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Category Theory, Watts' Theorem, and Homological Algebra

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Abstract

In this thesis, we develop the basics of category theory, both abstractly and through the use of examples from various fields in mathematics. We cover categories, functors, natural transformations, limits and colimits, and adjunctions. Using these categorical notions, we prove an important result in commutative algebra, called Watts' Theorem. This theorem states that the tensor product is the unique additive cocontinuous functor between module categories up to natural isomorphism. Finally we use a special class of categories called abelian categories to construct derived functors, which seek to extend left and right exact functors, and are used to generalize many (co)homological theories seen throughout topology. We end this last Chapter with a result that states that these derived functors can be computed by taking the homology of an acyclic resolution.

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0 Introduction

Throughout the twentieth century, it became clear that much of modern mathematics became reliant on thinking about algebraic and topological objects in terms of the mappings that connect them. In their paper *General Theory of Natural Equivalences* from 1945 [EM45], Samuel Eilenberg and Saunders Mac Lane first define the notions of categories, functors, and natural transformations, which are used to accommodate this modern view. The authors also apply this theory in the context of topology, namely for generalizing various (co)homological theories. Since then, category theory has gained much popularity throughout many fields of mathematics. Not just as a tool for generalizing numerous concepts from other mathematical fields, but also as a discipline of its own.

A category consists of two parts: a collection of objects, and a collection of morphisms. Each morphism has a domain and codomain object, and we can compose two morphisms if the codomain of the first matches the domain of the second. Moreover, each object has a designated identity morphism which acts as an identity under the composition operation. The standard examples of categories are the ones with structured sets as objects, and structure preserving functions between these objects as morphisms. There are categories of groups with group-homomorphisms, rings with ring-homomorphisms, vector spaces with linear maps, smooth manifolds with smooth maps, and many more. There are also numerous examples of categories which do not fit in this framework.

Category theory is not just useful for generalizing the properties that these categories have, it also provides a way to compare different categories with one another using functors. An example of such a functor is the fundamental group; this is a functor from the category of topological spaces and continuous maps (where we give each space a designated basepoint, and require the morphisms to preserve this basepoint) to the category of groups. What makes the fundamental group a functor is that a continuous map $(X, x) \to (Y, y)$ induces a homomorphism of groups $\pi_1(X, x) \to \pi_1(Y, y)$ in a way that preserves composition of morphisms.

In the first Chapter of this thesis, we explore how categories and functors are used to define and generalize many common constructions throughout different fields of mathematics. Key among these are limits, colimits, and adjunctions. Limits and colimits are special objects in the codomain of specific functors that satisfy a certain universal property. An adjunction is a pair of opposite pointing functors that encode a special duality relation between the morphisms in both categories.

Chapter 2 is mostly done in the category of R-modules, where R is a commutative ring with identity. We build the theory of R-modules up to prove a result called Watts' Theorem, which first appeared in the 1950s papers *Abstract Description of some Basic Functors* and *Intrinsic Characterizations of some Additive Functors* by Samuel Eilenberg and Charles Watts [Eil60, Wat60]. The theorem states that any functor between module categories that is 'nice enough' is naturally isomorphic to the tensor product functor. We also discuss how this result can be used in the theory of module-localization. Localizing a module looks like introducing fractions, where the numerators are elements of the module, and the denominators are elements of a certain subset of the underlying ring. As it turns out, the functor that takes a module to its localization is indeed 'nice enough' and so can be described using a tensor product.

Specifically, a functor is 'nice enough' if it preserves the zero module, addition of module-homomorphisms, direct sums of modules, and cokernels of module-homomorphisms. This last property is equivalent to the functor being right exact, meaning it takes a short exact sequence

$$0 \to A \to B \to C \to 0$$

to an exact sequence

$$FA \to FB \to FC \to 0.$$

The final Chapter considers so-called abelian categories, which are categories that resemble the category of abelian groups to such an extent that ideas like kernels, cokernels, exact sequences, the first isomorphism theorem, and more actually make sense. The main goal of this Chapter is to construct derived functors. These are functors that help us to extend exact sequences either on the left or the right, and can also be used as a measure of how close a functor is to being exact (meaning it preserves a short exact sequence on both sides).

This Chapter ends with an example from differential geometry: We show how the classical definition of de Rham cohomology coincides with the derived functors of a functor from the category of sheaves to the category of abelian groups. A full exploration of the underlying sheaf theory is out of the scope of this thesis, but Appendix A gives a short outline of the necessary definitions and results.

A word on conventions: In this thesis, we assume the Axiom of Choice as described at the beginning of the fifth chapter of [Jec07]. Unless stated otherwise, we assume all rings have an identity, and that ring-homomorphisms preserve this identity (that is, we presume the conventions of [LOT17]). In Chapter 2, we also assume all rings are commutative.

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1 Category Theory

The language of categories is affectionately known as 'abstract nonsense,' so named by Norman Steenrod. This term is essentially accurate and not necessarily derogatory: categories refer to 'nonsense' in the sense that they are all about the 'structure,' and not about the 'meaning,' of what they represent.

–Paolo Aluffi [Alu09]

This Chapter introduces the basic notions from category theory that we need for the rest of the thesis. We start by defining what a category is in this Section, along with some basic constructions like subcategories, isomorphisms and initial/terminal objects. The Section after this one defines *functors*, which are akin to mappings between categories. After this we define *natural transformations*, which in some sense are mappings between functors. The next Section covers *limits and colimits*, which are special objects that encompass many constructions we see in mathematics like products, kernels, direct sums and more. Finally we cover *adjunctions*, which consist of a pair of opposite pointing functors that have some special properties, most notably is that of preserving (co)limits.

A large focus throughout this entire Chapter is on examples. Truly understanding category theory requires understanding the numerous things it generalizes. Many of the examples are not necessary for the two Chapters on Watts' Theorem and derived functors, and are also taken from non-algebraic contexts like set theory, topology, and even analysis. Any specific examples needed for the later Chapters are highlighted, and redefined in more detail in those Chapters.

Most of the content of this first Chapter is adapted from Emily Riehl's *Category Theory in Context* [Rie16], which is a textbook that covers almost all the basics of category theory. The basic definitions, examples, and most of the notation is originally from this book.

1.1 Beginnings

Definition 1.1.1. A *category* consists of a collection of *objects*, and a collection of *morphisms* between these objects. Each morphism has a specified *domain* and *codomain* object. We typically denote a morphism f with domain A and codomain B as $f : A \to B$.
Along with this, every object A has an *identity morphism* $1_A : A \to A$. Given morphisms $f : A \to B$ and $g : B \to C$, there is a *composite morphism* $g \circ f : A \to C$. This composition law satisfies the following two axioms:

- For any $f : A \to B$, the composites $1_B \circ f$ and $f \circ 1_A$ are equal to f.
- If $f : A \to B$, $g : B \to C$ and $h : C \to D$ are morphisms, then the compositions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal, and thus can be denoted $h \circ g \circ f : A \to D$. We say composition is *associative*. ▼

Notation. In a category C, we denote the collection of objects as $\mathrm{Ob}(\mathsf{C})$, and the morphisms between objects A and B as $\mathsf{C}(A, B)$ or $\mathrm{Hom}(A, B)$ (named that way after the homomorphisms which appear in many algebraic categories). In the interest of clarity, we may denote the composition $g \circ f$ of morphisms as gf.

Often, it might be easier to display information about the composition of morphisms in a *commutative diagram*. This is a directed graph where the vertices represent objects and the arrows individual morphisms.

What makes a diagram commutative is that all paths with the same initial and terminal vertex through the directed graph yield the same resulting composite morphism. As an example, saying that the following diagram commutes is the same as saying the composition law in a category is associative: any path from A to D should yield the same composite morphism, so $(h \circ g) \circ f = h \circ (g \circ f)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & B \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

When drawing a commutative diagram, we often leave out identity morphisms and compositions that are implicitly included. We use a dashed '$\dashrightarrow$' arrow to draw the attention to a specific morphism, similar to how one might *italicize* words to emphasize them in a text.

Remark. For set-theoretic reasons, the objects and morphisms of a category cannot always exist in a set, otherwise we might encounter constructions like a 'set of all sets' which cannot exist. See [Shu08] for details. We mostly ignore this technical hiccup in this thesis, and refer to vague 'collections' of objects and morphisms instead. We call a category *small* if its morphisms actually do form a set, and *locally small* if, for all objects A and B, the morphisms from A to B form a set. A category is said to be *large* if it is not locally small.

The following is a (non-exhaustive) list of categories. Not all of them are necessary to understand the later sections, but many of them return to help aid other examples in this Chapter. Any examples are necessary to know for Chapters 2 and 3 are denoted by a dagger (†).

Example 1.1.2. Many categories fall in the class where the objects are sets with a certain structure, and the morphisms are functions between these sets that preserve this structure. These are called *concrete Categories*. There are also a lot of 'exotic' categories which do not fit this description, a few of which are also highlighted here.

(i) (†) The category of sets, denoted Set, has sets as objects, and functions between sets as morphisms. Identity morphisms are given by the identity maps, and composition of morphisms is just the composition of functions.

(ii) Top has topological spaces as objects, and continuous maps as morphisms.

(iii) Eucl has open subsets of Euclidean spaces as objects, and continuously differentiable maps as morphisms.

(iv) Man has smooth real manifolds as objects, and smooth maps as morphisms.

(v) $\mathsf{Set}_*, \mathsf{Top}_*, \mathsf{Eucl}_*, \mathsf{Man}_*$ are the categories of *pointed* sets, topological spaces, Euclidean spaces and smooth manifolds. The objects are the same as their non-pointed counterpart, but each object has a designated basepoint. The morphisms are the same as well, with the stipulation that a morphism maps the basepoint of its domain to the basepoint of its codomain. In all of these categories, we denote the objects as (X, x), where X is an object of the non-pointed category, and x is an element of X.

(vi) (†) The categories Group, Ring, Field and Monoid have groups, rings, fields and monoids[1] as objects respectively. The morphisms are group–, ring–, field– and monoid-homomorphisms. This is where

[1] A *monoid* is a set equipped with a binary operation that is associative, and has an identity element. A monoid-homomorphism is a map $f : M \to N$ that preserves the binary operation, as well as the identity element. The nonnegative integers $\mathbb{Z}_{\geqslant 0}$ with the addition operation form the prototypical example of a monoid.

the name 'morphism' originally came from. In this thesis, we assume all rings are unitary and ring-homomorphisms preserve this unit, unless otherwise stated.

(vii) (†) For a ring R, the category Mod_R has left R-modules as objects, and R-module-homomorphisms as morphisms. A special case of this is $\mathsf{Mod}_{\mathbb{Z}}$, which is 'the same'[2] as Ab, the category of abelian groups with group-homomorphisms. In a similar vein, Mod_K for a field K is the same as Vect_K, the category of K-vector spaces with linear maps between them. We define ${}_R\mathsf{Mod}$ to be the category of right R-modules.

(viii) The category Quiver has *quivers* as objects. A quiver is a directed graph, but the vertices are allowed to have more than one arrow between them. Specifically, a quiver consists of a set of vertices V and a set of arrows E, along with two functions $s : E \to V$ and $t : E \to V$ which give the start and target of an arrow respectively. A morphism $m : (V, E, s, t) \to (V', E', s', t')$ of quivers consists of two maps $m_V : V \to V'$ and $m_E : E \to E'$ that are compatible with the source and target maps. Compatibility means that $m_V \circ s = s' \circ m_E$, and $m_V \circ t = t' \circ m_E$.

(ix) (†) For any category C, we can construct its *opposite category* C^{op}. This category has the same objects as C, but for every morphism $f : A \to B$ in C, the opposite category has an opposite morphism $f^{\mathrm{op}} : B \to A$ instead. Composition of morphisms is defined via $f^{\mathrm{op}} \circ g^{\mathrm{op}} := (g \circ f)^{\mathrm{op}}$ for morphisms f and g in C.

(x) For a ring R, we can consider the category Mat_R where the objects are positive integers, and the set of morphisms from n to m is the set of $m \times n$ matrices. The composition of morphisms is given by matrix multiplication, and the identity and associativity axioms are satisfied using the identity matrix and associativity of matrix multiplication. That is, if A is a matrix of size $m \times n$, and B is one of size $k \times m$, then we can form the matrix BA of size $k \times n$. Displayed in a commutative diagram, we have

$$\begin{tikzcd} n \arrow[r, "A"] \arrow[dr, "BA"'] & m \arrow[d, "B"] \\ & k. \end{tikzcd}$$

The opposite category can be viewed as having $n \times m$ matrices from n to m instead.

(xi) Given a group (or more generally, a monoid) G, we can construct the small category $\mathsf{B}G$. This category has a single object, denoted $\bullet$, and the set of morphisms $\mathsf{B}G(\bullet, \bullet)$ is just the set of elements of G. Composition of morphisms is given by the multiplication of elements of G. The identity morphism is the identity element of G, and associativity is guaranteed from the definition of a group. The opposite category coincides with the idea of the *opposite group* G^{op}, where the elements are the same as those of G, but multiplication is defined by $g \cdot_{\mathrm{op}} h := h \cdot g$.

(xii) A poset[3] $(P, \leqslant)$ can be viewed as a small category, where the objects are elements of P. For $p, q \in P$, there is a single morphism $p \to q$ if $p \leqslant q$, and no morphism from p to q otherwise. Transitivity of

[2] *Technically* the two categories are not the same but isomorphic, as defined in Section 1.2, but the difference is so minute that we may as well say they are equal.

[3] A *poset* is a set with a partial ordering $\leqslant$, i.e. not all elements are comparable with one another. This ordering has to satisfy $p \leqslant p$ (reflexivity), $p \leqslant q \leqslant r \implies p \leqslant r$ (transitivity), and $p \leqslant q \leqslant p \implies p = q$ (antisymmetry) for all p, q, r in the poset. A nice example of a poset is $(O(X), \subseteq)$, which is the set of open subsets of a topological space X with the subset-ordering.

the ordering makes the composition of morphisms possible. The opposite category also has a tangible meaning here, where now there is a morphism $p \to q$ if and only if $p \geqslant q$.

(xiii) The category Htpy has topological spaces as objects, and *homotopy classes* of continuous maps as morphisms. That is, given spaces X and Y, if two continuous maps $f, g \in \mathsf{Top}(X, Y)$ have a homotopy between them (as defined in e.g. [Arm83, definition 5.1, p.88]), then we consider f and g to be the same morphism in Htpy.

This category also has a 'pointed' version Htpy_*, with homotopy classes of continuous maps which keep the basepoint fixed.

(xiv) Any set can be turned into a category, where the objects of the category are elements of the set, and the only morphisms are the identity morphisms. We call a category with only identity morphisms a *discrete category*. A category is *indiscrete* if, for all its objects A and B, $\mathrm{Hom}(A, B)$ contains exactly one morphism.

(xv) (†) There is an empty category, denoted $\mathbf{0}$, with no objects or morphisms. The category $\mathbf{1}$ has one object and only an identity morphism. The category $\mathbf{2}$ has two objects (labelled 1 and 2) with identities, and a single morphism $1 \to 2$. Generally, we define the category $\mathbf{n}$ (for $n \in \mathbb{N}$) to be the poset $(\{1, \ldots, n\}, \leqslant)$, viewed as a category.

Definition 1.1.3. Given two categories C and D, we can form their *product category* $\mathsf{C} \times \mathsf{D}$. Objects in this category are pairs (A, B), where A is an object of C and B an object of D. A morphism $(A_1, B_1) \to (A_2, B_2)$ is given by a pair (f, g) where $f : A_1 \to A_2$ is a morphism in C and $g : B_1 \to B_2$ is a morphism in D. Composition is done component-wise: $(f_1, g_1) \circ (f_2, g_2) := (f_1 \circ f_2, g_1 \circ g_2)$, and identities are defined by $1_{(A,B)} := (1_A, 1_B)$.
The *disjoint union* of C and D, denoted $\mathsf{C} \amalg \mathsf{D}$, is a category where an object is either an object of C, or D. A morphism in this category is a morphism of either C, or D. ▼

Another important class of categories are those *generated by quivers*. These categories are useful for the construction of categories that have a specific 'shape', which we want to construct *limits* over, which we do in Section 1.4.

Definition 1.1.4. Let Q be a quiver. The *category generated by* Q, denoted $C(Q)$, has vertices of Q as objects, and arrows of Q as morphisms. Along with this, every object gains an identity morphism, and all possible compositions of arrows are added as morphisms as well. ▼

Example 1.1.5. Consider the following quiver Q with two vertices and two arrows:

$$\bullet \rightleftarrows \bullet.$$

The category $C(Q)$ it generates has two objects, say, A and B, and at least two morphisms $f : A \to B$ and $g : B \to A$:

$$A \underset{g}{\overset{f}{\rightleftarrows}} B.$$

This does not define a category yet though. Both objects still need an identity morphism, and we also need each of the compositions $fg, fgf, fgfg, \ldots$ and $gf, gfg, gfgf, \ldots$. With these morphisms included, $C(Q)$ is actually a category.

It is often useful to consider the sub-structures that many mathematical structures contain. This is no different for categories:

Definition 1.1.6. A category D is a *subcategory* of a category C if $\mathrm{Ob}(\mathsf{D}) \subseteq \mathrm{Ob}(\mathsf{C})$, and $\mathsf{D}(A, B) \subseteq \mathsf{C}(A, B)$ for all objects A and B in D.
We call D a *full subcategory* of C if, for all objects A and B in D, we have $\mathsf{D}(A, B) = \mathsf{C}(A, B)$. That is, if A and B are objects in the subcategory, then we require *all* morphisms between them in C to be included in the subcategory. ▼

Example 1.1.7. Some examples of subcategories include:

(i) $\mathsf{Vect}_K^{\mathrm{fd}}$, the category of finite-dimensional K-vector spaces with linear maps between them is a full subcategory of the category of K-vector spaces.

(ii) (†) Similarly, Ab is a full subcategory of Group, and $\mathsf{Set}^{\mathrm{fin}}$, the category of finite sets, is a full subcategory of Set.

(iii) The category of commutative rings CRing is a full subcategory of Ring, which in itself is a non-full subcategory of Rng, the category of (not necessarily unitary) rings with (not necessarily unit-preserving) ring-homomorphisms.

(iv) If G is a group and H a subgroup of G, then the one-object category $\mathsf{B}H$ is a (generally non-full) subcategory of $\mathsf{B}G$.

A common theme in mathematics, especially in algebra, is study objects in a category 'up to isomorphism'. We can define this concept with the use of a special class of morphisms:

Definition 1.1.8. A morphism $f : A \to B$ in a category is an *isomorphism*, or is *invertible*, if there is another morphism $g : B \to A$, such that $gf = 1_A$ and $fg = 1_B$. In this case, we say the objects A and B are *isomorphic* and write $A \cong B$. The morphism g is called the *inverse* of f, and is often denoted by f^{-1}, ▼

Proposition 1.1.9. *The inverse of an isomorphism is an isomorphism itself and is unique as well. Moreover, the identity morphism of any object is an isomorphism, as is the composition of isomorphisms.*

Proof. Let g be an inverse of an isomorphism f. This inverse g is an isomorphism because f is an inverse of it, which follows immediately from the definition. Regarding uniqueness, if g' is also an inverse of f, then we have $gf = 1_A = g'f$. Composing with g on the right gives, by associativity,

$$g(fg) = g'(fg) \implies g1_B = g'1_B \implies g = g',$$

so the inverse of f is unique.

For any object A, the identity 1_A is an isomorphism because it is its own inverse. Namely $1_A \circ 1_A = 1_A$ by definition of being an identity.

Now let $h : A \to B$ and $k : B \to C$ be two isomorphisms. Note that, by way of associativity,

$$(kh)(h^{-1}k^{-1}) = k(hh^{-1})k^{-1} = kk^{-1} = 1_C.$$

Similarly, we have $(hk)(k^{-1}h^{-1}) = id_A$, showing that $kh : A \to C$ is an isomorphism, with inverse $(kh)^{-1} = h^{-1}k^{-1}$. □

Example 1.1.10. Many examples of isomorphisms in the categorical sense coincide with those we are familiar with.

(i) (†) Isomorphisms in Set are bijections, isomorphisms in Top are homeomorphisms, and isomorphisms in $\mathsf{Group}, \mathsf{Ring}, \mathsf{Field}, \mathsf{Mod}_R$ are the familiar bijective homomorphisms.

(ii) Every morphism in the one-object category $\mathsf{B}G$ of a group G is an isomorphism. We call a category a *groupoid* if all of its morphisms are isomorphisms. With this terminology, one could define a group as a groupoid with a single object.

(iii) For any category C, we can define C^{iso} (sometimes called the *maximal groupoid* of C) to be the subcategory consisting of all the objects of C, but only keeping the isomorphisms and dropping the other morphisms. A consequence of the second part of Proposition 1.1.9 is that the maximal groupoid is a well-defined subcategory, as it contains identities and composition of the isomorphisms.

(iv) Given a ring R, the isomorphisms in Mat_R are exactly the invertible matrices. Since invertible matrices are square, this implies that any two distinct natural numbers are not isomorphic in this category.

(v) The isomorphisms in Htpy are exactly the classes of maps that define a homotopy equivalence between two topological spaces.

This highlights an important point in category theory: though different categories may share certain objects, their structure is defined by the morphisms between these objects. For example, in Set, the sets $\mathbb{Z}$ and $\mathbb{Q}$ are isomorphic, as they are both countable sets and thus have a bijection between them. While in Group or Ring, these objects are not isomorphic at all.

A logical next step would be to generalize the concepts of injective and surjective functions, which we do as follows:

Definition 1.1.11. We call a morphism $f : A \to B$ in a category:

- a *monomorphism* (or simply a *mono*, or *monic*) if, for all morphisms $g_1, g_2 : X \rightrightarrows A$,[4] we have that $fg_1 = fg_2$ implies $g_1 = g_2$. In other words, a monomorphism is *left-cancallable.*
- an *epimorphism* (or simply an *epi* or *epic*) if, for all morphisms $h_1, h_2 : B \rightrightarrows Y$, we have that $h_1 f = h_2 f$ implies $h_1 = h_2$. In other words, an epimorphism is *right-dancallable.*

Some more jargon for special kind of morphisms include *endomorphisms*, which are morphisms form an object to itself, and *automorphisms*, which are isomorphisms from an object to itself. ▼

One can show that any isomorphism is both monic and epic; namely composing with the inverse isomorphism proves the required implications. Monomorphisms in Set are exactly the injective functions, while epimorphisms are exactly the surjective ones. This idea also holds in other categories: monomorphisms and epimorphism in $\mathsf{Top}, \mathsf{Group}$ and Mod_R are exactly the corresponding injective and surjective morphisms repsectively. This comparison is not always accurate however. For example, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in Ring both monic and epic, but it is not surjective[5].

[4] The notation $g_1, g_2 : X \rightrightarrows Y$ means that the two morphisms are *parallel*, i.e. they have the same domain and codomain.

[5] The inclusion being monic follows from injectivity. As for it being epic, let $f, g : \mathbb{Q} \rightrightarrows R$ be two ring-homomorphisms. The image $f(a/b)$ is equal to $f(a \cdot (1/b)) = f(a) \cdot f(b)^{-1}$, so is completely determined by where it takes integers a and b, and the same holds for g. Thus, denoting $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ as the inclusion, we have that $f \circ \iota = g \circ \iota$ implies $f = g$, making the inclusion epic.

Some, but not all, monomorphisms $f : A \to B$ are left-cancellable because there is another morphism $k : B \to A$ such that $kf = 1_A$. These are called *split monomorphisms*. Similarly, we can define *split epimorphisms* as those that are right-cancellable by way of a morphism $h : B \to A$ such that $fh = 1_B$. In Set, all monomorphisms are split, as are all epimorphisms (assuming the Axiom of Choice).

One more categorical notion to define generalizes the concept of the trivial group object in Group. Since we usually cannot 'look inside' the objects like we can in concrete categories, we have to define this notion through morphisms as well.

Definition 1.1.12. We call an object A in a category C:

- *Initial* if, for all objects Y in C, there is a *unique* morphism $A \to Y$ (i.e. $\mathsf{C}(A, Y)$ is a singleton set for each Y);
- *Terminal* if, for all objects X in C, there is a *unique* morphism $X \to A$ (i.e. $\mathsf{C}(X, A)$ is a singleton set for each X);
- A *zero object* if it is both initial and terminal. ▼

Example 1.1.13. The following are examples of initial and terminal objects in different categories.

(i) (†) The empty set is initial in Set, with the only function $\varnothing \to S$ for a set S being the, admittedly vacuous, empty function. Any singleton set is terminal, as the only function from a set S to a singleton set is the one that maps all elements of S to the unique element in the singleton.

(ii) (†) The trivial group, denoted 0, is a terminal object in Group for the same reasons as in Set. Since group homomorphisms preserve the group identity element, we also have that there is only a single morphism from 0 to any other group. Therefore the trivial group is initial as well, and thus a zero object. Similarly, the zero module is a zero object in Mod_R for any ring R.

(iii) The category Field has no initial or terminal objects, since there are no morphisms between fields of different characteristic. However, if we consider the full subcategory Field_p of fields with fixed characteristic $p \geqslant 0$, then the prime field (which is isomorphic to $\mathbb{Q}$ if $p = 0$ and $\mathbb{F}_p$[6] if $p > 0$) forms an initial object.[7]

(iv) If a poset $(P, \leqslant)$ has a minimal element, then that element is an initial object when we view the poset as a category. If the poset has a maximal element, that element is terminal. If the category has a zero object, then we necessarily have that P contains a single element (this follows from antisymmetry of the $\leqslant$ relation).

As we highlighted above, not every category has initial and/or terminal objects. However, if a category does have these kind of objects, those objects are what we call 'essentially unique', meaning unique up to isomorphism.

Proposition 1.1.14. *The initial (resp. terminal, zero) object is unique up to isomorphism, if it exists*

[6] The field $\mathbb{F}_p$ is the finite field with p elements. For a prime p, this field is usually seen as $\mathbb{Z}/p\mathbb{Z}$.

[7] In any characteristic, a homomorphism $P \to L$ from the prime field to another field is fully determined by the image of $1 \in P$ [LOT17, section VIII.1]. Thus since field-homomorphisms fix the multiplicative unit, we are locked into a single possible homomorphism, making P initial. Existence is guaranteed by the field-homomorphism $P \to L$ defined by mapping P into the prime subfield of L.

Proof. Let I_1 and I_2 be two initial objects in C. By definition, there are unique morphisms $f_1 : I_1 \to I_2$ and $f_2 : I_2 \to I_1$. Composing these morphisms leaves us with an endomorphism $f_1 \circ f_2 : I_2 \to I_2$. Now since I_2 is initial, there can only be a single morphism from it to another object, including itself. Because C is a category, we require I_2 to have an identity morphism, and so uniqueness of this endomorphism tells us $f_1 \circ f_2 = 1_{I_2}$. Similarly, we also have $f_2 \circ f_1 = 1_{I_1}$. Therefore, as per Definition 1.1.8, I_1 and I_2 are isomorphic.

This argument can be *dualized* to show that terminal objects are unique up to isomorphism. This means that the same proof strategy works, except we reverse the direction of the morphisms involved. Another way to see it is that by replacing the category C above by C^{op} (see Example 1.1.2(ix)), we prove the statement that initial objects in C^{op} are unique up to isomorphism. Since initial objects in C^{op} are terminal in C, this shows that terminal objects are unique up to isomorphism.

Combining the two results shows that zero objects are unique up to isomorphism as well. □

Remark. It should be noted that we have so far only scratched the surface of the concept of *duality*. Almost every categorical definition or result has some kind of 'dual' variant, where everything is the same, except that the 'direction of the arrows have been reversed'. As we have seen, an object is initial/terminal in a category if and only if it is terminal/initial in its opposite category. Similarly, if $f : A \to B$ is a monomorphism, then its opposite $f^{\mathrm{op}} : B \to A$ is an epimorphism in the dual category. This idea of duality returns more substantially in the next section.

While it is interesting to generalize some of the notions of set theory and abstract algebra, the real power of category theory is being able to connect these notions between different categories. We start on this journey in the next section.

1.2 Functors and Variance

This Section covers *functors*, which can be seen as mappings between categories. These mappings allow one to 'change their perspective' and look at a category through a different lens. Important mathematical constructions like the fundamental group, Jacobian matrix, group actions, tensor products and more can be described using functors. This often gives a deeper and more general connection between distinct categories than if these concepts were discussed without the use of functors at all. These functors come in two flavours: the ones that preserve the domains and codomains of morphisms, and those that swap the domains and codomains. We call this distinction between the functors their *variance*.

Definition 1.2.1. Given categories C and D, a *covariant functor* $F : \mathsf{C} \to \mathsf{D}$ consists of the following data:

- A mapping $\mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$. We denote the image of an object A under F by $F(A)$ or FA.
- For all objects A and B in C, F induces a mapping $\mathsf{C}(A, B) \to \mathsf{D}(F(A), F(B))$. The image of a morphism f is denoted $F(f)$ or Ff.

These mappings are also required to preserve the composition law and identities. That is, given composable morphisms f and g in C, we have $F(f \circ g) = F(f) \circ F(g)$. Moreover, for any object A in C, we have $F(1_A) = 1_{F(A)}$. These two requirements are also often called *functoriality*, we say F maps objects in C to objects in D *functorially*.

A *contravariant functor* $F : \mathsf{C} \to \mathsf{D}$ works the same on objects as a covariant one, but has a mapping $\mathsf{C}(A, B) \to \mathsf{D}(F(B), F(A))$ that preserves identities but *reverses* compositions. Namely, for f and g composable in C, we have $F(f \circ g) = F(g) \circ F(f)$. ▼

Remark. A contravariant functor $\mathsf{C} \to \mathsf{D}$ corresponds exactly to a covariant functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ (or, equivalently, a covariant functor $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$). In the interest of brevity, we may write that $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ is a functor; the variance is clear from the notation.

Just as how group-homomorphisms are functions that preserve the inner group structure (the group operation), and continuous maps are functions that preserve the structure of topological spaces (the open sets), functors can be seen as functions that preserve the structure of categories. What determines the structure of categories are its morphisms: their domain/codomain, as well as compositions and identities.

Example 1.2.2. The following is a list of examples of functors. Again, most of these are not be used in the later Chapters, but seeing more examples helps to make it more clear why one might find functoriality important.

(i) The power set is a covariant functor $P : \mathsf{Set} \to \mathsf{Set}$. It maps a set A to its power set $P(A)$, and a function $f : A \to B$ to the 'forward image' function $f_* : P(A) \to P(B)$. This map takes a subset $S \subseteq A$ and sends it to the set $f(S) = \{b \in B \mid f(s) = b \text{ for some } s \in S\}$.

We can also view the power set contravariantly, namely by sending a function $f : A \to B$ to the 'pre-image' function $f^{-1} : P(B) \to P(A)$, which sends a subset $T \subseteq B$ to $f^{-1}(T) = \{a \in A \mid f(a) \in T\}$.

(ii) The dual of a vector space can be viewed as a functor $(-)^* : \mathsf{Vect}_K^{\mathrm{op}} \to \mathsf{Vect}_K$ (note the variance!) which sends a K-vector space V to its dual space $V^* := \{f : V \to K \mid f \text{ is linear}\}$. The functor sends a linear map $L : V \to W$ to its dual (or transpose) $L^* : W^* \to V^*$. Functoriality tells us that the transpose of the identity map is again an identity, and that composable linear maps L_1 and L_2 satisfy $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$.

(iii) There is a functor $O : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$ that sends a topological space X to its set of open subsets $O(X)$ (i.e. its *topology*). A continuous function $f : X \to Y$ is sent to the pre-image function $f^{-1} : O(Y) \to O(X)$. This function indeed maps open subsets of Y to open subsets of X, precisely by the definition of continuous maps.

A similar functor is $C : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$, that sends a space X to its set of closed subsets. Continuity guarantees that this is well-defined, by the fact that the pre-image of a closed set under a continuous map is closed.

(iv) Given a topological space X, its set of open sets $O(X)$ is a poset with respect to the '$\subseteq$' relation, and can thus be seen as a category (see Example 1.1.2 (xii)). A functor $(O(X), \subseteq)^{\mathrm{op}} \to \mathsf{Set}$ is exactly a *presheaf of sets* on X. The prototypical example of such a presheaf is $C^0(-, \mathbb{R})$, which sends an open subset U to the set of continuous functions $U \to \mathbb{R}$. The morphism that encodes the relation $U \subseteq V$ is sent to the *restriction function* $C^0(V, \mathbb{R}) \to C^0(U, \mathbb{R})$ that sends a continuous function $f : V \to \mathbb{R}$ to the restriction $f|_U : U \to \mathbb{R}$.

Appendix A goes more into detail of the theory of (pre)sheaves, specifically where the codomain category Set is replaced by Ab.

(v) The fundamental group is a covariant functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Group}$. Given a topological space X with a specified basepoint $x \in X$, the fundamental group $\pi_1(X, x)$ is the group of homotopy classes of paths $\alpha : [0, 1] \to X$, such that $\alpha(0) = \alpha(1) = x$. The group operation is given by concatenating two loops, see [Arm83, chapter 5] for details. A continuous and basepoint preserving map $f : (X, x) \to (Y, y)$ induces a group homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ that sends a homotopy class of paths $[\alpha]$ to the class $[f \circ \alpha]$. In fact, for any positive integer n, there is a functor $\pi_n : \mathsf{Top}_* \to \mathsf{Group}$ that sends a pointed topological space to its n-th homotopy group.

(vi) The *jacobian matrix* is a covariant functor $D : \mathsf{Eucl}_* \to \mathsf{Mat}_{\mathbb{R}}$ from pointed Euclidean spaces to real matrices. It sends a Euclidean space $U \subseteq \mathbb{R}^n$ to its dimension n, and a basepoint preserving C^1 map $f : (U,p) \to (V,q)$ to the Jacobian matrix Df_p, evaluated at p. Functoriality is given by the *Chain Rule*, which states that $D(f \circ g)_p = Df_{g(p)} Dg_p$.

(vii) In a similar vein, the *tangent space* of a smooth manifold is a covariant functor $T : \mathsf{Man}_* \to \mathsf{Vect}_{\mathbb{R}}$ that sends a pointed smooth manifold (M,p) to the tangent space T_pM. A smooth map $f : (M,p) \to (N,q)$ is sent to the differential $df_p : T_pM \to T_qN$. The *cotangent space* is also a funtor, though it is contravariant.

(viii) There is a covariant functor $Q : \mathsf{Domain}^{\mathrm{inj}} \to \mathsf{Field}$ from the category of integral domains(i.e. commutative rings with no zero divisors) with injective ring-homomorphisms to the category of fields. It sends a domain R to its *field of fractions* $Q(R)$, as defined in [LOT17, section I.3]. An injective ring-homomorphism $\varphi : R \to S$ is sent to a field homomorphism $\bar{\varphi} : Q(R) \to Q(S)$, with $\bar{\varphi}(a/b) := \varphi(a)/\varphi(b)$. Because φ is injective, the denominator of the image of $\bar{\varphi}$ is never 0, which makes the functor well-defined.

(ix) (†) There is a family of *forgetful functors* from a concrete category to Set, which 'forgets' the additional structure of the objects and just looks at them as sets. For example, the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ takes a group G and sends it to the underlying set, which we denote by UG. A group-homomorphism is sent to the underlying set-function.

(x) The *free group* is a covariant functor $F : \mathsf{Set} \to \mathsf{Group}$. It sends a set S to the free group $\langle S \rangle$, which consists of finite strings of elements of S, along with formal inverses of these elements, where concatenation of strings is the group operation. A function $f : S \to T$ induces a group homomorphism $\langle S \rangle \to \langle T \rangle$ that sends a string of elements of S to the string of the images of those elements. There is a nice connection between this free functor F and the forgetful functor U, which we see in more detail in Section 1.5.

(xi) Given groups G and H, a functor $F : \mathsf{B}G \to \mathsf{B}H$ is exactly a group-homomorphism on morphisms, since preserving composition of morphisms in these categories coincides with preserving the group operation.

More generally, a covariant functor $X : \mathsf{B}G \to \mathsf{Set}$ maps the object of the domain to some set X, and the group-homomorphisms to automorphisms of X.[8] This is what we call a *group action*, as defined in [DF04, section 1.7]. Functoriality tells us that, if we denote the action by $\cdot : G \times X \to X$, we have $(gh) \cdot x = g \cdot (h \cdot x)$ and $e \cdot x = x$ for all elements $g, h \in G$, identity $e \in G$, and x in the set X that G acts on. Similarly, a functor $\mathsf{B}G \to \mathsf{Vect}_K$ is a *representation* of the group G as a subgroup of the automorphism group of some K-vector space.

This can be generalized further: given a quiver Q, viewed as a category it generates as per Definition 1.1.4, a *quiver representation* is a covariant functor $C(Q) \to \mathsf{Vect}_K$.

(xii) (†) The identity $1_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$, which sends an object and morphism to itself, is a covariant functor. Given an object D of D, the constant functor $D : \mathsf{C} \to \mathsf{D}$, which sends every object to D and every morphism

[8] This is a consequence of functors preserving isomorphisms. That is, if f is an isomorphism in C with inverse f^{-1}, and $F : \mathsf{C} \to \mathsf{D}$ is a functor, then $F(f)$ is an isomorphism, with inverse $F(f^{-1})$ This is immediate from the axioms of functoriality. And indeed, the isomorphisms in $\mathsf{C}(X,X)$ for any object X form a group under composition.

to the identity 1_D, is a covariant functor as well. If C is a subcategory of D, there is a straightforward inclusion functor $I : \mathsf{C} \hookrightarrow \mathsf{D}$.

(xiii) (†) Given an object A in a locally small category C, we can construct the covariant and contravariant *Hom-functors* represented by A. The covariant functor $\mathrm{Hom}(A, -) : \mathsf{C} \to \mathsf{Set}$ sends an object B to the set of morphisms $\mathrm{Hom}(A, B)$. A morphism $f : B \to C$ is sent to the *pushforward* function $f_* : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$. This pushforward takes a morphism $g : A \to B$, and left-composes it with f to make $f_*(g) := f \circ g : A \to C$.

The contravariant Hom-functor $\mathrm{Hom}(-, A) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ takes an object B and sends it to $\mathrm{Hom}(B, A)$. A morphism $f : B \to C$ is sent to the *pullback* $f^* : \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ which takes a morphism $g : C \to A$ and right-composes it with f to make $f^*(g) := g \circ f : B \to A$.

$$\mathsf{C} \xrightarrow{\mathrm{Hom}(A,-)} \mathsf{Set} \qquad\qquad \mathsf{C}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,A)} \mathsf{Set}$$

$$\begin{tikzcd} B \arrow[d, "f"'] & & \mathrm{Hom}(A,B) \arrow[d, "f_* \ (f_*(g)=f\circ g)"] \\ C \arrow[r, phantom, ""] & {} \arrow[u, phantom, ""] & \mathrm{Hom}(A,C) \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "f"'] & & \mathrm{Hom}(B,A) \\ C & & \mathrm{Hom}(C,A) \arrow[u, "f^* \ (f^*(g)=g\circ f)"'] \end{tikzcd}$$

More generally, the *hom-bifunctor* $\mathrm{Hom}(-, -) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$ is a functor that takes two objects to the set of morphisms between them. It is contravariant in the first argument and covariant in the second.

(xiv) (†) For any ring R, and R-module T, the tensor product $T \otimes_R (-) : \mathsf{Mod}_R \to \mathsf{Mod}_R$ is a covariant functor that sends an R-module N to the tensor product $T \otimes_R N$. An R-module-homomorphism $\varphi : N \to P$ is sent to the homomorphism $1_T \otimes \varphi : T \otimes_R N \to T \otimes_R P$, which acts on elementary tensors by $(1_T \otimes \varphi)(t \otimes n) = t \otimes \varphi(n)$. We define the tensor product in more detail in 2.

Given functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$, we can form their composition $GF : \mathsf{C} \to \mathsf{E}$, which sends an object A in C to $G(F(A))$, and a morphism f to $G(F(f))$. This composition has some interesting properties:

Proposition 1.2.3. *Given functors $F : \mathsf{A} \to \mathsf{B}$ and $G : \mathsf{B} \to \mathsf{C}$, the following hold:*

(a). *The composition $GF : \mathsf{A} \to \mathsf{C}$ is a functor. It is covariant if and only if F and G have the same variance;*

(b). *The compositions $1_{\mathsf{B}}F$ and $F1_{\mathsf{A}}$ are both equal to F;*

(c). *Compostition of functors is associative.*

Proof. (a). For GF to be a functor, it needs to preserve identities and composition. Let A be an object in A, then note $GF(1_A) = G(F(1_A)) = G(1_{F(A)}) = 1_{GF(A)}$. As for composition, first assume both F and G are covariant, and let $f : A \to A'$ and $g : A' \to A''$ be morphisms in A. Then, we have

$$GF(g \circ f) = G(Fg \circ Ff) = GFg \circ GFf \quad \text{(covariant)}.$$

If F and G are both contravariant, we have

$$GF(g \circ f) = G(Ff \circ Fg) = GFg \circ GFf \quad \text{(covariant)}.$$

Now assume the functors have distinct variance, say F is covariant while G is contravariant. Then the composition evaluates to

$$GF(g \circ f) = G(Fg \circ Ff) = GFf \circ GFg \quad \text{(contravariant)}.$$

The same also holds if F is contravariant and G is covariant.

