematical manner. To do this we describe our information as a graph with vertices and edges. Each agent is a vertice with an opinion on the scale [0,1] and if the agents are close enough to interact, then there exists an edge between the two vertices. Thus the Neighbour set describes all the agents within a certain confidence bound of the focus agent. Then the opinion at the next time step is an average of the opinion states in the Neighbour set and the focus agent's own opinion value. The model can be represented by a state dependent graph such as the following:

Vertices V = $\{$1, ...,N$\}$ with edges $E(\xi(t)) = \{(i,j) \mid i \neq j \mid \xi_i(t) - \xi_j(t) \mid \leq d\}$.

The vertices represent the agents and the edges represent whether they are within range to influence each other. $\xi_i \in [0,1]$ represents agent i's opinion and d here represents the boundary level which one has to be in to have an influence. This model is homogeneous as every agent has the same d value. The model has to calculate the various opinion states at different time steps by taking the average of the opinion values. It is important to note that this also includes the original agent's opinion as this is important for the average. The model thus looks like this:

$$\xi_i(t+1) = \frac{1}{\#N_i(\xi_i(t))} \sum_{j \in N_i(\xi(t))} \xi_j(t). \tag{1}$$

We then rewrite this model as a piecewise linear model as $\xi(t+1) = B(\xi(t))\xi(t)$ with B defined as below as $N \times N$ matrix:

$$B_{ij}(\xi(t)) = \begin{cases} \frac{1}{\#N_i(\xi(t))} & \text{if j} \in N_i(\xi(t)) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is apparent that there are some significant limitations to the discrete time model. Firstly, all agents have the same confidence bound or the model is homogeneous. It is a universal value that determines how close you are to someone else. Furthermore, the discrete time model has no direction to the influence interactions. This is not representative of how opinions evolve in real life. Different people have different personality types, confidence or meekness, stubbornness or open-mindedness, and this would mean different people have different requirements to be influenced. Additionally, some people may influence another person without being influenced themselves. Examples of this include, a politician on television or just a confident opinion leader, who expresses their stances but is not moved by rebuttals. Some papers found in the literature review investigated precisely these opinion leaders and their influences on group conformity [15].

Finally, due to it being a mere average of opinions we can find quite simple convergence within a certain range. In fact, the previous paper [5] found an upper bound for how many time steps it takes before the results converge for the discrete time model. This is again not representative of society, we have not seen complete convergence of opinion and we presumably never will.

5.2 Example

Example 1: On the right is an example of the discrete time model of five agents. This image has the opinion values at time step zero as $\xi(0) = (0.1, 0.4, 0.7, 1, 0.5)$.

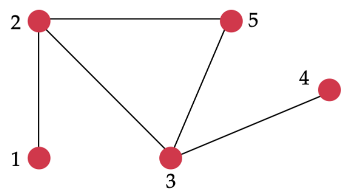

Figure 1: Simple network with 5 agents

This image has boundary level $d = 0.3$ so we can translate this into the diagram on the right. We use $\mid \xi_i(t) - \xi_j(t) \mid \leq d$ for $t = 0$ and $d = 0.3$. The edges on the graph show the influences between the agents.

We see Agent one is only close enough to interact with agent two and agent four is only close enough to agent three. Agents two, three and five are all close enough to influence each other. Note that each agent is always influencing themselves in this model. Thus the Neighbour sets look like:

- $N_1(0) = \{1, 2\}$
- $N_2(0) = \{1, 2, 3, 5\}$

- $N_3(0) = \{2,3,4,5\}$
- $N_4(0) = \{3,4\}$
- $N_5(0) = \{2,3,5\}$

The evolution over the future time steps is calculated like this for each agent. Below is agent one's opinion calculated at time step one.

$$\xi_1(1) = \frac{1}{\#N_1(\xi_1)} \sum_{j \in N_1(\xi)} \xi_j(0) = \frac{1}{2}(0.1+0.4) = \frac{1}{4}.$$

We thus calculate this for each agent and find:

$$\xi(1) = (\frac{1}{4}, \frac{1}{4}(0.1+0.4+0.7+0.5), \frac{1}{4}(0.4+0.7+1+0.5), \frac{1}{2}(0.7+1), \frac{1}{3}(0.4+0.7+0.5)).$$

Which becomes $\xi(1) = (0.25, 0.425, 0.65, 0.85, 0.5\dot{3})$. Note that each of the agent's opinions here shift towards 0.5 aside from agent five who is pulled away from the centre as agent three has a more extreme opinion than agent two. This is showing the expected behaviour when all agents are connected somehow, which is that over time they will all converge past a specific time step.

5.3 Continuous Time Model

To improve on the discrete time model, the model is changed into an ordinary differential equation. This means that the state variable is defined at any time interval. We also introduce a more complicated boundary system to determine whether agents are 'neighbours' or not. Each agent now has two boundary values for each other individual agent. They have different boundaries for different agents to represent how an agent's malleability can change depending on who they are interacting with. An agent may respect another agent's opinion more than someone they dislike even if this opinion is further away from their own. An example of this would be being friends with someone changes how seriously you take their opinion [14]. One might be homophobic but upon discovering a friend is gay their opinion on similar issues may shift. Thus they need different boundaries for each agent. They have two values for each individual to represent their malleability in different directions. One may be a centrist, but be more open to being influenced by a 'left' position than a 'right' position for example. That is why each agent has two boundary values d_{ij}^C which symbolises competitiveness and d_{ij}^G symbolising generosity, where i and j are agent numbers.

Now it is required to alter these boundary ranges into a binary system i.e. whether or not the agent is influenced. To do this, influence functions are used. These are functions that assess two agents. They return one if agent j's opinion is close enough to that of agent i and zero if otherwise. This allows us to 'weigh' the influence of each agent with respect to the original agent. Within this, there are now the unique influence boundaries discussed above for each agent. The influence functions look like this:

$$\phi_{ij}(\xi_i,\xi_j)=\begin{cases}1 & \text{if} -d_{ij}^G \leq \xi_j-\xi_i \leq d_{ij}^C\\ 0 & \text{if otherwise.}\end{cases} \tag{3}$$

This means the corresponding dynamics are now given by the following equation. Note that values of ϕ_{ii} are taken as zero and not included because we are no longer accounting for agents influencing their own opinions.

$$\dot{\xi}_i=\sum_{j=1}^{N}\phi_{ij}(\xi_i,\xi_j)(\xi_j-\xi_i). \tag{4}$$

This equation takes into account the distance between the two agents if they are close enough to interact. Here the time dependency of $\dot{\xi}_i$ is excluded to maintain simplicity. Now the evolution from the discrete time model is clear as now there are directions to the interactions and we have boundary levels that are more representative of how people actually behave. Another difference is we can now exclude the agent's opinion when calculating their evolution as we are no longer using the previous method. This formulation still takes into account the distance between opinions when predicting the shift as an interaction with a more extreme opinion would shift you more than a moderate one. In general, this means that convergence is now more complicated.

