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Regularity Tests for Pattern DAEs

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Contents

Abstract

This thesis will use a graph theoretical approach in the framework of the regularity of DAEs. The DAEs considered will be of the pattern matrix form. The method used in this thesis will follow one of the literature on strong structural controllability, where they also used a graph theoretical approach. Some necessary and sufficient conditions surrounding regularity will be discussed. Beforehand, some preliminaries on regular DAE, Weierstrass Canonical Form, Nilpotent DAE, graph theory, and pattern matrix will be introduced. Then, a rank test based on graph theory will be introduced. And lastly, a necessary condition on the nilpotency of a DAE along with some sufficient conditions on the regularity of a DAE will be introduced.

Introduction

Differential Algebraic Equations or DAE is a term used to describe any differential equations with algebraic constraints. DAEs prove to be useful in modeling in different kinds of disciplines. They are used in physics, economics, and even medicine among many others. DAE is derived using the attributing laws that govern the system which is going to be modeled. For example, suppose that we want to find out the DAE of an LC electrical circuit in Figure 1

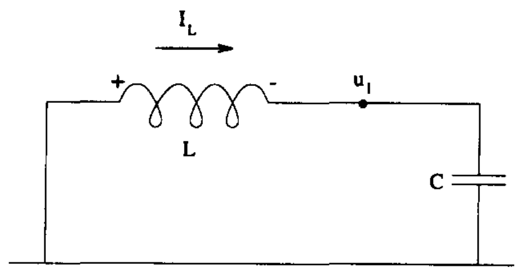

Figure 1: LC Circuit taken from [9]

The attributing law involved is the Kirchoff Law. The law would produce the following differential equations:

$$-I_L + I_C = 0 \tag{1}$$
$$U_L + u_1 = 0 \tag{2}$$
$$U_C - u_1 = 0 \tag{3}$$
$$C\dot{U}_C - I_C = 0 \tag{4}$$
$$L\dot{I}_L - U_L = 0 \tag{5}$$

or equivalently,

$$\begin{bmatrix} C & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}_C \\ \dot{I}_L \\ \dot{I}_C \\ \dot{U}_L \\ \dot{u}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} U_C \\ I_L \\ I_C \\ U_L \\ u_1 \end{bmatrix} = 0 \tag{6}$$

with inductance L, capacitance C, U_C and I_C the voltage and current through the capacitor, respectively, U_L and I_L the voltage and current through the inductor, respectively, and a potential u_1. The inductance L and capacitance C are assumed to be positive but unknown. In order to express the family of DAE for all values of C and L, we can use pattern matrices.

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}_C \\ \dot{I}_L \\ \dot{I}_C \\ \dot{U}_L \\ \dot{u}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 \\ * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} U_C \\ I_L \\ I_C \\ U_L \\ u_1 \end{bmatrix} = 0 \tag{7}$$

where the entry "*" denotes a non-zero real number entry and the entry "?" is used to denote an arbitrary zero or non-zero entry. This is done to anticipate an entry taking any real value, which could be the case for some parameters. Denote a DAE being modeled with pattern matrices to be Pattern DAE.

Pattern DAE represents a family of DAE with real-valued matrices. DAE has a unique solution if it is regular. Pattern DAE is called regular if all possible DAE representations of it are regular. Besides regularity, there is another property of DAE that we are interested in, namely nilpotency. As with the regular case, Pattern DAE is called nilpotent if all possible DAE representations of it are nilpotent. A nilpotent DAE is a simpler case of a regular DAE, in terms of solvability. Investigating regular DAEs is generally more complex than investigating nilpotent DAEs. Certainly, it is useful to know if the system that is being modeled is nilpotent or otherwise. We can characterize the nilpotency of a DAE by a certain rank condition on the matrix pencil of the DAE.

Due to its nature, the matrix pencil of a Pattern DAE will involve also pattern matrices in its corresponding matrix pencil.

[1] proposes a rank test to also check the same rank condition, although for a slightly different matrix pencil. This rank test test will serve as the main tool in this thesis. This thesis will also use a similar approach as was done in [1] to characterize a necessary graph condition for nilpotency of a Pattern DAE. Moreover, some minor results on sufficient graph conditions of the regularity of a Pattern DAE will be derived in a similar manner. As a remark, [8] has introduced a necessary and sufficient graph condition for the regularity of DAEs, although not for Pattern DAE. [8] used weighted graphs to analyze the DAE, which unfortunately cannot be used in this Pattern DAE case.

In the Preliminaries section, the concept of DAE, regular DAE, WCF, and nilpotent DAE will be introduced along with graph and pattern matrix theory. Then in the Results section, the rank test will be introduced along with its applications on nilpotency and regularity of Pattern DAE.

Preliminaries

DAE(Differential Algebraic Equations)

DAEs are equations of the form $F(x, \dot{x}, t) = 0$. Although, the scope of this thesis only deals with its linear form,

$$E\dot{x} = Ax + f \tag{8}$$

E and A are matrices of $\mathbb{R}^{m \times n}$ and f is some inhomogeneous input. A notion (E, A) is used to denote such a system. Like ODE(Ordinary Differential Equations), a general solution for DAE can be obtained. However, due to a more complex nature of DAE, a different approach than what was done for ODE needs to be taken. In general, DAE can be categorized into 2 types; regular DAE and singular DAE.

Definition 1. *The matrix pencil $sE - A \in \mathbb{R}^{m \times n}[s]$ is called regular if, and only if, $n = m$ and $det(sE - A)$ is not the zero polynomial. The matrix pair (E,A) and the corresponding DAE is called regular whenever $sE - A$ is regular.*

DAE which are not regular are called singular DAE. In [2], solutions of both regular and singular DAE are discussed. For regular DAE, the existence and uniqueness of solution is guaranteed for all initial values and input f. On the other hand, singular DAE does not guarantee to have these properties.

Example 1. *Consider the DAE:*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + f \tag{9}$$

The DAE is singular, since $det(sE - A) = 0$. The DAE gives a couple of relations, namely $x_2 = -f_2$, $x_1 = \dot{x_2} - f_1 = -\dot{f_2} - f_1$ and $f_3 = 0$.

Not for all inhomogeneity input f the solution exists. Let $f = [a \;\; b \;\; c]$ with $a, b, c \in \mathbb{Z}_{\neq 0}$, then since $f_3 = c \neq 0$, the solution does not exist. Given that the value x_3 is not restricted, the uniqueness of solution is also not present.

In order to demonstrate the solvability of regular DAE, the concept of Weierstrass Canonical Form will be introduced in the next section.

Weierstrass Canonical Form(WCF)

In [4], the concept of equivalence of matrix pairs is used to obtain classical solutions of DAE,

Definition 2. $(E_1, A_1) \cong (E_2, A_2) \Leftrightarrow \exists$ *matrices $S \in \mathbb{R}^{m \times m}$ and $T \in \mathbb{R}^{n \times n}$ both invertible such that $(E_1, A_1) = (SE_2T, SA_2T)$*

An observation of the consequence of this equivalence relation is that existence and uniqueness of solution of one DAE transfers to another. If x solves $E_1\dot{x} = A_1x + f$, then let $z = Tx$, it can be easily seen that z solves $E_2\dot{z} = A_2z + S^{-1}f$.

Theorem 1. *The matrix pencil $sE - A \in \mathbb{R}^{n\times n}[s]$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{C}^{n\times n}$ such that $sE - A$ is transformed into the Weierstrass Canonical Form (WCF)*

$$S(sE - A)T = s\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \tag{10}$$

where $J \in \mathbb{C}^{n1\times n1}$, $N \in \mathbb{C}^{n2\times n2}$, $n1 + n2 = n$, are matrices in Jordan canonical form and N is nilpotent.

