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Searching for TeV-scale leptoquarks in Grand Unified Theories

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Abstract

The standard model of particle physics is expected to be an incomplete description. For instance, it provides no natural explanations for the observed baryon asymmetry and the hierarchy problem. Possible extensions of the standard model are grand unified theories (GUTs). Many GUTs predict the existence of leptoquarks: hypothetical particles that directly couple quarks to leptons. This thesis discusses the possible mass scales of leptoquarks in GUTs based on the groups $SU(5)$, $SO(10)$ and $SU(3)_C \times SU(3)_L \times SU(3)_R$ (trinification). Minimal $SU(5)$ contains twelve vector leptoquarks (X/Y bosons) and a single scalar leptoquark T^α, all of which mediate proton decay. The X/Y bosons lie at the GUT scale $M_U \approx 10^{15}$ GeV, while T^α must have a mass of at least 10^{11} GeV. However, minimal $SU(5)$ predicts a proton lifetime that is incompatible with experimental limits, so it is not considered a viable GUT. Minimal $SO(10)$ contains vector leptoquarks that mediate proton decay (A/Y bosons), as well as ones that do not (X bosons). If the $SO(10)$ symmetry is broken using a single intermediate symmetry scale, X bosons lie at $M_I \approx 10^{11}$ GeV, whereas A/Y bosons lie at $M_U \approx 10^{16}$ GeV. If three intermediate symmetry scales are included, the masses of X bosons can naturally be lowered to the TeV scale. Minimal $SO(10)$ also contains many scalar leptoquarks that are believed to lie at the GUT scale, but scalar-mediated proton decay is suppressed. Trinification forbids gauge-mediated proton decay by assigning quarks and leptons to separate irreducible representations. The scalar leptoquarks must have masses of at least 10^{11} GeV, since they mediate proton decay. Without intermediate symmetry scales, the unification scale is $M_U \approx 10^{14}$ GeV. If an intermediate $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ symmetry scale is used, one obtains $M_I \approx 10^{11}$ GeV and $M_U \approx 10^{16}$ GeV. Among the theories that we have considered in this thesis, the model with the lowest leptoquark masses is an $SO(10)$ GUT with three intermediate symmetry scales. In this scenario, TeV-scale vector leptoquarks are naturally possible. As far as we are aware, such a scenario has not been explored yet in any existing literature.

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Chapter 1

Introduction

The standard model is the current best description of all interactions between elementary particles. It is a gauge theory based on the group $SU(3)_C \times SU(2)_L \times U(1)_Y$. However, the theory has several shortcomings. For example, the standard model provides no explanation for the baryon asymmetry problem: the imbalance between matter and antimatter in the universe. It also does not naturally explain why the weak force is so much stronger than gravity, a problem known as the hierarchy problem. Apart from phenomenological issues, the standard model is unsatisfactory from a theoretical point of view. The theory has many free parameters, all of which have to be measured experimentally. Furthermore, the charges of particles appear to be assigned rather arbitrarily. For instance, the exact equality of the magnitudes of the electric charges of protons and electrons seems very coincidental. The standard model does not offer an explanation for such relations between charges.

Many of these issues are naturally solved in the context of grand unified theories (GUTs). The idea of grand unification is that beyond some high energy scale, all interactions can be described by a single coupling constant. Hence, all interactions would be unified. The relative strength of each interaction would be fixed solely by factors that the GUT group provides. Only at low energies the symmetry reduces to the standard model group, at which point the electroweak and strong interactions become separate again. Contrary to the standard model, GUTs do offer an explanation for relations between quantum numbers of fermions, because several different fermions can be combined in irreducible representations. These relations once again follow from the GUT group. Thus, GUTs allow us to reduce the amount of arbitrary and coincidental aspects that the theory contains.

Besides this, GUTs predict new phenomena. The standard model does not contain any bosons carrying both color and weak isospin/hypercharge. Consequently, there are no vertices involving both quarks and leptons. GUTs, on the other hand, are based on larger symmetry groups, which necessarily introduce new particles and hence new interactions. These interactions may lead to direct couplings between quarks and leptons. The particles that mediate them are referred to as leptoquarks. If their interactions violate baryon number, they may lead to proton decay (Figure 1.1). Many GUTs predict the existence of such particles, so an important test of GUTs is the observation of proton decay.

The main purpose of this thesis is to find out at which energy scales leptoquarks can be expected in various GUTs. The GUTs that will be considered are subgroups of the exceptional group E_6. Many of its subgroups (Figure 1.2) are candidates for grand unification. Strictly

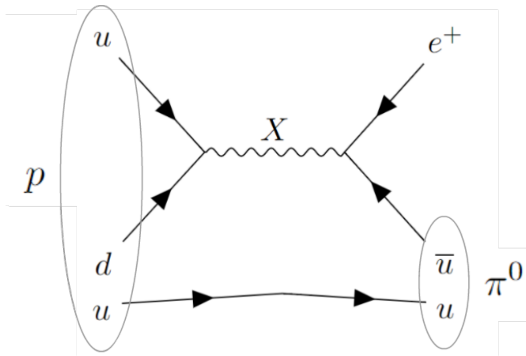

Figure 1.1: An example of a process where a proton decays into a neutral pion and a positron. The particle X here, is a leptoquark.

speaking, a symmetry group and the corresponding GUT based on that symmetry are not the same. Nevertheless, in literature these are used interchangeably and in this thesis we will also do this. Typically, though not always, we will be looking at minimal versions of GUTs. In minimal GUTs, particles are assigned to the smallest representations necessary to make the GUT phenomenologically acceptable.

E_6 contains $SU(5)$, one of the first and simplest GUTs proposed. However, $SU(5)$ in its minimal form turns out to be in disagreement with limits on proton decay. A different subgroup of E_6 that can potentially supersede $SU(5)$ is $SO(10)$, which contains $SU(5)$. E_6 can also be seen to contain the trinification group $SU(3)_C \times SU(3)_L \times SU(3)_R$. In this thesis, these three GUT groups will be considered.

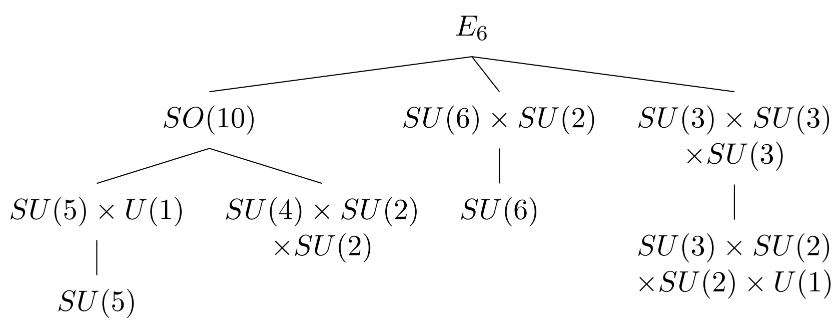

Figure 1.2: Various subgroups of E_6 [1].

Chapter 2 discusses several concepts that appear in the standard model, which are necessary to understand more complicated theories. In GUTs, the standard model is recovered by spontaneously breaking the GUT symmetry to the standard model group. Depending on the subgroups that a GUT contains, this may happen in one or more stages, which each have an associated energy scale. The masses of leptoquarks are intimately tied to these energy scales. For this reason, chapter 2 details the mechanism of spontaneous symmetry breaking. Chapter 3 covers $SU(5)$ grand unification. Even though minimal $SU(5)$ has been refuted, it allows us to see how leptoquarks and other features of GUTs appear in a tangible manner. Many of the same ideas and issues appear in other GUTs as well. Chapter 4 discusses $SO(10)$

grand unification. $SO(10)$ has the aesthetic benefit that all fermions can be placed into a single irreducible representation. Moreover, the mass scales of leptoquarks are high enough to sufficiently suppress proton decay. $SO(10)$ is large enough to allow a symmetry breaking pattern with multiple stages. As a result, the scale at which new physics occurs can, at least in theory, be lowered to the TeV scale. Lastly, chapter 5 covers trinification, which is based on an $SU(3)_C \times SU(3)_L \times SU(3)_R$ symmetry together with a $\mathbb{Z}_3$ symmetry to ensure gauge coupling unification. Trinification avoids the issue of gauge-mediated proton decay altogether by imposing baryon number conservation on its gauge sector. As we will see, this means that the constraints on the theory are much less stringent.

Chapter 2

Standard model preliminaries

Many of the ideas appearing in the standard model are present in GUTs as well. For instance, the types of interactions that particles can have, are still based on symmetry principles and fermions and gauge bosons acquire their masses through the Higgs mechanism. In this sense, the standard model serves as a basis for understanding more complicated theories. Moreover, any extension has to be compatible with the standard model. So at low enough energies, a GUT should reduce to the standard model. A good understanding of this theory is therefore essential. The first section discusses a very common group in particle physics: $SU(N)$. Next, we discuss the transformation properties of SM particles and how this determines the types of interactions they have. After that, we cover how particles can acquire mass through the Higgs mechanism. In the final section we turn to the renormalization group equations, which are used to impose gauge coupling unification in GUTs. The information presented here is largely based on refs. [2–5].

2.1 Representations of SU(N)

One of the most common groups in particle physics is $SU(N)$. In the standard model, for instance, color and weak isospin are desribed by an $SU(3)_C$ and a $SU(2)_L$ symmetry. Moreover, many GUTs are based on symmetries involving $SU(N)$ groups: $SU(5)$, $SU(3)_C \times SU(3)_L \times SU(3)_R$, the Pati-salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$ and so on. Understanding the structure of $SU(N)$ is therefore very useful. $SU(N)$ is defined as the group of unitary $N \times N$ matrices with unit determinant. The two conditions, unitarity and unimodularity, leave $N^2 - 1$ free parameters. So a general $SU(N)$ transformation U can be written in terms of $N^2 - 1$ generators T^a:

$$U = e^{-i\alpha^a T^a}, \tag{2.1}$$

where α^a are real parameters and all generators are hermitian and traceless. The N-dimensional fundamental representation of $SU(N)$ is denoted as N. The basis of this representation consists of N complex numbers ψ^i. In some cases there is also an antifundamental representation $\overline{N}$, obtained by taking the complex conjugate of the fundamental representation. A basis for this representation is formed by ψ^{i*}. For convenience, lower indices are used to denote complex conjugation: $\psi_i \equiv \psi^{i*}$. The transformation rules for ψ^i and ψ_i are then

$$\psi^i \to \psi^{i\prime} = U^i_j \psi^j, \tag{2.2}$$

$$\psi_i \to \psi_i' = \psi_j (U^\dagger)^j_i. \tag{2.3}$$

Note that we do not use a similar convention for the transformation matrices U; there is no difference between upper and lower indices for them. An important example where there is no antifundamental representation is $SU(2)$, whose representations are all real. So the 2 and the $\overline{2}$, in particular, are equivalent. To see this, consider the generators of the 2: $T^a = \sigma^a/2$, where σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

The 2 of $SU(2)$ is real if there exists a matrix S such that

$$S^{-1} T_a S = -T_a^*, \quad a = 1, 2, 3. \tag{2.5}$$

This is satisfied by $S = i\sigma_2$. So ψ^* and $i\sigma_2\psi$ transform in the same way under $SU(2)$. Hence, there is no antifundamental representation for $SU(2)$.

Higher dimensional representations of $SU(N)$ can be obtained by taking tensor products of (anti) fundamental representation. This is known as the tensor method. An $SU(N)$ tensor is defined as any object that transforms in the same way as a product of the objects in Eq. (2.2). Since we are only interested in the transformation properties, the only thing that matters is the number of upper and lower indices. A tensor with n lower indices and m upper indices is then written as $\psi^{i_1 \dots i_m}_{j_1 \cdots j_n}$, which transforms as

$$\psi^{i_1 \dots i_m}_{j_1 \cdots j_n} \to \left(\psi^{i_1 \dots i_m}_{j_1 \cdots j_n}\right)' = U^{i_1}_{k_1} \dots U^{i_m}_{k_m} \psi^{k_1 \dots k_m}_{l_1 \dots l_n} (U^\dagger)^{l_1}_{j_1} \dots (U^\dagger)^{l_n}_{j_n}. \tag{2.6}$$

This representation is generally reducible. To find out how $SU(N)$ tensors can be decomposed into irreps, a few observations have to be made. The first follows from the fact that all upper (lower) indices transform the same way. So if a tensor were to have a symmetry in its upper or lower indices, this symmetry will be preserved under transformations. For instance, consider a second rank tensor ψ^{ij}. From this we can create a symmetric tensor S^{ij} and an antisymmetric tensor A^{ij} as follows:

$$S^{ij} = \frac{1}{2}(\psi^{ij} + \psi^{ji}), \quad A^{ij} = \frac{1}{2}(\psi^{ij} - \psi^{ji}). \tag{2.7}$$

Under a transformation, S^{ij} would go to $U^i_k U^j_l S^{kl}$, which is still symmetric in i and j. In the same manner, the antisymmetry of A^{ij} is preserved. Thus, a general tensor can be reduced by forming linear combinations with certain permutation symmetries.

So far, no information about $SU(N)$ has been used, so this holds for tensor products of any group. The two properties $U^\dagger U = I$ and $\det U = 1$, can be used to further reduce a tensor. Because of unitary, contractions between upper and lower indices are left invariant. Another way of saying this is that the Kronecker delta δ^i_j is invariant:

$$\delta^i_j \to U^i_k \delta^k_l (U^\dagger)^l_j = U^i_k (U^\dagger)^k_j = \delta^i_j. \tag{2.8}$$

Hence, δ^i_j is an invariant tensor. Now suppose we use the Kronecker delta to contract an upper and a lower index of a rank $(n+m)$ tensor:

$$\psi^{i_1 i_2 \ldots i_m}_{i_1 j_2 \ldots j_n} = \delta^{j_1}_{i_1} \psi^{i_1 \ldots i_N}_{j_1 \ldots j_N}. \tag{2.9}$$

Then because of Eq. (2.8), this will transform like a tensor of rank $(n+m-2)$. If a tensor has just as many upper as lower indices, contracting all of them would give an invariant.

The determinant of $SU(N)$ transformations being equal unity, gives us two other invariant tensors: the totally antisymmetric symbols $\epsilon_{i_1 \ldots i_N}$ and $\epsilon^{i_1 \cdots i_N}$. These transform as

$$\begin{aligned}\epsilon_{i_1 \ldots i_N} \to U^{j_1}_{i_1} \ldots U^{j_N}_{i_N} \epsilon_{j_1 \ldots j_N} \\ = \epsilon_{i_1 \ldots i_N} U^{j_1}_{1} \ldots U^{j_N}_{N} \epsilon_{j_1 \ldots j_N} \\ = \epsilon_{i_1 \ldots i_N} \mathrm{det} U \\ = \epsilon_{i_1 \ldots i_N},\end{aligned} \tag{2.10}$$

and likewise for $\epsilon^{i_1 \cdots i_N}$. So to reduce $SU(N)$ tensors, $\epsilon_{i_1 \ldots i_N}$ can be contracted with upper indices and $\epsilon^{i_1 \cdots i_N}$ with lower indices.

Now we discuss a few examples that often show up. First consider the $N \times N$ tensor product, which is furnished by a tensor with two upper indices ψ^{ij}. We can reduce this representation by symmetrizing and antisymmetrizing the two indices. ψ^{ij} then decomposes into a symmetric and an antisymmetic part as

$$\psi^{ij} = \frac{1}{2}(\psi^{ij} + \psi^{ji}) + \frac{1}{2}(\psi^{ij} - \psi^{ji}). \tag{2.11}$$

The symmetric part has $N(N+1)/2$ independent components, while the antisymmetric part has $N(N-1)/2$ independent components. Thus, in terms of dimensions the $N \times N$ tensor product decomposes into irreps as

$$N \times N = \frac{1}{2}N(N+1) + \frac{1}{2}N(N-1). \tag{2.12}$$

Next, consider the $N \times \overline{N}$ tensor product, which is furnished by a tensor with one upper and one lower index: ψ^i_j. Since, there is only one upper and one lower index, we cannot reduce this representation by symmetrizing or antisymmetrizing indices. We can only contract the two indices, which gives an invariant tensor. ψ^i_j can therefore be decomposed into two irreps as follows:

$$\psi^i_j = (\psi^i_j - \frac{1}{N}\psi^k_k) + \frac{1}{N}\psi^k_k, \tag{2.13}$$

In terms of dimensions this means that

$$N \times \overline{N} = (N^2 - 1) + 1. \tag{2.14}$$

The $N^2 - 1$ dimensional representation can be recognized as the adjoint representation.

As a final example we consider the $N \times N \times \overline{N}$ tensor product. This is furnished by a tensor with two upper indices and one lower index: ψ^{ij}_k. First, to reduce the tensor we can symmetrize and antisymmetrize the two upper indices. this produces a tensor S_k^{ij}, which is symmetric in i in j, and a tensor A_k^{ij}, which is antisymmetric in i and j:

$$S_k^{ij} = \frac{1}{2}(\psi_k^{ij} + \psi_k^{ji}), \tag{2.15}$$

$$A_k^{ij} = \frac{1}{2}(\psi_k^{ij} - \psi_k^{ji}). \tag{2.16}$$

Then, for every k, S_k^{ij} contains $N(N+1)/2$ independent components and A_k^{ij} contains $N(N-1)/2$ independent components. Since k runs from 1 to N, this means that S_k^{ij} furnishes a representation of dimension $N^2(N+1)/2$, while A_k^{ij} furnishes a representation of dimension $N^2(N-1)/2$. Both tensors can be reduced further by contracting an upper and a lower index. Note that it does not matter which upper index we pick, because S_k^{ij} and A_k^{ij} are symmetric/antisymmetric. If we separate out the traceless parts in S_k^{ij} and A_k^{ij}, we obtain the following decompositions.

$$S_k^{ij} = (S_k^{ij} - \frac{1}{N}S_j^{ij}) + \frac{1}{N}S_j^{ij}, \tag{2.17}$$

$$A_k^{ij} = (A_k^{ij} - \frac{1}{N}A_j^{ij}) + \frac{1}{N}A_j^{ij}, \tag{2.18}$$

where the first term in each expression is traceless. The trace in each expression has just one free upper index, so it furnishes the N representation. The traceless part of S_k^{ij} therefore has dimension $N^2(N+1)/2 - N = N(N+2)(N-1)/2$ and the traceless part of A_k^{ij} has dimension $N^2(N-1)/2 - N = N(N-2)(N+1)/2$. Thus, the $N \times N \times \overline{N}$ tensor product can be decomposed into four irreps:

$$N \times N \times \overline{N} = \frac{1}{2}N(N+2)(N-1) + \frac{1}{2}N(N-2)(N+1) + N + N. \tag{2.19}$$

2.2 Particle representations

In the standard model, all particles transform according to representations of $SU(3)_C \times SU(2)_L \times U(1)_Y$ and the Lorentz group. The Lagrangian has to be invariant under these groups, so the transformation properties of a particle restrict the types of interactions it can have. In this section we review these transformation properties. First, the spin 1/2 fermions. Spin 1/2 fermions transform according to the four-dimensional spinor representation of the Lorentz group:

$$\psi \to \psi' = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi, \tag{2.20}$$

where $\omega_{\mu\nu}$ is antisymmetric and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. This representation is reducible into two parts. These are the chiral spinors

$$\psi_L = P_L\psi = \frac{1}{2}(1+\gamma_5)\psi, \quad \psi_R = P_R\psi = \frac{1}{2}(1-\gamma_5)\psi. \tag{2.21}$$

We can find another object which transforms in the same way as ψ. Consider the charge conjugate spinor $\psi^c = C\gamma^0\psi^*$, where $C = \gamma^2\gamma^0$ is the charge conjugation matrix. C satisfies

$$C = C^{-1} = -C^* = -C^T = C^\dagger, \tag{2.22}$$

$$C\gamma^0\gamma^{\mu *} = -\gamma^\mu C\gamma^0, \tag{2.23}$$

and hence,

$$C\gamma^0(\sigma^{\mu\nu})^* = -C\gamma^0\frac{i}{2}[\gamma^{\mu *},\gamma^{\nu *}] = -\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]C\gamma^0 = -\sigma^{\mu\nu}C\gamma^0. \tag{2.24}$$

Then, under a Lorentz transformation ψ^c transforms as

$$\psi^c \to C\gamma^0 e^{\frac{i}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})^*}\psi^* = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}C\gamma^0\psi^* = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi^c. \tag{2.25}$$

So ψ^c transforms in the same way as ψ. An additional property of the charge conjugate spinor is that it interchanges left and right handedness:

$$(\psi_L)^c = \frac{1}{2}C\gamma^0(1+\gamma_5^*)\psi^* = \frac{1}{2}(1-\gamma_5)C\gamma^0\psi^* = (\psi^c)_R, \tag{2.26}$$

and likewise $(\psi_R)^c = (\psi^c)_L$. So for example, if the subscript L would be assumed implicit, then we can denote left handed fields by ψ and right handed fields by ψ^c. A final common usage of ψ^c is to rewrite the bilinear $\overline{\psi}\phi$. ψ^c can be used in favor of $\overline{\psi}$ as follows:

$$\overline{\psi}\phi = -\psi^\dagger C\gamma^0 C\phi = -(C\gamma^0\psi^*)^T C\phi = -(\psi^c)^T C\phi. \tag{2.27}$$

Now, to write down a fermion mass term, we couple a spinor to its charge conjugate spinor.

So far we have covered the transformations of fermions under space-time symmetries. Now we discuss the transformation properties under the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The bases of representations of the gauge group are the particles themselves. So the irrep that a particle is in, determines how it couples to other particles. The gauge interactions, in particular, are completely fixed once we specify the transformation properties of the fermions. To see this, suppose we have n irreps R^i of dimension d^i, $i = 1\ldots n$. So in total we can fit $\sum_i d^i$ particles in them. We combine the d^i basis states of the irrep R^i in a column vector Ψ^i:

$$\Psi^i = \begin{pmatrix} \psi_1^{(i)} \\ \psi_2^{(i)} \\ \vdots \\ \psi_{d_i}^{(i)} \end{pmatrix}. \tag{2.28}$$

The interactions between fermions and gauge bosons are then given by a Lagrangian of the form

$$\mathcal{L}_{\text{gauge}} = \sum_i \overline{\Psi}^i i\not{D}\Psi^i, \tag{2.29}$$

where

$$D_\mu\psi^i = \left[\partial_\mu - ig_a T^a(R^i)A_\mu^a\right]\Psi^i. \tag{2.30}$$

The fields A_μ^a are the gauge fields and $T^a(R^i)$ are the generators of the irrep R^i. So we see that corresponding to every generator of a group, there is a gauge boson. It is the form of that generator that determines which fermions A_μ^a couples to. More precisely, the component

$\psi_j^{(i)}$ of Ψ^i will only couple to A_μ^a if it transforms under a transformation in the direction of $T^a(R^i)$. Only then will the product $T^a(R^i)\Psi^i$ (for fixed i) contain $\psi_j^{(i)}$. Thus, the irrep that a given particle is in, directly determines to which gauge bosons it couples.

Starting with the leptons e^- and ν_e, their left handed components form an $SU(2)_L$ doublet with hypercharge $-1/2$:

$$L_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L . \tag{2.31}$$

The right handed electron e_R^- is an isoscalar with hypercharge -1. A right handed neutrino is absent in the SM. The colour components of the u and d quarks each transform as triplets under $SU(3)_C$ and their left handed components form an $SU(2)_L$ doublet with hypercharge 1/6:

$$Q_L = \begin{pmatrix} u_1 & u_2 & u_3 \\ d_1 & d_2 & d_3 \end{pmatrix}_L . \tag{2.32}$$

The right handed components of the u and d quarks are isoscalars with hypercharges 2/3 and $-1/3$ respectively. For product groups, such as the SM group, it is customary to summarize the transformation properties by giving the dimensions of the irreps of each of the factor groups. The irreps of abelian factors are necessarily one-dimensional, so only the eigenvalue of the generator is given. The transformation properties of fermions under $SU(3)_C \times SU(2)_L \times U(1)_Y$ are then given by

$$\begin{aligned} L_L &\sim (1, 2, -1/2), \\ e_R &\sim (1, 1, -1), \\ Q_L &\sim (3, 2, 1/6), \\ u_R &\sim (3, 1, 2/3), \\ d_R &\sim (3, 1, -1/3). \end{aligned} \tag{2.33}$$

Note that this only includes the fermions of the first generation. The other two generations have the same transformation properties. Then in total the standard model contains fifteen chiral fermions per generation.

As shown before, every generator of the gauge group has an associated gauge boson. The standard model group has twelve generators and hence twelve gauge bosons associated with them. These are the eight gluons $G_\mu^1 \dots G_\mu^8$, the $W_\mu^\pm$ and Z_μ^0 bosons and the photon A_μ. Gauge bosons transform according to the adjoint representation of the group, so their transformation properties are given by

$$\begin{aligned} G_\mu^1 \dots G_\mu^8 &\sim (8, 1, 0), \\ W_\mu^+, Z_\mu^0, W_\mu^- &\sim (1, 3, 0), \\ A_\mu &\sim (1, 1, 0). \end{aligned} \tag{2.34}$$

Aside from fermions and gauge bosons, there are also scalar particles that have their own transformation properties. These are discussed in section 2.3.

2.3 The Higgs mechanism

Fermions and gauge bosons acquire their masses through the Higgs mechanism. The main ingredient of the Higgs mechanism is spontaneous symmetry breaking. In many cases, if the Lagrangian is invariant under a transformation U, the ground state is also invariant under U, which is to say that $U|0\rangle = |0\rangle$. If, on the other hand, the ground state is not invariant, then $U|0\rangle$ will be a different state with the same energy. We could therefore pick this state to be the ground state as well. However, if we were to choose a preferred ground state and describe the system in terms of it, the symmetries of the Lagrangian would no longer be manifest. In that case, the symmetry is broken spontaneously. The consequence of this is that a scalar field ϕ can obtain a non-vanishing vacuum expectation value (vev) $\langle\phi\rangle$, which can be used to label the different ground states. For continuous symmetries this non-vanishing vev leads to the appearance of massless particles. When combined with gauge invariance, these particles are what gives mass to the various gauge bosons.

2.3.1 Spontaneous symmetry breaking

Consider a set of N real scalar fields ϕ_i whose Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\vec{\phi})\cdot(\partial^\mu\vec{\phi}) - V(\vec{\phi}) \tag{2.35}$$

To illustrate the concept of spontaneous symmetry breaking we assume that the fields transform according to the fundamental representation of $O(N)$. A potential invariant under $O(N)$ is

$$V(\vec{\phi}) = -\frac{\mu^2}{2}\vec{\phi}^2 + \frac{\lambda}{4}(\vec{\phi}^2)^2, \quad \lambda > 0. \tag{2.36}$$

For $\mu^2 < 0$, $\vec{\phi}$ can be interpreted to give rise to spin 0 bosons with mass $\sqrt{-\mu^2}$ and the quartic term in the potential leads to interactions. Moreover, the ground state is just $\vec{\phi} = 0$. For $\mu^2 > 0$, the same interpretation would not make sense anymore and the ground state is not given by $\vec{\phi} = 0$. Minimizing $V(\vec{\phi})$ gives the condition

$$\vec{\phi}^2 = \frac{\mu^2}{\lambda} \equiv v^2. \tag{2.37}$$

There are many values of $\vec{\phi}$ that satisfy this, all of which are connected by an $O(N)$ transformation. Suppose we pick a ground state that points along the $i = 1$ direction:

$$\langle\vec{\phi}\rangle = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.38}$$

With this choice, only rotations and reflections in planes orthogonal to the $i = 1$ axis will leave the groundstate invariant. The $O(N)$ symmetry has therefore been broken to an $O(N-1)$ symmetry. Infinitesimally this means that some generators do not annihilate the vev. This can be seen as follows. Under an infinitesimal symmetry transformation, the fields transform as

$$\phi_i \to \phi'_i = \phi_i + \Delta(\phi_i) = \phi_i - i\alpha^a (T^a)_{ij}\phi_j, \tag{2.39}$$

where α^a are real infinitesimal parameters and T^a are the generators of the fundamental representation of $O(N)$. If the groundstate does not have the same symmetries as the Lagrangian, there are values of α^a such that $\langle\vec{\phi}\rangle \neq \langle\vec{\phi}'\rangle$. In other words, there exists at least some linear combination S of the generators that does not annihilate the vev:

$$S\langle\vec{\phi}\rangle \neq 0. \tag{2.40}$$

The linear combinations for which this is the case, are referred to as broken generators.