Now we look to express $\dot{\xi}_i$ in matrix form using the following matrix.

$$L_s=\begin{bmatrix}\sum_{j=1}^{N}\phi_{1j} & -\phi_{12} & \dots & -\phi_{1N}\\ -\phi_{21} & \sum_{j=1}^{N}\phi_{2j} & \dots & -\phi_{2N}\\ \vdots & \vdots & & \vdots\\ -\phi_{N1} & -\phi_{N2} & \dots & \sum_{j=1}^{N}\phi_{Nj}\end{bmatrix}. \tag{5}$$

Here S is the total combinations of influence functions and s=1,2... S. This means s is dependent on ξ's position in the space. This leads to the following piecewise linear form for the model, where ξ is each of the opinion values in vector form:

$$\dot{\xi}=-L_s\xi. \tag{6}$$

In simple terms, this model takes agent interactions to formulate a matrix. Then by multiplying this matrix by the initial opinion positions of the N agents we receive back a vector direction in which the agent will travel.

5.4 Dimension Reduction

The workings of the model in three dimensions are already hard to picture. To combat this we reduce the dimensions of the model to the intuitive two dimensional model. This is done by centring the last agent and measuring the other agents with respect to this last agent. This adds an equilibrium point at the origin, so agent N is given opinion state zero and previous agents' values are now their distance to the opinion of agent N given variable x_i.

$$x_i = \xi_i - \xi_N, \tag{7}$$

$$x_i - x_j = (\xi_i - \xi_N) - (\xi_j - \xi_N) = \xi_i - \xi_j. \tag{8}$$

As we can see the differences between all agents are conserved within this dimension reduction. It is also important to note that $\phi_{ij}(\xi_i, \xi_j) = \phi_{ij}(x_i, x_j)$. The next step to formulate the linear piecewise model is to find the derivative of $\dot{x}_i$ and we use equation (4) of $\dot{\xi}_i$.

$$\begin{aligned}
\dot{x}_i &= \dot{\xi}_i - \dot{\xi}_j \\
&= \sum_{j=1}^{N} \phi_{ij}(\xi_i, \xi_j)(\xi_j - \xi_i) - \sum_{j=1}^{N-1} \phi_{Nj}(\xi_N, \xi_j)(\xi_j - \xi_N). \\
&= \sum_{j=1}^{N} \phi_{ij}(x_i, x_j)(x_j - x_i) - \sum_{j=1}^{N-1} \phi_{Nj}(x_N, x_j)(x_j).
\end{aligned}$$

This leads to the following definition for the derivative $\dot{x}_i$:

$$\dot{x}_i = -(\sum_{j=1}^{N} \phi_{ij} + \phi_{Ni})x_i + (\sum_{j=1, j\neq i}^{N-1} \phi_{ij} - \phi_{Nj})x_j. \tag{9}$$

The influence functions are discontinuous by nature, meaning they jump immediately from value 0 to 1. This means that we have to partition the state space $\mathbb{D}$ = [-1, 1]$^{N-1}$ into polyhedral regions X_s where S is the total number of influence functions. Now we apply the formulas found during the dimension reduction to try and define the state space in 2 dimensions and leading to the following definition for X_s and the state space.

$$X_s = \{x \in \mathbb{R}^{N-1} \mid H_s x + g_s \leq 0\}. \tag{10}$$

In this definition $H_s \in \mathbb{R}^{(N_s+2N-2)\times(N-1)}$ and $g_s \in \mathbb{R}^{N_s+2N-2}$. This comes from the number of influence function inequalities combined with the 2(N-1) inequalities from the state boundaries giving $-1 \leq x \leq 1$ for all $i = 1, 2 ... N-1$.

This is useful to investigate the three agent case which is what this paper focuses on. The three agent case can now be visualised on two axes. The previous matrix L_s now becomes A_s:

$$A_s = \begin{bmatrix} -\sum_{j=1}^{N}\phi_{1j} - \phi_{N1} & \phi_{12} - \phi_{N2} & \cdots & \phi_{1,N-1} - \phi_{N,N-1} \\ \phi_{21} - \phi_{N1} & -\sum_{j=1}^{N}\phi_{2j} - \phi_{N2} & \cdots & \phi_{2,N-1} - \phi_{N,N-1} \\ \vdots & \vdots & & \vdots \\ \phi_{N-1,1} - \phi_{N1} & \phi_{N-1,2} - \phi_{N2} & \cdots & -\sum_{j=1}^{N}\phi_{N-1,j} - \phi_{N,N-1} \end{bmatrix}. \quad (11)$$

The model is now given in this new, reduced dimension, piecewise linear form:

$$\dot{x} = A_s x. \quad (12)$$

This matrix shifts as the solution crosses boundaries and thus alters as the solution evolves. This is important because with the dimension reduction we have complicated some issues. For example, if x_1 is close enough to interact with x_3 then this shifts x_2 because x_2 is measured with respect to x_3 so now the movement is more complex than in the discrete time case. We can also write s as s(x) such that $\dot{x} = F(x)$ because s changes dependent on the position of x. Later we discuss what happens when x is on a boundary and then F(x) is not well defined.

5.5 State Space

This is an example of the state space for the following boundary matrices:

$$[d_{ij}^G] = \begin{bmatrix} 0 & 0.5 & 0.6 \\ 0.4 & 0 & 0.5 \\ 0.3 & 0.4 & 0 \end{bmatrix} [d_{ij}^C] = \begin{bmatrix} 0 & 0.3 & 0.2 \\ 0.3 & 0 & 0.1 \\ 0.2 & 0.1 & 0 \end{bmatrix}.$$

Each of the zones are labelled on the figure. 31 indicates that agent 3 is being influenced by agent 1 in the scenario, i.e. $\phi_{31} = 1$. The axes are given by x_1 and x_2 which are measured with respect to x_3. That means x_3 in this depiction is at [0,0]. The graph is now measured from -1 to 1 in the axis because within [0,1] x_1 can be both +1 and -1 from agent 3 so we need to expand the domain. This does mean that in the top right and bottom left corners we have no solutions past the dotted lines 0 to 0 in the top left and bottom right directions. However, this definition of the state space is problematic because the boundary lines are not well defined. This is because, as mentioned before when x is on a boundary the $s(x)$ can be the dynamics on either side. That is, when a point is precisely on the boundary, it can actually choose which zone dynamics to act with. That means that when boundaries are investigated for sliding conditions the points along the boundaries have to be considered with the dynamics of each of the adjacent zones.