Proof. See [3] □

Recall that the matrix pair (E, A) is regular whenever $sE - A$ is regular. To illustrate the theorem in the sense of the matrix pair, the matrix pair (E, A) is regular given the existence of invertible matrices $S, T \in \mathbb{C}^{n\times n}$ such that $(SET, SAT) = (E_e, A_e)$ such that

$$E_e = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \text{ and } A_e = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \tag{11}$$

with matrices J and N as described in Theorem [1]. This can be seen by looking at how the matrix pencil $sE - A$ is transformed into the matrix pencil $sSET - SAT$ by these invertible matrices $S, T \in \mathbb{C}^{n\times n}$. Therefore, given $sE - A$ regular, by Definition 2, $(E, A) \cong (E_e, A_e)$. Therefore, the problem of showing the solvability of a regular DAE reduces to proving the existence and uniqueness of the solution of (E_e, A_e).

Consider now the matrix pair (E_e, A_e) with E_e and A_e as in (13). In terms of (8), (E_e, A_e) can be written as

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \dot{x} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} x + f \tag{12}$$

for some x and f such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{13}$$

with x having dimension n, x_1 and x_2 will in turn have dimensions n_1 and n_2 such that $n_1 + n_2 = n$. The same applied to f with dimension n such that f_1 and f_2 have dimension n_1 and n_2, respectively. This in turn will decouple the DAE (E_e, A_e) into an ODE and another simpler DAE called a Nilpotent DAE,

$$\dot{x_1} = Jx_1 + f_1 \tag{14}$$

$$N\dot{x_2} = x_2 + f_2 \tag{15}$$

(14) is the ODE and (15) is the Nilpotent DAE. ODE is already guaranteed to have a unique solution. What remains to be seen is the existence and uniqueness of solution of a Nilpotent DAE. In the next section, Nilpotent DAE will be discussed.

Nilpotent DAE

Nilpotent DAE is a special case of DAE where in terms of the matrix pair it can be written as (N, I). Nilpotent DAE is a regular DAE, since it can be transformed into the Weierstrass Canonical Form by setting the ODE part of the decoupling 0. Explicitly, Nilpotent DAE is of the form:

$$N\dot{x} = x + f \tag{16}$$

for some f inhomogeneous input and N Nilpotent matrix. A nilpotent matrix N is a matrix $N \in \mathbb{R}^{n\times n}$ such that there exists an integer $v \in \mathbb{R}$ with this relation:

$$N^v = 0 \tag{17}$$

w is called the index of nilpotency of the corresponding DAE if w is the smallest possible integer such that $N^w = 0$. The index of nilpotency describes how close is the Nilpotent DAE to an ODE. In more complicated cases of DAE, possibly nonlinear, there are other forms of index. Amongst them are Kronecker index, differentiation index, perturbation index, tractability index, geometric index, and strangeness index. These indexes in the framework of this thesis are equivalent, since we are considering only linear DAE.

Due to the unique property of a nilpotent matrix, a solution formula for a nilpotent DAE can be formed. Consider the Nilpotent DAE (N, I) with index of nilpotency n. Apply $N\frac{d}{dt}$ into both sides of (16) successively up to n times

$$N\dot{x} = x + f \tag{18}$$

$$N^2\ddot{x} = N\dot{x} + N\dot{f} = x + f + N\dot{f} \tag{19}$$

$$N^3\dddot{x} = N^2\ddot{x} + N^2\ddot{f} = x + f + N\dot{f} + N^2\ddot{f} \tag{20}$$

$$\vdots \tag{21}$$

$$N^n x^{(n)} = x + \sum_{i=0}^{n-1} N^i f^{(i)} \tag{22}$$

Since $N^n = 0$, the last iteration of the above operation will result in:

$$x = -\sum_{i=0}^{n-1} N^i f^{(i)} \tag{23}$$

This shows that the solution x is determined by the inhomogeneity input f uniquely. Therefore, the solution of Nilpotent DAE exists and is unique.

Graph Theory

A graph is $G(V, E)$ such that V is the set of vertices(or nodes) and E the set of edges with $V = 1, 2, ..., p$. p corresponds to the total number of vertices. An edge is an unordered $\{i, j\}$ or ordered (i, j) pair of vertices such that $i, j \in V$. There are many ways that we can construct a graph. Consider one type of graph, namely the directed nonsimple graph. A directed graph is a graph where the edge is an ordered pair. In a directed graph, an edge (i, j) corresponds to an arrow that goes from vertex i to vertex j. A nonsimple graph is a graph where multiple edges and loops are allowed. A loop is an edge (i, j) where $i = j$. In the graph drawing, a loop corresponds to an arrow that goes from a vertex to itself. In the directed nonsimple graph, there are also the notions of out-neighbor and in-neighbor. For a vertex $v \in V$, the set of out-neighbors and in-neighbors of v are denoted by $O(v)$ and $I(v)$, respectively. They are defined by:

$$O(v) = \{w|(v, w) \in E\}. \text{ and } I(v) = \{w|(w, v) \in E\}. \tag{24}$$

Example 2. *Example of a directed nonsimple graph $G(V, E)$ with $V = \{1, 2, 3, 4\}$ is given in Figure 1.*

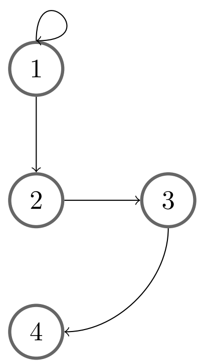

Figure 2: Graph G(V,E) with V={1,2,3,4}

This graph has edges $E = \{(1, 2), (1, 1), (2, 3), (3, 4)\}$. $O(1) = \{2\}$, $I(2) = \{1\}$. The graph has directed edges and also a loop, therefore it is a directed nonsimple graph.

Pattern Matrix

Let $\mathcal{M}$ be a matrix such that $\mathcal{M} \in \{0, *, ?\}^{m \times n}$. * denotes an arbitrary non zero entry and ? denotes an arbitrary zero or non-zero entry. This $\mathcal{M}$ matrix is called a pattern matrix. Examples of 3×3 pattern matrices:

$$\mathcal{M}_1 = \begin{pmatrix} * & 0 & * \\ 0 & ? & * \\ * & * & * \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} * & * & ? \\ ? & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{M}_3 \begin{pmatrix} * & ? & ? \\ ? & 0 & * \\ ? & ? & * \end{pmatrix}$$

For a given pattern matrix $\mathcal{M} \in \{0, *, ?\}^{m \times n}$, the pattern class of $\mathcal{M}$ is defined as:

Definition 3. $\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{m\times n} | M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0 \text{ and } M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *\}$

Example 3. *Take a pattern matrix $\mathcal{M}_1$ in one of the examples of 3×3 pattern matrices. Here are some elements of $\mathcal{P}(\mathcal{M}_1)$:*

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 10 & -5 & 7 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 1 \\ 0 & 5 & -2 \\ 10 & 10 & 10 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Next, a definition of a nilpotent pattern matrix is introduced:

Definition 4. *a pattern matrix $\mathcal{N} \in \{0, *, ?\}^{n\times n}$ is nilpotent if for all $N \in P(\mathcal{N})$, N is nilpotent*

Graph Representation of a Pattern Matrix

Next, an association between a graph and a pattern matrix will be introduced. Let $G(V, E)$ be a graph representation for a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{m\times n}$ with $m \leq n$. Then, $V = \{1, 2, ..., n\}$ and $E \subseteq V \times V$. Denote the graph representation of matrix $\mathcal{M}$ as $G(\mathcal{M})$. An edge $(j, i) \in E$ if and only if $\mathcal{M}_{ij} = *$ or $\mathcal{M}_{ij} =?$. To further characterize the matrix from the graph, we define the new sets E_* and $E_?$ as

Definition 5. $E_* = \{(j, i) | M_{i,j} = *\}$ *and* $E_? = \{(j, i) | M_{i,j} =?\}$

E_* and $E_?$ will contain the set of edges corresponding to the * entries of the matrix and the ? entries of the matrix, respectively. The set of edges E of the pattern matrix will be partitioned into E_* and $E_?$ such that $E = E_* \cup E_?$ and $E_* \cap E_? = \emptyset$. This is because an entry can only be either * or ? and the edges are defined as ordered pairs. This way, the graph will preserve all the information that the pattern matrix has. An example for a graph representation of a pattern matrix will be given.