2.3.2 Massless particles

The interpretation of the Lagrangian in Eq. (2.35) becomes clear if we describe all fields relative to the groundstate $\vec{\phi} = \langle\vec{\phi}\rangle$. To do this we introduce shifted fields $\vec{\phi}'$, defined through $\vec{\phi} = \langle\vec{\phi}\rangle + \vec{\phi}'$. In terms of ϕ' fields the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\vec{\phi}')\cdot(\partial^\mu\vec{\phi}') - \mu^2\phi_1'^2 + \ldots \tag{2.41}$$

The field ϕ'_1 now has a mass $\sqrt{2}\mu$, whereas all other particles are massless. So because the $O(N)$ symmetry is spontaneously broken, one massive and $N-1$ massless bosons have appeared. The masses of the particles may also be calculated directly from the potential as follows. A general power series expansion of the potential $V(\vec{\phi})$ around $\vec{\phi} = \langle\vec{\phi}\rangle$ is

$$V(\vec{\phi}) = V(\langle\vec{\phi}\rangle) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\langle\vec{\phi}\rangle}(\phi_i - \langle\phi_i\rangle)(\phi_j - \langle\phi_j\rangle) + \ldots \tag{2.42}$$

There are no first order terms because $\langle\vec{\phi}\rangle$ is assumed to minimize $V(\vec{\phi})$. The fields $\phi_i - \langle\phi_i\rangle$ are the shifted fields ϕ'_i, so their masses can be obtained by diagonalizing the matrix

$$\left(M_H^2\right)^{ij} \equiv \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\langle\vec{\phi}\rangle}, \tag{2.43}$$

In our case, M_H^2 has a 0 eigenvalue with a degeneracy of $N-1$. More generally, the Goldstone theorem states that for every spontaneously broken generator of a continuous symmetry in a relativistic theory, there will be a massless particle (or Goldstone boson). So if a group G is spontaneously broken to a subgroup H, there will be $\dim(G) - \dim(H)$ Goldstone bosons. Since the $O(N)$ symmetry was broken to $O(N-1)$, there will be $N(N-1)/2 - (N-1)(N-2)/2 = N-1$ Goldstone bosons, precisely the amount that we found.

We can find out which states correspond to Goldstone bosons by considering an infinitesimal transformation of the potential:

$$V(\phi_i) \to V(\phi_i + \Delta(\phi_i)) = V(\phi_i) - \frac{\partial V}{\partial\phi_j}\Delta(\phi_j). \tag{2.44}$$

Since $V(\phi)$ is invariant, this means that

$$\frac{\partial V}{\partial \phi_j}\Delta(\phi_j) = 0. \tag{2.45}$$

If we take the derivative w.r.t. ϕ_i and evaluate the resulting expression at $\vec{\phi} = \langle\vec{\phi}\rangle$, we obtain

$$\left.\frac{\partial V}{\partial \phi_j}\right|_{\langle\vec{\phi}\rangle} \left.\frac{\partial \Delta(\phi_j)}{\partial \phi_i}\right|_{\langle\vec{\phi}\rangle} + \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\langle\vec{\phi}\rangle} \Delta(\langle\phi_j\rangle) = 0. \tag{2.46}$$

Since $\langle\vec{\phi}\rangle$ minimizes $V(\phi)$, the first term is zero. Hence,

$$M_H^2 \Delta(\langle\vec{\phi}\rangle) = 0. \tag{2.47}$$

Now suppose some linear combination S of the generators is broken. That is, $S\langle\vec{\phi}\rangle \neq 0$. Eq. (2.47) then says that the vector $S\langle\vec{\phi}\rangle$ is an eigenvector of M_H^2 with eigenvalue 0. $S\langle\vec{\phi}\rangle$ would therefore be a massless state. Thus, the Goldstone bosons lie in the directions of the vectors $\tilde{S}\langle\vec{\phi}\rangle$, where $\tilde{S}$ is any broken generator.

2.3.3 Spontaneous symmetry breaking in gauge theories

In the previous sections we imposed a global symmetry on the Lagrangian and this led to the emergence of massless spin 0 bosons. In gauge theories, where local symmetries are imposed, the interpretation is different, as we will see. To obtain a gauge invariant Lagrangian, we replace the partial derivative with the covariant derivative:

$$\mathcal{L} = \frac{1}{2}(D_\mu\vec{\phi}) \cdot (D^\mu\vec{\phi}) - V(\phi), \tag{2.48}$$

$$\text{with } D_\mu = \partial_\mu - igT^a A_\mu^a. \tag{2.49}$$

If we insert $\vec{\phi} = \langle\vec{\phi}\rangle + \vec{\phi}'$, the part containing the covariant derivative up to terms second order in the fields is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\vec{\phi}') \cdot (\partial^\mu\vec{\phi}') - ig(T^a A_\mu^a\langle\vec{\phi}\rangle) \cdot (\partial^\mu\vec{\phi}') - \frac{1}{2}g^2(T^a A_\mu^a\langle\vec{\phi}\rangle) \cdot (T^b A_\mu^b\langle\vec{\phi}\rangle) + \cdots . \tag{2.50}$$

The third term is second order in the gauge fields and by itself, this would give mass to the gauge bosons. However, the second term shows that the gauge fields are not independent modes: they also mix with ϕ'. This term consists of a dot product between $T^a A_\mu^a\langle\vec{\phi}\rangle$ and $\partial^\mu\vec{\phi}'$. So A_μ^a only couples to those fields that are in the direction of $T^a\langle\vec{\phi}\rangle$. But as we saw before, if $T^a\langle\vec{\phi}\rangle \neq 0$, this vector lies in the direction of the Goldstone bosons. Thus, after spontaneous symmetry breaking, the gauge fields only mix with these particles. However, in gauge theories, the fields of the Goldstone bosons are unphysical. It can be proved [6] that there always exists a gauge in which $\vec{\phi}$ does not contain any Goldstone bosons. More precisely, this means that for every broken generator S we would have

$$(S\langle\vec{\phi}\rangle) \cdot \vec{\phi}' = 0. \tag{2.51}$$

The gauge in which this holds is known as the unitary gauge. In this gauge, the second term in Eq. (2.50) would vanish. The Lagrangian then takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi}) \cdot (\partial^\mu \vec{\phi}) - \frac{1}{2}(M_G^2)^{ab} A_\mu^a A^{b\mu} + \cdots , \tag{2.52}$$

where M_G^2 is the mass matrix for gauge bosons:

$$\left(M_G^2\right)^{ab} = g^2 (T^a \langle\vec{\phi}\rangle) \cdot (T^b \langle\vec{\phi}\rangle). \tag{2.53}$$

This is a symmetric matrix, so it can always be diagonalized by an orthogonal transformation O, under which $M_G^2 \to M_G^{2\prime} = O M_G^2 O^T$. If we include indices, the diagonal components of $M_G^{2\prime}$ are

$$\left(M_G^{2\prime}\right)^{aa} = g^2 (O^{ab} T^b \langle\vec{\phi}\rangle) \cdot (O^{ac} T^c \langle\vec{\phi}\rangle) = g^2 (T^{a\prime} \langle\vec{\phi}\rangle)^2, \tag{2.54}$$

where $T^{a\prime} = O^{ab} T^b$. The gauge bosons A_μ^a transform as

$$A_\mu^a \to A_\mu^{a\prime} = O^{ab} A_\mu^b. \tag{2.55}$$

The mass eigenstates are therefore the states $A_\mu^{a\prime}$ whose masses are given by

$$M_{A_\mu^{a\prime}} = \sqrt{\left(M_G^{2\prime}\right)^{aa}} = g \left| T^{a\prime} \langle\vec{\phi}\rangle \right| . \tag{2.56}$$

So the field $A_\mu^{a\prime}$ only acquires mass if $T^{a\prime}\langle\vec{\phi}\rangle \neq 0$. In other words, the generator $T^{a\prime}$ must be broken. We can interpret this generator as follows. Each gauge boson A_μ^a couples to particles via the product $T^a A_\mu^a$ in the covariant derivative. For this reason, the generator T^a is said to be associated with A_μ^a. But we may equally well use a different basis of gauge bosons, such as the mass eigenstates $A_\mu^{a\prime}$. From the fact that $O^{ab} O^{ac} = \delta^{bc}$ it follows that

$$T^a A_\mu^a = T^b O^{ab} O^{ac} A_\mu^c = T^{a\prime} A_\mu^{a\prime}. \tag{2.57}$$

Hence, $T^{a\prime}$ has the interpretation that it is the generator associated with the mass eigenstate $A_\mu^{a\prime}$. Only if $T^{a\prime}$ is broken, $A_\mu^{a\prime}$ becomes massive. So associated with every broken generator, there is a massive gauge boson.

2.3.4 The electroweak theory

To illustrate the aforementioned ideas, we show how they are used in the standard model to give mass to the gauge bosons. The Lagrangian of the standard model is invariant under $SU(3)_C \times SU(2)_L \times U(1)_Y$. The gauge bosons associated with it are the eight gluons, the $W^\pm$ and Z^0 bosons and the photon. Of the twelve gauge bosons, only the $W^\pm$ and Z^0 bosons acquire mass, while the gluons and photon remain massless. Hence, after spontaneous symmetry breaking, color and electric charge must still be explicitly conserved. We are therefore looking for a Higgs mechanism that yields the following symmetry breaking pattern:

$$SU(3)_C \times SU(2)_L \times U(1)_Y \to SU(3)_C \times U(1)_Q, \tag{2.58}$$

where Q is the electric charge operator. In the standard electroweak theory, an isospin doublet of complex scalar fields with hypercharge 1/2 is used to perform the symmetry breaking:

$$\Phi = \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}. \tag{2.59}$$

So the standard model Higgs particles transform according to the $(1, 2, 1/2)$ representation of the SM group. The scalar sector of the SM Lagrangian is given by

$$\mathcal{L} = (D_\mu \Phi)^\dagger D^\mu \Phi - V(\Phi), \tag{2.60}$$

where

$$D_\mu = \partial_\mu - igT^a W_\mu^a - ig' Y B_\mu, \tag{2.61}$$

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2, \quad \mu^2, \lambda > 0. \tag{2.62}$$

Here, $T^a = \sigma^a/2$ are the generators of the fundamental representation of $SU(2)_L$ and $Y = 1/2$ is the hypercharge generator. The stationary points of $V(\Phi)$ are determined by the condition

$$\Phi^\dagger \Phi = \frac{\mu^2}{2\lambda}. \tag{2.63}$$

All solutions are related by an $SU(2)_L \times U(1)_Y$ transformation, so any choice will have the same physical consequences. We can for instance pick the vev

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \text{ with } v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{2.64}$$

A general field Φ is obtained by expanding around the ground state:

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_1(x) + i\eta_2(x) \\ v + \sigma(x) + i\eta_3(x) \end{pmatrix}, \tag{2.65}$$

where the four real fields σ and η_i have been introduced. Consider the vector of these real fields $\vec{\phi} = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \sigma \end{pmatrix}^T$. From Eq. (2.43), the mass matrix for Higgs particles in this basis can be calculated to be

$$M_H^2 = 2\mu^2 \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}. \tag{2.66}$$

So the fields η_i are the anticipated massless Goldstone bosons, while σ has a mass $\sqrt{2}\mu$. After switching to the unitary gauge, the η_i will be eliminated. The remaining field σ is the Higgs field, which can be interpreted to be a real spin 0 boson. At this point the gauge bosons have acquired mass. The mass matrix for gauge bosons can be worked out to be

$$M_G^2 = \frac{v^2}{4} \begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix}. \tag{2.67}$$

The (normalized) eigenvectors give the mass eigenstates and the eigenvalues their masses. From W^1_μ and W^2_μ we can form a particle-antiparticle pair:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right), \ m_W = \frac{vg}{2}. \tag{2.68}$$

The neutral states contain the Z^0 and the massless photon:

$$Z^0_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gW^3_\mu - g'B_\mu\right), \ m_Z = \frac{v}{2}\sqrt{g^2+g'^2}, \tag{2.69}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'W^3_\mu + gB_\mu\right), \ m_\gamma = 0. \tag{2.70}$$

Since there is one massless state, there must also be an associated unbroken generator. This generator can be found by writing the covariant derivative in terms of mass eigenstates:

$$D_\mu = \partial_\mu - \frac{igg'}{\sqrt{g^2+g'^2}}(T_3+Y)A_\mu + \dots \tag{2.71}$$

So the photon couples to particles via the operator

$$Q = T_3 + Y. \tag{2.72}$$

The coupling constant can be identified with the elementary charge e:

$$e = \frac{gg'}{\sqrt{g^2+g'^2}}. \tag{2.73}$$

We can also check explicitly that Q is unbroken by acting on $\langle\Phi\rangle$ with an arbitrary linear combination of $SU(2)_L \times U(1)_Y$ generators:

$$(\alpha^a T^a + \beta Y)\langle\Phi\rangle = \frac{1}{2\sqrt{2}}\begin{pmatrix}\alpha^3+\beta & \alpha^2 - i\alpha^3 \\ \alpha^2 + i\alpha^3 & \beta - \alpha^3\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} = \frac{v}{2\sqrt{2}}\begin{pmatrix}\alpha^2 - i\alpha^3 \\ \beta - \alpha^3\end{pmatrix} \tag{2.74}$$

The unbroken generators are those linear combinations that annihilate the vev. This happens only if

$$\alpha^2 = \alpha^3 = 0, \quad \alpha^3 = \beta. \tag{2.75}$$

But this gives precisely the linear combinations that are proportional to Q. Thus, by employing a single Higgs representation, the $SU(2)_L \times U(1)_Y$ symmetry has been broken to the electromagnetic group $U(1)_Q$. The standard model is special in this regard, since only one vev parameter was necessary to break the symmetry. The masses of the gauge bosons that acquired mass, are all related to this single parameter, so the vev v sets the energy scale of these particles: the electroweak scale $M_{\text{EW}} \approx 100$ GeV. We will see that for different models, such as grand unified theories, multiple stages may be necessary to break the symmetry, resulting in a hierarchy of mass scales.

2.4 Renormalization group equations

GUTs are based on the assumption that all coupling constants unify at some scale. To be able to calculate the value of this scale, as well as any intermediate scales, we first have to understand how the coupling constants evolve. Many calculations in quantum field theories give inifinity as a result. The procedure to get rid of the infinities is known as renormalization. The standard method of renormalization involves redefining the parameters of the theory in such way that the infinities are absorbed. If this procedure is carried out, the parameters become functions of the energy scale. The precise manner in which the parameters evolve from one energy scale to another, is governed by the renormalization group equations (RGEs). Suppose the value of a parameter is given at some energy scale μ_1 and we want to know its value at a different scale μ_2. The RGEs can then be used to 'run' the parameter from the scale μ_1 to μ_2. Most of this section is based on [7].

2.4.1 The running of the couplings

In general, the RGEs depend on the entire particle content of the theory. However, it often happens that some particles have a mass much larger than the scale we are interested in. In that case, the theory can be described completely by a Lagrangian containing only the light particles. The resulting theory is referred to as an effective field theory. Consequently, the RGEs at a scale μ depend, to a good approximation, only on particles with a mass $m < \mu$. If the relevant coupling constants at some scale μ are g_i, the RGE for g_i up to order $O(g_i^3)$ is given by

$$\frac{dg_i}{d\ln\mu} = \beta_i(g_i) = b_i\frac{g_i^3}{(4\pi^2)}. \tag{2.76}$$

The function $\beta_i(g_i)$ is known as the β-function and the corresponding coefficient b_i is the β-coefficient. Conventionally, these equations are expressed in terms of the fine structure constants $\alpha_i = g_i^2/(4\pi)$. The RGE then becomes

$$\frac{d\alpha_i}{d\ln\mu} = b_i\frac{\alpha_i^2}{2\pi}, \tag{2.77}$$

which can be solved to give

$$\alpha_i^{-1}(\mu_2) = \alpha_i^{-1}(\mu_1) - \frac{b_i}{2\pi}\ln\frac{\mu_2}{\mu_1}. \tag{2.78}$$

Determining the evolution of the fine structure constants comes down to calculating the β-coefficients. Their values depend on certain properties of the representations according to which the relevant particles transform:

$$b_i = -\frac{11}{3}C_2(G_i) + \frac{4}{3}\kappa S_2(F_i) + \frac{1}{6}\eta S_2(S_i). \tag{2.79}$$

$C_2(G_i)$ is the quadratic Casimir of the gauge group associated with g_i. This is given by

$$C_2(G) = \begin{cases} 0 & \text{for } U(1), \\ N & \text{for } SU(N). \end{cases} \tag{2.80}$$

$S_2(R)$ is the Dynkin index of an irreducible representation R, defined through

$$\mathrm{Tr}[T^a T^b] = S_2(R)\delta^{ab}, \tag{2.81}$$

where T^a are the generators of R. Then, $S_2(F_i)$ is short for the sum of the Dynkin indices of all fermion irreps. Likewise, $S_2(S_i)$ means the sum of the Dynkin indices of all scalar irreps. S_2 is also sometimes referred to as the normalization of an irrep.

There is some ambiguity in the definitions for $C_2(G_i)$ and $S_2(F_i)$, because we can always rescale $T^a \to \alpha T^a$ and $g_i \to g_i/\alpha$. The end result would be the same. Usually for $SU(N)$ the generators are normalised such that

$$S_2(R) = \begin{cases} 1/2 & \text{for fundamental representations,} \\ N & \text{for adjoint representations.} \end{cases} \tag{2.82}$$

There is no convention for $U(1)$ groups. In general, if the generator of an irrep of $U(1)_\lambda$ is λ, the Dynkin index is

$$S_2(R) = \lambda^2 \text{ for } U(1)_\lambda. \tag{2.83}$$

Eq. (2.79) also contains the factors κ and η, which specify the kinds of spinor or scalar fields we are dealing with. Their values are given by

$$\kappa = \begin{cases} 1/2 & \text{for chiral spinors,} \\ 1 & \text{for Dirac spinors,} \end{cases} \tag{2.84}$$

$$\eta = \begin{cases} 1 & \text{for real scalars,} \\ 2 & \text{for complex scalars.} \end{cases} \tag{2.85}$$

2.4.2 The running above and below electroweak symmetry breaking

To illustrate the use of Eq. (2.79), we calculate the β-coefficients above and below electroweak symmetry breaking (EWSB), which occurs at the energy scale M_{EW}. Below M_{EW} there is a $SU(3)_C \times U(1)_Q$ symmetry. We denote the fine structure constants associated with them as α_s and α_Q. At low enough energies ($\mu \ll m_e$) the only relevant particles would be photons, gluons and neutrinos, since these are assumed to be massless in the SM. Once the energy scale is increased, we would start noticing the effects of quarks and leptons. Each time another particle becomes relevant, this affects the running of the couplings. Suppose we are at a scale where, apart from the top quark, all quarks and leptons are relevant. Say, at $\mu \approx 10$ GeV. Then up to M_{EW}, the running of the couplings is essentially fixed. First we show how to calculate the β-coefficient for $SU(3)_C$. There are a total of five quarks present, which each transform according to the fundamental representation of $SU(3)_C$. These are all 4-component spinors, so $\kappa = 1$. The β-coefficient for $SU(3)_C$ below M_{EW} is then given by

$$b_s = -\frac{11}{3} \cdot 3 + \frac{4}{3} \cdot \frac{1}{2} \cdot 5 = -\frac{22}{3}. \tag{2.86}$$

For $U(1)_Q$ we need to include all electrically charged particles. This includes two up-type quarks with charge $+2/3$, three down-type quarks with charge $-1/3$ and the three charged leptons with charge -1. Moreover, the contribution from the quarks needs to be multiplied by three, because each quark has three color components.

Next we calculate β-coefficient for $U(1)_Q$. As mentioned before, there is no convention for the normalisation of $U(1)$ groups. In GUTs, on the other hand, all generators that appear in the Lagrangian have the same normalisation. It is the couplings associated with them that are assumed to unify at the GUT scale. In $SU(5)$, $SO(10)$ and trinification GUTs (see the corresponding chapters) it turns out that the properly normalised electric charge and hypercharge generators are always

$$Q' = \sqrt{\frac{3}{8}}Q, \quad Y' = \sqrt{\frac{3}{5}}Y. \tag{2.87}$$

This is the convention that we use. The appropriate Dynkin indices are therefore $3Q^2/8$ and $3Y^2/5$. The β-coefficient for $U(1)_Q$ below M_{EW} is then given by

$$b_Q = \frac{4}{3}\cdot\frac{3}{8}\left[\left(\frac{2}{3}\right)^2\cdot 2\cdot 3 + \left(-\frac{1}{3}\right)^2\cdot 3\cdot 3 + (-1)^2\cdot 3\right] = \frac{10}{3}. \tag{2.88}$$

Above M_{EW} the symmetry is $SU(3)_C \times SU(2)_L \times U(1)_Y$ and all SM particles become relevant. This means that we also get contributions from the Higgs field Φ. This is a complex field, so $\eta = 2$. Moreover, the chiral components of fermions transform differently under $SU(2)_L \times U(1)_Y$, so we should also treat them separately. Consequently, $\kappa = 1/2$ for the couplings associated with $SU(2)_L$ and $U(1)_Y$. We begin by calculating b_Y, the β-coefficient corresponding to $U(1)_Y$. The hypercharges for fermions were given in Eq. (2.33). Note that Q_L, for instance, contains six states with $Y = 1/6$, so its contribution should be multiplied by a factor six. The total contribution from the fermions in one generation should be multiplied by three, since we have three generations. b_Y is then given by

$$\begin{aligned} b_Y = \frac{4}{3}\cdot\frac{1}{2}\cdot 3\cdot\frac{3}{5}&\left[6\cdot\left(\frac{1}{6}\right)^2 + 3\cdot\left(-\frac{2}{3}\right)^2 + 3\cdot\left(\frac{1}{3}\right)^2 + 2\cdot\left(-\frac{1}{2}\right)^2 + 1^2\right] \\ &+ \frac{1}{6}\cdot 2\cdot 2\cdot\frac{3}{5}\cdot\left(\frac{1}{2}\right)^2 = \frac{41}{10}. \end{aligned} \tag{2.89}$$

The β-coefficients for $SU(3)_C$ and $SU(2)_L$ are calculated similarly:

$$b_{2L} = -\frac{11}{3}\cdot 2 + \frac{4}{3}\cdot\frac{1}{2}\cdot 3\cdot\frac{1}{2}\cdot 4 + \frac{1}{6}\cdot 2\cdot\frac{1}{2} = -\frac{19}{6}, \tag{2.90}$$

$$b_{3C} = -\frac{11}{3}\cdot 3 + \frac{4}{3}\cdot 6\cdot\frac{1}{2} = -7. \tag{2.91}$$

The running of the couplings is constrained by the fact that their energy-dependence is continuous. So even though the β-coefficients are different above and below M_{EW}, there cannot be a sudden jump at M_{EW} in the value of the fine structure constant associated with a single group. Then, at M_{EW} we necessarily have

$$\alpha_s(M_{\text{EW}}) = \alpha_{3C}(M_{\text{EW}}) \tag{2.92}$$

This is an example of a matching condition: a condition that needs to be satisfied by the fine structure constants when we move to an energy scale with a different symmetry. For α_Q,

α_{2L} and α_Y the situation is different, because Q is a linear combination of T_3 and Y (Eq. 2.72). There will then also be a relation between α_Q, α_{2L} and α_Y. To find this we need to relate the coupling constants associated with $U(1)_Q$, $SU(2)_L$ and $U(1)_Y$ to each other. This was already done in Eq. (2.73). However, the coupling constants e and g' correspond to the unnormalized generators Q and Y, respectively. The coupling constants corresponding to the normalized generators (Eq. 2.87) are $g_Q = \sqrt{3/8}e$ and $g_Y = \sqrt{3/5}g'$. Substituting this into Eq. (2.73) gives

$$\frac{1}{g_Q^2} = \frac{3}{8}\frac{1}{g^2} + \frac{5}{8}\frac{1}{g_Y^2}, \tag{2.93}$$

or

$$\alpha_Q^{-1} = \frac{3}{8}\alpha_{2L}^{-1} + \frac{5}{8}\alpha_Y^{-1}. \tag{2.94}$$

Notice the difference between Eqs. (2.92) and (2.94). The reason for this is that $U(1)_Q$ is embedded into $SU(2)_L \times U(1)_Y$ through Eq. (2.72). So α_Q 'splits' into two fine structure constants: α_{2L} and α_Y. This does not mean that the energy-dependence of the couplings is discontinuous, because unlike α_s and α_{3C}, the fine structure constants α_Q, α_{2L} and α_Y each correspond to a different group.

Now we will make a graph of the running of the couplings. We have already calculated the β-coefficients, which determine how the couplings evolve from one energy scale to another. The matching conditions then tell us what happens to the couplings ones we reach a scale at which a different symmetry holds. The only piece of information we are missing is the initial conditions. For this we use the values of the fine structure constants at the electroweak scale [8]:

$$\alpha_{3C}^{-1}(M_{\text{EW}}) \approx 8.45, \tag{2.95}$$

$$\alpha_{2L}^{-1}(M_{\text{EW}}) \approx 29.61, \tag{2.96}$$

$$\alpha_Y^{-1}(M_{\text{EW}}) \approx 58.97. \tag{2.97}$$

That this completely fixes the running of the couplings can be seen as follows. We have a total of five fine structure constants: α_Q, α_s, α_{3C}, α_{2L} and α_Y. Each of these contains one parameters, which is its value at some reference scale. The values of the fine structure constants at M_{EW} provide three conditions and Eqs. (2.92) and (2.94) provide two more. So in total there are five unknowns and five conditions, which fixes the running. Figure 2.1 shows a plot of the energy-dependence of the fine structure constants. Important to note here is that at each scale we show the values of the fine structure constants corresponding to the symmetry that holds at that scale. So below M_{EW} α_Q and α_s are shown and above M_{EW} α_{3C}, α_{2L} and α_Y are shown.

Figure 2.1 also clearly shows the effect of the matching conditions. At M_{EW} we see that the line corresponding to $SU(3)_C$ remains continuous. However, the line corresponding to $U(1)_Q$ splits into two lines corresponding to $SU(2)_L$ and $U(1)_Y$. Both of these features are consequences of the matching conditions Eqs. (2.92) and (2.94).

The running up to $\mu \approx M_{\text{EW}}$ has been measured experimentally. The part of the graph beyond that is merely a prediction of the standard model. Judging from the slopes of the

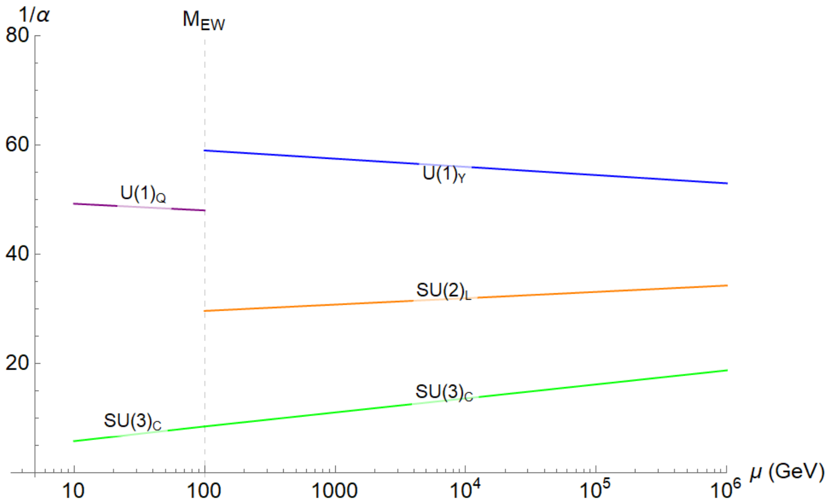

Figure 2.1: Running of the fine structure constants above and below EWSB.

three lines corresponding to $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$, it appears as though their couplings will unify at some scale. This is a point to which we will return when discussing $SU(5)$ grand unification.

2.4.3 Matching conditions in general

Beyond the electroweak scale a different symmetry holds and different coupling constants are relevant. However, the requirement of continuity relates the coupling constants above and below the electroweak scale to each other. This resulted in the matching conditions Eqs. (2.92) and (2.94). We can generalize this. Suppose that at some energy scale M_I a group H gets embedded into a larger group G. We assume that the group H is simple, so only a single coupling constant h is associated with it. G, on the other hand, can be a product of different groups, so several coupling constants might be associated with it. Since all generators T^a_H of H are associated with the same coupling constant, we can focus on just one of them. For instance T^1_H. This generator can be written as a linear combination of the generators T^b_G of G:

$$T^1_H = \sum_b c_b T^b_G, \tag{2.98}$$

where c_b are real constants. It can be proved [8] that at M_I the relation between the coupling constant h and the coupling constants g_b associated with T^b_G, is given by

$$\frac{1}{h^2(M_I)} = \sum_b c_b^2 \frac{1}{g_b^2(M_I)}. \tag{2.99}$$

So Eq. (2.92) is just a special case where $c_b = 1$ for only one value of b and all others are zero. The matching condition in Eq. (2.94) also follows from Eq. (2.99), because the expression for the electric charge operator in terms of normalized generators is

$$Q' = \sqrt{\frac{3}{8}} T_3 + \sqrt{\frac{5}{8}} Y'. \tag{2.100}$$

If we substitute the coefficients in front of T_3 and Y' into Eq. (2.99), we obtain Eq. (2.94).

