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Colouring Celtic knotted trivalent graphs

Bachelor's Project Mathematics

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Abstract

This thesis presents research on Celtic knots using a framework of knotted trivalent graphs (KTGs) with the aim of finding a construction method of Celtic knots and finding a knot invariant of KTGs. We give some basic knot theory and theory on KTGs necessary for understanding the thesis. Additionally, we discuss the colouring of knots and KTGs and prove theorems which give a relation between knot colouring and knot invariance, as well as between KTG colouring and KTG invariance. We use the theory to colour several Celtic knots and analyse the construction of Celtic designs as KTGs.

Contents

1 Introduction

Despite what the name might suggest, Celtic knots are not unique to Celtic culture [5]. The Celts were inspired by knot-work of the Romans, Greeks and Vikings. The Celtic knots were drawn to decorate the borders of manuscripts, but they have also been used as decoration for churches [6]. The illustrations in manuscripts often feature zoomorphic motives, which are knot-work designs featuring animals.

While these Celtic designs can be found in multiple places, the techniques used to make them is unknown to us. In the twentieth century an artist named George Bain developed a method which allowed us to recreate the Celtic knots that have been found [3]. This method is explored later in this thesis.

The mathematics of knots was not explored until the nineteenth century and since the Celts did not leave behind any sources documenting the construction of Celtic knots, we cannot know how much mathematical theory was used for making these knots. We can however use the knot theory developed in the last three centuries to analyse Celtic knots and to discover the mathematics hidden in these centuries-old creations.

With Celtic knots or Celtic designs we mean any Celtic artwork consisting of intertwined bands, many of which are knots. However, not all Celtic designs are knots. Figure 1 shows a 'knot' from the book of Kells. In the bottom left three arcs are connected at some point, which implies that this is not a knot. Hence another framework must be used to analyse knots such as this one. An object that allows us to do this is a knotted trivalent graph (KTG), which can contain forks: three arcs connected at a point.

Figure 1: A Celtic 'knot' from the book of Kells.

The grid method made by Bain is used to construct alternating knots, but KTGs allow us to find many other designs. We hence want to research how the grid method can be adjusted for KTGs.

Besides constructing knots we can also study if two knots are equivalent to each other: knot invariance. We would like to analyse how concepts from knot invariance can be adjusted to find if two KTGs are equivalent to each other.

In this thesis we hence want to answer the following question:

How can we build Celtic knots using KTGs and prove invariance of KTGs?

In order to answer this question we start by giving a brief history on Celtic knots and the research done by George Bain. This is followed by a chapter on basic knot theory and the theory of KTGs. Next knot colouring is explained and we prove how knot colouring is related to knot invariance. This theory is then adjusted so it can be used for colouring KTGs. We then use KTG colouring to colour several Celtic knots and find KTG invariants. The final chapter shows how to construct Celtic KTGs using a method based on the grid method by George Bain.

2 Historical background

The Book of Kells, the Lindisfarne Gospels and the Lichfield Gospels are all famous examples of gospels made during the early medieval period. These books are of great art historical importance because of their illuminated pages, which showcase the art style commonly used by the Celtic peoples living in Britain and Ireland at the time. This art style is referred to as “Insular art”, coming from “insula”, the Latin word for island. The complex designs of the illuminated pages feature initials and borders filled with intricate knotwork.

The knots used in these designs are alternating knots, based on weaving techniques used for the creation of for example baskets. Although these knots are referred to as “Celtic knots”, they do not have a Celtic origin. Rather, they are based on an interlacing pattern made by the Romans, which was then adapted and altered by Germanic artists [12]. The interlacing pattern we see in Celtic art is hence not exclusive to Celtic culture, as it can be found in art ranging from Roman art to Islamic art [9]. The designs also have features of animals, which are sometimes used in knots. They are so-called “zoömorphic”, see for example the right image of figure 2.

Figure 2: Left: Incipit page of the Gospel of Matthew in the Lindisfarne Gospels. Right: Detail of an illuminated page in the Book of Kells. Images taken from [1] and [11].

The knowledge on the construction of Celtic knots was ultimately lost and hence we cannot determine how exactly the Celtic artists made these complicated designs. However, in 1951 George Bain published a book in which he explained a possible method of constructing Celtic knots, which he calls “the grid method”. This method has since then been used to replicate many of the designs that have been found in Celtic art [8].

An example of how a Celtic knot can be constructed using Bain’s method can be found in figure 3. After taking a grid, we create a secondary grid (the red and blue dots respectively). We can then create a plait indicated by the red lines in (d) and add external weaving. Since Celtic knots are alternating, we must indicate where the knot has overlap (Figure 3(f)). Finally the band is drawn and the grid is removed, leaving us with a knot [8].

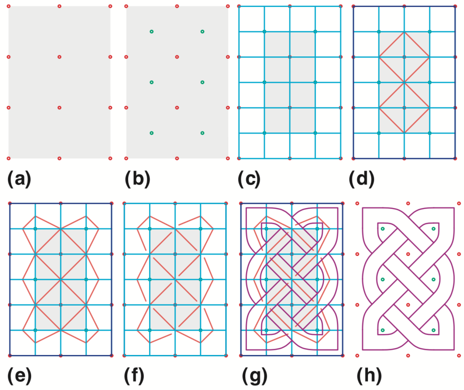


Figure 3: Bain's grid method. Image taken from [8].

The aforementioned method can be further complicated by adding more strands or by indicating breaklines in the initial grid which are not allowed to be crossed by the strands. This allows for the creation of more complicated Celtic knots, see figure 4.

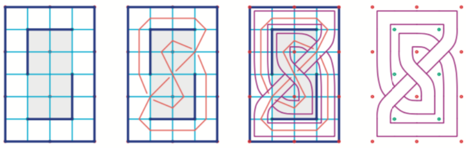


Figure 4: The grid method using breaklines. Image taken from [8].

Throughout time multiple researches have been done into identifying different types of Celtic knots. Romilly Allen identified eight of the most commonly used Celtic knots, which are shown in figure 5. The knots shown in the figure are a part of a larger decoration in which several knots, either of the same or a different type, are tied together. This can for example be done to fill a rectangular space such as a frieze, but it can be used to fill spaces of other shapes. A famous example would be the knotwork used in a Celtic cross.

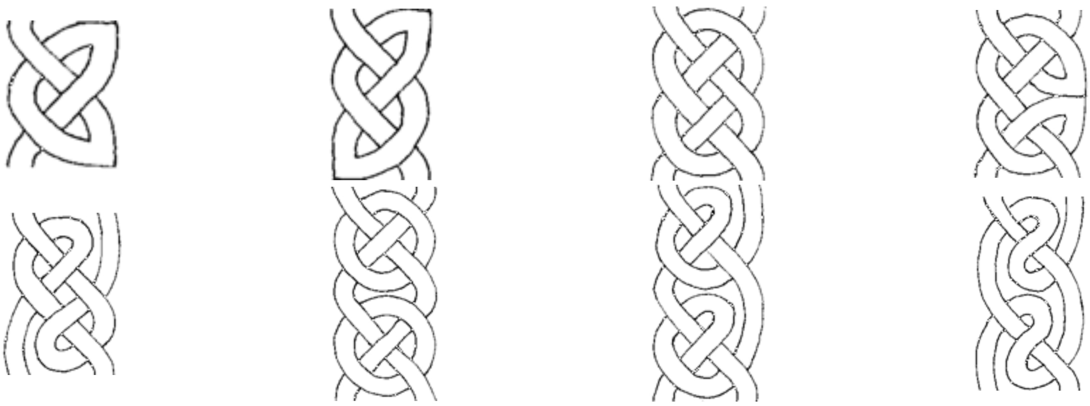


Figure 5: Eight most common Celtic knots designs. Images taken from [7].

3 Theory

This chapter presents some basic concepts of knot theory. Moreover, an introduction to the theory of knotted trivalent graphs is given.

3.1 Knots

Definition 3.1. Given two topological spaces X, Y, an injective and continuous map $f : X \to Y$ with $f : X \to f(X)$ a homeomorphism, then f is an embedding. Note that $f(X)$ has the subspace topology.

Definition 3.2. A knot is a closed, non-self-intersecting curve smoothly embedded in S^3.

Note that $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ is compact. Knots can be visualised by considering them as objects in $\mathbb{R}^3$ with an additional point at ∞. With "smoothly embedded" we mean that the knot k is a smooth embedding (C^∞) $k : S^1 \to S^3$. This is required to exclude wild knots, which consist of an infinite amount of knots tied one after the other [2]. An example of a wild knot is depicted in figure 6.

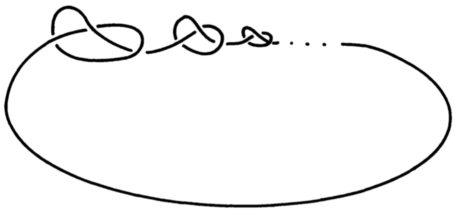
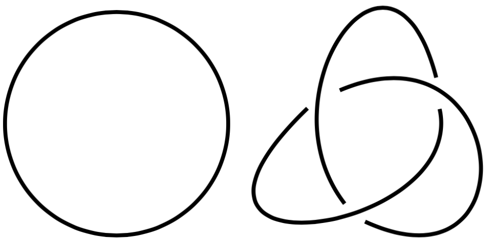


Figure 6: Example of a wild knot. Image taken from [2].

In order to draw a knot we can project it onto S^2. In the drawing of a knot we indicate a crossing by leaving some space around the overcrossing. Figure 6 for example shows such a projection.

The simplest knot is the unknot, depicted in figure 7a. This is a closed loop without any 'knots' in it.

A part of the knot going from one undercrossing to the next is called an arc. The trefoil knot (figure 7b) has 3 arcs.



(a) Unknot. (b) Trefoil knot.

Definition 3.3. A link is a finite number of circles smoothly embedded in S^3.

