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Relation between different notions of monodromy

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ABSTRACT

Monodromy is a concept which arises in various areas of mathematics, including integrable systems and isolated critical points of holomorphic functions. It is concerned with the behavior of functions or geometric objects as they are moved around in space, specifically a singularity for our purposes. We will be looking more in the direction of Liouville theorem for integral systems and Picard-Lefschetz formula for holomorphic functions. Monodromy comes into play for Liouville theorem as it is an obstruction to the "global" version of this theorem. The Picard-Lefschetz formula can be used to explicitly describe monodromy around an isolated critical point of a holomorphic function. In the end we describe how these two notions of monodromy are related.

1 INTRODUCTION

Monodromy is a concept that arises in various areas of mathematics, including algebraic geometry, complex analysis, and topology. At its core, monodromy is concerned with the behavior of functions or geometric objects as they are moved around in space. Specifically, it studies the transformations that occur when a path is taken around a singularity or other geometric feature of an object. Monodromy has deep connections to many other areas of mathematics, including group theory, differential equations, and algebraic topology, and has important applications in physics, engineering, and computer science. In this way, monodromy plays a critical role in understanding the structure and behavior of complex systems across a wide range of fields.

We will be looking more the direction of integral systems and holomorphic functions. Specifically, the Liouville theorem and Picard-Lefschetz theory.

Integrable systems are a class of mathematical models that arise in many areas of physics and mathematics. These systems are characterized by the existence of a large number of conserved quantities, which allow for their solutions to be written down explicitly in terms of elementary functions. The study of integrable systems is closely related to monodromy, as the latter plays a critical role in understanding the behavior of solutions of these systems. Specifically, the monodromy properties of an integrable system are related to the symmetries of its solutions, which in turn are related to the algebraic and geometric properties of the system itself.

Picard–Lefschetz theory is a powerful tool in algebraic topology that is intimately related to the study of monodromy. Specifically, it concerns the behavior of certain types of singularities under deformation, and provides a way to compute topological invariants of algebraic varieties using the monodromy action. Picard–Lefschetz theory provides a way to overcome the difficulties of non-isolated critical values and non-smooth level sets by studying the topology of a complex algebraic variety V near the critical values of a holomorphic function f on V using the monodromy action. Specifically, it shows that the topology of the level sets of f changes in a predictable way as we move around the critical values of f in the complex plane. This change is related to the monodromy action of the vanishing cycles associated with the critical points of f.

2 INTEGRABLE SYSTEMS

In this section we will be working toward proving the Liouville Theorem. We will start with some definitions we will need to understand what the theorem is about. All definitions and theorems from this section can be found in [1].

Definition 1. Symplectic manifold
A symplectic manifold is a manifold which is endowed with a symplectic structure. This symplectic structure is a 2-form ω with the following 2 properties:

1. $d\omega = 0$, meaning that ω is closed
2. ω is non-degenerate.

Furthermore, a symplectic manifold M has the following properties:

1. M is even-dimensional
2. M is orientable

Being even-dimensional follows from ω being non-degenerate. We can define the orientation by the symplectic volume form which is of maximal rank and vanishes nowhere. Being of positive orientation corresponds to a positive volume form.

Definition 2. Skew-symmetric gradient
Let M be a symplectic manifold and let H be a smooth function on M. Then the Skew-symmetric gradient, or sgrad H is defined by:

$$\omega(\nu, \text{sgrad } H) = \nu(H)$$

With ν being an arbitrary tangent vector on M

Next we will define the Poisson brackets and some of their properties. These will be relevant for the proof of the Liouville Theorem.

Definition 3. Poisson brackets
Let f, g be smooth functions on a symplectic manifold M. We define $\{f, g\} := \omega(\text{sgrad } f, \text{sgrad } g) = (\text{sgrad } f)(g)$

Now for some properties of the Poisson bracket

Proposition 1. Properties of the Poisson bracket
The following are some properties of the Poisson bracket, not all of them will be used.

1) Bilinearity: {af + bg, h} = a{f, h}+b{g,h}, {h,af + bg} = a{h, f}+b{h,g}
2) Skew-symmetric: {f,g} = -{g,f}
3) Jacobi's identity: {g, {f, h}} + {h, {g, f}}+ {f, {h, g}}
4) The Leibniz rule: {fg, h} = f{g, h} + g{f, h}
5) Homeomorphism between the Lie algebra and smooth vector fields:
   $\text{sgrad } \{f, g\} = [\text{sgrad } f, \text{sgrad } g]$
6) Function f is a first integral of Hamiltonian vector field $\nu = \text{sgrad } H \iff \{f, H\} = 0$

We will not give a full proof of all of the properties, but we will give some directions to how they are proven. A full proof can be found in [1].

Proof. The bilinearity, skew-symmetry and first integral property of the Poisson bracket are evident from the definition of the Poison bracket.

Jacobi's identity follows from the Cartan formula:

$$s\omega(\xi, \eta, \zeta) = \xi\omega(\eta, \zeta) - \omega([\xi, \eta], \zeta) + \text{(cyclic permutation)}$$

where ω is an arbitrary 2-form, and ξ, η, ζ are vector fields. In the case that ω is a symplectic structure and $\xi = \text{sgrad } f$, $\eta = \text{sgrad } g$ and $\zeta = \text{sgrad } h$. This can be written into the Jacobi's identity.

The Leibniz rule follows from the similar rule for the skew-symmetric gradient:

$$\text{sgrad } fg = f\text{sgrad } g + g\text{sgrad } f$$

For property 5, we can differentiate a function h along the vector field sgrad $\{f, g\}$ to obtain:

$$\begin{aligned}
\text{sgrad } \{f, g\}(h) &= \{\{f, g\}, h\} \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\} \\
&= \text{sgrad } f(\text{sgrad } g(h)) - \text{sgrad } g(\text{sgrad } f(h)) \\
&= [\text{sgrad } f, \text{sgrad } g](h)
\end{aligned}$$

This covered all properties. □

The next two definitions are about manifolds and their (Liouville) integrability.