Example 4. *Let $\mathcal{M} \in \{0, *, ?\}^{3\times 3}$ be a pattern matrix such that:*

$$\mathcal{M} = \begin{pmatrix} 0 & * & 0 \\ 0 & ? & * \\ * & 0 & 0 \end{pmatrix}$$

Figure 3 depicts the graph representation of the matrix $\mathcal{M} := G(\mathcal{M})$:

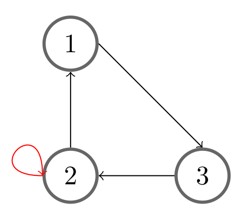

Figure 3: Graph G(M)

The black arrows and the red arrows denote E_* and $E_?$, respectively. This will remain the consensus for further graph drawing.

Pattern DAE

Recall that a DAE is often denoted as a matrix pair (E, A) with $E, A \in \mathbb{R}^{m\times n}$ such that

$$E\dot{x} = Ax + f \tag{27}$$

Instead of using real-numbered matrices, consider a Pattern DAE. Roughly speaking, Pattern DAE is a DAE described with pattern matrices. To be precise,

Definition 6. *Pattern DAE $(\mathcal{E}, \mathcal{A})$ is a family of DAEs such that $\forall (E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$, each matrix pair (E, A) corresponds to a DAE (E, A).*

Pattern DAE $(\mathcal{E}, \mathcal{A})$ has a characteristic "A" if for all $(E, A) \in P(\mathcal{E})\times P(\mathcal{A})$, the corresponding DAE (E, A) has characteristic "A". The characteristic "A" can be taken as regularity, or nilpotency, etc. Formally, here is an example of when the characteristic "A" is taken as regularity,

Definition 7. *Let $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{m\times n}$, then the Pattern DAE $(\mathcal{E}, \mathcal{A})$ is regular $\Leftrightarrow$ for all $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$, the DAE (E, A) is regular.*

For this reason, the term family does not mean that the DAEs have similar properties other than belonging to the same pattern class. For example,

Example 5. *Consider a regular DAE (E, A) such that,*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + f \tag{28}$$

Observe that the matrix $E \in \mathbb{R}^{3\times 3}$ is nilpotent with $E^n = 0$ and $A \in \mathbb{R}^{3\times 3}$ is the identity matrix. The DAE above is regular since it takes the form of a Nilpotent DAE. Consider the Pattern DAE $(\mathcal{E}, \mathcal{A})$ such that,

$$\mathcal{E} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & ? \end{pmatrix}$$

It can be seen easily that $(E, A) \in P(\mathcal{E})\times P(\mathcal{A})$. Moreover, the pair $(E, A) \in P(\mathcal{E})\times P(\mathcal{A})$ is regular even though $(\mathcal{E}, \mathcal{A})$ is not, since there exists at least an element of the pattern class $P(\mathcal{E}) \times P(\mathcal{A})$ which is not regular, namely the general DAE in (9).

Considering Pattern DAE allows for easier ways to determine the properties of a system with variable constraints. For example, (7) is the Pattern DAE representation for the DAE description of the electrical circuit. By examining this Pattern DAE, any attributing properties attached to this Pattern DAE will also hold true for all its corresponding DAE pattern classes.

Results

Rank Test

Given a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{m \times n}$ with $m \leq n$, [1] introduces a rank test using $G(\mathcal{M})$. This rank test will determine if the pattern matrix $\mathcal{M}$ has full row rank. The rank test will rely on the notion of colorability. $G(\mathcal{M})$ will start off with all white nodes. Then, a certain procedure will be applied to change the color of some of the nodes black. The procedure is as follows:

1. if a node i has exactly one white out-neighbor j and $(i, j) \in E_*$, we change the color of j to black;
2. repeat step 1 until no more color changes are possible.

Recall that j is an out-neighbor of i if $(i, j) \in E$ and that a loop is also an edge. After applying the procedure, a graph $G(M)$ for a pattern matrix $M \in \{0, *, ?\}^{m \times n}$ is colorable if and only if the nodes $1, 2, ..., m$ are colored black following the procedure above. By graph theory, the nodes $m + 1, ..., n$ are not out-neighbors of any nodes $1, ..., n$, therefore they will never be colored black. An example of colorability will be given below:

Example 6. *Let us check the colorability property of the graph $G(\mathcal{M})$ in Figure 3. Recall that this graph corresponds to the matrix $\mathcal{M} \in \{0, *, ?\}^{3 \times 3}$ in Example 4. Node 1 has exactly 1 out-neighbor, namely Node 3. Furthermore, Node 3 is the only white out-neighbor of Node 1 and $(1, 3) \in E_*$. Therefore, Node 1 colors Node 3 black.*

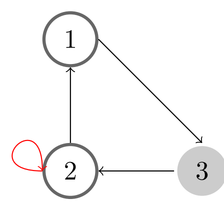

Figure 4: Graph G(M)

By similar reasoning, Node 3 colors Node 2 black.

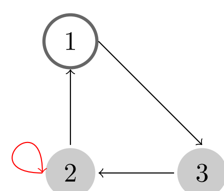

Figure 5: Graph G(M)

Node 2 has 2 out-neighbors, namely node 2 and node 1. However, Node 1 is the only white out-neighbor of Node 2 and $(2, 1) \in E_*$. Therefore, Node 2 colors Node 1 black.

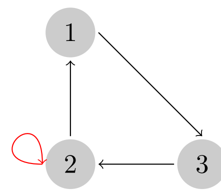

Figure 6: Graph G(M)

Since the Nodes 1,2,3 are colored black following the procedure, therefore the graph is colorable.

Following the procedure, now we will be able to determine if $G(\mathcal{M})$ is colorable given a pattern matrix $\mathcal{M}$. The next theorem will relate the notion of colorability with the row rank of the pattern matrix.

Theorem 2. *Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leq q$. Then, $\mathcal{M}$ has full row rank if and only if $G(\mathcal{M})$ is colorable.*

Proof. In order to prove Theorem 2, an additional lemma needs to be introduced,

Lemma 3. *Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leq q$. Consider the directed graph $G(\mathcal{M})$. Suppose that each node is colored white or black. Let $D \in \mathbb{R}^{p \times p}$ be the diagonal matrix defined by*

$$D_{kk} = \begin{cases} \text{1,} & \text{if node } k \text{ is black} \\ \text{0,} & \text{otherwise} \end{cases}$$

Suppose further that $j \in \{1, 2, ..., p\}$ is a node for which there exists a node $i \in \{1, 2, ..., p\}$, possibly identical to j, such that j is the only white out-neighbor of i and $(i, j) \in E_*$. Then, for all $M \in P(\mathcal{M})$, we have that $[M \;\; D]$ has full row rank if and only if $[M \;\; D + e_j e_j^T]$ has full row rank, where e_j denotes the jth column of I.

The proof of this lemma can be found in [1]. Now we can start to proof Theorem 2. *"sufficient"* Suppose that $G(\mathcal{M})$ is colorable. Let $M \in P(\mathcal{M})$. Apply lemma 3 to $G(\mathcal{M})$ with all nodes $1, ..., p$ colored white. Then,

$$[M] \text{ has full row rank} \Leftrightarrow [M \;\; 0 + e_{p_1} e_{p_1}^T] \text{ has full row rank} \tag{30}$$

for some $p_1 \in 1, 2, ..., p$. p_1 exists since $G(\mathcal{M})$ is colorable. Again, apply lemma 3, but now to $G(\mathcal{M})$ with all nodes $1, 2, ..., p$ colored white except the node p_1, which are colored black. Then,

$$[M \;\; 0 + e_{p_1} e_{p_1}^T] \text{ has full row rank} \Leftrightarrow [M \;\; 0 + e_{p_1} e_{p_1}^T + e_{p_2} e_{p_2}^T] \text{ has full row rank} \tag{31}$$

