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A Fourier-Series representation of periodic solutions in the Lid-Driven Cavity

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Abstract

The Lid-Driven Cavity is a popular benchmark for fluid simulation programs and computers. It has been observed that, after a critical Reynolds Number, the solutions showcase periodic behavior. In this paper we attempt to take advantage of this periodicity to find a Fourier-Series representation of the solutions, which could lead to better computational efficiency. We take a Galerkin approach to the time discretization and propose some solution algorithms.

1 Introduction

The Lid-Driven Cavity (LDC) is a problem in incompressible viscous fluid dynamics ([1]). It consists of a fluid within a rectangular container (the *cavity*), of which one of the walls moves at a constant velocity in one direction (the *lid*). This movement causes shear stress, which drives the flow within the cavity ([1]).

This problem is well known in the field of numerical mathematics and fluid simulations. It is frequently used as a benchmark to compare the accuracy and efficiency of different methods and equipment to simulate fluid flow, as well as providing a simple system to understand viscous flows driven by shear stress in small containers ([3]). Examples of such methods include an extended system method and standard time integration ([5]).

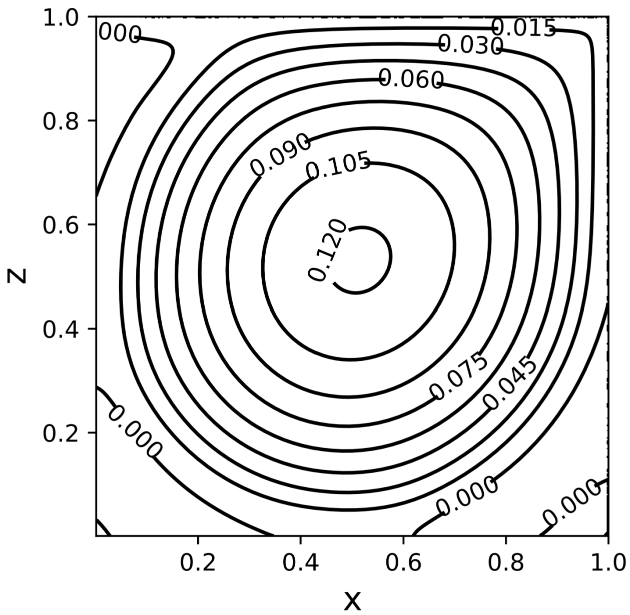

Figure 1: Example of a simulated LDC. The lid is the wall on top. moving towards the right.

There are multiple variations of this problem. For example, the number of dimensions that are accounted for: In a 2-dimensional case, the cavity is a rectangular plane section and the walls are lines, while in a 3-dimensional case, the cavity is a rectangular prism and the walls are rectangular planes. This thesis focuses on a 2-dimensional cavity.

Motivation

The Reynolds Number is an important dimensionless constant that helps identify certain characteristics of a fluid flow ([2]). In the LDC, it is determined by the aspect ratio of the container, the velocity of the lid and the viscosity of the fluid. Performing a bifurcation analysis on the Reynolds Number reveals a Hopf bifurcation at around a critical $Re_{\text{crit}} \approx 8200$ ([5]). Before this Reynolds Number there is only a steady, stable state. For Reynolds Numbers greater than the critical value, however, periodic solutions appear; plus another steady, though unstable, solution ([5]).

Past this critical Reynolds Number, the total kinetic energy of the LDC over time can be computed ([5]), leading to a 1-dimensional quantity that makes periodicity simple to verify, as in Figure (2). Note the periodic behavior in one of the solutions, and the steady behavior of the other one.

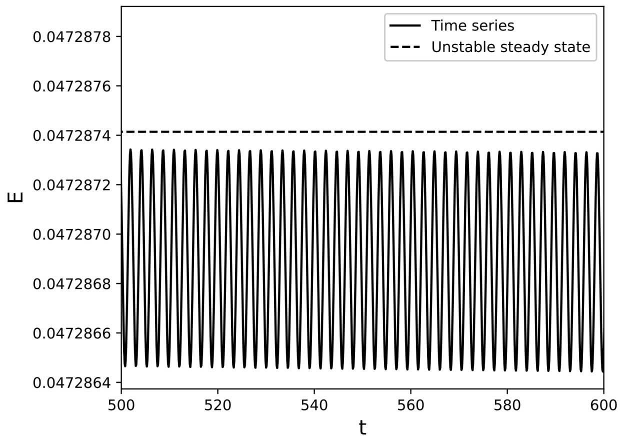

Figure 2: Total kinetic energy over time.

This periodic behavior raises the question of whether it is possible to represent these solutions with a Fourier Series and how this representation would look like. Particularly, we're interested in the possibility of using the periodic nature of the Fourier Series to represent the solutions with only a few terms of the Series; which could reduce the time, processing power and storage required to simulate the solutions.

Given this, we propose some tentative goals for research on *Fourier-Series representations of periodic solutions in the Lid-Driven Cavity*:

1. Derive a time-discretization using a Fourier Series.

2. Implement this discretization.
3. Compute Fourier coefficients and compare it to other methods.

Due to time constraints, this thesis will only focus on goal 1.

2 Discretization

We recall that we are working with viscous flow. Therefore, the flow must satisfy the (2-dimensional, incompressible) Navier-Stokes equations ([5]):

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} & = -\frac{\partial vv}{\partial x} + \frac{\partial wv}{\partial y} - \frac{\partial p}{\partial x} + \frac{\Delta v}{Re} \\ \frac{\mathrm{d}w}{\mathrm{d}t} & = -\frac{\partial vw}{\partial x} + \frac{\partial ww}{\partial y} - \frac{\partial p}{\partial y} + \frac{\Delta w}{Re} \\ 0 & = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \end{cases} . \tag{1}$$

Here, v, w represent the velocity of flow with respect to the x, y coordinates, respectively; Δ is the Laplace operator and Re is the Reynolds Number.

The Navier-Stokes equations are a well known set of Partial Differential Equations (PDEs) that give complications when attempting to find analytic solutions outside of very specific cases, which further motivates us to employ numerical methods for their resolution. The equations will be discretized in multiple steps; first a simple discretization in the space coordinates, and later the discretization in time using a Fourier Series.

2.1 Space Discretization

As the main focus of this article is the discretization of the time variable, the choice for a space discretization is not very relevant. In preliminary implementations of the methods presented in this paper, a Finite Volume Method was employed, through the program TransiFlow–BIMAU ([4]), though other discretization methods are not expected to have a significant impact on the output.