Definition 4. Smooth submanifolds
For (M^{2n}, ω) a symplectic manifold, $N \subset M$ is called a symplectic submanifold if the restriction of ω onto N is non-degenerate. $N \subset M$ is called Lagrangian if dim $N = n$ and the restriction of ω onto N vanishes identically.

Definition 5. Liouville integrable
Let M^{2n} be a smooth symplectic manifold, let H be a smooth function and let $v = \text{sgrad } H$ be a Hamiltonian system. A Hamiltonian system is called Liouville integrable if smooth functions $f_1, \dots, f_n$ exist such that the following 4 statements hold:

1) $f_1, \dots, f_n$ are integrals of v
2) $f_1, \dots, f_n$ are linearly independent on M almost anywhere
3) For any $i, j \in 1, 2, \dots n$, we have $\{f_i, f_j\} = 0$
4) The vector fields sgrad f_i are complete

The last definition definition we need, before tackling the main theorem is about Liouville foliation.

Definition 6. Liouville foliation
The Liouville foliation corresponding to integrable system $v = \text{sgrad } H$ is a decomposition of the manifold M^{2n} into connected components of common level surfaces of the integrals $f_1, \dots, f_n$. These integrals $f_1, \dots, f_n$ are preserved by the flow v, therefore every leaf is an invariant surface.

Liouville foliation consists of two types of leaves, regular leaves and singular leaves. The latter fills a set of zero measure. Now that we have all the pieces, we can start presenting the main theorem of this section. This Theorem describes the Liouville foliation near regular leaves.

Theorem 1. *J. Liouville*
Let $v =$ sgrad H be a Liouville integrable Hamiltonian system on M^{2n}, and let T_ξ be a regular level surface of the integrals $f_1, \dots, f_n$. Then

1) *T_ξ is a smooth Lagrangian submanifold that is invariant with respect to the flow $v =$ sgrad H and sgrad $f_1, \dots,$ sgrad f_n;*
2) *if T_ξ is connected and compact, then T_ξ is diffeomorphic to the n–dimensional torus T^n (this torus is called the Liouville torus);*
3) *the Liouville foliation is trivial in some neighborhood of the Liouville torus, that is, a neighborhood U of the torus T_ξ is the direct product of the torus T^n and the disc D^n;*
4) *in the neighborhood $U = T^n \times D^n$ there exists a coordinate system $s_1, \dots, s_n, \varphi_1, \dots, \varphi_n$ (which is called the action-angle variables), where $s_1, \dots, s_n$ are coordinates on the disc D^n and $\varphi_1, \dots, \varphi_n$ are standard angle coordinates on the torus, such that*
   a) *$\omega = \sum d\varphi_i \wedge ds_i$,*
   b) *the action variables s_i are functions of the integrals $f_1, \dots, f_n$,*
   c) *in the action-angle variables $s_1, \dots, s_n, \varphi_1, \dots, \varphi_n$, the Hamiltonian flow v is straightened on each of the Liouville tori in the neighborhood U, that is, $\dot{s}_i = 0, \dot{\varphi}_i = q_i(s_1, \dots, s_n)$ for $i = 1, 2, \dots, n$ (this means that the flow v determines the conditionally periodic motion that generates a rational or irrational rectilinear winding on each of the tori).*

Proof. 1). As we are in a Liouville integrable Hamiltonian system, the integrals $f_1, \dots, f_n$ are functionally independent. Therefore $[f_i, f_j] = f_i f_j - f_j f_i = f_i f_j - f_i f_j = 0$. Hence they commute. Therefore they are first integrals for $v =$ sgrad H and for each of the flows sgrad f_i. As they are first integrals, their common level surface T_ξ is invariant under these flows. As the flows are linearly independent, the vector fields sgrad $f_1, \dots,$sgrad f_n form a basis in every tangent plane of T_ξ. As $\omega(\text{sgrad } f_i, \text{sgrad } f_j) = \{f_i, f_j\} = 0$, we have that T_ξ is indeed a Lagrangian submanifold.

2) As the flows sgrad $f_1, \dots,$sgrad f_n pairwise commute and are complete, we can define an action φ from $\mathbb{R}^n$ to our manifold M^{2n}. This action φ will be generated by shifts along the flows sgrad $f_1, \dots,$sgrad f_n. This can be written explicitly as:

$$\varphi(t_1, \dots, t_n) = g_1^{t_1} g_2^{t_2} \dots g_n^{t_n}$$

Where g_i^t is the diffeomorphism shifting all the points of M^{2n} along the integral trajectories of the field sgrad f_i. Before continuing this proof, we will need to introduce some lemmas.

Lemma 1. If the submanifold T_ξ is connected, then it is an orbit of the $\mathbb{R}^n$-action .

Proof. We will consider the image of the group $\mathbb{R}^n$ in M under the following mapping:

$$A_x : (t_1, \dots, t_n) \mapsto \varphi(t_1, \dots, t_n)(x)$$

Here, x is a point in T_ξ. As the fields sgrad f_i are independent, this mapping is a local diffeomorphism onto the image. Therefore, the image of $\mathbb{R}^n$ is open in T_ξ. For a contradiction, assume that the submanifold T_ξ is not a single orbit of the group $\mathbb{R}^n$, then it is an union of at least two orbits. We know that both of them have to be open, therefore T_ξ becomes disconnected. but this contradicts our assumption of the submanifold T_ξ being connected. Hence, our lemma is proven. □

Now for the next lemma.

Lemma 2. An orbit $O(x)$ of maximal dimension of the action of the group $\mathbb{R}^n$ is the quotient space of $\mathbb{R}^n$ with respect to some lattice $\mathbb{Z}^k$. If $O(x)$ is compact, then $k = n$, and $O(x)$ is diffeomorphic to the $n-$dimensional torus.