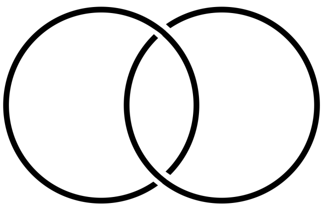


Figure 8: Hopf link.

It is important to establish rules regarding invariance of knots and KTGs, because we want to determine which Celtic knots are equivalent to each other.

Definition 3.4. A diffeomorphism is a smooth homeomorphism.

Definition 3.5. A diffeomorphism φ is isotopic to the identity if there exists a homotopy $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism for all $t \in [0,1]$, $h_0 = \mathrm{id}_{\mathbb{R}^3}$ and $h_1 = \varphi$.

Definition 3.6. Two knots k_1 and k_2 are equivalent if there exists a diffeomorphism φ such that $\varphi(k_1) = k_2$ and φ is isotopic to the identity map.

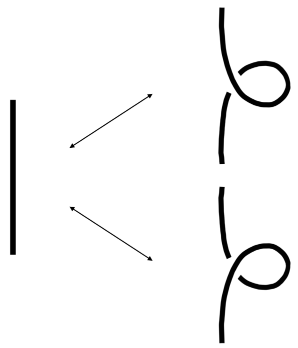
By requiring φ to be isotopic we ensure that two mirror images of a knot are not necessarily equivalent.

One method to determine if knots are equivalent is to use Reidemeister moves. The Reidemeister moves of type I, II and III do not change the knot, but only its projection.

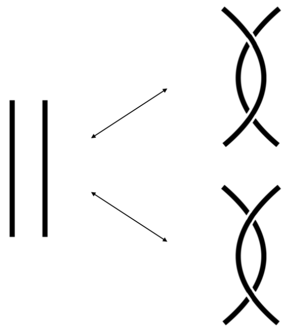
(I) Adding or removing a twist in the knot (figure 9a).

(II) Adding or removing two crossings (figure 9b).

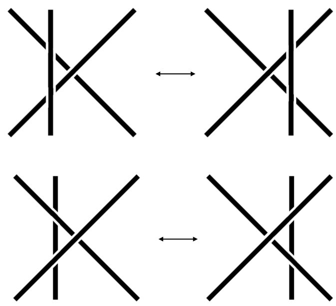
(III) Sliding a strand of the knot from one side to another side of the crossing (figure 9c)



(a) Type I



(b) Type II



(c) Type III

Figure 9: Reidemeister moves

Figure 9 shows two cases for every Reidemeister move. Note that each of these cases follows from the other.

Theorem 3.1. Two knot projections are equivalent if and only if they are related by a finite sequence of Reidemeister moves [10].

3.2 Knotted trivalent graphs

Definition 3.7. A fat graph is a finite 1-dimensional simplicial complex together with an embedding into a two-dimensional manifold with boundary.

We can picture a fat graph as a 'thickened' graph. The vertices of the graph are disks and the edges are bands.

Definition 3.8. A knotted trivalent graph (KTG) is a trivalent fat graph smoothly embedded as a surface in S^3 and considered up to isotopy.

Note that a circle, a single edge without vertices, is a KTG.

An example of a KTG and its simplified representation is shown in figure 10. We see that the bands are drawn as lines. Half twists are indicated by drawing a short line crossing the band lines in the direction of the twist.

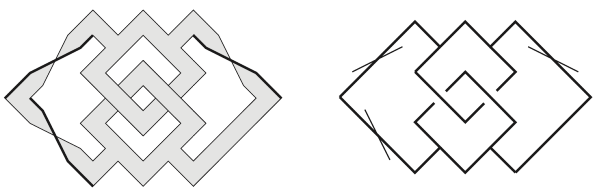
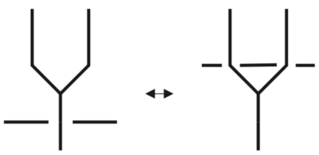


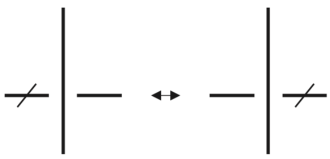
Figure 10: A KTG diagram and its simplified representation. Image taken from [13].

Isotopic KTGs can be related through a series of moves, similarly to what was done for knots using Reidemeister moves. These moves are called trivalent isotopic moves and are depicted in figure 11.

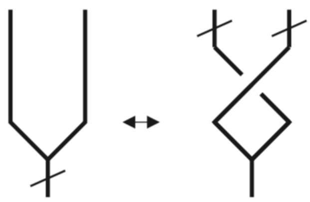
- Fork slide: we can slide a band from one side of the crossing of a fork to the other side of this crossing (figure 11a).
- Twist slide: a half twist in a band can be slid from one side of a crossing to the other side (figure 11b).
- Trivalent twist: if we have a half twist in a band which later splits into a fork, then the two bands of this splitting can overlap and the half twists are moved upward (figure 11c).
- Addition of twists 1: if a band has two half twists in opposite directions, we can remove them. Similary, we can add two half twists without changing the KTG (figure 11d).
- Addition of twists 2: a band with two half twists in the same direction is isotopic to that band having a loop (figure 11e).



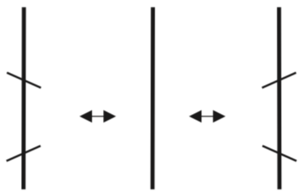
(a) Fork slide



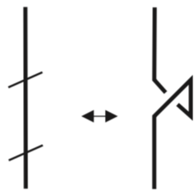
(b) Twist slide



(c) Trivalent twist



(d) Addition of twists 1



(e) Addition of twists 2

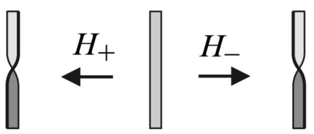
Figure 11: Trivalent isotopy moves. Image taken from [13].

Definition 3.9. Two KTGs G_1, G_2 are equivalent if there exists a diffeomorphism φ such that $\varphi(G_1) = G_2$ and φ is an isotopy.

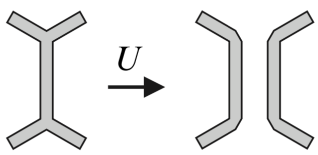
Theorem 3.2. Two KTGs are equivalent if they are related by a finite sequence of trivalent isotopy moves.

KTGs can be altered using a set of moves:

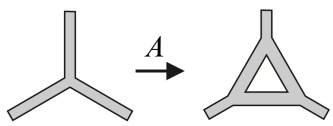
- Adding a left hand or right hand half twist (figure 12a).
- Unzipping a band (figure 12b).
- Adding a triangle in a forked band (figure 12c).



(a) KTG moves H_+ and H_-.



(b) KTG move U.



(c) KTG move A.

Figure 12: KTG moves. Image taken from [13].

Lemma 3.1. Suppose there exists a finite sequence of KTG moves that transform the theta graph into some KTG with H_+, H_-, U, A used ℓ_-, ℓ_+, m, n times respectively. Then we can first perform the move A n-times, then H_+, H_- n_+, n_- times and finally U m times.

Proof. Let us take an arbitrary KTG G such that there exists a sequence of KTG moves which transform the theta graph into G.

- If U is applied to an edge which was created by a move A, then A is applied before U.If U is used on a different edge then the moves U and A do not influence each other and hence A can be applied before U.
- If we unzip an edge with a half twist, then H_+ or H_- must have been applied to the edge before the unzip. If we give a half twist to an edge which is never unzipped, then the moves H_+, H_-, U do not influence each other, thus H_+, H_- can be used before U.
- Finally, if an edge created by A is given a half twist, then A must be applied before H_+ or H_-. Alternatively, if we give a half twist to an edge not created by A, then these moves do not affect each other and we can thus apply A before H_+, H_-.

Hence we can first apply A n times, then H_+, H_- ℓ_+, ℓ_- times and finally all U m times. ☐

Theorem 3.3. Every KTG can be constructed by using the moves H_+, H_-, U, A on the theta graph (figure 13) [13].

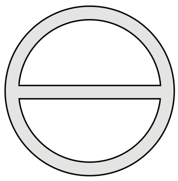


Figure 13: Theta graph.

As an example of the construction of a KTG we consider the KTG in figure 10. Figure 14 shows all steps, starting at the theta graph and finishing with the desired KTG.

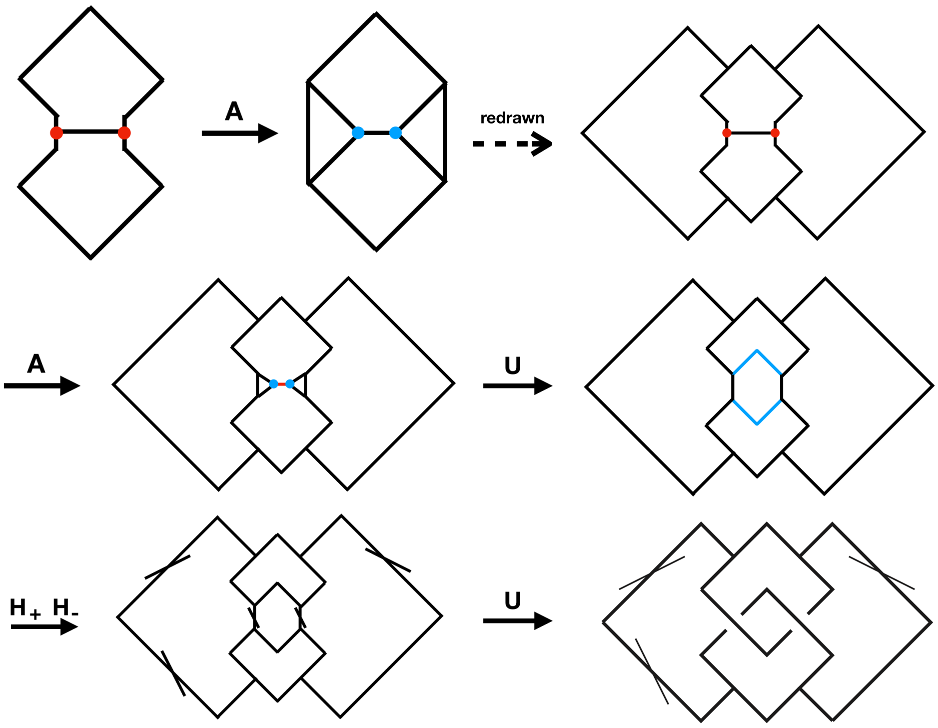


Figure 14: The construction of a KTG. Red indicates that the next move operates on this element. In the next step such elements are blue.