The discretization in time will assume that we have a space discretization with n_x and n_y steps in the x- and y-axes respectively. Thus, the total size of the discretization will be $n = n_x n_y$.

It is expected that the space discretization yields a *state vector* of the following form:

$$u = (v_1, w_1, p_1, v_2, w_2, p_2, \cdots, v_n, w_n, p_n)^\mathsf{T},$$

where v_i, w_i, p_i are the velocity of flow in the x, y axes and pressure respectively, computed at the ith point of the discretization. Since we work with 3 variables, we have $u \in \mathbb{R}^{3n}$. The points of the discretization are assumed to be ordered left to right, then top to bottom. For example, for $n_x = n_y = 3$, the top-middle point is $i = 2$ and the left-middle point is $i = 4$.

After this discretization, equation (1) now looks as follows, for $1 \leq i \leq n$:

$$\begin{cases} \frac{\mathrm{d}v_i}{\mathrm{d}t} &= -\frac{\partial v_i v_i}{\partial x_i} + \frac{\partial w_i v_i}{\partial y_i} - \frac{\partial p_i}{\partial x_i} + \frac{\Delta v_i}{Re} \\ \frac{\mathrm{d}w_i}{\mathrm{d}t} &= -\frac{\partial v_i w_i}{\partial x_i} + \frac{\partial w_i w_i}{\partial y_i} - \frac{\partial p_i}{\partial y_i} + \frac{\Delta w_i}{Re} \\ 0 &= \frac{\partial v_i}{\partial x_i} + \frac{\partial w_i}{\partial y_i} \end{cases}.$$

Note that the v_i, w_i, p_i are defined as above, x_i, y_i are the points of the discretization grid and, most remarkably, $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ are discrete derivatives, according to the chosen discretization method.

Now, in order to simplify the look of the equations we shall define the following operators:

- M – Acting as a *mass matrix*, it is defined in a way that creates an appropriate left hand side, as it is multiplied by the state vector u:

$$M_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} M_1 & 0 & \cdots \\ 0 & M_2 & \\ \vdots & & \ddots \end{pmatrix} \quad \text{for } 1 \leq i \leq n.$$

- $a(\cdot, \cdot)$ – A bilinear function that accounts for the bilinear derivatives when computed with the state vector as both arguments (u, u):

$$(a(u, v))_{i:i+2} = \begin{pmatrix} -\frac{\partial u_i v_i}{\partial x_i} + \frac{\partial u_{i+1} v_i}{\partial y_i} \\ -\frac{\partial u_i v_{i+1}}{\partial x_i} + \frac{\partial u_{i+1} v_{i+1}}{\partial y_i} \\ 0 \end{pmatrix} \quad \text{for } i\%3 = 1.$$

  Regarding the notation on the left hand side, $v_{i:i+j}$ refers to all entries of the vector v between, and including, entries number i and $i + j$. We use % as the modulo operator.

- $\mathcal{L}$ – A linear map that takes care of the remaining terms of the right hand side, when computed at the state vector u:

$$\mathcal{L}_i = \begin{pmatrix} \frac{\Delta}{Re} & 0 & -\frac{\partial}{\partial x_i} \\ 0 & \frac{\Delta}{Re} & -\frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial x_i} & \frac{\partial}{\partial y_i} & 0 \end{pmatrix}$$
$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & 0 & \cdots \\ 0 & \mathcal{L}_2 & \\ \vdots & & \ddots \end{pmatrix} \text{ for } 1 \le i \le n.$$

After this, the equations can be written in the following, simpler form:

$$M\frac{\mathrm{d}u}{\mathrm{d}t} = a(u,u) + \mathcal{L}u =: F(u).$$

Then, we can introduce an existing CCC, unstable steady state $\bar{u}$, so one that satisfies $F(\bar{u}) = 0$ in order to further simplify the equations:

$$M\frac{\mathrm{d}u}{\mathrm{d}t} = T(Ju + a(u,u)), \tag{2}$$

where, as we recall, T is the period of the solutions, $u \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$, $J = \mathcal{J}(\bar{u} \in \mathbb{R}^{n\times n}$ is the Jacobian of F with respect to space, computed at the steady state, and $a(\cdot,\cdot)$ is a bilinear map $(\mathbb{R}^n)^2 \to \mathbb{R}^n$. The calculations performed to reach this result can be found in the Appendix.

3 Time Discretization

After a space discretization is given, we may proceed to the Fourier Series discretization of the equations. To begin, we define the sine-cosine space:

Definition 1 (Discretized, m-dimensional sine-cosine space) *Let $\varphi_i(t)$ be basis functions defined as follows:*

$$\varphi_i(t) = \begin{cases} \sin(2\pi \left(\frac{i}{2}\right) t) & i \text{ even} \\ \cos(2\pi \left(\frac{i-1}{2}\right) t) & i \text{ odd} \end{cases}.$$

Then, the m-dimensional sine-cosine space is defined as the function space "spanned[1]" by the first m of such functions:

$$\mathcal{V} = \left\{ v \mid v(t) = \sum_{i=1}^{m} q_i \varphi_i(t), q_1, \cdots, q_m \in \mathbb{R}^{3n} \right\}.$$

For most intents and purposes, m should be only an odd number, to have exactly a sine and a cosine for each used frequency, plus the initial constant function.

We want to solve equation (2). Thus, we take the residual:

$$\mathbf{r}(u) := M\frac{du}{dt} - T\left(\mathcal{J}(\bar{u})u + a(u,u)\right),$$

and then perform a Galerkin projection on $\mathcal{V}$. For this, we will need another definition:

Definition 2 (Discretized inner product) *Let $f, g \in \mathcal{V}$. Then, the discretized inner product is defined as follows:*

$$\langle f(t), g(t) \rangle_t = \int_0^1 f(t)g(t)\mathrm{d}t$$

Note that we integrate over the period, rather than the whole domain. Furthermore, similar as in the "span" used above, this is not a proper inner product, as it yields a vector in $\mathbb{R}^{3n}$, rather than a scalar.

The product between the (vector valued) functions is element-wise, resulting in an identically shaped vector, which allows for abuses of notation. Occasionally, we use scalar valued functions instead, and the inner product is computed in a similar way, but with scalar multiplication in place of the mentioned element-wise multiplication.

[1]Here, we use "span" as an *abuse of terminology*, as the "coefficients" in the sum are vectors, rather than scalars.