Proof. Every orbit $O(x)$ of a smooth action of $\mathbb{R}^n$ is a quotient space of $\mathbb{R}^n$ with respect to the stationary subgroup H_x of the point x. As A_x is a local diffeomorphism, the subgroup H_x is discrete. A discrete subgroup has no accumulation points and inside a bounded set, there are only a finite number of elements in this subgroup.

Now we will proof by induction that H_x is a lattice $\mathbb{Z}^k$. Starting with $n = 1$, We can take a non-zero element e_1 of H_x on the line $\mathbb{R}^1$ such that e_1 is nearest to the origin. Then every element had to be a multiple of e_1. Suppose there is an element e which is not a multiple of e_1, then, for some k, we have:

$$ke_1 < e < (k+1)e_1$$

But this would mean that $e - ke_1$ is closer to the origin then e_1, which is a contradiction. Hence H_x is the lattice generated by e_1.

Now Suppose $n = 2$, for e_1 we choose a non-zero element such that it is closest to the origin and consider the straight line $l(e_1)$ generated by it. For our previous proof, we know that all element of H_x which are on the line $l(e_1)$ are multiples of e_1. It is possible for all elements to lie on $l(e_1)$, in which case the proof is complete. Otherwise there has to exist elements which are not on $l(e_1)$. Then, let e_2 be the element nearest to $l(e_1)$, which is non-zero and not on $l(e_1)$. Next we want to prove that all elements of H_x are a linear combination of e_1, e_2 with integer coefficients. For this we assume the contrary. This means that there exists element $h \in H_x$ which is not a linear combination of e_1, e_2 with integer coefficients. Using e_1, e_2 we can generate parallelograms on the plane and h has to be in one of these parallelograms. Moreover, h can not be on a vertex of a parallelogram. It should be clear that we can move h by an integer combination of e_1, e_2 to find an element closer to $l(e_1)$ than e_2. This gives a contradiction. Hence H_x is indeed a lattice generated by e_1, e_2.

We can continue this reasoning by induction to come to the conclusion that there exists a basis $e_1, e_2, \dots, e_k$ in the subgroup H_x such that each element is a unique linear combination of the basis vectors with integer coefficients. One can also say that H_x is a lattice generated by $e_1, e_2, \dots, e_k$.

If $K < n$, we have that the quotient space $\mathbb{R}^n/\mathbb{Z}^k$ is a cylinder. This cylinder is also a direct product $T^k \times \mathbb{R}^{n-k}$, where T^k is a k-dimensional torus. In particular, the orbit is compact for $n = k$ only. Therefore, $O(x)$ is diffeomorphic to the torus T^n. □

This proofs the second item.

3) This follows from the implicit function theorem. Suppose $f : M \mapsto N$ is a smooth mapping of smooth manifolds and $y \in N$ is a regular value of f. In other words, for each point of the preimage $f^{-1}(y)$, the rank of df is equal to the dimension of N. In particular, dim $M \geqslant$ dim N. If we additionally assume that $f^{-1}(y)$ is compact, then there exists a neighborhood D of a point y in N such that its preimage $f^{-1}(D)$ is diffeomorphic to the direct product $D \times f^{-1}(y)$. Furthermore, this structure is compatible with the mapping f in the sense that $f : D \times f^{-1} \mapsto D$ is the natural projection. From this it follows that each set $f^{-1}(z)$, with $z \in D$, is diffeomorphic to $f^{-1}(y)$.

4) Now we will start the construction of the action-angle variables. We consider a

neighborhood of the Liouville torus $U(T_\xi) = T_\xi \times D^n$. Then we choose a point x on each tori T depending smoothly on the torus. Consider T as $\mathbb{R}^n \setminus H_x$ and fix a basis $e_1, \dots e_n$ in the lattice H_x. Note that this is similar to what we did in second statement. This basis will depend smoothly on x. The coordinates of the basis vector $e_i = (t_1, \dots, t_n)$ are the solutions of the equation $\varphi(t_1, \dots, t_n)x = x$, where x is regarded as a parameter. The solutions of this equation depend on x smoothly as a result of the implicit function theorem. We can use this theorem as its assumptions holds that $\frac{\partial}{\partial t_j}\varphi(t)x = \text{sgrad } f_j(\varphi(t)x)$ and the vector fields sgrad f_j are linearly independent.

Now we will define the angle coordinates $(\psi_1, \dots, \psi_n)$ on the torus T_ξ int he following way. If $y = \varphi(a)x$, with $a = a_1e_1 + \dots + a_ne_n \in \mathbb{R}^n$, then $\psi_1(y) = 2\pi a_1 \mod 2\pi, \dots, \psi_n(y) = 2\pi a_n \mod 2\pi$. We have that these coordinates satisfy the property that the vector fields $\partial/\partial\psi_1, \dots, \partial/\partial\psi_n$ and sgrad $f_1, \dots,$ sgrad f_n are connected with a linear change of with constant coefficients. These constant coefficients are: $\partial/\partial\psi_i = \sum c_{ik}\text{sgrad } f_k$.

Now we will write ω in the coordinates $(f_1, \dots, f_n, \psi_1, \dots, \psi_n)$.

$$\omega = \sum_{i,j} \tilde{c}_{ij}\, df_i \wedge d\psi_j + \sum_{i,j} b_{ij}\, df_i \wedge df_j$$

As the Liouville tori are Lagrangian, the terms of the form $a_{ij}\, d\psi_i \wedge d\psi_j$ vanish. We claim that the coefficients $\tilde{c}_{ij}$ coincide with the coefficients c_{ij} and do not depend on $\psi_1, \dots, \psi_n$. We have that:

$$\begin{aligned}
\tilde{c}_{ij} &= \omega\left(\frac{\partial}{\partial f_i}, \frac{\partial}{\partial \psi_j}\right) \\
&= \omega\left(\frac{\partial}{\partial f_i}, \sum c_{kj}\text{sgrad } f_k\right) \\
&= \sum c_{kj}\,\omega\left(\frac{\partial}{\partial f_i}, \text{sgrad } f_k\right) \\
&= \sum c_{kj}\frac{\partial f_k}{\partial f_j} \\
&= c_{ij} \\
&= c_{ij}(f_1, \dots, f_n)
\end{aligned}$$

Next we will show that the functions b_{ij} do not depend on $(\psi_1, \dots, \psi_n)$. As ω is closed, we have:

$$\frac{\partial b_{ij}}{\partial \psi_k} = \frac{\partial c_{kj}}{\partial f_i} - \frac{\partial c_{ki}}{\partial f_j}$$

Function b_{ij} is 2π-periodic as it is a function on a torus, but its derivative, $\frac{\partial b_{ij}}{\partial \psi_k}$, is not dependent on ψ_k. Therefore b_{ij} does not depend on ψ_k.