Since $G(\mathcal{M})$ is colorable, Repeat this process for all nodes $1, 2, ..., p$. The last iteration would look like:

$$[M \;\; 0 + e_1 e_1^T + ... + e_{p-1} e_{p-1}^T] \text{ has full row rank} \Leftrightarrow [M \;\; 0 + e_1 e_1^T + ... + e_p e_p^T] \text{ has full row rank} \tag{32}$$

Combining all the results together,

$$[M] \text{ has full row rank} \Leftrightarrow [M \;\; 0 + e_{p_1} e_{p_1}^T] \text{ has full row rank} \Leftrightarrow ...$$
$$\Leftrightarrow [M \;\; 0 + e_1 e_1^T + ... + e_p e_p^T] \text{ has full row rank} \quad (33)$$

By definition of e_j, observe that the matrix $[0 + e_1 e_1^T + ... + e_p e_p^T]$ is nothing but an identity matrix $I \in \mathbb{R}^{p \times p}$. Therefore,

$$[M] \text{ has full row rank} \Leftrightarrow [M \;\; I] \text{ has full row rank} \quad (34)$$

Observe that the matrix $[M \;\; I]$ has a full row rank. Therefore, we can conclude that $[M]$ has a full row rank for all $M \in P(\mathcal{M})$ and that $\mathcal{M}$ has full row rank. *"necessary"* Suppose that $\mathcal{M}$ has full row rank, but $G(\mathcal{M})$ is not colorable. Let C be the set of nodes that are colored black by repeated application of the color change rule until no more color changes are possible. Then, C is a subset of $1, 2, ..., p$ with a dimension stricly smaller than p. Thus, after reordering the nodes, $\mathcal{M}$ can be partitioned as

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \quad (35)$$

with $\mathcal{M}_1 \in \{0, *, ?\}^{p_1 \times q}$ containing all the rows corresponding to the subset C and $\mathcal{M}_2 \in \{0, *, ?\}^{p_2 \times q}$ the remaining rows of $\mathcal{M}$ such that $p_1 + p_2 = p$. Observe that none of the columns of $\mathcal{M}_2$ containing only one * entry, while all other entries of the column are 0. Therefore, the structure of the column of $\mathcal{M}_2$ will be either one of the following cases:

1. all the entries are 0.
2. One of the entries is ? with the rest of the entries 0.
3. At least two entries belong to the set $\{*, ?\}$.

Take a matrix $M_2 \in P(\mathcal{M}_2)$ such that the entries of the columns are:

1. all 0 for a column corresponding to case 1
2. Take 0 as the entry for ? for a column corresponding to case 2
3. if the column is corresponding to case 3, then set the pattern entries as real numbers such that their sum is 0. Observe that this is possible, since there are at least 2 pattern entries belonging to the set $\{*, ?\}$

Observe that the matrix $M_2 \in P(\mathcal{M}_2)$ has its row sums zero, that is $\mathbf{1}^T M_2 = 0$, where $\mathbf{1}$ denotes the vector of all ones with size p_2.

Take such M_2 and any $M_1 \in \mathcal{M}_1$, then

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \in P\left(\begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}\right) = P(\mathcal{M}) \quad (36)$$

such that $[\mathbf{0}^T \;\; \mathbf{1}^T]M = 0$ with $\mathbf{0}$ denotes the vector of all zeros with size p_1 and $\mathbf{1}$ denotes the vector of all ones with size p_2.

By definition, M does not have full rank. Therefore, $\mathcal{M}$ does not have full rank, which is a contradiction. Therefore, $G(\mathcal{M})$ is colorable. □

Example 7. *Let $\mathcal{M} \in \{0, *, ?\}^{3\times 3}$ be a pattern matrix in Example 4. Then, by Example 6, the graph $G(\mathcal{M})$ is colorable. By Theorem 2, $\mathcal{M}$ has full row rank.*

Example 8. *Let $\mathcal{M} \in \{0, *, ?\}^{10\times 10}$ such that*

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & ? \\ 0 & 0 & 0 & 0 & 0 & ? & 0 & 0 & * & 0 \\ * & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \end{bmatrix} \tag{37}$$

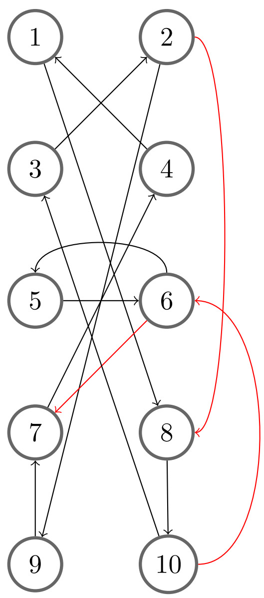

Figure 7: Graph G($\mathcal{M}$)

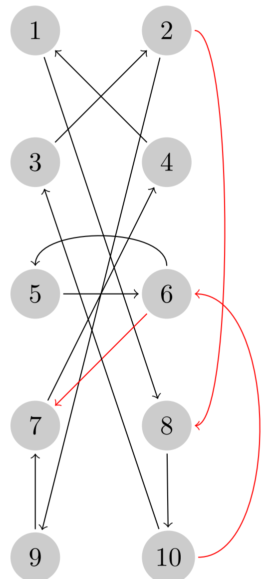

Figure 8: Graph G($\mathcal{M}$) after the rank test procedure. The sequence of coloring goes as follows: 1 colors 8, 2 colors 9, 3 colors 2, 4 colors 1, 5 colors 6, 7 colors 4, 8 colors 10, 9 colors 7, 10 colors 3, 6 colors 5

.

Since all the nodes of $1, 2, ..., 10$ *are colored black,* $G(\mathcal{M})$ *is colorable and by Theorem 2,* $\mathcal{M}$ *has full row rank.*

From the example, observe that the red edges do not have the power to color a node. Therefore its existence will not be able to increase the chance of a graph being colorable. Furthermore, red edges also prevent colorability. Take for example this graph,

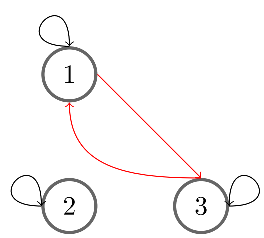

Figure 9: Graph $G(\mathcal{M})$ for some pattern matrix $\mathcal{M}$

Observe that if the red edges are removed, then the graph will be colorable. This can also be seen when looking at one of the requirements of the colorability procedure which states that the in-neighbor of a black-colored node has to be exactly 1. The existence of red edges might increase the number of in-neighbors of a node.

Based on these observations, we can conclude that if a graph $G(\mathcal{M})$ is colorable then $G(\mathcal{M})$ with its red edges removed is colorable. This is because $G(\mathcal{M})$ is still colorable despite the existence of possibly more than 1 red edge, which can prevent colorability. However, the inverse is not true also because the existence of red edges might prevent colorability.

Rank Test with Nilpotent DAE

As we have seen above, a regular DAE can be partitioned into 2 separate algebraic equations. One of which is nilpotent DAE. Knowing a regular DAE is nilpotent tells us that the DAE has no ODE part on its Weierstrass Canonical Form (WCF). This would allow an easier computation of the WCF.

A rank criterion to determine if a regular DAE is nilpotent will be introduced. This will give way for us to put the rank test to use.