4 Colouring

As explained in section 3.1 two knots are equivalent if we can transform one into the other using Reidemeister moves. It can however be useful to apply other methods of determining whether two knots are equivalent, because projections of knots can be complicated. Using Reidemeister moves to show the equivalence of two knots can then lead to a rather messy process. As an example, consider the knot in figure 15, which is a more complex projection of the unknot.

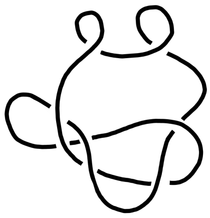


Figure 15: An alternative projection of the unknot.

There are several methods used to determine whether two knots are equivalent, such as using knot polynomials and knot colouring. Sections 4.2, 4.3 and 4.4 focus on the latter. We first however discuss the fundamental group of knots and KTGs.

4.1 Knot group and KTG group

Definition 4.1. Let X be a topological space and $p \in X$. The fundamental group $\pi_1(X, p)$ is the set of all equivalence classes of continuous paths $\alpha : [0, 1] \to X$ such that $\alpha(0) = \alpha(1) = p$.

In order to present a knot K we consider the fundamental group of the knot complement $S^3 \backslash K$. If we want to do this we must assign an orientation to the knot. This is the direction we would follow if we were to stand on the knot. Adding an orientation to the knot gives rise to positive and negative crossings as shown in figure 16.

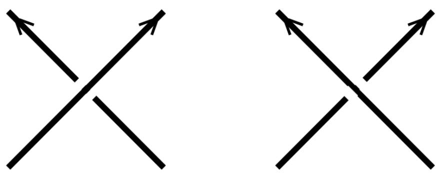
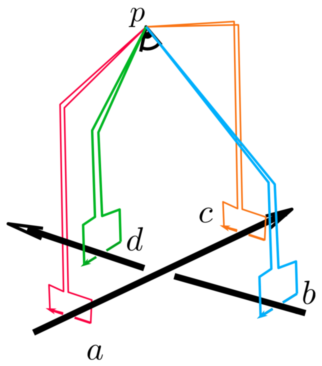


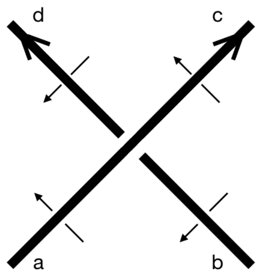
Figure 16: Positive crossing (left) and negative crossing (right).

After giving the knot an orientation we can find the fundamental group $\pi_1\left(S^3 \backslash K, p\right)$. K is projected onto a plane and as a point p we take a point in a plane above K. From the point p we have paths running around the arcs of the knot. The direction of these paths is determined by the right hand rule. We consider the paths close to the crossings of a knot in order to obtain a representation of the knot, which is depicted in figure 17a. Figure 17b shows the upper view of this crossing, where the small arrows indicate the paths around the arcs. For the positive crossing we have the relation $cd = ba$. Keeping in mind that a and c are the same arc we get $ad = ba$ and hence $d = a^{-1}ba$. In case of a negative crossing (figure 17c) we have $d = aba^{-1}$.

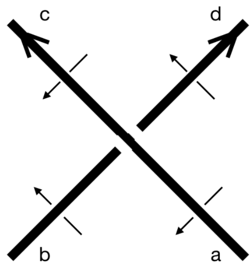
Rather than memorising this relation, we can also use a simple trick. We consider the crossing in figure 17b, keeping in mind that a and c are the same arc. Following along a path around d is equivalent to first following the path around a in opposite direction to the orientation of the path and then following the paths around b and a, as shown in figure 17d. This gives the relation $d = a^{-1}ba$.



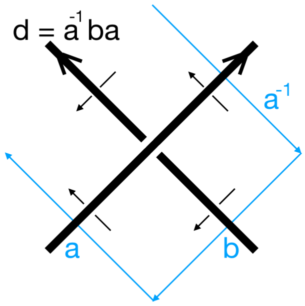
(a) Paths running around the arcs of a knot.



(b) Paths running around a positive crossing.



(c) Paths running around a negative crossing.



(d) The relation of paths running around the arcs.

Figure 17: Depiction of the fundamental group of the knot complement around a crossing. Leftmost image taken from [4].

We always take a point p above the knot, hence we can leave it out of our notation for the fundamental group of the knot complement. We can write $\pi_1\left(S^3\backslash K\right)$ instead of $\pi_1\left(S^3\backslash K, p\right)$ and refer to this as the knot group.

As an example we consider the knot shown in figure 18.

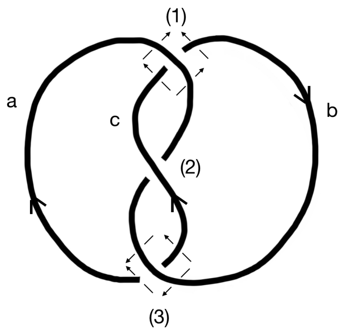


Figure 18: Example of a knot with paths running around the arcs.

Using figure 17 we get the following equalities from the four crossings:

$$(1)\ ac = ba$$
$$(2)\ cb = ac$$
$$(3)\ ba = cb$$

By these equalities we can derive the relations between a, b, c and obtain an expression for the knot. Equation (1) gives $c = a^{-1}ba$ and by substituting this into (2) we obtain the relation $aba = bab$. Hence for the knot of figure 18 we have $\pi_1\left(S^3\backslash K\right) = \langle a, b \mid aba = bab\rangle$.

In a similar way we can present KTGs, where we need to account for forks in the graphs. This is done by considering paths running around the bands of the KTG, close to the fork, see figure 19. The paths start and end at a point p above the KTG and are elements of the fundamental group of the KTG complement $\pi_1(G, p)$. The direction of the paths can be derived using the right hand rule.

Similarly to the knot group we can leave out the point in $\pi_1(G, p)$ and refer to this group as the KTG group.

In the fork of figure 19 we have the relation $c = ba$. This is because crossing the arc c is equivalent to crossing arcs b and a consecutively.

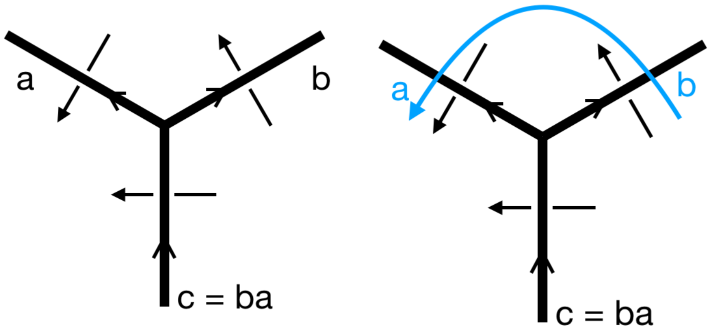


Figure 19: Fork of a KTG with paths running around the edges.

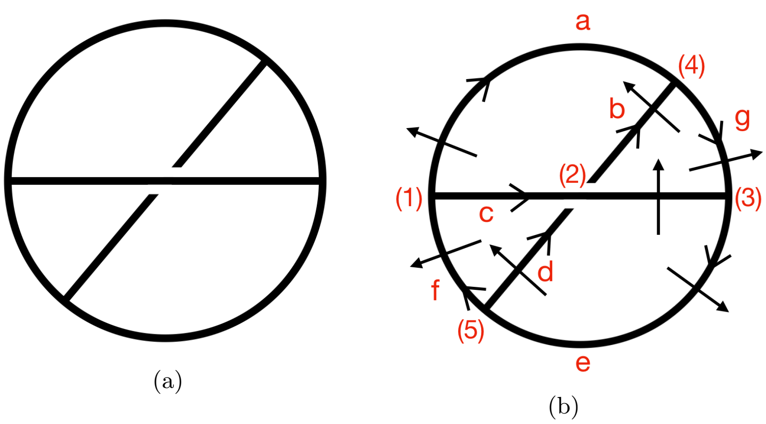
As an example we consider the KTG G in figure 20. We have five arcs labelled a, b, c, d, e, f, two forks $1), (3)$ and one crossing (2). Using figures 17 and 19 we get the following equalities:

$$\begin{aligned}
&(1)\ f = ca \\
&(2)\ b = c^{-1}dc \\
&(3)\ e = cg \\
&(4)\ g = ba \\
&(5)\ e = df
\end{aligned}$$

Thus the KTG group presentation is

$$\pi_1\left(S^3\backslash G\right) = \langle a, b, c, d, e, f \mid b = c^{-1}dc,\ e = cg = df,\ f = ca\rangle$$

Note that the relation $g = ba$ is left out of this expression because it follows from the other relations.



(a) (b)

Figure 20: Left: KTG G. Right: KTG G with labels for the arcs and crossings and arrows indicating orientation and direction of paths.

If two KTGs are equivalent then they have the same KTG group, implying that the KTG group can be used to determine if two KTGs are invariant. This is not explicitly explored in this thesis, but the next chapters on the colouring of knots and KTGs discuss the relation between the KTG group and KTG colouring as knot invariants.

4.2 Knot colouring

Definition 4.2. For $n \geq 3$, an n-colouring of a knot assigns to every arc of a knot a 'colour' $0, 1, \ldots, n-1 \in \mathbb{Z}/n\mathbb{Z}$ such that the relation shown in figure 21 holds.