These statements imply the following important corollary. We can write the form ω in the following way:

$$\begin{aligned}
\omega &= \left(\sum c_{ij}\, df_j\right) \wedge d\psi_i + \sum b_{ij}\, df_i \wedge df_j \\
&= \sum \omega_i \wedge d\psi_i + \beta,
\end{aligned}$$

Where $\omega_i = \sum c_{ij}\, df_j$ and $\beta = \sum b_{ij}\, df_i \wedge df_j$ are forms on the disc D^n. Therefore as ω is closed, ω_i and β are closed. The next lemma will be about the exactness of ω.

Lemma 3. In the neighborhood $U(T_\xi)$, the form ω is exact, which means that there exists a 1-form α such that $d\alpha = \omega$.

Proof. This Lemma is a special case of the following more general statement. Let Y be a submanifold of X and let there exists a mapping $f : X \mapsto Y \subset X$ which is homotopic to the identity mapping $id : X \mapsto X$. Then a closed form χ is exact on X if and only if its restriction χ_Y onto Y is exact. In our case we have that X is a neighborhood of the Liouville torus and Y is the Liouville torus itself. Furthermore, we have the stronger condition that $\omega_{T_\xi} = 0$ as T_ξ is Lagrangian. Hence, ω is exact. □

Next we will consider the functions $s_1 = s_1(f_1, \dots, f_n), \dots, s_n = s_n(f_1, \dots, f_n)$ and show that they are independent. From the formula $\omega = \sum ds_i \wedge d\psi_i + \beta$, we get that the matrix of the symplectic structure Ω is of the following form

$$\Omega = \left(\begin{array}{c|c} 0 & c_{ij} \\ \hline -c_{ij} & b_{ij} \end{array}\right)$$

Here, $c_{ij} = \frac{\partial s_i}{\partial f_i}$. The determinant of Ω is therefore $(\det C)^2$ and $\det C \neq 0$. Here, C is the Jacobi matrix of the transformation $s_1 = s_1(f_1, \dots, f_n), \dots, s_n = s_n(f_1, \dots, f_n)$. We can now consider the new system of independent coordinates $(s_1, \dots, s_n, \varphi_1, \dots, \varphi_n)$.

Now, we are going to represent χ in the form $\chi = g_i ds_i$ and let $\varphi_i = \psi_i - g_i(s_1, \dots, s_n)$. This has the effect of changing the initial points of reference for the angle coordinates on the Liouville tori. The level lines and even basis vector fields are not changed by this.

Lastly, we will show that the system of action-angle variables we constructed, $(s_1, \dots, s_n, \varphi_1, \dots, \varphi_n)$, are canonical.

$$\begin{aligned}
\sum ds_i \wedge d\varphi_i &= \sum ds_i \wedge d(\psi_i - g_i(s_1, \dots, s_n)) \\
&= \sum ds_i \wedge d\psi_i + \sum dg_i(s_1, \dots, s_n) \wedge ds_i \\
&= \sum ds_i \wedge d\psi_i + d\chi \\
&= \sum ds_i \wedge d\psi_i + \beta \\
&= \omega
\end{aligned}$$

Now we have constructed the action-angle coordinates. The final thing to proof is that the flow $v = \text{sgrad } H$ straightens on Liouville tori in coordinates $(s_1, \dots, s_n, \varphi_1, \dots, \varphi_n)$. As $\text{sgrad } s_i = \frac{\partial}{\partial \varphi_i}$, we have that $\frac{\partial H}{\partial \varphi_i} = \text{sgrad } s_i(H) = \{s_i(f_1, \dots, f_n), H\} = 0$. In other words, H is a function of only $s_1, \dots, s_n$. Therefore we have:

$$v = \text{sgrad } H = \sum_i \frac{\partial H}{\partial s_i} \text{sgrad } s_i = \sum_i \frac{\partial H}{\partial s_i} \frac{\partial}{\partial \varphi_i}$$

Furthermore, the coefficients $\frac{\partial H}{\partial s_i}$ depend only on the action variables $(s_1, \dots, s_n)$. In other words, they are constant on Liouville tori. This completes the proof of the Liouville theorem. □

Note that this theorem generally only works locally. When there is a singularity in play, this theorem start to fall apart. This is because monodromy come into play then. Monodromy is an obstruction to a global version of this theorem. We will cover more on what this means exactly later on.

3 HOLOMORPHIC FUNCTIONS

In this section we will take a look at focus-focus singularities and mainly use the lectures notes by J.P. Chassé [2]. Definitions and theorems in this section come from these notes, unless specified otherwise.

To start, we need to define the integrable system with two degrees of freedom that we will be working with.

Definition 7. Let (M, ω) be a 4-dimensional symplectic manifold, equipped with the Poisson bracket $\{\cdot, \cdot\}$ that we defined in the previous chapter (definition 3 and proposition 1). Let it also satisfy that any smooth function H on M gives rise to a Hamiltonian vector field denoted by X_H.