Theorem 4. *Let $E, A \in \mathbb{R}^{n\times n}$ and (E, A) be regular, then $rank(\lambda E - A) = n\ \forall \lambda \in \mathbb{C}$ if and only if $(E, A) \cong (N, I)$ for some matrix $N, I \in \mathbb{R}^{n\times n}$ such that N is nilpotent and I is the identity matrix.*

Proof. *"sufficient"* Let $rank(\lambda E - A) = n\ \forall \lambda \in \mathbb{C}$. Assume that $(E, A) \ncong (N, I)$ for some matrix $N, I \in \mathbb{R}^{n\times n}$ such that N is nilpotent and I is the identity matrix. Since (E, A) is regular, by Theorem [1], there exist $S, T \in \mathbb{C}^{n\times n}$ invertible matrices such that (E, A) is transformed into the Weierstrass Canonical Form,

$$n = rank(\lambda E - A) = rank(\lambda SET - SAT) = rank(\lambda I - J) + rank(\lambda N - I) \tag{38}$$

with $J \in \mathbb{C}^{n_1\times n_1}$ and $N \in \mathbb{C}^{n_2\times n_2}$ such that $n = n_1 + n_2$. Let N_n be the Jordan Canonical Form of the matrix N. Then, $rank(\lambda N - I) = rank(\lambda N_n - I)$. Let m be a positive integer such that $N^m = 0$ and $\lambda \in \mathbb{C}$ is an aribtrary eigenvalue of N with its corresponding eigenvector $x \in \mathbb{C}^n$, then

$$\lambda^m x = N^m x = 0 \tag{39}$$

Since x is an eigenvector, λ is therefore 0. This shows that eigenvalues of N are all 0. Therefore, N_n will be:

$$N_n = \begin{pmatrix} 0 & x & & \\ & \ddots & \ddots & \\ & & \ddots & x \\ & & & 0 \end{pmatrix} \tag{40}$$

with $x \in \{1, 0\}$. Therefore,

$$\lambda N_n - I = \begin{pmatrix} -1 & y & & \\ & \ddots & \ddots & \\ & & \ddots & y \\ & & & -1 \end{pmatrix} \tag{41}$$

with $y \in \{\lambda, 0\}$. Since $\lambda N_n - I$ is an upper triangular matrix with non-zero entries in the diagonal, therefore $\lambda N_n - I$ has a full rank. In other words, $rank(\lambda N - I) = rank(\lambda N_n - I) = n_2$. On the other hand, for all λ eigenvalues of the matrix J, $rank(\lambda I - J) < n_1$. Therefore, $rank(\lambda E - A) = n_1 + n_2 < n$ and we reach a contradiction. Therefore, $(E, A) \cong (N, I)$. *"sufficient"* Let $(E, A) \cong (N, I)$. From the *"necessary"* proof, $\lambda N - I$ has full rank. Due to the equivalence relation, there exist invertible matrices $S, T \in \mathbb{C}^{n \times n}$ such that

$$\lambda E - A = \lambda SNT - ST \tag{42}$$

Multiplication of a matrix with invertible matrices from the right or left or both does not change its rank, since the invertible linear transformations conserve its number of linearly independent vectors. Therefore,

$$rank(\lambda E - A) = rank(\lambda SNT - ST) = rank(\lambda N - I) \tag{43}$$

Since $\lambda N - I$ has full rank, $rank(\lambda E - A) = n \ \forall \lambda \in \mathbb{C}$ □

By checking the rank of the corresponding matrix pencil, Theorem 4 tells us that we can determine if the regular DAE is a nilpotent DAE or otherwise. An important observation is that the idea of Theorem 4 can be extended to Pattern DAE. First, we define the equivalence relation for the nilpotency of Pattern DAEs,

Definition 8. *Let $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{n \times n}$ such that the Pattern DAE $(\mathcal{E}, \mathcal{A})$ is regular, then $(\mathcal{E}, \mathcal{A}) \cong (\mathcal{N}, I)$ for some matrices $\mathcal{N} \in \{0, *, ?\}^{n \times n}$ nilpotent and $I \in \mathbb{R}^{n \times n}$ the identity matrix $\Leftrightarrow$ for all $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$, $(E, A) \cong (N, I)$ for some nilpotent DAE (N, I)*

Then, the corollary below will give rank conditions for a Pattern DAE to be nilpotent.

Corollary 1. *Let $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $(\mathcal{E}, \mathcal{A})$ be regular, then for an arbitrary pair $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$, $rank(\lambda E - A) = n \ \forall \lambda \in \mathbb{C}$ if and only if $(\mathcal{E}, \mathcal{A}) \cong (\mathcal{N}, I)$ for some matrices $\mathcal{N} \in \{0, *, ?\}^{n \times n}$ and $I \in \mathbb{R}^{n \times n}$ such that $\mathcal{N}$ is nilpotent and I is the identity matrix.*

Proof. Similar to Theorem 4. □

In [1], checking the rank of a matrix pencil is also considered. Although, being in the framework of controllability, they considered a different kind of matrix pencil. In order to check controllability, they use the Hautus test,

Theorem 5. *Given an ODE such that $\dot{x} = \mathcal{A}x + \mathcal{B}u$ with the state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, $\mathcal{A} \in \{0, *, ?\}^{n \times n}$, $\mathcal{B} \in \{0, *, ?\}^{n \times m}$, the ODE is controllable if and only if $rank(\lambda I - A \ \ B) = n$ $\forall \lambda \in \mathbb{C}$ for all $(A, B) \in P(\mathcal{A}) \times P(\mathcal{B})$*

[1] then proposed a theorem that together with the rank test will be able to determine controllability for the Pattern ODE. A similar theorem as in [1] (Theorem 6) is introduced below for checking nilpotency of a Pattern DAE:

Theorem 6. *Assume $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{n \times n}$. $rank(\lambda E - A) = n$ for all $\lambda \in \mathbb{C}$ and $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A}) \Rightarrow$ The conditions below are satisfied*

1. *A has full rank*
2. *$\bar{A}$ has full rank with $\bar{A} = E + A$*

for all $A \in P(\mathcal{A})$ and $\bar{A} \in P(\bar{\mathcal{A}})$ such that $\bar{\mathcal{A}} = \mathcal{E} + \mathcal{A}$.

Proof. To proof the 1^{st} condition, take $\lambda = 0$ on the assumption. This will result in A having a full rank for all $A \in P(\mathcal{A})$. To proof the second condition, take an arbitrary $\bar{A} \in P(\bar{\mathcal{A}})$. Consider a matrix $\bar{E} \in \mathbb{R}^{n \times n}$ such that

1. $\bar{E}_{ij} = 1$ if $\bar{\mathcal{A}}_{ij} =?$, $\mathcal{A}_{ij} =?$ and $\mathcal{E}_{ij} \in \{*, ?\}$
2. $\bar{E}_{ij} = -\bar{A}_{ij}$ if $\bar{\mathcal{A}}_{ij} = *$ and $\mathcal{A}_{ij} = 0$
3. $\bar{E}_{ij} = -\bar{A}_{ij}$ if $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} = 0$
4. $\bar{E}_{ij}$ such that $\bar{E}_{ij} \neq -\bar{A}_{ij}$ and $\bar{E}_{ij} \neq 0$ if $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} \neq 0$ and $\mathcal{E}_{ij} = *$
5. $\bar{E}_{ij} \neq -\bar{A}_{ij}$ if $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} \neq 0$ and $\mathcal{E}_{ij} =?$
6. $\bar{E}_{ij} = 0$ otherwise

for $i, j \in \{1, 2, ..., n\}$. Claim: $\bar{E} \in P(\mathcal{E})$. Proof: By tracing back the pattern of $\mathcal{E}$ from $\bar{\mathcal{A}}$ and $\mathcal{A}$, it can be checked if the matrix $\bar{E}$ follows the pattern of $\mathcal{E}$ from these cases: Let $i, j \in \{1, 2, ..., n\}$,

1. If $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} =?$, and $\mathcal{E}_{ij} \in \{?, *\}$. The entry $\bar{E}_{ij} = 1$ satisfies the pattern entry of $\mathcal{E}_{ij}$ in both possible cases, since $1 \in *$ and $1 \in ?$.
2. If $\bar{\mathcal{A}}_{ij} = *$ and $\mathcal{A}_{ij} = 0$, then this implies that $\mathcal{E}_{ij} = *$. The entry $\bar{E}_{ij} = -\bar{A}_{ij}$ satisfies the pattern entry of $\mathcal{E}_{ij}$, since $-\bar{A}_{ij} \in *$.
3. If $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} = 0$, then this implies that $\mathcal{E}_{ij} =?$. The entry $\bar{E}_{ij} = -\bar{A}_{ij}$ satisfies the pattern entry of $\mathcal{E}_{ij}$, since $-\bar{A}_{ij} \in ?$.
4. If $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} \neq 0$ and $\mathcal{E}_{ij} = *$, the entry $\bar{E}_{ij}$ such that $\bar{E}_{ij} \neq -\bar{A}_{ij}$ and $\bar{E}_{ij} \neq 0$ satisfies the pattern entry of $\mathcal{E}_{ij}$, since $\bar{E}_{ij}$ is a nonzero entry.
5. If $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} \neq 0$ and $\mathcal{E}_{ij} =?$, the entry $\bar{E}_{ij} \neq -\bar{\mathcal{A}}_{ij}$ satisfies the pattern entry of $\mathcal{E}_{ij}$, since $-\bar{A}_{ij} \in ?$.
6. The rest of the conditions deal with the identity mapping from $\mathcal{A}_{ij}$ to $\bar{\mathcal{A}}_{ij}$, aside from case 1, in terms of patterns. This implies that $\mathcal{E}_{ij} = 0$. Since also $\bar{E}_{ij} = 0$, therefore $\bar{E}_{ij}$ satisfies the pattern entry of $\mathcal{E}_{ij}$.