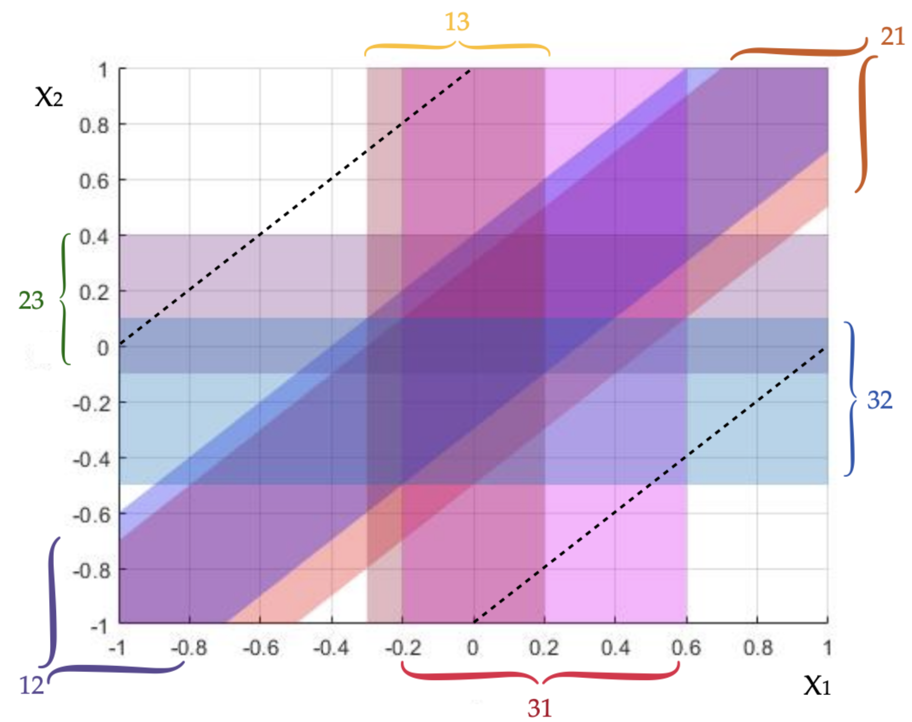

Figure 2: Example of State Space for Asymmetric Model

5.6 Example

Here we see a simple example of the 3 agent case. On the state space image in figure 2, this interaction would place us in the orange diagonal polygon where the zones 31, 13, 32 and 21 intersect. Using the boundaries above let us take the opinions states to be $[x_1, x_2, x_3]^T = [0.1, -0.3, 0]^T$. This would give us precisely the relationship given in figure 3. We can also calculate the direction that this solution will take in future time steps. The matrix defining this relationship is given below:

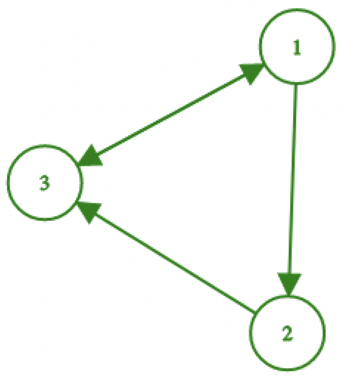

Figure 3: Example of sliding solutions

$$A_s = \begin{bmatrix} -\sum_{j=1}^{3} \phi_{1j} - \phi_{31} & \phi_{12} - \phi_{32} \\ \phi_{21} - \phi_{31} & -\sum_{j=1}^{3} \phi_{2j} - \phi_{32} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A_s \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.3 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.2 \end{bmatrix}.$$

This shows that the solution will move with respect to vector $\begin{bmatrix} -0.5 \\ 0.2 \end{bmatrix}$, so x_1 will decrease in value and x_2 will become larger albeit at a slower rate. This solution will move towards the boundary where 2 will begin to influence 1 and then the matrix A will change to include the interaction 12. Then they will converge to opinion value 0 because they will interact with agent 3 who is centred at 0. This is how the model simulates the evolution of opinions in the continuous time case.

6 Results

The formulation of the continuous time model has given us many benefits as has just been detailed. However, some problems arise with this new model. The influence function ϕ only has values from 0 to 1 and very small differences in opinion can switch the result from 1 to the other. This gives the function a discontinuous right hand side, and it means the model is now only piecewise linear. This causes problems for the solution set of the model.

Solutions over a piecewise linear model now have possibilities of sliding and Zeno behaviour [30]. Suddenly, the existence of a global Lyapunov function is not guaranteed and finding it becomes more difficult. This is why the paper that precedes this one [5] assumed the non-existence of sliding and Zeno behaviours. However, this assumption is not guaranteed, thus next this paper will discuss sliding solutions in the Hegselmann-Krause Model.

6.1 Sliding Solutions

In this section, we will introduce different solution types and how they can be spotted. Note that in this section we are referring to the model form:

$$\dot{x} = A_s x. \tag{13}$$

Using this formulation, we have that each partition X^s is a domain subset that is both closed and convex. Then we can define the index set of current modes at $x \in \mathbb{D}$ where $\mathbb{D} = [\text{-}1,\ 1]^{N-1}$. This is defined by $\Sigma^x := \{s \in \Sigma \mid x \in X^s\}$, as in [31]. This is using that $\Sigma = 1, ..., S$. Thus Σ^x contains the index of the polygon for all x within the interior of X^s, but when x lies on the boundary, Σ^x includes the indices of all polygons which contain that point, or the polygons on both sides. Now we recall these 2 definitions from the Preliminaries to discuss them.

Definition 6.2. *Caratheodory Solution*

The map $\xi : [t_0, T) \to \mathbb{R}$ where $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, is referred to as Caratheodory solution of the system if

1. *ξ is absolutely continuous on $[t_0, T)$*
2. *for almost all $t \in [t_0, T)$, $\dot{\xi}(t) \in \{A_s \xi(t) \mid s \in \Sigma^{\xi(t)}\}$.*

Definition 6.3. *The map $\xi : [t_0, T) \to \mathbb{R}$ where $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, is referred to as Classical solution of the system if and only if*

1. *ξ is differentiable on $[t_0, T)$,*
2. *for all $t \in [t_0, T)$, $\dot{\xi}(t) \in \{A_s \xi(t) \mid s \in \Sigma^{\xi(t)}\}$.*

Here we see that a classical solution is also a Caratheodory solution, as everywhere is a stronger condition than almost everywhere and if ξ differentiable on $[t_0, T)$ then it is continuous on $[t_0, T)$. These are the solutions that are focused on for the proof of Asymptotic Stability. It is important to note that in our state space a classical solution is quite rare as it requires that the solution continues over the boundary without losing differentiability at that point. So for this model we mainly focus on Caratheodory solutions. However, it has been shown [30] that the existence of Caratheodory solutions for all initial values is not guaranteed even in Piecewise systems with maximal Caratheodory solutions. This paper avoids this issue by convexifying the differential inclusion. This also has the added advantage that all solutions become global. However, this is where we encounter different types of solutions.

Definition 6.4. *The map $\xi : [t_0, T) \to \mathbb{R}$ where $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, is referred to as a Filippov solution of the system if*

1. *ξ is absolutely continuous*
2. *for almost all $t \in [t_0, T), \dot{\xi}(t) \in conv\{A_s\xi(t) \mid s \in \Sigma^{\xi(t)}\}$*

Definition 6.5. *The map $\xi : [t_0, T) \to \mathbb{R}$ is sliding is referred to as a sliding solution of the system if and only if*

1. *it is not a Caratheodory solution on $[t_0, T)$*
2. *there exists an index set $\Sigma^{\xi(\cdot)}_{slide} \subseteq \Sigma$ such that $\Sigma^{\xi(\cdot)}_{slide} = \Sigma^{\xi(t)}$ for all $t \in (t_0, T)$*
3. *$\dot{\xi}(t) \in conv\{A_s\xi(t) \mid s \in \Sigma^{\xi(t)}\}$ for almost all $t \in [t_0, T)$.*

Here *conv* indicates the convex hull of the set. We see that Caratheodory solutions are Filippov solutions, but sliding solutions are explicitly not Caratheodory. In less formal terms, Filippov solutions are absolutely continuous curves, where directions of the vector field in the neighbourhood of a point x are applied to the value. This is useful as it applied directly to the concept of the state space that was previously introduced.