Lemma 4. (Complex Morse lemma) Let $f : M \mapsto \mathbb{C}$ be a holomorphic function, and $z \in M$ be a nondegenerate critical point of f. Then, there exist local holomorphic coordinates $z_1, \dots, z_n$ centered at z such that

$$f(z_1, \dots, z_n) = f(0) + \sum_{i=1}^{n} z_i^2$$

We will not provide the proof here, however the full proof can be found in [3]. A consequence of this Lemma is that there are no local invariant of critical points of complex Morse functions.

Now we will take a look at the monodromy of a non-degenerate critical point. Let M be a complex manifold of dimension $n = 2$, and $f : M \mapsto \mathbb{C}$ be a holomorphic function with a non-degenerate critical point $z \in M$. We can assume, without loss of generality, that $f(z) = 0$. Take $\epsilon > 0$ such that the complex Morse theorem holds for $|z_1|^2 + |z_2|^2 \leqslant 4\epsilon^2$. At the price of a multiplication of f and the z_i's with $\epsilon^{\pm 1}$, we can assume that $\epsilon = 1$. We will fix these local coordinates for the rest of the section and we will denote by B_r, $r < 2$, the closed 2n-ball in these coordinates. As we took $n = 2$, this becomes the closed 4-ball. Furthermore, we restrict f to this ball for this section, $f = f\,|_{B_2}$.

As there are no critical points of f on the boundary of B_2, $f\,|_{B_2}$ is a submersion onto D_2, which is the disk of radius r in $\mathbb{C}$, with a compact domain. This fulfills the conditions of Ehresmann's theorem, which then concludes that it is a fiber bundle over D_2. Ehresmann's theorem will not be covered here, but more information about this theorem can be found at [4]. Furthermore, as D_2 is contractible, the bundle is trivial. By the same argument, f gives a potentially nontrivial fiber bundle $B_2 \setminus f^{-1}(0) \mapsto D_2 \setminus \{0\}$.

Let $F_\lambda = f^{-1}(\lambda)$ be the fiber in B_2 over $\lambda \in D_2$. If λ is a regular value of f ($\lambda \neq 0$), then F_λ is a compact complex manifold of dimension $n-1$ and its boundary is $\partial F_\lambda = F_\lambda \cap \partial B_2$. Now we will consider the loop $\gamma : [0, 1] \mapsto D_2 \setminus \{0\}$ based at 1. As the bundle $f\,|_{S_2}$: $S_2 \mapsto D_2$ is trivial, we can take diffeomorphisms $g_t = g(-, t) : \partial F_1 \mapsto \partial F_{\gamma(t)}$ such that $g_1 = \mathbb{1}_{F_1}$. Because of the relative homotopy lifting property of fibrations, we know there now exists a map $\Gamma : F_1 \times [0, 1] \mapsto B_2 \setminus F_0$, which makes the following diagram commute:

$$\begin{tikzcd}
(F_1 \times \{0\}) \cup (\partial F_1 \times [0,1]) \arrow[r, "\iota \cup g"] \arrow[d, hook] & B_2 \setminus F_0 \arrow[d, "f"] \\
F_1 \times [0,1] \arrow[r] \arrow[ur, dotted, "\Gamma"] & D_2 \setminus \{0\}
\end{tikzcd}$$

Here, the bottom arrow does nothing with the F_1 variable, but is equal to γ in the $[0,1]$ one. Specifically, we have that $\Gamma_t := \Gamma(0,t)$ sends F_1 to $F_{\gamma(t)}$. Also, up to homotopy, Γ only depends on the homotopy class of γ as a loop in $D_2 \setminus \{0\}$.

Definition 8. The transformation $h_\gamma := \Gamma_1 : F_1 \mapsto F_1$ is called the monodromy of γ, whilst the induced morphism $(h_\gamma)_\star$ on homology (with integer coefficients) is called the monodromy operator.

When you take the monodromy operator of a loop $\gamma : [0,1] \mapsto D_2 \setminus \{0\}$, it induces a morphism $H_\bullet(F_1, \partial F_1) \mapsto H_\bullet(F_1)$ as follows. Let δ be a relative cycle of $(F_1, \partial F_1)$, this means that $\delta \in C_\bullet(F_1)$ and $\partial\delta \in C_{\bullet-1}(\partial F_1)$. Any element of $H_\bullet(F_1, \partial F_1)$ can be represented by such a chain. By our construction we have that $h_\gamma |_{\partial F_1} = \mathbb{1}_{\partial F_1}$. Therefore, $h_\gamma \delta - \delta$ is actually a cycle of F_1 and it thus defines a class $\text{var}_\gamma[\delta]$ in $H_\bullet(F_1)$. With a direct calculation one can show that $\text{var}_\gamma[\delta]$ does not depend on the relative cycle representing the homology class.

Definition 9. The ensuing group homomorphism

$$\text{var}_\gamma : H_{n-1}(F_1, \partial F_1) \mapsto H_{n-1}(F_1)$$

is called the variation operator of γ

This gives rise to the following relations:

$$(h_\gamma)_\star = \mathbb{1} + (\text{var}_\gamma) j \quad \text{and} \quad (h_\gamma^{(r)})_\star = \mathbb{1} + j(\text{var}_\gamma) \tag{1}$$

Here, $h_\gamma^{(r)}$ is similar to h_γ, but it is seen as a relative map $(F_1, \partial F_1) \mapsto (F_1, \partial F_1)$, and $j : C_\bullet(F_1) \mapsto C_\bullet(F_1, \partial F_1)$ is the canonical map. These operators are well-behaves with respect to concatenation of loops:

$$\begin{aligned} (h_{\gamma_1\gamma_2})_\star = (h_{\gamma_2})_\star (h_{\gamma_1})_\star, \quad (h^{(r)}_{\gamma_1\gamma_2})_\star = (h^{(r)}_{\gamma_2})_\star (h^{(r)}_{\gamma_1})_\star \\ \text{var}_{\gamma_1\gamma_2} = \text{var}_{\gamma_1} + \text{var}_{\gamma_2} + \text{var}_{\gamma_2} j \text{var}_{\gamma_1} \end{aligned} \tag{2}$$

The first two formulas follow from the fact that $\text{var}_{\gamma_1\gamma_2}$ can be chosen to be the concatenation of var_{γ_1} and var_{γ_1}, which is homotopically relative to ∂F_1 to the composition of the two. The last formula follows from the first two.