Therefore, $\bar{E} \in P(\mathcal{E})$. Another claim is that: $\hat{A} = \bar{A} + \bar{E} \in P(\mathcal{A})$. Proof: Let $i, j \in \{1, 2, ..., n\}$,

1. If $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} =?$, then $\bar{A}_{ij} + \bar{E}_{ij} \in \mathcal{A}_{ij}$.
2. If $\bar{\mathcal{A}}_{ij} = *$ and $\mathcal{A}_{ij} = 0$, then by construction, $\bar{E}_{ij} = -\bar{A}_{ij}$. Therefore, $\bar{A}_{ij} + \bar{E}_{ij} = 0$. Therefore, $\bar{A}_{ij} + \bar{E}_{ij} \in \mathcal{A}_{ij}$.
3. If $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} = 0$, then by construction, $\bar{E}_{ij} = -\bar{A}_{ij}$. Therefore, $\bar{A}_{ij} + \bar{E}_{ij} = 0$. Therefore, $\bar{A}_{ij} + \bar{E}_{ij} \in \mathcal{A}_{ij}$.
4. If $\bar{\mathcal{A}}_{ij} =?$ and $\mathcal{A}_{ij} \neq 0$, then by construction, $\bar{E}_{ij} \neq -\bar{A}_{ij}$. Therefore, $\bar{A}_{ij} + \bar{E}_{ij} \neq 0$. Therefore, $\bar{A}_{ij} + \bar{E}_{ij} \in \mathcal{A}_{ij}$.
5. For the rest of the other conditions, as seen from the previous proof of $\bar{E} \in P(\mathcal{E})$, $\bar{A}_{ij} + \bar{E}_{ij} = \bar{A}_{ij} \in \mathcal{A}_{ij}$, since the pattern does not change for $\bar{\mathcal{A}}_{ij}$ from $\mathcal{A}_{ij}$.

Therefore, $\hat{A} = \bar{A} + \bar{E} \in P(\mathcal{A})$. Since $\hat{A} \in P(\mathcal{A})$ and $\bar{E} \in P(\mathcal{E})$, therefore $rank(\lambda\bar{E} - \hat{A}) = n$ for all $\lambda \in \mathbb{C}$. By taking $\lambda = 1$:

$$n = rank(\bar{E} - \hat{A}) = rank(\bar{E} - \bar{A} - \bar{E}) = rank(-\bar{A}) = rank(\bar{A}) \tag{44}$$

Therefore, $\bar{A}$ has full rank for all $\bar{A} \in P(\bar{\mathcal{A}})$. □

Integrating the theorem with the rank test will result in the following corollary:

Corollary 2. *Assume $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{n\times n}$. $rank(\lambda E - A) = n$ for all $\lambda \in \mathbb{C}$ and $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A}) \Rightarrow$ The conditions below are satisfied*

1. *$G(\mathcal{A})$ is colorable.*
2. *$G(\bar{\mathcal{A}})$ is colorable.*

for $\bar{\mathcal{A}} = \mathcal{E} + \mathcal{A}$.

Proof. Assume that $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{n\times n}$ and $rank(\lambda E - A) = n$ for all $\lambda \in \mathbb{C}$ and $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$. Then, by Theorem 6, A and $\bar{A}$ both have full rank for all $A \in P(\bar{A})$ and $\bar{A} \in P(\bar{\mathcal{A}})$ such that $\bar{\mathcal{A}} = \mathcal{E} + \mathcal{A}$. In other words, $\mathcal{A}$ and $\bar{\mathcal{A}}$ both have full rank. Since $\mathcal{A}$ and $\bar{\mathcal{A}}$ are square matrices, having full rank is equivalent to having full row rank. By Theorem 2, $G(\mathcal{A})$ and $G(\bar{\mathcal{A}})$ are colorable □

The other direction unfortunately has not been able to be fully proven yet. However, we could still say something for $\lambda \in \mathbb{R}$,

Theorem 7. *Assume $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{n\times n}$. $rank(\lambda E - A) = n$ for all $\lambda \in \mathbb{R}$ and $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A}) \Leftarrow$ The conditions below are satisfied*

1. *$G(\mathcal{A})$ is colorable.*
2. *$G(\bar{\mathcal{A}})$ is colorable.*

for $\bar{\mathcal{A}} = \mathcal{E} + \mathcal{A}$.

Proof. Consider 2 cases. For $\lambda = 0$, then $rank(\lambda E - A) = rank(-A) = n$ for all $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$ since $G(\mathcal{A})$ is colorable and by Theorem 2, $-\mathcal{A} = \mathcal{A}$ has full rank (due to $\mathcal{A}$ being a square matrix, row rank corresponds to rank). For $\lambda \neq 0$, since $\lambda E \in P(\mathcal{E})$, therefore $\lambda E - A \in P(\bar{\mathcal{A}})$. By assumption, $G(\bar{\mathcal{A}})$ is colorable and by Theorem 2, $\bar{\mathcal{A}}$ has full rank (due to $\bar{\mathcal{A}}$ being a square matrix, row rank corresponds to rank). Therefore, $rank(\lambda E - A) = n$ for all $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$ □

Showing Theorem 7 for all $\lambda \in \mathbb{C}$ instead of just $\lambda \in \mathbb{R}$ would produce a stronger result such that if paired with Corollary 1, we could determine if a Pattern DAE $(\mathcal{E}, \mathcal{A})$ is Nilpotent or otherwise by checking the graphs of the pattern matrices proposed.

Nevertheless, by using Corollary 1 and Corollary 2, we could use the rank test to check a necessary condition for nilpotency of a regular Pattern DAE. One of its applications is shown below:

Example 9. *Take $S, T \in \mathbb{R}^{10\times 10}$ invertible matrices such that $(SNT, SIT) = (E, A)$ for some matrices $N, I, E, A \in \mathbb{R}^{10\times 10}$ with N nilpotent and I the identity matrix. By the equivalence relation, $(E, A) \cong (N, I)$. Recall that $(E, A) \cong (N, I)$ implies that (E, A) is a Nilpotent DAE.*

$$S = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{45}$$

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{46}$$

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

Then, take one Pattern DAE $(\mathcal{E}, \mathcal{A})$ representation of (E, A) such that $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$.