Taking a quick glance at this definition, makes it clear that it is no more than the standard function inner product, but in multiple dimensions. This results in the following lemma:

Lemma 1 *The discretized inner product $\langle \cdot, \cdot \rangle_t$ is bilinear.*

Thus, we try to find a $v \in \mathcal{V}$ such that $\langle \hat{v}, \mathbf{r}(v) \rangle_t = 0$ for all $\hat{v} \in \mathcal{V}$ CCC. We rewrite the vectors as an expansion of the basis functions:

$$v = \sum_{i=1}^{m} q_i \varphi_i := Q\Phi, \hat{v} = \sum_{i=1}^{m} \hat{q}_i \varphi_i := \hat{Q}\Phi,$$

where $Q = (q_1, q_1, \cdots, q_m)^\mathsf{T}$ is a 2-dimensional vector, similarly for $\hat{Q}$, and $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_m)^\mathsf{T}$. As they must be in the same dimension as $v, \hat{v}$, we must have $q, \hat{q} \in \mathbb{R}^{3n}$. Working the equation out, taking advantage of bilinearity, yields:

$$\begin{aligned}
0 = \left\langle \hat{Q}\Phi, \mathbf{r}(Q\Phi) \right\rangle_t &= \left\langle \sum_{i=1}^{m} \hat{q}_i \varphi_i, \mathbf{r}(Q\Phi) \right\rangle_t \\
&= \sum_{i=1}^{m} \hat{q}_i \left\langle \varphi_i, \mathbf{r}(Q\Phi) \right\rangle_t \\
&\Rightarrow \left\langle \varphi_i, \mathbf{r}(Q\Phi) \right\rangle_t = 0 \text{ for all } 1 \le i \le m.
\end{aligned}$$

Moving from the second line to the third, we use the fact that the equation must be satisfied for arbitrary $\hat{q} \in \mathbb{R}^{3n}$. We continue, solving for $1 \le i \le m$:

$$\begin{aligned}
0 = \left\langle \varphi_i, \mathbf{r}(Q\Phi) \right\rangle_f &= \left\langle \varphi_i, M\frac{d(Q\Phi)}{dt} - T\left(\mathcal{J}(\bar{u})(Q\Phi) - a((Q\Phi), (Q\Phi))\right) \right\rangle_t \\
&= \left\langle \varphi_i, M\frac{d(\Phi Q)}{dt} \right\rangle_f tT \left\langle \varphi_i, \mathcal{J}(\bar{u})(\Phi Q) \right\rangle_t - T \left\langle \varphi_i, a((\Phi Q), (\Phi Q)) \right\rangle_t \\
&= \left\langle \varphi_i, M\frac{d(\sum_{j=1}^{m} \varphi_j q_j)}{dt} \right\rangle_t - T \left\langle \varphi_i, \mathcal{J}(\bar{u}) \sum_{j=1}^{m} \varphi_j q_j \right\rangle_t - \\
&- T \left\langle \varphi_i, a\left(\sum_{j=1}^{m} \varphi_j q_j, \sum_{k=1}^{m} \varphi_k q_k \right) \right\rangle_t \\
&= M \sum_{j=1}^{m} q_j \left\langle \varphi_i, \partial_t \varphi_j \right\rangle_t - T\mathcal{J}(\bar{u}) \sum_{j=1}^{m} q_j \left\langle \varphi_i, \varphi_j \right\rangle_t - \\
&- T \sum_{j=1}^{m} \sum_{k=1}^{m} a\left(q_j, q_k\right) \left\langle \varphi_i, \varphi_j \varphi_k \right\rangle_t
\end{aligned}$$

$$=: M\sum_{j=1}^{m} q_j\alpha_{i,j} - T\mathcal{J}(\bar{u})\sum_{j=1}^{m} q_j\beta_{i,j} - T\sum_{j=1}^{m}\sum_{k=1}^{m} a(q_j, q_k)\gamma_{i,j,k} \tag{3}$$

In equation (3), all inner products are constant in time and space, and renamed as α, β, γ.

Inner products with basis functions

Now we work out the inner products α, β, γ defined above. Here, only the final results are displayed. Calculations can be found in the Appendix.

- $\alpha_{i,j}$: The value of $\alpha_{i,j}$ is zero for most values of i, j. They are only nonzero whenever $|i-j| = 1$. The table below shows the results for $|i-j| \leq 1$:

i\j	even	odd
even	0	-1
odd	1	0

  Thus, $\alpha_{i,j}$ can be expressed as a sort of "shifted Kronecker delta", with different signs for odd and even i, j:

$$\alpha_{i,j} = (i\%2)\delta_{i,i-1}\frac{i-1}{2} - ((i+1)\%2)\delta_{i,i+1}\frac{i}{2}$$

- $\beta_{i,j}$: These values are mostly zero as well, except when $i = j$ precisely. The computed results are, for $|i-j| \leq 1$:

i\j	even	odd
even	$\frac{1}{2}$	0
odd	0	$\frac{1}{2}$

  This can also be expressed as:

$$\beta_{i,j} = \frac{\delta_{i,j}}{2}$$

  with the Kronecker delta.

- $\gamma_{i,j,k}$: Similarly, most of these are zero as well. The exceptions are:

$$\gamma_{1,1,1} = 1.$$

  Otherwise:

- $k = 1 + i - j, k = 1 - i + j$:

$$\begin{cases} \frac{1}{2} & \text{if } i = j \text{ or } i = 1 \text{ or } j = 1 \\ \frac{1}{4} & \text{otherwise} \end{cases}.$$

- $k = 1 + i + j$:

$$\begin{cases} -\frac{1}{4} & \text{if } i, j \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$

- $k = -1 + i + j$:

$$\begin{cases} \frac{1}{2} & \text{if } i = 1 \text{ or } j = 1 \\ \frac{1}{4} & \text{otherwise} \end{cases}.$$

Given the large amount of Kronecker deltas, we can infer that most of these constants are actually zero. As such, it may be useful to rewrite equation (3) using a Kronecker matrix product $\otimes$.