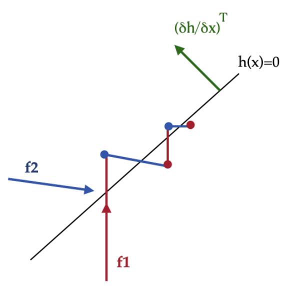

Figure 4: Example of sliding solutions

At a boundary there are 4 different scenarios to consider. Let us take the boundary in question to be a function $h(x) = 0$. When $h(x) < 0$ we have $f_1(x)$ as the value of our solution and when $h(x) > 0$ we take $f_2(x)$. This is precisely what happens on both sides of a boundary in our state space. The state space here is divided into the two subspace polyhedrons on either side of the boundary, Σ^1 for $f_1(x)$ and Σ^2 for $f_2(x)$. The different scenarios are: f_1 and f_2 both push away from the boundary; f_1 pushes towards the boundary while f_2 pushes away; vice versa; finally f_1 and f_2 both push towards the boundary. The last option is where sliding occurs. It should be noted that either f_1 or f_2 can move in the same direction as the boundary, but this does not pose a problem either.

As suggested above a sliding solution is caused by opposite pressure in two zones on both sides of the boundary. If both zones push towards the boundary line then as soon as the solution enters the other zone it is pushed back into the first zone and so on. This causes sliding behaviour across the switching surface. This is not Caratheodory solution behaviour and so it poses a problem. There is an easy way to check for sliding behaviour. We use the gradient transpose of the boundary which is the green vector. Then the inner product of the gradient and the f_i solution will be positive if they travel in the same planar direction. Thus for sliding we need f_1 to have a positive inner product with the normal vector and f_2 to have a negative inner product. This explanation is often referred to as the Lie derivatives. Formally written:

$$[\frac{dh}{dx}^T f_1(x)] > 0,$$
$$[\frac{dh}{dx}^T f_2(x)] < 0.$$

.

This causes attractive sliding motion. Filippov's book [27] states that this is a solution to the differential equation:

$$x'(t) = \begin{cases} f_1(x(t)) & h(x) < 0 \\ f_F(x(t)) & h(x) = 0 \\ f_2(x(t)) & h(x) > 0. \end{cases} \tag{14}$$

We see that this is applicable to our model. He has that the vector field at the surface is the convex hull (*conv*) of f_1 and f_2 such that:

$$f_F(x(t)) = [\alpha f_2(x) + (1-\alpha) f_1(x)].$$

Here $\alpha \in [0,1]$ chosen such that $\frac{dh}{dx}^T f_F(x) = 0$. Thus $\alpha(x)$ is given by:

$$\alpha(x) = \frac{\frac{dh}{dx}^T f_1(x)}{\frac{dh}{dx}^T (f_1(x) - f_2(x))}.$$

.

This is how Filippov defines his sliding solutions and this method of opposite inner products is the method this paper will use to investigate the existence of said solutions in the HK model.

6.6 Super Symmetric Model

6.6.1 Preliminaries

Here are some lemmas, claims and basic notation that will be useful in the following section.

Notation

- Zone where agent one is influenced by agent two is denoted by zone 12.
- Zone where agents one and two influence each other is denoted by zone 121.
- Zone where agent one influenced by agent two and agent three is given as zone 12+13.
- The vectors [1,1], [1,0] or [0,1] all refer to the lines in the direction of these vectors through the state space point (0,0).

Lemma 2. *: If the boundary points $u = (x_1, y_1)$ and $v = (x_2, y_2)$ multiplied by their zone matrix and the normal vector (n) to the boundary are positive, then the whole boundary is positive with respect to the normal vector.*

Proof

Assume $n^T Au > 0$ and $n^T Av > 0$.

Any point along the boundary can be written as $\alpha u + (1 - \alpha)v$ where $\alpha \in [0, 1]$.

If $n^T Au > 0$ and $n^T Av > 0$ then
$n^T A(u + (1 - \alpha)v) = n^T Au + (1 - \alpha)n^T Av$ which is > 0. □

Claim

When zone a has matrix A and zone b has matrix B, then the zone of a overlapping b has matrix A+B, where a and b have none of the same interactions. This can be seen from the formulation of $\dot{x}$ and A_s.

Lemma 3. *The addition of 2 positive matrices with respect to a normal vector n, is still positive with respect to n.*

Assume $n^T Au > 0$ and $n^T Bv > 0$ for all u in the zone for A and v for the zone with B.

Then we have $n^T (A + B)w > 0$ where w is in the zone of both interactions.

Proof

Assume $n^T Au > 0$ and $n^T Bv > 0$.

Then we have $n^T(A+B)w = n^T Aw + n^T Bw$ which is > 0 because w is within zone for the A matrix and the zone for the B matrix. Thus $n^T Aw > 0$ and $n^T Bw > 0$ for all w and the addition of them is positive also. $\square$

Claim

There is a symmetry of agents 2 and 1 with respect to 3.

The matrices and vector spaces of interactions between agents 2 and 1 with respect to 3 are symmetrical as they are both measured from the point of equilibrium. On our graph one interaction is represented by a horizontal hyperplane and the other by a vertical hyperplane. However, the behaviour within them should be the same and for this to occur the matrices have the following vector swap.

$$13 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, 23 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$31 = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, 32 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}.$$

This is also true for 21 with respect to 12.

$$21 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, 12 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

6.6.2 Boundary Testing

The first scenario of the model to investigate is what is called the Super Symmetric model. This model has that if agent i influences agent j, then agent j influences agent i. With these conditions the continuous time model starts to look like the discrete time model. There are no one way influences and this reduces the zones and boundary points of the state space.

The first method we approach is to test each of the boundary points using the reduced dimension model and multiplying the zonal matrix A by the points for x_1 and x_2, which again are the distances of agents 1 and 2 from agent 3. We thus get the direction of movement of these boundary points which, as we proved above, summarizes the whole boundary if they are both positive with respect to the normal vector.

In Figure 5, an example of the super symmetrical model is given where every boundary value is equal to 0.5. For a model to be super symmetric note that not all boundaries have to be equal as long as the d^C_{ij} and the d^G_{ij} matrices are symmetric (equal to their transpose) and equal to each other. The arrows on the diagram are the directions of the

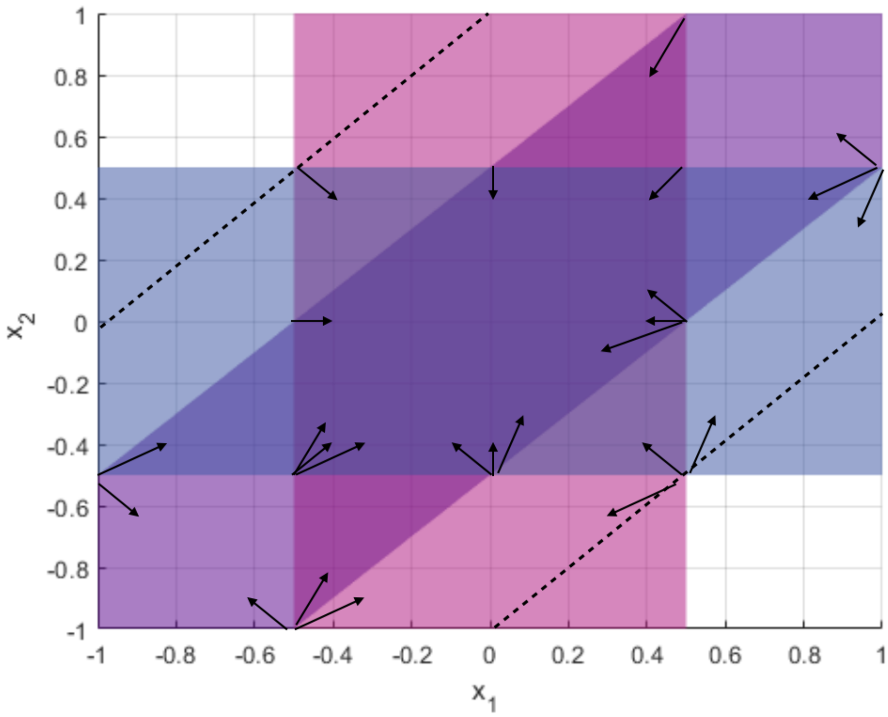

Figure 5: Super symmetric model, $d_{ij}^G = d_{ij}^C = 0.5$

boundary points with respect to each zone. However, each of these boundary points has multiple directions depending on which zone's dynamics are chosen at the boundary. The multiple directions are shown only for points beneath the [1,1] vector through (0,0). This is because with this level of symmetry we have simply reflective behaviour.