If all the generators in Eq. (2.98) have the same normalisation we obtain a useful relation. To derive it, suppose that the generators of G and H are normalized such that

$$\mathrm{Tr}\left[T_G^a T_G^b\right] = \mathrm{Tr}\left[T_H^a T_H^b\right] = \Delta\delta^{ab}, \tag{2.101}$$

for some Δ. Then from Eq. (2.98) it follows that

$$\Delta = \mathrm{Tr}\left[(T_H^1)^2\right] = \sum_{a,b} c_a c_b \mathrm{Tr}\left[T_G^a T_G^b\right] = \sum_{a,b} c_a c_b \Delta\delta^{ab} = \Delta \sum_a c_a^2, \tag{2.102}$$

and hence,

$$\sum_a c_a^2 = 1. \tag{2.103}$$

So the sum of the squares of the coefficients in Eq. (2.98) is equal to 1. This is satisfied by Eq. (2.94) because we only worked with generators that have the same normalisation. Eq. (2.103) is also useful if we want to normalise linear combinations of normalised generators. For instance, suppose we have two generators T^1 and T^2 that have the same normalisation and we want to normalise the combination $T^1 + T^2$. Then Eq. (2.103) tells us that this must be $(T^1 + T^2)/\sqrt{2}$, since the sum of the squares of the coefficients is equal to 1.

Another scenario that may occur is when several simple groups H_i, $i = 1 \ldots n$, are embedded into a single simple group G at an energy scale M_I. Since G is simple, it only has one coupling constant g. H_i becomes part of G so the coupling constant h_i of H_i has to be equal to g at M_I. Thus, at M_I we have the matching conditions

$$g(M_I) = h_1(M_I) = h_2(M_I) = \ldots = h_n(M_I). \tag{2.104}$$

A scenario like this can, for instance, occur at the unification scale, where all coupling constants are assumed to be equal to each other.

Chapter 3

SU(5) grand unification

One of the first grand unification groups that were proposed is $SU(5)$ [9]. It is the smallest $SU(N)$ group that can contain the standard model. $SU(5)$ has dimension 24, so it has twelve generators more than the standard model and thus twelve new gauge bosons associated with them. Some of these new bosons may carry both color and weak isospin/hypercharge. As it turns out, this is the case for all new gauge bosons in $SU(5)$, which leads to the appearance of vector leptoquarks.

The smallness of $SU(5)$ and the ease with which all fermions can be fitted into irreps, made it an attractive candidate for grand unification. However, several issues exclude minimal $SU(5)$. The first of these is that gauge coupling unification turns out to be impossible. So minimal $SU(5)$ is inconsistent with the hypothesis of grand unification. Unification is only possible if we take some experimentally measured parameters to be a free parameter. As we will see, this yields a unification scale $M_U \approx 10^{15}$ GeV. But, as we will also see, the masses of all vector leptoquarks lie at this scale and they mediate proton decay. So M_U is directly related to the proton's lifetime. The prediction for the proton's lifetime is incompatible with experimental limits on proton decay. Lastly, the minimal Yukawa sector predicts wrong relations between the masses of quark and leptons.

We start by discussing which representations are necessary to fit the fermions into $SU(5)$. Next, the phenomenon of proton decay, as mediated by vector leptoquarks, is discussed. After that, we cover how $SU(5)$ must be broken to the standard model and show that minimal $SU(5)$ also contains a scalar leptoquark. This will lead to a yet unsolved issue known as the doublet-triplet splitting problem. Lastly, we discuss gauge coupling unification in minimal $SU(5)$. Throughout, several issues are emphasized that lead to the exclusion of minimal $SU(5)$. The information presented here about $SU(5)$ appears in many places. For this chapter, refs. [2, 8, 10] were mostly used.

3.1 The 5 and 10 dimensional irreducible representations

The hypothesis of $SU(5)$ grand unification is that at some high energy scale, the GUT scale M_U, all interactions exhibit an $SU(5)$ symmetry. At that point the gauge interactions would be described by a single coupling constant g. In such a scenario, all particles transform according to irreps of $SU(5)$. In the minimal version of the model [9], five of the fermions are placed in the 5-dimensional fundamental representation and the other 10 are placed in the 10-dimensional irrep. This choice is not arbitrary, because the goal is to extend the standard model. So at the very least, the new interactions should contain all interactions present in the standard model. As we will see, $SU(5)$ contains the standard model group as a subgroup. So a subset of the generators of $SU(5)$, generates $SU(3)_C \times SU(2)_L \times U(1)_Y$. To ensure that we recover the interactions from the standard model, it is required that the fermions transform in the same way under the SM subgroup as they do in the SM (see Eq. 2.33 for these transformation properties). In this section we look at the structure of the 5- and 10-dimensional irreps of $SU(5)$ and we will see how the fermions can be placed in them.

3.1.1 The generators

The generators of the 5-dimensional fundamental representation consist of all Hermitian, traceless 5×5 matrices. The first eight generate an $SU(3)$ subgroup.

$$\lambda^a = \begin{pmatrix} & & & 0 & 0 \\ & \lambda^a & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a = 1 \ldots 8. \tag{3.1}$$

Note that we use the name λ^a both for $SU(3)$ and $SU(5)$ generators. However, it should always be clear from context which ones are being referred to. The next twelve generators are:

$$\lambda^9 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^{10} = \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

$$\lambda^{17} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^{19} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \lambda^{20} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}.$$

Three others generate an $SU(2)$ subgroup:

$$\lambda^{20+a} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{σ^a}} \\ 0 & 0 & 0 & & \end{pmatrix}, \qquad a = 1, 2, 3. \tag{3.3}$$

The last generator is:

$$\lambda^{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}. \tag{3.4}$$

The hypercharge generator Y is related to λ^{24} through

$$Y = \sqrt{\frac{5}{12}}\lambda^{24} = \begin{pmatrix} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1/2 & \\ & & & & 1/2 \end{pmatrix}. \tag{3.5}$$

Note that the normalization of λ^a is chosen such that $\mathrm{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$. In the Lagrangian, one uses the generators $T^a = \lambda^a/2$, which have normalization $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$, as is customary for fundamental representations of $SU(N)$.

Now we are in a position to see that the SM group is indeed a subgroup of $SU(5)$. Recall that the SM group is a direct product of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$. We have seen which generators generate each subgroup separately, but if the direct product of these subgroups is also a subgroup of $SU(5)$, then their generators should commute with each other. From Eqs. (3.1) and (3.3) it is clear that the generators of the $SU(3)_C$ and $SU(2)_L$ subgroups act on different subspaces. So they commute with each other. Moreover, the hypercharge generator (Eq. 3.5) is diagonal with respect to the generators of both $SU(3)_C$ and $SU(2)_L$, so it commutes with them as well. Thus, $SU(5)$ contains the SM group as a subgroup.

3.1.2 Placing fermions in the 5

Having established that the SM group is contained in $SU(5)$, we can now find out which particles can be placed into the fundamental representation. To do so, we need to know how it decomposes into irreps of the SM group. From the block-diagonal forms of the generators of the SM subgroup, it follows that the fundamental representation can be decomposed into two irreps: a color triplet with hypercharge $-1/3$ and an isospin doublet with hypercharge $1/2$,

$$5 = (3, 1, -1/3) + (1, 2, 1/2). \tag{3.6}$$

In the first term the transformation properties of the right handed down quark d_R can be recognized. The second term resembles the transformation properties of the lepton doublet L_L, but with opposite hypercharge. To flip the sign of the hypercharge, the charge conjugate of the fields can be used:

$$L_L^c = \begin{pmatrix} \nu_e^c \\ e^c \end{pmatrix}_R \tag{3.7}$$

Note that we used the fact that $(\psi_L)^c = (\psi^c)_R$ to rewrite L_L^c in terms of right handed fields. We do this so that all fields in the fundamental representation are right handed. Simply replacing the fields with their charge conjugates would, however, mean that $(\nu_e^c)_R$ and $(e^c)_R$ transform as if they have $T_3 = 1/2$ and $T_3 = -1/2$, respectively. This is wrong, because a conjugated field should transform according to the complex conjugate of the representation. So all quantum numbers should flip sign. Fortunately, $SU(2)$ representations are real, so to do this we can simply switch to a different basis. We already found this basis in section 2.1: it was $i\sigma_2 L_L^c$. Thus, the following choice gives the right transformation properties under $SU(2)_L$:

$$L^e = SL_L^c = \begin{pmatrix} e^c \\ -\nu_e^c \end{pmatrix}_R. \tag{3.8}$$

Then L_L^c transforms infinitesimally as $\delta L_L^c = S^{-1}\delta L^e = -i\alpha_i S^{-1}\sigma_i L^e = i\alpha_i \sigma_i^* L_L^c$, confirming that L_L^c has the right transformation properties. The five basis states d_i and L^e can be combined into an $SU(5)$ vector:

$$\Psi = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^c \\ -\nu_e^c \end{pmatrix}_R, \tag{3.9}$$

which transforms according to the 5 of $SU(5)$. Occasionally, the conjugate of Eq. (3.9) is necessary. This will be denoted Ψ^c:

$$\Psi^c = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu_e \end{pmatrix}_L, \tag{3.10}$$

which transforms according to the $\overline{5}$ of $SU(5)$. For $SU(5)$ tensors, it will be convenient to denote color indices by greek letters $\alpha, \beta, \ldots = 1, 2, 3$ and isospin indices by $a, b, \ldots = 4, 5$. Then Ψ consists of two sets of states:

$$\Psi^\alpha \sim (3, 1, -1/3), \quad \Psi^a \sim (1, 2, 1/2). \tag{3.11}$$

3.1.3 Placing fermions in the 10

Now we turn to the 10-dimensional representation. Unlike the fundamental representation, we do not list the generators of the 10-dimensional irrep, but instead use tensor methods to examine its structure. In section 2.1 we showed how the $N \times N$ tensor product of $SU(N)$ decomposes into a symmetric and an antisymmetric irrep. For $SU(5)$ this gives

$$5 \times 5 = 15 + 10, \tag{3.12}$$

where the 10 is antisymmetric. So we can describe the 10 with a two-indexed tensor χ^{ij} that is antisymmetric in its indices. To see how the 10 decomposes into irreps of the SM group, the notation that was introduced for color and isospin indices will be useful. The components of a general 5×5 tensor ψ^{ij} can be divided into four parts: $\psi^{\alpha\beta}$, $\psi^{a\alpha}$, $\psi^{\alpha a}$ and ψ^{ab}. The first of these, $\psi^{\alpha\beta}$, can be written in terms of a symmetric and an antisymmetric part as

$$\psi^{\alpha\beta} = \frac{1}{2}(\psi^{\alpha\beta} + \psi^{\beta\alpha}) + \frac{1}{2}\epsilon^{\alpha\beta\gamma}\epsilon_{\gamma\mu\nu}\psi^{\mu\nu}. \tag{3.13}$$

The antisymmetric part contains a tensor $\epsilon_{\gamma\mu\nu}\psi^{\mu\nu}$ (under the SM group). But notice that it has one lower color index γ so it transforms according to the $\overline{3}$ of $SU(3)_C$. The tensor has no isospin indices, so it is a singlet under $SU(2)_L$. The hypercharge quantum numbers add, because under a hypercharge transformation a product state $\psi_1^i\psi_2^j$ transforms like

$$\psi_1^i\psi_2^j \to e^{-iaY_1}\psi_1^i e^{-iaY_2}\psi_2^j = e^{-ia(Y_1+Y_2)}\psi_1^i\psi_2^j. \tag{3.14}$$

So the hypercharge of $\psi^{\alpha\beta}$ is $-1/3 - 1/3 = -2/3$. We can therefore conclude that $\chi^{\alpha\beta}$, which is the antisymmetric part of $\psi^{\alpha\beta}$, transforms according to the $(\overline{3}, 1, -2/3)$ irrep of the SM group. But this is precisely how $(u_R)^c = (u^c)_L$ would transform. So we can identify $(u^c)_{\alpha L} = \epsilon_{\alpha\beta\gamma}\psi^{\beta\gamma}/\sqrt{2}$. The factor $1/\sqrt{2}$ accounts for the fact that the particle appears twice. We can then write $\chi^{\alpha\beta} = \epsilon^{\alpha\beta\gamma}(u^c)_{\gamma L}/\sqrt{2}$.

The components $\psi^{a\alpha}$ and $\psi^{\alpha a}$ can be decomposed in terms of symmetric and antisymmetric parts as

$$\psi^{a\alpha} = \frac{1}{2}(\psi^{a\alpha} + \psi^{\alpha a}) + \frac{1}{2}(\psi^{a\alpha} - \psi^{\alpha a}), \tag{3.15}$$

$$\psi^{\alpha a} = \frac{1}{2}(\psi^{a\alpha} + \psi^{\alpha a}) - \frac{1}{2}(\psi^{a\alpha} - \psi^{\alpha a}) \tag{3.16}$$

The antisymmetric part $\chi^{a\alpha} = (\psi^{a\alpha} - \psi^{\alpha a})/2$ has one upper color index and one upper isospin index. So it transforms according to the 3 of $SU(3)_C$ and the 2 of $SU(2)_L$. Its hypercharge is $-1/3 + 1/2 = 1/6$. Thus, $\chi^{a\alpha}$ transforms according to the $(3, 2, 1/6)$ irrep of the SM group, which is how Q_L transforms. We identify $\chi^{a\alpha} = Q_L^{a\alpha}/\sqrt{2}$. Finally, there are the components ψ^{ab}. Written in terms of its symmetric and antisymmetric part, this is

$$\psi^{ab} = \frac{1}{2}(\psi^{ab} + \psi^{ba}) + \frac{1}{2}\epsilon^{ab}\epsilon_{cd}\psi^{cd}. \tag{3.17}$$

Strictly speaking, the only non-zero components of ϵ^{ij} are ϵ^{12} and ϵ^{21}. However, since isospin indices can only have the values 4 and 5, it will be convenient to redefine ϵ^{ab} such that $\epsilon^{45} = -\epsilon^{54} = 1$.

The antisymmetric part of Eq. (3.17) contains the tensor $\epsilon_{cd}\psi^{cd}$, which is both a color singlet and an isospin singlet. Its hypercharge is $1/2 + 1/2 = 1$. The tensor therefore transforms according to the $(1, 1, 1)$ irrep of the SM group, which is how $(e_R)^c = (e^c)_L$ transforms. We identify $(e^c)_L = \epsilon_{cd}\psi^{cd}/\sqrt{2}$. Then $\chi^{ab} = \epsilon^{ab}(e^c)_L/\sqrt{2}$.

To summarize, the 10-dimensional irrep of $SU(5)$ can be decomposed into three irreps of the SM group:

$$10 = (\overline{3}, 1, -2/3) + (3, 2, 1/6) + (1, 1, 1). \tag{3.18}$$

We describe it using an antisymmetric second rank tensor χ^{ij}, whose components are given by

$$\chi^{\alpha\beta} = \frac{1}{\sqrt{2}}\epsilon^{\alpha\beta\gamma}(u^c)_{\gamma L}, \quad \chi^{a\alpha} = \frac{1}{\sqrt{2}}Q_L^{a\alpha}, \quad \chi^{ab} = \frac{1}{\sqrt{2}}\epsilon^{ab}(e^c)_L. \tag{3.19}$$

In matrix notation this is

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & e^c \\ d_1 & d_2 & d_3 & -e^c & 0 \end{pmatrix}_L . \tag{3.20}$$

Thus, we have succeeded in placing all fifteen fermions into irreps of $SU(5)$, in such a way that their transformation properties under the SM group are retained. This ensures that all gauge interactions present in the standard model also appear in our $SU(5)$ GUT.

So far, only one generation of fermions has been placed into representations. In the standard model, each generation has exactly the same transformation properties as other generations. This principle is assumed to hold in GUTs as well. The Lagrangian therefore contains three copies of the 5 and 10 representations, one for each generation.

3.2 The X and Y vector leptoquarks

That some of the gauge bosons of $SU(5)$ are leptoquarks, can already be seen in their transformation properties. The gauge bosons transform according to the adjoint representation of $SU(5)$, which decomposes in terms of irreps of the SM group as [8]

$$24 = (8, 1, 0) + (1, 3, 0) + (1, 1, 0) + (3, 2, -5/6) + (\overline{3}, 2, 5/6). \tag{3.21}$$

The first three multiplets correspond to the twelve standard model gauge bosons. The eight gluons belong to the color octet $(8, 1, 0)$, the $W^{\pm}$ and Z^0 to the isospin triplet $(1, 3, 0)$ and the photon to the singlet $(1, 1, 0)$. The two other sets of six bosons each, are new. The fact that they carry colour, weak isospin and hypercharge opens up the possibility that quarks and leptons couple directly to each other. Since these bosons originate from the gauge sector, they transform as Lorentz vectors and are consequently categorized as vector leptoquarks. In

$SU(5)$-based grand unified theories, these are denoted as X and Y bosons. We assign them to the $(\overline{3}, 2, -5/6)$ irrep as follows:

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix} \sim (\overline{3}, 2, 5/6). \tag{3.22}$$

With this notation, colour indices run horizontally and isospin indices vertically. Then X is the isospin up component and Y the isospin down component. Their antiparticles $\overline{X}$ and $\overline{Y}$ are assigned to the $(3, 2, 5/6)$ irrep:

$$\begin{pmatrix} \overline{X}_1 & \overline{X}_2 & \overline{X}_3 \\ \overline{Y}_1 & \overline{Y}_2 & \overline{Y}_3 \end{pmatrix} \sim (3, 2, -5/6). \tag{3.23}$$

To see precisely what kinds of interactions the X and Y bosons mediate, we write down the gauge sector of the Lagrangian. Since the fermions were placed into two irreps, Ψ and χ, the gauge sector will also consist of two parts. One part only contains couplings to Ψ and the other only contains couplings to χ. The part containing Ψ is

$$\mathcal{L}_\Psi = \overline{\Psi} i \not{D} \Psi, \tag{3.24}$$

where the covariant derivative D_μ is

$$D_\mu = \partial_\mu - igT^a A^a_\mu, \tag{3.25}$$

It will be convenient to define the gauge boson matrix $A_\mu = T^a A^a_\mu$. We can divide it into a part that leads to SM couplings and one that leads to leptoquarks couplings:

$$A_\mu = A^{\text{SM}}_\mu + A^{\text{LQ}}_\mu. \tag{3.26}$$

For A^{SM}_μ we run over all generators associated with the SM subgroup:

$$\begin{aligned} A^{\text{SM}}_\mu &= \sum_{a=1}^{8} A^a_\mu T^a + \sum_{a=21}^{23} A^a_\mu T^a + A^{24}_\mu T^{24} \\ &= \frac{1}{2} \begin{pmatrix} \sum_{a=1}^{8} G^a_\mu \lambda^a & \\ & \sum_{i=1}^{3} W^a_\mu \sigma^a \end{pmatrix} + B_\mu Y. \end{aligned} \tag{3.27}$$

Note that in the second line, we related the gauge bosons A^a_μ from $SU(5)$ to G^a_μ, W^a_μ and B_μ from the standard model:

$$G^1_\mu \dots G^8_\mu = A^1_\mu \dots A^8_\mu, \tag{3.28}$$

$$W^1_\mu \dots W^3_\mu = A^{21}_\mu \dots A^{23}_\mu, \tag{3.29}$$

$$B_\mu = \sqrt{\frac{12}{5}} A^{24}_\mu. \tag{3.30}$$

Since A_μ^{SM} consists of a 3×3 block that acts on Ψ^α and a 2×2 block that acts on Ψ^a, it cannot couple Ψ^α to Ψ^a. A_μ^{SM} will therefore not produce any leptoquark couplings. To obtain such couplings we need a gauge boson matrix with non-zero components outside the 3×3 and 2×2 blocks. These are contained in A_μ^{LQ}. For A_μ^{LQ} we run over the remaining generators:

$$A_\mu^{\text{LQ}} = \sum_{a=9}^{20} A_\mu^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} & & & \overline{X}_{1\mu} & \overline{Y}_{1\mu} \\ & & & \overline{X}_{2\mu} & \overline{Y}_{2\mu} \\ & & & \overline{X}_{3\mu} & \overline{Y}_{3\mu} \\ X_{1\mu} & X_{2\mu} & X_{3\mu} & & \\ Y_{1\mu} & Y_{2\mu} & Y_{3\mu} & & \end{pmatrix}. \tag{3.31}$$

So the X and Y vector leptoquarks appear as linear combinations of $A_\mu^9 \ldots A_\mu^{20}$.

Before we can write down the Lagrangian involving χ, we have to know how the generators act on it. Recall that χ is the antisymmetric part of a 5×5 tensor. So whereas Ψ transform as $\Psi \to \Psi' = U\Psi$, χ transforms as

$$\chi \to \chi' = U\chi U^T = \chi - i\alpha^a T^a \chi - i\alpha^a \chi (T^a)^T + \ldots \tag{3.32}$$

The action of the covariant derivative on χ is then

$$D_\mu \chi = \partial_\mu \chi - ig\left[A_\mu^a T^a \chi + \chi A_\mu^a (T^a)^T\right]. \tag{3.33}$$

The couplings between gauge bosons and χ are then contained in

$$\mathcal{L}_\chi = \text{Tr}\left[\overline{\chi}^T i\not{D}\chi\right]. \tag{3.34}$$

Now that we have the explicit form of the gauge sector, we can find out which leptoquark couplings it contains. The parts of $\mathcal{L}_\Psi$ and $\mathcal{L}_\chi$ containing couplings to X and Y bosons are [10]

$$\begin{aligned} \mathcal{L}_{X,Y} &= g\left(\overline{\Psi}\not{A}^{\text{LQ}}\Psi + \text{Tr}\left[\overline{\chi}^T \not{A}^{\text{LQ}}\chi\right] + \text{Tr}\left[\overline{\chi}^T \gamma^\mu \chi \left(A_\mu^{\text{LQ}}\right)^T\right]\right) \\ &= \frac{g}{\sqrt{2}}\Big(-\epsilon_{\alpha\beta\gamma}\overline{u_{\alpha L}}\not{X}_\beta u_{\gamma L}^c + \overline{e_L^c}\not{X}_\alpha d_L^\alpha - \overline{e_R^c}\not{X}_\alpha d_R^\alpha \\ &\quad - \epsilon_{\alpha\beta\gamma}\overline{d_{\alpha L}}\not{Y}_\beta u_{\gamma L}^c - \overline{e_L^c}\not{Y}_\alpha u_L^\alpha + \overline{\nu_{eR}^c}\not{Y}_\alpha d_R^\alpha\Big) + \text{h.c.} \end{aligned} \tag{3.35}$$

Evidently, the X and Y bosons have diquark as well as direct lepto-quark couplings. This shows that by exchanging a single X or Y boson many baryon number violating processes are possible. The ones that contribute to proton decay are shown in Figure 3.1. Diagrams 3.1a-d contribute to $p \to e^+\pi^0$, whereas 3.1e contributes to $p \to \overline{\nu}_e \pi^+$. These interactions do not preserve baryon number B and lepton number L separately. But perhaps there might be another quantum number that is conserved by them. This turns out to be the combination $B - L$ [8]. From Eq. (3.35) we can see that the $B-L$ quantum numbers of the X and Y bosons are

$$(B-L)_X = (B-L)_Y = \frac{2}{3}. \tag{3.36}$$

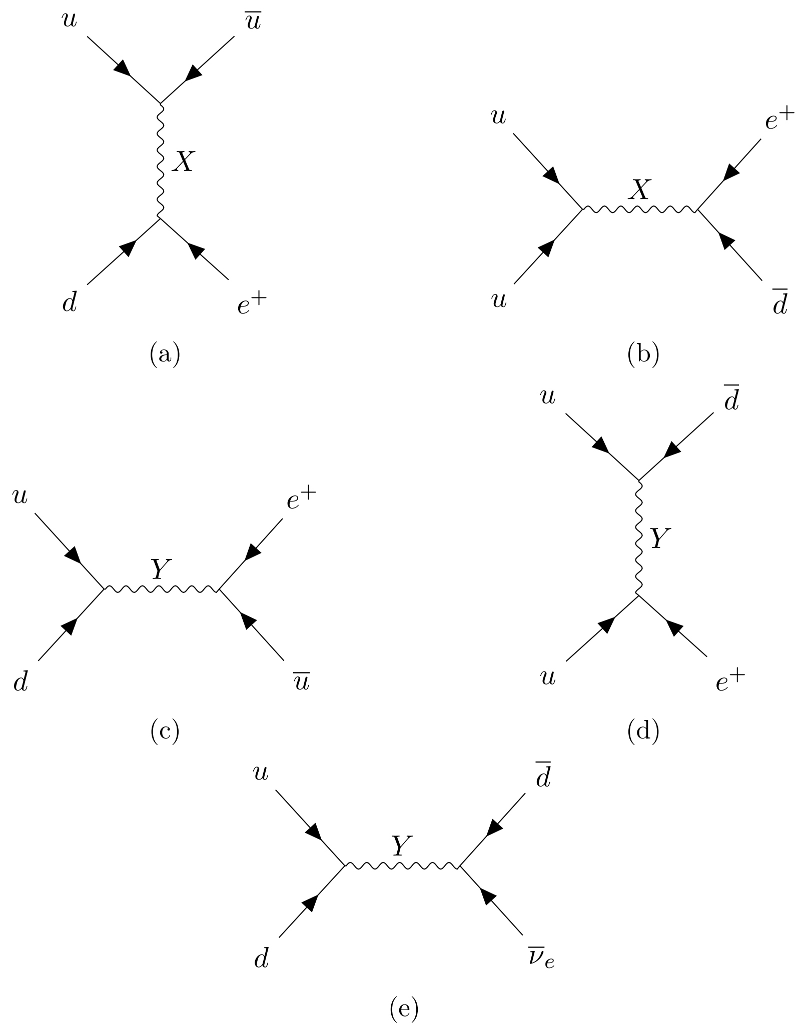

Figure 3.1: Proton decay diagrams in minimal $SU(5)$.

So while a process may alter both B and L, only interactions with $\Delta B = \Delta L$ are allowed. This is why proton decay in $SU(5)$ can only produce the antileptons e^+ and $\overline{\nu_e}$, since we always have $\Delta B = -1$.

To conclude, the gauge sector of the Lagrangian has shown that $SU(5)$ contains two vector leptoquarks which mediate the decays $p \to e^+\pi^0$ and $p \to \overline{\nu}_e\pi^+$. However, this does not tell us anything about the masses of the leptoquarks, so we cannot calculate the decay rates of the processes. Which mass scale are possible, is discussed in the next sections.

3.3 Symmetry breaking

Just as in the standard model, particles obtain their masses by spontaneously breaking the symmetry. In the standard model the masses of the $W^\pm$ and Z^0 bosons were all related to one vev parameter. This parameter sets the electroweak scale M_{EW}. In GUTs, it is assumed that there is at least one larger scale M_U, at which the GUT symmetry holds. The vector leptoquarks X and Y mediate proton decay, which has never been observed. So it is natural to expect them to lie at a scale larger than M_{EW}. The kinds of mass scales that are possible depend on which symmetry breaking patterns we choose. Nevertheless, if the theory is to be compatible with experimental data, we have to make sure that there is an electroweak scale with an $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry. This symmetry must then be broken further to $SU(3)_C \times U(1)_Q$. Moreover, the scalar particles that are used to break the symmetry have to include the Higgs field from the standard model. This section discusses the most minimal way in which the $SU(5)$ symmetry can be broken, while adhering to the aforementioned constraints.