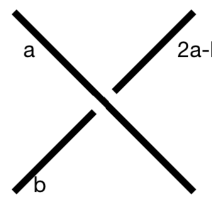


Figure 21: Relation of the colours of the strands for an n-colouring.

Figure 22 shows an n-colouring of the trefoil knot for $n = 3$. One can check that the relation of figure 21 holds for all crossings of the trefoil knot.

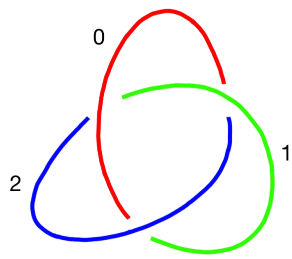
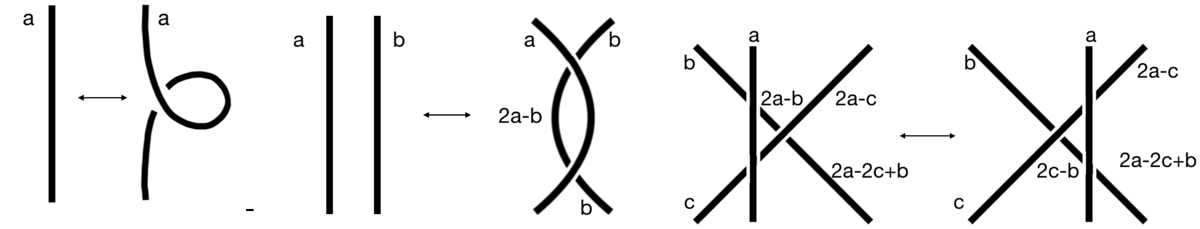


Figure 22: A tricolouring of the trefoil knot.

Theorem 4.1. The number of n-colourings of a link diagram D, $col_n(D)$, is preserved by Reidemeister moves.

Consequently, if two knots have a different number of n-colourings then they are not equivalent.

Proof. Let $n \in \mathbb{Z}$, $n \geq 3$ be arbitrary. Next we pick distinct arbitrary colours $a, b, c \in \mathbb{Z}/n\mathbb{Z}$.
For the first Reidemeister move we have one crossing with the overstrand and the right understrand both labeled a. As a result of definition 4.2 the left understrand has label $2a - a = a$, which is exactly as expected. Since both the knot with and without strand is coloured using only a, they have the same number of colourings.
A type II Reidemeister move creates two crossings. Label the two loose strings with a and b. The understrand on the left of the upper crossing has colour $2a - b$. As a result the understrand on the right of the lower crossing is labeled with $2a - (2a - b) = b$. By assumption $a \neq b$. a and b generate the new colour $2a - b$, which implies there is a one to one correspondence between these colours: one colouring of the left image corresponds to one colouring of the right image and vice versa. As a result the images on the left and right of figure 23b have the same amount of colourings.
Finally, a type III Reidemeister move has three crossings and we pick three colours a, b, c to colour the strands, as depicted in figure 23c. We see that the same colours are used in the left and right image. Thus there is a one to one correspondence in the colours used in these two images. Hence they have the same amount of colours.
We conclude that all Reidemeister moves preserve the number of n-colourings of a link diagram. $\square$



(a) n-colouring of type I Reidemeister move.

(b) n-colouring of type II Reidemeister move.

(c) n-colouring of type III Reidemeister move.

Figure 23: n-colourings of Reidemeister moves.

The arcs of a knot can also be labelled using elements of the dihedral group, D_{2n}.

Definition 4.3. The dihedral group is the symmetry group of a regular polygon.

$$D_{2n} = \{r, s \mid r^n = s^2 = 1,\ sr = r^{-1}s\}$$

Specifically, we assign elements sr^{α} to the arcs. This is for example done for the trefoil knot in figure 24 for the case $n = 3$. One can see the similarities in the colouring of this figure and the n-colouring in figure 22. We can prove that there is a relation between this method of knot colouring and the n-colouring method explained above, namely that there exists a bijection between the amount of n-colourings of a knot and the homomorphisms that map $\pi_1(K)$ to D_{2n}.

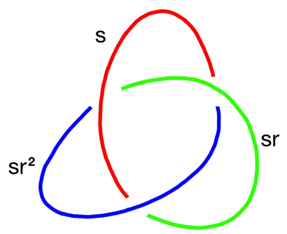


Figure 24: A tricolouring of the trefoil knot with labels from the dihedral group.

Lemma 4.1. Let K be a knot and C a colouring of the knot. C assigns elements $\alpha \in \mathbb{Z}/n\mathbb{Z}$ to arcs of the knot. The map h which assigns a reflection to every arc and is defined as $h : \pi_1(K) \to D_{2n}$, $[a] \to sr^{\alpha}$, $\alpha \in \mathbb{Z}/n\mathbb{Z}$ is a homomorphism.

Proof. Let $[a], [b] \in \pi_1(K)$ be arbitrary with $h([a]) = sr^{\alpha}$, $h([b]) = sr^{\beta}$.

$$\begin{aligned} h([b]^{-1}[a][b]) &\overset{(1)}{=} sr^{2\alpha-\beta} \\ &= ssr^{\beta-\alpha}sr^{\alpha} \\ &= sr^{\alpha}sr^{\beta}sr^{\alpha} \\ &= (sr^{\alpha})^{-1}sr^{\beta}sr^{\alpha} \\ &= h([b]^{-1})h([a])h([b]) \end{aligned}$$

Thus h is a homomorphism.
(1) Figure 25 shows why this equality holds. □

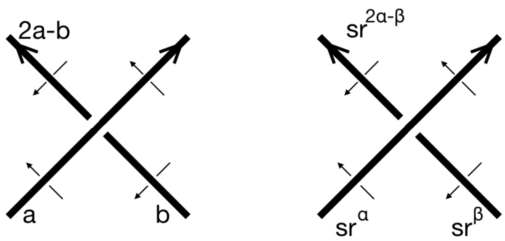


Figure 25: The 'normal' n-colouring of a crossing and the colouring using te dihedral group.

We should take care to assign an orientation to our knots as theory from fundamental groups (section 4.1) is used. However, the results of colouring a positive or a negative crossing is the same. One can also check that $h\left([a][b][a]^{-1}\right) = h\left([a]^{-1}[b][a]\right)$.

If we consider the relations of the knot group and an n-colouring of the knot, we can see why assigning elements $sr^\alpha \in D_{2n}$ leads to an n-colouring. Namely, we see that the relations between the generators of the fundamental group are preserved by the n-colouring.

The only thing left is to prove the relation between the n-colourings and the homomorphisms from lemma 4.1.

Theorem 4.2. Let C be the set of n-colourings of a knot K and H be the set of group homomorphisms $h : \pi_1(K) \to D_{2n}$ that map equivalence classes in $\pi_1(K)$ to reflections. There exists a bijection between C and H.

Proof. Define $\varphi : C \to H$ by $c \mapsto h$, where h is a homomorphism mapping $[a] \to sr^\alpha$. In order to prove that this is injective, suppose we have $\varphi(c) = \varphi(\tilde{c})$, then $h = \tilde{h}$, thus $h, \tilde{h}$ both map $[a] \to sr^\alpha$. This means $h, \tilde{h}$ give the same colouring of the knot, thus $c = \tilde{c}$.
Next take any arbitrary homomorphism $h \in H$. By its definition h maps $[a] \to sr^\alpha$, which means that h assigns a reflection to every arc of the knot. We can assign the elements α in the exponents of sr^α to the arcs. This then gives a colouring of the knot, because definition 4.2 is satisfied. Thus there exists $c \in C$ such that $\varphi(c) = h$, hence φ is surjective. We conclude that φ is bijective. □

4.3 KTG colouring

The aforementioned method of colouring arcs with elements $sr^\alpha \in D_{2n}$ cannot be applied to KTGs because it breaks when we colour forks, which is illustrated in figure 28. Suppose we colour the two outgoing arcs with sr^α and sr^β, then using the KTG group we get that the incoming arc is coloured with $sr^\alpha sr^\beta$. However, $sr^\alpha sr^\beta = r^{\beta-\alpha}$ is not a reflection. We would hence like to prove that we can colour KTGs using both reflections and rotations.

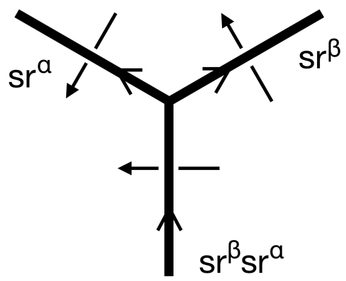


Figure 26: n-colouring of a fork using definition 4.2.

The introduction of rotations creates different relations between colours of arcs at crossings. Figure 27a depicts the relation of arcs meeting at a crossing as determined by the KTG group, with figure 27b showing how this relation is derived. An example of a colouring of a crossing can be found in figure 27c.

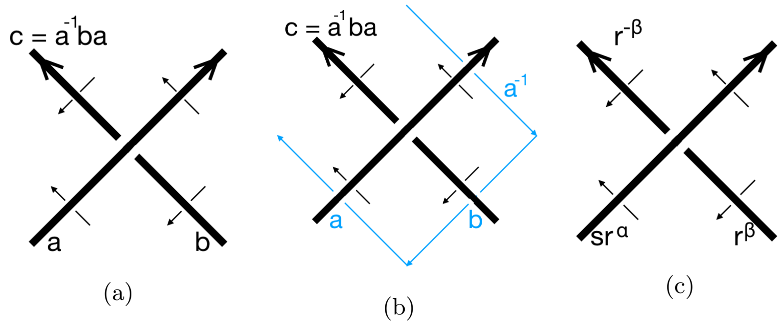


Figure 27: Relation and a colouring of crossings in a KTG.

Another difference between knots and KTGs is the forks found in KTGs but not in knots. We can use the KTG group to determine how we colour the arcs meeting at a fork. The relation of the paths running around these arcs is shown in figure 28a and an example of a fork colouring is found in figure 28c.