At the start of this section we set $n = 2$ for the dimension of our complex manifold, this allows us to somewhat visualise what we have been talking about so far. When $n = 2$ we can make the following change of variables $x = z_1 + iz_2$ and $y = z_1 - iz_2$ on what we got from the Complex Morse Lemma. This allows us to transform $z_1^2 + z_2^2 = \lambda$ into $xy = \lambda$. In the real 2-ball in these coordinates, This fibre can take on the shapes according to the following picture:

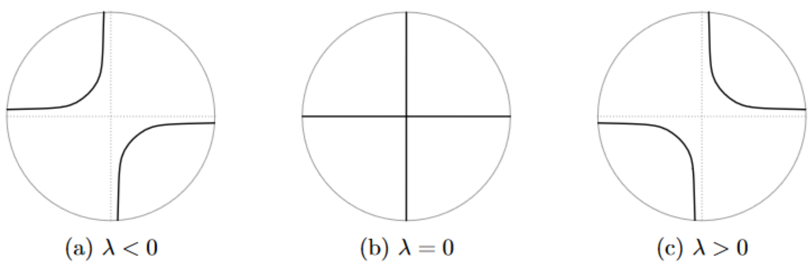
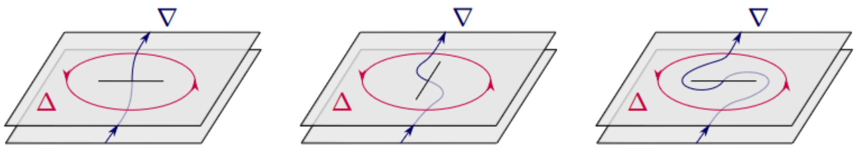


Figure 1: Real pictures of F_λ, Picture can also be found in [2]

One can think of an apposing pair of points in these pictures as being the intersection of a circle in $\mathbb{C}^2$ with $\mathbb{R}^2$. This makes the fibers corresponding to (a) and (c) cylinders and makes the fibers corresponding to (b) a cone.

To make this more rigorous, we can identify F_λ with the Riemann surface corresponding to the holomorphic function $w = \sqrt{\lambda - z^2}$ over D_2. When $\lambda \neq 0$, this surface can be obtained by making a cut in two copies of D_2 along the the line from $-\sqrt{\lambda}$ to $\sqrt{\lambda}$ and gluing these disks together along the boundary of the cut we just created. This is illustrated in the following picture where the line in the middle of the sheets is the cut we created.



(a) $t = 0$ (b) $t = \frac{1}{2}$ (c) $t = 1$

Figure 2: Image of Δ and ∇ under Γ_t.[2]

We can consider the loop $\gamma(t) = e^{2\pi i t}$ in $D_2 \setminus \{0\}$. As $[\gamma]$ generates $\pi_1(D_2 \setminus \{0\})$, it follows from (2) that we only need to look at this loop to understand the monotonicity operator. We can take its lift $\Gamma_t : F_1 \mapsto F_{\gamma(t)}$ to be

$$\Gamma_t(z_1, z_2) = \exp(\pi i t \chi(|z_1|^2 + |z_2|^2))(z_1, z_2)$$

Here, $\chi : [0, \inf) \mapsto [0, 1]$ is a smooth map such that $\chi \mid_{[0,2]} \equiv 1$ and $\chi \mid_{[3,\inf]} \equiv 0$. Then we take representatives Δ and ∇ of the generator of $H_1(F_1)$ and $H_1(F_1, \partial F_1)$ respectively, such that their intersection number $\Delta \cdot \nabla = 1$. This way we can see the effect of Γ_t on them.

From figure 2, we see that Δ stays unchanged while ∇ gets twisted in the opposite direction of Δ. This can be written down more explicitly in homological terms:

$$(h_\gamma)_\star(\Delta) = \Delta \quad \text{and} \quad (h_\gamma^{(r)})_\star(\nabla) = \nabla - j(\Delta)$$

therefore

$$\mathrm{var}_\gamma(\nabla) = -\Delta$$

Note that we have used the same symbol for the homology class and its representative here. The variation operator is trivial in other degrees, since $H_0(F_1, \partial F_1) = 0$ and $H_2(F_1) = 0$. One can think of this variation operator as "How does it change?". Here we have that ∇ changes by $-\Delta$ to become $\nabla - \Delta$ after having gone around the singularity.

The $(n-1)$-sphere or radius 1 embeds $\lambda \in D_2 \setminus \{0\}$ into the fiber F_λ as the following set:

$$S(\lambda) := \left\{ (z_1, \dots, z_n) \in B_2 \mid z_j = \sqrt{|\lambda|} \exp(\frac{i}{2} \arg(\lambda)) x_j, x_j \in \mathbb{R}, \sum_{j=1}^{n} x_j^2 = 1 \right\}$$

Furthermore, this embedding depends smoothly on $\lambda \in D_2 \setminus [-2, 0]$.

Definition 10. The homology class $\Delta \in H_{n-1}(F_1)$ represented by $S(1) \subseteq F_1$ is called the vanishing cycle

The following Lemma will show us that the chosen nomenclature is well-chosen: Δ is precisely the homology class of F_λ that vanishes as $\lambda \to 0$.

Lemma 5. The embedding $S^{n-1} \hookrightarrow F_\lambda$ extends to a diffeomorphism from a disk subbundle of TS^{n-1} onto F_λ.