Once the GUT symmetry is broken, we are always left with a symmetry group that is one of its subgroups. For $SU(5)$ the largest subgroups are $SU(3)_C \times SU(2)_L \times U(1)_Y$ and $SU(4) \times U(1)$ [11]. The latter can only be broken further to $SU(3)_C \times U(1)_Q$. This is unacceptable because there has to be a scale with the SM symmetry. So $SU(5)$ must be broken directly to the SM group. To achieve this, we need a scalar representation that contains at least one SM singlet. In other words: a state that is chargeless w.r.t the SM. The simplest representation that can break $SU(5)$ to the SM, is the 24-dimensional adjoint [9], whose decomposition was given in Eq. (3.21). This scalar multiplet is denoted ϕ. If the $(1,1,0)$ singlet obtains a vev, the symmetry will be reduced to that of the standard model. To reduce it further to $SU(3)_C \times U(1)_Q$ we need a colorless state, carrying no electric charge. Such a state is contained in the $(1,2,1/2)$ component of the fundamental representation. Hence, the symmetry can be broken to $SU(3)_C \times U(1)_Y$ using a 5-dimensional Higgs [9], denoted as H. In this case, the $(1,2,1/2)$ doublet corresponds to the SM Higgs doublet. The full symmetry breaking mechanism then consists of two steps:

$$SU(5) \xrightarrow{24} SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow{5} SU(3)_C \times U(1)_Q. \tag{3.37}$$

Recall that a gauge boson only becomes massive if the generator associated with it is broken. So in the first step, the leptoquarks X and Y obtain their masses, since the generators associated with them do not generate the SM subgroup. These masses are related to the vev $\langle\phi\rangle$ obtained by ϕ. Thus, $\langle\phi\rangle$ sets the mass scale of the vector leptoquarks. The adjoint of $SU(5)$ corresponds to the traceless part of the $5 \times \overline{5}$ tensor product. So to expand this tensor in terms of fields that transform according to the adjoint representation, we need to find 24 traceless matrices. Moreover, if we take ϕ to be hermitian, these matrices need to be hermitian as well. Therefore, ϕ can be written as a linear combination of the generators of the fundamental representation:

$$\phi = \sum_{i=1}^{24} \phi_i \lambda_i. \tag{3.38}$$

That this really furnishes the adjoint representation, can be seen from an infinitesimal transformation of ϕ:

$$\begin{aligned} \phi \to \phi' &= U\phi U^\dagger \\ &= (I - ia_j\lambda_j)\phi_i\lambda_i(I + ia_j\lambda_j) + \dots \\ &= \phi_i\lambda_i - ia_j\phi_i[\lambda_j, \lambda_i] + \dots \\ &= \phi_i\lambda_i - a_j\phi_k f^j_{ik}\lambda_i + \dots . \end{aligned} \tag{3.39}$$

Thus, the induced transformation on ϕ_i is

$$\phi_i \to \phi_i' = \phi_i - ia_j \left(-if_{ik}^j\right) \phi_k + \dots, \tag{3.40}$$

which shows that ϕ_i transforms according to the adjoint representation. The next step is to find a potential $V(\phi)$, which allows us to spontaneously break the symmetry to the standard model. As mentioned before, this occurs if the SM singlet contained within the adjoint obtains a vev. To make sure that the symmetry is only broken spontaneously, the potential has to consist of $SU(5)$ invariant terms. Up to terms quartic in ϕ, the most general potential is [10]

$$V(\phi) = -\frac{1}{2}\mu^2 \mathrm{Tr}\left[\phi^2\right] + \frac{a}{4}\left(\mathrm{Tr}\left[\phi^2\right]\right)^2 + \frac{b}{2}\mathrm{Tr}\left[\phi^4\right], \quad \mu^2 > 0. \tag{3.41}$$

If the vev is to be invariant under the SM group, it needs to commute with $\lambda_1 \dots \lambda_8$, and $\lambda_{21} \dots \lambda_{24}$. This can only be λ_{24}, since it is the only generator that is diagonal w.r.t both the $SU(3)$ and $SU(2)$ generators. This fixes the form of the vev:

$$\langle\phi\rangle = -\frac{\sqrt{15}}{2} v\lambda_{24} = v \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3/2 & \\ & & & & -3/2 \end{pmatrix}. \tag{3.42}$$

Minimizing the potential with this vev gives [10]

$$v = \sqrt{\frac{2\mu^2}{15a + 7b}}. \tag{3.43}$$

So this only has a real solution if $a > -(7/15)b$. The full Lagrangian containing the Higgs field ϕ is [10]

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[(D_\mu\phi)^\dagger(D^\mu\phi)] - V(\phi), \tag{3.44}$$

$$\text{with } D_\mu\phi = \partial_\mu\phi + ig[A_\mu, \phi]. \tag{3.45}$$

This allows us to calculate the masses of the leptoquarks X and Y. The mass terms originate from the covariant derivative:

$$\begin{aligned} \mathcal{L}_M^\phi &= \frac{1}{2}\mathrm{Tr}[(D_\mu\langle\phi\rangle)^\dagger(D^\mu\langle\phi\rangle)] \\ &= -\frac{1}{2}g^2\mathrm{Tr}\left([A_\mu, \langle\phi\rangle]^\dagger[A^\mu, \langle\phi\rangle]\right) \\ &= -\frac{25}{16}g^2v^2\mathrm{Tr}\left|\begin{pmatrix} & & & -\overline{X}_{1\mu} & -\overline{Y}_{1\mu} \\ & & & -\overline{X}_{2\mu} & -\overline{Y}_{2\mu} \\ & & & -\overline{X}_{3\mu} & -\overline{Y}_{3\mu} \\ X_{1\mu} & X_{2\mu} & X_{3\mu} & & \\ Y_{1\mu} & Y_{2\mu} & Y_{3\mu} & & \end{pmatrix}\right|^2 \\ &= -\frac{25}{8}g_5^2v^2\sum_i\left(|X_i|^2 + |Y_i|^2\right). \end{aligned} \tag{3.46}$$

Therefore, the masses of the leptoquarks X and Y are [10]

$$M_X = M_Y = \frac{5}{2\sqrt{2}} g v. \tag{3.47}$$

This expression obtains a small correction if the Higgs multiplet H is included, which is necessary to reduce the symmetry to $SU(3)_C \times U(1)_Q$ below the electroweak scale. Just as we did for the fermions in the fundamental representation, we can divide H into two parts: a color triplet T^α and an isospin doublet D^a,

$$H = \begin{pmatrix} T^\alpha \\ D^a \end{pmatrix}. \tag{3.48}$$

The breaking to $SU(3)_C \times U(1)_Q$ can be achieved by giving the fifth component of H a vev:

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_0 \end{pmatrix}, \tag{3.49}$$

which leaves the $SU(3)_C$ generators unbroken as well as the combination

$$Q = T_{23} + Y, \tag{3.50}$$

which is electric charge. The potential that allows such a vev, is analogous to the SM Higgs potential:

$$V(H) = -\frac{\nu^2}{2} H^\dagger H + \frac{\lambda}{4} (H^\dagger H)^2, \quad \nu^2, \lambda > 0 \tag{3.51}$$

Note that this would mean that all components of H lie at the same mass scale. This cannot be true, because the doublet D^a is the Higgs doublet from the SM, whereas the triplet T^α has never been observed. T^α must therefore lie at a larger mass scale than D^a. We return to this issue in the next section, where we discuss the couplings of T^α in more detail.

The contributions to the masses of the gauge bosons due to H are

$$\begin{aligned} \mathcal{L}_M^H &= \frac{1}{2} \mathrm{Tr}[(D_\mu \langle H \rangle)^\dagger (D^\mu \langle H \rangle)] \\ &= -\frac{1}{2} g^2 \mathrm{Tr}[(A_\mu \langle H \rangle)^\dagger A_\mu \langle H \rangle] \\ &= -\frac{1}{8} g^2 v_0^2 \left| \begin{pmatrix} Y_{1\mu} \\ Y_{2\mu} \\ Y_{3\mu} \\ \frac{1}{\sqrt{2}} \left(W_\mu^3 - \sqrt{\frac{3}{5}} B_\mu \right) \\ W_\mu^+ \end{pmatrix} \right|^2 \\ &= -\frac{1}{8} g^2 v_0^2 \left(\sum_i |Y_i|^2 + \frac{4}{5} \left| \sqrt{\frac{5}{8}} W_\mu^3 - \sqrt{\frac{3}{8}} B_\mu \right|^2 + W_\mu^+ W^{\mu -} \right). \end{aligned} \tag{3.52}$$

So both the $W^{\pm}$ and Z_0 bosons as well as the Y boson obtain contributions to their masses of order gv_0. Thus, the Higgs multiplet H slightly lifts the degeneracy between the masses of the leptoquarks.

3.4 The color triplet scalar leptoquark

The Higgs multiplet H can be coupled to fermions in the Yukawa sector. The terms appearing in the Yukawa sector consist of the minimal set that is necessary to generate masses for the fermions. A mass term has the form $-m\overline{\psi}_R\psi_L = m(\psi^c)_L C\psi_L$, plus its hermitian conjugate. So to generate masses for the down quark and the electron, we need to couple the $\overline{5}$ to the 10. This can then be coupled to another $\overline{5}$, the Higgs field $\overline{H}$, to obtain an $SU(5)$ invariant expression:

$$\mathcal{L}_Y^5 = Y_5(\Psi^c)^T C\chi\overline{H} + \text{ h.c.} \tag{3.53}$$

Both u_L and u_L^c are contained in the 10, so the 10 needs to be coupled to itself to obtain a mass for the up quark. To make this $SU(5)$ invariant, these two factors can be coupled to H and the resulting tensor should be antisymmetrized:

$$\mathcal{L}_Y^{10} = Y_{10}\epsilon^{ijklm}\chi_{ij}C\chi_{kl}H_m + \text{ h.c.} \tag{3.54}$$

The masses can be found by setting H to its expectation value $\langle H\rangle$. For $\mathcal{L}_Y^5$ this results in $\mathcal{L}_Y^5(\langle H\rangle) = -Y_5 v_0\left(\overline{d}d + \overline{e}e\right)$. This means that the masses of the down quark and the electron are equal:

$$m_e = m_d = Y_5 v_0 \tag{3.55}$$

This mass relation is, however, scale dependent. So it is only exact when the $SU(5)$ symmetry holds. Nevertheless, even when extrapolated down to lower energy scales, minimal $SU(5)$ yields wrong mass relations [8]. This is yet another reason why minimal $SU(5)$ does not work.

Now we turn to the color triplet T^α. If we restrict to the parts in $\mathcal{L}_Y^5$ containing T^α, we obtain

$$\begin{aligned}\mathcal{L}_Y^5 &= \frac{Y_5}{\sqrt{2}}\begin{pmatrix} d_1^c & d_2^c & d_3^c & e^- & -\nu_e\end{pmatrix}_L C\begin{pmatrix}\overline{T}_2u_3^c - \overline{T}_3u_2^c \\ \overline{T}_3u_1^c - \overline{T}_1u_3^c \\ \overline{T}_1u_2^c - \overline{T}_2u_1^c \\ \overline{T}_1u_1 + \overline{T}_2u_2 + \overline{T}_3u_3 \\ \overline{T}_1d_1 + \overline{T}_2d_2 + \overline{T}_3d_3\end{pmatrix}_L + \text{ h.c.} \\ &= \frac{Y_5}{\sqrt{2}}\left(\epsilon^{\alpha\beta\gamma}d_{\alpha L}^c C\overline{T}_\beta u_{\gamma L}^c + e_L^- C\overline{T}_\alpha u_L^\alpha - \nu_{eL}C\overline{T}_\alpha d_L^\alpha\right) + \text{ h.c..}\end{aligned} \tag{3.56}$$

These interaction terms lead to the same type of diagrams as the vector leptoquarks X and Y. Thus, T^α is a scalar leptoquark that mediates proton decay. This further reinforces that T^α must be much heavier than the SM Higgs doublet contained in H. In [12], for instance, it is estimated that to keep the proton from decaying to quickly, a lower bound on the mass M_T of T^α is

$$M_T > 3\cdot 10^{11}\text{ GeV.} \tag{3.57}$$

The question is then, how can such a mass hierarchy be incorporated in the theory? This is known as the doublet-triplet splitting problem [13]. First of all, the potential in Eq. (3.51) cannot produce a mass hierarchy, since both the doublet and triplet are treated the same way. However, since two Higgs multiplets, H and ϕ, are present, a potential involving couplings between the two is also possible [10]:

$$V(\phi, H) = \alpha H^\dagger H \mathrm{Tr}\left[\phi^2\right] + \beta H^\dagger \phi^2 H. \tag{3.58}$$

This gives additional contributions to the masses of each component of H. The essence of the doublet-triplet splitting problem can already be illustrated if we include only the vev obtained by ϕ. The first three diagonal components of $\langle\phi\rangle$ are different than the last three, so the couplings in Eq. (3.58) treat the triplet and the doublet differently, leading to mass splitting. If we denote the doublet as D^a, the part of the full potential containing H at $\phi = \langle\phi\rangle$ can be written as [14]

$$V(T, D) = \left(-\frac{\nu^2}{2} + \frac{15\alpha}{2}v^2 + \beta v^2\right) T^\dagger T + \left(-\frac{\nu^2}{2} + \frac{15\alpha}{2}v^2 + \frac{9\beta}{4}v^2\right) D^\dagger D + \dots, \tag{3.59}$$

where higher order terms have been omitted, since they do not contribute to the mass. Thus, the masses of the doublet and triplet are given by

$$m_D^2 = -\frac{\nu^2}{2} + \frac{15\alpha}{2}v^2 + \frac{9\beta}{4}v^2 = -\frac{\nu^2}{2} + \frac{15\alpha + 9\beta}{15a + 7b}\mu^2, \tag{3.60}$$

$$m_T^2 = -\frac{\nu^2}{2} + \frac{15\alpha}{2}v^2 + \beta v^2 = -\frac{\nu^2}{2} + \frac{15\alpha + 2\beta}{15a + 7b}\mu^2. \tag{3.61}$$

To solve the problem, we would have to choose values for the parameters such that m_D lies around 100 GeV and m_T lies at a much larger scale. However, the scales of μ^2 and ν^2 can be very different so the terms in Eqs. (3.60) and (3.61) are generally dominated by the larger one. The scales of the masses would therefore be the same. The only way out would be to rely on precise cancellations between the two contributions.

3.5 Gauge coupling unification

Section 3.3 showed that minimal $SU(5)$ contains two non-zero mass scales: the electroweak scale M_{EW} where the $W^\pm$ and Z^0 bosons lie, and the unification scale M_U where the vector leptoquarks lie. The magnitude of M_{EW} is known from experiments to be around 100 GeV. The magnitude of the unification scale, if there even is one, has not been measured experimentally. Nevertheless, if $SU(5)$ grand unification is correct, all coupling constants should be equal to each other at M_U. As we will see, this places constraints on the value of M_U.

In minimal $SU(5)$, there is an $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry from M_{EW} up to M_U. Moreover, of the scalar particles only the Higgs doublet D^a is assumed to lie around the electroweak scale. The particles that are relevant from M_{EW} up to M_U, are therefore the same as in the standard model. Before we discuss running of the couplings in $SU(5)$, we first show that the normalized hypercharge and electric charge generators agree with the expressions given in Eq. (2.87). In $SU(5)$ the normalization of the generators is always given by $\mathrm{Tr}(T^aT^b) = \delta^{ab}/2$. This is not the case for the hypercharge and electric charge generators.

For instance, for the fermions in the fundamental representation the normalization of Y and Q are

$$\mathrm{Tr}(Y^2) = 3 \cdot (1/3)^2 + 2 \cdot (1/2)^2 = 5/6, \tag{3.62}$$
$$\mathrm{Tr}(Q^2) = 3 \cdot (-1/3)^2 + 1^2 + 0^2 = 4/3. \tag{3.63}$$

So to obtain the same normalization as the $SU(5)$ generators we must multiply Y by $\sqrt{3/5}$ and Q by $\sqrt{3/8}$. Thus the properly normalized hypercharge and electric charge generators are:

$$Q' = \sqrt{\frac{3}{8}}Q, \quad Y' = \sqrt{\frac{3}{5}}Y, \tag{3.64}$$

in agreement with Eq. (2.87). We can therefore conclude that below M_U the running of the couplings in minimal $SU(5)$ is identical to the standard model.

The running above and below EWSB was shown in Figure 2.1 from section 2.4. Judging from the graph, it seems as though the coupling constants will unify at some scale. However, if we extrapolate the values we obtain the graph shown in Figure 3.2. The couplings associated with $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ do not meet at a common point. Instead, they have three distinct intersections at energy scales of roughly 10^{13}, 10^{14} and 10^{17} GeV. The gaps between these scales is large enough to conclude that gauge coupling unification is impossible in minimal $SU(5)$.

One reason that unification is impossible is that $SU(5)$ can only be broken directly to the SM. So if no particles are added at the electroweak scale, the running of the couplings is fixed by the SM. Thus, one possible way of extending $SU(5)$ would be to find a larger symmetry group that can be broken to the SM in multiple stages. Nevertheless, to get an

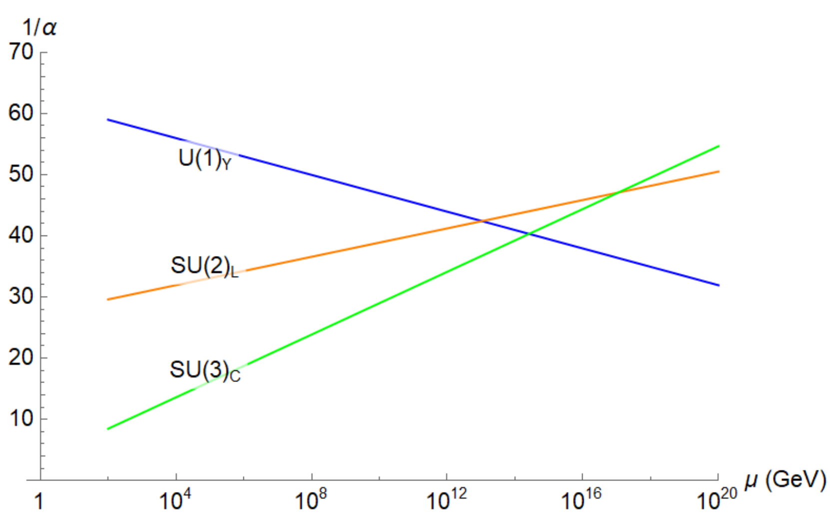

Figure 3.2: Running of the fine structure constants in minimal $SU(5)$.

idea of the magnitude of M_U, we can try to adjust the low energy parameters such that grand unification is possible. The values of the fine structure constants $\alpha = e^2/4\pi$ and α_{3C} at the electroweak scale have been determined very precisely. Following [15], these two values will be used as input:

$$\alpha^{-1}(M_{\text{EW}}) \approx 128, \tag{3.65}$$

$$\alpha_{3C}^{-1}(M_{\text{EW}}) \approx 8.45. \tag{3.66}$$

Note that α is the fine structure constant associated with the unnormalized Q. In $SU(5)$, this generator is related to T_{23} and Y' through

$$Q = T_{23} + \sqrt{\frac{5}{3}}Y', \tag{3.67}$$

which implies that

$$\alpha^{-1}(M_{\text{EW}}) = \alpha_{2L}^{-1}(M_{\text{EW}}) + \frac{5}{3}\alpha_Y^{-1}(M_{\text{EW}}). \tag{3.68}$$

Together with the condition for unification,

$$\alpha_U \equiv \alpha_Y(M_U) = \alpha_{2L}(M_U) = \alpha_{3C}(M_U), \tag{3.69}$$

this yields four equations in the four unknowns $\alpha_Y(M_{\text{EW}})$, $\alpha_{2L}(M_{\text{EW}})$, $\alpha_{3C}(M_{\text{EW}})$ and M_U. Solving the system of equations results in the values (see Figure 3.3):

$$M_U \approx 7.7 \cdot 10^{15} \text{ GeV}, \tag{3.70}$$

$$\alpha_U^{-1} \approx 41.5, \tag{3.71}$$

$$\alpha_Y^{-1}(M_{\text{EW}}) \approx 60.9, \tag{3.72}$$

$$\alpha_{2L}^{-1}(M_{\text{EW}}) \approx 26.6, \tag{3.73}$$

So with the assumption of unification in $SU(5)$, we would expect the masses of the X and Y bosons to lie around 10^{15} GeV. If minimal $SU(5)$ is a viable theory, these masses should be consistent with limits on proton decay rates. One of the most recent limits comes from the Super-Kamiokande Collaboration, which reported a lower bound $\tau_p > 2 \cdot 10^{34}$y [16]. We can compare this to the $SU(5)$ prediction. The lifetime of the proton is approximately given by [17]

$$\tau_p \approx \frac{M_U^4}{g^4 m_p^5} \stackrel{\text{SI}}{=} \frac{\hbar}{16\pi^2 c^2 \alpha_U^2 m_p}\left(\frac{M_U}{m_p}\right)^4 \approx 1.8 \cdot 10^{-34}\text{y} \cdot \alpha_U^{-2}\left(\frac{M_U}{\text{GeV}}\right)^4. \tag{3.74}$$

So if this is to be consistent with the experimental limit, we need $M_U > 2 \cdot 10^{16}$ GeV. Clearly, the predicted unification scale is too low. Minimal $SU(5)$ is therefore also ruled out by limits on proton decay rates.

3.6 Conclusions

In this chapter we reviewed how leptoquarks appear in one of the simplest grand unified theories: minimal $SU(5)$. Minimal $SU(5)$ contains the twelve standard model gauge bosons as well as twelve new gauge bosons that have lepto-quark couplings. These latter bosons are the vector leptoquarks of $SU(5)$ and are conventionally denoted as X and Y bosons.

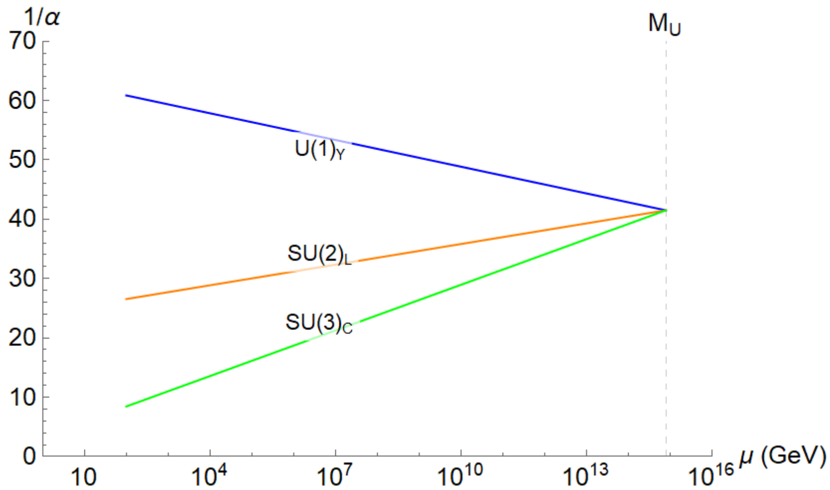

Figure 3.3: Running of the fine structure constants in $SU(5)$ if unification is assumed.

The gauge sector of the Lagrangian showed that they mediate the proton decay processes $p \to e^+\pi^0$ and $p \to \overline{\nu}_e\pi^0$. These processes violate both baryon number B and lepton number L, but they preserve the combination $B - L$.

The most minimal way to break the $SU(5)$ symmetry involved two scalar multiplets, ϕ and H, which transform according to adjoint representation and the fundamental representation of $SU(5)$, respectively. The vev of ϕ breaks the $SU(5)$ symmetry to $SU(3)_C \times SU(2)_L \times U(1)_Y$ at the unification scale M_U. The vev of H then further breaks the symmetry to $SU(3)_C \times U(1)_Q$ at the electroweak scale M_{EW}. All vector leptoquarks lie at the unification scale. However, unification turned out to be impossible in minimal $SU(5)$ because the gauge couplings do not meet at a common point. Nevertheless, if one adjust the parameters such that unification is possible, one obtains $M_U \approx 10^{15}$ GeV. The vector leptoquarks would therefore have masses around 10^{15} GeV. That being said, leptoquark masses in this range are inconsistent with limits on proton decay. Moreover, the minimal Yukawa sector yields wrong relations between the masses of quarks and leptons. Minimal $SU(5)$ is therefore not a viable model.

The scalar multiplet H contains both the SM Higgs doublet and a color triplet. The minimal Yukawa sector showed that this color triplet mediates the same proton decay processes as the vector leptoquarks. Minimal $SU(5)$ therefore also contains a scalar leptoquark. To keep the proton from decaying too quickly, its mass is estimated to be at least 10^{11} GeV. The triplet in H must then somehow become much heavier than the doublet; a problem known as the doublet-triplet splitting problem. Minimal $SU(5)$, therefore, does not contain any leptoquarks below 10^{11} GeV.

Chapter 4

SO(10) grand unification

A possible grand unified theory beyond $SU(5)$ is based on $SO(10)$ [18]. In minimal $SU(5)$, the fermions were placed into two different irreps: the 5 and 10. In minimal $SO(10)$, however, all fermions can be placed into a single 16 dimensional spinor representation. $SO(10)$ contains various other GUT groups as its subgroup. One of them is $SU(5)$ [8]. In terms of $SU(5)$ irreps, the spinor representation decomposes as [2]

$$16 = 10 + \overline{5} + 1. \tag{4.1}$$

The 10 and $\overline{5}$ can be recognized from $SU(5)$ and they contain the standard model fermions. But apparently, $SO(10)$ comes with an extra singlet, leaving room for a right handed neutrino. $SO(10)$ also contains the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$ as one of its subgroups [8]. $SO(10)$ is, therefore, a left-right symmetric theory. In the standard model parity is explicitly broken, because left hand and right handed particles transform differently. But because $SO(10)$ is left-right symmetric, parity is only implicitly broken at low energies.

Gauge coupling unification failed in minimal $SU(5)$, partly because it had to be broken to the standard model in on step. Due to the many subgroups of $SO(10)$, the breaking can occur in multiple steps. As we will see, this means that gauge coupling unification can be achieved in $SO(10)$ by means of one or more intermediate scales. Moreover, the resulting unification scale is found to be consistent with proton decay limits.

The first section discusses the spinor representation of $SO(10)$ and how fermions are placed into it. The second section provides an overview of which gauge bosons $SO(10)$ contains and what their associated generators are. Here we also examine which gauge bosons are leptoquarks and whether they can mediate proton decay. Then, the third section discusses how the $SO(10)$ symmetry can be broken to the standard model and we explore the energy scales this yields. Finally, the last section covers the Yukawa sector and the scalar leptoquarks that it contains.

4.1 The spinor representation of SO(10)

The group $SO(10)$ is generated by $10 \cdot 9/2 = 45$ antisymmetric matrices Σ_{ab} satisfying the Lie algebra

$$[\Sigma_{ab}, \Sigma_{cd}] = \delta_{ad}\Sigma_{bc} + \delta_{bc}\Sigma_{ad} - \delta_{ac}\Sigma_{bd} - \delta_{bd}\Sigma_{ac}, \quad a, b, c, d = 1 \ldots 10. \tag{4.2}$$

This generates representations consisting of orthogonal matrices. But there exist more representations that satisfy the Lie algebra in Eq. (4.2). The spinor representation of $SO(10)$ falls into this category [19]. This representation is based on a local isomorphism between $SO(10)$ and its double cover Spin(10). The method used to build this representation is a generalization of the way in which a local isomorphism between $SO(3)$ and $SU(2)$ is established. Before we move on to the spinor representation of $SO(10)$ We first review the local isomorphism between $SO(3)$ and $SU(2)$.

4.1.1 The local isomorphism between SO(3) and SU(2)

Consider a hermitian, traceless 2×2 matrix X. This can be written as a linear combination of the Pauli matrices σ_i:

$$X = \vec{x} \cdot \vec{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}. \tag{4.3}$$

The determinant of X is related to the norm x^2 of $\vec{x}$:

$$\text{det}X = -x_1^2 - x_2^2 - x_3^3 = -x^2 \tag{4.4}$$

Now suppose that X transforms as

$$X \to X' = UXU^\dagger, \tag{4.5}$$

where U is an $SU(2)$ transformation. Since X is hermitian and traceless, so is X'. X' can therefore also be written as

$$X' = \vec{x}' \cdot \vec{\sigma}. \tag{4.6}$$

The norm x'^2 is then related to x^2 by

$$x'^2 = -\text{det}X' = -\text{det}(UXU^\dagger) = -\text{det}X = x^2. \tag{4.7}$$

So under an $SU(2)$ transformation, the norm x^2 is preserved. The vector $\vec{x}$ must then transform according to some orthogonal representation:

$$\vec{x} \to \vec{x}' = O\vec{x}, \tag{4.8}$$

where O is an orthogonal matrix. To find the transformation we use that U can in general be parameterized as

$$U(\phi, \hat{n}) = e^{-i\phi\hat{n}\cdot\vec{\sigma}} = I\cos\frac{\phi}{2} - i\hat{n}\cdot\vec{\sigma}\sin\frac{\phi}{2}. \tag{4.9}$$

Suppose we pick $\hat{n}$ to point in the z direction. In that case

$$U\sigma_1 U^\dagger = \sigma_1\cos\phi + \sigma_2\sin\phi, \tag{4.10}$$

$$U\sigma_2 U^\dagger = -\sigma_1\sin\phi + \sigma_2\cos\phi, \tag{4.11}$$

$$U\sigma_3 U^\dagger = \sigma_3. \tag{4.12}$$

From this it follows that

$$X' = (x_1 \cos\phi - x_2 \sin\phi)\sigma_1 + (x_1 \sin\phi + x_2 \cos\phi)\sigma_2 + x_3\sigma_3. \tag{4.13}$$

Thus, the orthogonal transformation O is given by

$$O(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.14}$$

which is an $SO(3)$ transformation. But notice that in $U(\phi)$ there appears the half angle $\phi/2$, whereas in $O(\phi)$ there is only the full angle ϕ. This means that when ϕ runs from 0 to 4π we cover all $SU(2)$ transformations exactly once, but we cover all $SO(3)$ transformations twice. For this reason $SU(2)$ is said to be the double cover of $SO(3)$. We can then only say that $SU(2)$ and $SO(3)$ are locally isomorphic to each other. This local isomorphism can be made more explicit by using the trace identities for the Pauli matrices:

$$\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}. \tag{4.15}$$

Then the components of $\vec{x}'$ are given by

$$x'_i = \frac{1}{2}\mathrm{Tr}(\sigma_i X') = \frac{1}{2}\mathrm{Tr}(\sigma_i U X U^\dagger) = \frac{1}{2}\mathrm{Tr}(\sigma_i U \sigma_j U^\dagger) x_j. \tag{4.16}$$

Thus, the orthogonal transformation O is related to U through

$$O_{ij} = \frac{1}{2}\mathrm{Tr}(\sigma_i U \sigma_j U^\dagger), \tag{4.17}$$

Both U and $-U$ can be seen to give the same matrix O. Therefore, $SU(2)$ and $SO(3)$ are only locally isomorphic to each other.