A last notable difference between knots and KTGs is the presence of half twists in KTGs. However, these do not affect the colouring of a KTG, as the arc is given the same colour on both sides of the half twist.

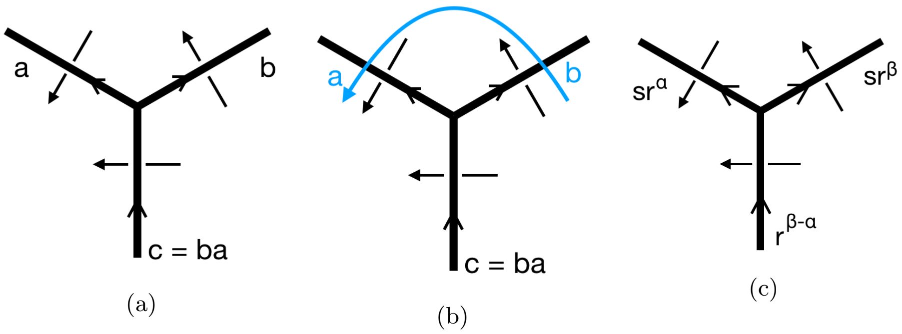


Figure 28: Relations of the arcs and a colouring of a fork.

For KTGs too we find that there is relationship between the n-colourings of a KTG and the dihedral group, where we again make use of homomorphisms between the fundamental group and the dihedral group.

Lemma 4.2. Let G be a KTG and $[a] \in \pi_1\left(S^3 \backslash G\right)$. $h : \pi_1\left(S^3 \backslash G\right) \to D_{2n}$ is a homomorphism.

This can be proven by making case distinctions and using the same method as in the proof of lemma 4.1. The full proof can be found in the appendix.

Theorem 4.3. Let C be the set of n-colourings of a KTG G and H be the set of group homomorphisms $h : \pi_1\left(S^3 \backslash G\right) \to D_{2n}$. There exists a bijection between C and H.

The proof of this theorem is given in the appendix.

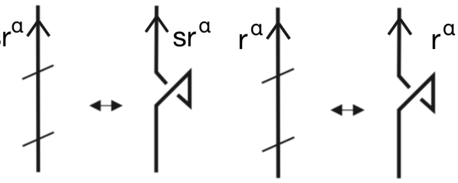
Having proven the relation between the amount of n-colourings of KTGs and the dihedral group we give a proof to show that the amount of n-colourings of a KTG is invariant under the trivalent isotopy moves.

Theorem 4.4. The number of n-colourings of a KTG, $col_n(KTG)$, is preserved by trivalent isotopy moves.

As a result of this theorem two KTGs with a different number of n-colourings are not equivalent.

Proof. Half twists do not influence the colouring of a KTG. Hence all steps of the twist slide (figure 11b) can be coloured in the exact same way, which implies that they have the same number of n-colourings. The same holds for the addition of twists 1 (figure 11d).

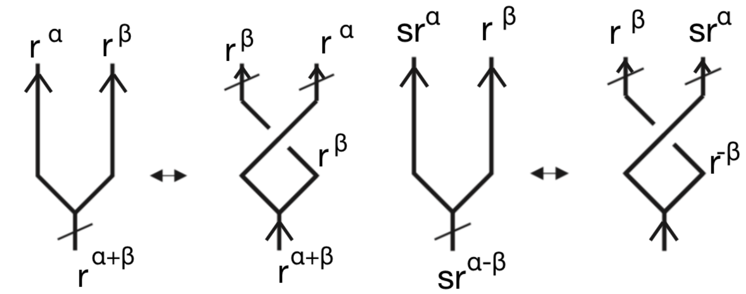
For the remaining three trivalent isotopy moves we consider different cases for different colourings of these moves. The addition of twists 2 gives two different colourings, which are depicted in figure 29. Since the image on the left and right use the same colour, they have the same number of n-colourings. The other case for the addition of twists 2 replaces r^{α} with sr^{α} and gets to the same conclusion.



(a) (b)

Figure 29: Colouring of the addition of twists 2.

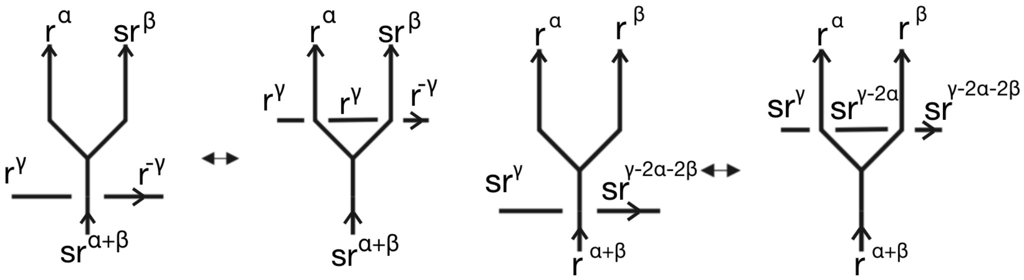
The next move to be considered in the trivalent twist. We know that at a fork we have two reflections and a rotation or three rotations, which gives four different cases. Two of them are shown in figure 30 The remaining two cases are left as an exercise to the reader. The right image in figure 30a has one extra colour in comparison to the left image, but it is generated by the three colours from the left image. We hence have a one to one correspondence of the colours. This holds for all possible colourings of the trivalent twist, thus the amount of n-colourings is preserved by this move.



(a) (b)

Figure 30: Two possible colourings of the trivalent twist.

Finally we consider the fork slide. There are four ways of colouring the fork and two ways of colouring the understrand, leading to a total of eight colourings. Two cases can be found in figure 31. The other cases are again left as an exercise to the reader. The part of the understrand between two bands of the fork is given a colour which is generated by the colours of the other arcs. The remaining colours of the right image correspond to those of the left image. As a result there is a one to one correspondence between the colours in both pictures, leading to the preservation of the amount of n-colourings by the fork slide.
Hence all trivalent isotopy moves preserve the amount of n-colourings.



(a) (b)

Figure 31: Two possible colourings of the fork slide.

□

To finish this section we discuss the relation between KTG colouring and the KTG group. Section 4.1 mentioned that two equivalent KTGs have the same KTG group, thus the KTG group is a KTG invariant. In this chapter on KTG colouring we based the colouring of KTGs on the relations between the arcs of the KTG using the KTG group. Combining these two facts it becomes clear why KTG colouring is a KTG invariant.

4.4 Colouring Celtic knots

Having proven that the number of n-colourings of a knot is preserved by trivalent isotopy moves, we would like to apply this theorem to some Celtic designs, for example the one mentioned in the introduction. This same design is shown in figure 32 together with a simplified representation.

Figure 32: A Celtic 'knot'from the book of Kells and its simplified representation.

The simplified representation shows that there are several loose ends in the design, which makes this a tangle. We make this into a KTG by drawing a circle around the design which hits the loose ends, see figure 33.

Let us find the number of tricolourings for this Celtic KTG. In order to do so we start by assigning a, b, c to some of the arcs, as indicated in the figure. Using this we can compute the colours assigned to the other arcs of the KTG, as well as find relations for the forks created by the circle attached to the loose ends.

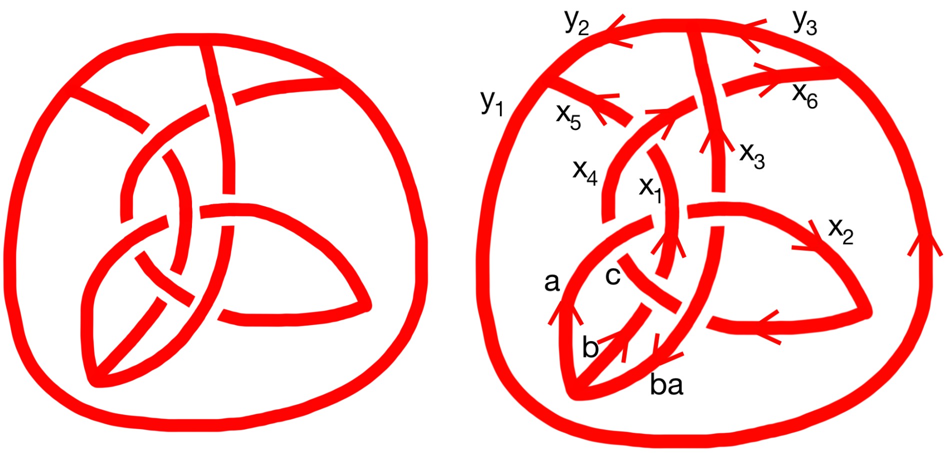


Figure 33: The Celtic design as KTG.

$$
\begin{aligned}
x_1 &= cbc^{-1} \\
x_2 &= x_1 a x_1^{-1} = a^{-1}b^{-1}cba \\
x_3 &= x_2^{-1} b a x_2 \\
x_4 &= a^{-1}ca \\
x_5 &= x_4^{-1} x_1 x_4 \\
x_6 &= x_3^{-1} x_4 x_3 \\
y_1 &= y_2 x_5 \\
y_3 &= y_2 x_3 \\
y_3 &= y_1 x_6
\end{aligned}
$$

We have two expressions for y_3. Setting these equal to each other gives

$$
\begin{aligned}
y_2 x_3 &= y_1 x_6 \\
\iff y_2 x_3 &= y_2 x_5 x_6 \\
\iff x_3 &= x_5 x_6
\end{aligned}
$$