(b). Let A be an object of A, then note $1_{\mathsf{B}}F(A) = 1_{\mathsf{B}}(F(A)) = F(A)$ and $F1_{\mathsf{A}}(A) = F(1_{\mathsf{A}}(A)) = F(A)$. For the same reason, the compositions take a morphism f to $F(f)$. Thus indeed $1_{\mathsf{B}}F = F1_{\mathsf{A}} = F$.

(c). Let A be an object of A, then we have

$$(H(GF))(A) = H(GF(A)) = H(G(F(A))) = (HG)(F(A)) = ((HG)F)(A).$$

Replacing A above by some morphism f, we also find $(H(GF))(f) = ((HG)F)(f)$. Thus indeed it follows that $H(GF) = (HG)F$. $\square$

Example 1.2.4. This proposition implies that we can form some kind of 'category of categories', where the objects are categories and the morphisms are functors between them. Defining it like this actually conflicts with size issues, as we might say that this category includes the 'category of categories that don't contain themselves', which runs into Russel's paradox. So instead, we define Cat to be the category of *small* categories, with functors between them. This category is large itself, so it is not an object of itself and thus avoids the problematic paradox.

There is a functor $\mathrm{Ob} : \mathsf{Cat} \to \mathsf{Set}$ that sends a small category to its set of objects, and a functor to the underlying function between the sets of objects. The forgetful functor $U : \mathsf{Cat} \to \mathsf{Quiver}$ sends a category to its underlying quiver, where we forget the fact that morphisms (arrows in the quiver) can be composed with one another. The empty category $\mathbf{0}$ is an initial object, with a single 'empty functor' to every other small category. The singleton category $\mathbf{1}$ is terminal, with only the constant functor from another category to it. Cat has a full subcategory $\mathsf{Groupoid}$ with small groupoids as objects.

The isomorphisms in this category are exactly the functors that are invertible:

Definition 1.2.5. A functor $F : \mathsf{C} \to \mathsf{D}$ is an *isomorphism of categories* if there is another functor $G : \mathsf{D} \to \mathsf{C}$ such that $FG = 1_{\mathsf{D}}$ and $GF = 1_{\mathsf{C}}$. We say C and D are *isomorphic*, and write $\mathsf{C} \cong \mathsf{D}$. ▼

This concept of isomorphism between categories is strong, and it is often useful to use a weaker notion of 'equivalence', which we define in Section 1.3.

There are two more properties of functors which correspond, in a certain sense, to the notion of 'local injectiveness' and 'local surjectiveness'.

Definition 1.2.6. Given a covariant (resp. contravariant) functor $F : \mathsf{C} \to \mathsf{D}$ and objects A and B in C, we call the functor

- *faithful* if the map $\mathsf{C}(A, B) \to \mathsf{D}(F(A), F(B))$ (resp. $\mathsf{C}(A, B) \to \mathsf{D}(F(B), F(A))$) is injective;
- *full* if the map $\mathsf{C}(A, B) \to \mathsf{D}(F(A), F(B))$ (resp. $\mathsf{C}(A, B) \to \mathsf{D}(F(B), F(A))$) is surjective. ▼

Example 1.2.7. Here we list some examples of full and faithful functors:

(i) The action of a group G on a set X is faithful (as defined in [DF04, section 4.1]) if and only if the corresponding functor $\mathsf{B}G \to \mathsf{Set}$ is faithful.

(ii) If C is a subcategory of D, then the corresponding inclusion functor $\mathsf{C} \hookrightarrow \mathsf{D}$ is faithful and injective on objects. The inclusion is full if and only if C is a full subcategory of D.

(iii) We define a *concrete category* to be a category C with a faithful functor $U : \mathsf{C} \to \mathsf{Set}$. These are usually the evident forgetful functors from Example 1.2.2(ix).

A functor that is full and faithful is called *fully faithful*. An important property of these functors is that they reflect isomorphisms:

Proposition 1.2.8. *Let $F : \mathsf{C} \to \mathsf{D}$ be a fully faithful functor, and let X and Y be objects of C. If $FX \cong FY$, then $X \cong Y$.*

Proof. Let $g : FX \to FY$ be the isomorphism and $g^{-1} : FY \to FX$ its inverse. F being fully faithful implies that g and g^{-1} have unique corresponding maps $f : X \to Y$ and $f' : Y \to X$ such that $Ff = g$ and $Ff' = g^{-1}$. To verify that f is invertible, with f' as its inverse, Note that the composition $f'f : X \to X$ is mapped to $F(f'f) = g^{-1}g = 1_{FX}$. Similarly, the identity 1_X is also mapped to 1_{FX}. Faithfulness of F implies that $f'f = 1_X$. The same argument can be used to show $ff' = 1_Y$. Thus, f is an isomorphism between X and Y. $\square$

1.3 Natural Transformations and Equivalence

One way to motivate the definition of a natural transformation is as a mapping between functors. Given a pair of functors $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$, and a morphism $f : A \to B$ in C, the functors F and G map this morphism to the following two morphisms respectively:

$$\begin{tikzcd} FA \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB & GB. \end{tikzcd}$$

There are many ways to define some a relation from F to G, but the way we displayed the images of the functors above hints to a nice way to do so. 'Completing' the diagram above by adding morphisms $FA \to GA$ and $FB \to GB$ is exactly what a natural transformation is.

As it turns out, these natural transformations do not just give ways to compare functors, but also the objects that they map to. As an example from finite-dimensional linear algebra, the vector spaces V, V^*, and $V^{**} := (V^*)^*$ are all isomorphic because they have the same dimension. However the isomorphism $V \cong V^{**}$ is 'special' in that the isomorphism $v \mapsto \mathbf{ev}_v$ (with $\mathbf{ev}_v(f) := f(v)$ for $f \in V^*$) feels more natural than the basis-dependent isomorphism $V \cong V^*$. This Section defines this idea in more detail using *natural isomorphisms*.

Definition 1.3.1. Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors of the same variance between categories C and D. A *natural transformation* $\eta : F \Rightarrow G$ consists of a collection of morphisms $\eta_A : FA \to GA$ in D for every object A in C. These morphisms are called the *components* of the natural transformation. We require that, for all morphisms

$f : A \to B$ in C, the components satisfy $Gf \circ \eta_A = \eta_B \circ Ff$, i.e. the *naturality square*

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"'] & GB \end{tikzcd}$$

commutes. If each η_A is an isomorphism in D, we call η a *natural isomorphism* and write $F \cong G$. We say the objects FA and GA are *naturally isomorphic* in this case. ▼

The main result from Chapter 2 is statement about a natural isomorphism, so it is important we have a good grasp of this concept. As such, the following examples are given in more detail than we have done so far.

Example 1.3.2. As alluded before, the functors $1_{\mathsf{Vect}_K^{\mathrm{fd}}}$ and $(-)^{**}$ from $\mathsf{Vect}_K^{\mathrm{fd}}$ to itself are naturally isomorphic for any field K. The double dual sends a linear map $L : V \to W$ to the double transpose $L^{**} : V^{**} \to W^{**}$. This map takes a functional $\mu : V^* \to K$ and sends it to $L^{**}(\mu) : W^* \to K$. This functional is defined on functionals $f \in W^*$ by $L^{**}(\mu)(f) := \mu(f \circ L) \in K$.

The natural isomorphism $\mathbf{ev} : 1_{\mathsf{Vect}_K^{\mathrm{fd}}} \Rightarrow (-)^{**}$ is defined component-wise as $\mathbf{ev}_V : V \to V^{**}$, by taking a vector v and sending it to $\mathbf{ev}_{V,v} : V^* \to K$. This morphism takes a functional $f : V \to K$ and sends it to $f(v)$. Now, let V and W be finite-dimensional K-vector spaces, and $L : V \to W$ a linear map. To prove naturality, we verify that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}_V"] \arrow[d, "L"'] & V^{**} \arrow[d, "L^{**}"] \\ W \arrow[r, "\mathrm{ev}_W"'] & W^{**}. \end{tikzcd}$$

To that end, let v be a vector in V and f a functional in W^{**}. The top path of the square is given by $L^{**}(\mathbf{ev}_V(v)) = L^{**}(\mathbf{ev}_{V,v})$, which acts on f by

$$L^{**}(\mathbf{ev}_{V,v})(f) = \mathbf{ev}_{V,v}(f \circ L) = f(L(v)).$$

The other path of the square is $\mathbf{ev}_W(L(v)) = \mathbf{ev}_{W,L(v)}$, which acts on f by $\mathbf{ev}_{W,L(v)}(f) = f(L(v))$, which is exactly what we wanted. Since v and f were picked arbitrarily, we have $L^{**} \circ \mathbf{ev}_V = \mathbf{ev}_W \circ L$, proving naturality.

In the category of all K-vector spaces, these components give a natural transformation $\mathbf{ev} : 1_{\mathsf{Vect}_K} \Rightarrow (-)^{**}$. But in finite dimensions, it is an isomorphism as well. To see this, note that if $\mathbf{ev}_{V,v} = \mathbf{ev}_{V,v'}$ for some $v, v' \in V$, then $f(v) = f(v')$ for all $f \in V^*$. Using linearity, we find that $f(v - v') = 0$ for *every* functional $f : V \to K$, meaning that $v - v' = 0$ necessarily.[9] Thus $v = v'$ and the map $\mathbf{ev}_V$ is injective. Since we are dealing with finite dimensional spaces, a consequence of the Rank-Nullity Theorem states that $\mathbf{ev}_V$ is surjective too, thus an isomorphism. This argument holds for all finite-dimensional vector spaces V, and so $\mathbf{ev}$ is a natural isomorphism between the identity and double dual functors.

All of the above arguments also follow for arbitrary vector spaces (including the basis part, a consequence of the Axiom of Choice is that every vector space has a (potentially infinite) basis [Bar14, lemma 3.1, p.5].).

[9] If $v - v'$ were nonzero, then $\{v - v', v_2, \ldots, v_n\}$ forms a basis of V given some vectors $v_2, \ldots, v_n$. Now we can define a functional $g : V \to K$ so that $g(v - v') = 1$ and $g(v_i) = 0$. But this is a functional on which $v - v'$ does not vanish, contradiction!

The problem is that $\mathbf{ev}_V$ being injective does not imply it is surjective in infinite dimensions. Regardless, for these spaces, there is still a natural transformation $\mathbf{ev} : 1_{\mathsf{Vect}_K} \Rightarrow (-)^{**}$. The category of *reflexive* vector spaces is a full subcategory containing exactly the vector spaces for which $\mathbf{ev}$ is a natural isomorphism.

Example 1.3.3. The topology of a space X can be described using its open sets, but can just as well be described by its closed sets. The same holds true for many other topological properties. This can be made more formal by the fact that the sets $O(X)$ of open subsets and $C(X)$ of closed subsets are not just isomorphic as sets, but that this isomorphism is 'natural in X'. What we mean by this is that the functors $O, C : \mathsf{Top}^{\mathrm{op}} \rightrightarrows \mathsf{Set}$ as described in Example 1.2.2(iv) are naturally isomorphic.

Now by definition, a subset of X is closed if its complement is open. This suggests a natural choice of function $O(X) \to C(X)$, namely we take the complement of an open set with respect to X. Thus, given a continuous map $f : X \to Y$, we wish to show the following square commutes (recall that O and C are contravariant!):

$$\begin{tikzcd} O(Y) \arrow[r, "Y\setminus -"] \arrow[d, "f^{-1}"'] & C(Y) \arrow[d, "f^{-1}"] \\ O(X) \arrow[r, "X\setminus -"'] & C(X). \end{tikzcd}$$

To that end, let U be an open subset of Y. The top half of the square is evaluated to be

$$f^{-1}(Y \setminus U) = \{x \in X \mid f(x) \in Y \setminus U\} = \{x \in X \mid f(x) \notin U\} = \{x \in X \mid x \notin f(U)\} = X \setminus f^{-1}(U),$$

which is exactly what the bottom half of the square is equal to.

Every component is invertible, following from the fact that $X \setminus (X \setminus U) = U$. Thus the complement is a natural isomorphism between O and C.

Example 1.3.4. There are two functors GL_n and $(-)^{\times} : \mathsf{CRing} \rightrightarrows \mathsf{Group}$ from the category of commutative rings to the category of groups. Given a positive integer n, GL_n takes a ring R to the group of invertible $n \times n$ matrices with entries in R, $\mathrm{GL}_n(R)$. A ring-homomorphism $\varphi : R \to S$ is sent to the group-homomorphism $\bar{\varphi} : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ that takes a matrix $A = (a_{ij})$ and applies φ to each entry to obtain a matrix $\bar{\varphi}(A) = (\varphi(a_{ij}))$. Note that if A is invertible, $\bar{\varphi}(A)$ is invertible too, with inverse $\bar{\varphi}(A^{-1})$.

The functor $(-)^{\times}$ takes a ring R and sends it to its group of units $R^{\times}$. A ring homomorphism $\varphi : R \to S$ is sent to the restriction $\varphi|_{R^{\times}} : R^{\times} \to S^{\times}$. Given a unit r with inverse r^{-1} in $R^{\times}$, $\varphi(r)$ is invertible with inverse $\varphi(r^{-1})$ because φ is a ring-homomorphism. So $\varphi|_{R^{\times}}$ is a group-homomorphism from $R^{\times}$ to $S^{\times}$.

As is shown in e.g. [DF04, theorem 11.4.30, p.440], a square matrix is invertible if and only if its determinant is a unit in R. This suggests a relationship between invertible matrices and the group of units of a ring via the determinant. Indeed, the determinant gives a natural transformation $\det : \mathrm{GL}_n \Rightarrow (-)^{\times}$. For every commutative ring R, the components of the transformation are given by the determinant-homomorphisms $\det_R : \mathrm{GL}_n(R) \to R^{\times}$. The commutativity of the naturality square

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\det_R"] \arrow[d, "\bar{\varphi}"'] & R^{\times} \arrow[d, "\varphi|_{R^{\times}}"] \\ \mathrm{GL}_n(S) \arrow[r, "\det_S"'] & S^{\times} \end{tikzcd}$$

is rather straightforward to verify. Given an invertible matrix $A = (a_{ij}) \in \mathrm{GL}_n(R)$, the top path of the square evaluates to $\varphi(\det(A))$, while the bottom path evaluates to $\det(\bar{\varphi}(A))$. Now, the determinant of A is a combination of the entries of A using the operations on R, all of which is preserved by φ. So $\varphi(\det(A))$ is the same as computing the determinant of the matrix $(\varphi(a_{ij}))$, which is exactly $\det(\bar{\varphi}(A))$.

The determinant is generally *not* a natural isomorphism (the determinant is usually not injective: different matrices can have the same determinant), but it still highlights a connection between invertible matrices and invertible elements of the underlying ring. One that feels rather canonical if we were to write

$$\mathrm{GL}_n(R) = \{A \in M_n(R) \mid \det(A) \in R^\times\}.$$

Now if $n = 1$, then GL_1 and $(-)^\times$ *are* naturally isomorphic. This corroborates the common notion that (1×1)-matrices over R are just elements of R.

Like functors, natural transformations can also be composed. This can be done both vertically and horizontally:

Definition 1.3.5. • Given functors $F, G, H : \mathsf{C} \to \mathsf{D}$ and natural transformations $\eta : F \Rightarrow G$ and $\theta : G \Rightarrow H$, we can form their *vertical composition* $\theta \circ \eta : F \Rightarrow H$ component-wise by defining $(\theta \circ \eta)_A := \theta_A \circ \eta_A$ for all objects A in C. The term vertical composition comes from the following diagram, which displays the natural transformations vertically.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=50, "H"'{name=L}] & \mathsf{D}
\arrow[Rightarrow, from=U, to=M, "\eta"] \arrow[Rightarrow, from=M, to=L, "\theta"]
\end{tikzcd}$$

• If $F_1, G_1 : \mathsf{C} \rightrightarrows \mathsf{D}$ and $F_2, G_2 : \mathsf{D} \rightrightarrows \mathsf{E}$ are functors, with natural transformations $\eta : F_1 \Rightarrow G_1$ and $\theta : F_2 \Rightarrow G_2$, their *horizontal composition* $\theta * \eta : F_2F_1 \Rightarrow G_2G_1$ is constructed component-wise by defining $(\theta * \eta)_A$ to be the diagonal composition of the following commutative square:

$$\begin{tikzcd}
F_2F_1A \arrow[r, "\theta_{F_1A}"] \arrow[d, "F_2(\eta_A)"'] \arrow[dr, dashed] & G_2F_1A \arrow[d, "G_2(\eta_A)"] \\
F_2G_1A \arrow[r, "\theta_{G_1A}"'] & G_2G_1A.
\end{tikzcd}$$

The square itself commutes by naturality of θ, applied to the morphism $\eta_A : F_1A \to G_1A$, which makes $(\theta * \eta)$ well-defined. The term horizontal composition becomes evident when we display the functors involved as follows:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "F_1"{name=A}] \arrow[r, bend right, "G_1"'{name=B}] & \mathsf{D} \arrow[r, bend left, "F_2"{name=C}] \arrow[r, bend right, "G_2"'{name=D}] & \mathsf{E}
\arrow[Rightarrow, from=A, to=B, "\eta"] \arrow[Rightarrow, from=C, to=D, "\theta"]
\end{tikzcd}$$

• Given categories C and D, we can form their *functor category* with functors $\mathsf{C} \to \mathsf{D}$ as objects, natural transformation as morphisms, and composition is given by the vertical composition described above. This category is denoted as $[\mathsf{C}, \mathsf{D}]$. The identity natural transformation for a functor $F : \mathsf{C} \to \mathsf{D}$ is the natural transformation $1_F : F \Rightarrow F$ which is defined component-wise by $(1_F)_X := 1_{FX}$ for objects X in C. ▼

Remark. Using natural transformations, the category Cat can be viewed as a so-called 2-*category*. Such a category consists of objects, '1-morphisms' between those objects and '2-morphisms' between the 1-morphisms, satisfying a few composition laws we do not cover here. Indeed, Cat has small categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms. For details, see [Mac98, section XII.3].

A common theme in category theory is that the best way to study an object is to study its relation to other objects. *Representable functors* are special functors that follow this philosophy more closely.

Definition 1.3.6. A covariant (resp. contravariant) functor $F : \mathsf{C} \to \mathsf{Set}$ is *represented* by an object X of C if F is naturally isomorphic to the Hom-functor $\mathrm{Hom}(X, -)$ (resp. $\mathrm{Hom}(-, X)$). ▼

Example 1.3.7. The following are examples of representable functors:

(i) The forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ is represented by the group $\mathbb{Z}$. Namely, given a group G, its set of elements UG is in natural bijection with the set $\mathrm{Hom}(\mathbb{Z}, G)$; any group-homomorphism $\mathbb{Z} \to G$ is uniquely determined by the image of $1 \in \mathbb{Z}$, which can be sent to any element of G. Similarly, the forgetful functors from Ring and Mod_R to Set are represented by the ring $\mathbb{Z}[x]$ and the R-module R respectively.

(ii) The functor $\mathrm{Ob} : \mathsf{Cat} \to \mathsf{Set}$ that takes a small category to its set of objects is represented by the category $\mathbf{1}$. Indeed, functors $\mathbf{1} \to \mathsf{C}$ are in natural correspondence with objects of C, because such functors 'choose' an object of C. Similarly, the functor $\mathrm{Mor} : \mathsf{C} \to \mathsf{Set}$ that sends a small category to its set of morphisms is represented by $\mathbf{2}$.

(iii) The composition $U(-)^* : \mathsf{Vect}_K^{\mathrm{op}} \to \mathsf{Set}$ that takes a vector space to its set of dual vectors is represented by the vector space K. This follows by definition of the dual space: the elements of V^* are *exactly* linear maps $V \to K$, which gives an equality $UV^* = \mathrm{Hom}(V, K)$.

Though it is nice to know if a functor F is representable, it is also helpful to know how we might find a natural isomorphism between F and the corresponding Hom-functor. Another question is that of uniqueness of representing objects. That is, if F is represented by both X and X', can we guarentee that X and X' are isomorphic? These questions, among others, can be answered by the famous Yoneda Lemma:

Proposition 1.3.8 (Yoneda Lemma)**.** *Let* C *be a small category, and let* $F : \mathsf{C} \to \mathsf{Set}$ *be a covariant functor. For every object* X *of* C*, there is an isomorphism*

$$\mathsf{Nat}(\mathsf{C}(X, -), F) \cong F(X)$$

between the set of natural transformations between the Hom-functor $\mathsf{C}(X, -)$ *and* F*, and the set* $F(X)$*. Moreover, this isomorphism is natural in both* X *and* F*. The contravariant Yoneda Lemma states that for functors* $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$*, and any object* X *of* C*, there is a natural isomorphism*

$$\mathsf{Nat}(\mathsf{C}(-, X), F) \cong F(X).$$

A full proof of this result is given in [Rie16, theorem 2.2.4, p.57]. One of the most useful corollaries of the Yoneda Lemma is the following:

Corollary 1.3.9. *Let* C *be as above. The functor* よ $: \mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ *that takes an object* Y *of* C *to the functor* $\mathsf{C}(-, Y)$ *is fully faithful. Dually, the functor* $よ^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to [\mathsf{C}, \mathsf{Set}]$ *that takes an object* X *to the functor* $\mathsf{C}(X, -)$ *is also fully faithful.*[10]

The reason this is useful comes from Proposition 1.2.8, which combines with the previous corollary to state that, if $\mathsf{C}(-, X)$ and $\mathsf{C}(-, X')$ are naturally isomorphic, then X and X' are isomorphic as objects. Dually, if $\mathsf{C}(X, -)$ and $\mathsf{C}(X', -)$ are naturally isomorphic, then X and X' are isomorphic as well. This also implies that the representing object of a representable functor is unique up to isomorphism. The Yoneda Lemma and this corollary reflects a more general philosophy in the field of category theory: To study an object is to study its relation (i.e. morphisms) to the objects around it.

As was mentioned in section 1.2, the idea of categories being isomorphic is very strict, and the following weaker notion is more common:

Definition 1.3.10. A functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence of categories* if there is another functor $G : \mathsf{D} \to \mathsf{C}$, and natural isomorphisms $FG \cong 1_{\mathsf{C}}$ and $GF \cong 1_{\mathsf{D}}$. We say the categories C and D are *equivalent*, and write $\mathsf{C} \simeq \mathsf{D}$. ▼

Any isomorphism of categories is an equivalence as well, namely by letting the natural isomorphisms just be the identity transformations.

Example 1.3.11. Examples of equivalence of categories include:

(i) The category $\mathbf{1}$ with a single object and only an identity morphism and the category D with two objects A and B, and two non-identity morphisms $f : A \to B$ and $g : B \to A$ satisfying $gf = 1_A$ and $fg = 1_B$ are equivalent. The equivalence $\mathbf{1} \to \mathsf{D}$ sends the one object of $\mathbf{1}$ to any of the two objects in D, while the inverse equivalence $\mathsf{D} \to \mathbf{1}$ is the constant functor.

$$\begin{tikzcd}
\mathbf{1} & \simeq & \mathsf{D} \\
\bullet \arrow[loop left, "1_\bullet"] & \simeq & A \arrow[loop right, "1_A"] \arrow[d, shift left, "f"] \\
 & & B \arrow[loop right, "1_B"] \arrow[u, shift left, "g"]
\end{tikzcd}$$

More generally, if a groupoid has at least one morphism between any two objects, it is equivalent to the automorphism group of any of its objects, seen as a one-object category. We call such a groupoid *connected*. To prove this, let C be a connected groupoid, and $G := \mathsf{C}(A, A)$ be the automorphism group of an object A of C. The inclusion functor $\mathsf{B}G \hookrightarrow \mathsf{C}$ sending the only object of the domain to A in C, and an element of G to itself, is an equivalence of categories as a consequence of Proposition 1.3.12 proven below.

(ii) Given a topological space X, we can construct its *fundamental groupoid* $\Pi_1(X)$.. The objects of this category are points of X, and the morphisms between two points are endpoint-preserving homotopy classes of paths between the two points. This also defines a functor from Top to $\mathsf{Groupoid}$. If the space X is path-connected, meaning there is a path between any pair of points, then $\Pi_1(X)$ is connected as a groupoid. Thus it is equivalent to the automorphism group of any of its objects.

[10] The functors よ and $よ^{\mathrm{op}}$ are called the *Yoneda embeddings*. The symbol used is the Japanese hiragana for the mora 'yo' which appears in name Nobuo Yoneda, who the Yoneda Lemma is named after.

For a point $x \in X$, the automorphism group in $\Pi_1(X)$ of this point is exactly the fundamental group $\mathsf{B}\pi_1(X, x)$ as a one-object category. These fundamental groups are equivalent to $\Pi_1(X)$ for any basepoint in X, so we have that $\mathsf{B}\pi_1(X, x) \simeq \mathsf{B}\pi_1(X, y)$ for all $x, y \in X$. An equivalence of one-object categories is the same as an isomorphism (the relevant natural isomorphisms consist of a single component). Therefore, if X is path-connected, then the fundamental group of X is independent of the basepoint, as they all give isomorphic fundamental groups.

(iii) For any field K, the categories Mat_K and $\mathsf{Vect}_K^{\mathrm{fd}}$ are equivalent. The equivalences are given by functors $K^{(-)} : \mathsf{Mat}_K \to \mathsf{Vect}_K^{\mathrm{fd}}$ which sends a natural number n to the vector space K^n, and an $n \times m$ matrix $A : m \to n$ to the linear map $K^m \to K^n$ that it induces with respect to the standard bases of K^m and K^n. The functor $G : \mathsf{Vect}_K^{\mathrm{fd}} \to \mathsf{Mat}_K$ chooses a basis for each vector space V, and sends it to its dimension $\dim V \in \mathbb{N}$. A linear map $\varphi : V \to W$ is sent to the matrix $[\varphi] : \dim V \to \dim W$ formed with respect to the chosen bases of V and W. Note that the choice of bases is not canonical at all, so the inverse of an equivalence of categories may not be unique.

The two categories are not isomorphic, as there are uncountably many more vector spaces than natural numbers, but they are equivalent. This highlights the connection any undergraduate student comes across between 'concrete' linear algebra with numbers and matrices, and 'abstract' linear algebra with vector spaces and linear maps.

Despite being a weaker notion than isomorphism, two equivalent categories share many of the same properties that isomorphic categories do. One way to think about it is that equivalent categories are structurally the same, except in the 'total number' of objects that are in a single isomorphism class (i.e. a collection of objects that are isomorphic to one another).

We call a functor $F : \mathsf{C} \to \mathsf{D}$ *essentially surjective on objects* if, for any object X of C, there is an object Y of D such that FX is isomorphic to Y. This notion is used to fully characterize equivalences, and is helpful for proving certain properties are preserved under equivalent functors:

Proposition 1.3.12.

(a). *A functor is an equivalence of categories if and only if it is fully faithful and essentially surjective on objects.*

(b). *If F is an equivalence of categories, and f is a monomorphism (resp. epimorphism), then Ff is a monomorphism (resp. epimorphism) too.*

(c). *If F is an equivalence of categories, and X is an initial (resp. terminal, zero) object, then FX is initial (resp. terminal, zero) as well.*

Proof. (a). The proof for the 'if' direction is quite long, so we do not write it here fully, see [Rie16, theorem 1.5.9, p.31] for the complete proof. The idea is to let $F : \mathsf{C} \to \mathsf{D}$ be fully faithful and essentially surjective on objects, and to use the axiom of choice to construct objects GY such that $F(GY) \cong Y$ by essential surjectivity, for any object Y of D. After this one proves that the assignment $Y \mapsto GY$ is actually functorial, and that we can find a natural isomorphism $GF \cong 1_{\mathsf{C}}$ as well.

For the 'only if' direction, let $F : \mathsf{C} \to \mathsf{D}$ be an equivalence of categories, with G the inverse equivalence. Now, let $f, g \in \mathsf{C}(A, B)$ be two morphisms in C such that $Ff = Fg$. Both f and g are morphisms so that the naturality square

$$\begin{tikzcd} A \arrow[r, "\eta_A", "\cong"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & GFA \arrow[d, "GFf=GFg"] \\ B \arrow[r, "\cong", "\eta_B"'] & GFB \end{tikzcd}$$

commutes. Here $\eta : 1_{\mathsf{C}} \Rightarrow GF$ is the natural isomorphism that makes F an equivalence. Now, by commutativity we have $f = \eta_B^{-1} \circ GFf \circ \eta_A = g$. So the mapping $\mathsf{C}(A,B) \to \mathsf{D}(FA,FB)$ is injective, meaning F is faithful. An analogous argument can be used to show that G is faithful as well.

Given a morphism $h : FA \to FB$ in C, the morphism $Gh : GFA \to GFB$ defines a morphism $k : A \to B$, given by $k := \eta_B^{-1} \circ Gh \circ \eta_A$. Now by naturality, both Gh and GFk should make the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A", "\cong"'] \arrow[d, dashed, "k"'] & GFA \arrow[d, shift right, "GFk"'] \arrow[d, shift left, "Gh"] \\ B \arrow[r, "\cong", "\eta_B"'] & GFB \end{tikzcd}$$

commute. Using similar arguments as before, we can conclude that $GFk = Gh$. Because G is faithful, we have that $Fk = h$. This proves that the mapping $\mathsf{C}(A,B) \to \mathsf{D}(FA,FB)$ is surjective. Thus, F is full.

Now finally, let Y be an object of D, then the natural isomorphism $FG \cong 1_{\mathsf{D}}$ tells us that $FGY \cong Y$, thus F is essentially surjective on objects. The proof for F being contravariant is completely dual: the order and direction of the morphisms would change but other than that the proof is the same.

(b). Again, let G be the inverse equivalence to F, and $\eta : 1_{\mathsf{C}} \Rightarrow GF$ the natural isomorphism. Let $f : A \to B$ be a monomorphism in C. To show that $Ff : FA \to FB$ is a monomorphism in D, let $g, h : X \rightrightarrows FA$ be two morphisms in D so that $Ff \circ g = Ff \circ h$. Left-composing both sides with G and applying functoriality gives $GFf \circ Gg = GFf \circ Gh$. Now by naturality, we have $GFf = \eta_B \circ f \circ \eta_A^{-1}$. So it follow that

$$\eta_B \circ f \circ \eta_A^{-1} \circ Gg = \eta_B \circ f \circ \eta_A^{-1} \circ Gh.$$

Left-composing with η_B^{-1}, using that f is monic, and left-composing with η_A gives $Gg = Gh$. Now G is an equivalence, so it is faithful by (a), and we find $g = h$. Because $Ff \circ g = Ff \circ h$ implies $g = h$ for all such morphisms g and h, we conclude that Ff is a monomorphism. The proof for epimorphisms is dual.

(c). Now let X be an initial object in C. We wish to show that FX is initial as well. To that end, let A be any object in D. We wish to show $\mathsf{D}(FX, A)$ has a single element. As per (a), F is essentially surjective, so there is an object Y of C so that $FY \cong A$, by some isomorphism $g : FY \to A$. Because F is fully faithful by part (a), there is a bijection of sets $\mathsf{C}(X,Y) \cong \mathsf{D}(FX,FY)$. Note that both of these are actually sets, because X is initial so there can only be one morphism from X to Y.

Denoting $f : X \to Y$ as the unique morphism from the initial object to Y, the morphism $Ff : FX \to FY$ is unique between FX and FY because of the bijection. Composing with the isomorphism g gives a morphism $g \circ Ff : FX \to A$. This morphism is also unique, because if there were another $\bar{g} : FX \to A$, we could left-compose with g^{-1} to obtain a new morphism $FX \to FY$, which is impossible. Because there is a unique morphism $FX \to A$ for any object A of D, we conclude that FX is an initial object of D. The proof for terminal objects is dual, and can be combined with the proof above to prove the statement for zero objects. $\square$

Remark. Not all properties are shared among equivalent categories. For example, a category being discrete does not imply an equivalent one is discrete as well. A category being small also does not imply an equivalent category is. Rather humorously, some category theorists call a categorical construction 'evil' if it is not shared among equivalent categories.

1.4 Limits and Colimits

Many algebraic constructions can be defined as objects satisfying a certain *universal property*. Loosely stated, an object in a category satisfies a universal property if there are some morphisms going into, or out of that object, in such a way that if there is another object with those morphisms, there is a unique morphism between this object and the object with the universal property. One can define universal properties more carefully with the Yoneda Lemma (see [Rie16, definition 2.3.3, p.63] for details), but here we focus on a special class of universal properties: Limits and colimits.

Definition 1.4.1. Let J be a small[11] category, and C another category.

- A functor $J : \mathsf{J} \to \mathsf{C}$ is called a *diagram* of shape J. We call the diagram *finite* if J contains finitely many objects and morphisms. This category J is often thought of as a quiver, indexing a collection of objects and morphisms of C by the use of J.

- A *cone* over the diagram J, denoted (N, ψ), consists of an object N (called the *apex*), and morphisms $\psi_A : N \to JA$ (called the *legs* of the cone) for each object A in J. This satisfies the property that for each $f : A \to B$ in J, the following diagram commutes:

$$\begin{tikzcd} & N \arrow[dl, "\psi_A"'] \arrow[dr, "\psi_B"] & \\ JA \arrow[rr, "Jf"'] & & JB. \end{tikzcd}$$

  Dually, a *cocone* under the diagram J, denoted (M, φ), consists of an object M (called the *nadir*), and morphisms $\varphi_A : JA \to M$ for each object A in J. This satisfies the property that for each $f : A \to B$ in J, the following diagram commutes:

$$\begin{tikzcd} JA \arrow[rr, "Jf"] \arrow[dr, "\varphi_A"'] & & JB \arrow[dl, "\varphi_B"] \\ & M. & \end{tikzcd}$$

- A *limit* of the diagram J is a cone $(\lim J, \psi)$ over J such that if (N, ψ') is another cone over J, there

[11] One could also define non-small diagrams, cones, and limits, but we do not consider these in this thesis.

exists a unique *universal morphism* $u : N \to \lim F$ such that

$$\begin{tikzcd}
& N \arrow[ddl, bend right, "\psi'_A"'] \arrow[ddr, bend left, "\psi'_B"] \arrow[d, dashed, "\exists! u"] & \\
& \lim F \arrow[dl, "\psi_A"'] \arrow[dr, "\psi_B"] & \\
JA \arrow[rr, "Ff"'] & & JB
\end{tikzcd}$$

commutes. This is the *universal property of the limit.*

Dually, a *colimit* of J is a cocone $(\operatorname{colim} J, \varphi)$ under J such that if (M, φ') is another cocone under J, there exists a unique *universal morphism* $u : \operatorname{colim} J \to M$ such that

$$\begin{tikzcd}
JA \arrow[rr] \arrow[dr, "\varphi_A"] \arrow[ddr, bend right, "\varphi'_A"'] & & JB \arrow[dl, "\varphi_B"'] \arrow[ddl, bend left, "\varphi'_B"] \\
& \operatorname{colim} J \arrow[d, dashed, "\exists! u"] & \\
& M' &
\end{tikzcd}$$

commutes. This is the *universal property of the colimit.* ▼

Limits and colimits are special kinds of universal properties, namely one where the 'property' is having morphisms to or from each object in the image of J making each triangle commute. Before moving to examples, we should first show that these limits are unique up to isomorphism:

Proposition 1.4.2. *If the limit (resp. colimit) of a diagram $J : \mathsf{J} \to \mathsf{C}$ exists, it is unique up to isomorphism. That is, if (N, ψ) and (N', ψ') are limits (resp. colimits) of J, then N and N' are isomorphic.*

Proof. Since (N, ψ) and (N', ψ') are both limits of J, they are also both cones over J. Thus by the universal property of the limit, there are unique morphisms $u : N \to N'$ and $u' : N' \to N$. Now we can consider their composition $u'u : N \to N$. Since N is a cone over J, there is a unique morphism from N to N. By the definition of a category, we know that this morphism is required to be the identity, so $u'u = 1_N$. Similarly, we find that $uu' = 1_{N'}$, making N and N' isomorphic. The proof for the colimit of J is completely dual. □

Remark. Note that the isomorphism $u : N \to N'$ in the proof above is unique. We say the limits of J are *unique up to unique isomorphism.* This is an inherently stronger notion than just being unique up to isomorphism, because there is some canonical isomorphism between the two limits.

This proof can be nearly copied for any other universal property, showing that two objects satisfying the same universal property have a unique isomorphism between them.

Example 1.4.3. The following is a list of examples of limits and colimits, as well as examples of specific limits in certain categories. In all of these, J is the indexing category and J is a functor from J to some other category.

(i) If J is empty, then the limit of $J : \mathsf{J} \to \mathsf{C}$ is a terminal object in C. A cone over an empty diagram is just an object, and universality says that for every object N, there is a unique morphism $N \to \lim J$. This is the definition of $\lim J$ being a terminal object. Dually, the colimit of an empty diagram is an initial object. This is why the proof of Proposition 1.1.14 and that of Proposition 1.4.2 are so similar; the former is a special case of the latter.

(ii) (†) If J is a discrete category, then a diagram $J : \mathsf{J} \to \mathsf{C}$ is a collection of objects X_i in C indexed by J. The limit of this diagram is the *product* of the X_i, and is denoted $\prod_i X_i$. The definition of the limit gives, for every i, *projection morphisms* $\pi_i : \prod_i X_i \to X_i$ such that, for any other object Y with morphism $f_i : Y \to X_i$, there is a unique morphism $u : Y \to \prod_i X_i$ making

$$\begin{tikzcd}
\prod_i X_i \arrow[d, "\pi_i"'] & Y \arrow[l, dashed, "\exists! u"'] \arrow[dl, "f_i"] \\
X_i &
\end{tikzcd}$$

commute. This limit appears in many concrete categories as the cartesian product, or something similar to it:

- In Set, the product of two sets X_1 and X_2 is the set $X_1 \times X_2$ consisting of ordered pairs of elements of X_1 and X_2. The projections are given by $\pi_1(x_1, x_2) = x_1$ and $\pi_2(x_1, x_2) = x_2$. Given another set Y with functions f_1 and f_2 from Y to X_1 and X_2 respectively, there is a unique function $f : Y \to X_1 \times X_2$ defined by $f(y) = (f_1(y), f_2(y))$. This can of course be extended to the product of arbitrarily many sets. Similarly, products and projection maps also appear in $\mathsf{Top}, \mathsf{Group}, \mathsf{Ring}$, and Mod_R, where the product of two objects gives an object consisting of ordered pairs of elements from the two original objects. In these categories, infinite products may not be as well-behaved as the finite ones.
- Given a poset $(P, \leqslant)$, viewed as a category. The product of a collection of elements $\{p_i\}_i$ in P is the infimum of the elements p_i, if it exists. This is because $\inf_i p_i$ is smaller than or equal to every p_i, and any other element $q \leqslant p_i$ for all i is smaller than $\inf_i p_i$.
- The product of two small categories in Cat is exactly the product category, as defined in Definition 1.1.3.
- Products do not exist in every category, for example the product does not exist in Field. Say the product $\mathbb{Q} \times \mathbb{F}_p$ is an object in Field. This has a field-homomorphism to $\mathbb{Q}$, which implies the characteristic of this field is 0. But it should also have a field homomorphism to $\mathbb{F}_p$, which implies it has characteristic $p > 0$. This is impossible of course, hence the product of fields does not exist, at least not for fields of different characteristic.

The colimit of this diagram is called the *coproduct* of the objects X_i and is denoted $\coprod_i X_i$. This coproduct comes with *inclusion morphisms* $\iota_i : X_i \to \coprod_i X_i$.

- The coproduct of sets is exactly their disjoint union. The inclusion maps are just inclusions. The same is also true for topological spaces. The disjoint union of two sets or spaces is denoted $X \amalg Y$.
- This is different for groups, in Group, the coproduct of two groups G and H is their *free product* $G * H$. This group consists of elements of the form $g_1 h_1 g_2 h_2, \ldots, g_n h_n$ where each $g_i \in G$ and $h_i \in H$. In Ab, the coproduct is given by the direct sum, which is also a product actually. Given abelian groups A and B, the direct sum $A \oplus B$ has projection homomorphisms $(a, b) \mapsto a$ and $(a, b) \mapsto b$, and inclusion homomorphisms $a \mapsto (a, 0)$ and $b \mapsto (0, b)$. The same is also true in

Mod_R, and part of defining *additive categories* in Chapter 3 assumes those categories have a similar coinciding product and coproduct.