Proof. We will only prove this for the case that $\lambda = 1$, as the general case follows similarly. Writing $z_j = u_j + iv_j$, for $u_j, v_j \in \mathbb{R}$, we have

$$
\begin{aligned}
F_1 &= \left\{ \sum_{j=1}^{n} z_j^2 = 1, \sum_{j=1}^{n} |z_j|^2 \leqslant 4 \right\} \\
&= \left\{ \sum_{j=1}^{n} (u_j^2 - v_j^2) = 1, \sum_{j=1}^{n} u_j v_j = 0, \sum_{j=1}^{n} (u_j^2 + v_j^2) \leqslant 4 \right\} \\
&= \left\{ \sum_{j=1}^{n} x_j^2 = 1, \sum_{j=1}^{n} x_j y_j = 0, \sum_{j=1}^{n} y_j^2 \leqslant \frac{3}{2} \right\}
\end{aligned}
$$

where $x_k = u_k / \sqrt{\sum_j u_j^2}$ and $y_k = v_k$. But the last set naturally identifies with the radius $\sqrt{\frac{3}{2}}$ disk subbundle of $TS^{n-1} \subseteq \mathbb{R}^{2n}$. the inverse morphism

$$
(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \left(x_1 \sqrt{1 + \sum_j y_j^2} + iy_1, \ldots, x_n \sqrt{1 + \sum_j y_j^2} + iy_n, \ldots \right)
$$

is therefore the diffeomorphism we were initially looking for. $\square$

Corollary 1. *We have:*

$$
H_k(F_1) = \begin{cases} \mathbb{Z} \textit{ if } k = 0, n-1 \\ 0 \textit{ otherwise} \end{cases} \qquad \textit{and } H_k(F_1, \partial F_1) = \begin{cases} \mathbb{Z} \textit{ if } k = n-1, 2(n-1) \\ 0 \textit{ otherwise} \end{cases}
$$

The variation operator is zero on all degrees except $n-1$ for any path in $\mathbb{D}_2 \setminus \{0\}$.

As we are mostly interested in the $n-1$ case, we have that $H_{n-1}(F_1) = \mathbb{Z}$ and $H_{n-1}(F_1, \partial F_1) = \mathbb{Z}$ for our purposes.

Proof. The calculation of the homology of F_1 follows from the fact that a disk bundle deformation retracts onto the image of the zero section. This is then naturally identified with S^{n-1}. The one on the relative homology of $(F_1, \partial F_1)$ follows from Poincare-Lefschetz duality. That part will not be covered here, but more information on it can be found in [5]. $\square$

The variation operator can be nontrivial in degree $n-1$. In order to find its relation, we will fix the generator ∇ of $H_{n-1}(F_1, \partial F_1)$ such that $\nabla \cdot \Delta = 1$. Here, $\cdot$ denoted the intersection product. Furthermore, because of the relations specified in (2), we only need to look at the case $\gamma(t) = e^{2\pi i t}$. Therefore, it will be omitted from the equation.

Theorem 2.

$$
var(\nabla) = (-1)^{\frac{n(n+1)}{2}} \Delta
$$

We can use the fact that $H_{n-1}(F_1) = \mathbb{Z} \cdot \Delta$, $H_{n-1}(F_1, \partial F_1) = \mathbb{Z} \cdot \nabla$ and the relations defined in (1) to get the formulas for the monodromy and variation operators described in the next corollary.

Corollary 2. *(Picard-Lefschetz formulas)*
For any $a \in H_{n-1}(F_1, \partial F_1)$ and $b \in H_{n-1}(F_1)$, we have

$$
\begin{aligned}
var(a) &= (-1)^{\frac{n(n+1)}{2}} (a \cdot \Delta) \Delta \\
h_\star^{(r)}(a) &= a + (-1)^{\frac{n(n+1)}{2}} (a \cdot \Delta) j(\Delta) \\
h_\star(b) &= b + (-1)^{\frac{n(n+1)}{2}} (b \cdot \Delta) \Delta
\end{aligned}
$$

4 BACK TO INTEGRABLE SYSTEMS

Now that we have talked about Picard-Lefschetz, we will continue on integrable systems and start connecting the dots on monodromy among these subjects.

Definition 11. Momentum Map
Let M^{2n} be a symplectic manifold with an integrable Hamiltonian system $v =$ sgrad H , and $f_1, \dots, f_n$ be its independent integrals in involution. Let us define the smooth mapping

$$F : M^{2n} \mapsto \mathbb{R}^n, \quad \text{where } F(x) = (f_1(x), \dots, f_n(x))$$

This mapping is called the momentum mapping.

Let L be a singular leaf of a Liouville foliation and let $y = F(L)$ be its image under the momentum mapping. Consider a circle γ_ϵ, with with small radius ϵ, centered at the point y and its preimage $Q_{\gamma_\epsilon} = F^{-1}(\gamma_\epsilon)$. This can be seen illustrated in figure (3). The 3−manifold Q_{γ_ϵ} is a fiber bundle over the circle γ_ϵ whose fibers are Liouville tori T^2. This fiber bundle is completely determined by the monodromy group.

Definition 12. Monodromy Group
The Monodromy Group is the group of automorphisms of the fundamental group of a fiber $\pi_1(T^2)$ corresponding to closed loops on the base. As $\pi_1(T^2) = \mathbb{Z} \bigoplus \mathbb{Z}$ and the base is the circle γ_ϵ, the monodromy group is a cyclic subgroup in the automorphism of $\mathbb{Z} \bigoplus \mathbb{Z} \subset \text{SL}(2,\mathbb{Z})$

Note that monodromy for integrable systems can also be defined similarly to (8) with the homotopic lifting property. This is using the same commutative diagram except that we do not have a boundary in this case.