For x_3, x_5, x_6 we obtain the following expressions:

$$
\begin{aligned}
x_3 &= x_2^{-1} b a x_2 \\
&= a^{-1}b^{-1}c^{-1}babaa^{-1}b^{-1}cba \\
&= a^{-1}b^{-1}c^{-1}bacba
\end{aligned}
$$

$$
\begin{aligned}
x_5 &= x_4^{-1} x_1 x_4 \\
&= a^{-1}c^{-1}acbc^{-1}a^{-1}ca
\end{aligned}
$$

$$
\begin{aligned}
x_6 &= x_3^{-1} x_4 x_3 \\
&= a^{-1}b^{-1}c^{-1}a^{-1}b^{-1}cbaa^{-1}caa^{-1}b^{-1}c^{-1}bacba \\
&= a^{-1}b^{-1}c^{-1}a^{-1}b^{-1}cbcb^{-1}c^{-1}bacba
\end{aligned}
$$

Using these expressions and the equality $x_3 = x_5 x_6$ we obtain

$$
\begin{aligned}
a^{-1}b^{-1}c^{-1}bacba &= a^{-1}c^{-1}acbc^{-1}a^{-1}caa^{-1}b^{-1}c^{-1}a^{-1}b^{-1}cbcb^{-1}c^{-1}bacba \\
\iff 1 &= c^{-1}acbc^{-1}a^{-1}cb^{-1}c^{-1}a^{-1}b^{-1}cbc \\
\iff 1 &= acbc^{-1}a^{-1}cb^{-1}c^{-1}a^{-1}b^{-1}cb
\end{aligned}
$$

We make case distinctions to find what colours can be given to the arcs a, b, c in order for the equality above to hold. Before doing so we can exclude several cases. We note that the left hand side of the equality is a rotation. The right hand side contains the terms a, a^{-1} three times, b, b^{-1} four times and c, c^{-1} five times. If we colour a with a rotation r^α and b, c with reflections r^β, r^γ, then the right hand side of the equality is a reflection, but then the equality does not hold. There are a total of four cases leading to the same result.

- a is coloured with a reflection and b, c with rotations.
- a, b are coloured with rotations and c with a rotation.

- a, b are coloured with a reflections and c with a rotation.
- a is coloured with a rotations and b, c with reflections.

This leaves us to consider four cases. Here we use $sr^\alpha, r^\alpha, sr^\beta r^\beta, sr^\gamma, r^\gamma$ to denote the reflections and rotations given to a, b, c. Here $\alpha, \beta, \gamma \in \mathbb{Z}/3\mathbb{Z}$.

Case 1: a, b, c are coloured with rotations. Using $1 = acbc^{-1}a^{-1}cb^{-1}c^{-1}a^{-1}b^{-1}cb$ we have

$$\begin{aligned} 1 &= r^\alpha r^\gamma r^\beta r^{-\gamma} r^{-\alpha} r^\gamma r^{-\beta} r^{-\gamma} r^{-\alpha} r^{-\beta} r^\gamma r^\beta \\ &= r^{-\alpha+\gamma} \end{aligned}$$

This gives $-\alpha + \gamma = 0$, thus $\alpha = \gamma$. a and c are hence given the same rotation. We can pick one of three rotations, $1, r, r^2$, thus there are three colour options for a, c and three for b. Case 1 gives a total of $3^2 = 9$ colourings.

Case 2: a, c are coloured with rotations and b with a reflection.

$$\begin{aligned} 1 &= r^\alpha r^\gamma sr^\beta r^{-\gamma} r^{-\alpha} r^\gamma r^{-\beta} sr^{-\gamma} r^{-\alpha} r^{-\beta} sr^\gamma sr^\beta \\ &= sr^{\beta-2\alpha-2\gamma+\gamma-\beta} sr^{-\gamma-\alpha-\beta} r^{-\gamma+\beta} \\ &= sr^{-2\alpha-\gamma} sr^{-2\gamma-\alpha} \\ &= r^{\alpha-\gamma} \end{aligned}$$

Hence $\alpha - \gamma = 0$. Case 2 also gives $3^2 = 9$ tricolourings.

Case 3: a, c are coloured with reflections and b with a rotation.

$$\begin{aligned} 1 &= sr^\alpha sr^\gamma r^\beta r^{-\gamma} sr^{-\alpha} ssr^\gamma r^{-\beta} r^{-\gamma} sr^{-\alpha} sr^{-\beta} sr^\gamma r^\beta \\ &= sr^\alpha sr^\gamma r^\beta r^{-\gamma} sr^{-\alpha} r^\gamma r^{-\beta} r^{-\gamma} sr^{-\alpha} sr^{-\beta} sr^\gamma r^\beta \\ &= r^{\gamma+\beta-\gamma-\alpha} r^{-\alpha+\alpha-\gamma+\beta+\gamma} r^{\gamma+2\beta} \\ &= r^{-\alpha+4\beta+\gamma} \end{aligned}$$

This gives us $-\alpha + 4\beta + \gamma = 0$. We can freely choose two of the colours which then determine the third colour by this equality. There are hence $3^2 = 9$ tricolourings.

Case 4: a, b, c are coloured with reflections.

$$\begin{aligned} 1 &= sr^\alpha sr^\gamma sr^\beta r^{-\gamma} sr^{-\alpha} ssr^\gamma r^{-\beta} sr^{-\gamma} sr^{-\alpha} sr^{-\beta} ssr^\gamma sr^\beta \\ &= sr^\alpha sr^\gamma sr^\beta r^{-\gamma} sr^{-\alpha+\gamma-\beta} sr^{-\gamma} sr^{-\alpha} sr^{-\beta+\gamma} sr^\beta \\ &= r^{\gamma-\alpha} r^{-\alpha+2\gamma-2\beta} r^{-\alpha+\gamma} r^{2\beta-\gamma} \\ &= r^{3\gamma-3\alpha} \end{aligned}$$

Thus we have $3\gamma - 3\alpha = 0$, which holds for all $\alpha, \gamma \in \mathbb{Z}/3\mathbb{Z}$. In case 4 we can choose any combination of reflections for the arcs a, b, c, giving $3^3 = 27$ tricolourings.

In total there are 54 tricolourings of the Celtic KTG of figure 32.

Two similar looking Celtic designs and their simplified representations are depicted in figure 34. The number of tricolourings of both KTGs can be computed with a similar method as used above. These computations can be found in the appendix. For the left KTG we find that there are 216 tricolourings, but for the right KTG we find 108. By the contrapositive of theorem 4.4 we conclude that these KTGs are different.

Figure 34: Celtic designs from the Gospel of Saint Luke in the Quatuor evangelia.

5 Constructing Celtic knots

As we have seen in chapter 2 there are a variety of different Celtic knots, some consisting of the patterns from figure 5 tied together, others being a link of different knots. The majority of Celtic knots, however, seem to have a common characteristic: they are alternating knots.

In chapter 2 a construction of Celtic knots was given based on the work of George Bain. However, for the sake of this thesis we want to make a more concrete, algorithmic method for drawing Celtic knots, as described in [9].

The Celtic knot is drawn in a $2m \times 2n$ grid with $n, m \in \mathbb{Z}_{>0}$. We place a dot at a lattice point $(x, y) \in [0, 2n] \times [0, 2m]$ if $x + y$ is odd. Two diagonal intersecting line segments are drawn at each of the dots. For x odd we have an overlapping line from the top left to the top right (upper image of figure 35b). If x is even, the overlapping line runs from the top right to the bottom left, as in the lower image of figure 35b. Finally we connect these line segment, which can be done in one of two ways. Firstly, we can connect two line segments from squares diagonally adjacent to each other, which is indicated with green in figure 35a. The second option is to extend the segments along the border of the grid, which is shown in blue in 35a. This construction leads to a so-called barrier-free Celtic knot and is the same as what was constructed in figure 3.

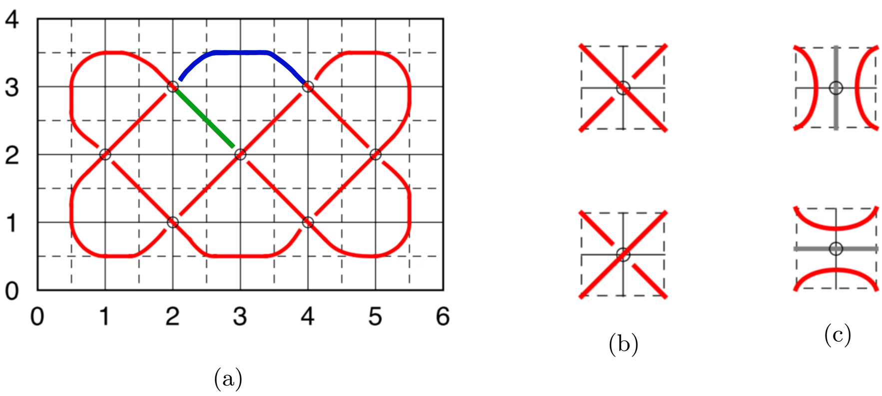


Figure 35: From left to right: a drawing of a Celtic knot, crossing subgrids and barrier subgrids. Image taken from [9] and edited.

Similarly to what was done in chapter 2, we can make other Celtic knots by placing barriers. These barriers are placed through the dots in our grid, which removes the crossings through the dots. We get one of the two situations shown in figure 35c. An example of a Celtic knot design using barriers is shown in figure 36.

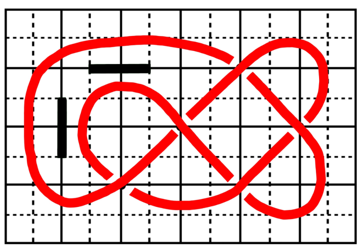


Figure 36: Celtic knot in a grid with barriers.

Based on this construction of Celtic knots can create a method of making Celtic designs with KTGs. We again have a grid with dots as in the method for knots. Next horizontal or vertical barriers can be placed at one or more dots. The theta graph is then drawn in the grid with each half of the theta corresponding to one of the 'circles 'which we obtain if we make barrier-free Celtic knot. If we have barriers then we cannot always pick our theta halves like this and must instead break the circles at the barriers to create a larger circle. To illustrate this idea figure 37 shows two thetas that can be used to make the Celtic knot of figure 36.