4.1.2 Spinor representations

The ideas presented in the previous section can be extended to obtain the spinor representation of $SO(2n)$. Now one looks for $2n$ matrices Γ_i that anticommute with each other [19]:

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}, \quad i,j = 1\ldots 2n. \tag{4.18}$$

For $n = 1$, this is satisfied by the Pauli matrices σ_1 and σ_2. Solutions for higher n can be obtained by iteratively taking tensor products. Suppose one has found $2n$ matrices Γ_i^n that satisfy Eq. (4.18). Then the $2n+2$ matrices [20]

$$\begin{aligned} \Gamma_i^{(n+1)} &= \Gamma_i^n \otimes \sigma_3, \qquad i = 1, 2\ldots 2n \\ \Gamma_{2n+1}^{(n+1)} &= I_{2n\times 2n} \otimes \sigma_1, \\ \Gamma_{2n+2}^{(n+1)} &= I_{2n\times 2n} \otimes \sigma_2, \end{aligned} \tag{4.19}$$

also satisfy Eq. (4.18). So after each iteration, the dimension of the representation is doubled. The dimension of the spinor representation of $SO(2n)$ is therefore equal to 2^n. The generators are obtained by taking commutators [19]:

$$\Sigma_{ij} = -\frac{i}{2}[\Gamma_i, \Gamma_j] = \begin{cases} -i\Gamma_i\Gamma_j, & \text{for } i \neq j, \\ 0, & \text{for } i = j, \end{cases} \tag{4.20}$$

which is antisymmetric in i and j. So in total there are $2n(2n-1)/2$ generators for $SO(2n)$. The resulting representations are reducible into two pieces of equal size. To see this, consider the product of all Γ-matrices:

$$\Gamma_F = (-i)^n \Gamma_1 \Gamma_2 \ldots \Gamma_{2n} = \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3. \tag{4.21}$$

Since Γ_F anticommutes with all Γ_i, it commutes with all the generators. Furthermore, $\Gamma_F^2 = 1$. This allows us to form the projection operators $P_\pm = (1 \pm \Gamma_F)/2$. A general spinor Ψ can therefore be decomposed into two irreducible left- and right-handed spinors as $\Psi = \Psi_L + \Psi_R$, with

$$\Psi_L = P_- \Psi, \qquad \Psi_R = P_+ \Psi. \tag{4.22}$$

The chirality of a spinor is obtained by acting on it with Γ_F: $\Gamma_F \Psi_L = -\Psi_L$ and $\Gamma_F \Psi_R = -\Psi_R$. From the form of Γ_F it follows that it is diagonal and contains just as many 1's as -1's. So the two irreducible spinors have the same size. Therefore, the 2^n dimensional representation decomposes into two irreps of size 2^{n-1}. In some cases, these two irreps are related to each other. Consider the charge conjugation matrix B, which transforms the representation to its complex conjugate:

$$B \sigma_{ij} B^{-1} = -\sigma_{ij}^*. \tag{4.23}$$

Note that in this case, B acts on $SO(2n)$ spinors, and not on Dirac spinors. For $n = 1$ one can take $B_1 = i\sigma_2$. For general n, B_n can be obtained recursively [2]:

$$B_{n+1} = \begin{pmatrix} 0 & B_n \\ (-1)^n B_n & 0 \end{pmatrix} = i\sigma_2 \otimes \sigma_1 \otimes \ldots \otimes i\sigma_2 \otimes \sigma_1. \tag{4.24}$$

Since σ_3 anticommutes with both σ_1 and σ_2, B_n and Γ_F commute if n is even and anticommute if n is odd:

$$\Gamma_F B_n = (-1)^n B_n \Gamma_F. \tag{4.25}$$

and hence,

$$P_\pm B_n = \begin{cases} B_n P_\pm, & \text{for } n \text{ even}, \\ B_n P_\mp, & \text{for } n \text{ odd}. \end{cases} \tag{4.26}$$

Now consider an infinitesimal transformation of Ψ_L:

$$\delta\Psi_L = -\frac{i}{4}\omega_{ij}\Sigma_{ij} P_- \Psi_L. \tag{4.27}$$

Under charge conjugation this becomes

$$\begin{aligned}\delta(B^{-1}\Psi_L^*) &= -\frac{i}{4}\omega_{ij}B^{-1}\Sigma_{ij}^*P_-\Psi_L^* \\ &= \frac{i}{4}\omega_{ij}\begin{cases}\Sigma_{ij}P_-(B^{-1}\Psi_L^*), & \text{for } n \text{ even}, \\ \Sigma_{ij}P_+(B^{-1}\Psi_L^*), & \text{for } n \text{ odd}.\end{cases}\end{aligned} \tag{4.28}$$

For $SO(10)$, $n = 5$ is odd, meaning that a charge conjugated spinor transforms as a spinor with opposite chirality. In other words, the two irreducible representations are each other's conjugates. Hence, the spinor representation of $SO(10)$, which has a dimension of $2^5 = 32$, decomposes as

$$32 = 16 + \overline{16}. \tag{4.29}$$

Sometimes the names 16_L and 16_R, under which ψ_L and ψ_R transform, are used to denote the 16 and $\overline{16}$ representations. When broken to the standard model, the 16 decomposes as [20]

$$\begin{aligned}16 = (3, 2, 1/6) + (1, 2, -1/2) + (\overline{3}, 1, 1/3) \\ + (\overline{3}, 1, -2/3) + (1, 1, 1) + (1, 1, 0),\end{aligned} \tag{4.30}$$

where the transformation properties of Q_L, L_L, $(d^c)_L$, $(u^c)_L$, $(e^c)_L$ and $(\nu^c)_L$, respectively, can be recognized. Then, in the basis defined by Eq. (4.19), the fermions can be inserted into Ψ_L and Ψ_R as follows [20]:

$$\Psi_L = \begin{pmatrix} u_1 \\ \nu_e \\ u_2 \\ u_3 \\ -\nu_e^c \\ -u_1^c \\ -u_2^c \\ -u_3^c \\ d_1 \\ e^- \\ d_2 \\ d_3 \\ e^c \\ d_1^c \\ d_2^c \\ d_3^c \end{pmatrix}_L, \quad \Psi_R = \begin{pmatrix} u_1 \\ \nu_e \\ u_2 \\ u_3 \\ -\nu_e^c \\ -u_1^c \\ -u_2^c \\ -u_b^c \\ d_1 \\ e^- \\ d_2 \\ d_3 \\ e^c \\ d_1^c \\ d_2^c \\ d_3^c \end{pmatrix}_R, \quad \Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}. \tag{4.31}$$

4.2 The gauge sector

In this section we turn to the gauge sector of $SO(10)$, with an emphasis on the leptoquark interactions. First we discuss which gauge bosons $SO(10)$ contains and the generators that correspond to them. We will also note some of the subgroups they generate. The generators can in general be expressed as linear combinations of Σ_{ij}. But unlike in $SU(5)$, where the generators were 5×5 matrices, the generators of the spinor representation are considerably more complex, since they are 32×32 in size. Nevertheless, all the appropriate linear combinations have been identified in [20]. Here we simply state the findings or quote the linear combinations we need. Lastly, we construct the Lagrangian of the leptoquarks to see what kinds of interactions they mediate.

4.2.1 The gauge bosons

We first start with the Pati-Salam subgroup $SU(4)_C \times SU(2)_L \times SU(2)_R$. The $SU(4)_C$ factor has $4^2 - 1 = 15$ generators. These are linear combinations of the generators Σ_{ij}, with $i, j = 1 \dots 6$, which have $6 \cdot 5/2 = 15$ independent generators. Eight of the linear combinations generate an $SU(3)_C$ subgroup and the physical fields corresponding to them are the eight gluons:

$$G_1 \dots G_8, \text{ with generators } U_{G_1} \dots U_{G_8}. \tag{4.32}$$

Six other generators correspond to the fields $X_1 \dots X_3$ and $\overline{X}_1 \dots \overline{X}_3$:

$$X_1 \dots X_3, \ \overline{X}_1 \dots \overline{X}_3, \text{ with generators } U_{X_1} \dots U_{X_6}. \tag{4.33}$$

Later on, we will see that these are leptoquarks. The last generator is proportional to $B - L$, and the corresponding field is denoted X_{B-L}:

$$X_{B-L}, \text{ with generator } U_{B-L}. \tag{4.34}$$

So whereas in $SU(5)$, $B - L$ was an accidental symmetry of the Lagrangian, it is now part of the gauge symmetry.

The $SU(2)_L \times SU(2)_R$ subgroup is generated by the generators with $i, j = 7 \dots 10$. The bosons associated with the $SU(2)_L$ subgroup are the left handed W bosons:

$$W_L^1, W_L^2, W_L^3, \text{ with generators } L_1, L_2, L_3. \tag{4.35}$$

Their right handed counterparts are associated with the $SU(2)_R$ subgroup:

$$W_R^1, W_R^2, W_R^3, \text{ with generators } R_1, R_2, R_3. \tag{4.36}$$

The linear combination corresponding to hypercharge is also given in [20] and turns out to be a combination of $SU(4)_C$ and $SU(2)_R$ generators:

$$Y = \frac{U_{B-L}}{2} + R_3. \tag{4.37}$$

$SO(10)$ also contains $SU(5)$ as a subgroup. As we know from the chapter about $SU(5)$, a subset of the $SU(5)$ generators generates the SM group. These generators are $U_{G_1} \dots U_{G_8}$, $L_1 \dots L_3$ and Y that were mentioned before. As we will see, the gauge bosons associated with the remaining twelve generators are leptoquarks, just as in minimal $SU(5)$. There we called them X and Y bosons, but in the context of $SO(10)$ they are called Y and Y' bosons.

The generators associated with them are denoted as $D_{Y_\alpha}, D_{Y'_\alpha}$ and the generators associated with their antiparticles are $D_{\overline{Y}_\alpha}, D_{\overline{Y}'_\alpha}$:

$$\begin{matrix} Y_1 \dots Y_3,\ \overline{Y}_1 \dots \overline{Y}_3, \\ Y'_1 \dots Y'_3,\ \overline{Y'}_1 \dots \overline{Y'}_3, \end{matrix} \quad \text{with generators } D_{Y_\alpha}, D_{Y'_\alpha}, D_{\overline{Y}_\alpha}, D_{\overline{Y}'_\alpha}. \tag{4.38}$$

There are twelve remaining generators that generate neither the Pati-Salam subgroup nor the $SU(5)$ subgroup. The gauge bosons associated with these generators are new leptoquarks, as we will see, and they are denoted as A and A':

$$\begin{matrix} A_1 \dots A_3,\ \overline{A}_1 \dots \overline{A}_3, \\ A'_1 \dots A'_3,\ \overline{A'}_1 \dots \overline{A'}_3 \end{matrix} \quad \text{with generators } D_{A_\alpha}, D_{A'_\alpha}, D_{\overline{A}_\alpha}, D_{\overline{A}'_\alpha}. \tag{4.39}$$

To identify the leptoquarks, let us see how the gauge bosons transform under the SM group. The 45 decomposes into irreps of the SM group as [20]

$$\begin{aligned} 45 = {} & (8,1,0) + (1,3,0) + (1,1,0) + (3,2,-5/6) + (\overline{3},2,5/6) \\ & + (3,2,1/6) + (\overline{3},2,-1/6) + (3,1,1/3) + (\overline{3},1,-1/3) \\ & + (1,1,1/2) + (1,1,0) + (1,1,-1/2). \end{aligned} \tag{4.40}$$

In the first three irreps we can recognize the transformation properties of the gluons, $W^\pm$ and Z^0 bosons and the photon, respectively. The last three terms, which are all color and isospin singlets, contains the transformation properties of the right handed bosons W^a_R. The states of the remaining six irreps all carry both color and weak isospin/hypercharge. These are the vector leptoquarks of $SO(10)$. The $(3,2,-5/6)$ multiplet also appeared in $SU(5)$ and this is how the Y and Y' bosons transform. The A and A' transform according to the $(3,2,1/6)$ irrep. Finally, the X and $\overline{X}$ bosons can be assigned to the $(3,1,1/3)$ irrep. The transformation properties of the leptoquarks can then be summarized as

$$\begin{gathered} \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y'_1 & Y'_2 & Y'_3 \end{pmatrix} \sim (\overline{3},2,5/6), \quad \begin{pmatrix} A_1 & A_2 & A_3 \\ A'_1 & A'_2 & A'_3 \end{pmatrix} \sim (3,2,1/6), \\ \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} \sim (3,1,1/6), \end{gathered} \tag{4.41}$$

where color indices run horizontally and isospin indices vertically. The antiparticles of the leptoquarks are all assigned to the complex conjugate representations.

4.2.2 The Lagrangian

Now we turn to the Lagrangian of the gauge sector to see which leptoquark couplings it contains. But first we make sure that the generators we use are properly normalized. Conventionally, the generators of $SU(N)$ irreps are normalized to $\mathrm{Tr}[T^a T^b] = \delta^{ab}/2$. But in $SO(10)$, the 32 dimensional spinor representation contains a total of eight color triplets and also eight isospin doublets. So the generators Σ_{ij} should be normalized to $8 \cdot 1/2 = 4$. From the definition of Σ_{ij} it follows that for $i \neq j$

$$\mathrm{Tr}[\Sigma_{ij}^2] = -\mathrm{Tr}[\Gamma_i \Gamma_j \Gamma_i \Gamma_j] = \mathrm{Tr}[\Gamma_i \Gamma_i \Gamma_j \Gamma_j] = \mathrm{Tr}[I_{32\times 32}] = 32. \tag{4.42}$$

The normalization of Σ_{ij} can then be summarized as

$$\mathrm{Tr}[\Sigma_{ij}\Sigma_{kl}] = 32(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{4.43}$$

Hence, the properly normalized generators are $\Sigma'_{ij} = \Sigma_{ij}/2\sqrt{2}$. Having normalized the generators, we can now write the Lagrangian of the gauge sector as

$$\mathcal{L} = \overline{\Psi} i\gamma^{\mu}\left(\partial_{\mu} - igW^{ij}_{\mu}\frac{\Sigma_{ij}}{2\sqrt{2}}\right)\Psi, \tag{4.44}$$

where the gauge boson matrix is [20]

$$\begin{aligned} W^{ij}\frac{\Sigma_{ij}}{2\sqrt{2}} = G\cdot U_G + X\cdot U_X + \sqrt{\frac{3}{8}}X_{B-L}\cdot U_{B-L} \\ + W_L\cdot L + W_R\cdot R + (A_{\alpha}\cdot D_{A_{\alpha}} + Y_{\alpha}\cdot D_{Y_{\alpha}} \\ + A'_{\alpha}\cdot D_{A'_{\alpha}} + Y'_{\alpha}\cdot D_{Y'_{\alpha}} + \text{ h.c.}). \end{aligned} \tag{4.45}$$

Focusing on the leptoquarks special to $SO(10)$, the relevant parts of the gauge boson matrix are those involving X and A bosons. For X bosons this yields the following interaction Lagrangian [20]:

$$\mathcal{L}_X = \frac{g}{\sqrt{2}}(-\overline{d^{c\alpha}_L}\not{X}_{\alpha}e^c_L - \overline{u^{c\alpha}_L}\not{X}_{\alpha}\nu^c_L + \overline{d_{\alpha L}}\overline{\not{X}^{\alpha}}e_L + \overline{u_{\alpha L}}\overline{\not{X}^{\alpha}}\nu_L) + \text{ h.c.}. \tag{4.46}$$

Evidently, X bosons are always coupled to a single quark and $\overline{X}$ bosons to a single anti-quark. Hence, if we were to assign a baryon number of $-1/3$ to X bosons, $\mathcal{L}_X$ conserves baryon number. Proton decay is therefore not mediated by gauge interactions in Pati-Salam models. The other leptoquarks, $A/\overline{A}$ and $A'/\overline{A'}$, do mediate proton decay, as can seen in their interaction Lagrangians [20]:

$$\mathcal{L}_A = \frac{g}{\sqrt{2}}(-\epsilon_{\alpha\beta\gamma}\overline{d_{\alpha L}}\not{A}_{\beta}d^c_{\gamma L} + \overline{\nu^c_{eL}}\not{A}_{\alpha}u^{\alpha}_L - \overline{u^{c\alpha}_L}\not{A}_{\alpha}\nu_{eL}) + \text{ h.c.}, \tag{4.47a}$$

$$\mathcal{L}_{A'} = \frac{g}{\sqrt{2}}(-\epsilon_{\alpha\beta\gamma}\overline{u_{\alpha L}}\not{A}'_{\beta}d^c_{\gamma L} + \overline{\nu^c_{eL}}\not{A}'_{\alpha}d^{\alpha}_L - \overline{u^{c\alpha}_L}\not{A}'_{\alpha}e_L) + \text{ h.c.} \tag{4.47b}$$

The possible proton decay diagrams are shown in Figure 4.1. More diagrams can be generated for A' bosons than for A bosons, because A bosons mostly mediate neutron decay.

4.3 Symmetry breaking and gauge coupling unification

As $SU(5)$ was the smallest candidate for grand unification, it contained no subgroups that could act as an intermediate stage. Consequently, the running of the couplings is determined solely by low energy parameters. And as it has turned out, the standard model does not automatically unify. To truly achieve gauge coupling unification at the GUT scale, one would either have to add particles at the electroweak scale or implement at least one intermediate symmetry. In contrast to $SU(5)$, $SO(10)$ contains several maximal subgroups that can serve as intermediate symmetries, so $SO(10)$ can be broken to the standard model in multiple stages. These maximal subgroups are $SU(5)\times U(1)$ and the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$ [8]. Using an intermediate $SU(5)\times U(1)$ symmetry scale would, however, lead to a prediction for the proton's lifetime that is even lower than that of minimal $SU(5)$ [8]. An $SU(5)$ intermediate scale in $SO(10)$ is therefore impossible. Consequently, in this section we only consider symmetry breaking patterns with an intermediate Pati-Salam scale.

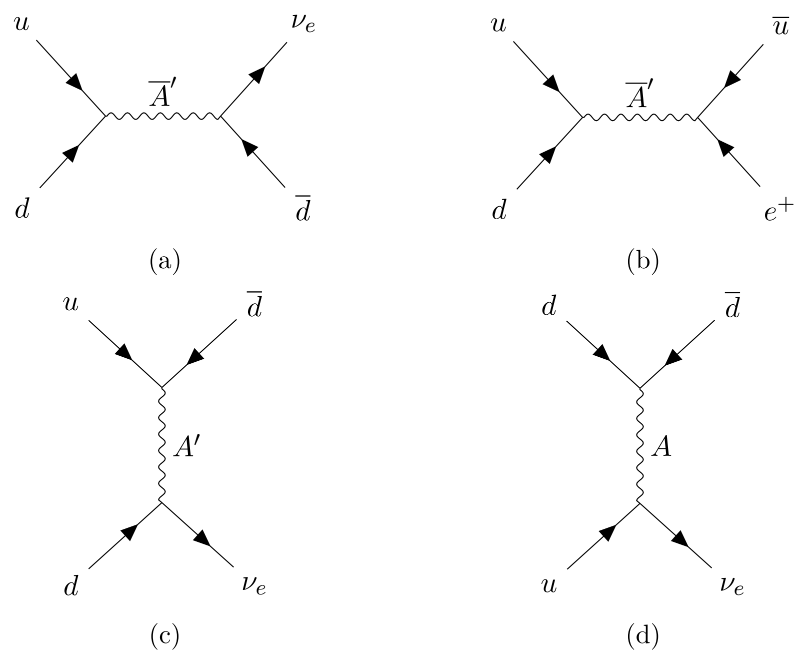

Figure 4.1: Proton decay diagrams in $SO(10)$, mediated by A and A' bosons.

4.3.1 A single intermediate Pati-Salam symmetry scale

Including an intermediate Pati-Salam scale allows for many symmetry breaking patterns. The simplest possibility is a breaking mechanism where the Pati-Salam scale is the only intermediate scale. Then all leptoquarks that were found to mediate proton decay, lie at the GUT scale. The ones associated to the Pati-Salam subgroup, which do not mediate proton decay, lie at the intermediate scale. Two candidate scalar representations that break the symmetry to this scale are a 54 and a 210 [21]. If the 54 is used, this also leaves a $\mathbb{Z}_2$ symmetry that forces the $SU(2)_L$ and $SU(2)_R$ couplings to be equal [22, 23]. This symmetry is more commonly referred to as D-parity. However, this leads to a unification scale of $\sim 10^{15}$ GeV [22, 23], which is in disagreement with proton decay limits. The 210, on the other hand, breaks D-parity, leading to a higher unification scale consistent with proton decay limits [22]. The next step, where the symmetry is broken to the standard model, can be performed by a 126 Higgs [22, 23]. This representation also plays a role in the Yukawa sector. Finally, the symmetry is broken further to $SU(3)_C \times U(1)_Q$ by a 10 Higgs. Under $SU(3)_C \times SU(2)_L \times U(1)_Y$ this decomposes as

$$10 = (1,2,1/2) + (1,2,-1/2) + (3,1,-1/3) + (\overline{3},1,1/3). \tag{4.48}$$

The (1, 2, 1/2) contains the standard model Higgs, so it must stay at the electroweak scale. To summarize, the complete breaking mechanism is

$$\begin{aligned} SO(10) &\xrightarrow{210} SU(4)_C \times SU(2)_L \times SU(2)_R \\ &\xrightarrow{126} SU(3)_C \times SU(2)_L \times U(1)_Y \\ &\xrightarrow{10} SU(3)_C \times U(1)_Q. \end{aligned} \tag{4.49}$$

The values of the intermediate and unification mass scales follow from gauge coupling unification. In addition to the RGEs, there are several matching conditions that need to be imposed. Continuity requires that at the Pati-Salam scale,

$$\alpha_{3C}(M_I) = \alpha_{4C}(M_I), \tag{4.50}$$

$$\alpha_{2L}(M_I) = \alpha'_{2L}(M_I). \tag{4.51}$$

Note that α_{2L} is associated with the $SU(2)_L$ symmetry at the electroweak scale, while α'_{2L} corresponds to the $SU(2)_L$ symmetry at the intermediate scale. In general, whenever a symmetry group appears multiple times in a breaking pattern, we add primes to the fine structure constant and the β-coefficient of that group at higher scales.

At the Pati-Salam scale $U(1)_Y$ is embedded into $SU(4)_C \times SU(2)_R$ through Eq. (4.37). To impose boundary conditions in this case, we must work with normalized generators. In terms of the generators Σ_{ij}, the expressions for Y and U_{B-L} are [20]

$$U_{B-L} = \frac{1}{3}(\Sigma_{12} + \Sigma_{34} + \Sigma_{65}), \tag{4.52}$$

$$Y = \frac{1}{6}(\Sigma_{12} + \Sigma_{34} + \Sigma_{65}) + \frac{1}{4}(\Sigma_{78} + \Sigma_{10,9}). \tag{4.53}$$

From Eq. (4.43), their square traces are found to be 32/3 and 20/3, respectively. So the normalized generators are

$$U'_{B-L} = \sqrt{\frac{3}{8}}U_{B-L}, \quad Y' = \sqrt{\frac{3}{5}}Y. \tag{4.54}$$

In the terms of normalized generators, the epxression for the hypercharge generator is then

$$Y' = \sqrt{\frac{2}{5}}U'_{B-L} + \sqrt{\frac{3}{5}}R'_3. \tag{4.55}$$

U'_{B-L} is a generator of $SU(4)_C$ and R_3 is a generator of $SU(2)_R$. Hence, the matching condition is

$$\alpha_Y^{-1}(M_I) = \frac{2}{5}\alpha_{4C}^{-1}(M_I) + \frac{3}{5}\alpha_{2R}^{-1}(M_I). \tag{4.56}$$

Finally, at the unification scale M_U we have

$$\alpha_{4C}(M_U) = \alpha_{2L}(M_U) = \alpha_{2R}(M_U). \tag{4.57}$$

The values of the fine structure constants and the β-coefficients at the electroweak scale were given in section 2.4. The β-coefficients between the scales M_I and M_U are [17]

$$b'_{2L} = 2, \quad b_{2R} = 26/3, \quad b_{4C} = -7/3. \tag{4.58}$$

The total system of equations then consists of five matching conditions and five unknowns: $\alpha_{4C}(M_I), \alpha'_{2L}(M_I), \alpha_{2R}(M_I), M_I$ and M_U. This yields a unique solution for the intermediate and unification scales:

$$M_I \approx 3.1 \cdot 10^{11} \text{ GeV}, \tag{4.59}$$

$$M_U \approx 2.5 \cdot 10^{16} \text{ GeV}. \tag{4.60}$$

The full evolution of the fine structure constants is shown in Figure 4.2. Judging by the order of magnitude of M_I, no new physics is to be expected around the TeV scale.

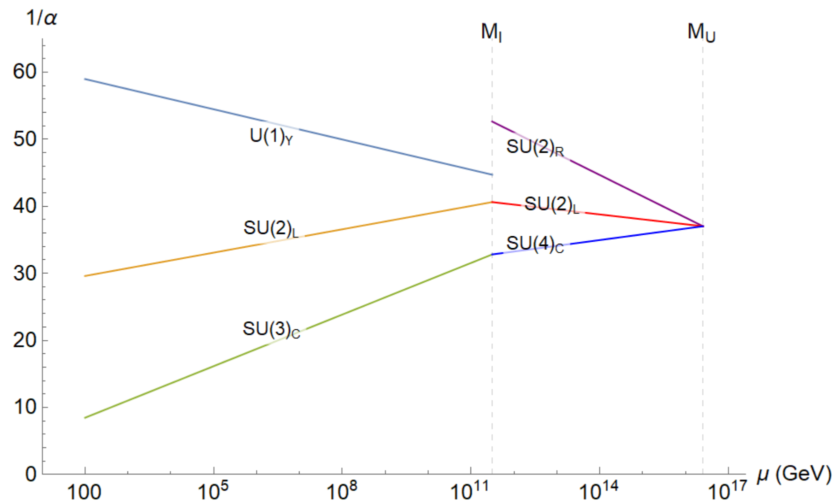

Figure 4.2: Running of the fine structure constants in $SO(10)$ from the electroweak scale up to the unification scale with one intermediate Pati-Salam scale.

4.3.2 Three intermediate symmetry scales

To lower the scale at which new physics occurs, multiple intermediate stages can be implemented. One such model is discussed in [24]. Here, the symmetry is first broken to a Pati-Salam symmetry with D-parity by a 54 Higgs. A 210 Higgs then breaks D-parity at an intermediate scale M_{D_P}. The Pati-Salam symmetry is broken again by a 210 Higgs to an asymmetric left-right symmetry based on $SU(3)_C \times SU(2)_L \times U(1)_R \times U(1)_{B-L}$. At this stage the right-handed W_R boson obtains its mass, with mass scale M_{W_R}. Since the $SU(4)_C$ symmetry is broken to $SU(3)_C$, this also sets the mass scale of the leptoquarks $X_1 \dots X_3$ and $\overline{X}_1 \dots \overline{X}_3$. A 16 Higgs breaks the symmetry to the standard model, at which point the right-handed Z_R obtains its mass, with mass scale M_{Z_R}. As before, the final breaking stage to $SU(3)_C \times U(1)_Q$ is performed by a 10 Higgs. The total breaking mechanism thus contains three intermediate stages:

$$\begin{aligned} SO(10) &\xrightarrow{54} SU(4)_C \times SU(2)_L \times SU(2)_R \times D \\ &\xrightarrow{210} SU(4)_C \times SU(2)_L \times SU(2)_R \\ &\xrightarrow{210} SU(3)_C \times SU(2)_L \times U(1)_R \times U(1)_{B-L} \\ &\xrightarrow{16} SU(3)_C \times SU(2)_L \times U(1)_Y \\ &\xrightarrow{10} SU(3)_C \times U(1)_Q. \end{aligned} \tag{4.61}$$

The β-coefficients in the energy range between M_{Z_R} and M_U are summarized in Table 4.1.