Definition 3 (Kronecker Product) *The Kronecker product is a matrix operation that consists of multiplying every entry of the left side operand by the entire right side operand:*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & & & \vdots \\ \vdots & & & \\ a_{n1}B & \cdots & & a_{nn}B \end{pmatrix}.$$

We will use the *vectorization* of the 2-diensional state vector $\overline{Q}$. This simply means "piling up" all vectors on top of each other, as a tall vector, rather than having them sorted column wise. A, B, J are defined as $(A)_{ij} = \alpha_{i,j}, (B)_{ij} = \beta_{i,j}, J = \mathcal{J}(\bar{u})$ respectively.

$$M \sum_{j=1}^{m} q_j \alpha_{i,j} - TJ \sum_{j=1}^{m} q_j \beta_{i,j} - T \sum_{j=1}^{m} \sum_{k=1}^{m} a(q_j, q_k) \gamma_{i,j,k} = 0$$

$$\Rightarrow \quad \Big((A \otimes M) \overline{Q} - T (B \otimes J) \overline{Q} \Big)_i - T \sum_{j=1}^{m} \sum_{k=1}^{m} a(q_j, q_k) \gamma_{i,j,k} = 0$$

$$\Rightarrow \quad \left(\left(\frac{1}{T} (A \otimes M) - (B \otimes J) \right) \overline{Q} \right)_i = \sum_{j=1}^{m} \sum_{k=1}^{m} a(q_j, q_k) \gamma_{i,j,k}.$$

To get something nicer looking, we assemble another *big vector* from the right hand side, defined as:

$$D_i(Q) = \sum_{j=1}^{m} \sum_{k=1}^{m} a(q_j, q_k)\gamma_{i,j,k}.$$

Note that each D_i is not 1-dimensional, but rather $3n$-dimensional, the same as the q_is. The equations now become:

$$\left(\frac{1}{T}\left(A \otimes M\right) - \left(B \otimes J\right)\right)\overline{Q} = D(Q) \tag{4}$$

4 Solution Algorithms

Since our discretization stems from a system of nonlinear PDEs, equation (1), it should be no suprise that the final equation is nonlinear as well. As such, a very convenient method to solve the equation is one of the many variants of Newton's method.

From ([5]), we take the suggestion to employ a Newton-Picard iteration; using the fact that, in equation (4), the left hand side depends linearly on Q while the right hand side is bilinear in that aspect. We will provide an initial guess $Q^{(0)}$ which then will be iterated as follows:

$$\left(\frac{1}{T}(A \otimes M) - (B \otimes J)\right)\overline{Q^{(l+1)}} = D(Q^{(l)}). \tag{5}$$

As a final note, it is remarkable that the left hand side matrix, $\left(\frac{1}{T}(A \otimes M) - (B \otimes J)\right)$, is very sparse, due to the vast amount of zero cases of $\alpha_{i,j}, \beta_{i,j}$. In fact, looking at their definitions, it turns out that it is a block-diagonal matrix composed of blocks of size $2n \times 2n$:

$$\left(\frac{1}{T}(A \otimes M) - (B \otimes J)\right)_{2i:2(i+1)-1} = -\frac{1}{2}\begin{pmatrix} J & -iM \\ iM & J \end{pmatrix}.$$

Thus, it would suffice to solve the $2n \times 2n$ linear system $m/2$ times, instead of a $mn \times mn$ system once. The resulting complexity is thus $2mn^2$, rather than the usual m^2n^2.

4.1 Computing the initial guess

Approximate solutions near the critical Reynolds Number can be computed with the aid of the known steady state and the eigenvectors associated with it ([5]). Thus, we can try to use such vectors as an starting basis for the Newton Iterations. Computation of the initial guess can be found in the Appendix.

5 Conclusions

Bringing together the results from all sections, we can say the following:

- Solution algorithms for periodic solutions near Re_{crit} in Fourier Series form can be derived.
- Approximate guesses can be based on eigenvectors near Re_{crit}.

However, there are multiple loose ends that can be tied up through future work. We suggest some of the following:

- Implement the method in a computer program and run the algorithm.
- Observe the outcome and compare with existing methods. How many nonzero terms does the Series have?
- Perform a convergence analysis on possible solution algorithms for the proposed method.
- Extend the equations with a phase condition and the period.
- Derive the relation between the magnitude of the eigenvectors for the initial guess and the distance $Re - Re_{\text{crit}}$.
- Research the stability of the periodic solutions.

References

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A Simplifying main equation

We want to simplify the equation:

$$M\frac{\mathrm{d}v}{\mathrm{d}t} = a(v,v) + \mathcal{L}v,$$

so we begin by rewriting $v(t) = \bar{u} + u(t)$ and substitute into the equation:

$$\begin{aligned} M\frac{\mathrm{d}(\bar{u}+u)}{\mathrm{d}t} &= a(\bar{u}+u(t), \bar{u}+u(t)) + \mathcal{L}(\bar{u}+u(t)) \\ \Rightarrow \quad M\frac{\mathrm{d}\bar{u}}{\mathrm{d}t} + M\frac{\mathrm{d}u}{\mathrm{d}t} &= a(\bar{u},\bar{u}) + a(\bar{u},u) + a(u,\bar{u}) + a(u,u) + \mathcal{L}\bar{u} + \mathcal{L}u \\ \Rightarrow \quad 0 + M\frac{\mathrm{d}u}{\mathrm{d}t} &= 0 + a(\bar{u},u) + a(u,\bar{u}) + a(u,u) + \mathcal{L}u \end{aligned}$$

Now we use $\mathcal{J}(\bar{u})u = a(\bar{u},u) + a(u,\bar{u}) + \mathcal{L}u$ to get:

$$M\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{J}(\bar{u})u + a(u,u).$$

Finally, we rescale time with $t_{\text{new}} = \frac{t}{T}$, where $T = \frac{2\pi}{\lambda}$ is the period of the orbit, computed with λ, the eigenvalue of the steady state problem given a Reynolds Number ([5]). Thus, we end up with:

$$M\frac{\mathrm{d}u}{\mathrm{d}t} = p\left(\mathcal{J}(\bar{u})u + a(u,u)\right).$$

B Computation of coefficients

This will require computing multiple cases for even or odd i, j, k, as well as handling the divisions by 0 that may arise. Henceforth, we will use % as the modulo operator, and $\delta_{i,j}$ as the Kronecker delta. Most calculations were performed with the aid of Wolfram Mathematica CCC.