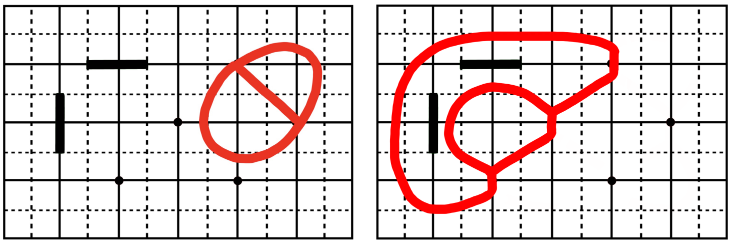


Figure 37: Two possible placements for the theta graph.

We continue by creating bubbles, triangles and squares until all the barrier-free dots are connected by double forks (figure 39). Finally half twists are used on the double forks, after which these double forks are unzipped. This is done such that the resulting knot is alternating. Alternatively, we can choose to leave several double forks unzipped leading to a KTG with both crossings and forks.

An example of this construction is shown in figure 38. In step b and c a bubble is created. Next, in figure 38d KTG move A is applied twice, after which we unzip the edge between the two triangles to create a 'rectangle'. In step f two triangles are again created, after which the edge connecting them in unzipped in step g. Then the double forks are given a half twist and are unzipped. These half twists must be placed such that when we unzip the double forks the resulting knot is alternating. The resulting Celtic knot is equivalent to the design found using the method from the beginning of this chapter.
The details on the creation of bubbles and rectangles can be found in the appendix.

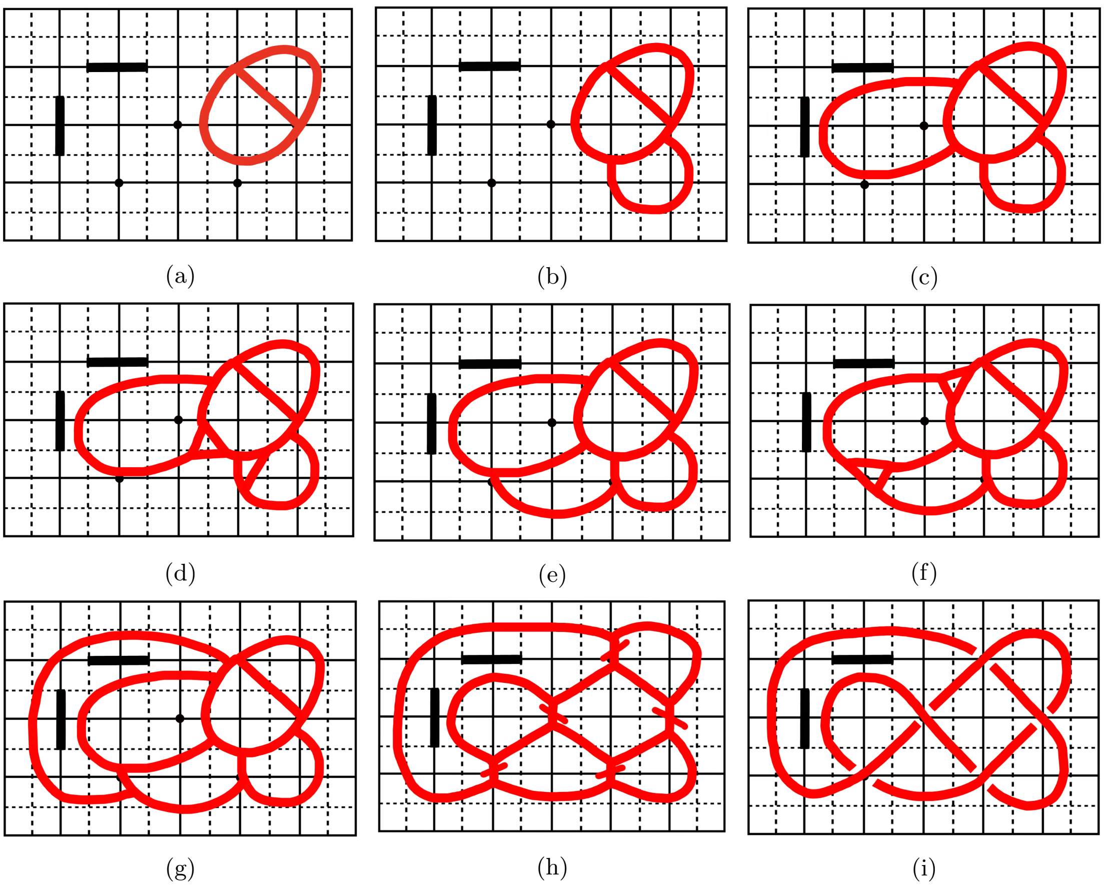


Figure 38: Construction of a Celtic design using KTG theory.

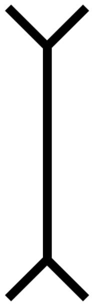


Figure 39: Double fork.

6 Conclusion

This thesis discussed Celtic knots within the framework of KTGs and how the concept of knot colouring can be adjusted for applying it to KTGs with the aim of answering the question

How can we build Celtic knots using KTGs and prove invariance of KTGs?

To find the answer to this question we first explored basic knot theory and the theory of KTGs.

Next it was explained how knots can be coloured and how this colouring is used for finding knot invariants. The relation between knot colouring and the dihedral group was given and proven. Moreover, we proved that KTGs can be coloured using reflections and rotations by showing that there exists a bijection between the n-colourings of a KTG and the dihedral group. Using this theory we have proven that the number of n-colourings is preserved by the trivalent isotopy moves, which allowed us to find KTG invariants. We then examined several Celtic KTGs from medieval manuscripts. Simplified representations of such KTGs were coloured using elements of the dihedral group and we applied the earlier results from the chapter to prove that some of the KTGs were not equivalent.

The final chapter discussed how Celtic KTGs can be created using the grid method by Gross and Tucker, which was an adjusted version of Bain's grid method. The Celtic KTGs were constructed starting from a theta graph, after which the KTG moves H_+, H_-, U, A were applied to adjust the theta graph. The final result of our method is the same as that of Gross and Tucker's method.

While this thesis answered the question "How can we build Celtic knots using KTGs and prove invariance of KTGs?" it also raises new questions for future research.

There are multiple knot invariants other than colouring, for example knot polynomials. In the future research could be done into finding KTG polynomials or a different KTG invariant. Another topic for research is formulating a method of denoting KTGs. A possible idea is to find names which allow us to read off what KTG moves were used to create the KTG when starting with the theta graph.

An issue that arose during this research was the complexity of KTG colouring. When computing the relations of the paths of the KTG group the expressions could become complicated, even for quite simple KTGs. It would hence be useful to program an algorithm which does these computations and gives the number of n-colourings of KTGs.

Finally, one could explore knotted graphs with vertices of degrees larger than three and adjust concepts proven in this thesis as well as other ideas.

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7 Appendix

Proof of lemma 4.2.

Lemma 4.2. Let G be a KTG and $[a] \in \pi_1\left(S^3 \backslash G\right)$. $h : \pi_1\left(S^3 \backslash G\right) \to D_{2n}$ is a homomorphism.

Proof. Let $[a], [b] \in \pi_1\left(S^3 \backslash G\right)$ be arbitrary with $h([a]) = s^i r^\alpha$ and $h([b]) = s^j r^\beta$. We now want to make some case distinctions.
Case 1: $i = j = 1$. See the proof of lemma 4.1.
Case 2: $i = 1, j = 0$.

$$\begin{aligned} h\left([a]^{-1}[b][a]\right) &= r^{-\beta} \\ &= r^{-\alpha-\beta+\alpha} \\ &= r^{-\alpha-\beta} s s r^\alpha \\ &= r^{-\alpha} s r^\beta s r^\alpha \\ &= \left(s r^\alpha\right)^{-1} r^\beta s r^\alpha \\ &= h\left([a]^{-1}\right) h([b]) h([a]) \end{aligned}$$

Case 3: $i = 0, j = 1$.

$$\begin{aligned} h\left([a]^{-1}[b][a]\right) &= s r^{2\alpha+\beta} \\ &= r^{-\alpha} s r^\beta r^\alpha \\ &= h\left([a]^{-1}\right) h([b]) h([a]) \end{aligned}$$

Case 4: $i = j = 0$.

$$\begin{aligned} h\left([a]^{-1}[b][a]\right) &= s r^{2\alpha+\beta} \\ &= r^{-\alpha} r^\beta r^\alpha \\ &= h\left([a]^{-1}\right) h([b]) h([a]) \end{aligned}$$

Thus h is a homomorphism. □

Proof of theorem 4.3

Theorem 4.3. Let C be the set of n-colourings of a KTG G and H be the set of group homomorphisms $h : \pi_1\left(S^3 \backslash G\right) \to D_{2n}$. There exists a bijection between C and H.

Proof. Define $\varphi : C \to H$ by $c \mapsto h$, where h is a homomorphism mapping $[a] \to s^i r^\alpha$, $i \in \{0, 1\}$, $\alpha \in \mathbb{Z}/n\mathbb{Z}$. To prove that φ is injective, suppose that we have $\varphi(c) = h$, $\varphi(\tilde{c}) = \tilde{h}$ and $\varphi(c) = \varphi(\tilde{c})$. This implies that $h = \tilde{h}$, thus $h, \tilde{h}$ map an element $[a]$ to the same $s^i r^\alpha \in D_{2n}$. But the $s^i r^\alpha$ determine the colouring of the KTG, hence $h, \tilde{h}$ give the same colouring of the KTG. Thus $c = \tilde{c}$.
Then we must show surjectivity. Let $h \in H$ be an arbitrary homomorphism which maps $[a] \to s^i r^\alpha$. h hence assigns a rotation or reflection to every arc of the KTG. By construction this mapping gives a colouring of the KTG. Hence there exists a colouring $c \in C$ such that $\varphi(c) = h$ and we conclude that φ is surjective. This concludes the prove of the theorem. □

Colouring knots from the St. Luke gospel in the Quatuor evangelia

We can place a circle around each design which hits the loose ends. This turns the Celtic designs into KTGs. The first of these KTGs is shown in figure 40. To determine the amount of tricolourings we start by labelling three arcs with a, b, c and compute the labels for the other arcs. We also find the relations of the arcs meeting at forks. These are the forks which are made by the loose ends hitting the circle around the KTG.