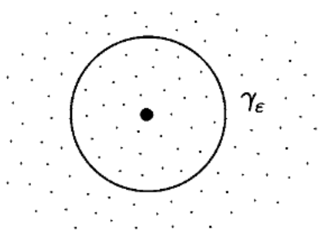


Figure 3: Loop around a critical point [1]

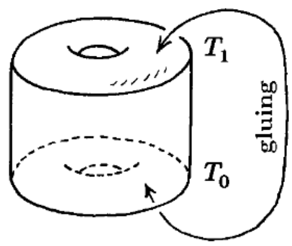


Figure 4: Gluing of tori [1]

Q_{γ_ϵ} can be represented as the result of identification of the boundary tori T_0 and T_1 of the 3-cylinder $T^2 \times [0, 1]$ by some diffeomorphism $\varphi : T_0 \mapsto T_1$. This is illustrated in 4. This automorphim is uniquely defined to be an integer unimodular matrix, which depends an the choice of basis on the torus. Its conjugacy class is a well-defined complete invariant of the fiber bundle $Q_{\gamma_\epsilon} \xrightarrow{T^2} \gamma_\epsilon$. This matrix is called the monodromy matrix.

5 RESULTS AND DISCUSSION

The next theorem will connect Picard-Lefschetz theory to integrable systems.

Theorem 3. *The Picard-Lefschetz formula implies that the monodromy of an integrable system on a symplectic 4-manifold M^4 around a focus-focus singularity is given by the matrix* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Definition 13. A momentum mapping $F = (f_1, f_2) : M \mapsto \mathbb{R}^2$ has a focus-focus singularity x_0 if there exists $c_1(f_1, f_2), c_2(f_1, f_2)$ such that:

$$\begin{cases} c_1 = q_1p_1 + q_2p_2 \\ c_2 = q_1p_2 - q_2p_1 \end{cases} \quad \text{And } (c_1, c_2) : \mathbb{R}^2 \mapsto \mathbb{R}^2 \text{ is a local diffeomorphism.}$$

We can without loss of generality assume that (c_1, c_2) is globally defined in a neighborhood of a focus-focus fiber. When we take $F = (c_1, c_2)$, and switch to using complex coordinates we can transform (c_1, c_2) to $(c_1 + ic_2)$. Then we can apply a chance of coordinates twice to get the following:

$$zw, \begin{cases} z = q_1 - iq_2 \\ w = p_1 + ip_2 \end{cases}$$

$$u^2 + v^2, \begin{cases} z = (u - iv) \\ w = (u + iv) \end{cases}$$

Notice that $zw = u^2 + v^2$ here as well. The coordinates in z, w are locally defined. Now we want to go from the locally defined coordinates on the cylinder from Picard-Lefschets Theory, to globally defined coordinates on the pinched torus from the Liouville Theorem.

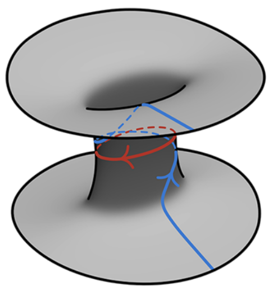


Figure 5: Cylinder from Picard-Lefschetz [6]

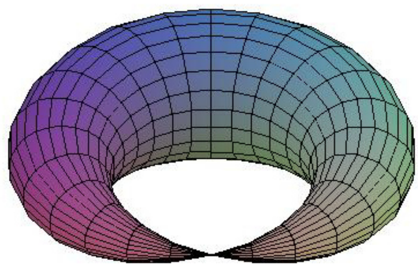


Figure 6: Pinched torus from Liouville Theorem [7]

Note that the pinched torus will not be pinched to a point for our purposes. This would happen only when the loop goes through the singularity and the loop goes around the singularity in our case.

From a cylinder one can make a torus by gluing the boundary together. Because any circle bundle over D^2 is trivial, we can do this without a problem. The absolute cycle of the cylinder will be the same as the cycle of the torus which gets pinched and the relative cycle will be the other absolute cycle of the torus.

When comparing the different notions of monodromy, (def 8) for Picard-Lefschetz theory and (def 12), we also have major similarities which are pointed out under definition (12).

There is in fact a more detailed description of what happens with the torus fibration around a focus-focus singularity. Specifically, one can show that one of the action coordinates, see Section 2, is given by

$$s_1 = \frac{1}{2\pi} \int \alpha = S(c) - \mathrm{Re}(c \ln c - c), \tag{3}$$

where $c = c_1 + ic_2$ (and $s_2 = c_2$) and α is and 1-from which is exact. The function $S(c)$ classifies such torus fibration near the focus-focus point in the precise sense of [8]. One can see the monodromy from the appearance of the ln in the above equation. We are dealing with complex numbers and therefore ln is multi valued. This implies the monodromy as after a rotation, you will not end up where you started.

For holomorphic functions we have the following result: When looking at figure 5, the red line (or the cycle Δ from figure 2) stays the same while the blue line (or the cycle ∇ from figure 2) gains a twist in the opposite direction of red line. This can also be explained from the Picard-Lefschetz formulas (2) when using $n = 2$. The last two equations then become:

$$h_\star^{(r)}(a) = a - j(\Delta)$$
$$h_\star(b) = b$$

The latter, which corresponds to the red line, stays constant and the former, which corresponds to the blue line, is transformed in the opposite direction of the red line.

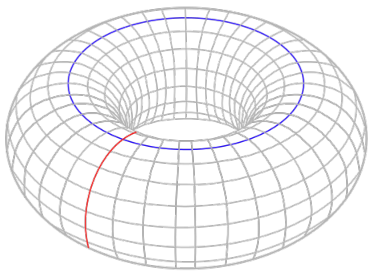


Figure 7: Here we have a torus with its two cycles coloured [9]

We obtain a similar result for the monodromy of an integrable system on a symplectic 4-manifold M^4. The Liouville tori undergo a similar transformation when they go around a focus-focus singularity. We also have here that the red cycle will stay the same while the blue cycle gains a twist in the opposite direction of the red cycle. The appearance of the ln in (3) is the reason this happens.

We can think of this transformation, which happens on both cases, as the following matrix transformation:

$$\begin{bmatrix} \Delta & \nabla \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Delta & \nabla + \Delta \end{bmatrix}$$

Here we have our monodromy matrix that acts on the cycles of the cylinder or torus. This matrix illustrates what happens to the cycles after having gone around the singularity. Δ stays the same while ∇ transforms into $\nabla + \Delta$. This completes the proof of Theorem 3. $\square$

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