For an example of these multiple dynamics, take point $[0,-0.5]$. This zone can choose multiple different dynamics. It can have the direction of zone 121, 131 or 323 or a mixture of the 4. We will explain why none of these boundary points are an issue.

First, here is an example of how the boundary point is calculated. Take the intersection point at $[x_1,x_2]^T = [0.5,-0.5]$. This intersection point can be the zone where agents 13 interact with each other and 23 interact with each other. It can also be in the zones of these 2 properties individually. It could also, in theory, be a point where no agents interact (the white corner), however, there is no movement there so there is no problem. Thus we take the matrix A_s from (11) and plug in N=3 and our interactions.

$$A_s = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}, [x_1,x_2]^T = [0.5,-0.5],$$

$$\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}.$$

This means that this point moves with gradient -1 towards the top left corner of the graph in this zone. This is shown by the arrow on the diagram. When this solution reaches a new zone the matrix will change and the behaviour will change until it reaches a stationary point. However, since it is a boundary point it can also be represented by the dynamics of the zones with merely 131 or merely 232. To be sure that no sliding solutions exist at this point we can test the boundary point with the 2 other zones that it is on.

$$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}.$$

In the first case the solution moves into the pink zone with the same gradient shown by the arrow on the graph and in the second case there is the same behaviour reflected over [-1,1]. Thus, the direction for 131 points away from both the vertical and horizontal boundary at that point and zone 232 behaves in the same manner. In fact, one example of the direction being away from the boundary is enough to discount sliding because for sliding to occur both vector fields need to push towards the boundary. Remember that also if two boundary points at the beginning and end of the boundary have the same direction then this is true for the whole boundary. This means all of the boundaries on the graph have sliding problems because all boundaries point within their own zone. This method can be used at every point on the graph to show no sliding occurs.

What is noted is that the 131 zone, the 121 zone and the 232 zone in isolation have counter flow about $x_1 = 0$ or $x_2 = 0$ and the [1,1] vector. The dynamics here make sense as agents 1 and 2 are close enough to interact without interacting with agent 3 so they become closer until they have the same opinion and then they stop shifting. Thus when points reach the central dividing line, they stop moving. So the point $[0, -0.5]$ with dynamics of the zone 131, doesn't move. This behaviour is true for all 3 of these simple zones. One might ask whether sliding can thus be formed by adding boundaries at these central dividing lines in an asymmetric case, however, despite this having opposite pressure it would not be a sliding solution. This is because the opposite pressure has to be at the boundary itself not just in the adjacent zones and at the boundary through 0, with dynamics 131/232/121, there is no movement. Even if there is movement, which can occur with more zones over this 0 boundary, the behaviour wouldn't be sliding it would simply travel along the boundary as a Caratheodory solution instead of having the infinite switching behaviour shown before.

This method is nice but it does not prove non-existence in other versions of the super symmetric model. To extend this we must make zonal observations as opposed to merely calculating boundary points. These zones have the same internal vector fields regardless of where they shift on the graph so if we can prove that none of these zones pose a sliding problem then we can show no sliding solutions exist for the super symmetric case, wherever the zones move.

6.6.3 Zonal proof

We have noticed that each of the vector fields point generally towards the line [1,1] through (0,0). This makes sense as when the opinions are close enough they start to converge towards the centre of our state space as shown in the previous thesis [5]. Other papers have also shown that the model converges once you are within a certain zonal depth of interaction [30]. If we can show each of these zones points towards this diagonal vector for all points $a,b \in [-1,1]$ then we can discount sliding solutions along the boundaries with gradient 1. Then again if we prove all vectors point towards a central diving vector [0,1] through (0,0) we can discount sliding solutions along vertical boundaries. We have also noted in previous sections that the relationships 23 and 32 are symmetric in behaviour to the relationships 13 and 31 as they are both agents measured with respect to our equilibrium point. Thus if no sliding solutions can occur along the vertical boundary then none can occur along the horizontal boundary. Another simplification is to consider only the points below [1,1] through (0,0) because the state space is symmetric over this line. Thus we can assume for all zones that $a > b$.

Below we show 3 more examples of the super symmetric model, to show that the zones only alter in size when the boundary values change but the internal vector fields do not and we can still divide the map by the [1,1] vector through (0,0). The boundary matrices of these spaces are also given below respectively.

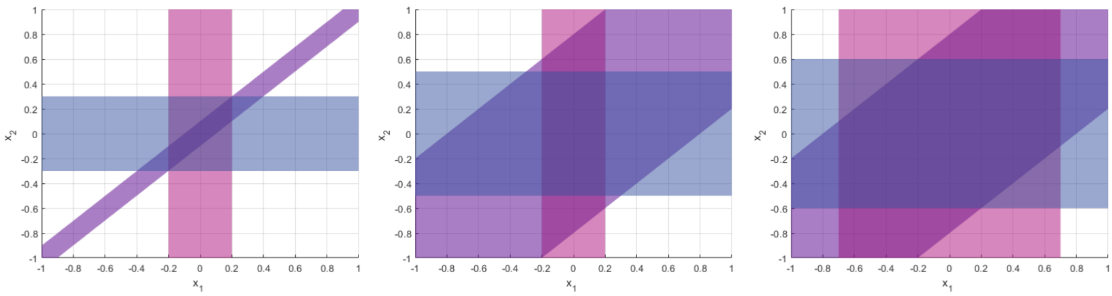

Figure 6: Three examples of the super symmetric model

$$[d_{ij}]^C = [d_{ij}]^G = \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0.1 & 0 & 0.3 \\ 0.2 & 0.3 & 0 \end{bmatrix}, [d_{ij}]^C = [d_{ij}]^G = \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0.8 & 0 & 0.5 \\ 0.2 & 0.5 & 0 \end{bmatrix}, [d_{ij}]^C = [d_{ij}]^G = \begin{bmatrix} 0 & 0.8 & 0.7 \\ 0.8 & 0 & 0.6 \\ 0.7 & 0.6 & 0 \end{bmatrix}.$$

In the super symmetric case, there are 3 hyperplanes, zones 121, 131 and 232. Every other zone is a different intersection made up of these 3. That means the total zones are $2^3 = 8$. This is because $\Sigma_{i=0}^{N}\binom{N}{i} = 2^N$ as shown in the preliminaries. The zones are:

$$131 = \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}, 232 = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix}, 121 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$121+232=\begin{bmatrix}-1 & 1\\ 0 & -3\end{bmatrix}, 121+131=\begin{bmatrix}-3 & 0\\ 1 & -1\end{bmatrix}, 232+131=\begin{bmatrix}-2 & -1\\ -1 & -2\end{bmatrix}$$

$$131+232+121=\begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}, \varnothing = 0_{2\times 2}.$$

The empty zone is the zone where no agents interact with each other and so trivially there is no movement here and the matrix is zeros. It was shown at the beginning that when the zones overlap the matrices are just simply added together. We also note a symmetry between zones 13 and 23 as can be expected, the matrices transpose both corners, a vector swap effectively and this means that they behave in an opposite fashion, which was also shown before.