- In Top_*, the coproduct of two pointed spaces (X, x) and (Y, y) is their *wedge sum* $X \vee Y := X \amalg Y / \sim$, where the equivalence relation is generated by defining $x \sim y$. This space can be seen as gluing the spaces X and Y along their basepoint, giving a new space with the basepoint being the identified common point. Given basepoint-preserving continuous maps $f_1 : X \to Z$ and $f_2 : Y \to Z$, the universal morphism $f : X \vee Y \to Z$ is defined as follows:

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X; \\ f_2(x), & \text{if } x \in Y; \\ f_1(x) = f_2(x), & \text{if } x \text{ is the basepoint of } X \vee Y. \end{cases}$$

- The coproduct of a collection of elements $\{p_i\}_i$ in a poset $(P, \leqslant)$ is the supremum of the elements p_i, if it exists.
- The coproduct of two small categories is their disjoint union, which is constructed by taking the disjoint union of their sets of objects as objects, and the disjoint union of their sets of morphisms as morphisms.

(iii) (†) Let J be the category generated by the quiver $\bullet \rightrightarrows \bullet$, with image under J in a category C denoted as $f, g : X \rightrightarrows Y$. The limit of this diagram is the so-called *equalizer* of f and g, denoted $\mathrm{Eq}(f, g)$. The components of the cone $(\mathrm{Eq}(f, g), \psi)$ are maps ψ_X and ψ_Y such that $f \circ \psi_X = \psi_Y = g \circ \psi_X$. The leg ψ_X is always a monomorphism, which follows immediately from the universal poperty of the equalizer. Usually the morphism ψ_Y is implied, and we only really care about what ψ_X is. Under this convention, the universal property of the equalizer is usually displayed as follows:

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, "\psi_X"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y. \\ E' \arrow[u, dashed, "\exists ! u"] \arrow[ur, "\psi'_X"'] & & \end{tikzcd}$$

- In Set, the equalizer of two functions $f, g : X \rightrightarrows Y$ is the set $\mathrm{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$, with ψ_X being the inclusion map into X. Universality tells us that if N is another set with a map $\psi'_X : N \to X$ such that $f \circ \psi'_X = g \circ \psi'_X$, then there is a unique function $u : N \to \mathrm{Eq}(f, g)$ so that $\varphi_X \circ u = \varphi'_X$. We can see the map $\psi_X : \mathrm{Eq}(f, g) \to X$ as identifying elements of N with elements of $\mathrm{Eq}(f, g)$ as a subset.
- An important example of equalizers is the *kernel*. In Mod_R (as well as many other algebraic categories), the kernel of a homomorphism $f : M \to N$ is defined as the equalizer of f and the zero map.[12] In this concrete category, we can interpret the map $\psi_M : \ker f \to M$ as the inclusion map, as this gives $f \circ \psi_M = 0$. More generally, the equalizer of two homomorphisms f, g is $\ker(f - g)$, where $f - g : x \mapsto f(x) - g(x)$.

The colimit of F is called the *coequalizer* of f and g. The leg $\varphi_Y : Y \to \mathrm{Coeq}(f, g)$ is always an epimorphism.

[12]The zero map $0 : M \to N$ takes everything in M to the zero element of N. Equivalently, the zero map may be defined as the composition $M \to 0 \to N$, which is unique because the zero module is both initial and terminal.

- The coequalizer of two functions $f, g : X \rightrightarrows Y$ in Set is the set $Y/\sim$, where $\sim$ is the smallest equivalence relation on Y such that $f(x_1) \sim f(x_2)$ for all $x_1, x_2 \in X$. The leg of the cocone $\varphi_Y : Y \to Y/\sim$ is the quotient map.
- The coequalizer of two R-module-homomorphisms $f, g : M \rightrightarrows N$ is the *cokernel* of the map $f - g$. In this category, the cokernel can be seen as $Y/\operatorname{im}(f-g)$ and is denoted $\operatorname{coker}(f-g)$. More generally, we can construct the quotient module M/N for any submodule N of M as the cokernel of the inclusion $N \hookrightarrow M$.

(iv) Let J be the category generated by the infinite quiver $\cdots \to \bullet \to \bullet$. The limit and colimit of a diagram $J : \mathsf{J} \to \mathsf{C}$ is called the *inverse limit* and *direct limit* respectively of the objects in the image of J.

- Given a commutative ring R, the ring of *formal power Series* $R[[x]]$ is the same as the inverse limit of the diagram

$$\cdots \to R[x]/x^3R[x] \to R[x]/x^2R[x] \to R[x]/xR[x]$$

in Ring. The homomorphisms

$$R[x]/x^iR[x] \to R[x]/x^{i-1}R[x]$$

are given by the projection that maps a polynomial of degree at most $i-1$ to one of degree at most $i-2$ by modding out the x^{i-1}-term. Elements of $R[[x]]$ are infinite polynomials, called *power series*, $\sum_{i\geqslant 0} a_i x^i$ with $a_i \in R$, where we do not worry about convergence and only their algebraic properties. The legs $R[[x]] \to R[x]/x^iR[x]$ are given by projecting a power series $\sum_{k\geqslant 0} a_k x^k$ to $\sum_{k=0}^{i-1} a_k x^k + (x^i)$. If $R = \mathbb{Z}/p\mathbb{Z}$, then $R[[x]]$ is isomorphic to the ring $\mathbb{Z}_p$, of *p-adic integers*. For details on the ring structure of $\mathbb{Z}_p$, see [DF04, exercise 7.6.11, p.269]. A p-adic number in $\mathbb{Z}_p$ is often displayed with positional notation as the infinite string $\ldots a_2a_1a_0$, with each $a_i \in \{0, \ldots, p-1\}$. The isomorphism $\mathbb{Z}/p\mathbb{Z}[[x]] \to \mathbb{Z}_p$ sends a power series $\sum_{i\geqslant 0} a_i x^i$ to the p-adic number $...a_2a_1a_0$.
- We can index the category J with the natural numbers, making it isomorphic to the poset category $(\mathbb{N}, \geqslant)$. In this case, the image of a covariant functor $a : (\mathbb{N}, \geqslant) \to (\mathbb{R}, \geqslant)$ is exactly a non-decreasing sequence of real numbers. This diagram has an inverse limit if and only if the corresponding sequence of real numbers converges. Namely, a real number a^* is a limit of a non-decreasing sequence (a_n) if and only if that sequence is bounded. This is the *monotone convergence theorem*, which is stated and proved in [Abb15, theorem 2.4.2, p.56]. If this is the case, we have that $a^* \geqslant a_n$ for all $n \in \mathbb{N}$, and that for any other b so that $b \geqslant a_n$, we have $b \geqslant a^*$ as well. This is exactly the universality of the limit of the diagram in this category.

  Dually, a functor $a : (\mathbb{N}, \geqslant) \to (\mathbb{R}, \geqslant)^{\mathrm{op}}$ corresponds to a non-increasing sequence, which as a diagram has an inverse limit if and only if the sequence has a limit, which happens if and only if it is bounded.

(v) The limit of a diagram of the form $\bullet \to \bullet \leftarrow \bullet$ is called a *pullback*. Denoting π_B and π_C as the projection morphisms from $B \times C$ to B and C respectively, the pullback of $B \xrightarrow{f} A \xleftarrow{g} C$ can be formed as the equalizer of $f\pi_B$ and $g\pi_C$. The colimit of a diagram $\bullet \leftarrow \bullet \to \bullet$ is called the *pushout*, and can be formed as the coequalizer of two morphisms from the middle object to the coproduct of the two outer objects.

- In Set, the pullback of $B \xrightarrow{f} A \xleftarrow{g} C$ is the set

$$B \times_A C := \{(b,c) \in B \times C \mid f(b) = g(c)\}.$$

The pushout of $B \xleftarrow{f} A \xrightarrow{g} C$ is $B \amalg C / \sim$, where the relation $\sim$ is generated by setting $f(b) \sim g(c)$ for all $b \in B$ and $c \in C$.

- The wedge sum of two pointed spaces (X, x) and (Y, y) can also be viewed as a pushout in Top. Specifically, the pushout of the diagram $X \leftarrow * \rightarrow Y$, with $*$ a one-point space and the arrows mapping its point to $x \in X$ and $y \in Y$, is the wedge sum $X \vee Y$.

As we have seen, not every diagram has a limit in every category. We define the categories that do as follows:

Definition 1.4.4. We call a category C *complete* if every diagram in C has a limit. We call C *cocomplete* if every diagram has a colimit.
A functor $G : \mathsf{C} \to \mathsf{D}$ is called *continuous* if it preserves all limits. That is, if $J : \mathsf{J} \to \mathsf{C}$ is a diagram with a limit $(\lim J, \psi)$ in C, then the diagram $GJ : \mathsf{J} \to \mathsf{D}$ has a limit $(\lim GJ, G\psi)$ in D. We say G is *cocontinuous* if $(\operatorname{colim} GJ, G\varphi)$ is a colimit in D whenever $(\operatorname{colim} J, \varphi)$ is a colimit in C. ▼

An important example of (co)continuous functors are the Hom-functors (or any representable functor), which are proven to preserve limits in [Mac98, theorem V.4.1, p.116]:

Proposition 1.4.5. *If the limit of $J : \mathsf{J} \to \mathsf{C}$ exists in C, then, for every object X in C, there is an isomorphism*

$$\operatorname{Hom}(X, \lim J) \cong \lim \operatorname{Hom}(X, J(-))$$

which is natural in X. Similarly, if the colimit of J exists, then

$$\operatorname{Hom}(\operatorname{colim} J, X) \cong \lim \operatorname{Hom}(J(-), X)$$

is a natural isomorphism in X as well.

It can seem daunting to check whether or not a category is (co)complete or not, but this is not actually the case! It turns out that (co)products and (co)equalizers are all we need to construct the (co)limit of a diagram.

Proposition 1.4.6. *If a category admits products and equalizers (resp. coproducts and coequalizers), it is complete (resp. cocomplete). Moreover, if a functor preserves products and equalizers (resp. coproducts and coequalizers), it is continuous (resp. cocontinuous).*

Proof. (Adapted from [Mes07, theorem 5.4, p.8]) Let $J : \mathsf{J} \to \mathsf{C}$ be a diagram. Our strategy is to form two products of objects in the diagram with two canonical maps between them. Then the equalizer of these maps is the limit of the diagram.

First define $A := \prod_j JX_j$ to be the product of all objects in the diagram, and $B := \prod_{\beta : \exists X_\beta \to X_\alpha} JX_\alpha$ to be the product of all objects that are the codomain of some morphism in J (this may include repeats).[13] Now let $f : X \to Y$ be any morphism in J. By the definition of the product, there is a projection morphism

[13] Note that, by assumption of J being small, it actually makes sense to index products over a condition like 'an object is a codomain'. If the category were large, this may not be a formally sound construction.

$\pi_Y : B \to JY$. Since Y is an object in J, there is also a projection morphism $\pi'_Y : A \to JY$, as well as the composition $A \xrightarrow{\pi'_X} JX \xrightarrow{Jf} JY$. These are both morphisms to objects in the product B, this can be done for *each* object JX_α in B, so by its universal property there are unique morphisms $u, v : A \rightrightarrows B$ that make the following triangle and square respectively commute:

$$\begin{tikzcd}
 & JY \\
A \arrow[ur, bend left, "\pi'_Y"] \arrow[r, dashed, shift left, "u"] \arrow[r, dashed, shift right, "v"'] \arrow[d, "\pi'_X"'] & B \arrow[u, "\pi_Y"'] \arrow[d, "\pi_Y"] \\
JX \arrow[r, "Jf"'] & JY,
\end{tikzcd}$$

that is, $\pi'_Y = \pi_Y \circ u$ and $Jf \circ \pi'_X = \pi_Y \circ v$.

Now let E be the equalizer of u and v, which comes with a morphism $e : E \to A$ such that $u \circ e = v \circ e$. We claim that E is the limit of J, and the legs of the cone are given by the morphisms $\psi_X := \pi'_X \circ e : E \to JX$.

$$\begin{tikzcd}
 & & JY \\
E \arrow[rru, bend left, "\psi_Y"] \arrow[r, dashed, "e"] \arrow[dr, bend right, "\psi_X"'] & A \arrow[ur, bend left, "\pi'_Y"] \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "\pi'_X"'] & B \arrow[u, "\pi_Y"'] \arrow[d, "\pi_Y"] \\
 & JX \arrow[r, "Jf"'] & JY
\end{tikzcd}$$

To show (E, ψ) is a cone, we want to show that $\psi_Y = Jf \circ \psi_X$ for any arbitrary $f : X \to Y$ in J. Starting with the right-hand side, we can use the definition of ψ_X to write $Jf \circ \psi_X = Jf \circ \pi'_X \circ e$. Using the defining property of v, we can write this as $Jf \circ \pi'_X \circ e = \pi_Y \circ v \circ e$. Because e is an equalizer, this becomes $\pi_Y \circ v \circ e = \pi_Y \circ u \circ e$. Now from how we defined u, we get that this is equal to $\pi_Y \circ u \circ e = \pi'_Y \circ e$, which is exactly ψ_Y by definition. Thus we find $\psi_Y = Jf \circ \psi_X$, like we wanted. This shows that (E, ψ) is a cone over J.

Finally, we want to show that if (Q, φ) is another cone over J, then there is a unique map $Q \to E$. Since Q is a cone, there are maps $\varphi_X : Q \to JX$ for each object X in J, thus by definition of the product there is a unique morphism $a : Q \to A$, as well as a unique morphism $b : Q \to B$. These morphisms satisfy $\varphi_X = \pi'_X \circ a$ for any object X and $\varphi_Y = \pi_Y \circ b$ for any codomain object Y. The plan is to show that $u \circ a = v \circ a$, which implies the existence of a morphism $Q \to E$ by the universal property of the equalizer.

The relevant morphisms fit in the following diagram:

$$
\begin{tikzcd}
 & & JY \\
A \arrow[urr, bend left, "\pi'_Y"] \arrow[rr, shift left, "u"] \arrow[rr, shift right, "v"'] \arrow[dd, "\pi'_X"'] & & B \arrow[u, "\pi_Y"'] \arrow[dd, "\pi_Y"] \\
 & Q \arrow[ul, dashed, "\exists ! a"] \arrow[ur, dashed, "\exists ! b"'] \arrow[dl, "\varphi_X"] \arrow[dr, "\varphi_Y"'] & \\
JX \arrow[rr, "Jf"'] & & JY.
\end{tikzcd}
$$

Note that this diagram is not guaranteed to be commutative! The morphisms a and b only satisfy the compositions given above, and not (yet) necessarily that $u \circ a = b$ for example.

Regardless, note that b is the *unique* morphism so that $\varphi_Y = \pi_Y \circ b$. To show $v \circ a$ and $u \circ a$ are equal, we show that they are both equal to b using this uniqueness. First, note that because $\pi_Y \circ v = Jf \circ \pi'_X$, we have that $\pi_Y \circ v \circ a = Jf \circ \pi'_X \circ a$. Then by what we know about a, we have $Jf \circ \pi'_X \circ a = Jf \circ \varphi_X$. Since (Q, φ) is a cone over J, we have $Jf \circ \varphi_X = \varphi_Y$. So, the morphism $v \circ a$ satisfies $\varphi_Y = \pi_Y \circ (v \circ a)$. But b is supposed to be unique with this property, which now implies $v \circ a = b$.

Now, we can use the defining property of u to find $\pi_Y \circ u \circ a = \pi'_Y \circ a$. The defining composition of a holds for *each* object of J. In particular, $\pi'_Y \circ a = \varphi_Y$. Again, we find $u \circ a = b$ by uniqueness of b. But this, combined with the previous part, shows that $u \circ a = v \circ a$. By the universal property of the equalizer, there is a unique morphism $s : Q \to E$ so that $\psi_X \circ s = \varphi_X$ for each object X of J. This is exactly what we wanted to show to guarantee that $E = \lim J$.

If a functor G from C to another category D preserves products and equalizers, it preserves the products A, B, and the equalizer E. So now the equalizer GE of the morphisms $Gu, Gv : GA \rightrightarrows GB$ is the same as the limit of GJ. Thus G preserves all limits.

The idea of the proof that coproducts and coequalizers are enough to form colimits is the same, except dualized. In this case we define the coproduct $\hat{A} := \coprod_j JX_j$ of all objects in the diagram, and we define $\hat{B} := \coprod_{\beta : \exists X_\alpha \to X_\beta} JX_\alpha$ the coproduct of all domains. We again consider two morphisms $\hat{B} \rightrightarrows \hat{A}$, and construct their coequalizer. Using a dual argument to the one above, this coequalizer is the colimit of the diagram J. Similarly, G preserving coproducts and coequalizers implies it preserves all colimits by the same argument as before. $\square$

Example 1.4.7. Some examples of complete and cocomplete categories include:

(i) The category Set is both complete and cocomplete. We have already seen how we construct products, coproducts, equalizers, and coequalizers in this category in Example 1.4.3. Similarly, Top is (co)complete as well. The underlying sets of the product, disjoint union, equalizer and coequalizer are the same as in Set, but with the topologies which make sure that the legs of the universal (co)cones are continuous maps. The pointed categories Set_* and Top_* are also complete and cocomplete.

(ii) The category of small categories, Cat is (co)complete as well. The product and coproduct have already been highlighted above. Given functors $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$, their equalizer is the subcategory E of C which consists of all objects and morphisms of C on which F and G agree. As for the coequalizer, we note that because D is small, we can impose an equivalence relation on $\mathrm{Ob}(\mathsf{D})$ and $\mathsf{D}(A, B)$ for all objects A and

B in D generated by stating that two objects or morphisms are equivalent if their image under F and G are the same. Taking the quotients of the set of objects, and of every Hom-set gives the coequalizer category Q.

(iii) The category $\mathsf{Set}^{\text{fin}}$ of finite sets is finitely complete and finitely cocomplete, meaning it admits (co)limits of every finite diagram, but not complete or cocomplete. For example, the infinite product $\prod_{i\in\mathbb{N}} S$ of any nonempty finite set S has infinitely many elements, and is thus not a product in the category.

Example 1.4.8. For any ring R, the category Mod_R is (co)complete as well. The product and coproduct are given by the direct product and direct sum respectively. The equalizer and coequalizer of two homomorphisms $f, g : M \rightrightarrows N$ are $\ker(f-g)$ and $\operatorname{coker}(f-g) := N/\operatorname{im} f$ respectively. Because this category has products, coproducts, equalizers, and coequalizers, it is complete and cocomplete.

Before moving on to adjunctions, there is one more result we highlight, the fact that being (co)continuous is *not* an evil property:

Proposition 1.4.9. *Let $F : \mathsf{C} \to \mathsf{D}$ be an equivalence of categories. If the limit (resp. colimit) of a diagram $J : \mathsf{J} \to \mathsf{C}$ exists, then F preserves this limit (resp. colimit).*

Proof. Let $(\lim J, \varphi)$ be the limit of the diagram J in C. We want to show that $(F \lim J, F\varphi)$ is a limit in D.

Note that $(F \lim J, F\varphi)$ is actually a cone over the diagram FJ. This follows from functoriality: if $f : X \to Y$ is a morphism in the image of J, then the legs of the cone (J, φ) satisfying $f \circ \varphi_X = \varphi_Y$ implies $Ff \circ F\varphi_X = F\varphi_Y$. We show that this cone is universal among all cones over FJ. To that end, let (C, ψ) be a cone in D over FJ as follows:

$$\begin{tikzcd}
 & C \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
 & F\lim J \arrow[dl, "F\varphi_X"'] \arrow[dr, "F\varphi_Y"] & \\
FX \arrow[rr, "Ff"'] & & FY
\end{tikzcd}$$

By essential surjectivity of F (see Proposition 1.3.12), there is an object Z of C such that $FZ \cong C$. Let $q : FZ \to C$ be an isomorphism. Now $(FZ, \psi_X \circ q)$ is a cone over FJ. Fullness of F allows us to write $\psi_X \circ q = Fh_X$ for some $h_X : Z \to X$ in C. Now we have a new cone (FZ, Fh) over FJ.

We can go back to C, where now (Z, h) forms a cone over N. To see why this is true, note that $Ff \circ Fh_X = Fh_Y$ in D implies, by faithfulness of F, that $f \circ h_X = h_Y$, making (Z, h) a cone over J. Now by the universality of $\lim J$, there is a unique morphism $u : Z \to \lim J$ making the following diagram commute:

$$\begin{tikzcd}
 & Z \arrow[ddl, bend right, "h_X"'] \arrow[ddr, bend left, "h_Y"] \arrow[d, dashed, "\exists! u"] & \\
 & \lim J \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Applying F again leaves us with a morphism $Fu : FZ \to F \lim J$ commuting with the legs of the cone, which is unique as well (this follows from the bijection $\mathsf{C}(Z, \lim J) \cong \mathsf{D}(FZ, F \lim J)$). Composing with the inverse of q gives a unique morphism $Fu \circ q^{-1} : C \to F \lim J$, which proves that $F \lim J$ is a limit over the diagram FJ. The proof for F preserving colimits is dual. $\square$

1.5 Adjunctions and Limit Preservation

We have seen before how forgetful functors allow us to remove the inner structure of objects to only look at the underlying sets. There are forgetful functors from $\mathsf{Group}, \mathsf{Vect}_K, \mathsf{Top}$ (and more) to Set. An interesting question may be if we can reverse this process? That is, given a set S, can we construct a group, vector space, or topological space from S in some general way? For most cases the answer is yes, and is done using a so-called *adjoint functors*. In this section we develop the tools necessary to define these kind of functors, and also see more general examples that do not fit in this class of functors that mirror the forgetful ones. Finally we discuss the most important property of adjoint functors: they always preserve limits or colimits.

Definition 1.5.1. Given functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$, if there is an isomorphism

$$\mathsf{D}(FX, Y) \cong \mathsf{C}(X, GY)$$

that is natural in both X and Y,[14] we say there is an *adjunction* between F and G. In this case, we say F is a *left adjoint (functor)* to G, and G is a *right adjoint (functor)* to F. We write $F \dashv G$ or $G \vdash F$. Under the natural bijection, we say corresponding morphisms

$$FX \xrightarrow{f} Y \qquad \text{and} \qquad X \xrightarrow{f^T} GY$$

are *transposes* of one another. ▼

Remark. There is no preference to the first morphism being the 'original' and the second the transposed morphism. We may also denote the transpose of a morphism $g : X \to GY$ as $g^T : FX \to Y$. A consequence of the bijection is that $(f^T)^T = f$.

As is detailed in [Rie16, section 4.1], expanding the definition of the natural isomorphism gives the fact that, for any $f : FX \to Y$, its transpose satisfies $Gk \circ f^T = (k \circ f)^T$ and $f^T \circ h = (f \circ Fh)^T$ for any morphism $k : GY \to Z$ in D and $h : W \to X$ in C.

As with many categorical constructions we have seen thus far, adjunctions are unique up to natural isomorphism:

Proposition 1.5.2. *Adjoint functors are unique up to natural isomorphism.*

Proof. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor, with two right adjoints $G, G' : \mathsf{D} \rightrightarrows \mathsf{C}$. By definition, for any objects X of C and Y of D, there are natural isomorphisms

$$\mathsf{C}(X, GY) \cong \mathsf{D}(FX, Y) \cong \mathsf{C}(X, G'Y).$$

Because these isomorphisms are natural in X, there is a natural isomorphism $\mathsf{C}(-, GY) \cong \mathsf{C}(-, G'Y)$. Proposition 1.2.8 and Corollary 1.3.9 imply that GY and $G'Y$ are isomorphic as objects, with some isomorphism $\eta_Y : GY \to G'Y$. In [nLa23, proposition 3.1], it is shown that these isomorphisms are also natural in Y, making the functors G and G' naturally isomorphic. $\square$

[14] Recall that this means that the bifunctors $\mathsf{D}(F-, -)$ and $\mathsf{C}(-, G-)$ from $\mathsf{C}^{\mathrm{op}} \times \mathsf{D}$ to Set are naturally isomorphic, as defined in Definition 1.3.1.

Example 1.5.3. The following are examples of adjoint functors in poset-categories.

(i) There are functors $\lceil - \rceil, \lfloor - \rfloor : (\mathbb{R}, \leqslant) \rightrightarrows (\mathbb{Z}, \leqslant)$ that take a real number to its ceiling and floor respectively. The inclusion functor $I : (\mathbb{Z}, \leqslant) \hookrightarrow (\mathbb{R}, \leqslant)$ forms a trio of adjoint functors $\lceil - \rceil \dashv I \dashv \lfloor - \rfloor$. For both categories, we have that $\#\operatorname{Hom}(x, y) = 1$ if $x \leqslant y$, and zero otherwise. Thus in practical terms, the first adjunction states that for a real number r and integer n, we have $\lceil r \rceil \leqslant n$ if and only if $r \leqslant n$. The other adjunction states that $n \leqslant r$ if and only if $n \leqslant \lfloor r \rfloor$.

(ii) A function $f : A \to B$ of sets induces two functors between the poset categories $(P(A), \subseteq)$ and $(P(B), \subseteq)$. The forward-image $f_* : (P(A), \subseteq) \to (P(B), \subseteq)$ and pre-image $f^{-1} : (P(B), \subseteq) \to (P(A), \subseteq)$ send a subset to their image and pre-image respectively. These functors form an adjunction. Namely, for subsets $A' \subseteq A$ and $B' \subseteq B$, we have that $f_*(A') \subseteq B'$ if and only if $A' \subseteq f^{-1}(B')$.

The pre-image also has a right adjoint $f_! : (P(A), \subseteq) \to (P(B), \subseteq)$ that takes a subset $A' \subseteq A$ to the set $f_!(A') := \{b \in B \mid f^{-1}(\{b\}) \subseteq A'\} \subseteq B$. The adjunction states that $f^{-1}(B') \subseteq A'$ if and only if $B' \subseteq f_!(A')$.

Example 1.5.4. There is a large family of adjunctions of the form $F \dashv U$, where U is a forgetful functor and F is some kind of 'free' functor.

(i) The forgetful functor $U : \mathsf{Vect}_K \to \mathsf{Set}$ has a left adjoint $K : \mathsf{Set} \to \mathsf{Vect}_K$ that sends a set S to the K-vector space $K[S]$ which has the set S as a basis. In other words, elements of this vector space are formal K-linear combinations of elements of S. The isomorphism

$$\mathsf{Vect}_K(K[S], V) \cong \mathsf{Set}(S, UV)$$

states that linear maps from $K[S]$ to V are completely and uniquely determined by where they map the basis of the domain. Specifically, the component

$$\eta_{S,V} : \mathsf{Vect}_K(K[S], V) \to \mathsf{Set}(S, UV)$$

sends a linear map L to the function $s \mapsto L(s)$. The inverse sends a function f to the linear map

$$\sum_{s \in S} k_s s \mapsto \sum_{s \in S} k_s f(s).$$

The beginning of chapter IV of [Mac98] goes into more details of this adjunction, as well as the naturality of the transformation η.

(ii) The forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ has the free group as its left adjoint (see Example 1.2.2(x)). The components of the natural isomorphism are maps

$$\eta_{S,G} : \mathsf{Set}(S, UG) \to \mathsf{Group}(FS, G)$$

that send a function $f : S \to UG$ to the group-homomorphism $FS \to G$ that sends a word $w = a_1 \dots a_n$ to the product $f(a_1) \cdot \ldots \cdot f(a_n)$ in G.

(iii) The forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$ has a left adjoint $D : \mathsf{Set} \to \mathsf{Top}$ that equips a set with the discrete topology. This forms an adjunction because any function $DS \to X$ is continuous, meaning the set of continuous maps from $DS \to X$ is in a natural bijection with the set of function $S \to UX$.

Similarly, the forgetful functor U also has a right adjoint in the functor $I : \mathsf{Set} \to \mathsf{Top}$ that equips a set with the indiscrete topology. Indeed, continuous maps $X \to IS$ are in bijection with functions $UX \to S$.

(iv) The forgetful functor $U : \mathsf{Ab} \to \mathsf{Set}$ has a left adjoint that takes a set S and sends it to the abelian group generated by elements of S. That is, it is sent to the direct sum $\bigoplus_{s \in S} \mathbb{Z}$.

(v) There is a forgetful functor $U : \mathsf{Ring} \to \mathsf{Mon}$ that sends a ring to the underlying monoid with respect to the multiplication operation. This functor has a left adjoint that sends a monoid M to the *free ring* $\mathbb{Z}[M]$. This is the ring of formal sums $\sum_{m \in M} r_m m$, where finitely many of the $r_m \in \mathbb{Z}$ are nonzero. Multiplication is done on monomials by $(rm) \cdot (r'm') = (rr')(mm')$, and extended to guarantee distributivity.

(vi) Any field-homomorphism is injective, so there is a forgetful functor $\mathsf{Field} \to \mathsf{Domain}^{\mathrm{inj}}$, where the codomain is the category of integral domains with injective ring-homomorphisms between them. This functor has a left adjoint in the field of fractions from Example 1.2.2(viii).

(vii) No forgetful functor U from Field to (e.g.) Set, Ring, or Ab has a left adjoint. To see this, note that for any fields K and L of *different* characteristic, there are morphisms in the aforementioned categories from $\mathbb{Z}$ to UK and from $\mathbb{Z}$ to UL. Thus whatever field an adjoint F sends $\mathbb{Z}$ to, the Hom-sets $\mathsf{Field}(F\mathbb{Z}, K)$ and $\mathsf{Field}(F\mathbb{Z}, L)$ both need to be nonempty. But this is impossible, since if the first set is nonempty, then the characteristic of $F\mathbb{Z}$ is the characteristic of K, which means there can be no field-homomorphisms from $F\mathbb{Z}$ to L. Therefore this adjoint F cannot exist.

Example 1.5.5. The following are examples related to Cat.

(i) The forgetful functor $U : \mathsf{Cat} \to \mathsf{Quiver}$ has a left adjoint. It takes a quiver Q and sends it to the category $C(Q)$ generated by Q, as defined in Example 1.1.4.

(ii) The object functor $\mathrm{Ob} : \mathsf{Cat} \to \mathsf{Set}$ has both left and right adjoints. The left adjoint takes a set S and sends it to the discrete category with elements of S as objects. The right adjoint takes a set S and sends it to the indiscrete category with elements of S as objects.[15]

(iii) The opposite category forms a functor $(-)^{\mathrm{op}} : \mathsf{Cat} \to \mathsf{Cat}$ that sends a small category to its opposite. This functor is *self-adjoint*, in the sense that $(-)^{\mathrm{op}} \dashv (-)^{\mathrm{op}}$ forms an adjoint pair. This means that for all small categories C and D, there is a natural correspondence between functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ and functors $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$. We have stated in Section 1.1 that a contravariant functor from C to D is 'the same' as a covariant functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ or $\mathsf{C} \to \mathsf{D}^{\mathrm{op}}$. This adjunction provides the necessary details to make this precise.

Example 1.5.6. Given a commutative ring R and an R-module M, the tensor product functor $M \otimes_R -$ is left adjoint to the covariant Hom-functor $\mathrm{Hom}(M, -)$. Note that in Mod_R, the set of homomorphisms between two R-modules is also an R-module, with pointwise addition and scalar multiplication. Thus $\mathrm{Hom}(M, -)$ is indeed a functor from Mod_R to itslef. This is also known as the *tensor-hom adjunction.* A full proof of this fact is given in the next chapter, where we also define the tensor product in detail.

[15] Recall that an indiscrete category is one where each Hom-set has exactly one morphism in it.

For most of the examples above, it should feel rather intuitive that the functors are adjoints, but rigorously proving that they are can take a lot more effort. Thankfully there is an equivalent way that is computationally more effective, though debatedly less intuitive:

Definition 1.5.7. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors. We say F and G form a *unit-counit adjunction* if there exist natural transformations $\eta : 1_{\mathsf{C}} \Rightarrow GF$ (called the *unit*) and $\varepsilon : FG \Rightarrow 1_{\mathsf{D}}$ (called the *counit*) that make the following diagrams commute:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "1_G"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G. \end{tikzcd}$$

That is, $1_F = \varepsilon F \circ F\eta$ and $1_G = G\varepsilon \circ \eta G$. ▼

Notation. To be clear, the composition in the Proposition is vertical composition of natural transformations, as in Definition 1.3.5. The functor-natural transformation compositions are defined component-wise by $(F\eta)_X := F(\eta_X)$ and $(\varepsilon F)_X := \varepsilon_{F(X)}$ for all objects X of C, and similarly for the compositions with G.

Proposition 1.5.8. *Two functors form an adjunction if and only if they form a unit-counit adjunction.*

The following Lemma turns out to be very useful in proving the above Proposition:

Lemma 1.5.9. *Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors, with F left adjoint to G. Let $f : FX \to Y$ and $g : FX' \to Y'$ be morphisms in D. Then, for all $h : X \to X'$ and $k : Y \to Y'$, the left square below commutes if and only if the right square does.*

$$\begin{tikzcd} FX \arrow[r, "f"] \arrow[d, "Fh"'] & Y \arrow[d, "k"] \\ FX' \arrow[r, "g"'] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f^T"] \arrow[d, "h"'] & GY \arrow[d, "Gk"] \\ X' \arrow[r, "g^T"'] & GY'. \end{tikzcd}$$

Proof. The proof consists of a straightforward diagram chase, making use of the remark after Definition 1.5.1. Assuming the lef square commutes, we compute the composite $Gk \circ f^T$:

$$Gk \circ f^T = (k \circ f)^T = (g \circ Fh)^T = g^T \circ h,$$

which shows that the right square commutes as well.

For the other direction, we assume the right-hand square commutes and compute $k \circ f$:

$$\begin{aligned} k \circ f &= ((k \circ f)^T)^T \\ &= (Gk \circ f^T)^T \\ &= (g^T \circ h)^T \\ &= ((g \circ Fh)^T)^T \\ &= g \circ Fh, \end{aligned}$$

which indeed shows that the left square commutes as well. □

With this Lemma in hand, we can prove Proposition 1.5.8:

Proof of Proposition 1.5.8. We start by proving that, given an adjunction $F \dashv G$, we can construct the unit and counit. We define the unit $\eta : 1_{\mathsf{C}} \Rightarrow GF$ as the natural transformation whose components $\eta_X : X \to GFX$ are the transposes of the identities $1_{FX} : FX \to FX$. To prove these components form a natural transformation, we are to prove that for any $f : X \to X'$ in C, the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GFX \arrow[d, "GFf"] \\ Y \arrow[r, "\eta_Y"'] & GFY \end{tikzcd}$$

commutes. This follows immediately from Lemma 1.5.9, seeing as the 'transposed' diagram

$$\begin{tikzcd} FX \arrow[r, "1_{FX}"] \arrow[d, "Ff"'] & FX \arrow[d, "Ff"] \\ FY \arrow[r, "1_{FY}"'] & FY \end{tikzcd}$$

definitively commutes.

Dually, we define the counit $\varepsilon : FG \Rightarrow 1_{\mathsf{D}}$ whose components $\varepsilon_Y : FGY \to Y$ are defined to be the transposes of the identity $1_{GY} : GY \to GY$. Next, we show that $1_F = \varepsilon F \circ F\eta$ and $1_G = G\varepsilon \circ \eta G$. We do this component-wise, by letting X be an object of C and Y an object of D. Consider the following pairs of transposed diagrams:

$$\begin{tikzcd} FX \arrow[r, "1_{FX}"] \arrow[d, "F\eta_X"'] & FX \arrow[d, "1_{FX}"] & & X \arrow[r, "\eta_X"] \arrow[d, "\eta_X"'] & GFX \arrow[d, "1_{GFX}"] \\ FGFX \arrow[r, "\varepsilon_{FX}"'] & FX & \Longleftrightarrow & GFX \arrow[r, "1_{GFX}"'] & GFX \end{tikzcd}$$

$$\begin{tikzcd} FGY \arrow[r, "1_{FGY}"] \arrow[d, "1_{FGY}"'] & FGY \arrow[d, "\varepsilon_Y"] & & GY \arrow[r, "\eta_{GY}"] \arrow[d, "1_{GY}"'] & GFGY \arrow[d, "G\varepsilon_Y"] \\ FGY \arrow[r, "\varepsilon_Y"'] & Y & \Longleftrightarrow & GY \arrow[r, "1_{GY}"'] & GY \end{tikzcd}$$

Note that the top-right and bottom-left diagrams commute, thus by Lemma 1.5.9, so do the transposed top-left and bottom-right respectively. Writing this out fully,

$$(1_F)_X = 1_{FX} = \varepsilon_{FX} \circ F\eta_X = (\varepsilon F \circ F\eta)_X.$$

Since this holds for each object X of C, we have that $1_F = \varepsilon F \circ F\eta$. Similarly, writing out the compositions of the bottom-right diagram tells us $1_G = G\varepsilon \circ \eta G$. Thus indeed, if F and G are adjoints, they form a unit-counit adjunction as well.

Now for the converse, assume that we have a unit $\eta : 1_{\mathsf{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow 1_{\mathsf{D}}$. To prove F is a left adjoint of G, we find a natural isomorphism $\mathsf{D}(F-,-) \cong \mathsf{C}(-,G-)$. To that end, let X and Y be objects of C and D respectively, and define a function $\Phi_{X,Y} : \mathsf{D}(FX, Y) \to \mathsf{C}(X, GY)$ by

$$\Phi_{X,Y}(f) := Gf \circ \eta_X.$$

For the other direction, define $\Psi_{X,Y} : \mathsf{C}(X, GY) \to \mathsf{D}(FX, Y)$ by

$$\Psi_{X,Y}(g) := \varepsilon_Y \circ Fg.$$

Now we compute the compositions $\Phi_{X,Y} \circ \Psi_{X,Y}$ and $\Psi_{X,Y} \circ \Phi_{X,Y}$. Given $g \in \mathsf{C}(X, GY)$, we find:

$$\begin{aligned}
(\Phi_{X,Y} \circ \Psi_{X,Y})(g) &= \Phi_{X,Y}(\varepsilon_Y \circ Fg) \\
&= G(\varepsilon_Y \circ Fg) \circ \eta_X \\
&= G\varepsilon_Y \circ GFg \circ \eta_X \\
&= G\varepsilon_Y \circ \eta_{GY} \circ g \\
&= (G\varepsilon \circ \eta G)_Y \circ g = 1_Y \circ g = g.
\end{aligned}$$

So indeed we have that $\Phi_{X,Y} \circ \Psi_{X,Y} = 1_{\mathsf{C}(X,GY)}$. From the third to the fourth line, we used the fact that η is a natural transformation (see the diagram below), from the fifth to the sixth line, we used the defining property of the unit and counit.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "g"'] & GFX \arrow[d, "GFg"] \\
GY \arrow[r, "\eta_{GY}"'] & GFGY
\end{tikzcd}$$

For the other composition, we take $f \in \mathsf{D}(FX, Y)$ arbitrary and note:

$$\begin{aligned}
(\Psi_{X,Y} \circ \Phi_{X,Y})(f) &= \Psi_{X,Y}(Gf \circ \eta_X) \\
&= \varepsilon_Y \circ F(Gf \circ \eta_X) \\
&= \varepsilon_Y \circ FGf \circ F\eta_X \\
&= f \circ \varepsilon_{FX} \circ F\eta_X \\
&= f \circ (\varepsilon F \circ F\eta)_X = f \circ 1_X = f.
\end{aligned}$$

Here we again used naturality of ε, as well as the defining property of units and counits. Finally, we end up with $\Psi_{X,Y} \circ \Phi_{X,Y} = 1_{\mathsf{D}(FX,Y)}$, thus $\mathsf{D}(FX, Y)$ and $\mathsf{C}(X, GY)$ are isomorphic as objects in Set.

The last part to show is that Φ and Ψ as we have defined them are actually natural transformations. To that end, we take $(f, g) : (X', Y) \to (X, Y')$ an arbitrary morphism in $\mathsf{C}^{\mathrm{op}} \times \mathsf{D}$, with the goal to show that the diagram

$$\begin{tikzcd}
\mathsf{D}(FX, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "{(Ff^*, g_*)}"'] & \mathsf{C}(X, GY) \arrow[d, "{(f^*, Gg_*)}"] \\
\mathsf{D}(FX', Y') \arrow[r, "\Phi_{X',Y'}"'] & \mathsf{C}(X', GY')
\end{tikzcd}$$

commutes. The morphism (Ff^*, g_*) is an abuse of notation, but to be precise it acts on morphisms $h \in \mathsf{D}(FX, Y)$ by $(Ff^*, g_*)(h) = g \circ h \circ Ff$, and similar for (f^*, Gg_*).