- $\alpha_{i,j}$:
  - i even:

* j even:

$$\begin{aligned}\alpha_{i,j} &= \langle \sin(\pi i t)\partial_t \sin(\pi j t)\rangle_f \\ &= j\int_0^1 \sin(\pi i t)\cos(\pi j t)dt \\ &= 0\end{aligned}$$

* j odd:

$$\begin{aligned}\alpha_{i,j} &= \langle \sin(\pi i t)\partial_t \cos(\pi (j-1) t)\rangle_f \\ &= -(j-1)\int_0^1 \sin(\pi i t)\sin(\pi (j-1) t)dt \\ &= \left[\frac{\sin(\pi(i-j+1)t)}{2\pi(i-j+1)} - \frac{\sin(\pi(i+j-1)t)}{2\pi(i+j-1)}\right]_{t=0}^{t=1} \\ &= 0\end{aligned}$$

Note that $i+j-1=0$ is not possible as $i,j \geq 1$. Thus, we only need to consider the exception $j = i+1$:

$$\begin{aligned}\alpha_{i,i+1} &= -i\int_0^1 \sin(\pi i t)\sin(\pi i t)dt \\ &= -i\left[\frac{t}{2} - \frac{\sin(2\pi i t)}{4\pi i}\right]_{t=0}^{t=1} \\ &= -\frac{i}{2}\end{aligned}$$

- i odd:

  * j even:

$$\begin{aligned}\alpha_{i,j} &= \langle \cos(\pi (i-1) t)\partial_t \sin(\pi j t)\rangle_f \\ &= j\int_0^1 \cos(\pi (i-1) t)\cos(\pi j t)dt \\ &= \left[\frac{\sin(\pi(j-i+1)t)}{2\pi(j-i+1)} - \frac{\sin(\pi(j+i-1)t)}{2\pi(j+i-1)}\right]_{t=0}^{t=1} \\ &= 0\end{aligned}$$

Note that $j+i-1=0$ is not possible as $i,j \geq 1$. Thus, we only need to consider the exception $j = i - 1$:

$$\begin{aligned}\alpha_{i,i-1} &= (i-1)\int_0^1 \cos(\pi(i-1)t)\cos(\pi(i-1)t)dt \\ &= (i-1)\left[\frac{t}{2} - \frac{\sin(2\pi(i-1)t)}{4\pi(i-1)}\right]_{t=0}^{t=1} \\ &= \frac{i-1}{2}\end{aligned}$$

Note again that $j = i - 1 = 0$ is not possible since $j \geq 1$.

* j odd:

$$\begin{aligned}\alpha_{i,j} &= \langle\cos(\pi(i-1)t)\partial_t \sin(\pi(j-1)t)\rangle_f \\ &= -(j-1)\int_0^1 \cos(\pi(i-1)t)\sin(\pi(j-1)t)dt \\ &= 0\end{aligned}$$

- $\beta_{i,j}$:
  - i even:
    * j even:

$$\begin{aligned}\beta_{i,j} &= \int_0^1 \sin(\pi i t)\sin(\pi j t)\mathrm{d}t \\ &= \frac{j\cos(j\pi)\sin(i\pi) - i\cos(i\pi)\sin(j\pi)}{i^2\pi - j^2\pi} \\ &= 0\end{aligned}$$

Except when dividing by zero, whenever $i = j$:

$$\begin{aligned}\beta_{i,i} &= \int_0^1 \sin^2(\pi i t)\mathrm{d}t \\ &= \frac{1}{2}\end{aligned}$$

    * j odd:

$$\begin{aligned}\beta_{i,j} &= \int_0^1 \sin(\pi i t)\cos(\pi(j-1)t)\mathrm{d}t \\ &= \frac{i + i\cos(i\pi)\cos(j\pi) + (j-1)\sin(i\pi)\sin(j\pi)}{(1+i-j)(-1+i+j)\pi} \\ &= 0\end{aligned}$$

Unless we divide by zero:

· $j = i + 1$:

$$\begin{aligned}\beta_{i,i+1} &= \frac{\sin^2(i\pi)}{2i\pi} \\ &= 0\end{aligned}$$

· $j = 1 - i$: Impossible, $i, j \geq 1$.

– i odd:

  * j even: By commutativity of multiplication, the same as the one above.
  * j odd:

$$\begin{aligned}\beta_{i,j} &= \int_0^1 \cos(\pi(i-1)t)\cos(\pi(j-1)t)\mathrm{d}t \\ &= \frac{1}{2\pi}\left(\frac{\sin((i-j)\pi)}{i-j} + \frac{\sin((-2+i+j)\pi)}{-2+i+j}\right) \\ &= 0\end{aligned}$$

Unless division by zero:

· $i = j$:

$$\begin{aligned}\beta_{i,i} &= \int_0^1 \cos^2(\pi(i-1)t)\mathrm{d}t \\ &= \frac{1}{4}\left(2 + \frac{\sin(2i\pi)}{(i-1)\pi}\right) \\ &= \frac{1}{2}\end{aligned}$$

Unless $i = 1$, then:

$$\beta_{i,i} = \int_0^1 \mathrm{d}t = 1.$$

· $j = i - 2$: Impossible, $i, j \geq 1$.

- $\gamma_{i,j,k}$: Since following the previous pattern would result in $2^3 = 8$ distinct cases, not accounting for handling divisions by zero, we will use commutativity of multiplication to simplify the calculations:

$$\gamma_{i,j,k} = \int_0^1 \varphi_i\varphi_j\varphi_k\mathrm{d}t = \int_0^1 \varphi_j\varphi_i\varphi_k\mathrm{d}t = \cdots$$

...and so on for all permutations. Thus, all that matters is the number of even/odd indices:

- 3 even, 0 odd:

$$\begin{aligned}
&\int_0^1 \sin(\pi i t)\sin(\pi j t)\sin(\pi k t)\mathrm{d}t = \\
&= \frac{1}{4\pi}\left(\frac{1}{i+j-k} + \frac{1}{i-j+k} - \frac{1}{i-j-k} - \frac{1}{i+j+k} + \frac{\cos((i-j-k)\pi)}{i-j-k}\right. \\
&\quad \left. - \frac{\cos((i+j-k)\pi)}{i+j-k} - \frac{\cos((i-j+k)\pi)}{i-j+k} + \frac{\cos((i+j+k)\pi)}{i+j+k}\right) \\
&= 0,
\end{aligned}$$

  since all arguments of the cosines are even multiples of π, thus the cosines all evaluate to 1. Division by 0 exceptions:

  * $k = i + j$:

$$\begin{aligned}
\gamma_{i,j,i+j} &= \frac{i^2 + ij + j^2 - j(i+j) - i(i+j) + ij}{8ij(i+j)\pi} \\
&= \frac{(i+j)^2 - j(i+j) - i(i+j)}{8ij(i+j)\pi} \\
&= \frac{i+j-j-i}{8ij} \\
&= 0
\end{aligned}$$

    In this case, another division by 0 is impossible as $i, j \geq 1$.