Several more matching conditions now dictate the evolution of the coupling constants. Eq. (4.56) turns into

$$\alpha_Y^{-1}(M_{Z_R}) = \frac{3}{5}\alpha_{B-L}^{-1}(M_{Z_R}) + \frac{2}{5}\alpha_{1R}^{-1}(M_{Z_R}), \tag{4.62}$$

Energy range	Symmetry	b_i
$M_{Z_R} - M_{W_R}$	$SU(3)_C \times SU(2)_L$ $\times U(1)_R \times U(1)_{B-L}$	$b'_{3C} = -7$ $b_{2L} = -3$ $b_{1R} = 53/12$ $b_{B-L} = 33/8$
$M_{W_R} - M_{D_P}$	$SU(4)_C \times SU(2)_L$ $\times SU(2)_R$	$b_{4C} = -19/3$ $b''_{2L} = -8/3$ $b_{2R} = 8$
$M_{D_P} - M_U$	$SU(4)_C \times SU(2)_L$ $\times SU(2)_R \times D$	$b_{4C} = -2$ $b_{2LR} = 8$

Table 4.1: β-coefficients in the energy range between M_{Z_R} and M_U. The values were taken from [24].

and continuity at $\mu = M_{Z_R}$ requires that

$$\alpha_{2L}(M_{Z_R}) = \alpha'_{2L}(M_{Z_R}), \tag{4.63}$$
$$\alpha_{3L}(M_{Z_R}) = \alpha'_{3L}(M_{Z_R}). \tag{4.64}$$

At $\mu = M_{W_R}$, $U(1)_{B-L}$ and $SU(3)_C$ are embedded into $SU(4)_C$ and $U(1)_R$ into $SU(2)_R$, resulting in the conditions

$$\alpha_{1R}(M_{W_R}) = \alpha_{2R}(M_{W_R}), \tag{4.65}$$
$$\alpha'_{2L}(M_{W_R}) = \alpha''_{2L}(M_{W_R}), \tag{4.66}$$
$$\alpha_{B-L}(M_{W_R}) = \alpha'_{3C}(M_{W_R}) = \alpha_{4C}(M_{W_R}). \tag{4.67}$$

At the scale at which D-parity holds, the couplings for $SU(2)_L$ and $SU(2)_R$ must be equal, resulting in the conditions:

$$\alpha''_{2L}(M_{D_P}) = \alpha_{2R}(M_{D_P}) = \alpha_{D_P}(M_{D_P}), \tag{4.68}$$
$$\alpha_{4C}(M_{D_P}) = \alpha'_{4C}(M_{D_P}). \tag{4.69}$$

Finally, at the unification scale

$$\alpha'_{4C}(M_U) = \alpha_{2LR}(M_U). \tag{4.70}$$

In total there are thirteen parameters and eleven conditions, leaving two free parameters. In the current model it is therefore possible that M_{Z_R} lies at the TeV scale. As in [24], taking $M_{Z_R} = 5$ TeV and $M_{W_R} = 10^{8.3}$ GeV results in

$$M_U \approx 1.6 \cdot 10^{16} \text{ GeV}, \tag{4.71}$$
$$M_{D_P} \approx 4.3 \cdot 10^{15} \text{ GeV}, \tag{4.72}$$
$$\alpha_U^{-1} \approx 42.2. \tag{4.73}$$

The running of the couplings is shown in Figure 4.3.

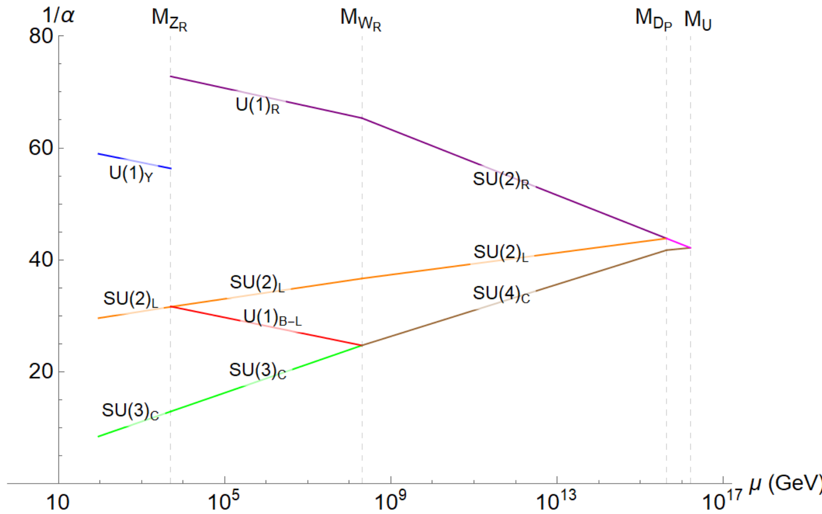

Figure 4.3: Running of the fine structure constants in $SO(10)$ from the electroweak scale up to the unification scale with three intermediate scales.

4.3.3 Comparison to experimental limits

To find out whether the scenarios discussed in the previous sections are viable, we need to compare the predicted proton lifetime to the experimental limit $\tau_p > 2 \cdot 10^{34}$y [16]. From Eq. (3.74) it followed that we roughly need $M_U > 10^{16}$ GeV. Strictly speaking, the predicted proton lifetime also depends on α_U^{-1}, but the value of α_U^{-1} typically does not change much (compare it to $SU(5)$, for which very similar values were found).

First we compare the estimated lower bound on M_U with the scenario where there is one intermediate Pati-Salam scale. In this scenario, the X leptoquarks lie at the intermediate scale of roughly 10^{11} GeV. These leptoquarks do not mediate proton decay, so their relatively low masses are not an issue. The A/A' and Y/Y' leptoquarks, which do mediate proton decay, lie the unification scale of roughly 10^{16} GeV. Thus, the scenario with one intermediate scale is not excluded.

Next we compare the lower bound to the scenario with three intermediate scales. Now, the X leptoquarks lie at the scale M_{W_R} at which the right handed W_R boson obtains its mass. The A/A' and Y/Y' leptoquarks still lie at the unification scale. In this scenario, imposing gauge coupling unification leaves two free parameters. In particular, this means that we can vary the two lowest intermediate scales: M_{Z_R} and M_{W_R}. In Eq. (4.71) it can be seen that the values for M_{Z_R} and M_{W_R} from [24] yield a prediction for the unification scale that is right at the estimated. So this scenario is not excluded.

However, there might be more values for M_{Z_R} and M_{W_R} that are consistent with the experimental limit. Figure 4.4 shows M_U as a function of M_{Z_R} for several values of M_{W_R} ranging between 10^4 GeV and 10^{14} GeV. A noteworthy feature is that M_U increases with M_{Z_R} but decreases with M_{W_R}. Thus, the proton decay limit only offers an upper bound on M_{W_R}, meaning that the masses of the X leptoquarks could in principle be low. From the graph it can be seen that M_{W_R} cannot exceed roughly 10^9 GeV in order to keep M_U above 10^{16} GeV. Lower bounds on the mass of the W_R boson come from experimental searches. For instance, the CMS experiment at CERN LHC excluded the mass range below 5 TeV [25]. So in the scenario where there are three intermediate scales, Tev scale leptoquarks are not excluded. An example of a scenario where M_{W_R} lies at the TeV scale is shown in Figure 4.5. Here we took $M_{Z_R} = 5$ TeV and $M_{W_R} = 20$ TeV. This results in $M_U \approx 2 \cdot 10^{17}$ GeV, which is well above the lower bound. As far as we are aware, scenarios like this have not yet been explored in any existing literature.

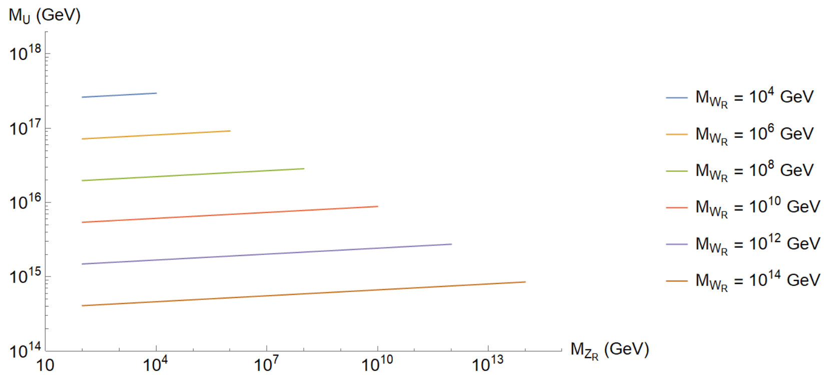

Figure 4.4: The unification scale M_U as a function of M_{Z_R} for several values of M_{W_R}. For each value of M_{W_R}, M_{Z_R} ranges between M_{EW} and M_{W_R}.

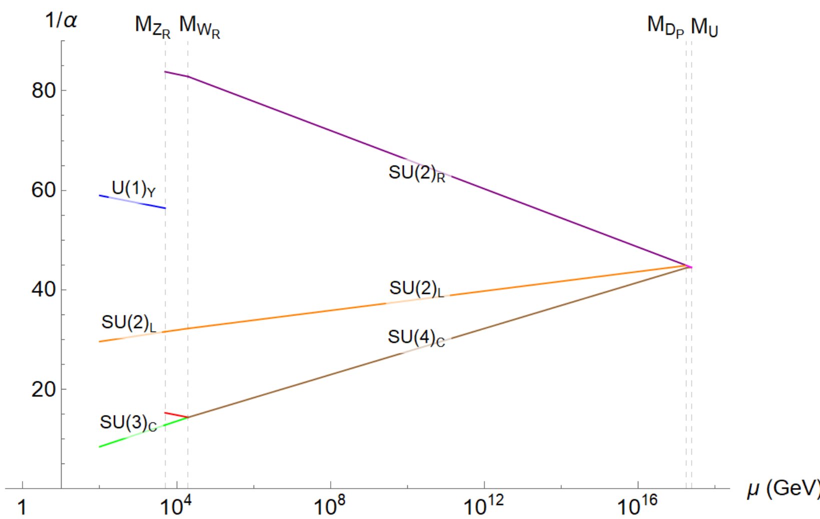

Figure 4.5: Example of a scenario where M_{W_R} lies at the TeV scale.

4.4 Scalar leptoquarks

In this section we will see that aside from vector leptoquarks, minimal $SO(10)$ also contains many scalar leptoquarks. A mass term takes the form $m(\psi^c)_L C \psi_L +$ h.c., where C is the Dirac charge conjugation matrix. So to obtain masses for the fermions, we need to couple left handed $SO(10)$ spinors to each other:

$$\mathcal{L}_Y = \Psi_L^T BC\Phi\Psi_L + \text{ h.c.} \tag{4.74}$$

Here, Φ is some yet to be determined Higgs field and B is the $SO(10)$ charge conjugation matrix introduced earlier. B is inserted because the combination $\Psi_L^T B$ transforms into $\Psi_L^T BU_R^\dagger$. To see this, consider a transformation U under which $\Psi \to U\Psi$. In that case the left handed $SO(10)$ spinor Ψ_L transforms like $\Psi_L \to U_L\Psi_L$, where $U_L = P_+U$. Then from Eq. (4.26) it follows that

$$(\Psi_L^T BC\Phi\Psi_L)' = \Psi_L^T U_L^T BC\Phi' U_L\Psi_L = \Psi_L^T BCU_R^\dagger \Phi' U_L\Psi_L. \tag{4.75}$$

To make this invariant under $SO(10)$ we need that Φ transforms as

$$\Phi' = U_R\Phi U_L^\dagger. \tag{4.76}$$

In other words, Φ is in the $16_R \times \overline{16}_L$ representation. But since the 16_L and 16_R representations are each other's conjugates, this is just the $\overline{16} \times \overline{16}$ representation of $SO(10)$, whose decomposition into irreps is [17]

$$\overline{16} \times \overline{16} = 10 + 120 + \overline{126}. \tag{4.77}$$

Two of these irreps, the 10 and the 126, already appeared in the symmetry breaking patterns. Apparently, the 120 can also be used to generate masses for the fermions. The most minimal choice would be to only use the smallest representation: the 10. But just like in $SU(5)$ this leads to wrong mass relations. At least one other irrep is necessary to obtain the correct fermion masses [26]. Thus, in general the Yukawa sector (the matrices B and C will be left out) takes the form

$$\mathcal{L}_Y = Y_{10}\Psi_L^T\Phi_{10}\Psi_L + Y_{120}\Psi_L^T\Phi_{120}\Psi_L + Y_{\overline{126}}\Psi_L^T\Phi_{\overline{126}}\Psi_L + \text{ h.c.}, \tag{4.78}$$

where Y_{10}, Y_{120} and Y_{126} are Yukawa matrices. Since the Higgs representations used are rather large, it is no surprise that many scalar leptoquarks are contained in them. This opens up the possibility for scalar-mediated proton decay in $SO(10)$.

Nevertheless, this type of proton decay is generally suppressed compared to those mediated by vector leptoquarks, because the Yukawa coupling of the first generation is relatively small: $Y_u/g \approx 10^{-4}$ [27]. This holds provided that the scalar and vector leptoquarks have similar mass scales. In principle, the masses of Higgs particles have a lot of freedom. A simple rule that is often used is the extended survival hypothesis [28], which says that only those Higgs fields that obtain a vev at some scale, obtain a mass at that scale. All other fields stay at the grand unification scale. Since leptoquarks necessarily carry color, none of them are allowed to obtain vevs. All leptoquarks are therefore assumed to obtain masses at the GUT scale. Consequently, these particles are at least as heavy as the vector leptoquarks.

In this section, the scalar leptoquarks appearing in each representation are discussed, based on the analysis in [27]. The decomposition of the 10 under the standard model group is given in Eq. (4.48). It contains two triplets carrying both color and hypercharge:

$$T^\alpha = (3, 1, -1/3), \qquad \overline{T}_\alpha = (\overline{3}, 1, 1/3). \tag{4.79}$$

These are the only leptoquarks in the 10. Their Lagrangian takes the form [27]

$$\begin{aligned}\mathcal{L}_{\mathrm{LQ}}^{10} \sim u^c_{\alpha L} T^\alpha e^c_L + \frac{1}{2}\epsilon_{\alpha\beta\gamma} u_L^{\alpha T} T^\beta d_L^\gamma - \frac{1}{2}\epsilon_{\alpha\beta\gamma} d_L^\alpha T^\beta u_L^\gamma \\ - \epsilon^{\alpha\beta\gamma} u^c_{\alpha L} \overline{T}_\beta d^c_{\gamma L} - u^\alpha_L \overline{T}_\alpha e_L + d^\alpha_L \overline{T}_\alpha \nu_{eL} + \text{ h.c.}\end{aligned} \tag{4.80}$$

Since T^α and $\overline{T}_\alpha$ have both di-quark couplings and direct lepto-quark couplings, they can mediate proton decay. Unlike the 10, the sizes of the 120 and 126 representations make them much less tangible. Generally, these representations are first decomposed in terms of $SU(5)$ irreps [29]. Here we discuss the existence of leptoquarks in terms of these irreps. The 126 decomposes as

$$126 = 1 + \overline{10} + 15 + 5 + \overline{45} + 50. \tag{4.81}$$

The one-dimensional irrep involves couplings to the right-handed neutrino, so it is irrelevant for proton decay. The $\overline{10}$ and 15 contain leptoquarks, but they conserve baryon number. The 5, $\overline{45}$ and 50 all contain a leptoquark with the same couplings as T^α, so they mediate proton decay. Similarly, the 120 decomposes as

$$120 = 5 + \overline{5} + 10 + \overline{10} + 45 + \overline{45}. \tag{4.82}$$

In this case the 5 does not couple to quarks. The triplet in the $\overline{5}$ however, has the same couplings as $\overline{T}_\alpha$. Leptoquarks can be found in both the 10 and $\overline{10}$, but none of them violate baryon number. The 45 and $\overline{45}$ contain two leptoquarks. They have di-quark couplings in the 45 and their conjugates have lepto-quark couplings in the $\overline{45}$, so they contribute to proton decay.

4.5 Conclusions

$SO(10)$ extends $SU(5)$ by including all fermions into a single 16-dimensional spinor representation. Since the SM only contains fifteen fermions, this means that $SO(10)$ predicts the existence of one extra fermion. This extra state transforms as a singlet under the SM group, so it may correspond to the right handed neutrino.

Two important subgroups of $SO(10)$ are the Pati-salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$ and $SU(5)$. The Pati-Salam subgroup has vector leptoquarks associated with it, which are denoted as X bosons. Their Lagrangian conserves baryon number, so they do not mediate proton decay. The $SU(5)$ subgroups also has vector leptoquarks associated with it, which are denoted as Y/Y' bosons. These bosons, which originally appeared in minimal $SU(5)$, do mediate proton decay. Finally, there are vector leptoquarks that are associated with neither the Pati-Salam subgroup nor the $SU(5)$ subgroup. These are denoted as A/A' bosons and they mediate proton decay.

Since $SO(10)$ contains many subgroups which in turn contain the SM group, the existence of one or more intermediate symmetry scales is allowed. The option where $SO(10)$ is first broken to $SU(5)$ is ruled out, as it would lead to a prediction for the proton's lifetime that is inconsistent with experimental data. The other option is to include an intermediate Pati-Salam scale. If only one intermediate scale is included, this results in $M_I \approx 10^{11}$ GeV and

$M_U \approx 10^{16}$. The X leptoquarks would therefore have a mass around 10^{11} GeV, while the Y/Y' and A/A' leptoquarks would lie around 10^{16} GeV.

If more intermediate stages are included, the scale at which new physics occurs can be lowered. One such model includes two additional scales M_{Z_R} and M_{W_R}, at which the right-handed Z_R and W_R bosons obtain masses. In this scenario M_{W_R} is also the scale at which the X leptoquarks lie, while the Y/Y' and A/A' leptoquarks still lie at the unification scale. Imposing gauge coupling unification, does not fix the values of M_{Z_R} and M_{W_R}. The only constraints stem from experimental measurements, such as proton decay limits and lower bounds on the W_R mass. This allows for the possibility that both M_{Z_R} and M_{W_R} lie at the TeV scale. Thus, the scenario with three intermediate scales allows for TeV scale vector leptoquarks.

The Yukawa sector consists of those scalar multiplets appearing in the tensor product $\overline{16} \times \overline{16} = 10 + 120 + \overline{126}$. Many leptoquarks are contained in these multiplets and some of them mediate proton decay. The ones that do not mediate proton decay could, in principle, be light. However, in accordance with the extended survival hypothesis, the masses of all scalar leptoquarks are generally believed to lie at the unification scale of at least 10^{16} GeV.

Chapter 5

Trinification

Both $SU(5)$ and $SO(10)$ GUTs were based on simple groups in which quarks and leptons appeared in the same irreducible representations. This led to gauge mediated proton decay, which has not been observed yet. In trinification this is avoided by asserting that the gauge group is $G_{333} = SU(3)_C \times SU(3)_L \times SU(3)_R$ [22]. The fermions are then placed in the fundamental 27 representation of the exceptional group E_6, which has G_{333} as one of its subgroups. As we will see, this means that quarks and leptons appear in different irreducible representations, which forbids gauge mediated proton decay. Scalar mediated proton decay, on the other hand, is possible. Since gauge-mediated proton decay is generally more important, this means that a lower unification scale is acceptable.

The three factor groups of the trinification model have three separate couplings g_C, g_L and g_R. Gauge coupling unification is ensured by imposing an additional $\mathbb{Z}_3$ symmetry that interchanges quarks with leptons, leptons with antiquarks and antiquarks with quarks [30]. Since $\mathbb{Z}_3$ is a finite and discrete group, no gauge bosons are associated with it.

An additional advantage of trinification is its left-right symmetry. In the standard model the asymmetry between left and right handed fields has to be inserted manually. So there is no natural explanation for parity violation. Trinification suggests that this assymmetry is a result of spontaneous symmetry breaking and only appears at low energies.

The setup of this chapter is as follows. The first section discusses how fermions are placed into the 27 representation and how we can deal with their transformations under G_{333}. The next section examines the gauge sector and shows that it conserves baryon number. Section 3 focuses on how the symmetry can be broken to the standard model and which energy scales are possible. We then turn our attention to the mass scales of fermions and their mixings within a generation. The final section discusses the scalar leptoquarks of the trinfication model and the consequences they have for proton decay.

5.1 The 27 representation of E_6

The fermions are placed in the fundamental 27 representation of E_6 [31]. This introduces twelve new fermions alongside the fifteen standard model fermions. The 27 decomposes into three irreps of G_{333}:

$$27 = (1, 3, \overline{3}) + (\overline{3}, 1, 3) + (3, \overline{3}, 1) = \psi_l + \psi_{q^c} + \psi_q. \tag{5.1}$$

Each multiplet is a cyclic permutation of others, in accordance with the $\mathbb{Z}_3$ symmetry. The fermions now have to be assigned, such that they possess the right transformation properties under the SM group. The understanding of these transformations can be simplified by noting that each multiplet is a direct product of a fundamental and an antifundamental representation of $SU(3)$. The generators of the three $SU(3)$ subgroups are denoted T^a_C, T^a_L and T^a_R. For the fundamental representation, these can be expressed in terms of the Gell-man matrices as $T^a_C = T^a_L = T^a_R = \lambda^a/2$, where

$$\begin{aligned}
&\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
&\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{5.2}$$

Note that the normalization of the generators is such that $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$. For the $\overline{3}$ representation, the generators are $\overline{T}^a_{C,L,R} = -(T^a_{C,L,R})^* = -(T^a_{C,L,R})^T$.

To illustrate how a general G_{333} transformation can be parameterized, we focus on the irrep $\psi_l \sim (1, 3, \overline{3})$. It transforms according to the direct product of the 3 of $SU(3)_L$ and the $\overline{3}$ of $SU(3)_R$. ψ_l thus consists of $3 \cdot 3 = 9$ states ψ_l^{ij}, $i, j = 1, 2, 3$. We use the convention from [22], where fundamental representations act on the first index and antifundamental representations act on the second. Then, given a 3 transformation $U_L \in SU(3)_L$ and a 3 transformation $U_R \in SU(3)_R$, ψ_l transforms as

$$\psi_l \to \psi_l' = U_L \psi_l U_R^\dagger \tag{5.3}$$

$$= \psi_l - i\alpha_a T^a_L \psi_l - i\beta_a \psi_l (\overline{T}^a_R)^T + \dots \tag{5.4}$$

$$= \psi_l - i\alpha_a T^a_L \psi_l + i\beta_a \psi_l T^a_R + \dots . \tag{5.5}$$

Note that we took the hermitian conjugate of U_R to obtain the $\overline{3}$ of $SU(3)_R$. The transformation properties of the other irreps, ψ_{q^c} and ψ_q, are obtained by cyclically permuting L, R and C. The action of a general G_{333} transformation on the fermion multiplets can then be parameterized as

$$\begin{aligned}
\psi_l \to \psi_l' &= U_L \psi_l U_R^\dagger = \psi_l - i\alpha_a T^a_L \psi_l - i\beta_a \psi_l (\overline{T}^a_R)^T + \dots, \\
\psi_{q^c} \to \psi_{q^c}' &= U_R \psi_{q^c} U_C^\dagger = \psi_{q^c} - i\beta_a T^a_R \psi_{q^c} - i\gamma_a \psi_{q^c} (\overline{T}^a_C)^T + \dots, \\
\psi_q \to \psi_q' &= U_C \psi_q U_L^\dagger = \psi_q - i\gamma_a T^a_C \psi_q - i\alpha_a \psi_q (\overline{T}^a_L)^T + \dots
\end{aligned} \tag{5.6}$$

Since we are dealing with direct product representations, it will be useful to introduce upper and lower indices. The convention for upper and lower indices is similar to the one discussed in section 2.1: fundamental representations are indicated by an upper index and antifundamental representations are indicated by a lower index. Since all G_{333} irreps contained in the

27 are direct products of a 3 and a $\overline{3}$ representation, each multiplet will have an upper and a lower index. The indices on the multiplets are then written as $(\psi_l)^i_j$, $(\psi_{q^c})^i_j$ and $(\psi_q)^i_j$. But note that whether an index refers to the $3(\overline{3})$ of $SU(3)_C$, $SU(3)_L$ or $SU(3)_R$, depends on the multiplet. For example, for ψ_l an upper index refers to the 3 of $SU(3)_L$ and a lower index refers to the $\overline{3}$ of $SU(3)_R$.

The transformation properties under the SM subgroup can be obtained if we restrict to the appropriate generators. The $SU(3)_C$ subgroup is generated by $T^1_C \ldots T^8_C$ and the $SU(2)_L$ subgroup by T^1_L, T^2_L and T^3_L. So from Eq. (5.6) we see that each column in ψ_l transforms as a color singlet, whereas in each row the first two entries form an isospin doublet and the last is a singlet. The rows in ψ_{q^c} are color antitriplets and isospin singlets. And finally, the columns in ψ_q are color triplets and the rows contain an isospin (anti) doublet and a singlet. To summarize, the decomposition of the irreps under $SU(3)_C \times SU(2)_L$ is

$$\psi_l \to 3(1,2) + 3(1,1), \tag{5.7}$$
$$\psi_{q^c} \to 3(\overline{3},1), \tag{5.8}$$
$$\psi_q \to (3,\overline{2}) + (3,1). \tag{5.9}$$

The hypercharge quantum numbers are obtained from the following combination of the generators (see appendix A):

$$Y = T^3_R - \frac{1}{\sqrt{3}}\left(T^8_L + T^8_R\right). \tag{5.10}$$

In expressions like these, $T^a_{C,L,R}$ is shorthand for whichever generator corresponds to the $SU(3)_{C,L,R}$ representation that a given multiplet is in. So whenever it is in the $\overline{3}$, the generators $\overline{T}^a_{C,L,R}$ are used or the contribution disappears if it is in the trivial representation. Moreover, the index that each generator acts on still follows the convention mentioned before. The action of Y on the different multiplets is then given explicitly by

$$Y\psi_l = -\frac{1}{\sqrt{3}}T^8_L\psi_l + \psi_l\left(\overline{T}^3_R - \frac{1}{\sqrt{3}}\overline{T}^8_R\right), \tag{5.11}$$
$$Y\psi_{q^c} = \left(T^3_R - \frac{1}{\sqrt{3}}T^8_R\right)\psi_{q^c}, \tag{5.12}$$
$$Y\psi_q = -\frac{1}{\sqrt{3}}\psi_q\overline{T}^8_L. \tag{5.13}$$

From this we can calculate the hypercharge quantum numbers of the states in each multiplet:

$$Y(\psi_l) = \begin{pmatrix} -1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y(\psi_{q^c}) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & -2/3 & -2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}, \\ Y(\psi_q) = \begin{pmatrix} 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \end{pmatrix}. \tag{5.14}$$

The expression for the electric charge operator is (see appendix A)

$$Q = T^3_L + Y = T^3_L + T^3_R - \frac{1}{\sqrt{3}}\left(T^8_L + T^8_R\right). \tag{5.15}$$

The electric charges of the states are then

$$Q(\psi_l) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q(\psi_{q^c}) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & -2/3 & -2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix},$$
$$Q(\psi_q) = \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \end{pmatrix}. \tag{5.16}$$

Having determined the $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ quantum numbers of the states, we can decompose the 27 into irreps of the SM group as follows:

$$\begin{aligned} \psi_l &\to (1,2,1/2) + 2(1,2,-1/2) + (1,1,1) + 2(1,1,0), \\ \psi_{q^c} &\to (\overline{3},1,-2/3) + 2(\overline{3},1,1/3), \\ \psi_q &\to (3,\overline{2},1/6) + (3,1,-1/3). \end{aligned} \tag{5.17}$$

We can now determine how the fermions can be placed in irreps [31]. In the decomposition of ψ_l we can recognize two lepton doublets with the same transformation properties as the doublet L_L from the standard model. However, we do not directly assign L_L to one of the irreps. This is because, as we will later see, the vev of some of the scalar fields that are responsible for generating masses, can in general be non-diagonal. As a result, some of states contained in ψ_l, ψ_{q^c} and ψ_q that we use, will mix with each other and are therefore not mass eigenstates. In anticipation of this, we use different notation for states that are not mass eigenstates.

The fields corresponding to two $(1,2,-1/2))$ multiplets are denoted $\mathscr{E}$ and $\mathscr{L}$. Out of these, two mass eigenstates can be formed, L and E. L is the familiar lepton doublet containing $(\nu_e)_L$ and $(e^-)_L$. The other (1, 2, 1/2) doublet with opposite hypercharge is the conjugate of E. To the $(1,1,1)$ we can assign the conjugate of the right handed electron $(e_R)^c = (e^c)_L$. The two remaining $(1,1,0)$ multiplets, denoted $\mathscr{N}_1$ and $\mathscr{N}_2$, are sterile w.r.t. the standard model. The mass eigenstates corresponding to these are N_1 and N_2. The field N_1 pairs up with $(\nu_e)_L$ to form a Dirac neutrino, so N_1 is the right handed neutrino.

In ψ_{q^c} we find the transformation properties of $(u^c)_L$ and twice that of $(d^c)_L$. The latter two multiplets are denoted $\mathscr{D}^c$ and $\mathscr{B}^c$. The conjugate of the usual down quark d^c will be a linear combination of the two and the field orthogonal to that is denoted B^c.

Finally, ψ_q contains the transformation properties of the quark doublet Q_L (up to a basis transformation) and the B field. The fields of each G_{333} multiplet can now be placed in 3×3 matrices as follows [31]:

$$\psi_l = \begin{pmatrix} (\mathscr{E}) & (E^c) & (\mathscr{L}) \\ \mathscr{N}_1 & e^c & \mathscr{N}_2 \end{pmatrix}, \qquad \psi_{q^c} = \begin{pmatrix} \mathscr{D}^c \\ u^c \\ \mathscr{B}^c \end{pmatrix},$$
$$\psi_q = \begin{pmatrix} -d & u & B \end{pmatrix}, \tag{5.18}$$

where all fields are left-handed and color indices have been omitted. If the fermions are assigned in this way, we do not directly use the quark doublet Q_L, but instead use the conjugate $\tilde{Q} = (Q_L)^T i\sigma_2 = \begin{pmatrix} -d & u \end{pmatrix}$. In this chapter, this will just be called Q.