Let $h : FX \to Y$ be an arbitrary morphism. The top half of the diagram evaluates to:

$$\begin{aligned}((f^*, Gg_*) \circ \Phi_{X,Y})(h) &= (f^*, Gg_*)(Gh \circ \eta_X) \\ &= Gg \circ Gh \circ \eta_X \circ f \\ &= Gg \circ Gh \circ GFf \circ \eta_{X'} \\ &= G(g \circ h \circ Ff) \circ \eta_{X'} \\ &= \Phi_{X',Y'}(g \circ h \circ Ff) \\ &= (\Phi_{X',Y'} \circ (Ff^*, g_*))(h),\end{aligned}$$

where we used the fact that η is a natural transformation from the second line to the third. What we end up with is exactly the bottom half of the diagram evaluated at h. Since h was chosen arbitrarily, the diagram indeed commutes, and thus $\Phi : \mathsf{D}(F-,-) \Rightarrow \mathsf{C}(-,G-)$ is a natural transformation. It is actually a natural isomorphism as well, because every component is invertable. Thus, there is a natural isomorphism $\mathsf{D}(F-,-) \cong \mathsf{C}(-,G-)$, which proves that $F \dashv G$ forms an adjunction. ☐

Example 1.5.10. Definition 1.5.1 is the most intuitive way to view adjunctions, but it is still worth it to see the unit and counit in actual examples:

(i) The left adjoint to the forgetful $U : \mathsf{Vect}_K \to \mathsf{Set}$ is the functor $K : \mathsf{Set} \to \mathsf{Vect}_K$ that sends a set S to the vector space $K[S]$ with elements of S as a basis. The unit of the adjunction has components $\eta_S : S \to UK[S]$ which map an element s to itself, which makes sense as s is an element of $K[S]$. The counit has components $\varepsilon_V : K[UV] \to V$ that maps a finite linear combination $\sum_{v_i \in UV} \lambda_i v_i$ to itself as an element of V.

(ii) The forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ and the free functor $F : \mathsf{Set} \to \mathsf{Group}$ form an adjunction. The components of the unit are set functions $\eta_S : S \to UFS$ that map an element $s \in S$ to the singleton string s, as an element of UFS. The counit has components $\varepsilon_G : FUG \to G$ that map a string $g_1 \dots g_n$ in the free group on UG to the product of the g_i in G.

(iii) Consider the adjunction $f_* \dashv f^{-1}$ of the forward-image and pre-image of a function of sets $f : A \to B$, as functors between the poset categories formed by the power sets of A and B. The definition of a unit-counit adjunction gives morphisms $f_*(A') \to f_*(f^{-1}(f_*(A'))) \to f_*(A')$ given by the components of the units at the object A'. These morphisms imply, in this category, that $f_*(A') = f_*(f^{-1}(f_*(A')))$ for any $A' \in P(A)$. Similarly, $f^{-1}(f_*(f^{-1}(B'))) = f^{-1}(B')$ for any $B' \in P(B)$. This is rather surprising, seeing as the inclusions $A' \subseteq f^{-1}(f_*(A'))$ and $f_*(f^{-1}(B')) \subseteq B'$ are not necessarily equalities in general.

One of the most important reasons we are interested in adjoint functors at all is their limit and colimit preserving properties:

Proposition 1.5.11. *Right adjoints preserve limits, and left adjoints preserve colimits.*

Proof. Let $F \dashv G$ be an adjoint pair, with $G : \mathsf{D} \to \mathsf{C}$ the right adjoint. Given $(\lim J, phi)$ a limit of the diagram $J : \mathsf{J} \to \mathsf{D}$, we show that $(G \lim J, G\varphi)$ is a limit of the diagram GJ in C.

First note that $(G \lim J, G\varphi)$ indeed forms a cone over GJ by functoriality, what remains is to show that it is actually a limit. To that end, let (Z, ψ) be another cone over GJ. Given any morphism $f : X \to Y$ in the image of J, there is a diagram

$$\begin{tikzcd} & Z \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\ & G\lim J \arrow[dl, "G\varphi_X"'] \arrow[dr, "G\varphi_Y"] & \\ GX \arrow[rr, "Gf"'] & & GY. \end{tikzcd}$$

We want to show there is a unique morphism $Z \to G\lim J$. To that end, we apply the transpose to all morphisms in the diagram to obtain a commutative diagram (as per Lemma 1.5.9)

$$\begin{tikzcd} & FZ \arrow[ddl, bend right, "\psi_X^T"'] \arrow[ddr, bend left, "\psi_Y^T"] & \\ & \lim J \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\ X \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

The universal property of the limit of J implies there is a unique morphism $u : FZ \to \lim J$ commuting with the legs of the cone. Now applying the transpose again, we obtain

$$\begin{tikzcd} & Z \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] \arrow[d, dashed, "\exists! u^T"] & \\ & G\lim J \arrow[dl, "G\varphi_X"'] \arrow[dr, "G\varphi_Y"] & \\ GX \arrow[rr, "Gf"'] & & GY. \end{tikzcd}$$

The morphism $u : FZ \to \lim J$ is unique, and the bijection $\mathsf{D}(FZ, \lim J) \cong \mathsf{C}(Z, G\lim J)$ implies that u^T is unique too. This proves that $G\lim J$ is a limit over the diagram $GJ : \mathsf{J} \to \mathsf{C}$. Because a limit over this diagram exists, it is canonically isomorphic to $\lim GJ$ as a consequence of Proposition 1.4.2. The proof of the statement that left adjoints preserve colimits is completely dual. $\square$

Example 1.5.12. This proposition leads to plenty of interesting corollaries, these include (but are not limited to):

(i) The forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ is a right adjoint, thus preserves products. Indeed, the product of groups $\prod_i G_i$ has, as an underlying set, the cartesian product of the underlying sets of the groups. On the other hand, the free functor $F : \mathsf{Set} \to \mathsf{Group}$ preserves coproducts (which are disjoint unions in Set and free products in Group). Thus, for sets S and T, we have that the free group $F(S \amalg T)$ is isomorphic to the free product $F(S) * F(T)$. This same idea holds for the other 'free $\dashv$ forgetful' adjunctions from Example 1.5.4.

(ii) The free functor $K : \mathsf{Set} \to \mathsf{Vect}_K$ is a left adjoint, and thus preserves colimits. In particular, given sets S and T, the vector space $K[S \amalg T]$ is isomorphic to $K[S] \oplus K[T]$. This generalizes the result from linear algebra that states $\mathrm{Span}\{v_1, \ldots, v_n\} \oplus \mathrm{Span}\{w_1, \ldots w_m\} \cong \mathrm{Span}\{v_1, \ldots, v_n, w_1, \ldots, w_m\}$.

(iii) The forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$ is both a left and right adjoint, and thus preserves all limits and colimits. Therefore any topological space formed as a limit has, as an underlying set, the same elements as the corresponding limit object in Set.

(iv) The ceiling function $\lceil - \rceil : (\mathbb{R}, \leqslant) \to (\mathbb{Z}, \leqslant)$ is a left adjoint, and thus preserves colimits. The coproduct of real numbers $\{x_i\}_i$ is their supremum, if it exists. Thus we obtain the fact that $\sup_i \lceil x_i \rceil = \lceil \sup_i x_i \rceil$. The ceiling does not preserve infima however. As an example, consider the sequence $\{x_i\}_{i \in \mathbb{N}}$ defined by $x_i = 1/i$. Then $\inf_i \lceil x_i \rceil = \inf_i 1 = 1$, but $\lceil \inf_i x_i \rceil = \lceil 0 \rceil = 0$.

(v) (†) Let M, A, and B be R-modules. Recall that the direct sum $A \oplus B$ is both a product and coproduct in Mod_R. Thus, using the Tensor-Hom adjunction $M \otimes_R - \dashv \operatorname{Hom}(M, -)$, we obtain the natural isomorphism

$$M \otimes_R (A \oplus B) \cong (M \otimes_R A) \oplus (M \otimes_R B).$$

The fact that adjoint functors are (co)continuous invites the opposite question: when is a continuous functor $G : \mathsf{D} \to \mathsf{C}$ a right adjoint of some other functor? One of the most well-known conditions is the *Freyd Adjoint Functor Theorem*, which first appeared as exercise 3.J (p.84) in [Fre64], with a proof given in [Mac98, theorem IV.6.2, p.121]. In modern categorical language, it states:

Theorem 1.5.13 (Freyd Adjoint Functor Theorem)**.** *Let $G : \mathsf{D} \to \mathsf{C}$ be a continuous functor, whose domain is complete and locally small. The functor G admits a left adjoint if and only if for every object X of C, there is a set of morphisms $\{f_i : X \to GA_i\}_i$ such that, for any morphism $f : X \to GA$, there is an i and some morphism $t : A_i \to A$ such that $f = Gt \circ f_i$.*

This ends this Chapter on category theory. We have seen how categories allow us to generalize concepts from many different fields of mathematics. In the next Chapter, we see how we can apply some of these categorical notions to prove a theorem regarding functors between categories of modules over commutative rings.

2 Watts' Theorem

Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial.

–Peter Freyd [Fre66]

This Chapter is focused on the Eilenberg-Watts' Theorem, first proved by Eilenberg and Watts independently in 1960 [Eil60, Wat60]. Despite this, the name of the theorem often simply goes by Watts' Theorem. The theorem states that any $F : \mathsf{Mod}_R \to \mathsf{Mod}_R$ that preserves colimits is naturally isomorphic to the tensor product functor $FR \otimes_R -$. This Chapter builds up the necessary background to understand the proof of the theorem. This includes a review of the basic theory of modules, which we do in the first Section. The second Section formally defines the *tensor product*, and proves some useful facts about it, including its adjunction to the Hom-functor. Following this, we define exact sequences and so-called module presentations in the third Section. In the penultimate Section we state and prove Watts' Theorem, and also discuss a few consequences, reformulations, and generalizations. The final Section takes a detour to cover *localization* of rings and modules, which function as a nice application of Watts' Theorem.

2.1 Modules and Direct Sums

This Section begins with a review of the basics of module theory. We define modules, module-homomorphisms, submodules, quotient modules, direct sums, and free modules. The Section is only meant as review, so most of the statements are not proven here. The theory itself is mostly adapted from [vGLOT17]. As before, we assume all rings are unitary, and all ring-homomorphisms preserve the multiplicative identity. For this Chapter, we also assume all rings are commutative, which we need for the Tensor-Hom adjunction in the second Section.

Definition 2.1.1. Let R be a ring. A *left R-module* M is an abelian group $(M, +, 0)$, along with an action of *scalar multiplication*, defined as a function $R \times M \to M$, by $(r, m) \mapsto rm$. This multiplication satisfies the following axioms for all $a, b \in R$ and $m, n \in M$:

- $a(m + n) = am + an$;
- $(a + b)m = am + bm$;
- $a(bm) = (ab)m$;
- $1m = m$ (here 1 denotes the multiplicative unit in R).

A *right R-module* is defined similarly, but with scalar multiplication as a function $M \times R \to M$, by $(m, r) \mapsto mr$ satisfying similar properties to that of left scalar multiplication. ▼

Remark. Left R-modules and right R-modules are quite similar, in the sense that the categories Mod_R of left R-modules is equivalent to ${}_{R^{\mathrm{op}}}\mathsf{Mod}$ of right R^{op}-modules.[16] Because we assume rings to be commutative in this Chapter, R and R^{op} are isomorphic, making the two categories isomorphic as well. As such, when talking about R-modules, we only consider left R-modules, unless otherwise stated. In the same way we denote Mod_R to be the category of R-modules, both the left and right variations.

[16] Recall that the ring R^{op} has the same elements and addition operation as R, but multiplication changes order, so $a \cdot_{\mathrm{op}} b := b \cdot a$ for $a, b \in R^{\mathrm{op}}$.

Definition 2.1.2. A function $f : M \to M'$ between R-modules is an *R-module-homomorphism* if

$$f(am + bn) = af(m) + bf(n)$$

for all $a, b \in R$ and $m, n \in M$. ▼

Put differently, an R-module-homomorphism is a homomorphism of the underlying abelian groups that commutes with scalar multiplication. A consequence of this definition is that

$$f(0) = f(0 - 0) = f(0) - f(0) = 0.$$

We call such an R-module-homomorphism an *isomorphism* if there is another R-module-homomorphism $g : M' \to M$ such that $fg = 1_{M'}$ and $gf = 1_M$. In Mod_R, isomorphisms are exactly bijective homomorphisms.

Definition 2.1.3. Let M be an R-module, and N a subset of M. We say N is a *submodule* of M if:

- $0 \in N$;
- $am + bn \in N$ for all $a, b \in R$ and $m, n \in N$.

In this case, there is an inclusion homomorphism $N \hookrightarrow M$ that sends an element to itself in M.

A submodule $N \subseteq M$ is also a subgroup of the underlying abelian group M. So it makes sense to talk about the *quotient module* M/N. Its elements are equivalence classes $m+N$ and inherits the additive structure from the quotient abelian group. Scalar multiplication is defined as $a(m + N) = am + N$ for $a \in R$ and $m \in M$. There is a canonical projection homomorphism $M \twoheadrightarrow M/N$ that sends an element to its equivalence class. ▼

Example 2.1.4. The following are examples of R-modules for various rings R.

(i) If R is a field, then an R-module is the same as a vector space over R. Homomorphisms of these modules are the same as linear maps between vector spaces. In this sense modules serve to generalize the concept of vector spaces.

(ii) A $\mathbb{Z}$-module is the same as an abelian group. The obvious $\mathbb{Z}$-action is $\mathbb{Z} \times M \to M$ by defining $zm := \operatorname{sign}(z)(\underbrace{m + \cdots + m}_{|z| \text{ times}})$. A $\mathbb{Z}$-module-homomorphism is the same as a homomorphism of abelian groups. From this perspective, modules generalize the concept of abelian groups.

(iii) Any ring R is an R-module over itself. Scalar multiplication is done with the multiplication operation of the ring. And if $\mathfrak{a}$ is an ideal of R, then $\mathfrak{a}$ is a submodule of R. The quotient ring $R/\mathfrak{a}$ is also a quotient module over R. This is another way in which modules generalize some algebraic concepts, namely rings and ideals.

(iv) For a ring R, we define $R[x_1, \ldots, x_n]$ to be the R-module of polynomials in n variables with coefficients in R. Addition and scalar multiplication is done term-by-term.

(v) For any smooth manifold M, the set of smooth real functions $C^\infty(M, \mathbb{R})$ is a ring, where addition and multiplication is done pointwise. The set of smooth vector fields $\mathfrak{X}(M)$ on M forms a module over this ring. See [Ser23, section 4.1] for details.

(vi) If $\varphi : R \to S$ is a ring-homomorphism, then any S-module M can be redefined as an R-module, by setting $rm := \varphi(r)m$ for $r \in R$ and $m \in M$. This is called *restriction of scalars*, and gathers into a functor $\mathsf{Mod}_S \to \mathsf{Mod}_R$, mapping an S-module to the corresponding R-module as above. As an example, if R is a subring of S, and φ the inclusion map, then we can 'restrict' the scalers of an S-module to only use scalars of R.

(vii) Over any ring R, the zero module, denoted 0, contains a single element. For any other R-module M, there are unique homomorphisms $M \to 0$ and $0 \to M$, making 0 a zero object in the category of R-modules. Between two modules M and M', there is a unique zero map $0 : M \to M'$ that maps everything to the zero element of M'. This map may also be defined as the unique composition $M \to 0 \to M'$.

(viii) (†) The set of R-module-homomorphism $\mathrm{Hom}(M, N)$, or $\mathrm{Hom}_R(M, N)$, is an R-module as well. Addition and scalar multiplication is done pointwise. Thus for homomorphisms $f, g \in \mathrm{Hom}_R(M, N)$, scalars $a \in R$, and elements $m \in M$, we have $(af + g)(m) := af(m) + g(m)$.

We have already defined the kernel and cokernel in general categories, but it is worth it to go over the definitions in this specific case, as we do not use the categorical definition in this Chapter for the most part.

Definition 2.1.5. Let $f : M \to M'$ be an R-module-homomorphism.

- The *kernel of* f, denoted $\ker f := \{x \in M \mid f(x) = 0\}$, is a submodule of M. It is governed by the following universal property:[17] there is a homomorphism $k : \ker f \to M$ with $f \circ k = 0$, such that for any other homomorphism $k' : K' \to M$ with $f \circ k' = 0$, there is a unique $u : K' \to \ker f$ making the following diagram commute:

$$\begin{tikzcd}
 & M \arrow[dr, "f"] & \\
 & \ker f \arrow[u, "k"] \arrow[r, "0"'] & M'. \\
K' \arrow[uur, bend left, "k'"] \arrow[ur, dashed, "\exists! u"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

The map k is usually the inclusion homomorphism. The map f is injective if and only if $\ker f = 0$.

- The *image of* f, denoted $\operatorname{im} f := \{y \in M' \mid f(x) = y \text{ for some } x \in M\}$ is a submodule of M'.

- The *cokernel of* f, denoted $\operatorname{coker} f := M' / \operatorname{im} f$, is governed by the following universal property, which is dual to that of the kernel: there is a homomorphism $q : M' \to \operatorname{coker} f$ with $q \circ f = 0$, such that for any other homomorphism $q' : M' \to Q'$ with $q' \circ f = 0$, there is a unique $u : \operatorname{coker} f \to Q'$ making the following diagram commute:

$$\begin{tikzcd}
 & M \arrow[d, "0"'] \arrow[dr, "f"] \arrow[ddl, bend right, "0"'] & \\
 & \operatorname{coker} f \arrow[dl, dashed, "\exists! u"] & M' \arrow[l, "q"] \arrow[dll, bend left, "q'"] \\
Q' & &
\end{tikzcd}$$

[17] This universal property is the same as the universal definition of the equalizer of f and 0, as defined in Section 1.4.

The map q is usually the canonical projection. The map f is surjective if and only if $\operatorname{coker} f = 0$. ▼

Remark. With the same notation as above, note that the image of f is exactly $\ker q$. Moreover, by the first isomorphism theorem, we have that the image of f is isomorphic to $\operatorname{coker} k = M/\operatorname{im} k = M/\ker f$ [vGLOT17, theorem VII.1.4 (a), p.60]. This is how we define the image in Chapter 3, as the kernel of the cokernel, or equivalently as the cokernel of the kernel.

One of the most common ways to create new modules out of smaller ones is by the direct sum, which we define now:

Definition 2.1.6. Let $\{M_i\}_{i\in I}$ be a set of modules for some indexing set I. We define their

- *direct product* $\prod_{i\in I} M_i := \{(m_i) \mid m_i \in M_i\}$ as the R-module of I-indexed sequences of elements of the modules. Addition and scalar multiplication is done component-wise.
- *direct sum* $\bigoplus_{i\in I} M_i$ to have the same elements as the direct product, but we stipulate that only finitely many of the entries in a sequence are nonzero. If I is finite, then the direct product and direct sum are one and the same.[18] ▼

As mentioned in Example 1.4.3(ii), for finite I, the direct sum is both a product and coproduct in Mod_R. Meaning that for all $j, k \in I$, there are maps

$$M_j \xrightarrow{\iota_j} \bigoplus_{i\in I} M_i \xrightarrow{\pi_k} M_k.$$

Here the inclusion maps an element m_j to the sequence with only zeroes, except the j-th entry having m_j. The projection maps a sequence to its k-th element. Note that these satisfy $\pi_k \circ \iota_j = 0$ unless $k = j$, in which case the composition is the identity map on M_j.

If the modules M_i are all submodules of some larger module M, with $M_i \cap M_j = \{0\}$ for all distinct $i, j \in I$, then we may define the *inner direct sum* as the R-module $\bigoplus_{i\in I} M_i$, containing finite sums of elements of each M_i. The fact that all modules intersect trivially implies that each element of the direct sum can be written in a *unique* way. As the notation may suggest, the inner direct sum is isomorphic to the direct sum as defined in Definition 2.1.6.

One helpful fact of linear algebra is that any vector space has a basis. This is not true in general of modules however. We call modules with a basis *free*:

Definition 2.1.7. Let M be an R-module, and $S \subseteq M$ some subset of elements. We call S a *generating set* of M if every element in M can be written as a finite linear combination of elements of S, with scalars in R. We say S *generates* M and write $M = \langle S\rangle$.
We say S is a *basis* of M if it generates M, and the elements of S are *linearly independent*. That is, given some finite subset $\{s_1, \ldots, s_n\} \subseteq S$, we have that $\sum_{i=1}^n r_i s_i = 0$ if and only if each r_i is zero. If M admits a basis, we call it *free*. The *rank* of a free module is the cardinality of the basis set S. ▼

Example 2.1.8. Some examples of free modules include:

(i) The direct sum $\bigoplus_{s\in S} R$ is free, with rank equal to the cardinality of S. In fact, every free R-module is isomorphic to a direct sum of copies of R. We often denote this repeated direct sum as $R^{\oplus S}$.

[18] If I is empty, we define both the direct product and direct sum to be the zero module.

(ii) If R is a commutative ring, then the R-module of polynomials $R[x]$ is free. Its basis is the set of monomials $\{1, x, x^2, \dots\}$. If f is a monic polynomial over R, then $R[x]/(f)$ is a free R-module, with rank equal to the degree of f.

(iii) Every vector space is free, with rank equal to its dimension. This is a consequence of the Axiom of Choice, which shows that every vector space can be given a basis [Bar14, lemma 3.1, p.5].

(iv) (†) We define the *torsion submodule* of an R-module M as the R-module

$$\operatorname{Tor} M := \{m \in M \mid rm = 0 \text{ for some } r \in R \setminus Z_R\},$$

where Z_R is the set of zero divisors of R. If M is finitely generated and R is a principal ideal domain, then by [DF04, theorem 12.1.5, p.462],

$$M \cong R^n \oplus R/(a_1) \oplus \cdots \oplus R/(a_t).$$

Here (a_i) is the ideal of the ring R generated by a_i, and these ideals satisfy $(a_i) \subseteq (a_{i+1})$ for all i. The module M is free if and only if $\operatorname{Tor} M \cong R/(a_1) \oplus \cdots \oplus R/(a_t) = 0$. If $R = \mathbb{Z}$, we obtain the well-known structure theorem for abelian groups, as given in [DF04, theorem 5.2.3, p.158].

Every free module satisfies the following universal property, which allows us to construct free modules over any ring, given any set of initial basis elements.

Proposition 2.1.9. *Let R be a ring, and S a set. The inclusion set-function $\iota : S \hookrightarrow R^{\oplus S}$ is universal in the sense that given some other set-function $f : S \to N$, where N is any other R-module, there is a unique R-module-homomorphism $\varphi : R^{\oplus S} \to N$ that makes the following diagram commute:*

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & R^{\oplus S} \arrow[d, dashed, "\exists!\varphi"] \\ & N. \end{tikzcd}$$

As with any universal property, this defines free modules up to unique isomorphism. The homomorphism φ extends f linearly, that is, it acts on finite linear combinations by

$$\varphi\left(\sum_i r_i s_i\right) = \sum_i r_i f(s_i).$$

2.2 Tensor Products and the Hom-Functor

This section is focused on the tensor product. The tensor product allows us to put two modules together while preserving linearity in both modules. One might suspect that the direct sum already does this, but this is not quite true. For example, we may want the element (rm, n) to be the same as (m, rn) in $M \oplus N$, but this is simply not true. We define the tensor product of M and N to be a module preserving exactly these relations. This is a bit of a hassle though, and we may prefer to utilize a certain universal property that defines the tensor product up to a canonical isomorphism. This is done with *bilinear maps*, which are functions $b : M \oplus N \to S$ such that that, for any $m \in M$ and $n \in N$, the functions $b(m, -) : N \to S$ and $b(-, n) : M \to S$ are R-module-homomorphisms.

Definition 2.2.1. Let M and N be R-modules. The *tensor product* of M and N consists of an R-module T and a bilinear map $\beta : M \oplus N \to T$ such that, for bilinear map $f : M \oplus N \to S$, there is a *unique* R-module-homomorphism $\varphi : T \to S$ such that the diagram

$$\begin{tikzcd} M \oplus N \arrow[r, "f"] \arrow[d, "\beta"'] & S \\ T \arrow[ur, dashed, "\exists!\varphi"'] & \end{tikzcd}$$

commutes. This only defines the tensor product up to isomorphism, but there is a 'natural' way to define it as follows:
The *tensor product* $M \otimes_R N$ contains finite sums of elements of the form $m \otimes n$ for $m \in M$ and $n \in N$. These elements are called *elementary tensors* and satisfy the following relations:

- $m \otimes n + m' \otimes n = (m + m') \otimes n$;
- $m \otimes n + m \otimes n' = m \otimes (n + n')$;
- $rm \otimes n = r(m \otimes n) = m \otimes rn$.

A priori, the elements of $M \otimes_R N$ do not satisfy any other relations. The bilinear map corresponding to the universal property is $\otimes : M \oplus N \to M \otimes_R N$ that sends a pair (m, n) to the elementary tensor $m \otimes n$. For a detailed construction and a proof of this module satisfying the universal property, see [DF04, section 10.1]. ▼

The universal property is great for proving certain facts about the tensor product. The following proposition gives some of these properties:

Proposition 2.2.2. *Let R be a ring and M and N be R-modules. Then the following hold:*

(a). $M \otimes_R N \cong N \otimes_R M$.

(b). $R \otimes_R M \cong M$.

(c). *If $f : M \to M'$ and $g : N \to N'$ are R-module-homomorphisms, then these maps induce a homomorphism $f \otimes g : M \otimes_R N \to M' \otimes_R N'$.*

Proof. (a). We could write down an isomorphism and check if these modules are in fact isomorphic, but it is good to see how one might prove it using the universal property. We show that $N \otimes_R M$ satisfies the universal property that $M \otimes_R N$ satisfies. Universality implies that these two are isomorphic. To that end, we need a bilinear map $\beta : M \oplus N \to N \otimes_R M$, which we define here as $\beta(m, n) := n \otimes m$.

Let $f : M \oplus N \to S$ be another bilinear map. We want to show there is a unique map $\varphi : N \otimes_R M \to S$ making

$$\begin{tikzcd} M \oplus N \arrow[r, "f"] \arrow[d, "\beta"'] & S \\ N \otimes_R M \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

commute. To that end, we define φ by $\varphi(n \otimes m) := f(m, n)$, and extending linearly. Because f is bilinear, this is indeed an R-module-homomorphism. Moreover, note that for $(m, n) \in M \oplus N$,

$$\varphi(\beta(m, n)) = \varphi(n \otimes m) = f(m, n),$$

which makes the diagram commute. Last is to show that φ is unique. To that end, assume there is some other homomorphism $\varphi' : N \otimes_R M \to S$ such that $\varphi' \circ \beta = f$. Now note that for any finite sum $\sum_i n_i \otimes n_i \in N \otimes_R M$,

$$\begin{aligned}
\varphi'\left(\sum_i n_i \otimes m_i\right) &= \sum_i \varphi'(n_i \otimes m_i) \\
&= \sum_i f(m_i, n_i) \\
&= \sum_i \varphi(n_i \otimes m_i) = \varphi\left(\sum_i n_i \otimes m_i\right).
\end{aligned}$$

Thus, $\varphi' = \varphi$, making φ unique. Therefore, since $N \otimes_R M$ satisfies the universal property of $M \otimes_R N$, these tensor products are isomorphic. The map β suggests an explicit isomorphism $M \otimes_R N \to N \otimes_R M$, namely the one defined on elementary tensors as by $m \otimes n \mapsto n \otimes m$, which is extended linearly.

(b). We show that M satisfies the universal property of $R \otimes_R M$. First define a bilinear map $\beta : R \oplus M \to M$ by $\beta(r, m) := rm$. Distributivity implies that β is indeed bilinear. Now let $f : R \oplus M \to S$ be another bilinear map. First of all, we show that the diagram

$$\begin{tikzcd}
R \oplus M \arrow[r, "f"] \arrow[d, "\beta"'] & S \\
M \arrow[ur, dashed, "\varphi"'] &
\end{tikzcd}$$

commutes for some homomorphism φ. We define this as $\varphi(m) := f(1, m)$, which is a homomorphism by the bilinearity of f. Now let $(r, m) \in R \oplus M$ and notice that

$$\varphi(\beta(r, m)) = \varphi(rm) = f(1, rm) = f(r, m),$$

where the last equality follows from bilinearity of f. Now that we have shown that the diagram commutes, we show this map φ is unique. Let $\varphi' : M \to S$ be another homomorphism such that $\varphi' \circ \beta = f$, and notice for all $m \in M$:

$$\varphi'(m) = \varphi'(\beta(1, m)) = f(1, m) = \varphi(m).$$

Therefore, $\varphi' = \varphi$, which means that M indeed satisfies the universal property of $R \otimes_R M$, making the two modules isomorphic. We can also write down the explicit isomorphism $R \otimes_R M \to M$, defined on elementary tensors as $r \otimes m \mapsto rm$ and extended linearly.

(c). We use the universal property to construct a homomorphism $f \otimes g : M \otimes_R N \to M' \otimes_R N'$. Let $b : M \oplus N \to M' \otimes_R N'$ be defined by $b(m, n) := f(m) \otimes g(n)$. This map is bilinear, because f and g are homomorphisms, and $- \otimes -$ is bilinear. Thus, there is a unique map $\varphi : M \otimes_R N \to M' \otimes_R N'$ such that $\varphi \circ (- \otimes -) = b$. By construction, this map is defined on elementary tensors by $\varphi(m \otimes n) = f(m) \otimes g(n)$, and extended linearly. We denote this map by $f \otimes g : M \otimes_R N \to M' \otimes_R N'$. $\square$

The last part helps us to formally define the tensor product as a functor. Given an R-module M, the functor $M \otimes_R - : \mathsf{Mod}_R \to \mathsf{Mod}_R$ is a functor that sends an R-module N to $M \otimes_R N$. A homomorphism $f : N \to P$ is sent to the tensor product $1_M \otimes f : M \otimes_R N \to M \otimes_R P$.

Before moving on to the Hom-functor, it is helpful to see examples of the tensor product in action.

Example 2.2.3.

(i) For any finite abelian group A, the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is isomorphic to the zero module. This can be seen by taking an arbitrary elementary tensor $q \otimes a$, and noticing we can rewrite this to

$$\operatorname{ord}(a)(\tfrac{q}{\operatorname{ord(a)}} \otimes a) = \tfrac{q}{\operatorname{ord(a)}} \otimes \operatorname{ord}(a)a = \tfrac{q}{\operatorname{ord(a)}} \otimes 0.$$

By bilinearity, tensoring anything with zero gives the zero element of the module, so every elementary tensor in $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is zero, and so $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$.[19]

(ii) More generally, if R is a domain with field of fractions $Q(R)$, then for any R-module M, it follows that $Q(R) \otimes_R \operatorname{Tor} M = 0$ by a similar argument as before.

(iii) Let R be a commutative ring. If M is a free R-module with basis S and N a free R-module with basis T, then $M \otimes_R N$ is free as well, with basis $\{s \otimes t \mid s \in S, t \in T\}$. If M and N both have finite rank, then the tensor product has rank equal to the product of the ranks of M and N. For a proof of this fact, see [vGLOT17, proposition VII.3.11, p.69]. A consequence of this is that the tensor product of polynomial modules $R[x] \otimes_R R[y]$ is isomorphic to $R[x, y]$.

(iv) Let K be a field, and $f : V \to V'$ and $g : W \to W'$ linear maps between K-vector spaces of finite dimension. If we equip V, V', W, and W' with some basis, then the matrix corresponding to the linear map $f \otimes g : V \otimes_K W \to V' \otimes_K W'$ is the *kronecker product* of the matrices corresponding to f and g. For information on applications of the Kronecker product, see [Loa00].

(v) Let R be a subring of a ring S. An R-module M can be extended to an S-module by way of the tensor product $S \otimes_R M$. It has a canonical S-action by $s(x \otimes m) := sx \otimes m$ for $s, x \in S$ and $m \in M$. This is called *extension of scalars* and is a sort of dual to the restriction of scalars we saw in 2.1.4(vi). This duality is actually an adjunction! In the sense that the functor $S \otimes_R - : \mathsf{Mod}_R \to \mathsf{Mod}_S$ is the left adjoint of the functor $\mathsf{Mod}_S \to \mathsf{Mod}_R$ that takes an S-module to its restricted R-module. The proof relies on the Tensor-Hom adjunction, and details are given in [Tae18, corollary 6.25, p.74].

We now move our attention to the Hom-functor. We have already seen its definition, but it is helpful to see it again in the context of R-modules.

Definition 2.2.4. Let R be a ring and M an R-module. The *Hom-functor*

$$\begin{gathered}
\operatorname{Hom}_R(M, -) : \mathsf{Mod}_R \to \mathsf{Mod}_R \\
N \mapsto \operatorname{Hom}_R(M, N) \\
(f : N \to P) \mapsto (f_* : \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, P))
\end{gathered}$$

takes a module N to the R-module of homomorphisms $\operatorname{Hom}_R(M, N)$. A homomorphism $f : N \to P$ is sent to the pushforward f_*, which acts on homomorphisms $g \in \operatorname{Hom}_R(M, N)$ by $f_*(g) := f \circ g \in \operatorname{Hom}_R(M, P)$. ▼

As stated before, $\operatorname{Hom}_R(M, N)$ has the structure of an R-module by pointwise addition and scalar multiplication. In some cases, the structure of this Hom-module can be explicitly computed, the following proposition gives a nice example of this:

[19] If an object of a category is isomorphic to the zero object, we often write it as an equality.

Proposition 2.2.5. *Let R be a ring and M an R-module. The Hom-module* $\mathrm{Hom}_R(R, M)$ *is isomorphic to M. Moreover, this isomorphism is natural in M.*

Proof. For an R-module M, define $\eta_M : \mathrm{Hom}_R(R, M) \to M$ by $\eta_M(f) = f(1)$. To show this is a homomorphism, take $f, g \in \mathrm{Hom}_R(R, M)$ and $r, s \in R$, and note:

$$\eta_M(rf + sg) = (rf + sg)(1) = rf(1) + sg(1) = r\eta_M(f) + s\eta_M(g),$$

by the R-module structure on $\mathrm{Hom}_R(R, M)$. Thus η_M is an R-module-homomorphism.

To show injectivity, note that $\eta_M(f) = 0$ if and only if $f(1) = 0$. Now because f is an R-module-homomorphism, for each r in R, it follows $f(r) = rf(1) = r0 = 0$. Thus, $f = 0$ and so η_M is injective.

For surjectivity, let $m \in M$ be arbitrary. We can define an R-module-homomorphism f by setting $f(1) = m$, and extending linearly. Indeed, $\eta_M(f) = m$, which shows that η_M is surjective. Together with injectivity, it follows that η_M defines an isomorphism $\mathrm{Hom}_R(R, M) \cong M$.

To show that the components form a natural isomorphism $\eta : \mathrm{Hom}_R(R, -) \Rightarrow 1_{\mathsf{Mod}_R}$, we show that for all R-modules M, M', and any homomorphism $g : M \to M'$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_R(R, M) \arrow[r, "\eta_M"] \arrow[d, "g_*"'] & M \arrow[d, "g"] \\
\mathrm{Hom}_R(R, M') \arrow[r, "\eta_{M'}"'] & M'.
\end{tikzcd}$$

To that end, let $f \in \mathrm{Hom}_R(R, M)$ and note:

$$g(\eta_M(f)) = g(f(1)) = (g \circ f)(1) = \eta_{M'}(g \circ f) = \eta_{M'}(g_*(f)),$$

which shows that $g \circ \eta_M = \eta_{M'} \circ g_*$, thus the diagram commutes. As each component η_M is an isomorphism, the functors $\mathrm{Hom}_R(R, -)$ and 1_{Mod_R} are naturally isomorphic.[20] $\square$

We are now ready to prove the Tensor-Hom adjunction, which is a useful result for the rest of the Chapter as well:

Proposition 2.2.6. *Let R be a ring and T be an R-module. The functor $T \otimes_R -$ is left adjoint to the functor* $\mathrm{Hom}_R(T, -)$.

Proof. We prove this using a unit-counit adjunction, as defined in Definition 1.5.7.. For clarity, write $F := T \otimes_R -$ and $G := \mathrm{Hom}_R(T, -)$. We define the unit of the adjunction as the natural transformation $\varepsilon : FG \Rightarrow 1_{\mathsf{Mod}_R}$ with components

$$\varepsilon_Z : FGZ = T \otimes_R \mathrm{Hom}_R(T, Z) \to Z, \quad \varepsilon_Z(t \otimes \varphi) := \varphi(t)$$

for any $t \in T$ and $\varphi \in \mathrm{Hom}_R(T, Z)$. This definition is extended to non-elementary tensors linearly. The counit is defined as the natural transformation $\eta : 1_{\mathsf{Mod}_R} \Rightarrow GF$ with components

$$\eta_Z : Z \to GFZ = \mathrm{Hom}_R(T, T \otimes_R Z), \quad \eta_Z(z) : t \mapsto t \otimes z$$

[20]A similar argument to this proof can be used to show that the isomorphism $R \otimes_R M \cong M$ from Proposition 2.2.2(b) is natural, with components $\eta_M(r \otimes m) := rm$.

for any $z \in Z$ and $t \in T$. Note that bilinearity of the tensor product implies that $\eta_Z(z)$ is indeed a homomorphism from T to $T \otimes_R Z$, and also that η_Z is a homomorphism from Z to $\mathrm{Hom}_R(T, T \otimes_R Z)$.

The next step is to show that both ε and η are actually natural transformations. Starting with ε, let $g : Z \to Z'$ be an R-module-homomorphism, with the goal of showing that the diagram

$$\begin{tikzcd} T \otimes_R \mathrm{Hom}_R(T, Z) \arrow[r, "\varepsilon_Z"] \arrow[d, "1_T \otimes g_*"'] & Z \arrow[d, "g"] \\ T \otimes_R \mathrm{Hom}_R(T, Z') \arrow[r, "\varepsilon_{Z'}"'] & Z' \end{tikzcd}$$

commutes. To that end, let $\sum_i t_i \otimes \varphi_i$ be an arbitrary tensor in $T \otimes_R \mathrm{Hom}_R(T, Z)$. The top path of the diagram evaluates to:

$$g\left(\varepsilon_Z\left(\sum_i t_i \otimes \varphi_i\right)\right) = \sum_i g\left(\varepsilon_Z(t_i \otimes \varphi_i)\right) = \sum_i g\left(\varphi_i(t_i)\right).$$

Here we used the linearity of both ε_Z and g. The other path of the diagram evaluates to:

$$\begin{aligned} \varepsilon_{Z'}\left((1_T \otimes g_*)\left(\sum_i t_i \otimes \varphi_i\right)\right) &= \sum_i \varepsilon_{Z'}((1_T \otimes g_*)(t_i \otimes \varphi_i)) \\ &= \sum_i \varepsilon_{Z'}(t_i \otimes g \circ \varphi_i) \\ &= \sum_i g(\varphi_i(t_i)). \end{aligned}$$

So indeed, $g \circ \varepsilon_Z = \varepsilon_{Z'} \circ (1_T \otimes g_*)$, which makes ε a natural transformation.

To show η is natural, we show that for any homomorphism $g : Z \to Z'$, the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "\eta_Z"] \arrow[d, "g"'] & \mathrm{Hom}_R(T, T \otimes_R Z) \arrow[d, "(1_T \otimes g)_*"] \\ Z' \arrow[r, "\eta_{Z'}"'] & \mathrm{Hom}_R(T, T \otimes_R Z'). \end{tikzcd}$$

To that end, let $z \in Z$. The image of z under both compositions is a homomorphism $T \to T \otimes_R Z'$, so to show they are equal, we take an arbitrary t in T. Now, evaluating the top path of the diagram at t, we find

$$\begin{aligned} \big(((1_T \otimes g)_* \circ \eta_Z)(z)\big)(t) &= (1_T \otimes g)(\eta_Z(z)(t)) \\ &= (1_T \otimes g)(t \otimes z) \\ &= t \otimes g(z). \end{aligned}$$

Now for the bottom path, again with arbitrary t in T:

$$\big(\eta_{Z'}(g(z))\big)(t) = t \otimes g(z),$$

which follows immediately from the definition of $\eta_{Z'}$. Now for both compositions, we took t arbitrary, meaning that the maps $((1_T \otimes g)_* \circ \eta_Z)(z)$ and $\eta_{Z'}(g(z))$ are equal. And thus, the diagram commutes, making η natural.

The last part of proving that ε and η form a unit-counit adjunction is to show that the following diagrams in the category $[\mathsf{Mod}_R, \mathsf{Mod}_R]$ commute:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "1_G"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G. \end{tikzcd}$$

Starting with the left one, we show that $(\varepsilon F \circ F\eta)_Z = 1_{FZ}$ for any R-module Z. These are homomorphisms from $T \otimes_R Z$ to itself, so let $\sum_i t_i \otimes z_i$ be an arbitrary tensor, and note that the left-hand side expands to[21]

$$\begin{aligned}(\varepsilon F \circ F\eta)_Z \left(\sum_i t_i \otimes z_i\right) &= \varepsilon_{T\otimes_R Z}\left(1_T \otimes \eta_Z \left(\sum_i t_i \otimes z_i\right)\right) \\ &= \varepsilon_{T\otimes_R Z}\left(\sum_i (1_T \otimes \eta_Z)(t_i \otimes z_i)\right) \\ &= \sum_i \varepsilon_{T\otimes_R Z}(t_i \otimes \eta_Z(z_i)) \\ &= \sum_i (\eta_Z(z_i))(t_i) \\ &= \sum_i t_i \otimes z_i = 1_{FZ}\left(\sum_i t_i \otimes z_i\right).\end{aligned}$$

Indeed, $(\varepsilon F \circ F\eta)_Z = 1_{FZ}$. Thus, because natural transformations are defined by their components, we have shown that $\varepsilon F \circ F\eta = 1_F$.