  * $k = j - i$:

$$\begin{aligned}
\gamma_{i,j,j-i} &= \frac{1}{8\pi}\left(\frac{1}{i} - \frac{1}{j} - \frac{1}{i-j} - \frac{\cos(2i\pi)}{i} + \frac{\cos(2(i-j)\pi)}{i-j} + \frac{\cos(2j\pi)}{j}\right) \\
&= \frac{1}{8\pi}(0) = 0
\end{aligned}$$

    Further exceptions impossible since $i, j, k \geq 0$.

  * $k = i - j$: Applying commutativity to above case:

$$\gamma_{i,j,i-j} = \gamma_{j,i,i-j} = 0.$$

  * $k = -i - j$: Impossible since $i, j, k \geq 1$.

- 2 even, 1 odd:

$$\begin{aligned}
&\int_0^1 \sin(\pi i t)\sin(\pi j t)\cos(\pi(k-1)t)\mathrm{d}t = \\
&= \frac{1}{4\pi}\left(\frac{\sin((1+i-j-k)\pi)}{1+i-j-k} + \frac{\sin((i+j-k)\pi)}{1+i+j-k}\right. \\
&\quad \left. + \frac{\sin((-1+i-j+k)\pi)}{-1+i-j+k} + \frac{\sin((i+j+k)\pi)}{1+i+j+k}\right) \\
&= 0,
\end{aligned}$$

since they are all sines evaluated at integers. Exceptions:

  * $k = 1+i-j$:

$$\begin{aligned}
\gamma_{i,j,1+i-j} &= \frac{1}{8}\left(2 + \frac{-\frac{\sin(2i\pi)}{i} + \frac{\sin(2(i-j)\pi}{i-j} - \frac{\sin(2j\pi)}{j}}{\pi}\right) \\
&= \frac{1}{4}
\end{aligned}$$

    Exception: $i = j$:

$$\begin{aligned}
\gamma_{i,i,1} &= \int_0^1 \sin(\pi i t)\sin(\pi i t)\mathrm{d}t \\
&= \frac{1}{2}
\end{aligned}$$

  * $k = 1+i+j$:

$$\begin{aligned}
\gamma_{i,j,1+i+j} &= \frac{1}{8}\left(-2 + \frac{\frac{\sin(2i\pi)}{i} - \frac{\sin(2(i+j)\pi}{i+j} + \frac{\sin(2j\pi)}{j}}{\pi}\right) \\
&= -\frac{1}{4}
\end{aligned}$$

    No exceptions.

  * $k = 1-i+j$: By commutativity, same as $k = 1+i-j$.
  * $k = -1-i-j$: Impossible since $k \geq 1$.

– 1 even, 2 odd:

$$\int_0^1 \sin(\pi i t)\cos(\pi(j-1)t)\cos(\pi(k-1)t)\mathrm{d}t =$$
$$= \frac{1}{4\pi}\left(\frac{1}{2+i-j-k} + \frac{1}{i+j-k} + \frac{1}{i-j+k} + \frac{1}{-2+i+j+k} - \frac{\cos((i+j-k)\pi)}{i+j-k}\right.$$
$$\left. - \frac{\cos((2+i-j-k)\pi)}{2+i-j-k} - \frac{\cos((i-j+k)\pi)}{i-j+k} - \frac{\cos((-2+i+j+k)\pi)}{-2+i+j+k}\right)$$
$$= 0$$

Except:

* $k = 2+i-j$:

$$\gamma_{i,j,2+i-j} = \frac{1}{8\pi}\left(\frac{1}{i} + \frac{1}{1+i-j} + \frac{1}{j-1} - \frac{\cos(2i\pi)}{i}\right.$$
$$\left. - \frac{\cos(2(1+i-j)\pi)}{1+i-j} - \frac{\cos(2j\pi)}{j-1}\right)$$
$$= 0$$

Exception: $j = 1+i$

$$\gamma_{i,1+i,1} = \int_0^1 \sin(\pi i t)\cos(\pi i t)\mathrm{d}t$$
$$= 0$$

* $k = i+j$:

$$\gamma_{i,j,i+j} = \frac{1}{8\pi}\left(\frac{1}{i} + \frac{1}{1-j} + \frac{1}{-1+i+j} - \frac{\cos(2i\pi)}{i}\right.$$
$$\left. + \frac{\cos(2j\pi)}{j} + \frac{\cos(2(-1+i+j)\pi)}{-1+i+j}\right)$$
$$= 0$$

Exception:$j = 1$

$$\gamma_{i,1,i+1} = \int_0^1 \sin(\pi i t)\cos(\pi i t)\mathrm{d}t$$
$$= 0.$$

* $k = j - i$:

$$\gamma_{i,j,j-i} = \frac{1}{8\pi}\left(\frac{1}{i} + \frac{1}{1+i-j} + \frac{1}{j-1} - \frac{\cos(2i\pi)}{i} - \frac{\cos(2(1+i-j)\pi)}{1+i-j} - \frac{\cos(2j\pi)}{j-1}\right)$$
$$= 0$$

Exception: $j = 1 + i$

$$\gamma_{i,1+i,1} = \int_0^1 \sin(\pi i t)\cos(\pi i t)\mathrm{d}t$$
$$= 0$$

* $k = 2 - i - j$: Impossible since $i, j, k \geq 0$

- 0 even, 3 odd:

$$\int_0^1 \cos(\pi(i-1)t)\cos(\pi(j-1)t)\cos(\pi(k-1)t)\mathrm{d}t =$$
$$= \frac{1}{4\pi}\left(\frac{\sin((1+i-j-k)\pi)}{1+i-j-k} + \frac{\sin((-1+i+j-k)\pi)}{-1+i+j-k} + \frac{\sin((-1+i-j+k)\pi)}{-1+i-j+k} + \frac{\sin((-3+i+j+k)\pi)}{-3+i+j+k}\right)$$
$$= 0$$

Exceptions:

* $k = 1 + i - j$:

$$\gamma_{i,j,1+i-j} = \frac{1}{8}\left(\frac{\sin(2i\pi)}{(i-1)\pi} + \frac{2+\sin(2(i-j)\pi)}{(i-j)\pi} + \frac{\sin(2j\pi))}{(j-1)\pi}\right)$$
$$= \frac{1}{4}$$

Exceptions:

· $j = 1$:

$$\begin{aligned}\gamma_{i,1,i} &= \int_0^1 \cos(\pi i t)\cos(\pi i t)\mathrm{d}t \\ &= \frac{1}{4}\left(2 + \frac{\sin(2i\pi)}{(i-1)\pi}\right) \\ &= \frac{1}{2}\end{aligned}$$

Unless $i = 1$, then $i = j = k = 1$ which is shown later.