5.2 Baryon number conservation in the gauge sector

Generally baryon number violating processes mediated by gauge bosons are the largest contribution to proton decay. The masses of these bosons lie around the unification scale, so this puts constraints on the magnitude of the unification scale. In trinification models, however, this constraint can be relaxed, because proton decay cannot be mediated by gauge bosons. This can be seen more explicitly by writing out the Lagrangian for the gauge sector. Trinification consists of three copies of $SU(3)$, so it contains $3 \cdot 8 = 24$ gauge bosons. Thus, trinification introduces twelve new gauge bosons. The generators of each of the three $SU(3)$ subgroups commute with those in the other subgroups, so the adjoint of G_{333} consists of three copies of the eight-dimensional adjoint of $SU(3)$:

$$24 = (8,1,1) + (1,8,1) + (1,1,8), \tag{5.19}$$

The first multiplet contains the transformation properties of the eight gluon fields $G^1 \dots G^8$. The second multiplet contains left handed gauge bosons $W_L^1 \dots W_L^8$. Trinification also introduces right handed gauge bosons $W_R^1 \dots W_R^8$, which are contained in the third multiplet. The photon, $W^\pm$ and Z^0 are linear combinations of left handed and right handed bosons. The assignment of the gauge bosons is therefore as follows:

$$\begin{aligned} G^1 \dots G^8 &\to (8,1,1), \\ W_L^1 \dots W_L^8 &\to (1,8,1), \\ W_R^1 \dots W_R^8 &\to (1,1,8). \end{aligned} \tag{5.20}$$

Now we need to find out how the covariant derivative acts on the multiplets ψ_l, ψ_{q^c} and ψ_q. However, since we already know how the generators $T^a_{C,L,R}$ act on these multiplets, we can write it as

$$\begin{aligned} D_\mu \psi_l &= \partial_\mu \psi_l - i g_L W^a_{L\mu} T^a_L \psi_l - i g_R W^a_{R\mu} \psi_l (\overline{T}^a_R)^T, \\ D_\mu \psi_{q^c} &= \partial_\mu \psi_{q^c} - i g_R W^a_{R\mu} T^a_R \psi_{q^c} - i g_C G^a_\mu \psi_{q^c} (\overline{T}^a_C)^T, \\ D_\mu \psi_q &= \partial_\mu \psi_q - i g_C G^a_\mu T^a_C \psi_q - i g_L W^a_{L\mu} \psi_q (\overline{T}^a_L)^T. \end{aligned} \tag{5.21}$$

The corresponding Lagrangian for the gauge sector is

$$\mathcal{L} = \mathrm{Tr}\left[\overline{\psi}_l \not{D} \psi_l\right] + \mathrm{Tr}\left[\overline{\psi}_{q^c} \not{D} \psi_{q^c}\right] + \mathrm{Tr}\left[\overline{\psi}_q \not{D} \psi_q\right]. \tag{5.22}$$

It follows from this expression that the different multiplets ψ_l, ψ_{q^c} and ψ_q do not mix in the gauge sector. We can therefore phase rotate each multiplet individually without changing the Lagrangian, meaning that the gauge sector has an accidental symmetry [31]:

$$U(1)_l \times U(1)_{q^c} \times U(1)_q. \tag{5.23}$$

The linear combination $q - q^c$ must be proportional to baryon number, so proton decay cannot be mediated by gauge bosons in the trinification model. Vector leptoquarks are absent in trinification. The only particles that do violate baryon number conservation are scalar Higgs particles.

5.3 Symmetry breaking and gauge coupling unification

In this section we explore the ways in which trinification can be broken to the standard model and the energy scales associated with each symmetry breaking pattern. Compared to the standard model, trinification contains many new particles. The scalar sector, as we will see, contains several new colorless scalar doublets in addition to the SM Higgs doublet. If these were to obtain low enough masses, they can change the running of the couplings such that one-step unification is possible [32]. The other option is to implement intermediate scales. Trinification has an $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R}$ subgroup [33], which can be used for this purpose. We consider both the cases with and without an intermediate scale.

5.3.1 The symmetry breaking scheme

The simplest way to break the symmetry involves two scalar multiplets, Φ^1 and Φ^2, transforming according to the 27 representation [22]. One scalar multiplet would not suffice, because its vev can always be diagonalized by a basis transformation. The theory would therefore always contain a left-right symmetry. If another scalar multiplet is added, both vevs cannot be diagonalized simultaneously, allowing us to break the left-right symmetry. The components of the scalar multiplets are denoted as follows [31]:

$$\Phi_l = \begin{pmatrix} (\phi_1) & (\phi_2) & (\phi_3) \\ S_1 & S_2 & S_3 \end{pmatrix}, \qquad \Phi_{q^c} = \begin{pmatrix} \mathscr{D}_H^c \\ \mathscr{U}_H^c \\ \mathscr{B}_H^c \end{pmatrix}, \qquad \Phi_q = \begin{pmatrix} -\mathscr{D}_H & \mathscr{U}_H & \mathscr{B}_H \end{pmatrix}. \tag{5.24}$$

Just as for the fermions, we can define a scalar doublet $Q_H = \begin{pmatrix} -\mathscr{D}_H & \mathscr{U}_H \end{pmatrix}$.

To leave $SU(3)_C$ unbroken, only the colorless components can obtain vevs. So the only components that play a role in the breaking mechanism are $\Phi_l^{1,2} \sim (1, 3, \overline{3})$. If the vev of Φ_l^1 is kept diagonalized, the most general expression for $\langle \Phi_l^1 \rangle$ is [30, 31]

$$\langle \Phi_l^1 \rangle = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & v_1 & 0 \\ 0 & 0 & M_1 \end{pmatrix}, \tag{5.25}$$

where M_1 lies at the unification scale, while v_1 and b_1 lie at the electroweak scale. Note that the doublet ϕ_2 transforms as $(1, 2, 1/2)$ under the SM subgroup, so ϕ_2 can be identified with the standard model Higgs doublet. Its isospin down component obtains the vev v_1.

The vev of Φ_L^2 can contain off-diagonal components as well. The most general expression for $\langle \Phi_l^2 \rangle$ that breaks G_{333} to $SU(3)_C \times U(1)_Q$ is [30, 31]:

$$\langle \Phi_l^2 \rangle = \begin{pmatrix} b_2 & 0 & b_3 \\ 0 & v_2 & 0 \\ M & 0 & M_2 \end{pmatrix}, \tag{5.26}$$

where M_2 lies at the unification scale, M lies at the intermediate scale and v_2, b_2 and b_3 lie at the electroweak scale. Precisely which generators are broken by the vevs at each scale is shown in Appendix A. Here we summarize the results. The parameters M_1 and

M_2 break the generators $T^4_{L,R}\dots T^8_{L,R}$ and leave only the combination $(T^8_L + T^8_R)$ unbroken. Thus, the symmetry that remains is $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R}$. M then breaks $SU(2)_R \times U(1)_{L+R}$ to $U(1)_Y$ at the intermediate scale. The expression for the hypercharge generator was given in Eq. (5.10). At the electroweak scale, the $SU(2)_L \times U(1)_Y$ symmetry is reduced further to $U(1)_Q$ by the remaining vev parameters v_1, v_2, b_1, b_2, b_3, which all break the same generators. Apart from T^a_C, this leaves only one combination of generators, which is electric charge (given in Eq. 5.15). The symmetry breaking chain can then be summarized as

$$\begin{aligned} G_{333} \times \mathbb{Z}_3 &\xrightarrow{M_1} SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R} \\ &\xrightarrow{M} SU(3)_C \times SU(2)_L \times U(1)_Y \\ &\xrightarrow{v_1} SU(3)_C \times U(1)_Q. \end{aligned} \tag{5.27}$$

5.3.2 The scalar potential

The most general potential, invariant under G_{333}, that can generate the vevs $\langle \Phi_l^{1,2} \rangle$, consists of many terms [22]. Since the trinification group only has $SU(N)$ factors, the concepts introduced in section 2.1 will be useful in finding the invariants. Recall that for $SU(3)$ there are three kinds of invariant tensors: the Kronecker deltaa δ^i_j and the antisymmetric symbols ϵ_{ijk} and ϵ^{ijk}. These three can be used to form invariants.

For the fermion multiplets we introduced upper and lower indices. Since the scalar fields transform according to the same representation, we do the same for the scalar multiplets. The fields are then written as $(\Phi^a_l)^i_j$, $(\Phi^a_{q^c})^i_j$ and $(\Phi^a_l)^i_j$. But as for the fermions, an upper (lower) index may refer to the $3(\overline{3})$ of either $SU(3)_C$, $SU(3)_L$ or $SU(3)_R$. When contracting indices with invariant tensors, we have to make sure that the indices refer to the same $SU(3)$ group. For instance consider combining two scalar multiplets as $(\Phi^{a*}_l)^i_j(\Phi^{b*}_l)^j_i = \mathrm{Tr}[(\Phi^a_l)^\dagger \Phi^b_l]$. This will then transform as

$$\begin{aligned} \mathrm{Tr}[(\Phi^a_l)^\dagger \Phi^b_l] \to \ &\mathrm{Tr}[U_R(\Phi^a_l)^\dagger U^\dagger_L U_L \Phi^b_l U^\dagger_R] \\ &= \mathrm{Tr}[U^\dagger_R U_R (\Phi^a_l)^\dagger \Phi^b_l] \\ &= \mathrm{Tr}[(\Phi^a_l)^\dagger \Phi^b_l]. \end{aligned} \tag{5.28}$$

So it is an invariant. But now consider for example $\mathrm{Tr}[(\Phi^a_l)^\dagger \Phi^b_q]$. This transforms as

$$\mathrm{Tr}[(\Phi^a_l)^\dagger \Phi^b_l] \to \mathrm{Tr}[U_R(\Phi^a_l)^\dagger U^\dagger_L U_C \Phi^b_q U^\dagger_L]. \tag{5.29}$$

This is not invariant, since in general $U^\dagger_L U_C \neq I$ and $U^\dagger_L U_R \neq I$. So we see that the indices we contract must refer to the same $SU(3)$ group. We then define

$$\overline{\Phi}^a_\alpha \Phi^b_\alpha \equiv (\Phi^{a*}_\alpha)^j_i (\Phi^b_\alpha)^i_j = \mathrm{Tr}\left[(\Phi^a_\alpha)^\dagger \Phi^b_\alpha\right], \tag{5.30}$$

where the subscript α can take any of the values l, q^c or q. The invariants that can be formed using two scalar multiplets are then

$$\overline{\Phi}^a_l \Phi^b_l, \quad \overline{\Phi}^a_{q^c} \Phi^b_{q^c}, \quad \overline{\Phi}^a_q \Phi^b_q \tag{5.31}$$

Now we consider combining three scalar multiplets. There are two options. First, we can contract pairs of indices using δ^i_j. However, this only gives an invariant if all three scalar multiplets are in different irreps:

$$\Phi^a_q \Phi^b_{q^c} \Phi^c_l \equiv (\Phi^a_q)^i_j (\Phi^b_{q^c})^j_k (\Phi^c_l)^k_i = \mathrm{Tr}\left[\Phi^a_q \Phi^b_{q^c} \Phi^c_l\right]. \tag{5.32}$$

This transforms as

$$\begin{aligned} \Phi^a_q \Phi^b_{q^c} \Phi^c_l \to & \ \mathrm{Tr}\left[U_R \Phi^a_{q^c} U_C^\dagger U_C \Phi^b_q U_L^\dagger U_L \Phi^c_l U_R^\dagger\right] \\ &= \mathrm{Tr}\left[\Phi^a_{q^c} \Phi^b_q \Phi^c_l\right] \\ &= \Phi^a_q \Phi^b_{q^c} \Phi^c_l, \end{aligned} \tag{5.33}$$

confirming that it is invariant. If the fields would not be in different irreps we would get products like $U_C^\dagger U_R$, which are not equal to the identity matrix.

The second option requires that all three fields are in the same irrep. In that case, we can antisymmetrize on all upper indices and all lower indices separately:

$$\Phi^a_\alpha \Phi^b_\alpha \Phi^c_\alpha \equiv \epsilon^{ijk} \epsilon_{rst} (\Phi^a_\alpha)^r_i (\Phi^b_\alpha)^s_j (\Phi^c_\alpha)^t_k, \tag{5.34}$$

where, as before, the subscript α may take any of the values l, q^c or q. The invariants that can be formed using three scalar multiplets are thus given by

$$\Phi^a_q \Phi^b_{q^c} \Phi^c_l, \quad \Phi^a_l \Phi^b_l \Phi^c_l, \quad \Phi^a_{q^c} \Phi^b_{q^c} \Phi^c_{q^c}, \quad \Phi^a_q \Phi^b_q \Phi^c_q. \tag{5.35}$$

Up to cubic terms the Higgs potential is then given by

$$\begin{aligned} V(\Phi^1, \Phi^2) = & -\mu^2_{1l} \overline{\Phi}^1_l \Phi^1_l - \mu^2_{2l} \overline{\Phi}^2_l \Phi^2_l + \gamma_1 \mathrm{det}\Phi^1_l + \gamma_2 \mathrm{det}\Phi^2_l \\ & + \gamma_3 \Phi^1_l \Phi^1_l \Phi^2_l + \gamma_4 \Phi^1_l \Phi^2_l \Phi^2_l + \gamma_5 \Phi^1_q \Phi^1_{q^c} \Phi^1_l + \gamma_6 \Phi^2_q \Phi^2_{q^c} \Phi^2_l \\ & + \gamma_7 \Phi^1_q \Phi^1_{q^c} \Phi^2_l + \gamma_8 \Phi^1_q \Phi^2_{q^c} \Phi^1_l + \gamma_9 \Phi^2_q \Phi^1_{q^c} \Phi^1_l \\ & + \gamma_{10} \Phi^2_q \Phi^2_{q^c} \Phi^1_l + \gamma_{11} \Phi^2_q \Phi^1_{q^c} \Phi^2_l + \gamma_{12} \Phi^1_q \Phi^2_{q^c} \Phi^2_l + \text{ cyclic} + \text{h.c.}, \end{aligned} \tag{5.36}$$

where 'cyclic' means any terms that are obtained by cyclically permuting the subscripts l, q^c and q. With only cubic terms and lower included, the quadratic terms determine the masses of the Higgs bosons and the cubic terms lead to additional mixing and mass splitting after spontaneous symmetry breaking. We could go on and include quartic terms as well, which would have additional contributions to the masses and mixing. For simplicity these are left out. As a final remark, it is worth noting that the notation introduced here for the different kinds of couplings, is not specific to scalar multiplets. Scalar particles and fermions have the same transformation properties in trinification, so the invariants we found work just as well for fermions. So for instance, some of the scalar fields can be replaced by fermions, as is done in the Yukawa sector.

5.3.3 Gauge coupling unification

In this section we use the RGEs to obtain the possible mass scales in trinification. First we consider the case where G_{333} is broken immediately to the standard model. In $SU(5)$

we saw that if the particle spectrum below the GUT scale stays the same, gauge coupling unification does not happen automatically. In trinification this can be fixed, by giving the six Higgs doublets in Φ_l^1 and Φ_l^2 a mass around the electroweak scale [32]. This modifies the β-coefficients in such a way that one-step unification might be possible. Nevertheless, only the SM Higgs doublet has been found at the electroweak scale. The five other Higgs doublets may possibly lie around the TeV scale.

Before we calculate the β-coefficients we first normalize the hypercharge generator. To do this the result from Eq. (2.103) can be used. In the expression for the hypercharge generator (Eq. 5.10), one can see that the sum of the squares of the coefficients is $1+1/3+1/3 = 5/3$. So the normalized hypercharge generator is

$$Y' = \sqrt{\frac{3}{5}}Y, \tag{5.37}$$

which agrees with Eq. (2.87).

Since Φ_l^1 and Φ_l^2 are color singlets, only the β-coefficients of $SU(2)_L$ and $U(1)_Y$ are changed: b_Y and b_{2L}. The coefficient for $SU(3)_C$ stays at $b_{3C} = -7$. If there are N_f fermion generations and N_H Higgs doublets, the general expressions for b_Y and b_{2L} are

$$b_Y = \frac{4}{3}N_f + \frac{1}{10}N_H, \tag{5.38}$$

$$b_{2L} = \frac{4}{3}N_f + \frac{1}{6}N_H - \frac{22}{3}. \tag{5.39}$$

For three generations and six Higgs doublets this gives

$$b_Y = \frac{23}{5}, \quad b_{2L} = -\frac{7}{3}. \tag{5.40}$$

So we see that b_Y has increased from 41/10 to 23/5 and b_{2L} has increased from $-19/6$ to $-7/3$). This means that the slopes of α_Y^{-1} and α_{2L}^{-1} will both decrease (note that β-coefficients appear with a minus sign in the RGEs), but the slope of α_{2L}^{-1} decreases more. If we plot the running (Figure 5.1) we see the couplings become very close to each other around 10^{14} GeV. Compare this to Figure 3.2 from the chapter about $SU(5)$. There we saw that the couplings intersected each other at energy scales that lied several orders of magnitude apart. So unification was impossible and the only way to make it work, was to tweak the parameters somewhat. But now that we have included five more Higgs doublets, the couplings intersect at scales that lie much closer to each other, without ever tweaking any parameters. While unification as shown Figure 5.1 is not exact, one has to keep in mind that this is only a leading order approximation (see section 2.4). Within the accuracy of this approximation, we can say that the couplings unify around 10^{14} GeV [32]. In a GUT this would normally be disastrously low, but trinification has the advantage that gauge bosons do not mediate proton decay.

Gauge coupling unification can also be ensured by including an intermediate $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R}$ scale. First we determine which particles are present at each scale. For fermions, we assume that only the SM fermions contribute to the RGEs at the electroweak scale. The right handed neutrino N_1 is an SM singlet, so it can only contribute past the intermediate scale. As in $SO(10)$, we assume that only those Higgs particles are

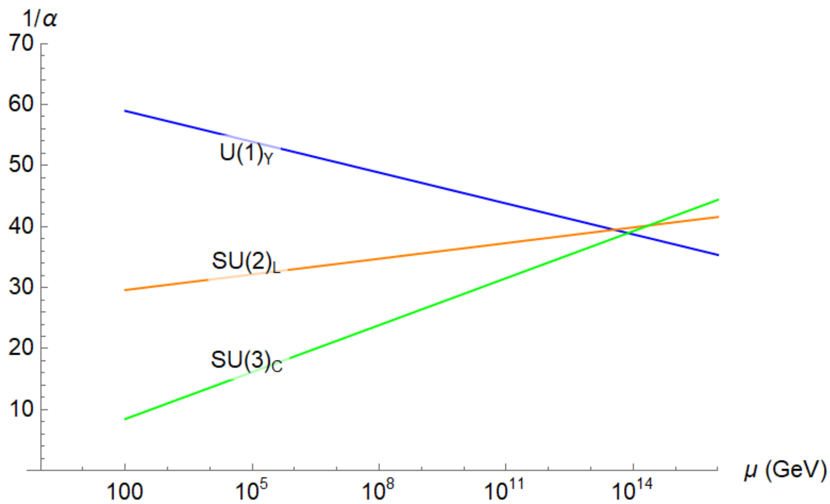

Figure 5.1: Running of the fine structure constants in trinification with six electroweak-scale Higgs doublets.

present that obtain vevs at or below a given scale. Looking at Eqs. (5.25) and (5.26), this means that there are a total of five Higgs doublets present at the electroweak scale.

Past the intermediate scale, the $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R}$ symmetry holds, so we will need to know the decomposition of the 27 under this group. The $SU(3)_C$ and $SU(2)_L$ quantum numbers do not change, but now there is also a right handed isospin group $SU(2)_R$, generated by $T^1_R \ldots T^3_R$. From the general transformation Eq. (5.6), it then follows that the first two states in each row of ψ_l form a right isospin doublet and so do the first two states in each column of ψ_{q^c}. All the states in ψ_q are $SU(2)_R$ singlets.

The $U(1)_{L+R}$ subgroup is generated by

$$T_{L+R} = \frac{1}{\sqrt{2}}(T^8_L + T^8_R) \tag{5.41}$$

The factor $1/\sqrt{2}$ ensures that it has the same normalization as the other generators of G_{333}. However, it is customary to normalize it as

$$T_{B-L} = \sqrt{\frac{8}{3}} T_{L+R}, \tag{5.42}$$

which corresponds to $B-L$ in a different basis [30]. For the RGEs, it is still the eigenvalues of T_{L+R} that are relevant, since the coupling constants associated with normalized generators are assumed to unify. We can now decompose the multiplets in the 27 in terms of $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ as follows:

$$\begin{gathered} \psi_l = \begin{pmatrix} (1,2,2,0) & (1,2,1,1) \\ (1,1,2,-1) & (1,1,1,0) \end{pmatrix}, \qquad \psi_{q^c} = \begin{pmatrix} (\overline{3},1,2,1/3) \\ (\overline{3},1,1,-2/3) \end{pmatrix}, \\ \psi_q = \begin{pmatrix} (3,2,1,-1/3) & (3,1,1,2/3) \end{pmatrix}. \end{gathered} \tag{5.43}$$

For the contribution from Higgs particles we should now also include the multiplet that obtains the intermediate scale vev M. The decompositions of the particles that contribute to the RGEs at each scale and the β-coefficients are summarized in Table 5.1.

Energy range	Symmetry	Fermions	Scalars	b_i
$M_{\text{EW}} - M_I$	$SU(3)_C \times SU(2)_L$ $\times U(1)_Y$	$(1,2,-1/2)$ $(1,1,-1)$ $(3,2,1/6)$ $(3,1,2/3)$ $(3,1,-1/3)$	$2(1,2,1/2)$ $3(1,2,-1/2)$	$b_{3C} = -7$ $b_{2L} = -5/2$ $b_Y = 9/2$
$M_I - M_U$	$SU(3)_C \times SU(2)_L$ $\times SU(2)_R \times U(1)_{B-L}$	$(1,2,1,1)$ $(1,1,2,-1)$ $(\overline{3},1,2,1/3)$ $(3,2,1,-1/3)$	$2(1,2,2,0)$ $(1,2,1,1)$ $(1,1,2,-1)$	$b_{3C} = -7$ $b_{2L} = -5/2$ $b_{2R} = -5/2$ $b_{L+R} = 9/2$

Table 5.1: Decompositions of the particles that contribute to the RGEs at each scale and the associated β-coefficients.

Finally, there are several matching conditions which need to be satisfied. At M_I, $U(1)_Y$ is embedded in $SU(2)_R \times U(1)_{L+R}$ through Eq. (5.10). In terms of normalized generators, this expression becomes

$$Y' = \sqrt{\frac{3}{5}} T_R^3 - \sqrt{\frac{2}{5}} T_{L+R}. \tag{5.44}$$

The corresponding matching condition is then

$$\alpha_Y^{-1}(M_I) = \frac{3}{5}\alpha_{2R}^{-1}(M_I) + \frac{2}{5}\alpha_{L+R}^{-1}(M_I), \tag{5.45}$$

At M_I, continuity requires that

$$\alpha_{2L}(M_I) = \alpha'_{2L}(M_I), \tag{5.46}$$
$$\alpha_{3C}(M_I) = \alpha'_{3C}(M_I). \tag{5.47}$$

Finally, at the unification scale M_U we have

$$\alpha'_{3C}(M_U) = \alpha'_{2L}(M_U) = \alpha_{2R}(M_U) = \alpha_{L+R}(M_U). \tag{5.48}$$

This yields six conditions on the six unknowns $\alpha_{2L}(M_I)$, $\alpha_{3C}(M_I)$, $\alpha_{2R}(M_I)$, $\alpha_{L+R}(M_I)$, M_I and M_U. Hence, we get a unique solution for the intermediate and unification scales:

$$M_I \approx 1.0 \cdot 10^{11} \text{ GeV}, \tag{5.49}$$
$$M_U \approx 1.3 \cdot 10^{16} \text{ GeV}. \tag{5.50}$$

The running of the couplings is shown in Figure 5.2. Note that since the β-coefficients for $SU(2)_L$ and $SU(2)_R$ at the intermediate scale are identical, their running overlaps. In conclusion, the current model predicts that the only new physics that can occur at the TeV scale, is the production of additional Higgs doublets. All other new particles lie at a mass scale of at least 10^{11} GeV.

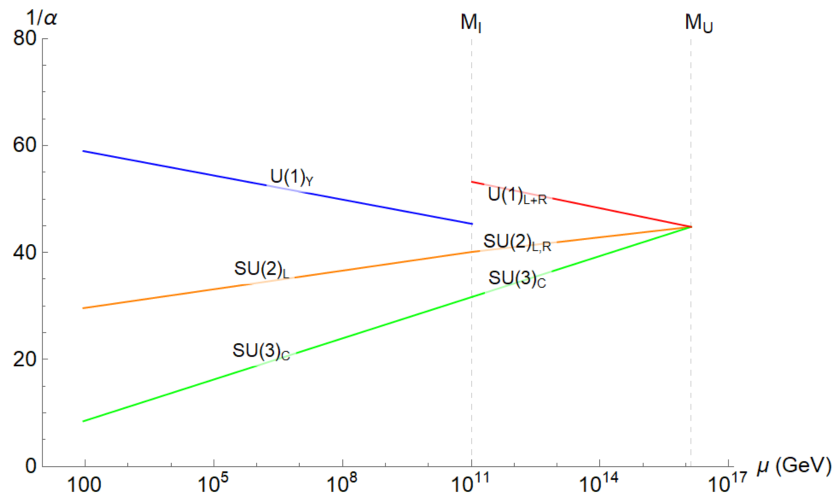

Figure 5.2: Running of the fine structure constants in trinification, with one intermediate scale.