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & * & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

$$\bar{\mathcal{A}} = \mathcal{E} + \mathcal{A} = \begin{bmatrix} 0 & * & * & 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & * & * & 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & * & 0 \\ * & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \end{bmatrix} \tag{49}$$

Next, use the rank test in order to determine if $\mathcal{A}$ and $\bar{\mathcal{A}}$ both have full rank. First, determine $G(\mathcal{A})$ and $G(\bar{\mathcal{A}})$:

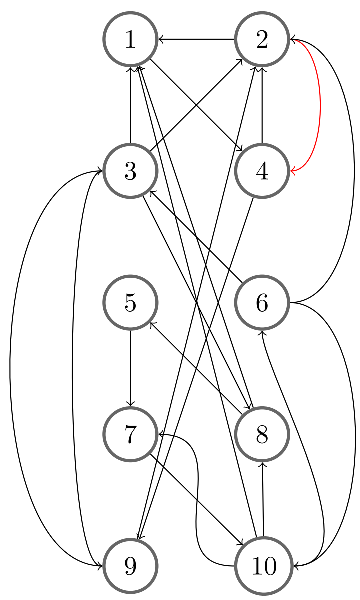

Figure 10: Graph G($\bar{\mathcal{A}}$)

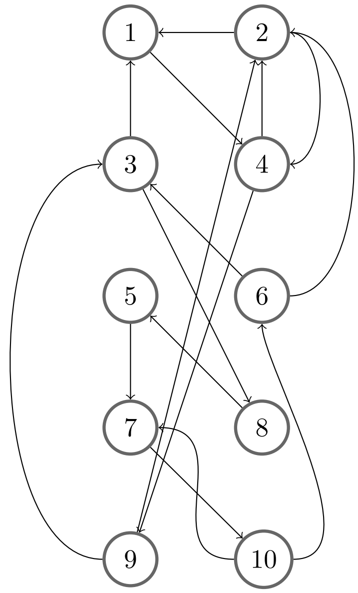

Figure 11: Graph G($\mathcal{A}$)

Then, apply the coloring procedure to both $G(\bar{\mathcal{A}})$ and $G(\mathcal{A})$.

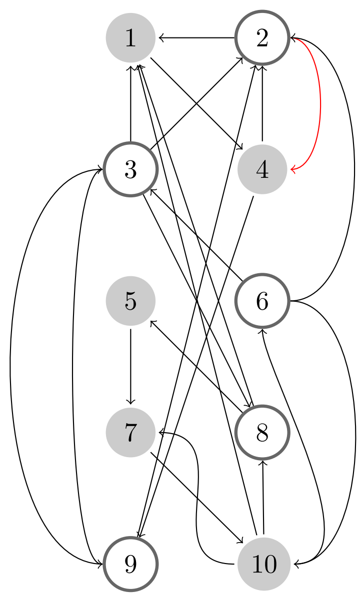

Figure 12: Graph G($\bar{\mathcal{A}}$) after the rank test procedure. The coloring procedure for $G(\bar{\mathcal{A}})$ is done in this sequence: 1 colors 4, 2 colors 1, 5 colors 7, 7 colors 10, 8 colors 5

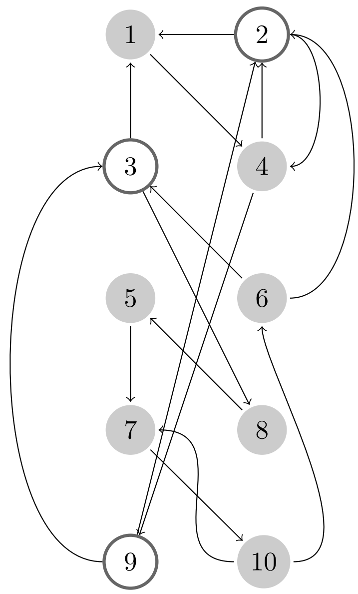

Figure 13: Graph G($\mathcal{A}$) after the rank test procedure. The coloring procedure for G($\mathcal{A}$) is done in this sequence: 1 colors 4, 2 colors 1, 3 colors 8, 5 colors 7, 7 colors 10, 8 colors 5, 10 colors 6

Since the nodes $1,..,10$ in both $G(\mathcal{A})$ and $G(\bar{\mathcal{A}})$ are not colored black following the procedure, therefore $G(\mathcal{A})$ and $G(\bar{\mathcal{A}})$ are not colorable. By taking the contrapositive of Corollary 2, rank$(\lambda E - A) < n$ for some $\lambda \in \mathbb{C}$ and $(E, A) \in P(\mathcal{E}) \times P(\mathcal{A})$. By applying Corollary 1, $(\mathcal{E}, \mathcal{A}) \ncong (\mathcal{N}, I)$ for any arbitrary Nilpotent Pattern DAE $(\mathcal{N}, I)$. In other words, $(\mathcal{E}, \mathcal{A})$ is not a Nilpotent DAE.

Now, one might wonder why $(\mathcal{E}, \mathcal{A})$ is not a Nilpotent DAE even though (E, A) is a Nilpotent DAE. This is a similar phenomena as the one happening in Example 5. By looking at where the rest of the nodes which are still colored white after the procedure for $G(\mathcal{A})$, namely the nodes $2, 9, 3$, the corresponding rows $2, 9, 3$ are clearly dependent, since

$$R_2 - R_3 - R_9 = 0 \tag{50}$$

Therefore, $\mathcal{A}$ is not invertible and $(\mathcal{E}, \mathcal{A}) \ncong (\mathcal{N}, I)$, since $\nexists S, T$ invertible matrices such that $S\mathcal{A}T = I$. This is due to rank$(I) \neq$ rank$(\mathcal{A}) =$ rank$(S\mathcal{A}T)$.

Rank Test with Regular DAE

Another possible application of the rank test is on showing the regularity of a DAE. As seen in the Preliminaries section, knowing that a DAE is regular guarantees the DAE has a unique solution. The approach done here will follow the subsection "Rank Test with Nilpotent DAE". Firstly, some rank conditions are introduced,

Theorem 8. *Let $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{m \times n}$ such that $(\mathcal{E}, \mathcal{A})$ is a Pattern DAE with $m = n$, if $\mathcal{A}$ is invertible then $(\mathcal{E}, \mathcal{A})$ is regular.*

Proof. Take arbitrary $E \in P(\mathcal{E})$ and $A \in P(\mathcal{A})$. Since $\mathcal{A}$ is invertible, $det(A) = det(-A) \neq 0$. Observe that $det(sE - A) \neq 0$ at $s = 0$, since $det(sE - A) = det(-A)$. Therefore, $det(sE - A)$ is not the zero polynomial. By definition, (E, A) is regular. Since the proof is done for arbitrary elements of $P(\mathcal{E})$ and $P(\mathcal{A})$, therefore $(\mathcal{E}, \mathcal{A})$ is regular. □

Theorem 9. *Let $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{m\times n}$ such that $(\mathcal{E}, \mathcal{A})$ is a Pattern DAE with $m = n$, if $\mathcal{E}$ is invertible then $(\mathcal{E}, \mathcal{A})$ is regular.*

Proof. Take arbitrary $E \in P(\mathcal{E})$ and $A \in P(\mathcal{A})$. Since $\mathcal{E}$ is invertible, $E^{-1} \in \mathbb{R}^{n\times n}$ exists such that $E^{-1} \in \mathbb{R}^{n\times n}$ is the inverse of E. From linear algebra,

$$rank(sI - \bar{E}^{-1}A) = n \text{ for all } s \notin \text{eigenvalues of } \bar{E}^{-1}A \tag{51}$$

Recall that multiplying a matrix with an invertible matrix does not change its rank. Multiply the matrix $(sI - E^{-1}A)$ from the right and left by E and I of appropriate dimensions, respectively. Therefore, for all $s \notin$ eigenvalues of $E^{-1}A$,

$$n = rank(E(sI - E^{-1}A)I) = rank(sE - A) \tag{52}$$

Therefore, $det(sE - A)$ is not the zero polynomial and by definition (E, A) is regular. Since the proof is done for arbitrary elements of $P(\mathcal{E})$ and $P(\mathcal{A})$, therefore $(\mathcal{E}, \mathcal{A})$ is regular. □

Secondly, integrating the rank test into Theorem 8 and Theorem 9,

Corollary 3. *Let $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{m\times n}$ such that $(\mathcal{E}, \mathcal{A})$ is a Pattern DAE with $m = n$, if either $G(\mathcal{A})$ or $G(\mathcal{E})$ is colorable then $(\mathcal{E}, \mathcal{A})$ is regular.*

Proof. Consider 2 cases:

1. Let $G(\mathcal{A})$ be colorable, then by Theorem 2, $\mathcal{A}$ has full row rank. Since $\mathcal{A}$ is a square matrix, then $\mathcal{A}$ has full rank. By Theorem 8, $(\mathcal{E}, \mathcal{A})$ is regular.
2. Let $G(\mathcal{E})$ be colorable, then by Theorem 2, $\mathcal{E}$ has full row rank. Since $\mathcal{E}$ is a square matrix, then $\mathcal{E}$ has full rank. By Theorem 9, $(\mathcal{E}, \mathcal{A})$ is regular.