· $i = 1$:

$$\gamma_{1,j,2-j} = \int_0^1 \cos(\pi j t)\cos(\pi j t + 2\pi t)\mathrm{d}t$$

Since the shift is a whole period, it's the same as above.

∗ $k = -1 + i + j$:

$$\begin{aligned}\gamma_{i,j,-1+i+j} &= \frac{1}{8}\left(2 + \frac{\frac{\sin(2i\pi)}{i-1} + \frac{\sin(2j\pi)}{j-1} + \frac{\sin(2(i+j)\pi)}{i+j-2}}{\pi}\right) \\ &= \frac{1}{4}\end{aligned}$$

Unless:$j = 1$:

$$\gamma_{i,1,i} = \frac{1}{2}$$

unless $i = 1$, then see last case.

∗ $k = 1 - i + j$: By commutativity, the same as two above.

∗ $k = 3 - i - j$: Only valid when $i = j = k = 1$:

$$\gamma_{1,1,1} = \int_0^1 1 \times 1 \times 1\mathrm{d}t = 1$$

C Initial guess

Given $V\cos(2\pi t) + W\sin(2\pi t)$ from the eigenvalue problem, we perform a Galerkin projection to the space $\{\bar{u}, V\cos(2\pi t) - W\sin(2\pi t)\}$:

$$\begin{cases}\int_0^1 \langle \bar{u}, \mathbf{r}(\psi\bar{u} + \omega(V\cos(2\pi t) - W\sin(2\pi t)))\rangle\, dt & = 0 \\ \int_0^1 \langle V\cos(2\pi t) - W\sin(2\pi t)), \mathbf{r}(\psi\bar{u} + \omega(V\cos(2\pi t) - W\sin(2\pi t)))\rangle\, dt & = 0\end{cases}$$

Here, $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product and ψ, ω are unknown real scalars; the target of these equations. V, W are given, more precisely, by $2*\mathfrak{R}, 2*\mathfrak{I}$ respectively, twice the real and imaginary parts of the eigenvector. For the first equation:

$$
\begin{aligned}
0 &= \int_0^1 \langle \bar{u}, \mathbf{r}(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t))) \rangle \, dt \\
&= \int_0^1 \langle \bar{u}, M\partial_t(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t))) \rangle \, dt \\
&- T\int_0^1 \langle \bar{u}, J(\psi\bar{u} + \omega(V\cos(2\pi t) - W\sin(2\pi t))) \rangle \, dt \\
&- T\int_0^1 \langle \bar{u}, a((\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t))), (\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t)))) \rangle \, dt \\
&=: \mathcal{A} - T\mathcal{B} - T\mathcal{C}
\end{aligned}
$$

- $\mathcal{A}$:

$$
\begin{aligned}
&\int_0^1 \langle \bar{u}, M\partial_t(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t))) \rangle \, dt \\
&= \int_0^1 \langle \bar{u}, M(0 + 2\pi\omega(-V\sin(2\pi t) + W\cos(2\pi t))) \rangle \, dt \\
&= \langle \bar{u}, M2\pi\omega(-V) \rangle \int_0^1 \sin(2\pi t)dt - \langle \bar{u}, M2\pi\omega W \rangle \int_0^1 \cos(2\pi t)dt \\
&= \langle \bar{u}, M2\pi\omega(-V) \rangle \, 0 - \langle \bar{u}, M2\pi\omega W \rangle \, 0 \\
&= 0
\end{aligned}
$$

- $\mathcal{B}$:

$$
\begin{aligned}
&\int_0^1 \langle \bar{u}, J(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t))) \rangle \, dt \\
&= \psi \langle \bar{u}, J\bar{u} \rangle \int_0^1 dt + \omega \langle \bar{u}, V \rangle \int_0^1 \cos(2\pi t)dt + \omega \langle \bar{u}, W \rangle \int_0^1 \sin(2\pi t)dt \\
&= \psi \langle \bar{u}, J\bar{u} \rangle \, 1 + \omega \langle \bar{u}, V \rangle \, 0 + \omega \langle \bar{u}, W \rangle \, 0 \\
&= \psi \langle \bar{u}, J\bar{u} \rangle
\end{aligned}
$$

- $\mathcal{C}$:

$$\int_0^1 \langle \bar{u}, a(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t)), \psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t)))\rangle\, dt$$
$$= \psi^2 \langle \bar{u}, a(\bar{u}, \bar{u})\rangle \int_0^1 dt + \psi\omega \langle \bar{u}, V\rangle \int_0^1 \cos(2\pi t) dt + \psi\omega \langle \bar{u}, a(\bar{u}, W)\rangle \int_0^1 \sin(2\pi t) dt$$
$$+ \omega\psi \langle \bar{u}, a(V, \bar{u})\rangle \int_0^1 \cos(2\pi t) dt + \omega^2 \langle \bar{u}, a(V, V)\rangle \int_0^1 \cos^2(2\pi t) dt$$
$$+ \omega^2 \langle \bar{u}, a(V, W)\rangle \int_0^1 \cos(2\pi t)\sin(2\pi t) dt$$
$$+ \omega\psi \langle \bar{u}, a(W, \bar{u})\rangle \int_0^1 \sin(2\pi t) dt + \omega^2 \langle \bar{u}, a(W, V)\rangle \int_0^1 \sin(2\pi t)\cos(2\pi t) dt$$
$$+ \omega^2 \langle \bar{u}, a(W, W)\rangle \int_0^1 \sin^2(2\pi t) dt$$
$$= \psi^2 \langle \bar{u}, a(\bar{u}, \bar{u})\rangle + \frac{\omega^2}{2} \langle \bar{u}, a(V, V) + a(W, W)\rangle$$

Thus, the first equation is:

$$\mathcal{A} - T\mathcal{B} - T\mathcal{C} = -T\psi \langle \bar{u}, J\bar{u}\rangle - T\psi^2 \langle \bar{u}, a(\bar{u}, \bar{u})\rangle - T\frac{\omega^2}{2} \langle \bar{u}, a(V, V) + a(W, W)\rangle$$

Since this is set equal to 0 and so is $\mathcal{A}$, the minus sings and the Ts may be discarded:

$$\psi \langle \bar{u}, J\bar{u}\rangle + \psi^2 \langle \bar{u}, a(\bar{u}, \bar{u})\rangle + \frac{\omega^2}{2} \langle \bar{u}, a(V, V) + a(W, W)\rangle$$