5.4 Fermion mixing within generations

The masses of the fermions originate from the Yukawa sector after spontaneous symmetry breaking. On top of mixing between generations, the minimal Yukawa sector also allows fermions within a generation to mix with each other. To be able to discuss their couplings to leptoquarks, it is important to know which combinations of states correspond to mass eigenstates. This section discusses which fermion fields mix with each other as well as their mass scales. Recall that the Higgs fields are in the same 27 representation of E_6 as the fermions, so the Yukawa couplings consist of triple products of the irreps in Eq. (5.1). In section 5.3, we found the types of invariants that can be formed from three fields in the 27 representation under G_{333}. To form a Yukawa coupling, we simply replace two of the fields by fermions. The first type of invariant consisted of traces between three fields in different irreps. Thus, we can write down the following Yukawa couplings (the charge conjugation matrix C is left out) [31, 32]:

$$\mathcal{L}_q = \psi_{q^c}\psi_q(g_1\Phi_l^1 + g_2\Phi_l^2) + \text{ cyclic } + \text{h.c.}, \tag{5.51}$$

where the $\mathbb{Z}_3$ symmetry ensures that the Yukawa coupling constants are the same for cyclic permutations. This Lagrangian leads to mass terms for the quarks. If we write out Eq. (5.32) we obtain [31]

$$\begin{aligned}\psi_{q^c}\psi_q\Phi_l^a = \mathscr{D}^c Q\phi_1^a + u^c Q\phi_2^a + \mathscr{B}^c Q\phi_3^a \\ + \mathscr{D}^c BS_1^a + u^c BS_2^a + \mathscr{B}^c BS_2^a.\end{aligned} \tag{5.52}$$

Since Φ_l^a are the only Higgs multiplets that obtain vevs, this is the relevant part as far as quark masses are concerned. The B quark acquires a mass at the unification scale and pairs up with a linear combination of $\mathscr{D}^c$ and $\mathscr{B}^c$. This can be seen as follows. Suppose only the vevs M_1, M_2 and M would be used. The part of the couplings containing these vevs is

$$\mathcal{L}_q \ni (g_1M_1 + g_2M_2)\mathscr{B}^c B + g_2M\mathscr{D}^c B + \text{ h.c.} \tag{5.53}$$

We can write this in terms of mass eigenstates by rotating the states $\mathscr{D}^c$ and $\mathscr{B}^c$ via an orthogonal transformation. One of them will be B^c and the linear combination orthogonal to that will be d^c:

$$\begin{pmatrix} d^c \\ B^c \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathscr{B}^c \\ \mathscr{D}^c \end{pmatrix}. \tag{5.54}$$

Filling this in and eliminating the cross terms gives $\tan\alpha = (g_1 M_1 + g_2 M_2)/g_2 M$. The mass of the B quark is then given by

$$\begin{aligned} m_B &= (g_1 M_1 + g_2 M_2)\sin\alpha + g_2 M\cos\alpha \\ &= \sqrt{(g_1 M_1 + g_2 M_2)^2 + (g_2 M)^2}, \end{aligned} \tag{5.55}$$

and other quarks remain massless at this stage. Hence, the B quark mass lies at the unification scale. This also shows that if M is an intermediate scale vev, there will barely be any mixing and its contribution to the masses is negligible. To obtain masses for the up and down quark, we need to include electroweak scale vevs as well. Inserting the vevs v_1 and b_1 then yields

$$m_u = g_1 v_1, \qquad m_d = g_1 b_1 \sin\alpha. \tag{5.56}$$

So the up and down quarks stay at the electroweak scale. Actually, b_1 slightly alters the mixing between $\mathscr{D}^c$ and $\mathscr{B}^c$, but this contribution is negligible. More generally, including all the vevs will lead to more complicated mixing between the states. The calculations shown here are only crude estimates that reveal the mass scales of the fermions. Masses for the leptons are generated by including antisymmetric products. The corresponding Lagrangian can then be written as [31, 32]

$$\mathcal{L}_l = \frac{1}{2}\psi_l\psi_l(h_1\Phi_l^2 + h_2\Phi_l^2) + \text{ cyclic } + \text{h.c.} \tag{5.57}$$

To see how this gives masses to the leptons, we restrict to the part containing Φ_l^a [31]:

$$\begin{aligned} \frac{1}{2}\psi_l\psi_l\Phi_l^a = &-(E^c\mathscr{N}_2 - \mathscr{L}e^c)\phi_1^a + (\mathscr{E}\mathscr{N}_2 - \mathscr{L}\mathscr{N}_1)\phi_2^a + (E^c\mathscr{N}_1 - \mathscr{E}e^c)\phi_3^a \\ &+ E^c\mathscr{L}S_1^a - \mathscr{E}\mathscr{L}S_2^a - E^c\mathscr{E}S_3^a, \end{aligned} \tag{5.58}$$

where a factor $i\sigma_2$ is implicitly placed between products involving two doublets, in keeping with the antisymmetry. When S_3^1, S_3^2 and S_1^2 obtain the vevs M_1, M_2 and M, respectively, we get the terms

$$\mathcal{L}_l \ni h_2 M E^c\mathscr{L} - (h_1 M_1 + h_2 M_2)E^c\mathscr{E} + \text{ h.c.} \tag{5.59}$$

So the fields $\mathscr{E}$ and $\mathscr{L}$ will mix with each other into the lepton doublet E. The orthogonal combination will be the lepton doublet L from the standard model. As before, $\mathscr{E}$ and $\mathscr{L}$ can be rotated into E and L by an orthogonal transformation:

$$\begin{pmatrix} E \\ L \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \mathscr{L} \\ \mathscr{E} \end{pmatrix}, \tag{5.60}$$

with $\tan\beta = (h_1 M_1 + h_2 M_2)/h_2 M$. The mass of E is then given by

$$\begin{aligned} M_E &= (h_1 M_1 + h_2 M_2)\sin\beta + h_2 M \cos\beta \\ &= \sqrt{(h_1 M_1 + h_2 M_2)^2 + (h_2 M)^2}. \end{aligned} \tag{5.61}$$

So apparently only the lepton doublet E stays at the unification scale. The other new fermions, $\mathcal{N}_1$ and $\mathcal{N}_2$, obtain masses at or below the electroweak scale, as we will see. If we include the vevs v_1 and b_1, and neglect couplings to the super massive lepton doublet E, we obtain

$$\mathcal{L}_l \ni -\sin\beta h_1 b_1 e^c e - h_1 v_1 \nu_e(\sin\beta \mathcal{N}_1 - \cos\beta \mathcal{N}_2) + \text{ h.c.} \tag{5.62}$$

So ν_e pairs up with a linear combination of the SM singlets $\mathcal{N}_1$ and $\mathcal{N}_2$. This is the right handed neutrino N_1 and the orthogonal combination is N_2:

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} \sin\beta & -\cos\beta \\ -\cos\beta & -\sin\beta \end{pmatrix} \begin{pmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{pmatrix}. \tag{5.63}$$

The masses are

$$m_e = h_1 b_1 \sin\beta, \qquad m_{v_e, N_1} = h_1 v_1. \tag{5.64}$$

The remaining SM singlet N_2 obtains a mass proportional to $b_1 v_1$, which leaves it at the eV scale. Thus, both of the unobserved fermions N_1 and N_2 are predicted to have masses at measurable energy scales. This poses a problem and has to be fixed by large radiative corrections to the masses [31]. On the other hand, from the expressions for the fermions masses we can see that there are no relations between them. So minimal trinification offers enough freedom to accommodate all fermion masses, which was impossible in minimal $SU(5)$.

5.5 Scalar mixing and proton decay

The general Higgs potential, Eq. (5.36), introduces many cross terms when the Higgs fields acquire their vevs. This leads to mixing and mass splitting between the Higgs fields. Since vector leptoquarks are absent in trinification, this mixing between scalar fields is especially relevant for proton decay. If two fields, which would otherwise be independent modes, mix with each other, this could lead to new interactions.

To obtain interactions that lead to proton decay, both $\mathcal{L}_q$ and $\mathcal{L}_l$ have to be included. This can be understood as follows [31]. If we would assign baryon numbers 0 to Φ_l, 1/3 to Φ_q and -1/3 to Φ_{q_c}, then $\mathcal{L}_q$ conserves baryon number. Similarly, if we assign baryon numbers 0 to Φ_l, -2/3 to Φ_q and 2/3 to Φ_{q_c}, $\mathcal{L}_l$ conserves baryon number. Including both Lagrangians would therefore violate baryon number conservation, since Φ_q and Φ_{q_c} do not have the same baryon numbers in each case. Φ_l, however, has the same baryon number in both cases, so only colored Higgs particles can mediate proton decay.

Since only the fields $\Phi_l^{1,2}$ acquire vevs, the cubic terms in the Higgs potential that contribute to the masses take the form $\Phi_{q^c}\Phi_q\Phi_l$ or $\Phi_l\Phi_l\Phi_l$. So the colored fields Φ_{q^c} and Φ_q mix with each other and the leptonic Higgs fields Φ_l mix separately. The colored Higgs fields mediate proton decay, so we focus on terms involving Φ_{q^c} and Φ_q. For the most general potential, the expressions for the masses and mass eigenstates would be very complicated. However, for simplicity we assume that the two Higgs fields Φ^1 and Φ^2 do not mix with each other. In that case, the only terms of interest are

$$V(\Phi^1,\Phi^2) \ni \gamma_5\Phi^1_q\Phi^1_{q^c}\Phi^1_l + \gamma_6\Phi^2_q\Phi^2_{q^c}\Phi^2_l + \gamma_7\Phi^1_q\Phi^1_{q^c}\Phi^2_l + \gamma_{10}\Phi^2_q\Phi^2_{q^c}\Phi^1_l + \text{ h.c.} \tag{5.65}$$

This would mean that the way Φ^1 and Φ^2 particles mix among themselves is identical. Only a few indices would be different, but the essential features stay the same. From here on out, only expressions pertaining to Φ^1 are shown and the index denoting whether its Φ^1 or Φ^2 is dropped. To simplify it even further we take all γ_i to be equal: $\gamma_i \equiv \gamma$. We also take $\mu_q = \mu_{q^c} \equiv \mu$. If we then insert the vevs M, M_1 and M_2, we obtain

$$V_M = \gamma(M_1\mathscr{B}^c_H\mathscr{B}_H + M\mathscr{D}^c_H + M_2\mathscr{B}^c_H)\mathscr{B}_H + \text{ h.c.}, \tag{5.66}$$

To find the mass eigenstates we first rewrite V_M as $V_M = a^\dagger(\Delta M^2_H)a$, where

$$a = \begin{pmatrix} \mathscr{B}_H \\ (\mathscr{D}^c_H)^\dagger \\ (\mathscr{B}^c_H)^\dagger \end{pmatrix}, \quad \Delta M^2_H = \gamma \begin{pmatrix} 0 & M & M_1+M_2 \\ M & 0 & 0 \\ M_1+M_2 & 0 & 0 \end{pmatrix}. \tag{5.67}$$

Since ΔM^2_H is symmetric it can be diagonalized by an orthogonal transformation. Its eigenvectors give the mass eigenstates $B_{1,2,3H}$:

$$\begin{pmatrix} B_{1H} \\ B_{2H} \\ B_{3H} \end{pmatrix} = \frac{\gamma}{\sqrt{2}} \begin{pmatrix} 1 & \alpha M & \alpha(M_1+M_2) \\ -1 & \alpha M & \alpha(M_1+M_2) \\ 0 & -\alpha\sqrt{2}(M_1+M_2) & \alpha\sqrt{2}M \end{pmatrix} \begin{pmatrix} \mathscr{B}_H \\ (\mathscr{D}^c_H)^\dagger \\ (\mathscr{B}^c_H)^\dagger \end{pmatrix}, \tag{5.68}$$

$$\alpha = \frac{1}{\sqrt{M^2 + (M_1+M_2)^2}}. \tag{5.69}$$

Thus, in general, the states $\mathscr{B}_H$, $\mathscr{D}^c_H$ and $\mathscr{B}^c_H$ mix with each other. The mass splitting follows from the eigenvalues:

$$\Delta m^2_1 = \gamma/\alpha, \tag{5.70}$$

$$\Delta m^2_2 = -\gamma/\alpha, \tag{5.71}$$

$$\Delta m^2_3 = 0. \tag{5.72}$$

So mixing increases the mass of B_{1H}, reduces that of B_{2H} and leaves the mass of B_{3H} unchanged. Therefore, the field B_{2H} will be relevant at a lower scale than the other colored scalar fields. In the context of proton decay, this scalar field will be of most interest. The full expression for the mass of B_{2H} is [31]

$$M^2_{B_{2H}} = \mu^2 - \frac{\gamma}{\alpha}. \tag{5.73}$$

If M would lie at an intermediate scale, αM would be negligibly small and $\alpha(M_1+M_2) \approx 1$. In that case the state $\mathscr{D}^c_H$ decouples from the other two and the mass eigenstates would be [31]

$$B_{1,2H} = \frac{1}{\sqrt{2}}(\pm\mathscr{B}_H + (\mathscr{B}^c_H)^\dagger), \quad B_{3H} = -\mathscr{D}^c_H. \tag{5.74}$$

Diagrams for proton decay can be generated either by putting together two Yukawa couplings involving the same colored Higgs field, or by connecting two different colored Higgs fields via the term $\Phi_{q^c}\Phi_q\Phi_l$ in the Higgs potential [31]. The latter would lead to the creation of a Φ_l particle. Both processes are depicted in Figure 5.3.

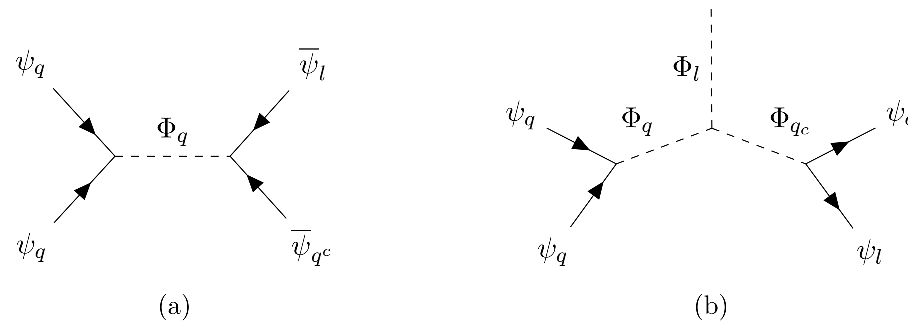

Figure 5.3: The two types of proton decay in the trinification model.

The relevant couplings for proton decay can be found by writing out the Yukawa couplings involving colored Higgs bosons. The terms in $\mathcal{L}_q$ give [31]

$$\begin{aligned}\psi_l\psi_{q^c}\Phi_q =& -(E^c u^c + \mathscr{E}\mathscr{D}^c + \mathscr{L}\mathscr{B}^c)Q_H \\ &+ (e^c u^c + \mathscr{N}_1\mathscr{D}^c + \mathscr{N}_2\mathscr{B}^c)\mathscr{B}_H,\end{aligned} \tag{5.75a}$$

$$\begin{aligned}\psi_q\psi_l\Phi_{q^c} =&(QE^c + Be^c)U_H^c + (Q\mathscr{E} + B\mathscr{N}_1)\mathscr{D}_H^c \\ &+ (Q\mathscr{L} + B\mathscr{N}_2)\mathscr{B}_H^c.\end{aligned} \tag{5.75b}$$

The terms in $\mathcal{L}_l$ give [31]

$$\psi_q\psi_q\Phi_q = QQ\mathscr{B}_H + BQQ_H, \tag{5.76a}$$

$$\psi_{q^c}\psi_{q^c}\Phi_{q^c} = \mathscr{D}^c u^c\mathscr{B}_H^c + u^c\mathscr{B}^c\mathscr{D}_H^c + \mathscr{B}^c\mathscr{D}^c\mathscr{U}_H^c. \tag{5.76b}$$

As mentioned before, the colored Higgs field with lowest mass is B_{2H}. With an intermediate scale, only $\mathscr{B}_H$ and $\mathscr{B}_H^c$ will mix into B_{2H} (Eq. 5.74), so the most relevant couplings are [31]:

$$\mathcal{L}_q \ni g\left[e^c u^c\frac{1}{\sqrt{2}}(B_{1H} - B_{2H}) + Q\mathscr{L}\frac{1}{\sqrt{2}}(B_{1H}^\dagger + B_{2H}^\dagger)\right] + \text{ h.c.}, \tag{5.77a}$$

$$\mathcal{L}_l \ni h\left[QQ\frac{1}{\sqrt{2}}(B_{1H} - B_{2H}) + \mathscr{D}^c u^c\frac{1}{\sqrt{2}}(B_{1H}^\dagger + B_{2H}^\dagger)\right] + \text{ h.c.} \tag{5.77b}$$

This yields three diagrams for the first type of proton decay that lead to the decays $p \to e^+\pi^0$ and $p \to \overline{\nu}_e\pi^+$ (shown in Figure 5.4).

To summarize, trinification exclusively contains scalar leptoquarks. When the symmetry is broken spontaneously, one scalar leptoquark obtains the lowest mass, which is B_{2H}. This field mediates several proton decay processes. In our simplified model, the mass splitting of the scalar leptoquarks depends on a single parameter γ. This parameter determines the strength of the cubic couplings in the scalar potential. Couplings of this type have never been observed. However, for scalar leptoquarks that mediate proton decay, their masses are constrained by the estimated lower bound 10^{11} GeV [12]. Thus, the lightest leptoquark that we can expect acccording to trinification, is the scalar B_{2H}, which has a mass of at least 10^{11} GeV.

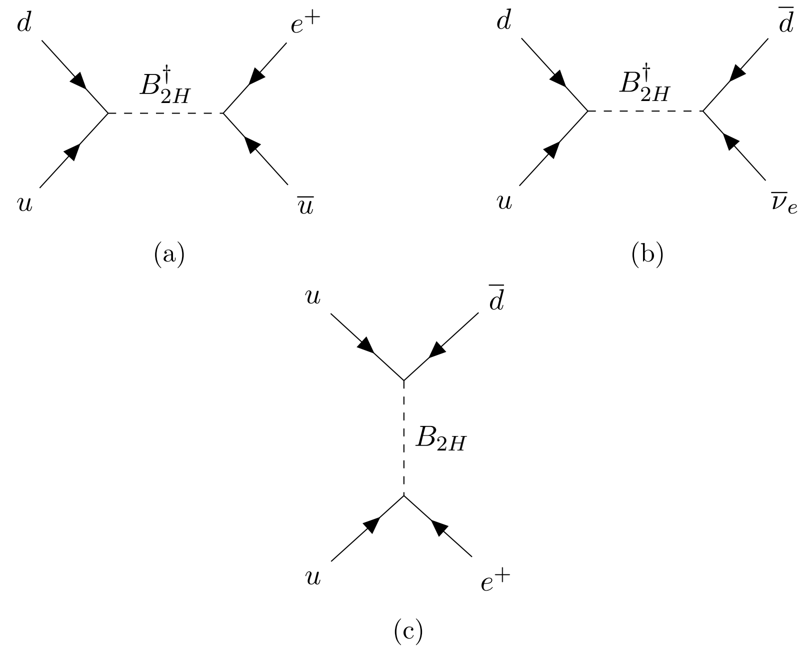

Figure 5.4: Proton decay diagrams in the trinification model.

5.6 Conclusions

This chapter discussed the leptoquarks that appear in trinification, a GUT based on the group $G_{333} = SU(3)_C \times SU(3)_L \times SU(3)_R$. This is a product of three simple groups, so gauge coupling unification does not happen automatically. It can be ensured by manually imposing a $\mathbb{Z}_3$ symmetry that interchanges quarks with leptons, leptons with antiquarks and antiquarks with quarks. The fermions are placed in the fundamental 27 representation of E_6, introducing twelve new fermions. This includes the field N_1, which may correspond to the right handed neutrino. A possible identity for the other new fermions is unknown.

When restricted to G_{333}, the 27 representation decomposes into three irreps. Quarks, antiquarks and leptons are each assigned separately to one of these irreps. As a consequence, the gauge sector conserves baryon number. Trinification therefore, does not contain any vector leptoquarks. Since gauge mediated proton decay is generally most important, this saves the theory from limits on proton decay.

In order to break the symmetry to the SM, two scalar multiplets that transform according to the 27 representation could be used. Gauge coupling unification could be achieved by including five extra Higgs doublets at the electroweak scale, resulting in a unification scale around 10^{14} GeV. The same scalar fields can be used to implement an intermediate $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R}$ scale. From the RGEs it followed that this intermediate scale could lie at 10^{11} GeV, whereas the unification scale lies at 10^{16} GeV.

Scalar mediated proton decay does occur in trinification, because the Yukawa sector has enough freedom to allow for baryon number violating processes. The Yukawa sector allowed two types of Yukawa couplings: $\psi_{q^c}\psi_q\Phi^a_l$ and $\psi_l\psi_l\Phi^a_l$ (plus cyclic permutations). Both types are necessary to violate baryon number conservation. Before any symmetry breaking the mass spectrum of each scalar multiplet is entirely degenerate. However, when they acquire vevs, the masses of the components split. Consequently, the field B_{2H} obtains the lowest mass among the colored scalar fields. From its Yukawa couplings to SM particles it followed that it mediates the proton decay processes $p \to e^+\pi^0$ and $p \to \overline{\nu}_e\pi^+$. Since B_{2H} mediates proton decay, its mass is subject to the estimated lower bound of 10^{11} GeV. Hence, the lightest leptoquark that we can expect according to trinification is the scalar B_{2H} whose mass must exceed 10^{11} GeV.

Chapter 6

Conclusions

In this thesis we have studied the leptoquarks that appear in several grand unified theories, with an emphasis on the possible mass scales that these particles can have. The standard model does not explain all observed phenomena and it contains many arbitrary parameters. A possible solution is provided by grand unified theories. Many GUTs predict the existence of leptoquarks, so it is of interest to carefully study their properties. This thesis discussed the leptoquarks appearing in GUTs based on the follwing E_6 subgroups: $SU(5)$, $SO(10)$ and $SU(3)_C \times SU(3)_L \times SU(3)_R$.

We first reviewed $SU(5)$ grand unification, one of simplest GUTs. $SU(5)$ contains twelve additional gauge bosons, commonly denoted as X and Y bosons, that carry both color and weak isospin/hypercharge. These are the vector leptoquarks of $SU(5)$. They mediate the proton decays channels $p \to e^+\pi^0$ and $p \to \overline{\nu}_e\pi^0$. These processes violate both baryon and lepton number. Only the combination $B - L$ is preserved. We reviewed how the $SU(5)$ symmetry is broken to the standard model group and then to $SU(3)_C \times U(1)_Q$ at a second stage. This required the use of a 24 and a 5 dimensional Higgs field. In the first step, the X and Y leptoquarks obtain their masses, so their mass scales coincide with the GUT scale M_U. From the RGEs it followed that gauge coupling unification is impossible in minimal $SU(5)$. Nevertheless, if the low energy parameters are adjusted slightly, this results in $M_U \approx 10^{15}$ GeV. From an estimate of the predicted proton lifetime, it was found that this is inconsistent with the experimental limit $\tau_p > 2 \cdot 10^{34}$y. In addition, the minimal Yukawa sector produces incorrect relations between the masses of quarks and leptons. Because of these issues, minimal $SU(5)$ is not considered a viable GUT. The Yukawa sector also showed that $SU(5)$ contains a scalar leptoquark, originating from the 5 dimensional Higgs field. Since it mediates proton decay, its mass has to exceed 10^{11} GeV.

$SO(10)$ has a richer spectrum of leptoquarks. It contains many subgroups, one of which is the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$. Several gauge bosons associated with it are leptoquarks, but the processes they mediate conserve baryon number. All 24 gauge bosons that lie outside the Pati-Salam subgroup are leptoquarks that mediate proton decay. In $SO(10)$, gauge coupling unification can be accomplished by using intermediate scales. If the Pati-Salam subgroup acts as the only intermediate stage, this results in $M_I \approx 10^{11}$ GeV and $M_U \approx 10^{16}$ GeV. So the masses of leptoquarks associated with the Pati-Salam group would lie around 10^{11} GeV, whereas all other leptoquarks lie around 10^{16} GeV. These energy scales are far outside the reach of particle accelerators, but they can be lowered by including more than one intermediate stage. We reviewed a symmetry breaking pattern with three intermediate stages: M_{D_P}, M_{W_R} and M_{Z_R}. M_{W_R} is the scale at which the baryon number conserving leptoquarks lie. In this scenario, imposing gauge coupling unification did not fix all energy scales. This left the possibility that M_{W_R} lies at the TeV scale. Thus, TeV-scale vector leptoquarks can occur in $SO(10)$ GUTs with several intermediate symmetry scales. $SO(10)$ also contains many scalar leptoquarks in 10, 120 and 126 dimensional representations. These are assumed to lie at the GUT scale. However, since the Yukawa couplings of the first generation are much smaller than gauge couplings, scalar-mediated proton decay is suppressed.

Trinification, on the other hand, only allows scalar mediated proton decay to occur. This GUT is based on the group $G_{333} = SU(3)_C \times SU(3)_L \times SU(3)_R$ with an additional $\mathbb{Z}_3$ symmetry to ensure gauge couplings unification. Fermions are placed in the 27 representation of E_6, which decomposes into three irreps under G_{333}. Quarks and leptons are each placed in separate irreps. Consequently, the gauge sector conserves baryon number. Trinification therefore does not contain any vector leptoquarks. The symmetry is broken to the standard model group by two 27 scalar multiplets. Their colored components mediate proton decay, so trinification does contain scalar leptoquarks. If six of the scalar doublets contained in the two scalar multiplets are present at the TeV scale, unification can occur in one step with $M_U \approx 10^{14}$ GeV. It is also possible to implement an intermediate $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ scale. This scale would then lie at 10^{11} GeV, whereas the unification scale lies at 10^{16} GeV. Trinification, therefore, does not predict the existence of leptoquarks at accessible energies.

To conclude, the GUTs we considered revealed that in most cases, leptoquarks reside at rather large scales (10^{11} GeV or higher). Nevertheless, we identified a single scenario where TeV-scale leptoquarks can naturally arise: an $SO(10)$ GUT with three intermediate symmetry scales. This provides theoretical motivation to continue the search for leptoquarks at accessible energy scales.

Appendix A

Broken symmetries in trinification

In this appendix we show which symmetries are broken at each energy scale in trinification. The symmetry breaking is performed by two scalar fields Φ_l^1 and Φ_l^2, which obtain the vevs

$$\langle\Phi_l^1\rangle = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & v_1 & 0 \\ 0 & 0 & M_1 \end{pmatrix}, \quad \langle\Phi_l^2\rangle = \begin{pmatrix} b_2 & 0 & b_3 \\ 0 & v_2 & 0 \\ M & 0 & M_2 \end{pmatrix}. \tag{A.1}$$

The vevs M_1 and M_2 lie at the unification scale, M lies at an intermediate scale and v_1, v_2, b_1, b_2, b_3 all lie at the electroweak scale.

The procedure used to find out which symmetries remain when we move to lower energies is as follows. First we consider the vevs at the highest scale (i.e. the unification scale) and determine which generators it breaks. The symmetry group that the unbroken generators generate, is the symmetry that remains. We then include the vevs at the next highest scale. But adding more vevs will only break more generators, so we only have to consider the action of the yet unbroken generators on the new vevs. We repeat this procedure until we have reached the lowest scale.

To see how the generators act on Φ_l, consider an infinitesimal transformation:

$$\Phi_l \rightarrow \Phi_l' = \Phi_l - i\alpha^a T_L^a \Phi_l + i\beta^a \Phi_l T_R^a \tag{A.2}$$

So the generators T_L^a act on the left of Φ_l, while the generators T_R^a act on the right of Φ_l with a minus sign. Note that since Φ_l^1 and Φ_l^2 are both singlets under $SU(3)_C$, the generators T_C^a always remain unbroken.

Let us first focus on the unification scale vevs M_1 and M_2. Since Φ_l^1 and Φ_l^2 have the same transformation properties, M_1 and M_2 break the same generators. So we focus just on M_1. The action of a general linear combination of the generators on M_1 is

$$\begin{aligned} &\sum_{a=1}^{8} \alpha^a T_L^a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_1 \end{pmatrix} \sum_{a=1}^{8} \beta^a T_R^a \\ &= \frac{M_1}{2} \begin{pmatrix} 0 & 0 & \alpha^4 - i\alpha^5 \\ 0 & 0 & \alpha^6 - i\alpha^7 \\ i\beta^5 - \beta^4 & i\beta^7 - \beta^6 & -2(\alpha^8 - \beta^8)/\sqrt{3} \end{pmatrix} \end{aligned} \tag{A.3}$$

We see that the linear combinations that annihilate the vev have

$$\alpha^4 \dots \alpha^7 = 0, \tag{A.4}$$

$$\beta^4 \dots \beta^7 = 0, \tag{A.5}$$

$$\alpha^8 = \beta^8 \tag{A.6}$$

So M_1 and M_2 leave the following generators unbroken:

$$T_L^1, T_L^2, T_L^3, \quad T_R^1, T_R^2, T_R^3, \quad T_L^8 + T_R^8. \tag{A.7}$$

Thus, the unbroken generators generate an $SU(2)_L \times SU(2)_R \times U(1)_{L+R}$ symmetry.

Next we include the intermediate scale vev M. The action of the yet unbroken generators on M is

$$\begin{aligned}\left(\sum_{a=1}^{3} \alpha^a T_L^a + \gamma^{L+R} T_L^8\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ M & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ M & 0 & 0 \end{pmatrix} \left(\sum_{a=1}^{3} \beta^a T_R^a + \gamma^{L+R} T_R^8\right) \\ = \frac{M}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\beta^3 - \sqrt{3}\gamma^{L+R} & -\beta^1 + i\beta^2 & 0 \end{pmatrix}\end{aligned} \tag{A.8}$$

To annihilate the vev we need

$$\beta^1 = \beta^2 = 0, \quad \beta^3 = -\frac{\gamma^{L+R}}{\sqrt{3}}, \tag{A.9}$$

which leaves the following generators unbroken:

$$T_L^1, T_L^2, T_L^3, \quad Y = T_R^3 - \frac{1}{\sqrt{3}}(T_L^8 + T_R^8), \tag{A.10}$$

where Y is the hypercharge generator. So at the intermediate scale $SU(2)_L \times SU(2)_R \times U(1)_{L+R}$ is broken to the electroweak symmetry group $SU(2)_L \times U(1)_Y$.

Finally, we include the electroweak scale vevs. Again, since Φ_l^1 and Φ_l^2 have the same transformation properties, $\langle\Phi_l^2\rangle$ breaks at least all generators that $\langle\Phi_l^1\rangle$ breaks. So we focus only on $\langle\Phi_l^2\rangle$. The action of the yet unbroken generators on v_2, b_2 and b_3 is

$$\begin{aligned}\left(\sum_{a=1}^{3} \alpha^a T_L^a - \frac{\gamma^Y}{\sqrt{3}} T_L^8\right) \begin{pmatrix} b_2 & 0 & b_3 \\ 0 & v_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \gamma^Y \begin{pmatrix} b_2 & 0 & b_3 \\ 0 & v_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(T_R^3 + \frac{1}{\sqrt{3}} T_R^8\right) \\ = \frac{1}{2} \begin{pmatrix} b_2(\alpha^3 - \gamma^Y) & v_2(\alpha^1 - i\alpha^2) & b_3(\alpha^3 - \gamma^Y) \\ b_2(\alpha^1 + i\alpha^2) & v_2(-\alpha^3 + \gamma^Y) & b_3(\alpha^1 + i\alpha^2) \\ 0 & 0 & 0 \end{pmatrix}\end{aligned} \tag{A.11}$$

So to annihilate the vev we need

$$\alpha^1 = \alpha^2 = 0, \quad \alpha^3 = \gamma^Y. \tag{A.12}$$

Note that including any one of the electroweak vevs would have led to these conditions, as can be seen in Eq. (A.11). So the electroweak vevs all break the same generators.

We are now left with a single unbroken generator, which is electric charge:

$$Q = T_L^3 + T_R^3 - \frac{1}{\sqrt{3}}(T_L^8 + T_R^8). \tag{A.13}$$

Thus, the symmetry that remains is $U(1)_Q$. The full symmetry breaking pattern can therefore be summarized as

$$\begin{aligned} G_{333} \times \mathbb{Z}_3 &\xrightarrow{M_1} SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R} \\ &\xrightarrow{M} SU(3)_C \times SU(2)_L \times U(1)_Y \\ &\xrightarrow{v_1} SU(3)_C \times U(1)_Q. \end{aligned} \tag{A.14}$$

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