□

As a remark, Corollary 3 only serves as sufficient conditions for regularity.

Example 10. *Let $(\mathcal{E}, \mathcal{A})$ be a Pattern DAE such that $\mathcal{E}, \mathcal{A} \in \{0, *, ?\}^{10\times 10}$,*

$$\mathcal{A} = \begin{bmatrix} 0 & ? & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ ? & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{E} = \begin{bmatrix} ? & * & 0 & ? & 0 & 0 & 0 & 0 & * & 0 \\ ? & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & ? & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & ? & 0 & 0 & 0 & * \\ 0 & 0 & ? & 0 & ? & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ? & 0 & 0 & * & 0 & ? & 0 & ? \\ 0 & * & 0 & 0 & 0 & 0 & ? & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ ? & 0 & ? & * & 0 & * & 0 & 0 & ? & * \end{bmatrix} \tag{53}$$

Apply the procedure of the rank test to $\mathcal{A}$ and $\mathcal{E}$,

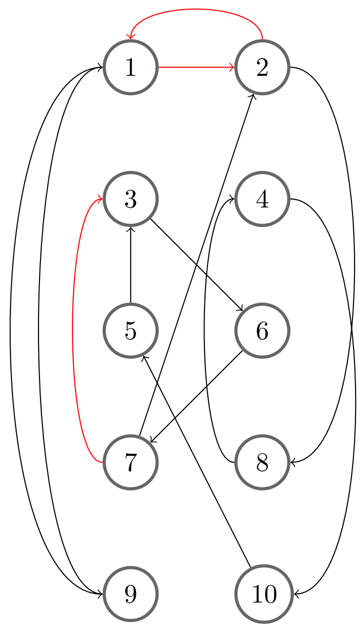

Figure 14: Graph G($\mathcal{A}$)

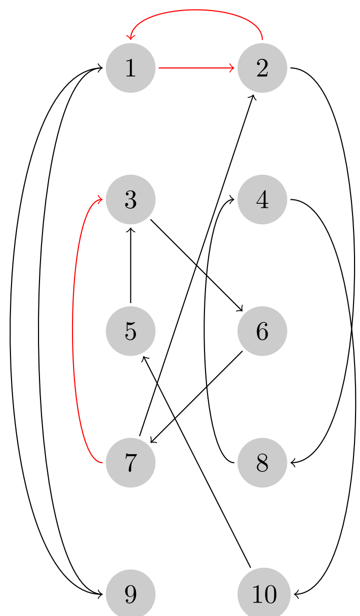

Figure 15: Graph G($\mathcal{A}$) after the rank test procedure. The sequence of the coloring is as follows: 3 colors 6, 4 colors 10, 5 colors 3, 6 colors 7, 7 colors 2, 8 colors 4, 9 colors 1, 10 colors 5, 1 colors 9, 2 colors 8

Since the nodes $1, 2, ..., 10$ *are colored black after the procedure is complete for* $G(\mathcal{A})$*,* $G(\mathcal{A})$ *is colorable. We can actually stop here and do not need to apply the rank test to* $\mathcal{E}$ *since by Corollary 3, if* $G(\mathcal{A})$ *is colorable, then* $(\mathcal{E}, \mathcal{A})$ *is regular.*

Conclusion

This thesis has tried to use graph theory used in the framework of controllability [1] to approach regularity. The notion of Pattern DAE is introduced and subjected to the graph theoretical approach of [1]. Then a rank condition on nilpotency is introduced for Pattern DAE. This allows for the application of the rank test. The first and main result is the necessary condition for a regular Pattern DAE to be nilpotent presented in Theorem 6. Some other results regarding sufficient conditions to prove the regularity of a Pattern DAE are also introduced in Theorem 8 and Theorem 9, by following the same method done for the nilpotent case. These theorems then are paired with the rank test in Corollary 2 and Corollary 3 as the full detailed application of the rank test. In each Results subsection, examples are also included in order to demonstrate the application of the theorems and corollaries.

For further research extension of the topic, one might want to attempt to prove the other direction of Theorem 6, namely giving a sufficient condition for a regular Pattern DAE to be nilpotent. As this thesis followed the idea of the proof of Theorem 6 in [1] for the necessary condition, it also might be a good place to start if one wants to attempt the other direction. Another idea to use graph theory in the framework of regularity is to introduce a sufficient and necessary condition for the regularity of Pattern DAE. One possible way to do this it to first introduce a similar rank condition as in Theorem 4. By definition, a DAE (E, A) with square matrices $E, A \in \mathbb{R}^{n\times n}$ is regular if and only if $det(sE - A)$ is not the zero polynomial. This is equivalent to

Theorem 10. *DAE (E, A) with square matrices $E, A \in \mathbb{R}^{n\times n}$ is regular if and only if rank(sE-A)=n for almost all $s \in \mathbb{C}$*

Proof. *"necessary"* Let the DAE (E, A) be regular. Then, $det(sE - A)$ is not the zero polynomial. Suppose that $det(sE - A) = P(s)$ for some nonzero polynomial $P(s)$. Consider cases of the consequences of this:

1. $det(sE - A) \neq 0$ for all $s \in \mathbb{C}$ NOT the root of $P(s)$
2. $det(sE - A) = 0$ for all $s \in \mathbb{C}$ the root of $P(s)$

Since $P(s)$ is a nonzero polynomial, $P(s)$ is of degree n integer with $n > 0$ or $P(s) = m$ with m strictly positive real value. If $P(s)$ is of degree n integer with $n > 0$, by the fundamental theorem of algebra, $P(s)$ has at least one root. Therefore, $det(sE - A) = 0$ for at least one $s \in \mathbb{C}$. if $P(s) = m$ with m strictly positive real value, then $det(sE - A) \neq 0$ for all $s \in \mathbb{C}$. Combining both "if statements", we can conclude that $det(sE - A) \neq 0$ for almost all $s \in \mathbb{C}$. In other words, $rank(sE - A) = n$ for almost all $s \in \mathbb{C}$. *"sufficient"* Let $rank(sE - A) = n$ for almost all $s \in \mathbb{C}$, then $det(sE - A) = P(s) \neq 0$ for some $s \in \mathbb{C}$. Therefore, $det(sE - A)$ is not the zero polynomial. By definition, (E, A) is regular. □

Theorem 10 can be extended into Pattern DAE $(\mathcal{E}, \mathcal{A})$ as was done in the Nilpotent Pattern DAE case. There are a couple of literature, such as [6],[7] that deal with checking similar rank conditions, although with a different type of matrix polynomial. For example, in [6], rank of $C_\lambda(sI - A_\lambda)B_\lambda$ is examined for almost all $\lambda \in \mathbb{R}$. [6] also make use of the graph theory in deriving their results.

In terms of pattern matrices, introducing signs(+ for positive non-zero numbers and - for negative non-zero numbers) instead of $*$ for all non-zero numbers might be worth looking into. Denote this by sign-definite pattern matrix. This would allow for more information retention when adding 2 pattern matrices together. Let $\mathcal{A}_{ij} = *$ and $\mathcal{B}_{ij} = *$ for some $A, B \in \{0, *, ?\}^{n\times n}$. Then $\mathcal{A}_{ij} + \mathcal{B}_{ij} =?$ if signs are not included. On the other hand, if signs are included, then

1. $* \bigoplus * =?$
2. $- \bigoplus + =?$
3. $+ \bigoplus - =?$
4. $+ \bigoplus + = +$
5. $- \bigoplus - = -$

Observe that not all the results of the nonzero elements addition between $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ are ? in otherwise would be all ? if only elements of $\{0, *, ?\}$ are considered. it also may be worth considering invertible pattern matrices $S, T \in \{0, *, ?\}^{n\times n}$ used in Weierstrass Canonical Form(WCF).

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