For the second diagram, we show $(G\varepsilon \circ \eta G)_Z = 1_{GZ}$ for any R-module Z. These are homomorphisms from $\mathrm{Hom}_R(T, Z)$ to itself, so let φ be a homomorphism in $\mathrm{Hom}_R(T, Z)$. Taking a t in T and expanding the left-hand side gives

$$\begin{aligned}\big((G\varepsilon \circ \eta G)_Z(\varphi)\big)(t) &= (\varepsilon_Z)_*\big((\eta_{\mathrm{Hom}_R(T,Z)}(\varphi))\big)(t) \\ &= \varepsilon_Z\big(\eta_{\mathrm{Hom}_R(T,Z)}(\varphi)(t)\big) \\ &= \varepsilon_Z(t \otimes \varphi) \\ &= \varphi(t).\end{aligned}$$

Therefore, because t was arbitrary, we conclude that $(G\varepsilon \circ \eta G)_Z(\varphi) = 1_{GZ}(\varphi)$. This proves that F and G form a unit-counit adjunction, and thus, by Proposition 1.5.8, also an adjoint pair $F \dashv G$. ☐

Corollary 2.2.7. *For a collection of R-modules $\{M_i\}_{i\in I}$, there is an isomorphism*

$$T \otimes_R \bigoplus_{i\in I} M_i \cong \bigoplus_{i\in I} (T \otimes_R M_i).$$

[21] Recall that this is vertical composition of natural transformations, so $(\varepsilon F \circ F\eta)_Z = \varepsilon_{FZ} \circ F\eta_Z$.

In particular,

$$T \otimes_R R^{\oplus I} \cong T^{\oplus I}$$

as a consequence of Proposition 2.2.2(b). Also, for any R-module homomorphism $\varphi : M \to N$, it follows that

$$T \otimes_R \operatorname{coker} \varphi \cong \operatorname{coker}(1_T \otimes \varphi).$$

Proof. This follows immediately from the fact that the direct sum and cokernel are colimits in Mod_R, and Proposition 1.5.11. □

This corollary is very useful for proving Watts' Theorem. Before we move on to that, we need one more topic, which is that of *exact sequences*. These are also important for Chapter 3.

2.3 Exact Sequences and Module Presentations

An exact sequence is a sequence of R-modules with R-module-homomorphisms between them, such that the image of every map is equal to the kernel of the subsequent one. These sequences allow us to specify injective and surjective homomorphisms, without relying on elements of the relevant domains and codomains. Another use of exact sequences is they help to define free module presentations, which in some way generalize presentations of groups.

Definition 2.3.1. Let R be a ring, and

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

be a (potentially infinite) sequence of R-modules with R-module-homomorphisms between them. We say this sequence is *exact in M_i* if $\operatorname{im} f_{i-1} = \ker f_i$. We call the sequence *exact* if it is exact in every module in the sequence. We call an exact sequence of the form

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

a *short exact sequence*. ▼

Remark. Note that in an exact sequence as above, $f_i \circ f_{i-1} = 0$ for any i. This is a necessary condition for the sequence being exact, but it is not sufficient. We call a sequence with this property a *chain complex*, which play a central role in Chapter 3.

To get a grasp on the relevance of exact sequences, it may be helpful to see examples:

Example 2.3.2.

(i) The sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

where we multiply an integer by n and then send an integer to its equivalence class modulo n, is exact. Exactness in the middle module follows from the fact that the image of $\cdot n$ is the set of integer multiples of n, which is exactly the kernel of the projection π. Exactness in the other two modules follows from the following two more general statements:

(ii) A sequence of the form

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact if and only if f is injective, as then the kernel of f is the same as the image of the zero map $0 \to M$, namely $\{0\} \subset M$. Dually, a sequence of the form

$$M \xrightarrow{f} N \longrightarrow 0$$

is exact if and only if f is surjective. Putting the two together, we see that

$$0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$$

is exact if and only if f is an isomorphism.

(iii) More generally, a sequence of the form

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P$$

is exact if and only if f is injective, and M is canonically isomorphic to the kernel of g. So not only is M isomorphic to $\ker g$ as R-modules, but the homomorphism f is exactly the one satisfying the universal property from Definition 2.1.5. The dual statement is that a sequence of the form

$$M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

is exact if and only if g is surjective, and P is canonically isomorphic to the cokernel of f.

(iv) For any submodule N of M, the sequence

$$0 \longrightarrow N \hookrightarrow M \longrightarrow M/N \longrightarrow 0$$

is exact, where the first nonzero homomorphism is the inclusion, and the second is the projection onto the quotient module.

(v) A short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is called *split* if N is isomorphic to the direct sum $M \oplus P$, in such a way that the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "1_M"'] & N \arrow[r] \arrow[d, "\cong"'] & P \arrow[r] \arrow[d, "1_P"'] & 0 \\
0 \arrow[r] & M \arrow[r, "\iota_M"'] & M \oplus P \arrow[r, "\pi_P"'] & P \arrow[r] & 0
\end{tikzcd}$$

commutes. Here ι_M and π_P are the inclusion and projection maps from M and onto P respectively. Not every short exact sequence is split, example (i) from before is not split for example, because $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ are not isomorphic as $\mathbb{Z}$-modules.

A useful result regarding exact sequences is the *five lemma*:

Lemma 2.3.3 (Five Lemma)**.** *For a ring R, consider the following commutative diagram of R-modules:*

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & M_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"'] & M_3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"'] & M_4 \arrow[r, "f_4"] \arrow[d, "\alpha_4"'] & M_5 \arrow[d, "\alpha_5"'] \\
N_1 \arrow[r, "g_1"'] & N_2 \arrow[r, "g_2"'] & N_3 \arrow[r, "g_3"'] & N_4 \arrow[r, "g_4"'] & N_5.
\end{tikzcd}$$

If both rows are exact sequences, α_2 and α_4 are isomorphisms, α_1 is surjective and α_5 is injective, then α_3 is an isomorphism.

Proof. To show α_3 is injective, take some x in $\ker \alpha_3$, with the goal of showing $x = 0$. Because $\alpha_3(x) = 0$, we have $g_3(\alpha_3(x)) = 0$. Applying commutativity gives $\alpha_4(f_3(x)) = 0$. The homomorphism α_4 is an isomorphism, so in particular it is injective, meaning that $f_3(x) = 0$, and so $x \in \ker f_3$.

By exactness, x is in the image of f_2, so there is some $m_2 \in M_2$ such that $f_2(m_2) = x$. Note that, by commutativity,

$$g_2(\alpha_2(m_2)) = \alpha_3(f_2(m_2)) = \alpha_3(x) = 0,$$

so $\alpha_2(m_2) \in \ker g_2 = \operatorname{im} g_1$ by exactness. As such there is some $n_1 \in N_1$ such that $g_1(n_1) = \alpha_2(m_2)$.

Now because α_1 is surjective, there is some $m_1 \in M_1$ such that $\alpha_1(m_1) = n_1$. Using commutativity, we find

$$\alpha_2(f_1(m_1)) = g_1(\alpha_1(m_1)) = g_1(n_1) = \alpha_2(m_2).$$

Because α_2 is an isomorphism, and thus injective, $f_1(m_1) = m_2$. By applying f_2 on both sides, we obtain

$$x = f_2(m_2) = f_2(f_1(m_1)) = 0,$$

by exactness. Therefore, α_3 is injective.

Next up is to show that α_3 is surjective. To that end, let $y \in N_3$, with the goal of showing that y is in the image of α_3. First of all, note that because α_4 is an isomorphism, and thus surjective, there is some $m_4 \in M_4$ such that $\alpha_4(m_4) = g_3(y)$. Applying g_4 on both sides, exactness, and commutativity implies

$$0 = g_4(g_3(y)) = g_4(\alpha_4(m_4)) = \alpha_5(f_4(m_4)).$$

Injectivity of α_5 implies that $f_4(m_4) = 0$, so $m_4 \in \ker f_4 = \operatorname{im} f_3$. So there is some $m_3 \in M_3$ such that $f_3(m_3) = m_4$.

Note that, by commutativity,

$$g_3(\alpha_3(m_3)) = \alpha_4(f_3(m_3)) = \alpha_4(m_4) = g_3(y).$$

As g_3 is an R-module-homomorphism, $g_3(\alpha_3(m_3) - y) = 0$. Because $\ker g_3 = \operatorname{im} g_2$, we can find an $n_2 \in N_2$ such that $g_2(n_2) = \alpha_3(m_3) - y$. The map α_2 is surjective, so there is some m_2 in M_2 such that $\alpha_2(m_2) = n_2$. Using commutativity, we can compute

$$\alpha_3(f_2(m_2)) = g_2(\alpha_2(m_2)) = g_2(n_2) = \alpha_3(m_3) - y.$$

Notice that y is in the image of α_3, namely because

$$\alpha_3(m_3 - f_2(m_2)) = \alpha_3(m_3) - \alpha_3(f_2(m_2)) = \alpha_3(m_3) - \alpha_3(m_3) + y = y.$$

Since we have found an element of M_3 such that α_3 evaluated at that element is y, it follows α_3 is surjective. Combined with the previous part, this completes the proof. □

A natural question is whether an exact sequence is preserved when a functor is applied to it. The general answer to this is no, but there is a special class of functors for which this is true:

Definition 2.3.4. Let R and S be rings and $F : \mathsf{Mod}_R \to \mathsf{Mod}_S$ a covariant functor. We call F *additive* if it preserves finite direct sums and the zero module.[22]
Let F be additive. Given a short exact sequence $0 \to M \to N \to P \to 0$ of R-modules, we call F:

- *left exact* if the induced sequence $0 \to FM \to FN \to FP$ is exact;
- *right exact* if the induced sequence $FM \to FN \to FP \to 0$ is exact;
- *exact* if it is both left and right exact, meaning that the sequence $0 \to FM \to FN \to FP \to 0$ is exact.

If F is contravariant, we say it is left exact if the induced sequence $0 \to FP \to FN \to FM$ is exact, and it is right exact if $FP \to FN \to FM \to 0$ is exact. ▼

Remark. As is proven in [Mac98, proposition 4, p. 197], a functor is additive if and only if it preserves addition of homomorphisms: so $F(f + g) = Ff + Fg$ for parallel homomorphisms f and g. Mac Lane proves this in more general categories where additivity and summation of morphisms makes sense, which include Mod_R. We define these categories in Chapter 3.

A helpful criterion to characterize left and right exactness uses Example 2.3.2(iii):

Proposition 2.3.5. *An additive covariant functor $F : \mathsf{Mod}_R \to \mathsf{Mod}_S$ is left exact (resp. right exact) if and only if it preserves kernels (resp. cokernels).*

Proof. Assume F is left exact and consider the exact sequence $0 \to \ker f \to M \to N$ for any homomorphism $f : M \to N$. Applying F, we obtain the exact sequence $0 \to F\ker f \to FM \to FN$. By exactness, there has to be some canonical isomorphism $F\ker f \cong \ker Ff$, by Example 2.3.2(iii) thus F preserves kernels.

Conversely, suppose that F preserves kernels and let $0 \to M \to N \to P \to 0$ be an exact sequence. Consider the induced sequence $0 \to FM \to FN \to FP$. The map $M \to N$ is injective and thus has trivial kernel, meaning that the induced map $FM \to FN$ has trivial kernel as well. Therefore $FM \to FN$ is injective too. Moreover, because $M \cong \ker(N \to P)$, there are isomorphims

$$FM \cong F\ker(N \to P) \cong \ker(FN \to FP).$$

Thus, the induced sequence is exact, making F a left exact functor.
The proof for right exactness being equivalent to cokernel-preservation is dual. □

Example 2.3.6. The following are examples of additive functors and their exactness:

[22] To be specific, because direct sums and the zero modules are limits and colimits of certain diagrams (see Examples 1.4.3(i) and (ii)), F preserves these if they preserve the corresponding limit cones in the sense of Definition 1.4.4.

(i) For any R-module T, the Hom-functor $\mathrm{Hom}_R(T, -)$ is left exact. To see this, note that for a homomorphism $g : M \to N$, the collection of homomorphisms $\mathrm{Hom}_R(T, \ker g)$ is the same as $\ker(g_*)$, where $g_* : \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(T, N)$ is the pushforward. Moreover, Proposition 1.4.5 implies that the Hom-functor preserves direct sums, and thus is additive. Therefore, it is left exact.

(ii) Corollary 2.2.7 immediately implies that $T \otimes_R -$ is right exact. In particular, if $g : M \to N$ is a surjective homomorphism, then $1 \otimes g : T \otimes_R M \to T \otimes_R N$ is surjective as well. If $T \otimes_R -$ is an exact functor, we call T a *flat module*. For example, any free module is flat, as is shown in [DF04, corollary 10.5.42, p.400].

(iii) The contravariant Hom-functor $\mathrm{Hom}_R(-, T) : \mathsf{Mod}_R^{\mathrm{op}} \to \mathsf{Mod}_R$ is left exact. This is a direct consequence of the second part of Proposition 1.4.5; the functor takes cokernels to kernels, which makes it left exact.

We now shift our focus to module presentations, which allows one to view a module as a free module, with some relations restricting it. This is a certain generalization of group presentations, as will become apparent.

Definition 2.3.7. Let R be a ring and M be an R-module. A *(free) presentation* of M is an exact sequence

$$R^{\oplus J} \longrightarrow R^{\oplus I} \longrightarrow M \longrightarrow 0$$

of two free modules and M. If the indexing sets I and J are finite, we call M *finitely presented*. ▼

Proposition 2.3.8. *Any module over any ring R admits a presentation.*

Proof. Let M be an R-module. Though it is not generally free, we can still form a generating set of M. In the most extreme case, this generating set may be M itself, but this has a lot of unnecessary repeats. For example if m is part of the generating set then we do not need to have $2m$ in that generating set.

Regardless, take some generating set S of M and consider the R-module-homomorphism $g : R^{\oplus S} \to M$ that sends a sequence $(r_s)_{s \in S}$ to the linear combination $\sum_s r_s s$. Because S generates M, this map is surjective. We can include the kernel of g to obtain the following exact sequence:

$$\ker g \xhookrightarrow{\iota} R^{\oplus S} \xrightarrow{g} M \longrightarrow 0.$$

As $\ker g$ is another R-module, we can construct a surjective homomorphism $f : R^{\oplus J} \to \ker g$ in the same way as we did before. The claim is that the sequence

$$R^{\oplus J} \xrightarrow{\iota \circ f} R^{\oplus S} \xrightarrow{g} M \longrightarrow 0$$

is exact. By Example 2.3.2(iii), together with the fact that g is surjective, we just need to show that M is the cokernel of the map $\iota \circ f$. To that end, note that

$$\mathrm{coker}(\iota \circ f) = R^{\oplus S} / \mathrm{im}(\iota \circ f) = R^{\oplus S} / \ker g \cong \mathrm{im}\, g = M.$$

Here the second equality followed from the fact that f is surjective, and that the image of ι is the kernel of g by exactness. The last equality holds because g is surjective. So M is the cokernel of $\iota \circ f$, which makes the above sequence exact, and we see that M has a presentation. □

Example 2.3.9. As eluded to before, module presentations give an alternative way to view group presentations. To recall, given an abelian group A, its *group presentation*, which we denote by $A = \langle g_i \mid r_j \rangle$ consists of a collection of generators g_i, and a collection of relations r_j. Any relation looks like some $\mathbb{Z}$-linear combination of the generators, and the implication is that any such combination is set to be zero in a quotient. For example, the abelian group $A = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has presentation $\langle g_1, g_2 \mid g_2 + g_2 \rangle$. We can view this as a presentation of $\mathbb{Z}$-modules with an exact sequence

$$\mathbb{Z} \xrightarrow{r} \mathbb{Z}^2 \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

The map g sends a pair (n, m) to $(n, m \bmod 2)$, which encodes the generators of A. The map r sends the integer 1 to the pair $(0, 2)$ and extends linearly. The above sequence being exact means that g is surjective, so the geneators indeed generate A. Exactness also implies that A is the cokernel of r, so the relation $2 = 0 \bmod 2$ is satisfied in the quotient.

2.4 Watts' Theorem and Variations

In this section, we state and prove Watts' Theorem. There have been many different formulations of this result over the last six decades, not the least of which are the original formulations in the two papers [Eil60, Wat60]. We state and prove the original formulation, and also discuss some related statements. Before that, we discuss some of the theory of bimodules, which are essentially modules over two rings.

Definition 2.4.1. Let R and S be rings. An (S, R)-*bimodule* is an abelian group M that is a left S-module, a right R-module, and

$$s(mr) = (sm)r$$

for any s in S, r in R, and m in M. ▼

This extra 'associativity' requirement really just states that the two module structures on M are compatible. There is one more lemma we need before we get to Watts' Theorem:

Lemma 2.4.2. *Let $F : \mathsf{Mod}_R \to \mathsf{Mod}_S$ be a covariant additive functor. For any left R-module M, the left S-module FM also exhibits the structure of a right R-module, turning it into an (S, R)-bimodule.*

Proof. For r in R and n in FM, define $nr := F(\mu_r)(n)$, where $\mu_r : M \to M$ is the *multiplication homomorphism* defined by $m \mapsto rm$. To show this action turns FM into a right R-module, note that for n, n' in FM and r, r' in R, we have

$$(n + n')r = F(\mu_r)(n + n') = F(\mu_r)(n) + F(\mu_r)(n') = nr + n'r,$$

and

$$n(r + r') = F(\mu_{r+r'})(n) = F(\mu_r + \mu_{r'})(n) = F(\mu_r)(n) + F(\mu_{r'})(n) = nr + nr',$$

by additivity of F. Moreover, note that because $\mu_{rr'} = \mu_r \circ \mu_{r'}$, functoriality of F implies that

$$n(rr') = F(\mu_{rr'})(n) = F(\mu_r)(F(\mu_{r'})(n)) = (nr)r'.$$

Finally, μ_1 is just the identity on M, so $n1$ is equal to n by functoriality. Thus, FM is a right R-module.

Finally note that for s in S, r in R, and n in FM, it follows

$$s(nr) = sF(\mu_r)(n) = F(\mu_r)(sn) = (sn)r,$$

since $F(\mu_r)$ is an S-module-homomorphism. Because of this, FM is an (S,R)-bimodule. $\square$

We are now ready for Watts' Theorem, which, loosely stated, says that any additive right exact functor that preserves direct sums is some form of tensor product.

Theorem 2.4.3 (Watts' Theorem)**.** *Let R and S be rings, and let $F : \mathsf{Mod}_R \to \mathsf{Mod}_S$ be a covariant additive functor. There is a natural transformation*

$$\theta : FR \otimes_R - \Rightarrow F$$

which is a natural isomorphism if and only if F preserves direct sums and is right exact.

Remark. The statement that θ is a natural isomorphism if and only if F preserves direct sums and is right exact can be replaced by stating that θ is a natural isomorphism if and only if F is cocontinuous. This is a consequence of Proposition 1.4.6, Proposition 2.3.5, and the fact that direct sums and cokernels are colimits in Mod_R. In [Hov09], the author phrases Watts' Theorem as stating that if F is additive and a left adjoint, then it is naturally isomorphic to the tensor product. Under this view, Watts' Theorem may be interpreted that, up to natural isomorphism, the tensor product is the only additive and left adjoint functor between module categories.

Proof. We first construct the transformation θ and prove it is natural. Let M be an R-module, and consider the mapping

$$\hat{\theta}_M : FR \oplus M \to FM$$

defined by $(n, m) \mapsto F(\rho_m)(n)$, where $\rho_m : R \to M$ is the R-module-homomorphism defined by $\rho_m(r) = rm$.[23] It is clear that $\hat{\theta}_M$ preserves sums in both arguments, what is less clear is that it is also linear in R. We use the right R-structure from Lemma 2.4.2 to show this. Namely, for any r in R, we have

$$\begin{aligned} \hat{\theta}_M(nr, m) &= F(\rho_m)(nr) \\ &= F(\rho_m)(F(\mu_r)(n)) \\ &= F(\rho_{rm})(n) \\ &= \hat{\theta}_M(n, rm). \end{aligned}$$

Here μ_r is the multiplication homomorphism from R to R. We can also rewrite $F(\rho_{rm})(n)$ to

$$F(\rho_{rm})(n) = F(\mu'_r \circ \rho_m)(n) = F(\mu'_r)(\hat{\theta}_M(n, m)) = \hat{\theta}_M(n, m)r,$$

where μ'_r is the multiplication map from M to M. Thus, $\hat{\theta}_M$ is bilinear in R, and extends to an R-module-homorphism

$$\theta_M : FR \otimes_R M \to FM$$

by the universal property of the tensor product. This map is defined as $\theta_M(n \otimes m) = F(\rho_m)(n)$ on elementary tensors. Note that $FR \otimes_R M$ is also a left S-module, by $s(n \otimes m) := sn \otimes m$, which makes θ_M an S-module-homomorphism as well. What we show next is that the components θ_M assemble into a natural transformation from $FR \otimes_R -$ to F.

[23] Note that ρ_m is a sort of 'dual' to the multiplication homomorphism in Lemma 2.4.2, in the sense that $\rho_m(r) = rm = \mu_r(m)$.

To that end, let $g : M \to M'$ be an R-module-homomorphism. We show that the naturality square

$$\begin{tikzcd}
FR \otimes_R M \arrow[r, "\theta_M"] \arrow[d, "1_{FR}\otimes g"'] & FM \arrow[d, "Fg"] \\
FR \otimes_R M' \arrow[r, "\theta_{M'}"'] & FM'
\end{tikzcd}$$

commutes, so let $\sum_i n_i \otimes m_i$ be a finite sum in $FR \otimes_R M$. The top path of the square evaluates to

$$\begin{aligned}
Fg\left(\theta_M\left(\sum_i n_i \otimes m_i\right)\right) &= \sum_i Fg(\theta_M(n_i \otimes m_i)) \\
&= \sum_i (Fg \circ F\rho_{m_i})(n_i) \\
&= \sum_i F(g \circ \rho_{m_i})(n_i) \\
&= \sum_i F(\rho_{g(m_i)})(n_i).
\end{aligned}$$

The bottom path of the square evaluates to

$$\begin{aligned}
\theta_{M'}\left((1_{FR} \otimes g)\left(\sum_i n_i \otimes m_i\right)\right) &= \sum_i \theta_{M'}((1_{FR} \otimes g)(n_i \otimes m_i)) \\
&= \sum_i \theta_{M'}(n_i \otimes g(m_i)) \\
&= \sum_i F(\rho_{g(m_i)})(n_i).
\end{aligned}$$

Thus, because both compositions through the square evaluate to the same homomorphism, θ is a natural transformation.

Now we are ready to prove the second part of the theorem. If θ is a natural isomorphism, then F preserves direct sums and is right exact, because $FR \otimes_R -$ is as well by Corollary 2.2.7 and Example 2.3.6(ii). For the converse, assume F preserves direct sums is right exact. We have already shown that θ is a natural transformation, so all we need to show now are that the components θ_M are isomorphisms for all M.

Before that though, it is useful to see how θ acts on free modules. If $M = R$, we find that the component $\theta_R : FR \otimes_R R \to FR$ acts on elementary tensors as

$$\theta_R(n \otimes r) = \theta_R(nr \otimes 1) = F(\rho_1)(nr) = nr.$$

Note that this map θ_R acts as the isomorphism $FR \otimes_R R \cong FR$ discussed in the proof of Proposition 2.2.2(a,b), but in the context of right R-modules. For arbitrary free modules, let I be a set, and note that we have the following commutative diagram, as a consequence of the tensor product and F preserving direct sunms, as well as naturality of θ:

$$\begin{tikzcd}
FR \otimes_R R \arrow[r, "\theta_R"] \arrow[d, "1_{FR}\otimes \iota_i"'] & FR \arrow[d, "F\iota_i"] \\
FR \otimes_R R^{\oplus I} \arrow[r, "\theta_{R^{\oplus I}}"] \arrow[d, "\cong"'] & F(R^{\oplus I}) \arrow[d, "\cong"] \\
\bigoplus_I (FR \otimes_R R) \arrow[r, "\oplus\theta_R"'] & \bigoplus_I FR.
\end{tikzcd}$$

The homomorphism $\oplus\theta_R$ is defined as applying θ_R to each entry of a sequence in the direct sum, which indeed makes the outer rectangle of the diagram commute. Because θ_R is an isomorphism, so is $\oplus\theta_R$. Using commutativity of the bottom square, we can write $\theta_{R^{\oplus I}}$ as the composition of three isomorphisms, meaning it is an isomorphism itself.

Now let M be an arbitrary R-module. By Proposition 2.3.8, there are sets I and J, and an exact sequence of free R-modules that present M:

$$R^{\oplus J} \longrightarrow R^{\oplus I} \longrightarrow M \longrightarrow 0.$$

Applying $FR \otimes_R -$ and F on this sequence, we obtain a commutative diagram (extended with zero modules)

$$\begin{tikzcd}
FR \otimes_R R^{\oplus J} \arrow[r] \arrow[d, "\theta_{R^{\oplus J}}"'] & FR \otimes_R R^{\oplus I} \arrow[r] \arrow[d, "\theta_{R^{\oplus I}}"'] & FR \otimes_R M \arrow[r] \arrow[d, "\theta_M"'] & 0 \arrow[r] \arrow[d, "0"'] & 0 \arrow[d, "0"'] \\
F(R^{\oplus J}) \arrow[r] & F(R^{\oplus I}) \arrow[r] & FM \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Commutativity follows from the naturality of θ. Note that because both the tensor product and F are right exact, both of the rows above are exact sequences. The maps $\theta_{R^{\oplus J}}$ and $\theta_{R^{\oplus I}}$ are isomorphisms, as are the zero maps between the zero modules. The Fve Lemma implies that θ_M is an isomorphism, which completes the proof. □

Remark. If F were additive and right exact, but not preserve arbitrary direct sums, the theorem would still hold in the subcategory containing only finitely presented modules. This is because additive functors preserve finite direct sums.

In [AK17, theorem 8.13, p.62], authors Altman and Kleiman prove a less general version of Watts' Theorem, where the rings R and S are the same. The proof of Theorem 2.4.3 could be copied directly, but the authors give a different proof, requiring the functor F to preserve scalar multiplication as well, so $F(rf) = rF(f)$ for $r \in R$ and any homomorphism f. In this setting, the component θ_M is defined using the homomorphism $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(FR, FM)$, which is an element of

$$\mathrm{Hom}_R(\mathrm{Hom}_R(R, M), \mathrm{Hom}_R(FR, FM)) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(FR, FM)) \cong \mathrm{Hom}_R(FR \otimes_R M, FM).$$

Unravelling the isomorphisms above gives the same map as we defined in the proof of Theorem 2.4.3.

The following is an example of an additive functor that does *not* satisfy the criteria for Watts' Theorem, and so is not naturally isomorphic to the tensor product with some module.

Example 2.4.4. Let R be an integral domain, and let $\mathrm{Tor} : \mathsf{Mod}_R \to \mathsf{Mod}_R$ be the functor that sends an R-module M to its torsion submodule

$$\mathrm{Tor}\, M := \{m \in M \mid rm = 0 \text{ for some } r \in R \setminus \{0\}\}, .$$

An R-module-homomorphism $f : M \to M'$ is sent to the restriction $f|_{\mathrm{Tor}\, M} : \mathrm{Tor}\, M \to \mathrm{Tor}\, M'$ (note that if $t \in \mathrm{Tor}\, M$, then $f(t) \in \mathrm{Tor}\, M'$, because if $rt = 0$, then $rf(t) = f(rt) = 0$ as well). This functor is indeed additive, and it preserves direct sums. To see this, let M and N be R-modules, and note that the modules $\mathrm{Tor}(M \oplus N)$ and $\mathrm{Tor}\, M \oplus \mathrm{Tor}\, N$ are not just isomorphic, but actually equal. Indeed, if $(t, s) \in \mathrm{Tor}(M \oplus N)$, then $r(t, s) = 0$ for some $r \neq 0$, so t and s are torsion elements, which implies $(s, t) \in \mathrm{Tor}\, M \oplus \mathrm{Tor}\, N$. Conversely, if $(t, s) \in \mathrm{Tor}\, M \oplus \mathrm{Tor}\, N$, then $rt = 0$ and $r's = 0$ for $r, r' \neq 0$. Now note that $rr'(t, s) = (0, 0)$ as well, thus we find $(t, s) \in \mathrm{Tor}(M \oplus N)$.

The torsion functor is not right exact however. As an example, consider the exact sequence of $\mathbb{Z}$-modules seen in Example 2.3.2(i):

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Note that as $\mathbb{Z}$-modules, the torsion of $\mathbb{Z}$ is zero, and the torsion of $\mathbb{Z}/n\mathbb{Z}$ is itself. Thus, applying the torsion functor gives the sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

This sequence is *not* exact however, since that would imply $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the zero module, which is not true at all if $n > 1$. More generally, the torsion fails to be right exact because if $f : M \to M'$ is surjective, that does not always imply $f|_{\mathrm{Tor}\, M} : \mathrm{Tor}\, M \to \mathrm{Tor}\, M'$ is.

Because the torsion is an additive functor that preserves direct sums, but is not right exact, it does not satisfy the hypotheses of Watts' Theorem. Thus, the natural transformation

$$\theta : \mathrm{Tor}\, R \otimes_R - \Rightarrow \mathrm{Tor}$$

is not an isomorphism. This can be seen more directly as well: The component θ_M of the natural transformation sends an elementary tensor $t \otimes m$ in $\mathrm{Tor}\, R \otimes_R M$ to the element tm in $\mathrm{Tor}\, M$ (note that $t \in \mathrm{Tor}\, R$, so $rt = 0$. This implies $rtm = 0$, making tm an element of the torsion of M). An inverse of θ_M would necessarily map an element m of the torsion of M to $1 \otimes m$, but the multiplicative unit of R is not torsion at all, so $1 \otimes m$ is not an element in $\mathrm{Tor}\, R \otimes_R M$, so the inverse homomorphism cannot exist. □

The original papers by Eilenberg and Watts [Eil60, Wat60] also discuss a dual theorem regarding contravariant additive functors:

Theorem 2.4.5 (Contravariant Watts' Theorem)**.** *Let R be a ring, and let $F : \mathsf{Mod}_R^{\mathrm{op}} \to \mathsf{Mod}_S$ be an additive functor. There is a natural transformation*

$$\theta : F \Rightarrow \mathrm{Hom}_R(-, FR)$$

which is a natural isomorphism if and only if F takes direct sums to direct products and is left exact.

In other words, an additive functor $F : \mathsf{Mod}_R^{\mathrm{op}} \to \mathsf{Mod}_S$ is representable, as in Definition 1.3.6, if and only if it takes direct sums to direct products and is left exact. The proof is in essence the same as the covariant theorem. The components of the relevant natural transformation are given by homomorphisms $\theta_M : FM \to \mathrm{Hom}_R(M, FR)$, where $n \in FM$ is sent to $\theta_M(n)$, which is defined as a map $M \to FR$ by $\theta_M(n)(m) = F(\rho_m)(n)$. Here ρ_m is the same as in the proof for the covariant Watts' Theorem.

In essence, what Watts' Theorem is really saying is that there is a correspondence between applying a linear cocontinuous functor, and tensoring with a bimodule. This connection between bimodules and these functors goes deeper than this actually:

Proposition 2.4.6. *Let R and S be rings. Let D denote the subcategory of $[\mathsf{Mod}_R, \mathsf{Mod}_S]$ of functors that are additive, preserve direct sums, and are right exact. The functor $\psi : B \mapsto B \otimes_R -$ is an equivalence of categories*

$$ {}_S\mathsf{Mod}_R \simeq \mathsf{D}, $$

where the domain is the category of (R, S)-bimodules.

Dually, the functor $\psi : B \mapsto \mathrm{Hom}_R(-, B)$ is an equivalence of categories

$$ {}_S\mathsf{Mod}_R \simeq \mathsf{D}', $$

where D' denotes the category of additive contravariant functors that take direct sums to direct products and are left exact.

Proof. To be clear, the functor ψ sends an (R, S)-bimodule B to the tensor product functor $B \otimes_R -$, which is indeed additive, preserves direct sums and is right exact. An (R, S)-bimodule-homomorphism $f : B \to B'$ is sent to the natural transformation

$$ f \otimes 1_{(-)} : B \otimes_R - \Rightarrow B' \otimes_R -, $$

defined on components by $(f \otimes 1_{(-)})_M := f \otimes 1_M$ for a bimodule M.

To show that ψ is an equivalence of categories, we find an inverse equivalence $\varphi : \mathsf{D} \to {}_S\mathsf{Mod}_R$ such that $\varphi\psi$ and $\psi\varphi$ are naturally isomorphic to the corresponding identity functors, following Definition 1.3.10. We define φ to send a functor $F : \mathsf{Mod}_R \to \mathsf{Mod}_S$ in D to the (R, S)-bimodule FR. A natural transformation $\eta : F \Rightarrow G$ is sent to the component $\eta_R : FR \to GR$.

First, let B be an (R, S)-bimodule, and note

$$ \varphi\psi(B) = \varphi(B \otimes_R -) = B \otimes_R R \cong B. $$

The isomorphism $B \otimes_R R \cong B$ is actually natural in B, so the functors $\varphi\psi$ and $1_{{}_S\mathsf{Mod}_R}$ are naturally isomorphic.

For the other composition, let F be a functor in D, and note that

$$ \psi\varphi(F) = \psi(FR) = FR \otimes_R - \cong F. $$

The natural isomorphism $FR \otimes_R - \cong F$ follows from Watts' Theorem. Thus, $\psi\varphi \cong 1_{\mathsf{D}}$ is a natural isomorphism.

The proof of the dual statement is similar. The inverse equivalence also sends a functor F to the bimodule FR. The fact that the compositions of these equivalences are naturally isomorphic to identity functors follows from Proposition 2.2.5 and Theorem 2.4.5. □

2.5 Localization: An Application of Watts' Theorem

The classical construction of the rational numbers is done by a quotient $(\mathbb{Z}\times\mathbb{Z}\setminus\{0\})/\sim$, where $(n,r)\sim(n',r')$ if $nr'=n'r$. A class $[n,r]$ corresponds to the rational number n/r. This construction can be generalized to the *field of fractions* $Q(R)$ of a domain R, as is done in e.g. [LOT17, section I.3]. This idea can be generalized further to the *localization of rings*. Loosely stated, the localization of a ring R by a so-called *multiplicative* subset A contains of fractions of the form r/a, with $r\in R$ and $a\in A$. This section also covers localizations of modules, and proves a theorem stating that localizing an R-module is is the same as localizing R and taking the tensor product.

Intuitively, the idea of localization is to take some non-invertible elements of a ring, and declare them to be invertible. To make this process well-defined however, we require the non-invertible elements to be part of a specific type of subset:

Definition 2.5.1. We call a subset A of a ring R *multiplicative* if it contains 1, and the product aa' is in A for $a,a'\in A$.
The *localization* of R by A, denoted $A^{-1}R$, is the ring $(R\times A)/\sim$, where $(r,a)\sim(r',a')$ if there exists an $x\in A$ such that $xa'r=xar'$. We denote the class of (r,a) by r/a or $\frac{r}{a}$. Addition and multiplication are done by

$$\begin{aligned}\frac{r}{a}+\frac{r'}{a'} &:= \frac{ra'+r'a}{aa'};\\ \frac{r}{a}\cdot\frac{r'}{a'} &:= \frac{rr'}{aa'}.\end{aligned}$$

The additive unit is $0/1$, and the multiplicative unit is $1/1$.

The localization $A^{-1}R$ is governed by the following universal property, which defines it up to unique isomorphism. There is a ring-homomorphism $\beta: R\to L$ such that $\beta(s)$ is a unit in L for all $a\in A$. Moreover, for any other $f: R\to Y$ that sends elements of A to units in Y, there is a unique ring-homomorphism $\varphi: L\to Y$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\beta"'] & Y \\ L. \arrow[ur, dashed, "\exists!\varphi"'] & \end{tikzcd}$$

For $L=A^{-1}R$, the map β sends an element $r\in R$ to $r/1$ in the localization. Indeed, the image of an element $a\in A$ is $a/1$, which is a unit with inverse $1/a$. The map φ is defined as $\varphi(r/a):=f(r)f(a)^{-1}$ (note that f sends elements of A to units in Y, so $f(a)^{-1}$ actually makes sense). ▼

Remark. If c is a nonzero zero divisor of R, with $cd=0$ for some $d\in A\subseteq R$, then in $A^{-1}R$, we have

$$\frac{c}{1}=\frac{cd}{d}=\frac{0}{d}=\frac{0}{1}.$$

Following the definition of the equivalence $\sim$, there is some $x\in A$ such that $cx=0$. This is why we require the element x in the definition of the equivalence relation. If we used the equivalence relation used to define the field of fractions, then we would have $c = 0$, which is a contradiction.

If R is an integral domain and A a multiplicative subset, then indeed $r/a = r'/a'$ if and only if $ra' = r'a$. In this case, the localization $A^{-1}R$ is a subring of the field of fractions $Q(R)$. In fact, the field of fractions of a domain is the localization of itself by the set of its nonzero elements.

Example 2.5.2. The following are examples of localizations of rings:

(i) If $A = \{1, a, a^2, a^3, \dots\}$ for some $a \in R$, then $A^{-1}R$ contains elements of the form r/a^n. This ring is isomorphic to the quotient ring
$$R[x]/(xa-1).$$
The isomorphism follows from the universal property, the map $\beta : R \to R[x]/(xa-1)$ sends r to the class $r + (xa-1)$. See the proof of [AK17, proposition 11.7, p.82] for details.

(ii) If $\mathfrak{p}$ is a prime ideal[24] of a ring R, then the set $R \setminus \mathfrak{p}$ is multiplicative. The localization of R by this set, denoted $R_{\mathfrak{p}}$ is the *local ring at* $\mathfrak{p}$.

(iii) The ring $A^{-1}R$ is the zero ring if and only if 0 is an element of A. Indeed, if 0 is an element of A, then $1/1 = 0/1$ by the equivalence relation defining ring localizations. Now for any other $r/a \in A^{-1}R$, it follows that
$$\frac{r}{a} = \frac{r}{a} \cdot \frac{1}{1} = \frac{r}{a} \cdot \frac{0}{1} = \frac{0}{1},$$
meaning every element is the zero element. Thus, $A^{-1}R$ is the zero ring. On the other hand, if $A^{-1}R$ is the zero ring, then $1/1 = 0/1$, implying there is some $x \in A$ such that $x(1 \cdot 1) = x(1 \cdot 0)$, so $0 = x \in A$.

(iv) If A only contains units of R, then the canonical map provided by the universal property $R \to A^{-1}R$ is an isomorphism, with inverse $r/a \mapsto ra^{-1}$.

Just like rings, we can also localize modules:

Definition 2.5.3. Let A be a multiplicative subset of a ring R. Given an R-module M, we define its *localization* by A to be the $A^{-1}R$-module $A^{-1}M$. Its elements are equivalence classes m/a for $m \in M$ and $a \in A$. Addition and scalar multiplication are done by
$$\frac{m}{a} + \frac{m'}{a'} := \frac{ma' + m'a}{aa'};$$
$$\frac{r}{a} \cdot \frac{m}{a'} := \frac{rm}{aa'}.$$
Equality is defined via a similar equivalence relation as for localizing rings. That is, $m/a = m'/a'$ if and only if there is some x in A such that $xa'm = xam'$. ▼

What we are about to prove, using Watts' Theorem, is that localizing a module is the same as taking a tensor product. Before that however, we need some more details about this localization:

Proposition 2.5.4. *Given a multiplicative subset A of a ring R, there is an additive, exact, and direct sum preserving functor $A^{-1}- : \mathsf{Mod}_R \to \mathsf{Mod}_{\mathsf{A^{-1}R}}$ that sends a module M to the localization $A^{-1}M$.*

Proof. Under $A^{-1}-$, an R-module-homomorphism $f : M \to N$ is sent to $\hat{f} : A^{-1}M \to A^{-1}N$, defined by $\hat{f}(m/a) := f(m)/a$. This is an $A^{-1}R$-module-homomorphism, because f is an R-module-homomorphism. To

[24] Recall that an ideal $\mathfrak{p} \trianglelefteq R$ is *prime* if $\mathfrak{p}$ is not equal to R, and $ab \in \mathfrak{p}$ implies either a or b is in $\mathfrak{p}$.

show functoriality, let $1_M : M \to M$ be an identity homomorphism. Its image under $S^{-1}-$ acts on elements $m/a \in A^{-1}M$ by

$$\widehat{1_M}(m/a) = 1_M(m)/a = m/a,$$

and thus, it is the identity on $A^{-1}M$. Now, if $f : M \to N$ and $g : N \to P$ are R-module-homomorphisms, we want to show $\hat{g}\hat{f} = \widehat{gf}$. To that end, let $m/a \in A^{-1}M$, and note

$$\widehat{gf}(m/a) = (gf)(m)/a = g(f(m))/a = \hat{g}\hat{f}(m/a).$$

Therefore, $A^{-1}-$ is indeed a functor.