Finally, we rename the known values for ease of resolution:

$$\psi\mathcal{B}' + \psi^2\mathcal{C}' + \omega^2\mathcal{C}''$$

The second equation:

$$
\begin{aligned}
0 &= \int_0^1 \langle V\cos(2\pi t) + W\sin(2\pi t), \mathbf{r}(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t)))\rangle\, dt \\
&= \int_0^1 \langle V\cos(2\pi t) + W\sin(2\pi t), M\partial_t(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t)))\rangle\, dt \\
&- T\int_0^1 \langle V\cos(2\pi t) + W\sin(2\pi t), J(\psi\bar{u} + \omega(V\cos(2\pi t) - W\sin(2\pi t)))\rangle\, dt \\
&- T\int_0^1 \langle V\cos(2\pi t) + W\sin(2\pi t), \\
&, a((\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t))), (\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t))))\rangle dt \\
&=: \mathfrak{A} - T\mathfrak{B} - T\mathfrak{C}
\end{aligned}
$$

- $\mathfrak{A}$:

$$
\begin{aligned}
&\int_0^1 \langle V\cos(2\pi t) + W\sin(2\pi t), M\partial_t(\psi\bar{u} + \omega(V\cos(2\pi t) + W\sin(2\pi t)))\rangle\, dt \\
&= \int_0^1 \langle V\cos(2\pi t) + W\sin(2\pi t), M\omega(-V\sin(2\pi t) + W\cos(2\pi t)))\rangle\, dt \\
&= \omega\left(-\langle V, MV\rangle 0 + \langle V, MW\rangle \int_0^1 \cos^2(2\pi t)dt + \langle W, MV\rangle \int_0^1 \sin^2(2\pi t)dt + \langle W, MW\rangle 0\right) \\
&= \frac{\omega}{2}\left(\langle V, MW\rangle - \langle W, MV\rangle\right)
\end{aligned}
$$

- $\mathfrak{B}$:

$$
\begin{aligned}
0 &= \int_0^1 \langle V\cos(2\pi t) + W\sin(2\pi t), J(\psi\bar{u} + \omega(V\cos(2\pi t) - W\sin(2\pi t)))\rangle\, dt \\
&= \psi\langle V, J\bar{u}\rangle 0 + \psi\langle W, J\bar{u}\rangle 0 \\
&+ \omega\langle V, JV\rangle \int_0^1 \cos^2(2\pi t)dt + \omega\langle V, JW\rangle 0 \\
&+ \omega\langle W, JV\rangle 0 + \omega\langle W, JW\rangle \int_0^1 \sin^2(2\pi t)dt \\
&= \frac{\omega}{2}\left(\langle V, JV\rangle + \langle W, JW\rangle\right)
\end{aligned}
$$

- $\mathfrak{C}$:

$$\int_0^1 \langle V\cos(2\pi t)+W\sin(2\pi t), a\Bigg((\psi\bar{u}+\omega(V\cos(2\pi t)+W\sin(2\pi t))),$$
$$(\psi\bar{u}+\omega(V\cos(2\pi t)+W\sin(2\pi t)))\Bigg)\rangle dt$$
$$=\psi^2\langle V,a(\bar{u},\bar{u})\rangle 0+\psi\omega\langle V,a(\bar{u},V)\rangle\int_0^1\cos^2(2\pi t)dt+\psi\omega\langle V,a(\bar{u},W)\rangle 0$$
$$+\omega\psi\langle V,a(V,\bar{u})\rangle\int_0^1\cos^2(2\pi t)dt+\omega^2\langle V,a(V,V)\rangle\int_0^1\cos^3(2\pi t)dt$$
$$+\omega^2\langle V,a(V,W)\rangle\int_0^1\cos^2(2\pi t)\sin(2\pi t)dt$$
$$+\omega\psi\langle V,a(W,\bar{u})\rangle 0+\omega^2\langle V,a(W,V)\rangle\int_0^1\sin(2\pi t)\cos^2(2\pi t)dt$$
$$+\omega^2\langle V,a(W,W)\rangle\int_0^1\cos(2\pi t)\sin^2(2\pi t)dt$$
$$+\psi^2\langle W,a(\bar{u},\bar{u})\rangle 0+\psi\omega\langle W,a(\bar{u},V)\rangle\int_0^1 0dt+\psi\omega\langle W,a(\bar{u},W)\rangle\int_0^1\sin^2(2\pi t)dt$$
$$+\omega\psi\langle W,a(V,\bar{u})\rangle 0+\omega^2\langle W,a(V,V)\rangle\int_0^1\sin(2\pi t)\cos^2(2\pi t)dt$$
$$+\omega^2\langle W,a(V,W)\rangle\int_0^1\sin^2(2\pi t)\cos(2\pi t)dt$$
$$+\omega\psi\langle W,a(W,\bar{u})\rangle\int_0^1\sin^2(2\pi t)dt+\omega^2\langle W,a(W,V)\rangle\int_0^1\sin^2(2\pi t)\cos(2\pi t)dt$$
$$+\omega^2\langle W,a(W,W)\rangle\int_0^1\sin^3(2\pi t)dt$$
$$=\frac{\psi\omega}{2}\left(\langle V,a(\bar{u},V)+a(V,\bar{u})\rangle+\langle W,a(\bar{u},W)+a(W,\bar{u})\rangle\right)$$

Adding the parts together, then yields:

$$\omega\mathfrak{A}'-\omega\mathfrak{B}'-\psi\omega\mathfrak{C}'$$

with

$$
\begin{aligned}
\mathfrak{A}' &= \frac{1}{2}(\langle V, MW\rangle - \langle W, MV\rangle) \\
\mathfrak{B}' &= \frac{T}{2}\left(\langle V, JV\rangle + \langle W, JW\rangle\right) \\
\mathfrak{C}' &= \frac{T}{2}\left(\langle V, a(\bar{u}, V) + a(V, \bar{u})\rangle + \langle W, a(\bar{u}, W) + a(W, \bar{u})\rangle\right)
\end{aligned}
$$

Thus, the equations become:

$$
\begin{cases}
\psi\mathcal{B}' + \psi^2\mathcal{C}' + \omega^2\mathcal{C}'' & = 0 \\
\omega\mathfrak{A}' - \omega\mathfrak{B}' - \psi\omega\mathfrak{C}' & = 0
\end{cases}
$$

It is worthy of note that there was a possibility to only use $V\cos(2\pi t) + W\sin(2\pi t)$ as the basis. However, the only possible result was zero, so it was discarded.