We note that most zones push towards the central dividing vector [1,1]. Thus we take the inner product of the zones beneath this vector and check if their inner product with the vector [-1,1] (the normal vector to [1,1]) is positive.

$$121=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-a+b\\ a-b\end{bmatrix} = 2a-2b$$

$$131+232=\begin{bmatrix}-2 & -1\\ -1 & -2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-2a-b\\ -a-2b\end{bmatrix} = a-b$$

$$131+121+232=\begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-3a\\ -3b\end{bmatrix} = 3a-3b.$$

These 3 zones cause no problems as each of these values is positive under the [1,1] vector because here $a > b$. The zones 131 and 232 individually we have already discussed and seen they do not pose a problem. Both of these point within themselves towards the [1,0] and the [0,1] vectors respectively. They, under certain conditions, push towards the boundary with each other, however, zone 131 + 232 causes no problems to our proof as shown above. The remaining 2 zones are shown below:

$$131+121=\begin{bmatrix}-3 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-3a+b\\ -b\end{bmatrix} = 3a-2b$$

$$232+121=\begin{bmatrix}-1 & 0\\ 1 & -3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-a\\ a-3b\end{bmatrix} = 2a-3b.$$

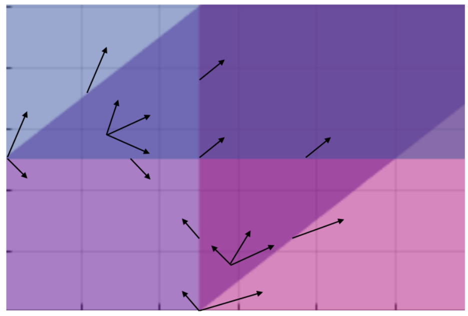

Figure 7: Zoomed view of problematic zones

These zones point towards or away from the [1,1] depending on whether or not a is positive. Remember, when the inner product is positive the vectors move in the same direction. For the first case, when $a > 0$ we have no problems so let us address when $a < 0$. In this scenario with a vector pointing away from [1,1] the solution would point towards the boundary with the 131 zone or the boundary with the 323 zone. We see the zoomed in figure 7. These two zones are the purple and blue triangles. We see that whichever boundary these zones point towards, all of the boundaries surrounding these zones point away from the boundary with the other dynamics. This means sliding is not possible in these zones.

Next, we investigate whether the zones point towards the central [0,1] dividing vector so we take the normal vector [-1,0] and take $a > 0$ as to the right and to the left of the vector we still have symmetry.

$$121 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -a+b \\ a-b \end{bmatrix} = a-b$$

$$131+232 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -2a-b \\ -a-2b \end{bmatrix} = 2a+b$$

$$131+121+232 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -3a \\ -3b \end{bmatrix} = 3a$$

$$131+121 = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -3a+b \\ -b \end{bmatrix} = 3a-b$$

$$232+121 = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -a \\ a-3b \end{bmatrix} = a.$$

We see here that for $a > 0$ and $a > b$ that all zones point in the positive direction with respect to the [0,1] vector through (0,0). This discounts opposite pressure on either side of a vertical boundary. By symmetry we know the horizontal boundaries behave the same way.

This overall shows that in the super symmetric case there are no possibilities for sliding solutions. This is because if all boundaries point in the same direction away from boundaries then we can never have sliding solutions. However, as discussed before, human interactions rarely follow the conditions within the super symmetric case. Thus we must try to extend this to more complex scenarios.

6.7 Asymmetric Model

6.7.1 Basic Zones

In the asymmetric case it is not clear that sliding solutions do not exist. Thus in the following section we try to construct one. We will first divide the state space under the vector $[1,1]$ $(x_1 > x_2)$ through $(0,0)$ into 3 sections shown below. Section 1 will be where $a = x_1$ is positive and $b = x_2$ is negative. Section 2 will have both as negative and section 3 will have both as positive.

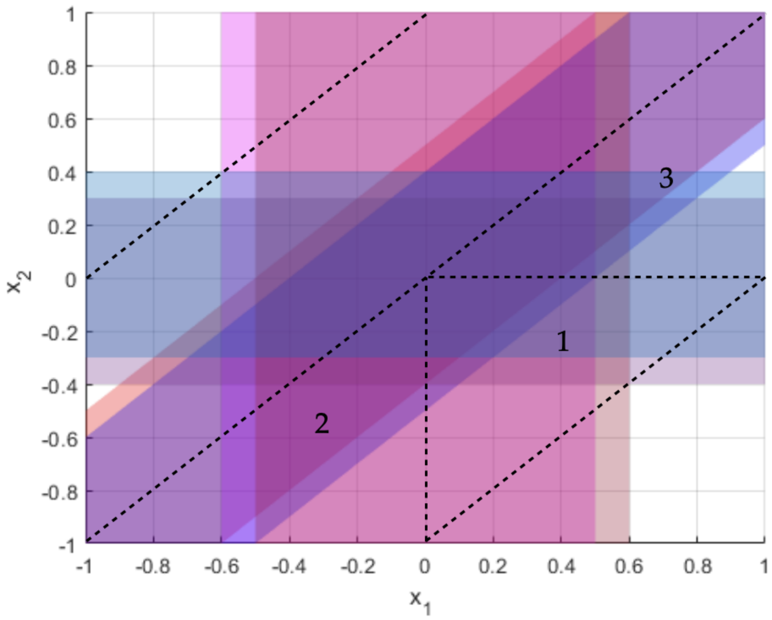

Figure 8: Example of Asymmetric Model (Figure 2) with sectors labelled

Different to the super symmetric case, we now have 6 base level zones.

$$12 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, 21 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, 13 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$31 = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, 23 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, 32 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}.$$

6.7.2 Section 1

We can check each of these zones with respect to section 1 where a is positive and b is negative.

$$12 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a - b$$

$$21 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a - b$$

$$13 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a$$

$$31 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$23 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -b$$

$$32 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

We see that all of those point towards the [1,1] vector as they have positive inner products, or are perpendicular. As has been proven at the beginning of the section, zones that are positive with respect to [-1,1] added are still positive with respect to [-1,1]. That means any combination of the 6 zones here will still have a positive inner product with the vector [-1,1]. However, one can still have sliding solutions along the vertical boundary. Thus we take the normal vector [-1,0] to check this.

$$12 \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a - b$$

$$21 \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$13 \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a$$

$$31 \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a$$

$$23 \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$32 \Rightarrow \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = b.$$

Here all values are positive in the segment aside from the 32 zone. However, we see that the 32 zone will point towards the 323 zone and towards the stability point in the centre of that zone, as shown in the super symmetric case. For sliding to occur, there needs to be alternate flow on either side of the boundary so this 32 zone that points to the right would need one of the zones pointing towards the centre to be on its right. However, this 32 section of the graph only has its own 323 zone in that direction. Thus there are no sliding solutions in this quadrant.