Now to show $A^{-1}-$ is additive, let $f, f' : M \rightrightarrows N$ be R-module-homomorphisms. Applying $A^{-1}-$ to their sum, applied to an element $m/a \in A^{-1}M$ evaluates to

$$(\widehat{f + f'})(m/a) = (f(m) + f'(m))/a = \hat{f}(m/a) + \hat{f}'(m/a).$$

Indeed, $A^{-1}-$ preserves sums of homomorphisms, and is thus additive.

By Proposition 2.3.5, $A^{-1}-$ is right exact if and only if there is a canonical isomorphism $A^{-1} \operatorname{coker} f \cong \operatorname{coker} \hat{f}$ for some R-module-homomorphism $f : M \to N$. In this case, we can send a class $(c + \operatorname{im} f)/a$ in $A^{-1} \operatorname{coker} f$ to $(c/a) + \operatorname{im} \hat{f}$. It is clear that this is a well-defined homomorphism, and has an inverse that sends $(c/a) + \operatorname{im} \hat{f}$ to $(c + \operatorname{im} f)/a$.

For left exactness, we want to show that if

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow A^{-1}M \xrightarrow{\hat{f}} A^{-1}N \xrightarrow{\hat{g}} A^{-1}P$$

is exact. By exactness of the original sequence, $\hat{g} \circ \hat{f} = 0$, so the image of $\hat{f}$ is contained in the kernel of $\hat{g}$. For the other direction, let n/a be an element of $\ker \hat{g}$. This implies that $g(n)/a = 0$, so $xg(n) = g(xn) = 0$ for some x in A. Now we have $xm \in \ker g$, which is the image of f, so there is some $m \in M$ such that $f(m) = xn$. Now, note that

$$\hat{f}\left(\frac{m}{xa}\right) = \frac{f(m)}{xa} = \frac{xm}{xa} = \frac{n}{a},$$

so n/a is in the image of f. Thus, since $\operatorname{im} \hat{f} = \ker \hat{g}$, we see that the above sequence is exact, making $A^{-1}-$ an exact functor.

Finally, $A^{-1}-$ also preserves direct sums. To prove this we show that, for some indexed collection $\{M_i\}_i$ of R-modules, $A^{-1} \bigoplus_i M_i$ satisfies the universal property of coproducts that $\bigoplus_i A^{-1}M_i$ does (see Example 1.4.3(ii)). First we let $\iota_i : A^{-1}M_i \to A^{-1} \bigoplus_i M_i$ be defined by

$$\iota_i(m_i/a) := \frac{(0, \ldots, m_i, \ldots, 0)}{a},$$

where the m_i is in the i-th entry. Now, for any other collection of homomorphisms $f_i : A^{-1}M_i \to N$, there is a unique $f : A^{-1} \bigoplus_i M_i \to N$ such that the diagram

$$\begin{tikzcd}
A^{-1}M_i \arrow[r, "\iota_i"] \arrow[dr, "f_i"'] & A^{-1}\bigoplus_i M_i \arrow[d, dashed, "f"] \\
& N
\end{tikzcd}$$

commutes for all i. We can define f by setting

$$f\left(\frac{(m_i)_i}{a}\right) := \sum_i f_i(m_i/a).$$

Note that this is indeed an $A^{-1}R$-module-homomorphism and satisfies $f \circ \iota_i = f_i$. Finally to prove uniqueness, let $g : A^{-1} \bigoplus_i M_i \to N$ be another homomorphism such that $g\iota_i = f_i$ for all i. Now note that

$$g\left(\frac{(m_i)_i}{a}\right) = g\left(\sum_i \iota_i(m_i/a)\right) = \sum_i g(\iota_i(m_i/a)) = \sum_i f_i(m_i/a) = f\left(\frac{(m_i)_i}{a}\right).$$

Therefore $g = f$, making f unique. Because $A^{-1} \oplus_i M_i$ satisfies the same universal property as $\bigoplus_i A^{-1}M_i$ does, it follows that $A^{-1}-$ preserves direct sums. $\square$

Corollary 2.5.5. *For an R-module M, there is a natural isomorphism $A^{-1}R \otimes_R M \cong A^{-1}M$.*

Proof. This is immediate from Proposition 2.5.4 and Watts' Theorem 2.4.3. $\square$

We call a property that an R-module M could satisfy *local* if M satisfies it if and only if $M_\mathfrak{p} := (R \setminus \mathfrak{p})^{-1}M$ satisfies it for all prime ideals $\mathfrak{p}$ of R. An important example of a local property is flatness. Recall that a module M is flat if $M \otimes_R -$ is an exact functor. As is proven in [DF04, proposition 10.5.40, p.400], M is flat if and only if, whenever $f : A \to B$ is injective, so is $1_M \otimes f : M \otimes_R A \to M \otimes_R B$.

Proving that flatness is a local property requires multiple steps, which consitute the following Proposition:

Proposition 2.5.6. *Let R be a ring. The following hold:*

(a). *Being the zero module is a local property.*

(b). *A homomorphism between two R-modules being injective and/or surjective is a local property.*

(c). *For R-modules M and N, and a prime ideal $\mathfrak{p}$ of R, there is a natural isomorphism of $R_\mathfrak{p}$-modules $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \cong (M \otimes_R N)_\mathfrak{p}$.*

(d). *Flatness is a local property.*

Proof. (a). To reiterate, the goal is to show that M is the zero module if and only if $M_\mathfrak{p}$ is zero for any prime ideal $\mathfrak{p}$ of R. If M is zero, then any element $m/a \in M_\mathfrak{p}$ is equal to $0/a = 0/1$, thus $M_\mathfrak{p}$ is the zero module.

Conversely, if $M_\mathfrak{p}$ is zero for all prime ideals $\mathfrak{p}$ of R, we consider some $a \in M$ and assume it is nonzero. Define

$$\mathrm{Ann}(a) := \{r \in R \mid ra = 0\}$$

to be the *annihilator* of a, which is an ideal of R. This ideal is contained in some maximal ideal[25] $\mathfrak{a}$ of R, because it is not the whole ring (e.g. $1 \in R$ is not in the annihilator since a is nonzero). By assumption, $M_\mathfrak{m}$ is zero, so $a/1 = 0/1$ in $M_\mathfrak{m}$, meaning there is some $x \in R \setminus \mathfrak{m}$ such that $xa = 0$. But this implies $x \in \mathrm{Ann}(a)$, which contradicts x not being in $\mathfrak{m}$. Therefore a is indeed zero, making M the zero module.

(b). Note that $f : M \to N$ is injective if and only if

$$0 \longrightarrow M \xrightarrow{f} N$$

is exact. It follows that

$$0 \longrightarrow M_\mathfrak{p} \xrightarrow{\hat{f}} N_\mathfrak{p}$$

is exact by Proposition 2.5.4. Exactness of the above sequence is equivalent to $\hat{f} : M_\mathfrak{p} \to N_\mathfrak{p}$ being injective.

Conversely, assume $\hat{f} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for all prime ideals $\mathfrak{p}$ of R. Now consider the exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N,$$

which becomes

$$0 \longrightarrow (\ker f)_\mathfrak{p} \longrightarrow M_\mathfrak{p} \xrightarrow{\hat{f}} N_\mathfrak{p}$$

after localizing by any prime ideal $\mathfrak{p}$. Since localization is exact, it preserves kernels, so we have a canonical isomorphism $(\ker f)_\mathfrak{p} \cong \ker \hat{f}$, which is zero by assumption of $\hat{f}$ being injective. So because $(\ker f)_\mathfrak{p}$ is zero for all prime ideals $\mathfrak{p}$ of R, so is $\ker f$ by part (a). Thus, it follows that f is injective, which proves that injectiveness is a local property. The proof of the fact that surjectivity is a local property is dual.

(c). For this part we use the fact that the tensor product is associative. That is, there is a natural isomorphism

$$M \otimes_R (M' \otimes_R M'') \cong (M \otimes_R M') \otimes_R M''$$

for R-modules M, M' and M''. This is proven in [AK17, theorem 8.8, p.61] using the universal property of the tensor product.

Now let M and N be R-modules, and $\mathfrak{p}$ some prime ideal of R, and note

$$\begin{aligned} M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} &\cong (M \otimes_R R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \\ &\cong M \otimes_R (R_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}) \\ &\cong M \otimes_R (R_\mathfrak{p} \otimes_R N) \\ &\cong R_\mathfrak{p} \otimes_R (M \otimes_R N) \cong (M \otimes_R N)_\mathfrak{p}. \end{aligned}$$

The first isomorphism follows from Corollary 2.5.5, the second from associativity of the tensor product, the third from Proposition 2.2.2(a) and the previously mentioned Corollary. The last two isomorphisms are a consequence of associativity and commutativity (see Proposition 2.2.2(b)) of the tensor product, and the same corollary again. All these isomorphisms are natural in M and N, which completes the proof.

(d). Let M be a flat R-module, and $\mathfrak{p}$ a prime ideal of R. Note that the functor $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} -$ is naturally

[25] Recall that an ideal $\mathfrak{m} \trianglelefteq R$ is *maximal* if it is not equal to R, and $\mathfrak{m} \subseteq \mathfrak{a} \subseteq R$ for some ideal $\mathfrak{a}$ implies $\mathfrak{a} = \mathfrak{m}$ or $\mathfrak{a} = R$. As is shown in [DF04, corollary 7.4.14, p.256], any maximal ideal is prime.

isomorphic to $(M \otimes_R -)_\mathfrak{p}$ by part (c) as functors from $\mathsf{Mod}_{R_\mathfrak{p}}$ to itself. It makes sense to have the same input variable for both functors, as an $R_\mathfrak{p}$-module N can also be seen as an R-module, where we restrict the scalar multiplication to elements of the form $r/1$. Localization by $\mathfrak{p}$, as well as $M \otimes_R -$ are exact functors by assumption, thus $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} -$ is as well, which implies $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module.

Conversely, if $M_\mathfrak{p}$ is flat for all prime ideals $\mathfrak{p}$ of R, then so is M. To prove this, we show that $M \otimes_R -$ preserves injective homomorphisms. Let $f : N \to N'$ be an injective R-module-homomorphism. By part (b), the induced map $\hat{f} : N_\mathfrak{p} \to N'_\mathfrak{p}$ is also injective, making

$$1_{M_\mathfrak{p}} \otimes \hat{f} : M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \to M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N'_\mathfrak{p}$$

injective as well by assumption of $M_\mathfrak{p}$ being flat. By part (c), this corresponds naturally to an injective homomorphism $(M \otimes_R N)_\mathfrak{p} \to (M \otimes_R N')_\mathfrak{p}$. Since this holds for *all* prime ideals of R, it follows that the corresponding map $M \otimes_R N \to M \otimes_R N'$ is injective, which proves that M is flat. $\square$

On this note, we conclude the Chapter on Watts' Theorem. We have seen how the basic notions of category theory can help to formalize certain concepts from commutative algebra. Including Watts' Theorem itself, which allows us to view a large and important class of functors in terms of a tensor product. In the next Chapter, we see how we can use the theory of *homological algebra* to extend a right or left exact functor to the left or right, respectively, to measure how far off it is to being exact. In the context of the tensor product functor, we can use this theory to measure how far off a module is from being flat.

3 Derived Functors

Je me borne à des cas simples, qui ne nécessitent aucune conjecture . . .
(Translation: I confine myself to simple cases, which require no conjecture . . .)

–Jean-Pierre Serre [Ser91]

This Chapter is focused on *derived functors*, which seek to answer the following question: Given an exact sequence $0 \to A \to B \to C \to 0$ in some 'nice' category (like Ab), after applying a right exact functor to obtain the exact sequence $FA \to FB \to FC \to 0$, is there a canonical way to extend this on the left to a long exact sequence? This is indeed possible, and is done using derived functors. This Chapter covers the background necessary to define these concepts. The first Section defines *abelian categories*, which are categories that resemble Ab to the extent to allow the definitions of concepts like exact sequences and homology to make sense. The second Section is about *chain complexes*, which are the basic building blocks for defining and proving certain properties of derived functors. In the third Section, we define derived functors using special chain complexes called *resolutions*. Finally, we look at an easier way to compute these derived functors through so-called *acyclic resolutions*. The theory in this Chapter is mainly adapted from [Fre64] with regard to abelian categories, and [HS97] and [Rot09] for the theory behind derived functors.

3.1 Additive and Abelian Categories

Loosely stated, an abelian category is a category in which each Hom-set is an abelian group. Along with this, the category has a zero object and zero morphisms, finite products and coproducts which coincide, and well-behaved kernels and cokernels. These are a lot of properties to consider however, so in this section we build up to abelian categories in two stages, and exhibit examples and properties along the way.

In the second half of this Section, we introduce *exact sequences* in general abelian categories. In principle they behave the same as exact sequences in Mod_R, except that the hypothesis of one morphism's image being equal to another's kernel needs to be weakened to a certain isomorphism.

Definition 3.1.1. A category A is called *additive* if:

- It has a zero object 0. The unique composition $A \to 0 \to B$ is the *zero morphism*, denoted 0 or 0_{BA}.
- Each Hom-set $\mathrm{Hom}(A, B)$ is an abelian group under an operation $+$. Moreover, we require,

  $$(f + g) \circ h = fh + gh, \quad \text{and} \quad k \circ (f + g) = kf + kg$$

  for all morphisms where this composition makes sense.[26]
- For all pairs of objects A_1 and A_2, there is an object $A_1 \oplus A_2$, called the *biproduct* of A_1 and A_2. This object has morphisms $\iota_i : A_i \to A_1 \oplus A_2$ and $\pi_i : A_1 \oplus A_2 \to A_i$ for $i = 1, 2$. These morphisms satisfy the following properties for $i \neq j$:

  $$\pi_i \circ \iota_j = 0, \quad \pi_i \circ \iota_i = 1_{A_i}, \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = 1_{A_1 \oplus A_2}. \quad ▼$$

[26] We say that A is *enriched* over Ab, meaning that every Hom-set is an object in the category of abelian groups. For more information on the enrichment of categories, see chapter 3 of [Rie14].

As one might suspect, adding the zero morphism to another morphism does not change anything. This is indeed the case:

Proposition 3.1.2. *In an additive category, the zero morphism $0_{BA} : A \to B$ is the identity element of the abelian group* $\mathrm{Hom}(A, B)$.

Proof. The morphism 0_{BA} is defined as the composition $0_{B0} \circ 0_{0A}$. Note that we can write the sum $0_{BA} + 0_{BA}$ as follows:

$$\begin{aligned} 0_{BA} + 0_{BA} &= 0_{B0} \circ 0_{0A} + 0_{B0} \circ 0_{0A} \\ &= (0_{B0} + 0_{B0}) \circ 0_{0A} \\ &= 0_{B0} \circ 0_{0A} = 0_{BA} \end{aligned}$$

Here we used the fact that 0 is initial, so the morphisms 0_{B0} and $0_{B0} + 0_{B0}$ are the same. Subtracting 0_{BA} on both sides tells us that, 0_{BA} is the identity element of the group $\mathrm{Hom}(A, B)$. $\square$

Remark. It is important to note the difference between the identity morphism $1_A : A \to A$, and the identity 0_{AA} of the abelian group $\mathrm{Hom}(A, A)$. Composing any morphism with a zero morphism leaves us with a zero morphism again, which is vastly different from how the identity morphism works.

If $1_A = 0_{AA}$, then A is a zero object. To see this, let f be any morphism to or from A. Composing this with 1_A leaves us with f again, but also the zero morphism, since $1_A = 0_{AA}$. Thus f is the zero morphism, which is unique. This makes A the zero object.

The name *biproduct*, along with the notation for its morphisms π_i and ι_i seem to hint at the following proposition:

Proposition 3.1.3. *In an additive category, the biproduct of a finite set of objects is a product and a coproduct.*

Proof. Let A_1 and A_2 be objects of an additive category. The definition of a biproduct already ensures the existence of morphisms $\pi_i : A_1 \oplus A_2 \to A_i$, so we only need to show that $A_1 \oplus A_2$ is universal among objects with morphisms to both A_i, which proves $A_1 \oplus A_2$ is a product. To that end, let C be another object with morphisms $f_i : C \to A_i$ for $i = 1, 2$. We construct a map $h : C \to A_1 \oplus A_2$ by defining $h := \iota_1 f_1 + \iota_2 f_2$.

$$\begin{tikzcd} & C \arrow[dl, "f_1"'] \arrow[d, "h"] \arrow[dr, "f_2"] & \\ A_1 & A_1 \oplus A_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & A_2 \end{tikzcd}$$

Note that this diagram indeed commutes, because

$$\pi_1 h = \pi_1(\iota_1 f_1 + \iota_2 f_2) = 1_{A_1} f_1 + 0 f_2 = f_1,$$

and similarly $\pi_2 h = f_2$. Finally, we show that this h is unique. Let $h' : C \to A_1 \oplus A_2$ be another morphism satisfying $\pi_i \circ h' = f_i$ for $i = 1, 2$. Then, we find

$$h' = 1_{A_1 \oplus A_2} \circ h' = (\iota_1 \pi_1 + \iota_2 \pi_2) h' = \iota_1 f_1 + \iota_2 f_2 = h.$$

So indeed, this h is unique among morphisms $C \to A_1 \oplus A_2$ making the product diagram commute. Therefore $A_1 \oplus A_2$ is a product of objects A_1 and A_2.

This proof can be extended by induction to prove that the biproduct of any finite amount of objects is also the product of those objects. The proof that the biproduct is a coproduct is dual. □

Example 3.1.4. The following are examples of additive categories:

(i) The category Ab of abelian groups is additive. The zero object is the trivial group 0, and the zero morphism $A \to B$ sends every element of A to $0 \in B$. Addition of morphisms is done pointwise, and finite biproducts are given by direct sums (or equivalently, direct products).

(ii) More generally, Mod_R is additive for any ring R, as is Vect_K for a field K.

(iii) For a ring R, the category of matrices Mat_R can be turned into an additive category by adding a zero object and zero morphisms. The abelian group structure of morphisms is given by addition of matrices, and the biproduct of two natural numbers (the objects of the category) is given by their sum.

(iv) The category CRing of commutative rings is not additive. Not only are the Hom-sets not abelian groups (the sum of two ring-homomorphisms does not preserve the multiplicative identity), this category also does not have a zero object. Though the zero ring is terminal, the ring of integers $\mathbb{Z}$ is initial. We require these two to be isomorphic in an additive category, which is not the case in CRing.

An *abelian* category is an additive category that has well-behaved kernels and cokernels. For clarity, we repeat the definition of those here:

Definition 3.1.5. For a morphism $f : A \to B$ in an additive category, we define its

- *kernel* as an object K, along with a morphism $k : K \to A$ such that $fk = 0$. Moreover, for any object K' with a morphism $k' : K' \to A$ with $fk' = 0$, there is a unique morphism $u : K' \to K$ such that the diagram

$$\begin{tikzcd}
& A \arrow[dr, "f"] & \\
& K \arrow[u, "k"] \arrow[r, "0"'] & B. \\
K' \arrow[uur, bend left, "k'"] \arrow[ur, dashed, "\exists! u"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

  commutes. This is the universal property of the kernel. We denote the kernel as $\ker f$.[27]

- *cokernel* as an object Q, along with a morphism $q : B \to Q$ such that $qf = 0$. Moreover, for any object Q' with a morphism $q' : B \to Q'$ with $q'f = 0$, there is a unique morphism $u : Q \to Q'$ such that the

[27] We invoke the Axiom of Choice to pick a specific object and morphism to denote $k : \ker f \to A$ as ‘the’ kernel of f. Though it should be noted that there is not a canonical choice for this in general. Any two choices of kernels are unique up to unique isomorphism however, because they are categorical limits.

diagram

$$\begin{tikzcd}
 & A \arrow[d, "0"'] \arrow[dr, "f"] \arrow[ddl, bend right, "0"'] & \\
 & Q \arrow[dl, dashed, "\exists! u"] & B \arrow[l, "q"] \arrow[dll, bend left, "q'"] \\
Q' & &
\end{tikzcd}$$

commutes. This is the universal property of the cokernel. We denote the cokernel as $\operatorname{coker} f$.

- *image* as the kernel of the morphism q as above, which we denote by $\operatorname{im} f$.
- *coimage* as the cokernel of the morphism k as above, which we denote by $\operatorname{coim} f$. $\blacktriangledown$

Remark. The kernel and cokernel of a morphism f (if they exist) do not just consist of the object K and Q, but the morphisms k and q as well. These morphisms play such a central role that we may call k and q the kernel and cokernel respectively, rather than the objects. With this convention, the image of f is the kernel of the cokernel of f, and the coimage is the cokernel of the kernel.

Any kernel $k : K \to A$ is a monomorphism, this follows from universality: If $kg = kh$ for morphisms $g, h : K' \rightrightarrows K$, then kg is a morphism from K' to A such that composing it with f gives the zero morphism. Thus there is a unique morphism from K' to K that, when composed with k, is equal to the morphism $kg = kh$. Both g and h have this property, and thus are necessarily equal. Dually, any cokernel $q : B \to Q$ is an epimorphism. We define *abelian categories* to be categories where the converse is always true:

Definition 3.1.6. An additive category A is called *abelian* if:

- Every morphism has a kernel and a cokernel.
- Every monomorphism $A \to B$ is the kernel of some morphism $B \to C$. And every epimorphism $B \to C$ is the cokernel of some morphism $A \to B$. $\blacktriangledown$

Abelian categories are, as the name suggests, generalizations of Ab. Many properties of this category are also present in abelian categories. One such property is that an abelian category admits all finite categorical limits and colimits. This follows from Proposition 1.4.6, whose proof can be modified to show that admitting finite products and equalizers is equivalent to admitting all finite limits. Just as in Ab, the equalizer of two morphisms f and g is simply the kernel of their difference. Dually, the same Proposition can be used to show that abelian categories admit all finite colimits, with the coequalizer of f and g being the cokernel of their difference.

More common properties from Ab include, but are not limited to:

Proposition 3.1.7. *In an abelian category, the following hold:*

(a). *A morphism is monic (resp. epic) if and only if its kernel (resp. cokernel) is the zero object.*

(b). *A morphism is an isomorphism if and only if it is monic and epic.*

(c). *The image and coimage of a morphism are the isomorphic.*

Proof. (a). Let $f : A \to B$ be a monomorphism, and consider its kernel $k : K \to A$. By definition of the kernel, we have that $fk = 0$, which itself is equal to $f \circ 0$. Since f is monic, it follows that k is the zero morphism. Composing with 1_K gives $k \circ 1_K = 0 = k \circ 0$, now we apply the fact that k is monic which implies $1_K = 0_{KK}$, which means K is the zero object.

For the converse, assume the kernel K of a morphism $f : A \to B$ is the zero object. By definition of zero objects, there is a single morphism $K \to A$, which is the zero morphism. Now let g and h be two morphisms from another object C to A such that $fg = fh$. Subtracting fh on both sides, we find $f(g - h) = 0$. Now because there is a morphism $g - h : C \to A$ that composes with f to the zero morphism, there is a unique morphism $u : C \to K$ such that $ku = g - h$. Now k is the zero morphism, so we get $0 = g - h$, which implies $g = h$. Thus f is monic.

The proof that a morphism is epic if and only if it has zero cokernel is dual.

(b). If $f : A \to B$ is an isomorphism, and $g, h : X \rightrightarrows A$ are morphisms such that $fg = fh$, then $g = h$ by composing with the inverse of f. Thus f is monic. The proof for f being epic is dual.

Let $f : A \to B$ be a mono and epimorphism. Because it is monic, it is the kernel of some morphism $g : B \to Y$. By definition of kerels, we have $gf = 0$, but because f is epic, this implies $g = 0$.

$$\begin{tikzcd}
B \arrow[dr, "g=0"] & \\
A = \ker g \arrow[u, "f"] \arrow[r, "0"'] & Y
\end{tikzcd}$$

Now note that the identity $1_B : B \to B$ also composes with g to make $g1_B = 0$, so there is a unique $u : B \to A$ such that $fu = 1_B$. On the other hand, the composition $uf : A \to A$ is necessarily the identity, since that is the unique morphism $v : A \to A$ such that $fv = f$, by universality of the kernel. Since there is a morphism u such that fu and uf are the relevant identity morphisms, f is an isomorphism.

(c). Let $f : A \to B$ be a morphism. The plan is to construct a morphism $\bar{f} : \operatorname{coim} f \to \operatorname{im} f$ and show it is an isomorphism. Let $k : \ker f \to A$ be the kernel of f and $c : A \to \operatorname{coim} f = \operatorname{coker} k$ its cokernel, and let $q : B \to \operatorname{coker} f$ be the cokernel of f and $i : \operatorname{im} f = \ker q \to B$ its kernel. Thus there is a commutative diagram:

$$\begin{tikzcd}
\ker f \arrow[r, "k"] \arrow[dr, "0"'] & A \arrow[r, "f"] \arrow[d, "c"] & B \arrow[r, "q"] & \operatorname{coker} f. \\
& \operatorname{coim} f & \operatorname{im} f \arrow[u, "i"] \arrow[ur, "0"'] &
\end{tikzcd}$$

Note that, by definition of the kernel, fk is the zero morphism. Thus by definition of the coimage (as the cokernel of k), there is a unique morphism $u : \operatorname{coim} f \to B$ such that $uc = f$. Similarly, because $qf = 0$, there is a unique $v : A \to \operatorname{im} f$ such that $iv = f$.

$$\begin{tikzcd}
\ker f \arrow[r, "k"] \arrow[dr, "0"'] & A \arrow[r, "f"] \arrow[d, "c"] \arrow[dr, dashed, "v"] & B \arrow[r, "q"] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[ur, dashed, "u"'] & \operatorname{im} f \arrow[u, "i"] \arrow[ur, "0"'] &
\end{tikzcd}$$

Note that because $uc = f$, it follows that $quc = qf = 0$. Because c is a cokernel, it is an epimorphism, which implies $qu = 0$. Now because $\operatorname{im} f$ is the kernel of q, there is a unique morphism $\bar{f} : \operatorname{coim} f \to \operatorname{im} f$ such that $i\bar{f} = u$. Note that this implies

$$i\bar{f}c = uc = f = iv.$$

Applying the fact that i is a kernel, and thus a monomorphism, it follows that $\bar{f}c = v$. Thus, the following diagram commutes:

$$\begin{tikzcd}
\ker f \arrow[r, "k"] \arrow[dr, "0"'] & A \arrow[r, "f"] \arrow[d, "c"] \arrow[dr, "v"] & B \arrow[r, "q"] & \operatorname{coker} f. \\
& \operatorname{coim} f \arrow[r, dashed, "\bar{f}"'] \arrow[ur, "u"] & \operatorname{im} f \arrow[u, "i"] \arrow[ur, "0"'] &
\end{tikzcd}$$

To show $\bar{f}$ is an isomorphism, we show it is monic and epic, and apply part (b). Before that, we first need to show that u is a monomorphism and v is an epimorphism.

To show u is monic, it suffices to take some $x : X \to \operatorname{coim} f$ such that $ux = 0$, and show that this implies $x = 0$.[28] Let $z : \operatorname{coim} f \to \operatorname{coker} x$ be the cokernel of x. Because ux is zero, there is a unique morphism $j : \operatorname{coker} x \to B$ such that $jz = u$.

$$\begin{tikzcd}
\ker f \arrow[r, "k"] & A \arrow[r, "f"] \arrow[d, "c"] & B \\
& \operatorname{coim} f \arrow[ur, "u"] \arrow[dr, "z"] & \\
& X \arrow[u, "x"] \arrow[r, "0"'] & \operatorname{coker} x \arrow[uu, dashed, "j"']
\end{tikzcd}$$

The morphisms c and z are both cokernels, and thus both epimorphisms. It follows that their composition zc is also an epimorphism. Therefore, it is the cokernel of some morphism $h : H \to A$. Note that the composition fh can be rewritten to

$$fh = uch = jzch = j0 = 0,$$

where we used that $zch = 0$ by zc being the cokerenel of h. Thus, by definition of the kernel of f, there is a unique $h' : H \to \ker f$ such that $kh' = h$. This gives the following commutative diagram:

$$\begin{tikzcd}
& H \arrow[dl, dashed, "h'"'] \arrow[d, "h"] & \\
\ker f \arrow[r, "k"] & A \arrow[r, "f"] \arrow[d, "c"] & B \\
& \operatorname{coim} f \arrow[ur, "u"] \arrow[dr, "z"] & \\
& X \arrow[u, "x"] \arrow[r, "0"'] & \operatorname{coker} x. \arrow[uu, "j"']
\end{tikzcd}$$

[28]If $ue_1 = ue_2$ for morphisms $e_i : X \to \operatorname{coim} f$, then we rewrite this to $u(e_1 - e_2) = 0$. The equality $e_1 = e_2$ is equivalent to $e_1 - e_2 = 0$, thus setting $x := e_1 - e_2$, and showing $ux = 0$ implies $x = 0$ proves that u is monic.

The composition ch can now be written as ckh', which is zero, since c is the cokernel of k. Now because zc is the cokernel of h, there is a unique morphism $c' : \operatorname{coker} x \to \operatorname{coim} f$ such that $c = c'(zc)$. Applying the fact that c is epic, it follows that $c'z$ is the identity on $\operatorname{coim} f$.

$$\begin{tikzcd}
 & H \arrow[dl, "h'"'] \arrow[d, "h"] & \\
\ker f \arrow[r, "k"] & A \arrow[r, "f"] \arrow[d, "c"] & B \\
 & \operatorname{coim} f \arrow[ur, "u"] \arrow[dr, bend left, "z"] & \\
 & X \arrow[u, "x"] \arrow[r, "0"'] & \operatorname{coker} x \arrow[uu, "j"'] \arrow[ul, dashed, "c'"]
\end{tikzcd}$$

Now, since z is the cokernel of x, we have $zx = 0$. Composing with c', we find $x = 0$, which proves that u is monic. The proof that v is epic is dual to the above proof.[29]

Now we can finally show that $\bar{f} : \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. To that end, assume $\bar{f}g = 0$ for some $g : G \to \operatorname{coim} f$. Composing with i gives $i\bar{f}g = 0$, which implies $ug = 0$ by definition of $\bar{f}$. Using the fact that u is monic, we obtain $g = 0$, thus making $\bar{f}$ a monomorphism as well. Similarly, using the fact that v is epic, it follows that $\bar{f}$ is an epimorphism. Part (b) of this proof implies that $\bar{f}$ is actually an isomorphism, which completes the proof. $\square$

Remark. We can use part (c) to write a morphism $f : A \to B$ as the composition of an epimorphism and monomorphism. Namely, because the image and coimage are isomorphic, we consider them to be the same object denoted $\operatorname{im} f$, with morphisms $c : A \to \operatorname{im} f$ and $i : \operatorname{im} f \to B$. The proof above implies that f is equal to the composition ic. This is the *epi-mono-factorization* of f, which always exists in abelian categories.

Example 3.1.8. The following are examples of abelian categories:

(i) The category Ab of abelian groups is abelian. The kernel and cokernel correspond to the usual kernel and cokernel of group-homomorphisms. If $m : A \to B$ is a group-monomorphism, then it is the kernel of the projection $B \to B/\operatorname{im} m$. So in essence, the fact that monomorphisms are kernels says that, in Ab, one can take the quotient of any subgroup of an abelian group. That is, every subgroup of an abelian group is normal. Dually, the fact that epimorphisms are cokernels is a reformulation of the fact that any quotient group is formed by taking the quotient of A with some normal subgroup. Part (c) of the above proposition states that, for any $f : A \to B$ in Ab, there is an isomorphism $A/\ker f = \operatorname{coim} f \cong \operatorname{im} f$, which is the first isomorphism theorem.

(ii) Similarly, Mod_R and Vect_K are abelian for any ring R and any field K.

(iii) If A is an abelian category, then its opposite A^{op} is too. The zero object in A is also zero in A^{op}. The biproduct of two objects in A^{op} is the same as in A, except now the projection is the opposite of the inclusion, and vice versa. Given a morphism $f : A \to B$ in A, its opposite $f^{\mathrm{op}} : B \to A$ in A^{op} has kernel equal to the opposite of the cokernel of f, and vice versa for the cokernel.

[29] It is not surprising that v is an epimorphism. In Ab, for example, the map $v : A \to \operatorname{im} f$ is the same as the morphism f itself, but with codomain restricted to just the image of f, which is definitively surjective.

(iv) The category $\mathsf{Ab}^{\text{tor-free}}$ of torsion-free abelian groups is not abelian. For example, consider the homomorphism $m : \mathbb{Z} \to \mathbb{Z}$ given by $m(1) = 2$. The cokernel of this homomorphism would be a map $g : \mathbb{Z} \to B$ such that $gm = 0$. However, this implies that

$$0 = g(m(1)) = g(2) = 2g(1),$$

which means that $g(1)$ is a torsion element of B, or g is the zero homomorphism. If g is zero, then the cokernel of m is zero, meaning m should be surjective which it is not the case. Therefore $g(1)$ is a nonzero torsion element of B, but that means the cokernel of m is not in $\mathsf{Ab}^{\text{tor-free}}$, making the category non-abelian.

Another concept from Ab and Mod_R we can generalize in abelian categories is *exact sequences*. Before that however, we need an important lemma:

Lemma 3.1.9. *Let $f : A \to B$ and $g : B \to C$ be morphisms in an abelian category. If the composition gf is the zero morphism, then there is a natural monomorphism $t : \operatorname{im} f \to \ker g$.*

Proof. Using the epi-mono-factorization of f, we can write the equality $gf = 0$ as $gic = 0$, where $c : A \to \operatorname{im} f$ and $i : \operatorname{im} f \to B$ are the morphisms described in the remark above. The morphism c is epic, so this implies $gi = 0$. The definition of the kernel of g ensures there is a unique morphism $t : \operatorname{im} f \to \ker g$ such that $i = kt$.

To show t is monic, we let $x : X \to \operatorname{im} f$ be another morphism such that $tx = 0$. Composing with the kernel $k : \ker g \to B$ gives $0 = ktx = ix$. The morphism i is monic, so this implies $x = 0$. Therefore, t is monic as well. $\square$

In a concrete category like Mod_R, this map $t : \operatorname{im} f \to \ker g$ is the inclusion map. This follows from $gf = 0$: the image of f is fully contained in the kernel of g. We can now define exact sequences for general abelian categories:

Definition 3.1.10. In an abelian category, we say a (potentially infinite) sequence of objects

$$\dots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \dots$$

is *exact* in A_i if $f_i f_{i-1} = 0$, and the natural morphism $\operatorname{im} f_{i-1} \to \ker f_i$ from Lemma 3.1.9 is an isomorphism. We say the sequence is *exact* if it is exact in every object in the sequence. We call an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

a *short exact sequence*. ▼

Example 3.1.11. Many examples and properties of exact sequences in Mod_R carry over to general abelian categories. For example, a sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if $f : A \to B$ is a kernel of of g. Dually, a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if $g : B \to C$ is a cokernel of f.

The following is a useful result that is readily proved in Mod_R, but may not be so immediate in general abelian categories.

Proposition 3.1.12. *In an abelian category, any sequence of the form*

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus B \xrightarrow{\pi_B} B \longrightarrow 0$$

is exact. Such a sequence is called a split exact sequence*.*

Proof. Following Example 3.1.11, exactness of the above sequence is equivalent to $\iota_A : A \to A \oplus B$ being the kernel of π_B, and $\pi_B : A \oplus B \to B$ being the cokernel of ι_A.

Note that $\pi_B \iota_A$ is already the zero morphism by definition of the biproduct. So all we need to show is that for any other $k' : K' \to A \oplus B$ with $\pi_B k' = 0$, there is a unique $u : K' \to A$ making

$$\begin{tikzcd}
& A \oplus B \arrow[dr, "\pi_B"] & \\
& A \arrow[u, "\iota_A"] \arrow[r, "0"'] & B \\
K' \arrow[uur, bend left, "k'"] \arrow[ur, dashed, "u"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

commute. Let $u := \pi_A k'$, which indeed satisfies $\iota_A u = k'$, by

$$\iota_A u = \iota_A \pi_A k' = (\iota_A \pi_A + \iota_B \pi_B) k' = k'.$$

Finally, let $v : K' \to A$ be another morphism such that $\iota_A v = k'$. It follows that

$$v = \pi_A \iota_A v = \pi_A k' = u,$$

and thus this u is unique. We conclude that $\iota_A : A \to A \oplus B$ satisfies the universal property of the kernel of π_B. A dual argument can be used to show that $\pi_B : A \oplus B \to B$ is the cokernel of ι_A. Therefore, the sequence

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus B \xrightarrow{\pi_B} B \longrightarrow 0$$

is exact. $\square$

Just like in Mod_R, there is an interest in functors that preserve the additive structure of abelian categories:

Definition 3.1.13. We call a functor $F : \mathsf{A} \to \mathsf{B}$ between abelian categories:

- *additive* if it preserves finite biproducts and zero objects.
- *left exact* (resp. *right exact*) if it is additive, and given a short exact sequence $0 \to A \to A' \to A'' \to 0$ in A, the sequence

  $$0 \to FA \to FA' \to FA'' \quad (\text{resp. } FA \to FA' \to FA'' \to 0)$$

  is exact. We say F is *exact* if it is both left and right exact. ▼

Remark. As noted before, [Mac98, proposition 4, p. 197] proves that a functor between additive (and in particular abelian) categories is additive if and only if it preserves the abelian group structure on Hom-sets. I.e., $F(f+g) = Ff + Fg$ for parallel morphisms f and g.

Example 3.1.14. Given an object A of an abelian category A, both Hom-functors $\operatorname{Hom}(A, -) : \mathsf{A} \to \mathsf{Ab}$ and $\operatorname{Hom}(-, A) : \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ are additive and left exact.

There are more properties of Mod_R that also hold in general abelian categories, some of which are covered throughout this Chapter. What may be surprising is that, in a certain sense, any abelian category is a subcategory of Mod_R for some ring R. This is the celebrated *Freyd-Mitchell Embedding Theorem*:

Theorem 3.1.15 (Freyd-Mitchell Embedding Theorem)**.** *Let A be a small abelian category. There exists a (not necessarily commutative) ring R and a fully faithful exact functor $F : \mathsf{A} \to \mathsf{Mod}_R$.*

The functor F defines an equivalence between A and a full subcategory of Mod_R. Exactness of F implies that kernels, cokernels, images, exact sequences, and biproducts in A can be seen as the corresponding concepts in Mod_R. Thus, a result like Proposition 3.1.7 can be proven by taking smallest abelian subcategory containing the relevant morphisms, and looking at it in terms of modules over a certain ring. This allows the convoluted diagram chase from part (c) of 3.1.7 for example to be proven as how one would prove the first isomorphism theorem in Mod_R. Proposition 1.2.8 implies that F reflects isomorphisms, so after proving the isomorphism in Mod_R, it can be taken back to A to conclude the proof.

Another example of this is the *Snake Lemma*, which can be proven in any arbitrary abelian category, as is done in e.g. [Wei94, lemma 1.3.2, p.12], using a proof in Mod_R, like the one in [AK17, lemma 5.10, p.33].

Lemma 3.1.16 (Snake Lemma)**.** *Consider the following commutative diagram with exact rows in an abelian category:*

$$\begin{tikzcd} & M \arrow[r, "f"] \arrow[d, "\alpha"'] & N \arrow[r, "g"] \arrow[d, "\beta"'] & P \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f'"'] & N' \arrow[r, "g'"'] & P'. & \end{tikzcd}$$

This induces an exact sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \xrightarrow{\delta} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma.$$

The proof for the embedding theorem itself is quite complicated, and outside the scope of this thesis. The seventh chapter of [Fre64] builds up to a proof of the theorem, which is theorem 7.34 (p.150) in the book. Note that Freyd uses much outdated language throughout his book, for example the embedding theorem is stated as saying any abelian category is 'fully abelian'.

3.2 Chain Complexes and Resolutions

This Section covers the theory of *chain complexes*. These are generalizations of exact sequences, where we do not require the image and kernel of two consecutive morphisms to be equal (or canonically isomorphic), but we still require consecutive morphisms to compose to zero. An important concept that we also define here is that of *homology*, which is a measure of how close a chain complex is to being exact. Finally we cover the theory of *resolutions*, which are special exact sequences that are used to define derived functors later on.

Definition 3.2.1. Let A be an abelian category.