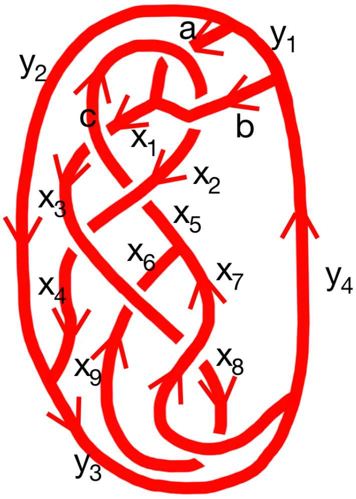

Figure 40: First design from the Gospel of Saint Luke as a KTG.

$$\begin{aligned}
x_1 &= cac^{-1}b \\
x_2 &= b^{-1}cb \\
x_3 &= c^{-1}x_1c \\
x_4 &= x_3x_2x_3^{-1} \\
x_5 &= x_2^{-1}cx_2 \\
x_8 &= x_7x_3x_7^{-1} \\
x_9 &= x_7^{-1}x_8x_7 = x_3 \\
x_5 &= x_7x_6 \\
y_1 &= y_2a \\
y_4 &= y_1b \\
y_3 &= y_4x_7 \\
y_3 &= y_2x_4
\end{aligned}$$

We have two expressions for y_3, giving us the equality

$$\begin{aligned}
y_4x_7 &= y_2x_4 \\
\iff y_1bx_7 &= y_2x_4 \\
\iff y_2abx_7 &= y_2x_4 \\
\iff abx_7 &= x_4
\end{aligned}$$

Using the expressions above we find the following for x_4.

$$\begin{aligned}
x_4 &= x_3x_2x_3^{-1} \\
&= c^{-1}x_1cb^{-1}cbc^{-1}x_1^{-1}c \\
&= c^{-1}cac^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1}c^{-1}c \\
&= ac^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1}
\end{aligned}$$

To find x_7 we must first compute x_6. From figure 40 we find that

$$x_3x_6x_3^{-1} = x_9 = x_7^{-1}x_8x_7$$

Here $x_7^{-1}x_8x_7 = x_3$. It follows that $x_6 = x_3$. This allows us to compute x_7.

$$\begin{aligned}
x_7 &= x_5x_6^{-1} \\
&= x_5x_3^{-1} \\
&= x_2^{-1}cx_2c^{-1}x_1^{-1}c \\
cac^{-1}b &= b^{-1}c^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1}c^{-1}c \\
&= b^{-1}c^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1}
\end{aligned}$$

We can now put these expression into the equality $abx_7 = x_4$.

$$\begin{aligned}
abb^{-1}c^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1} &= ac^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1} \\
\Longleftrightarrow \; ac^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1} &= ac^{-1}bcb^{-1}cbc^{-1}b^{-1}ca^{-1}
\end{aligned}$$

We obtain the same expression on the left and right hand side. Hence the colours assigned to a, b, c can be chosen arbitrarily. As we can choose one of six colours for a, b, c there are $6^3 = 216$ tricolourings.

The second design from figure 34 can also be made into a KTG, as shown in figure 41. We again begin by labelling three of the arcs a, b, c and proceed by finding the labels for the remaining arcs and the relations of the arcs meeting at forks.

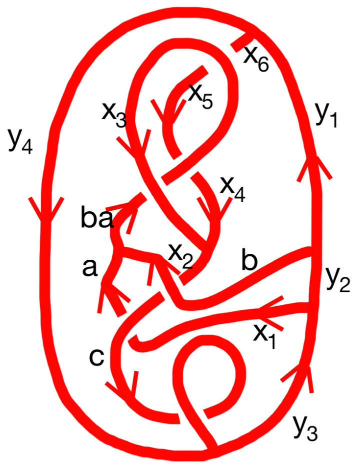

Figure 41: Second design from the Gospel of Saint Luke as a KTG.

$$\begin{aligned}
x_1 &= c^{-1}ac \\
x_2 &= bcb^{-1} \\
x_3 &= x_3^{-1}bax_3 \iff x_3 = ba \\
x_5 &= x_3^{-1}x_4x_3 \\
x_6 &= x_3x_5x_3^{-1} \\
x_2 &= x_3x_4 \\
y_1 &= y_4x_6 \\
y_2 &= y_1b \\
y_3 &= y_2a \\
y_3 &= y_4c
\end{aligned}$$

There are two expressions for y_3. Using them we find that

$$
\begin{aligned}
& y_2 a = y_4 c \\
\iff & y_1 ba = y_4 c \\
\iff & y_4 x_6 ba = y_4 c \\
\iff & x_6 ba = c
\end{aligned}
$$

By the expressions above we obtain the following for x_6:

$$
\begin{aligned}
x_6 &= x_3 x_5 x_3^{-1} \\
&= x_3 x_3^{-1} x_4 x_3 x_3^{-1} \\
&= x_4 \\
&= x_3^{-1} x_2 \\
&= a^{-1} b^{-1} bcb^{-1} \\
&= a^{-1} cb^{-1}
\end{aligned}
$$

Having obtained this expression we can use it in $x_6 ba = c$.

$$
\begin{aligned}
& x_6 ba = c \\
\iff & a^{-1} cb^{-1} ba = c \\
\iff & a^{-1} ca = c \\
\iff & ca = ac
\end{aligned}
$$

We thus only have restrictions on how arcs a, c are coloured. b can be any of the six colours. To find the amount of tricolourings we make four case distinctions. We use $sr^{\alpha}, r^{\alpha}, sr^{\gamma}, r^{\gamma}$ to denote the reflections and rotations given to a, c with $\alpha, \gamma \in \mathbb{Z}/3\mathbb{Z}$

Case 1: a, c are coloured with rotations. Using $ca = ac$ we get

$$
r^{\alpha+\gamma} = r^{\alpha+\gamma}
$$

This holds for any $\alpha, \gamma \in \mathbb{Z}/3\mathbb{Z}$. Recall that b could be given any of the six colours, hence case 1 gives $3^2 \cdots 6 = 54$ tricolourings.

Case 2: a is coloured with a reflection and c with a rotation.

$$
\begin{aligned}
& r^{\gamma} sr^{\alpha} = sr^{\alpha} r^{\gamma} \\
\iff & sr^{\alpha-\gamma} = sr^{\alpha+\gamma}
\end{aligned}
$$

Thus $\alpha - \gamma = \alpha + \gamma$ which implies $2\gamma = 0$. In $\mathbb{Z}/3\mathbb{Z}$ this holds for $\gamma = 0$. There are no restrictions on α, hence a can be coloured using any of s, sr, sr^2. There are hence $3 \cdot 6 = 18$ tricolourings.

Case 3: a is coloured with a rotation and c with a reflection.

$$
\begin{aligned}
& sr^{\gamma} r^{\alpha} = r^{\alpha} sr^{\gamma} \\
\iff & sr^{\gamma+\alpha} = sr^{\gamma-\alpha}
\end{aligned}
$$

Hence $\alpha + \gamma = \gamma - \alpha$ which holds for $2\alpha = 0$. Similarly to case 2 we find $\alpha = 0$ and that there are no restrictions on γ. This case gives $3 \cdot 6 = 18$ tricolourings.

Case 4: a, c are coloured with reflections.

$$
\begin{aligned}
& sr^{\gamma} sr^{\alpha} = sr^{\alpha} sr^{\gamma} \\
\iff & r^{\alpha-\gamma} = sr^{\gamma-\alpha}
\end{aligned}
$$

This equality holds if $2\alpha = 2\gamma$. The arcs a, c must hence be coloured with the same reflection. As a result we have $3 \cdot 6 = 18$ tricolourings for case 4.

In total there are 108 tricolourings for this KTG.

Construction of rectangles and bubbles in KTGs

Below are two figures showing the construction of a rectangle and of a bubble in a KTG. In each figure the KTG moves are applied to the red vertex or edge of the KTG.

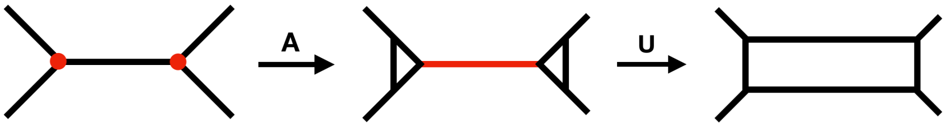

Figure 42: Construction of a rectangle in a KTG.

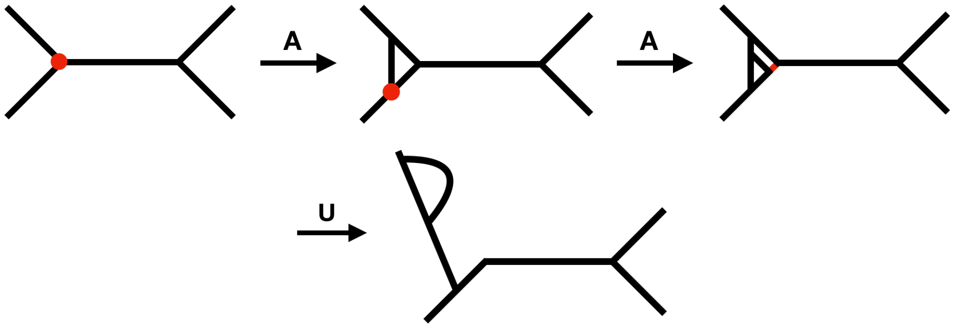

Figure 43: Construction of a bubble in a KTG.