6.7.3 Section 2

The other quadrant to consider is where both a and b are negative. If we return to the original 6 matrices, we see that again, only one zone is not positive with respect to [-1,1]. This zone is 13. This zone moves only left and right as the x_2 values disappear. Thus we see this zone will move towards 131 which we have already investigated in the super symmetric model. This time though, zone 131 cannot be excluded as for negative a, it too is negative with respect to [-1,1]. Thus we must test many zones: 13+21, 13+12, 13+ 212, 131+12, 131+21, 13+32, 13+23, 13+232, 131+32, 131+23 and 131+323 at least.

$$13+23 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a-b$$

$$13+232 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a-b$$

$$131+23 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a-b.$$

These zones offer us no issues. However, all the other zones do cause problems by pointing away from the [-1,1] vector. We start by breaking down some other issues. Firstly, note that each of the base zones have positive inner products with the normal vector [0,1], meaning they all point upwards in this quadrant.

$$12 \Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$21 \Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a-b$$

$$13 \Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$31 \Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -a$$

$$23 \Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -b$$

$$32 \Rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -b.$$

This means that the zones negative with respect to the [-1,1] vector are still moving upwards just at a lesser gradient than [1,1]. Next, we assess these zones with respect to the vector [1,1].

$$12 \Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -a+b$$

$$21 \Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a-b$$

$$13 \Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -1$$

$$31 \Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2a$$

$$23 \Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -b$$

$$32 \Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2b.$$

We can see here that each of these zones is positive with respect to [1,1] aside from 12. That means that for zones to have sliding aside from 12, it would have to be along the diagonal boundary line. With the functions negative to [-1,1] above and the functions positive to [-1,1] below. This means the zone above must contain 21 or 12. Thus there are only 3 functions that could possibly be below. One of those zones plus 21 or 12 must then be negative [-1,1].

$$13+23+12 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 2a-2b$$

$$13+232+12 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a-3b$$

$$131+23+12 \Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 3a-b.$$

Under the boundary [1,1] no values exist such that a can be 3 times or a third of b such that those values become negative. Thus there can be no sliding solutions along said boundary. Overall, we find that we cannot make any sliding solutions in this quadrant.

6.7.4 Section 3

In this section we look at where a and b are positive. This is the inverse symmetric zone of when a and b were negative so now instead of 13 being the problematic zone we see that 23 is the problematic zone. Thus all of the calculations made in the previous section still apply just with a vector swap to each of the matrices. We still cannot construct sliding solutions.

We have now investigated all 3 quadrants of the asymmetric case and we have not been able to construct a sliding solution in any of them. However, this is not a conclusive proof but we do use this information to assume no sliding solutions can be found in 3 dimensions. The only fool proof way to show this is to investigate all $2^6 = 64$ zones but it is too computationally extensive to test them all, the case with symmetric matrices does not reduce the different zone types either.

This finding is logical for 3 agents. The concept of your opinion shifting and gaining a new connection just to immediately lose it is not one that is intuitive to real life. Certainly not without accounting for certain opinions pushing one's opinion away for being too polarising.

We now consider how this extends to higher dimensions. This research finds no reason to assume sliding solutions cannot exist in higher dimensions. There is also no easy way to determine whether they exist either especially because the number of zones grows drastically in higher dimensions. In $\mathbb{R}^3$ we have $N(N-1)$ basic zones and $2^{N(N-1)}$ total zones for the asymmetric model. In 4 dimensions we then have $2^{12} = 4096$ zones and in 5 dimensions we then have $2^{20} = 1,048,576$ zones. This is clearly far too computationally difficult as an approach.

7 Discussion

7.1 Summary of Main Contributions

In this paper, the literature surrounding opinion dynamics and the Hegselmann-Krause model were discussed. A brief history of the origins of opinion dynamics and models to simulate the phenomena in question was delivered. Then the importance of opinion dynamics was evaluated and its applicability in the modern era due to the prevalence of social media. Some of the important political and societal applications were discussed such as shifting views on gender norms or the recent debates over COVID-19 policies that were all fought via the medium of targeted social media posts [12]. Companies such as Cambridge Analytica received significant funding precisely to shift public opinion on Brexit for example.

This paper then introduced the Hegselmann-Krause model for both the discrete time case and the continuous time version. The two models were then compared. The discrete time model had clear convergence and simple interactions between agents. It also had an upper limit for total convergence which was a useful result. However, it didn't simulate real behaviour particularly accurately as it was a homogeneous model where influences had no direction. The continuous time model added directionality to interactions and thus complicated the convergence properties to better simulate real life. However, in the conversion to the continuous time model, the differential equation developed a discontinuous right hand side. This discontinuity created the possibility of sliding solutions at the boundaries of the state space. Sliding solutions cause problems when proving the stability of the model. Thus this paper investigated the existence of said solutions in the HK Model.

We then approached 2 different cases of the model, the super symmetric case and the asymmetric case. For the super symmetric case, it was proven that there are no sliding solutions under any circumstances. For the asymmetrical case, no sliding solutions could be constructed. However, zones no longer have directions pointing within themselves and so the possibilities for sliding solutions increase, especially in higher dimensions. So far, no examples were found, however, there has been little evidence that they cannot exist at all in the HK model. In fact, specific literature has been designed looking to remove the discontinuities from the HK model [32], suggesting they may become a problem in higher dimensions. Thus we conclude sliding solutions are quite possible in the Hegselmann-Krause model in higher dimensions and they should be accounted for in future work.

7.2 Limitations

The limitations of this project are mainly related to computation power. An easy way to test the existence of sliding solutions was not found for the asymmetric case. There are 64 zones to test and thus many more boundaries if one wishes to be thorough. Some shortcut methods were found, along with some symmetries and some other useful tricks which could be applied more concretely.

This paper is also limited to the 3 dimensional case, as in higher dimensions one cannot picture the state space in the same manner and thus cannot perform the same zonal calculations at the boundaries. Furthermore, the number of zones grows quickly to a point where it is no longer feasible to compute them. As stated before, already 1 dimension higher is 4096 zones.

The existence of sliding solutions is a problem for the stability proofs of this model. However, some papers have tried to combat this by removing the discontinuous right hand side of the differential equation that causes the possibilities for these solutions. To do this instead of having the influence functions jump from 0 to 1 they add a function that smoothly translates the values from 0 to 1 [31]. This removes the jump discontinuities and thus the possibilities of sliding solutions. However, other issues arise. The boundary is no longer a single line, the boundary is now a zone in itself to retain continuity. However, that still means solutions can be lost inside. This is because within this zone we have continuously changing zonal matrices, instead of the usual finite amount. This infinite amount of matrices complicates the dynamics of the system and other proofs such as solution existence and stability.

7.3 Future Work

This paper found no evidence that sliding solutions cannot exist, especially in higher dimensions, so future research should try to construct examples of sliding solutions. This can be done by examining problematic zones found in three dimensions such as have been highlighted in this paper. Then, even if no problems arise in three dimensions, one can extrapolate to a similar zone in a higher dimension and investigate it there.