- A *chain complex* is an infinite sequence, indexed by integers,

$$\cdots \xrightarrow{d_{i+2}} A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} \cdots$$

  of objects and morphisms in A such that $d_{i+1}d_i = 0$ for all $i \in \mathbb{Z}$. We denote the complex as $(A_\bullet, d_\bullet)$, or just as $A_\bullet$. The morphisms d_i are often called *boundary morphisms*.

- A *chain map* $f : (A_\bullet, d_\bullet) \to (B_\bullet, d'_\bullet)$ between chain complexes is a collection of morphisms $f_i : A_i \to B_i$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+2}"] & A_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"'] & A_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & A_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"'] & \cdots \\
\cdots \arrow[r, "d'_{i+2}"'] & B_{i+1} \arrow[r, "d'_{i+1}"'] & B_i \arrow[r, "d'_i"'] & A_{i-1} \arrow[r, "d'_{i-1}"'] & \cdots .
\end{tikzcd}$$

- Given a chain complex A, we define its *i-th homology object* as $H_i(A_\bullet) := \operatorname{coker} t_i$, where t_i is the morphism from $\operatorname{im} d_{i+1}$ to $\ker d_i$, as defined in Lemma 3.1.9.

- Two chain maps $f, g : (A_\bullet, d_\bullet) \rightrightarrows (B_\bullet, d'_\bullet)$ are *homotopic* if there exists a collection of morphisms (called a *homotopy*) $\sigma_i : A_i \to B_{i+1}$ such that

$$f_i - g_i = d'_{i+1}\sigma_i + \sigma_{i-1}d_i$$

  for all $i \in \mathbb{Z}$. These may be portrayed in the following (non-commutative!) diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+2}"] & A_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, shift right, "f_{i+1}"'] \arrow[d, shift left, "g_{i+1}"] & A_i \arrow[r, "d_i"] \arrow[dl, "\sigma_i"'] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] & A_{i-1} \arrow[r, "d_{i-1}"] \arrow[dl, "\sigma_{i-1}"'] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] & \cdots \\
\cdots \arrow[r, "d'_{i+2}"'] & B_{i+1} \arrow[r, "d'_{i+1}"'] & B_i \arrow[r, "d'_i"'] & B_{i-1} \arrow[r, "d'_{i-1}"'] & \cdots
\end{tikzcd}$$

▼

Proposition 3.2.2. *Given an abelian category* A*, its chain complexes form an abelian category, denoted* $\mathsf{Ch}(\mathsf{A})$*, with chain complexes as objects, and chain maps as morphisms.*

Sketch of proof. (See [Wei94, theorem 1.2.3, p.7] for details) All constructions on a complex $(A_\bullet, d_\bullet)$ are done index-wise. Composition of chain maps is defined by $(fg)_i = f_i g_i$ for all $i \in \mathbb{Z}$. The zero object is the zero complex $\cdots \to 0 \to 0 \to \cdots$. Addition is defined by $(f + f')_i = f_i + f'_i$. The biproduct of two complexes $A_\bullet \oplus B_\bullet$ is defined by $(A_\bullet \oplus B_\bullet)_i = A_i \oplus B_i$. Given a chain map $f : A_\bullet \to B_\bullet$, its kernel is the chain map $k : (\ker f)_\bullet \to A_\bullet$, where $(\ker f)_i = \ker f_i$, and similar for the cokernel. Finally, a chain map f is monic (resp. epic) if and only if each f_i is monic (resp. epic). □

Remark. In the definition above, the boundary morphisms have their index going down. But in some contexts, it may be clearer to have the boundary morphisms going up, i.e. the morphisms go from A_i to A_{i+1}. These kind of complexes are *cochain complexes*, and their homology is instead called *cohomology*. The objects, boundary morphisms and chain maps usually have their index in a superscript. The distinction between chain and cochain complexes is only semantic, as the category of chain complexes is isomorphic to the category of cochain complexes. For completeness, both chain and cochain complexes are called chain complexes from here on out. In the general case, we assume the indices of the boundary maps go down, but they may go up in some specific cases (e.g. in defining injective resolutions in Definition 3.2.7)

The following are useful properties of homology. Importantly, it states that the n-th homology object defines a functor.

Proposition 3.2.3. *Let A be an abelian category. The following hold:*

(a). *For each n in $\mathbb{Z}$, the n-th homology defines an additive functor $H_n : \mathsf{Ch}(\mathsf{A}) \to \mathsf{A}$.*

(b). *If two chain maps f and g are homotopic, then the morphisms $H_n(f)$ and $H_n(g)$ are equal.*

(c). *A short exact sequence*

$$0_\bullet \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0_\bullet$$

of complexes in $\mathsf{Ch}(\mathsf{A})$ induces a long exact sequence in A:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+1}"] & H_n(A_\bullet) \arrow[r, "H_n(f)"] & H_n(B_\bullet) \arrow[r, "H_n(g)"] & H_n(C_\bullet) \arrow[dll, "\delta_n"] & \\
& H_{n+1}(A_\bullet) \arrow[r, "H_{n-1}(f)"'] & H_{n+1}(B_\bullet) \arrow[r, "H_{n-1}(g)"'] & H_{n+1}(C_\bullet) \arrow[r, "\delta_{n-1}"'] & \cdots
\end{tikzcd}$$

Proof. (a). It is clear how H_n acts on objects of $\mathsf{Ch}(\mathsf{A})$, but we still need to define $H_n(f)$ for a chain map $f : (A_\bullet, d_\bullet) \to (B_\bullet, d'_\bullet)$. First note that, for each n, there is a commutative diagram

$$\begin{tikzcd}
\ker d_n \arrow[r, "k"] \arrow[d, dashed] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r, "q"] \arrow[d, "f_{n-1}"'] & \operatorname{coker} d_n \arrow[d, dashed] \\
\ker d'_n \arrow[r, "k'"'] & B_n \arrow[r, "d'_n"'] & B_{n-1} \arrow[r, "q'"'] & \operatorname{coker} d'_n.
\end{tikzcd}$$

The morphism $\ker d_n \to \ker d'_n$ exists by the universal property of the kernel of d'_n, because

$$0 = f_{n-1} d_n k = d'_n f_n k,$$

which implies there is a unique morphism $\ker d_n \to \ker d'_n$ making the diagram commute. The morphism $\operatorname{coker} d_n \to \operatorname{coker} d'_n$ is constructed dually. By the same argument, there is a morphism $\operatorname{im} d_n \to \operatorname{im} d_{n+1}$ making the diagram

$$\begin{tikzcd}
\operatorname{im} d_n \arrow[r] \arrow[d, dashed] & A_{n-1} \arrow[r, "q"] \arrow[d, "f_{n-1}"'] & \operatorname{coker} d_n \arrow[d] \\
\operatorname{im} d'_n \arrow[r] & B_{n-1} \arrow[r, "q'"'] & \operatorname{coker} d'_n
\end{tikzcd}$$

commute. Because $H_n(A_\bullet)$ and $H_n(B_\bullet)$ are cokernels of $t_n : \operatorname{im} d_n \to \ker d_{n-1}$ and $t'_n : \operatorname{im} d'_n \to \ker d'_{n-1}$ respectively, there is a unique morphism $H_n(A_\bullet) \to H_n(B_\bullet)$, which we define to be $H_n(f)$, making the following diagram commute:

$$
\begin{tikzcd}
\operatorname{im} d_{n+1} \arrow[rr, "t_n"] \arrow[dr] \arrow[ddd, bend right] & & \ker d_n \arrow[r] \arrow[dl] \arrow[ddd, bend left] & H_n(A_\bullet) \arrow[ddd, dashed, "H_n(f)"] \\
A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"'] & \\
B_{n+1} \arrow[r, "d'_{n+1}"] & B_n \arrow[r, "d'_n"] & B_{n-1} & \\
\operatorname{im} d'_{n+1} \arrow[rr, "t'_n"'] \arrow[ur] & & \ker d'_n \arrow[r] \arrow[ul] & H_n(B_\bullet)
\end{tikzcd}
$$

More concisely, $H_n(f)$ is defined to be the unique morphism such that

$$
\begin{tikzcd}
\operatorname{im} d_{n+1} \arrow[r, "t_n"] \arrow[d] & \ker d_n \arrow[r, "q_n"] \arrow[d] & H_i(A_\bullet) \arrow[d, "H_n(f)"] \\
\operatorname{im} d'_{n+1} \arrow[r, "t'_n"'] & \ker d'_n \arrow[r, "q'_n"'] & H_n(B_\bullet)
\end{tikzcd}
$$

commutes.

Note that $H_n(1_{A_\bullet})$ is just the identity of $H_n(A_\bullet)$. This is because both of these morphisms make the relevant diagram commute, so uniqueness implies they are equal. Composition of morphisms is also preserved. If $f : A_\bullet \to B_\bullet$ and $g : B_\bullet \to C_\bullet$ are chain maps, then both $H_n(gf)$ and $H_n(g)H_n(f)$ make the diagram like the one above with chain map gf commute, thus they are equal.

Finally, additivity follows similarly. Let $q_n : \ker d_n \to H_n(A_\bullet)$ and $q'_n : \ker d'_i \to H_i(B_\bullet)$ be the horizontal morphisms displayed above. To show that $H_n(f+g) = H_n(f) + H_n(g)$ for parallel chain maps $f, g : A_\bullet \rightrightarrows B_\bullet$, denote the corresponding morphisms $\ker d_n \rightrightarrows \ker d'_n$ by $\hat{f}$ and $\hat{g}$ respectively. Note that

$$p'(\hat{f} + \hat{g}) = p'\hat{f} + p'\hat{g} = H_n(f)p + H_n(g)p = (H_n(f) + H_n(g))p.$$

So by uniqueness, $H_n(f+g)$ is equal to $H_n(f) + H_n(g)$. So H_n is indeed an additive functor.

(b). Because each H_n is additive, it suffices to show that if f is homotopic to the zero morphism, then $H_n(f) = 0$. To start, there is a collection of morphisms σ_n such that

$$f_n = d'_{n+1}\sigma_n + \sigma_{n-1}d_n.$$

Composing with $k : \ker d_n \to A_n$ gives

$$f_n k = d'_{n+1}\sigma_n k + \sigma_{n-1}d_n k = d'_{n+1}\sigma_n k.$$

Using the epi-mono factorization of $d'_{n+1} = jc$, there is a morphism $c\sigma_n k : \ker d_n \to \operatorname{im} d'_{n+1}$, which we denote by v, making the following diagram commute:

$$
\begin{tikzcd}
& \ker d_n \arrow[rr, "q_n"] \arrow[d, "k"] \arrow[ddd, dashed, bend left, "v"] & & H_n(A_\bullet) \arrow[ddd, "H_n(f)"] \\
A_{n+1} \arrow[r, "d_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "\sigma_n"'] & A_{n-1} & \\
B_{n+1} \arrow[r, "d'_{n+1}"] \arrow[dr, "c"'] & B_n \arrow[r, "d'_n"] & B_{n-1} & \\
& \operatorname{im} d'_{n+1} \arrow[u, "j"] \arrow[r, "t'_n"] & \ker d'_n \arrow[r, "q'_n"] & H_n(B_\bullet)
\end{tikzcd}
$$

The composition $q'_n t'_n v$ is zero, and by commutativity, so is $H_n(f)q_n$. Using the fact that q_n is an epimorphism (it is a cokernel), it follows that $H_n(f)$ is the zero morphism. This completes the proof.

(c). A proof is given in [Rot09, theorem 6.10, p.333], using the Freyd-Mitchell embedding theorem. The *connecting morphism* δ_n is obtained through a diagram chase in [Rot09, proposition 6.9, p.332], but can also be derived using the Snake Lemma. $\square$

Example 3.2.4. Let X be a topological space. An *n-simplex* is a continuous (and not necessarily injective) map $\sigma : \Delta^n \to X$, where Δ^n is the standard n-dimensional simplex in $\mathbb{R}^n$ (e.g. Δ^1 is the line segment $[0, e_1]$, Δ^2 is a triangle formed by the convex polygon $[0, e_1, e_2]$ and so on). We denote the image of an n-simplex as the set $[p_0, \dots, p_n] := [\sigma(0), \dots, \sigma(e_n)] \subseteq X$. The *boundary* of an n-simplex is defined as

$$\partial_n[p_0, \dots, p_n] = \sum_{k=0}^{n} (-1)^k [p_0, \dots, p_{k-1}, p_{k+1}, \dots, p_n],$$

where these sums are formal.[30] The collection of n-simplices on X generate a free abelian group $C_n(X)$, whose elements are called *n-chains*. If we stipulate that the boundary of an n-chain is the sum of the boundaries of the constituent simplices, then this forms a chain complex

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0.$$

A continuous function between topological spaces $f : X \to Y$ induces a chain map $C_\bullet(X) \to C_\bullet(Y)$, which sends n-chains $\sum_i \sigma_i$ to $\sum_i f \circ \sigma_i$. Its homology groups $H_n(X)$ are called the *singular homology* groups of X. In this context, they can be computed as

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1},$$

and are free abelian groups as well. Informally, the rank of these homology groups give a sense of the number

[30] The sum of two simplices *could* be seen in the context of differential forms. That is, if X is a smooth n-manifold, then the sum of two n-simplices $\sigma + \tau$ may be interpreted as a simplex satisfying, for all smooth n-forms ω, $\int_{(\sigma+\tau)(\Delta^n)} \omega = \int_{\sigma(\Delta^n)} \omega + \int_{\tau(\Delta^n)} \omega$.

of ‘holes’ X has. For example, if we consider the torus $T := S^1 \times S^1$, then its homology groups are

$$H_k(T) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z}^2 & \text{if } k = 1 \\ 0 & \text{if } k > 2. \end{cases}$$

This signifies that the torus consists of one path-connected component, and that it has two 1-dimensional holes which are enclosed by 1-simplices, as depicted below. It also has one 2-dimensional hole which is enclosed by the surface of the torus itself.

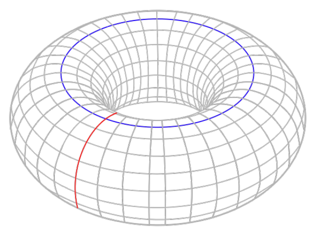

Figure 1: Two 1-simplices enclosing holes in a torus. Source: [Use14]

Chapter 2 of [Hat01] gives more details on the theory of singular homology. The proof of theorem 2.10 (p.112) of the book also gives an insight for why we define chain homotopy the way we do in Definition 3.2.1; homotopic continuous maps between topological spaces induce homotopic chain maps between their simplicial chain complexes.

Next we define a class of objects that are crucial for constructing derived functors: *projective* and *injective* objects.

Definition 3.2.5. Let A be an abelian category. An object P of A is *projective* if, for any epimorphism $e : A \to B$, and any morphism $f : P \to B$, there is a (not necessarily unique) *lift* $\bar{f} : P \to A$ such that the following diagram with an exact row

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \bar{f}"'] \arrow[d, "f"] & \\ A \arrow[r, "e"'] & B \arrow[r] & 0 \end{tikzcd}$$

commutes.

Dually, an object I of A is *injective* if, for any monomorphism $m : B \to A$, and any morphism $g : B \to P$, there is a *lift* $\bar{g} : A \to I$ such that the following diagram with an exact row

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & B \arrow[u, "g"] \arrow[r, "m"'] & A \arrow[ul, dashed, "\exists \bar{g}"'] \end{tikzcd}$$

commutes. ▼

Example 3.2.6. For clarity, it is helpful to see which objects are projective and injective in Mod_R:

- An R-module P is projective if and only if one of the following equivalent statements hold (for a proof, see [DF04, proposition 10.5.30, p.389]):
  - Every exact sequence $0 \to M \to N \to P \to 0$ splits (see Example 2.3.2(v) for the definition of a split exact sequence);
  - There is an R-module Q such that the direct sum $P \oplus Q$ is a free module;
  - The Hom-functor $\mathrm{Hom}_R(P, -)$ is exact (not just left exact).

  Some simple examples include the zero module, any free module, and any vector space. Finally if R is a PID (principal ideal domain), then a module is free if and only if it is projective. The $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not projective for $n > 1$. The reason for this is that the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ does not split.

- An R-module I is injective if and only if one of the following equivalent statements hold (for a proof, see [DF04, proposition 10.5.34, p.394]):
  - Every exact sequence $0 \to I \to M \to N \to 0$ splits;
  - For any R-module M containing I as a submodule, there is another submodule Q of M such that $Q \oplus I = M$.
  - The contravariant Hom-functor $\mathrm{Hom}_R(-, I)$ is exact.

  The zero module, any free module, and any vector space is injective. An abelian group is injective if and only it is *divisible*, meaning $nA = A$ for any nonzero integer n. For $n > 1$, it again follows that $\mathbb{Z}/n\mathbb{Z}$ is not injective. To see this, note that $n(\mathbb{Z}/n\mathbb{Z})$ is the trivial group, meaning the group is not divisible, and thus also not injective.

Now we move to defining resolutions, which are the building blocks to define derived functors.

Definition 3.2.7. Let A be an object in an abelian category. A *projective resolution* of A is a chain complex $P_\bullet$, with $P_n = 0$ for $n < 0$, and every P_i projective, together with a morphism $P_0 \to A$ such that

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

is an exact sequence. We denote such a resolution as $P_\bullet \to A \to 0$.
Dually, an *injective resolution* of A is a chain complex $I_\bullet$ (with increasing indices for notational convenience), with $I_n = 0$ for $n < 0$, and every I_i injective, together with a morphism $A \to I_0$ such that

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

is an exact sequence. We denote such a resolution as $0 \to A \to I_\bullet$.
We say an abelian category A has *enough projectives* (resp. *enough injectives*) if, for each object A, there is an epimorphism $P \to A$ (resp. monomorphism $A \to I$), where P is projective (resp. where I is injective). ▼

Remark. As is shown in e.g. [Rot09, corollary 6.3 and 6.5, p.326,327], if an abelian category has enough projectives (resp. enough injectives), then every object admits a projective (resp. injective) resolution. The idea of the proof for the projective case is to start with an epimorphism $d_0 : P_0 \to A$, then extend it by its kernel to obtain the exact sequence

$$\ker d_0 \xrightarrow{k_0} P_0 \xrightarrow{d_0} A \longrightarrow 0.$$

Now we repeat the process to obtain an epimorphism $P_1 \to \ker d_0$, and defining d_1 to be the composition of this morphism and k_0. By induction we obtain a projective resolution for A.

Before constructing derived functors in the next Section, the following result turns out to be quite helpful to make sure they are well-defined:

Proposition 3.2.8 (Comparison Theorem)**.** *Let A and B be objects in an abelian category, with projective resolutions $P_\bullet \to A \to 0$ and $Q_\bullet \to B \to 0$ respectively. A morphism $f : A \to B$ induces a chain map $f' : P_\bullet \to Q_\bullet$ such that the following diagram with exact rows commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "f'_2"] & P_1 \arrow[r] \arrow[d, dashed, "f'_1"] & P_0 \arrow[r] \arrow[d, dashed, "f'_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

Moreover, this chain map is unique up to homotopy.

Dually, if A and B admit injective resolutions $0 \to A \to I_\bullet$ and $0 \to B \to J_\bullet$, then the morphism $f : A \to B$ induces a chain map $f' : I_\bullet \to J_\bullet$ such that the following diagram with exact rows commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I_0 \arrow[r] \arrow[d, dashed, "f'_0"] & I_1 \arrow[r] \arrow[d, dashed, "f'_1"] & I_2 \arrow[r] \arrow[d, dashed, "f'_2"] & \cdots \\
0 \arrow[r] & B \arrow[r] & J_0 \arrow[r] & J_1 \arrow[r] & J_2 \arrow[r] & \cdots.
\end{tikzcd}$$

Moreover, this chain map is unique up to homotopy.

Proof. For $i \geqslant 0$, let $d_i : P_i \to P_{i-1}$ and $d'_i : Q_i \to Q_{i-1}$ denote the boundary morphisms of the projective resolutions, where $P_{-1} = A$ and $Q_{-1} = B$. We prove the statement by induction. The composition $fd_0 : P_0 \to B$ lifts to a morphism $f'_0 : P_0 \to Q_0$, because P_0 is projective, and $d'_0 : Q_0 \to B$ is an epimorphism by exactness of the projective resolution of B. By definition of this lift, it follows that $d'_0 f'_0 = fd_0$.

Now assume, for all $0 \leqslant i \leqslant n$, there is a morphism $f_i : P_i \to Q_i$ such that the diagram built so far

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] & P_n \arrow[r] \arrow[d, "f'_n"] & P_{n-1} \arrow[r] \arrow[d, "f'_{n-1}"] & \cdots \\
\cdots \arrow[r] & Q_{n+1} \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. Note that

$$d'_n f'_n d_{n+1} = f_{n-1} d_n d_{n+1} = f_{n-1} 0 = 0,$$

so there is a unique map $u : P_{n+1} \to \ker d'_n$ such that $ku = f'_n d_{n+1}$, where k is the kernel of d'_n. By exactness of the projective resolution of B, the morphism $t : \operatorname{im} d'_{n+1} \to \ker d'_n$ is invertible, so the composition $t^{-1}u$ is a morphism from P_{n+1} to $\operatorname{im} d'_{n+1}$. By the epi-mono-factorization $d'_{n+1} = ic$, the morphism $c : Q_{n+1} \to \operatorname{im} d'_{n+1}$

is an epimorphism, so by projectiveness of P_{n+1}, there is a lift $f'_{n+1} : P_{n+1} \to Q_{n+1}$ such that $cf'_{n+1} = t^{-1}u$. Note that

$$d'_{n+1}f'_{n+1} = icf'_{n+1} = it^{-1}u = ku = f'_n d_{n+1}.$$

Note that, by Lemma 3.1.9, t is defined such that $i = kt$, which implies $it^{-1} = k$. Therefore, the morphism f'_{n+1} makes the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, dashed, "f'_{n+1}"] & P_n \arrow[r] \arrow[d, "f'_n"] & P_{n-1} \arrow[r] \arrow[d, "f'_{n-1}"] & \cdots \\
\cdots \arrow[r] & Q_{n+1} \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commute. By induction, this process extends to any $f'_i : P_i \to Q_i$ for $i > 0$.

Now for uniqueness up to homotopy, let $g' : P_\bullet \to Q_\bullet$ be another chain map extending f like f' did. We construct a homotopy by induction as well. First let s_{-1} be the zero morphism from A to Q_0. Note that

$$d'_0(f'_0 - g'_0) = d'_0 f'_0 - d'_0 g'_0 = d_0 f - d_0 f = 0,$$

so there is a $u : P_0 \to \ker d'_0$ such that $ku = f'_0 - g'_0$. Like before, exactness implies that a morphism $t^{-1}u : P_0 \to \operatorname{im} d'_1$ exists, which lifts to a morphism $s_0 : P_0 \to Q_1$ such that $cs_0 = t^{-1}u$, where c is the epimorphism such that $d'_1 = ic$. Note that

$$d'_1 s_0 + d_0 s_{-1} = d'_1 s_0 = ics_0 = it^{-1}u = ku = f'_0 - g'_0,$$

so s_{-1} and s_0 already satisfy the requirements of being a homotopy.

Suppose, for all $0 \leqslant i \leqslant n$, there is a morphism $s_i : P_i \to P_{i+1}$ satisfying the definition of a homotopy between the chain maps f' and g'. Note that

$$\begin{aligned}
d'_{n+1}(f'_{n+1} - g'_{n+1} - s_n d_{n+1}) &= d'_{n+1}(f'_{n+1} - g'_{n+1}) - d'_{n+1} s_n d_{n+1} \\
&= (f'_n - g'_n) d_{n+1} - d'_{n+1} s_n d_{n+1} \\
&= (f'_n - g'_n - d'_{n+1} s_n) d_{n+1} \\
&= (s_{n-1} d_n) d_{n+1} = s_{n-1} 0 = 0.
\end{aligned}$$

So there is a morphism $u : P_{n+1} \to \ker d'_{n+1}$ such that $ku = f'_{n+1} - g'_{n+1} - s_n d_{n+1}$, where k is the kernel of d'_{n+1}. Again, using exactness this forms a morphism $t^{-1}u : P_{n+1} \to \operatorname{im} d'_{n+2}$, which lifts to a morphism $s_{n+1} : P_{n+1} \to Q_{n+2}$ such that $cs_{n+1} = t^{-1}u$, where $d'_{n+2} = ic$. The morphism s_{n+1} indeed satisfies the definition of a homotopy, because

$$\begin{aligned}
d'_{n+2} s_{n+1} + s_n d_{n+1} &= ics_{n+1} + (f'_{n+1} - g'_{n+1} - ku) \\
&= it^{-1}u + (f'_{n+1} - g'_{n+1} - ku) \\
&= ku + f'_{n+1} - g'_{n+1} - ku = f'_{n+1} - g'_{n+1}.
\end{aligned}$$

By induction, we can repeat this process to obtain morphisms $s_i : P_i \to Q_{i+1}$ for all $i > 0$, which forms a homotopy between f' and g'. This concludes the proof. The proof for injective resolutions is dual. $\square$

Remark. Note that it suffices for the sequences $Q_\bullet \to B \to 0$ and $0 \to B \to J_\bullet$ to be exact for the theorem to hold. However, we mainly use the comparison theorem in the case where it actually is a projective and injective resolution, respectively.

3.3 Derived Functors and Tor

In this chapter we define derived functors, which aim to extend right exact (resp. left exact) functors to the left (resp. right) to turn short exact sequences into long exact ones. For clarity, the main body of this section only covers definitions and results for *left* derived functors. All constructions for right derived functors are dual, and are stated at the end of the section.

Definition 3.3.1. Let A and B be abelian categories, with A having enough projectives, and let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor. Given an object A of A, let $P_\bullet \to A \to 0$ be a projective resolution and consider the *deleted resolution* $P_\bullet^A$, where A is removed:

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0.$$

The n-th *left derived functor* of F at A is defined as

$$L_n^{P_\bullet} F(A) := H_n(FP_\bullet^A),$$

where n is s nonnegative integer. ▼

Remark. Here $FP_\bullet^A$ denotes the chain complex obtained by applying F to each object in $P_\bullet^A$. Additivity of F guarentees that this is still a chain complex.

Given a morphism $f : A \to B$ in A, with projective resolutions $P_\bullet \to A \to 0$ and $Q_\bullet \to B \to 0$, for each $n \geqslant 0$ there is a morphism $f_n : P_n \to Q_n$ by the comparison theorem. Uniqueness up to homotopy and Proposition 3.2.3(ii) implies that the f_n extend to a unique morphism $H_n(Ff_n) : H_n(FP_\bullet^A) \to H_n(FQ_\bullet^B)$[31], which we denote by $L_n^{P_\bullet,Q_\bullet} F(f) : L_n^{P_\bullet} F(A) \to L_n^{Q_\bullet} F(B)$. Because both F and H_n are additive functors, $L_n^{P_\bullet,Q_\bullet} F$ is also an additive functor from A to B.

As one may hope, the construction of the left derived functor is independent of the choice of projective resolution, up to natural isomorphism:

Proposition 3.3.2. *Let $F : \mathsf{A} \to \mathsf{B}$ be as above. Given an object A of A and projective resolutions $P_\bullet \to A \to 0$ and $Q_\bullet \to A \to 0$, there is a canonical natural isomorphism $L_n^{P_\bullet} F(A) \cong L_n^{Q_\bullet} F(A)$.*

Proof. Consider the identity morphism $1_A : A \to A$. By the comparison theorem, this morphism lifts to chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$. These fit in the following commutative diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, shift right, "f_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, shift right, "f_1"'] & P_0 \arrow[r, "d_0"] \arrow[d, shift right, "f_0"'] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2'"'] \arrow[u, shift right, "g_2"'] & Q_1 \arrow[r, "d_1'"'] \arrow[u, shift right, "g_1"'] & Q_0 \arrow[r, "d_0'"'] \arrow[u, shift right, "g_0"'] & A \arrow[r] & 0.
\end{tikzcd}$$

[31] Note that because F is an additive functor, it preserves homotopies. So if f_n and g_n are homotopic morphisms, then $H_n(Ff_n)$ and $H_n(Fg_n)$ are equal.

Note that, for all $n \geqslant 0$,

$$d_n g_n f_n = g_{n-1} d'_n f_n = g_{n-1} f_{n-1} d_n,$$

so the composition of chain maps gf lifts $1_A : A \to A$ to form the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "g_2 f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "g_1 f_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "g_0 f_0"] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"'] & P_1 \arrow[r, "d_1"'] & P_0 \arrow[r, "d_0"'] & A \arrow[r] & 0.
\end{tikzcd}$$

But the chain identity $1_{P_\bullet} : P_\bullet \to P_\bullet$ also lifts 1_A, so by the comparison theorem, gf and $1_{P_\bullet}$ are homotopic. A similar argument can be used to show that fg and $1_{Q_\bullet}$ are homotopic.

Now deleting A from the resolutions, applying F, and taking homology, we get the $L_n^{P_\bullet} F(gf) = 1_{P_n}$ and $L_n^{Q_\bullet} F(fg) = 1_{Q_n}$, making $L_n^{P_\bullet} F(A)$ and $L_n^{Q_\bullet} F(A)$ isomorphic. Naturality of these isomorphisms is proved in [Rot09, proposition 6.20, p.346]. □

Notation. Because the choice of resolution ultimately does not matter, we omit the superscripts from the notation of left derived functors from here on out, and just write $L_n F$ as the left derived functor.

Proposition 3.3.3. *Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories, with A having enough projectives. The following hold:*

(a). *If F is right exact, then $L_0 F$ and F are naturally isomorphic.*

(b). *If F is exact, then $L_n F A = 0$ for all $n > 0$ and all objects A of A.*

(c). *If P is a projective object of A, then $L_n F P = 0$ for all $n > 0$.*

(d). *If $G : \mathsf{A} \to \mathsf{B}$ is a functor that is naturally isomorphic to F, then $L_n F$ and $L_n G$ are naturally isomorphic.*

Proof. (a). Let

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0$$

be a projective resolution of A. By definition, $L_0 F A$ is the zeroth homology of the deleted complex

$$\cdots \xrightarrow{Fd_2} FP_1 \xrightarrow{Fd_1} FP_0 \xrightarrow{0} 0,$$

which is the cokernel of the morphism $t_0 : \operatorname{im} Fd_1 \to \ker 0 = FP_0$. However, F is right exact, so the sequence

$$\cdots \xrightarrow{Fd_2} FP_1 \xrightarrow{Fd_1} FP_0 \xrightarrow{Fd_0} FA \longrightarrow 0$$

is exact. By exactness, there is an isomorphism $\operatorname{im} Fd_1 \cong \ker Fd_0$, and so $L_0 F A$ is isomorphic to the cokernel of $\ker Fd_0 \to FP_0$, which is the coimage of Fd_0 by definition. This is itself isomorphic to the image of Fd_0, which is FA. Thus, it follows that $L_0 F A$ is isomorphic to FA.

As for naturality, let $\eta_A : FA \to L_0 F A$ be the isomorphisms from above. The goal is to show that the following diagram commutes for any morphism $f : A \to B$ in A:

$$\begin{tikzcd}
FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & L_0FA \arrow[d, "L_0Ff"] \\
FB \arrow[r, "\eta_B"'] & L_0FB.
\end{tikzcd}$$

Let $Q_\bullet \to B \to 0$ be a projective resolution of B with boundary morphisms $d'_n : Q_n \to Q_{n-1}$. These fit into the following diagram where the rows are chain complexes (recall that L_0FA is the cokernel of the morphism $t_0 : \operatorname{im} Fd_1 \to FP_0$, and similar for L_0FB):

$$\begin{tikzcd}
 & & \operatorname{coker} t_0 \arrow[dd, bend left=60, "\widetilde{Ff}"] \\
\operatorname{im} Fd_1 \arrow[r, "t_0"] & FP_0 \arrow[r, "Fd_0"] \arrow[d, "Ff_0"] \arrow[ur, bend left, "q"] & FA \arrow[u, "\eta_A"'] \arrow[d, "Ff"] \\
\operatorname{im} Fd'_1 \arrow[r, "t'_0"'] & FQ_0 \arrow[r, "Fd'_0"'] \arrow[dr, bend right, "q'"'] & FB \arrow[d, "\eta_B"] \\
 & & \operatorname{coker} t'_0.
\end{tikzcd}$$

The morphism Ff_0 is the lift of Ff obtained from the comparison theorem, and $\widetilde{Ff}$ is the morphism constructed at the beginning of the proof of Proposition 3.2.3. By construction, the whole diagram above commutes if the morphisms η_A and η_B were left out. We compute the composition $\widetilde{Ff} \circ \eta_A \circ Fd_0$ as

$$\begin{aligned}
\widetilde{Ff} \circ \eta_A \circ Fd_0 &= \widetilde{Ff} \circ c \\
&= c' \circ Ff_0 \\
&= \eta_B \circ Fd'_0 \circ Ff_0 \\
&= \eta_B \circ Ff \circ Fd_0.
\end{aligned}$$

Note that, because F is right exact, Fd_0 is an epimorphism, so we obtain $\widetilde{Ff} \circ \eta_A = \eta_B \circ Ff$. Now by definition, $\widetilde{Ff} = L_0Ff$, so we indeed find that the above diagram, and thus the naturality square, commutes. Which proves that $\eta : F \Rightarrow L_0F$ is a natural isomorphism.

(b). If F is exact, then a projective resolution $P_\bullet \to A \to 0$ yields an exact sequence

$$\cdots \longrightarrow FP_{n+1} \xrightarrow{Fd_{n+1}} FP_n \xrightarrow{Fd_n} FP_{n-1} \longrightarrow \cdots \xrightarrow{Fd_0} FA \longrightarrow 0$$

For all $n > 0$, we compute L_nFA as

$$L_nFA = H_n(FP_\bullet^A) = \operatorname{coker}(\operatorname{im} Fd_{n+1} \to \ker Fd_n) = 0,$$

since the morphism $\operatorname{im} Fd_{n+1} \to \ker Fd_n$ is an isomorphism by exactness, hence an epimorphism, and thus has zero cokernel by proposition 3.1.7.

(c). If P is projective, then

$$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{1_P} P \longrightarrow 0$$

is a projective resolution for P. The deleted resolution is given by

$$\cdots \longrightarrow 0 \longrightarrow FP \longrightarrow 0.$$

Note that for $n > 0$, we compute $L_n FP$ as

$$L_n FP = \operatorname{coker}(\operatorname{im} 0 \to \ker 0) = \operatorname{coker}(0 \to 0) = 0,$$

which proves the statement.

(d). Let $\eta : F \Rightarrow G$ be the natural isomorphism relating F and G. Let A be an object of A with projective resolution $P_\bullet \to A \to 0$. We define a chain map $\eta_{P_\bullet} : FP_\bullet \to GP_\bullet$ by $(\eta_{P_\bullet})_n = \eta_{P_n}$. Now for every $n \geqslant 0$, define the natural transformation $L_n\eta : L_nF \Rightarrow L_nG$ defined on components by

$$(L_n\eta)_A := H_n(\eta_{P_\bullet}).$$

Because η is a natural isomorphism, each η_{P_n} is an isomorphism, and thus so is $(L_n\eta)_A$. Therefore, L_nF and L_nG are naturally isomorphic. $\square$

Before moving on, there is one more result we need to cover:

Lemma 3.3.4 (Horseshoe Lemma)**.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in an abelian category A with enough projectives, and let $P_\bullet \to A \to 0$ and $Q_\bullet \to C \to 0$ be projective resolutions. There exists a projective resolution $X_\bullet \to B \to 0$ such that the diagram*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r, dashed] \arrow[d] & X_1 \arrow[r, dashed] \arrow[d, dashed] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r, dashed] \arrow[d] & X_0 \arrow[r, dashed] \arrow[d, dashed] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commutes, and has short exact rows.

Sketch of proof. A full proof is given in [Rot09, proposition 6.24, p.349]. The projective resolution of B is defined as the biproduct $X_n = P_n \oplus Q_n$. The morphism $X_0 \to B$ is formed using the universal property of the coproduct, applied to the composite morphism $P_0 \to A \to B$, and the morphism $Q_0 \to B$ given by the definition of Q_0 being projective. The morphisms $X_n \to X_{n-1}$ are constructed by induction. Exactness of each row follows from Lemma 3.1.12. $\square$

Remark. It should be noted that, with the notation above, defining X_n as $P_n \oplus Q_n$, this does not imply that $X_\bullet = P_\bullet \oplus Q_\bullet$ as chain complexes. This is because the morphisms $X_n \to X_{n-1}$ may not be the same as the canonical morphisms $P_n \oplus Q_n \to P_{n-1} \oplus Q_{n-1}$, provided by the boundary morphisms of $P_\bullet$ and $Q_\bullet$ and the universal properties of the biproduct.

Now we state and prove the most important property of derived functors, which answer the question stated at the beginning of this Chapter: left derived functors extend the image of a short exact sequence under a right exact functor to a long exact sequence:

Theorem 3.3.5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in an abelian category A with enough projectives, and let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor. There is a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+1}"] & L_nFA \arrow[r] & L_nFB \arrow[r] & L_nFC \arrow[dll, "\delta_n"'] \\
& L_{n-1}FA \arrow[r] & L_{n-1}FB \arrow[r] & L_{n-1}FC \arrow[dll, dashed, "\delta_{n-1}", "\delta_1"'] \\
& L_0FA \arrow[r] & L_0FB \arrow[r] & L_0FC \arrow[r] & 0.
\end{tikzcd}$$

If F is right exact, then the sequence ends in

$$\cdots \longrightarrow L_1FC \xrightarrow{\delta_1} FA \longrightarrow FB \longrightarrow FC \longrightarrow 0.$$

Proof. Let $P_\bullet \to A \to 0$ and $Q_\bullet \to C \to 0$ be projective resolutions. By the Horseshoe Lemma, there is a projective resolution $X_\bullet \to B \to 0$ such that $0 \to P_n \to X_n \to Q_n \to 0$ is an exact sequence for all $n \geqslant 0$. Because we defined X_n to be the direct sum of P_n and Q_n, we find that F preserves the exactness. I.e. $0 \to FP_n \to FX_n \to FQ_n \to 0$ is exact. This follows from the fact that F is additive, so it preserves the biproduct and inclusion/projection morphisms into and out of each X_n. Lemma 3.1.12 implies that the resulting sequence is indeed exact.

Now, deleting the objects A, B, and C, we obtain an exact sequence of complexes

$$0 \longrightarrow FP_\bullet^A \longrightarrow FX_\bullet^B \longrightarrow FQ_\bullet^C \longrightarrow 0,$$

which, by Proposition 3.2.3(c), induces a long exact sequence in homology:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+1}"] & H_n(FP_\bullet^A) \arrow[r] & H_n(FX_\bullet^B) \arrow[r] & H_n(FQ_\bullet^C) \arrow[dll, "\delta_n"'] \\
& H_{n-1}(FP_\bullet^A) \arrow[r] & H_{n-1}(FX_\bullet^B) \arrow[r] & H_{n-1}(FQ_\bullet^C) \arrow[dll, dashed, "\delta_{n-1}", "\delta_1"'] \\
& H_0(FP_\bullet^A) \arrow[r] & H_0(FX_\bullet^B) \arrow[r] & H_0(FQ_\bullet^C) \arrow[r] & 0.
\end{tikzcd}$$

By definition of left derived functors, $H_n(FP_\bullet^A)$ is equal to L_nFA, and similar for B and C.[32] Therefore we obtain a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+1}"] & L_nFA \arrow[r] & L_nFB \arrow[r] & L_nFC \arrow[dll, "\delta_n"'] \\
& L_{n-1}FA \arrow[r] & L_{n-1}FB \arrow[r] & L_{n-1}FC \arrow[dll, dashed, "\delta_{n-1}", "\delta_1"'] \\
& L_0FA \arrow[r] & L_0FB \arrow[r] & L_0FC \arrow[r] & 0.
\end{tikzcd}$$

Note that the sequence terminates in 0, because any negative terms of a (deleted) projective resolution are defined to be zero objects and zero morphisms, which have zero homology.

If F is right exact, then there are natural isomorphisms $L_0FA \cong FA$, and $L_0FB \cong FB$, and $L_0FC \cong FC$ by Proposition 3.3.3(a). In this case, the long exact sequence indeed ends in

$$\cdots \longrightarrow L_1FC \xrightarrow{\delta_1} FA \longrightarrow FB \longrightarrow FC \longrightarrow 0,$$

which completes the proof. □

Left derived functors act as a measure of how close a right exact functor is to being exact. Indeed, by Proposition 3.3.3, the functor F is exact if and only if L_nF is the constant zero functor for every $n > 0$.

The prototypical example of a right exact functor is the *tensor product*. We now showcase some properties of the left derived functors of the tensor product.