Future work could also look to develop a code that tests the zones of the state space to see if they uniformly point towards the [1,1] vector and towards the [0,1] vector as this would disprove the existence of sliding solutions. Even if it does not disprove every zone, one would only have to test the outlier zones as opposed to every zone in the state space which would reduce the computational requirements of the task. One thing to consider would be whether your opinion can ever be pushed away from another agent due to them being too 'polarising' or having a dislike for the agent. This is a phenomenon that may play a part in real life opinion dynamics.

Another thing to assess would be to consider whether the proof of the nonexistence of sliding solutions in the super symmetric case extends to higher dimensions. One approach could also be to prove that every point in the space has a unique Caratheodory solution and thus that sliding solutions can be assigned measure zero in this case.

8 Conclusion

This paper has introduced and explained the Hegselmann-Krause Model for both discrete and continuous time. It has detailed how this model simulates opinion evolution and how applicable it is to real life. Then it introduced the problems with both models and offered solutions to the problems. This paper focused on one specific issue about the existence of sliding solutions and how this causes problems for stability proofs of the model. Then we proved that no sliding solutions exist in the super symmetric case and tried to construct a sliding solution in the asymmetric case to no avail. Overall, this paper is important to assess what can and cannot be assumed about solutions within the Hegselmann-Krause Model. This will aid stability studies and the general understanding of Fillipov solutions due to a differential equation with a discontinuous right hand side.

Bibliography

[1] N. Cooper, "Between knowledge and ignorance," *Phronesis*, vol. 31, no. 1-3, pp. 229–242, 1986.

[2] *The Oxford Dictionary*. Oxford University Press, 1989.

[3] C. Monti, G. De Francisci Morales, and F. Bonchi, "Learning opinion dynamics from social traces," in *Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pp. 764–773, 2020.

[4] R. Hegselmann, U. Krause, *et al.*, "Opinion dynamics and bounded confidence models, analysis, and simulation," *Journal of artificial societies and social simulation*, vol. 5, no. 3, 2002.

[5] T. Crean, *An Investigation into the Convergence Properties of the Hegselmann-Krause Model for Opinion Dynamics*. PhD thesis, University of Groningen, 2022.

[6] H. Xia, H. Wang, and Z. Xuan, "Opinion dynamics: A multidisciplinary review and perspective on future research," *International Journal of Knowledge and Systems Science (IJKSS)*, vol. 2, no. 4, pp. 72–91, 2011.

[7] P. G. Zimbardo, C. Haney, W. C. Banks, and D. Jaffe, *The Stanford prison experiment*. Zimbardo, Incorporated, 1971.

[8] S. McLeod, "The milgram experiment," *Simply Psychology*, 2007.

[9] S. E. Asch, "Effects of group pressure upon the modification and distortion of judgments," *Groups, leadership, and men*, pp. 177–190, 1951.

[10] V. L. Allen and J. M. Levine, "Social support, dissent and conformity," *Sociometry*, pp. 138–149, 1968.

[11] T. Willaert, P. Van Eecke, K. Beuls, and L. Steels, "Building social media observatories for monitoring online opinion dynamics," *Social Media+ Society*, vol. 6, no. 2, p. 2056305119898778, 2020.

[12] Y. Liu and Y. Yang, "A probabilistic linguistic opinion dynamics method based on the degroot model for emergency decision-making in response to covid-19," *Computers & Industrial Engineering*, vol. 173, p. 108677, 2022.

[13] A. Mutalieva, I. Savchuk, L. Orazgaliyeva, G. Jumazhanova, A. Zhumykbayeva, and E. Sheveleva, "Evaluation of psychological and pedagogical strategies for creating gender motivation in modern glossy magazines as an "overton window" in an electronic and digital society," in *SHS Web of Conferences*, vol. 88, p. 01027, EDP Sciences, 2020.

[14] Ö. M. Uluğ and J. C. Cohrs, “” if we become friends, maybe i can change my perspective:” intergroup contact, endorsement of conflict narratives, and peace-related attitudes in turkey.,” *Peace and Conflict: Journal of Peace Psychology*, vol. 23, no. 3, p. 278, 2017.

[15] S. Chen, D. H. Glass, and M. McCartney, “Characteristics of successful opinion leaders in a bounded confidence model,” *Physica A: Statistical Mechanics and its Applications*, vol. 449, pp. 426–436, 2016.

[16] C. A. Glass and D. H. Glass, “Social influence of competing groups and leaders in opinion dynamics,” *Computational Economics*, vol. 58, no. 3, pp. 799–823, 2021.

[17] J. R. French Jr, “A formal theory of social power.,” *Psychological review*, vol. 63, no. 3, p. 181, 1956.

[18] E. Katz and P. F. Lazarsfeld, *Personal influence: The part played by people in the flow of mass communications*. Routledge, 2017.

[19] M. H. DeGroot, “Reaching a consensus,” *Journal of the American Statistical association*, vol. 69, no. 345, pp. 118–121, 1974.

[20] N. Bellomo and L. Gibelli, “Crowd dynamics, volume 3,”

[21] U. Krause, “Soziale dynamiken mit vielen interakteuren. eine problemskizze,” *Modellierung und Simulation von Dynamiken mit vielen interagierenden Akteuren*, vol. 3751, no. 2, 1997.

[22] H. Rainer and U. Krause, “Opinion dynamics and bounded confidence: models, analysis and simulation,” 2002.

[23] A. Tversky and D. Kahneman, “Rational choice and the framing of decisions,” in *Multiple criteria decision making and risk analysis using microcomputers*, pp. 81–126, Springer, 1989.

[24] J. Hou, W. Li, and M. Jiang, “Opinion dynamics in modified expressed and private model with bounded confidence,” *Physica A: Statistical Mechanics and its Applications*, vol. 574, p. 125968, 2021.

[25] Q. Zha, G. Kou, H. Zhang, H. Liang, X. Chen, C.-C. Li, and Y. Dong, “Opinion dynamics in finance and business: a literature review and research opportunities,” *Financial Innovation*, vol. 6, no. 1, pp. 1–22, 2020.

[26] J. Wu, Y. Chen, T. Gai, Y. Liu, Y. Li, and M. Cao, “A new leader–follower public-opinion evolution model for maritime transport incidents: A case from suez canal blockage,” *Journal of Marine Science and Engineering*, vol. 10, no. 12, p. 2006, 2022.

[27] A. F. Filippov, *Differential equations with discontinuous righthand sides: control systems*, vol. 18. Springer Science & Business Media, 2013.

[28] C. E. Da Silva, P. R. Da Silva, and A. Jacquemard, “Sliding solutions of second-order differential equations with discontinuous right-hand side,” *Mathematical Methods in the Applied Sciences*, vol. 40, no. 14, pp. 5295–5306, 2017.

[29] L. Dieci and L. Lopez, “On filippov and utkin sliding solution of discontinuous systems,” in *Applied And Industrial Mathematics In Italy III*, pp. 323–330, World Scientific, 2010.

[30] R. Iervolino, S. Trenn, and F. Vasca, “Asymptotic stability of piecewise affine systems with filippov solutions via discontinuous piecewise lyapunov functions,” *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1513–1528, 2020.

[31] D. Tangredi, R. Iervolino, and F. Vasca, “Consensus stability in the hegselmann-krause model with coopetition and cooperosity,” *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 11920–11925, 2017.

[32] F. Ceragioli and P. Frasca, “Continuous and discontinuous opinion dynamics with bounded confidence,” *Nonlinear Analysis: Real World Applications*, vol. 13, no. 3, pp. 1239–1251, 2012.