Definition 3.3.6. Let R be a commutative ring with unity, and let T be an R-module. The left derived functors of $T \otimes_R - : \mathsf{Mod}_R \to \mathsf{Mod}_R$ are called the *Tor functors*, and are denoted $\mathrm{Tor}_n^R(T, -) := L_n(T \otimes_R -)$ for integers $n \geqslant 0$. ▼

Remark. Existence of the Tor functors relies on the fact that Mod_R is a category with enough projectives. This follows from the fact that every module has a free presentation, proven in Proposition 2.3.8. In particular, for any R-module M, the morphism $R^{\oplus I} \to M$ given in the proposition is a surjective homomorphism from a free module to M. Since free modules are projective, this proves that Mod_R has enough projectives.

Theorem 2.7.2 (p.58) of [Wei94] proves that, for all n, there is an isomorphism $\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^R(N, M)$ which is natural in M and N. This means that $\mathrm{Tor}_n^R(M, N)$ can be computed by a projective resolution of N, or a projective resolution of M.

Recall that we call the R-module T *flat* if $T \otimes_R -$ is an exact functor. Thus it follows that $\mathrm{Tor}_n^R(T, -)$ is zero for any $n > 0$ in this case. The following are more properties of the Tor functors, which also relate it to the torsion submodule:

Proposition 3.3.7.

[32] Note that the projective resolution $X_\bullet \to B \to 0$ may not be the same one used to define L_nB. However by Proposition 3.3.2, the derived functors are naturally isomorphic, so the distinction is not pertinent.

(a). *If R is a nonzero domain, and p is a nonzero element of R, then $\operatorname{Tor}_1^R(M, R/pR)$ is isomorphic to the* p-Torsion Submodule *defined as*

$$M[p] := \{m \in M \mid pm = 0\},$$

and $Tor_n^R(M, R/pR)$ is zero for $n > 1$. If R is a PID, this can be used to compute $\operatorname{Tor}_n^R(M, N)$ for any finitely generated N.

(b). *If R is a nonzero domain with field of fractions Q, then $\operatorname{Tor}_1^R(Q/R, M)$ is isomorphic to the torsion submodule of M (see Example 2.4.4 for details on the torsion submodule).*

(c). *If R is a PID, then $\operatorname{Tor}_n^R(M, N)$ is zero for all $n > 1$ and all R-modules M and N.*

Proof. (a). To compute $\operatorname{Tor}_1^R(M, R/pR)$, consider the exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot p} R \longrightarrow R/pR \longrightarrow 0$$

of R-modules where $R \to R/pR$ is the projection onto the quotient module. Because R is free, it is projective, making this a projective resolution of R/pR. To compute the Tor functors, we delete R/pR, apply $M \otimes_R -$ and compute the homology of the resulting chain complex. Note that $M \otimes_R R$ is naturally isomorphic to M, where $m \otimes r \mapsto rm$ is the isomorphism (see Proposition 2.2.2). The relevant complex is thus given by

$$0 \longrightarrow M \xrightarrow{\mu_p} M \longrightarrow 0,$$

where μ_p is the homomorphism sending m to pm. The first Tor functor, which is the first homology of the complex, is given by

$$\operatorname{Tor}_1^R(M, R/pR) = \ker \mu_p / \operatorname{im}(0 \to M) \cong \ker \mu_p = \{m \in M \mid pm = 0\},$$

which proves the statement. Because the other terms of the chain complex are zero, the higher Tor functors are zero as well.

If N is finitely generated, then there is an isomorphism

$$N \cong R^r \oplus R/a_1R \oplus \dots R/a_tR,$$

as is proven in [DF04, theorem 12.1.5, p.462], for $a_i \in R \setminus \{0\}$, and r and t nonnegative integers. Because the derived functors of an additive functor are additive as well, it follows that for all $n > 0$:

$$\operatorname{Tor}_n^R(M, N) \cong \operatorname{Tor}_n^R(M, R^r) \oplus \operatorname{Tor}_n^R(M, R/a_1R) \oplus \cdots \oplus \operatorname{Tor}_n^R(M, R/a_tR) \cong M[a_1] \oplus \cdots \oplus M[a_t].$$

Note that $\operatorname{Tor}_n^R(M, R^r)$ is zero because R^r is free.

(b). Consider the short exact sequence of R-modules

$$0 \longrightarrow R \longrightarrow Q \longrightarrow Q/R \longrightarrow 0$$

where the first map is the inclusion, and the second one the projection onto the quotient module. Theorem

3.3.5 implies there is a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^R(M,Q) \longrightarrow \mathrm{Tor}_1^R(M,Q/R) \xrightarrow{\delta_1} M\otimes_R R \longrightarrow M\otimes_R Q \longrightarrow M\otimes_R Q/R \longrightarrow 0.$$

The R-module R is free, and thus flat. By Proposition 2.5.6, it follows that Q, which is the localization $R_{(0)}$, is flat too.[33] Therefore $\mathrm{Tor}_1^R(M,Q)$ is zero. Using the fact that $M\otimes_R R$ is naturally isomorphic to M by Proposition 2.2.2, there is now an exact sequence (with removed final terms)

$$0 \longrightarrow \mathrm{Tor}_1^R(M,Q/R) \longrightarrow M \longrightarrow M\otimes_R Q.$$

Exactness of the sequence is equivalent to $\mathrm{Tor}_1^R(M,Q/R)\to M$ being the kernel of $M\to M\otimes_R Q$. Now because $Q=R_{(0)}$, we have that $M\otimes_R Q$ is naturally isomorphic to $M_{(0)}$ by Corollary 2.5.5. The isomorphism sends an elementary tensor $m\otimes r/a$ to $(rm)/a$.

Thus, finding the kernel of $M\to M\otimes_R Q$ is equivalent to finding the kernel of $M\to M_{(0)}$, which is the composition

$$m\mapsto m\otimes 1/1\mapsto m/1.$$

If the image $m/1$ is zero in $M_{(0)}$, there is an $x\in R\setminus(0)$ such that $xm=0$. In other words, m is an element of the torsion submodule of M. Conversely, if m is a torsion element with $xm=0$, then m is in the relevant kernel because $m/1=(xm)/x=0/x=0$.

So indeed, we conclude that $\mathrm{Tor}_1^R(M,Q/R)$ is the torsion submodule of M.

(c). Let M and N be modules over a PID R. As in the proof of Proposition 2.3.8, there is a surjective R-module-homomorphism $f:F\to N$ with F a free R-module. By including the kernel of this homomorphism, there is an exact sequence

$$0 \longrightarrow \ker f \longrightarrow F \xrightarrow{f} N \longrightarrow 0.$$

The kernel of f is a subgroup of F, and because F is free, so is $\ker f$. This follows from the fact that submodules of free modules are free over a PID, as is proven in detail in [AK17, theorem 4.12, p.29] (the proof in the case where the larger module is not finitely generated requires the Well-Ordering Theorem, which is equivalent to the Axiom of Choice, as is proven in [Bar14, theorem 2.11, p.2]).

The above sequence is, by freeness of the relevant terms, a projective resolution of N. Note that for any $n>1$, the n-th term of the resolution is zero. So $\mathrm{Tor}_n^R(M,N)$ is zero for these values of n. □

Historically, the Tor functors were introduced for abelian groups specifically. Given a free presentation $0\to F_1\to F_0\to A\to 0$ of an abelian group A (see part (c) above), the abelian group $\mathrm{Tor}(A,B)$ is defined as the kernel of $F_1\otimes_{\mathbb{Z}} B\to F_0\otimes_{\mathbb{Z}} B$. The name 'Tor' comes from 'torsion', which makes sense, as $\mathrm{Tor}(\mathbb{Z}/p\mathbb{Z},A)$ is a subgroup of the torsion group of A. The original name for $\mathrm{Tor}(A,B)$ is in fact the *torsion product* of A and B. See [CE56] for more historical context on the Tor functors.

The third part of the above proposition suggests that the Tor functors are a measure of 'how close' a module is to being flat. Over a PID, not every module is flat, but since the Tor functors only go up to degree 1, modules over PID's are not far off from being flat. This can be quantified by a ring's *Tor dimension*, which

[33] (0) is a prime ideal of R in this case. Indeed, if $ab=0$, then either a or b must be zero by definition of R being a domain.

is defined as

$$\operatorname{Tor} \dim R := \sup\{n \geqslant 0 \mid \operatorname{Tor}_n^R(M, N) \neq 0, \text{ for some } M, N \in \operatorname{Ob}(\mathsf{Mod}_R)\}.$$

(If there are modules over which none of the Tor functors vanish, we say the Tor dimension is infinity) With this terminology, part (c) of the above proposition states that the Tor dimension of a PID is at most 1.

To end this section, we run through the dual definitions and results, which regard *right derived functors*. These results are not proven, as the proofs are all dual to the corresponding results for left derived functors.

Definition 3.3.8. Let A and B be abelian categories, with A having enough injectives, and let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor. Given an object A of A, let $0 \to A \to I_\bullet$ be an injective resolution and consider the *deleted resolution* $I_\bullet^A$, where A is removed:

$$0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots .$$

The *n-th right derived functor* of F at A is defined as

$$R_n^{I_\bullet} F(A) := H_n(F I_\bullet^A),$$

where n is a nonnegative integer. ▼

Just like for left derived functors, right derived functors are independent of the chosen injective resolution, up to natural isomorphism:

Proposition 3.3.9. *Let $F : \mathsf{A} \to \mathsf{B}$ be as above. Given an object A of A and injective resolutions $0 \to A \to I_\bullet$ and $0 \to A \to J_\bullet$, there is a canonical natural isomorphism*

$$R_n^{I_\bullet} F(A) \cong R_n^{J_\bullet} F(A).$$

Proposition 3.3.10. *Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories, with A having enough injectives. The following hold:*

(a). *If F is left exact, then $R_0 F$ and F are naturally isomorphic.*

(b). *If F is exact, then $R_n F A = 0$ for all $n > 0$ and all objects A of A.*

(c). *If I is an injective object of A, then $R_n F I = 0$ for all $n > 0$.*

(d). *If $G : \mathsf{A} \to \mathsf{B}$ is a functor that is naturally isomorphic to F, then $R_n F$ and $R_n G$ are naturally isomorphic.*

The following is the result for which derived functors are the original motivation.

Theorem 3.3.11. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in an abelian category A with enough* *injectives, and let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor. There is a long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & R_0FA \arrow[r] & R_0FB \arrow[r] & R_0FC \arrow[dll, dashed, "\delta_0"{pos=0.2}, "\delta_{n-2}"{pos=0.8}] \\
& R_{n-1}FA \arrow[r] & R_{n-1}FB \arrow[r] & R_{n-1}FC \arrow[dll, "\delta_{n-1}"] \\
& R_nFA \arrow[r] & R_nFB \arrow[r] & R_nFC \arrow[r, "\delta_n"] & \cdots .
\end{tikzcd}$$

If F is left exact, then the sequence starts with

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \xrightarrow{\delta_0} R_1FA \longrightarrow \cdots .$$

It follows that the functor F is exact if and only if R_nF is the constant zero functor for every $n > 0$.

Definition 3.3.12. Let R be a commutative ring with unity, and let T be an R-module. The right derived functors of $\mathrm{Hom}_R(T, -) : \mathsf{Mod}_R \to \mathsf{Mod}_R$ are called the *Ext functors*, and are denoted $\mathrm{Ext}_n^R(T, -)$. Alternatively, $\mathrm{Ext}_n^R(-, T)$ are defined as the right derived functors of the contravariant Hom-functor, $\mathrm{Hom}_R(-, T)$. ▼

As highlighted in Example 3.2.6, a module P is projective if and only if $\mathrm{Hom}_R(P, -)$ is exact, which happens if and only if $\mathrm{Ext}_n^R(P, -)$ is zero for all $n > 0$ by Proposition 3.3.10(b). So where the Tor functors measure how close a module is to being flat, the Ext functors measure how close a module is to being projective.

Dually, a module I is injective if and only if $\mathrm{Hom}_R(-, I)$ is exact, which happens if and only if $\mathrm{Ext}_n^R(-, I)$ is zero for all $n > 0$. So the Ext functors can also be used to measure how close a module is to being injective.

There are many more derived functors between module categories to consider, but by Watts' Theorem (2.4.3 and 2.4.5), a large class of these are naturally isomorphic to the tensor product or Hom-functor. Proposition 3.3.3(d) implies that their derived functors can be computed using Tor and Ext.

3.4 Acyclic Resolutions and De Rham Cohomology

In many applications, finding projective and injective resolutions to compute derived functors can be quite a hassle. Thankfully there is an easier way to do so, namely through *acyclic resolutions*, which are resolutions where the objects vanish on derived functors. The main Theorem of this Section states that derived functors can be computed by the homology of a deleted acyclic resolution, after applying the functor. For this Section, we state and prove everything in the context of right derived functors, but as per usual, every definition and statement can be dualized for the context of left derived functors. At the end of this Section, we cover an example where this result is used in the field of *sheaf cohomology*, namely that the *de Rham cohomology* of a smooth manifold can be computed as the cohomology of a certain sheaf.

Definition 3.4.1. Let $F : \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories, with A having enough injectives. An object J in A is *right F-acyclic* or just *acyclic* if $R_nFJ = 0$ for all $n > 0$.

An exact sequence of the form

$$0 \longrightarrow A \longrightarrow J_0 \longrightarrow J_1 \longrightarrow J_2 \longrightarrow \cdots,$$

where each J_i is right F-acyclic is a *right F-acyclic resolution* of A, or just an *acyclic resolution* of A. ▼

Note that any injective object is right F-acyclic by Proposition 3.3.10(c). Before we can prove that derived functors can be proven using acyclic resolutions, we need a lemma that allows us to split an exact sequence apart along a cokernel:

Lemma 3.4.2. *Let $0 \to A \to X_0 \to X_1 \to \cdots$ be an exact sequence in an abelian category. There is a short exact sequence $0 \to A \to X_0 \to C \to 0$ and a long exact sequence $0 \to C \to X_1 \to X_2 \to \cdots$, where C is the cokernel of $A \to X_0$.*

Proof. Let $f : A \to X_0$ and $d_i : X_i \to X_{i+1}$ for $i \geqslant 0$ denote the above morphisms. Because f is monic, there is an exact sequence $0 \to A \xrightarrow{f} X_0 \xrightarrow{q} \operatorname{coker} f \to 0$.

Similarly, the other part of the exact sequence can be written as $0 \to \ker d_1 \xrightarrow{k} X_1 \to X_2 \to \cdots$. By exactness, $\ker d_1$ is naturally isomorphic to $\operatorname{im} d_0$. Now $\operatorname{im} d_0$ is naturally isomorphic to

$$\operatorname{coker}(\ker d_0) \cong \operatorname{coker}(\operatorname{im} f) \cong \operatorname{coker} f,$$

where the last natural isomorphism follows from theorem 2.11 (p.36) of [Fre64], which says that the kernel of a cokernel of a morphism is the original morphism again. In particular, the cokernel of the image of f is just the cokernel of f. Therefore, there is an exact sequence $0 \to \operatorname{coker} f \to X_1 \to X_2 \to \cdots$. □

We are now ready to state and prove the main result of this Section:

Theorem 3.4.3. *Let $F : \mathsf{A} \to \mathsf{B}$ be an additive left exact functor between abelian categories, with A having enough injections. Given an object A in A and a right F-acyclic resolution*

$$0 \longrightarrow A \xrightarrow{f} J_0 \xrightarrow{d_0} J_1 \xrightarrow{d_1} J_2 \xrightarrow{d_2} \cdots,$$

there is an isomorphism $R_n FA \cong H_n(FJ_\bullet^A)$, where $FJ_\bullet^A$ is the chain complex obtained by deleting A from the resolution, and applying F.

Proof. The case $n = 0$ is straightforward enough to verify. It follows from Proposition 3.3.10(a) that $R_0 FA \cong FA$. On the other hand, the zeroth homology of $FJ_\bullet^A$ is $\operatorname{coker}(0 \to \ker Fd_0)$. By left exactness of F applied to the original resolution, this is the same as $\operatorname{coker}(0 \to FA)$, which is just FA. Thus, we find $R_0 FA \cong FA \cong H_0(FJ_\bullet^A)$.

By Lemma 3.4.2, there are exact sequences

$$0 \longrightarrow A \xrightarrow{f} J_0 \xrightarrow{q} C \longrightarrow 0,$$

$$0 \longrightarrow C \xrightarrow{m} J_1 \xrightarrow{d_1} J_2 \xrightarrow{d_2} \cdots,$$

where C is the cokernel of f. Note that the second exact sequence is an acyclic resolution of C. By Theorem 3.3.5, there is a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & FA \arrow[r] & FJ_0 \arrow[r] & FC \arrow[dll] \\
& R_1FA \arrow[r] & R_1FJ_0 \arrow[r] & R_1FC \arrow[dll] \\
& R_2FA \arrow[r] & R_2FJ_0 \arrow[r] & R_2FC \arrow[dll] \\
& \cdots .
\end{tikzcd}$$

By acyclicity of J_0, each R_nFJ_0 is zero for $n > 0$. The exact sub-sequences $0 \to R_nFC \to R_{n+1}FA \to 0$ imply that R_nFC and $R_{n+1}FA$ are isomorphic for all $n > 1$.

We prove that $R_nFA \cong H_n(FJ_\bullet^A)$ for $n > 0$ using (strong) induction. Assume that, for *any* object B with acyclic resolution $0 \to B \to X_\bullet$, there is an isomorphism $R_nFB \cong H_n(FX_\bullet^B)$, where $n \in \{1, \ldots, N-1\}$ for some integer $N > 1$. It follows that

$$R_NFA \cong R_{N-1}FC \cong H_{N-1}(FX_\bullet^C) = H_N(FJ_\bullet^A),$$

where $0 \to C \to X_\bullet$ is the acyclic resolution of C, defined by $X_i = J_{i+1}$. So by induction, we obtain $R_nFA \cong H_n(FJ_\bullet^A)$ for all $n > 0$, assuming $R_1FA \cong H_1(FJ_\bullet^A)$, which we now show.

To prove $R_1FA \cong H_1(FJ_\bullet^A)$, note that we have an exact sequence $0 \to FA \to FJ_0 \to FC \to R_1FA \to 0$ by acyclicity of J_0, so R_1FA is naturally isomorphic to the cokernel of $Fq : FJ_0 \to FC$. The image of Fq is naturally isomorphic the image of Fd_0. This follows from

$$\operatorname{im} Fq \cong \operatorname{coker}(\ker Fq) \cong \operatorname{coker} Ff \cong \operatorname{coker} \ker Fd_0 = \operatorname{im} Fd_0,$$

where the second isomorphism follows from exactness of $0 \to FA \to FJ_0 \to FC$ by F being left exact, and the third isomorphism follows from exactness of $0 \to FA \to FJ_0 \to FJ_1$.

Using this, we can compute $H_1(FJ_\bullet^A)$ as:

$$\begin{aligned} H_1(FJ_\bullet^A) &= \operatorname{coker}(\operatorname{im} Fd_0 \to \ker Fd_1) \\ &\cong \operatorname{coker}(\operatorname{im} Fq \to FC) \\ &\cong \operatorname{coker}(Fq) \cong R_1FA. \end{aligned}$$

The isomorphism $\ker Fd_1 \cong FC$ follows from exactness of $0 \to FC \to FJ_0 \to FJ_1$. Now that we have shown $R_1FA \cong H_1(FJ_\bullet^A)$, the above induction argument implies that $R_nFA \cong H_n(FJ_\bullet^A)$ for all $n \geqslant 0$. $\square$

Now we discuss an example of where this Theorem may be useful, namely in the context of *sheaf cohomology*. We show that the *de Rham cohomology* of a smooth manifold can be computed as the cohomology of a certain sheaf over this manifold. See Appendix A for a brief summary of necessary concepts of sheaves and sheaf cohomology.

Example 3.4.4. Throughout this example, we follow the notation and conventions of elementary differential geometry from [Ser23]. This includes the definitions of smooth manifolds, coordinate charts, pullbacks, differential forms, and the exterior derivative. Let M be a (C^∞-)smooth real manifold of finite dimension n.

For any $k \geqslant 0$ and any open set U of M, we define $\Omega^k(U)$ as the abelian group (or $C^\infty(M,\mathbb{R})$-module) of smooth differential k-forms on U under addition. Note that $\Omega^0(U)$ is the same as the group $C^\infty(U,\mathbb{R})$ of smooth real functions on U.

For any k, the abelian groups $\Omega^k(U)$ assemble into a presheaf $\Omega^k : \mathsf{T}_M^{\mathrm{op}} \to \mathsf{Ab}$, where the restriction homomorphism $\Omega^k(V) \to \Omega^k(U)$ is the restriction of forms onto a smaller domain. Equivalently, if we denote $\iota : U \hookrightarrow V$ as the inclusion, then the restriction of a k-form ω on V is equal to the pullback $\iota^*\omega$. Differential forms are locally defined, in that two k-forms on U are equal if and only if they agree on all points of U. This immediately implies the locality condition for Ω^k to be a sheaf. As for gluing, if we have a collection of forms $\{\omega_i \in \Omega^k(U_i)\}_i$ for an open cover $\{U_i\}_i$ of U such that all ω_i and ω_j agree on the intersection of U_i and U_j, then we can glue these together by defining the form ω by

$$\omega(p) := \omega_i(p)$$

where U_i contains p. This is well-defined by assumption, and ω is actually smooth because we can restrict to a small subset fully contained in U_i where ω_i is smooth, making ω smooth at p. Thus, Ω^k is actually a sheaf on M.

The exterior derivative defines a sheaf morphism $d^k : \Omega^k \Rightarrow \Omega^{k+1}$, where the component d^k_U is the exterior derivative on $\Omega^k(U)$. A local computation on charts shows that $\iota^*(d\omega) = d(\iota^*\omega)$ for $\omega \in \Omega^k(V)$, where $\iota : U \hookrightarrow V$ is the inclusion. Because the exterior derivative commutes with the restriction, it follows that each d^k is actually a sheaf morphism.

One of the most important properties of the exterior derivative is that $dd\omega = 0$ for any form ω. Thus, for all open U there is a chain complex

$$\Omega^0(U) \xrightarrow{d^0_U} \cdots \xrightarrow{d^{k-2}_U} \Omega^{k-1}(U) \xrightarrow{d^{k-1}_U} \Omega^k(U) \xrightarrow{d^k_U} \Omega^{k+1}(U) \xrightarrow{d^{k+1}_U} \cdots \xrightarrow{d^{n-1}_U} \Omega^m(U).$$

Classically, the k-th *De Rham cohomology* group of M is defined as $H^k_{\mathrm{dR}}(M) := \ker d^k_M / \operatorname{im} d^{k-1}_M$. What we show now is that these groups can also be computed as the cohomology of a certain sheaf. This starts by first showing that

$$\Omega^0 \overset{d^0}{\Longrightarrow} \cdots \overset{d^{k-2}}{\Longrightarrow} \Omega^{k-1} \overset{d^{k-1}}{\Longrightarrow} \Omega^k \overset{d^k}{\Longrightarrow} \Omega^{k+1} \overset{d^{k+1}}{\Longrightarrow} \cdots \overset{d^{n-1}}{\Longrightarrow} \Omega^n$$

is an exact sequence of sheaves. To that end, the sequence is exact at Ω^k if and only if it is exact at the stalk Ω^k_p for all $p \in M$. I.e. for any $p \in M$, we consider the homomorphisms

$$\Omega^{k-1}_p \xrightarrow{d^{k-1}_p} \Omega^k_p \xrightarrow{d^k_p} \Omega^{k+1}_p$$

and we show that $\operatorname{im} d^{k-1}_p = \ker d^k_p$. This follows from Poincaré's Lemma (see [Ser23, corollary 10.0.17, p. 134]), which states that any k-form ω on M such that $d\omega = 0$, there is a $(k-1)$-form ψ such that $d\psi|_C = \omega|_C$, where C is a chart on M. Specifically, let (U, ω) be an element of $\ker d^k_p$. By how we defined stalks, this element is equal to (C, ω), where C is a chart of M containing p. By assumption,

$$d^k_p(C, \omega) = (C, d\omega) = (C, 0).$$

By Poincaré's Lemma, there is a $(k-1)$-form ψ such that $(C, \omega) = (C, d\psi)$, which is in the image of d^{k-1}_p.

Thus $\operatorname{im} d_p^{k-1} = \ker d_p^k$, as the other inclusion follows immediately from $dd\omega = 0$, and so

$$\Omega^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{k-2}} \Omega^{k-1} \xrightarrow{d^{k-1}} \Omega^k \xrightarrow{d^k} \Omega^{k+1} \xrightarrow{d^{k+1}} \cdots \xrightarrow{d^{n-1}} \Omega^n$$

is an exact sequence of sheaves.

We would like to compute the kernel of d^0. This is equal to the sheaf that sends an open subset U to $\ker(d_U^0)$, which consists of all 0-forms (i.e. smooth maps $U \to \mathbb{R}$) whose exterior derivatives vanish on U. If $C \subseteq U$ is a chart of M with coordinate maps $x^i : C \to \mathbb{R}$, then the exterior derivative of f on C is equal to $df = \sum_i \frac{\partial f}{\partial x^i}\, dx^i$. If df vanishes on U, then every partial derivative of f must vanish on any chart C in U. Smooth functions with this property are the *locally constant* ones, which are functions that are constant on each connected component of M. We denote the sheaf of locally constant smooth maps as $\underline{\mathbb{R}}$, which fits into the exact sequence as

$$0 \Longrightarrow \underline{\mathbb{R}} \Longrightarrow \Omega^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{k-2}} \Omega^{k-1} \xrightarrow{d^{k-1}} \Omega^k \xrightarrow{d^k} \Omega^{k+1} \xrightarrow{d^{k+1}} \cdots \xrightarrow{d^{n-1}} \Omega^n.$$

Extending on the right by the cokernel of d^{n-1} and an infinite amount of zero sheaves, the above becomes a resolution of $\underline{\mathbb{R}}$.

Each sheaf Ω^k is fine as well. This follows from theorem 1.4.6 (p.24) of [Ser23], which states that for any open cover $\{U_i\}_i$ of an open set U, there is a family of smooth maps $\rho_i : M \to \mathbb{R}$ satisfying the properties of a partition of unity, as defined in definition 1.4.4 of the same book. We can now define a family of sheaf morphisms $\{\eta_i : \Omega^k \Rightarrow \Omega^k\}_i$ by $(\eta_i)_U : \omega \mapsto \rho_i \omega$. This makes the η_i a sheaf partition of unity subordinate to the open cover $\{U_i\}_i$ of U, turning Ω^k into a fine sheaf.

Now finally, since fine sheaves are acyclic, $0 \Rightarrow \underline{\mathbb{R}} \Rightarrow \Omega^\bullet$ is an acyclic resolution of $\underline{\mathbb{R}}$ Thus, the sheaf cohomology of $\underline{\mathbb{R}}$ satisfies, for all integers $k \geqslant 0$,

$$H^k(M, \underline{\mathbb{R}}) := R_k \Gamma_M \cong H_k(\Omega^\bullet(M)) =: H^k_{\mathrm{dR}}(M),$$

where Γ_M is the global sections functor. This shows that the classical de Rham cohomology groups can be computed as the cohomology of a specific sheaf. □

The example above is adapted from [Mik20]. The same thesis also provides other examples where sheaf cohomology can be used to construct other cohomological theories. One such example is *singular cohomology*, which acts as a 'dual' theory to that of singular homology, as described in Example 3.2.4. In particular, it states the k-th singular cohomology group of a topological space X is isomorphic to the k-th sheaf cohomology group of the sheaf that assigns the abelian group $\mathbb{Z}$ to each open subset, and where each restriction homomorphism is the identity.

4 Discussion and Generalizations

In this thesis we have summarized the basic theory of categories. We have seen how categories generalize various other fields of mathematics, by studying the way objects relate to one another in terms of morphisms. Functors and natural transformations are the 'higher-level' analogues of this idea, allowing us to compare categories and functors respectively. In modern category theory, much of current research is spent trying to understand such higher-level structures. Namely, given a positive integer n, a *n-category* consists of a collection of objects, a collection of 1-morphisms between objects, and for every $j \in \{2, \dots, n\}$, a collection of j-morphisms between $(j-1)$-morphisms. All these morphisms have various composition rules that keep everything well-defined (at least up to equivalence, where we say two j-morphisms are equivalent if they are equal up to a $(j+1)$-morphism). The paper [Bae05] gives a more detailed introduction to n-categories, including some applications to homotopy theory and topological quantum field theory.

In a limiting sense, one can define an *∞-category* as a category with j-morphisms for any positive integer j, not just those with indices bounded by some n. The book [RV22] by Riehl and Verity developes the main concepts of ∞-category theory in much more detail than is possible here. The book also includes an appendix on 2-categories and 2-functors, which serves as a good summary of the topic.

The main result of Chapter 2, Watts' Theorem, can also be generalized in various ways. The paper [NS16] discuss a generalization of this theorem. Namely, given a commutative ring R, a cocomplete abelian category A enriched over Mod_R (see the footnote in Definition 3.1.1), we define an R-module in A as a pair (M, ρ), where M is an object of A and $\rho : R \to \mathsf{A}(M, M)$ is a homomorphism of R-algebras. In this setting, one can define the tensor product $- \otimes_R M$ as the unique (up to natural isomorphism) functor from Mod_R to A such that $R \otimes_R M \cong M$ and $- \otimes_R M$ is right exact and preserves direct sums. The fact that this is unique up to natural isomorphism is the general Watts' Theorem, but the main point of the paper cited above is to prove a result analogous to Proposition 2.4.6 in these more general categories.

Another generalization is done in [Hov09], where the author proves a version of Watts' Theorem for more general categories in which some form of homotopy can be done. These include, but are not limited to, (a subcategory of) Top, $\mathsf{Ch}(\mathsf{Ab})$, and Mod_R. Specifically, the paper works in certain classes of *closed symmetric monoidal categories*, which are categories M with a symmetric bifunctor $- \otimes - : \mathsf{M} \times \mathsf{M} \to \mathsf{M}$ satisfying a number of axioms (see, e.g., epilogue E2 of [Rie16] for detials). What makes a symmetric monoidal category *closed* is that the functor $M \otimes -$ has a specific right adjoint for every object M of M.

We have only scratched the surface of the theory of homological algebra, so there is much more to research and generalize. One concept not covered in the main text is that of *δ-Functors*. Given abelian categories A and B, a δ-functor is a collection of functors $\{T_i : \mathsf{A} \to \mathsf{B}\}_{i \in \mathbb{Z}_{\geqslant 0}}$, together with morphisms $\delta_n : T_n C \to T_{n-1} A$ for each short exact sequence $0 \to A \to B \to C \to 0$ in A. Along with this, we require that, given an exact sequence as above, there is a long exact sequence

$$\cdots \to T_{n+1} C \xrightarrow{\delta_{n+1}} T_n A \to T_n B \to T_n C \xrightarrow{\delta_n} T_{n-1} A \to \cdots ,$$

and a chain map between short exact sequences induces a chain map between the above long exact sequences. A nice result regarding these derived functors is that if $F : \mathsf{A} \to \mathsf{B}$ is an additive right exact functor, with A having enough projections, then the derived functors $L_i F$ form a 'universal' δ-functor, as defined in chapter III.1 of [Har77]. Moreover, if $\{T_i\}$ is a universal δ-functor, and T_0 is right exact, then T_i is naturally isomorphic to $L_i T_0$ for all i. Of course, this entire construction can be dualized to define δ-functors generalizing $R_i F$.

See [Wei94, chapter 2.1] for more details regarding δ-functors.

Another way to compute certain (co)homological theories on topological spaces is through *Čech cohomology*. This is another collection of cohomology groups that use the local data of a sheaf to give invariants of the topological space. See [Har77, chapter III.4] for details. The upshot is that Čech cohomology and sheaf cohomology coincide under certain conditions. This is helpful because, by its very nature, Čech cohomology lends itself to easier computations than sheaf cohomology does.

A Sheaves and their Cohomology

This appendix summarizes the basic concepts of sheaf theory and sheaf cohomology required for Example 3.4.4. Loosely stated, a *sheaf* is a collection of abelian groups corresponding to some local data of a topological space. Sheaves are useful because they allow us to make precise statements about certian local properties of a space. We do not prove any of the statements here, but we do provide citations to proofs whenever necessary. The definitions are adapted from [Har77] and [Rot09]. Throughout this appendix, X is a topological space, and $\mathsf{T}_X := (O(X), \subseteq)$ is its poset category of open subsets.

Definition A.1. A *presheaf* of abelian groups on X is a functor $F : \mathsf{T}_X^{\mathrm{op}} \to \mathsf{Ab}$.[34] The relation $U \subseteq V$ is mapped to the *restriction homomorphism* $r_{V,U} : FV \to FU$. The elements of the abelian group FV are called *sections* of F over V, and we denote the image of a section $s \in FV$ under the restriction as $s|_U := r_{V,U}(s)$. A *sheaf* is a presheaf F satisfying the following two conditions, where $\{U_i\}_{i\in I}$ is any open cover of an open subset U of X:

- (*locality*) Given sections s and t in FU, if $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
- (*gluing*) Given a collection of sections $\{s_i \in FU_i\}_{i\in I}$, if $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there is a section $s \in FU$ such that $s|_{U_i} = s_i$.

A *morphism* between sheaves F and G is a natural transformation $\eta : F \Rightarrow G$. We denote $\mathsf{Sheaf}(X)$ as the category of sheaves on X, which is a subcategory of $[\mathsf{T}_X^{\mathrm{op}}, \mathsf{Ab}]$. ▼

The main motivating examples of sheaves are ones that assign a set of functions to each open subset. For example, there is a sheaf $C^0(-,\mathbb{R})$ that assigns, to each open subset U, the abelian group of continuous functions $U \to \mathbb{R}$ with pointwise addition. The restriction homomorphism from $C^0(V,\mathbb{R}) \to C^0(U,\mathbb{R})$ sends a function $f : V \to \mathbb{R}$ to the restriction $f|_U : U \to \mathbb{R}$. Other examples include the sheaf $C^\infty(-,\mathbb{R})$ of smooth real-valued functions if X is a smooth manifold, and the sheaf O of holomorphic functions if X is a complex manifold.

Note that the locality condition implies that the section s from the gluing condition is unique. The two sheaf conditions are equivalent to saying that the following is an exact sequence in Ab:

$$0 \longrightarrow FU \longrightarrow \textstyle\prod_i FU_i \longrightarrow \prod_{i,j} F(U_i \cap U_j),$$

where $FU \to \prod_i FU_i$ sends a section s to the i-indexed sequence $(s|_{U_i})_i$, and $\prod_i FU_i \to \prod_{i,j} F(U_i \cap U_j)$ sends a sequence $(s_i)_i$ to $(s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$.

Definition A.2. Let p be a point in X. The *stalk* of a sheaf F at p, denoted F_p, is the colimit of the diagram $F \circ I : \mathsf{T}_{X,p}^{\mathrm{op}} \to \mathsf{Ab}$, where $\mathsf{T}_{X,p}$ is the full subcategory of T_X containing only the open sets that contain p, and $I : \mathsf{T}_{X,p}^{\mathrm{op}} \to \mathsf{T}_X^{\mathrm{op}}$ is the inclusion functor.

More concretely, elements of F_p are pairs (U, s), with $s \in U$, subject to the relation that $(U, s) = (U', s')$ if and only if there is an open subset $W \subseteq U \cap U'$ such that $s|_W = s'|_W$. The legs of the cocone of the colimit are homomorphisms $FU \to F_p$ that send a section s to the pair (U, s). ▼

A morphism $\eta : F \Rightarrow G$ of sheaves induces a homomorphism $\eta_p : F_p \to G_p$ on the stalks, defined by taking a pair (U, s) to $(U, \eta_U(s))$. An important property of these induced maps is that the morphism η is an

[34] One could also consider functors from $\mathsf{T}_X^{\mathrm{op}}$ to any category C. For example a functor $\mathsf{T}_X^{\mathrm{op}} \to \mathsf{Ring}$ is a presheaf of rings. However, here we only consider presheaves of abelian groups.

isomorphism of sheaves if and only if every η_p is an isomorphism of abelian groups for all $p \in X$, which is proven in [Har77, proposition II.1.1, p.63].

Definition A.3.

- The *zero sheaf* on X, denoted 0, associates the trivial group to every open subset of X, and $r_{V,U}$ is the zero homomorphism for all open set U and V.
- Given sheaves F and G on X, and morphisms η and ε from F to G, their *sum* $\eta + \varepsilon$ is defined on components as $(\eta + \varepsilon)_U := \eta_U + \varepsilon_U$, where the latter is the sum of homomorphisms in Ab. This turns $\mathsf{Sheaf}(X)(F, G)$ into an abelian group.
- Given sheaves F and G on X, their *direct sum*, denoted $F \oplus G$, is defined on open subsets U of X by the direct sum of abelian group $(F \oplus G)(U) := FU \oplus GU$.
- Given a morphism $\eta : F \Rightarrow G$ of sheaves on X, we define its *kernel* as the sheaf $\ker \eta : U \mapsto \ker \varphi_U$. ▼

Remark. In general, the presheaf $U \mapsto \operatorname{coker} \eta_U$ is not a sheaf, so instead we define the *cokernel* of a morphism of sheaves η as the *sheaf associated with the presheaf*, as defined and proven to exist in [Har77, definition-proposition II.1.2, p.64].

The definitions above are enough to show the following, which is proven in [Rot09, theorem 5.91, p.309].

Proposition A.4. *$\mathsf{Sheaf}(X)$ is an abelian category.*

As an abelian category, definitions of images, coimages, exact sequences, chain complexes, injective and projective sheaves carry over from arbitrary abelian categories. A useful result is that exactness of sheaves is a very local property, as is proven in [Rot09, theorem 5.85, p.300]:

Proposition A.5. *The sequence*

$$\cdots \Longrightarrow F_{n-1} \Longrightarrow F_n \Longrightarrow F_{n+1} \Longrightarrow \cdots$$

in $\mathsf{Sheaf}(X)$ is exact if and only if the induced sequence on stalks

$$\cdots \longrightarrow (F_{n-1})_p \longrightarrow (F_n)_p \longrightarrow (F_{n+1})_p \longrightarrow \cdots$$

is exact in Ab for all points $p \in X$.

The functor we use to define sheaf cohomology is the following:

Definition A.6. Given an open subset $U \subseteq X$, the *global sections functor* is a functor $\Gamma_X : \mathsf{Sheaf}(X) \to \mathsf{Ab}$ that sends a sheaf F to $\Gamma_X(F) := FX$, and a morphism of sheaves $\eta : F \Rightarrow G$ to the group-homomorphism $\eta_X : FX \to GX$. ▼

This global sections functor is left exact, as is shown in [Rot09, lemma 6.68, p.378]. Proposition 5.97 (p.314) of the same book shows that $\mathsf{Sheaf}(X)$ has enough injections, meaning we can nicely define the right derived functors of Γ_X:

Definition A.7. We define the n-th *sheaf cohomology functor* $H^n(X, -)$ to be the n-th right derived functor of Γ_X. ▼

We would like to compute this sheaf cohomology using Theorem 3.4.3, which means we need a characterization of acyclic objects in $\mathsf{Sheaf}(X)$. One particularly nice class of acyclic sheaves are the following:

Definition A.8. A sheaf F is *fine* if, for any locally finite[35] open cover $\{U_i\}_{i\in I}$ of an open set U, there is a family of sheaf morphisms $\{\eta_i : F \Rightarrow F\}_{i\in I}$ such that:

- For all $i \in I$, the set $\{p \in X \mid (\eta_i)_p \neq 0\}$, called the *support* of η_i, is contained in U_i (here $(\eta_i)_p$ denotes the homomorphism of stalks $F_p \to F_p$);
- For all $p \in X$, the sum of homomorphisms $\sum_{i\in I} (\eta_i)_p$ is the identity homomorphism on the stalk F_p.

Such a family of sheaf morphisms is called a *sheaf partition of unity subordinate to* $\{U_i\}_i$. ▼

As is shown in [Wel07, propositions 3.5, 3.11, p.53, 56], any fine sheaf is also a so-called *soft* sheaf, and any soft sheaf is acyclic. Thus, if F is fine, then $H^n(X, F) = 0$ for all $n > 0$.

Remark. Usually, fine sheaves are only defined if X is a *paracompact space*, which are spaces for which every open cover can be refined to a locally finite cover. In Example 3.4.4, we consider the case where X is a smooth manifold, which is always paracompact by the requirement of a manifold being second-countable.

[35] An open cover is *locally finite* if for all points $p \in X$, there is an open neighbourhood $\tilde{U}$ of p such that $\tilde{U}$ only intersects finitely many of the covering sets. The requirement of such a covering being locally finite guarentees that the sum of homomorphisms in the definition is well-defined.

References

[Abb15] S. Abbott. *Understanding Analysis.* Springer New York, NY, 2nd edition, 2015. ISBN: 978-1-4939-2711-1.

[AK17] A. Altman and S. Kleiman. *A Term of Commutative Algebra.* Worldwide Center of Mathematics, 1st edition, 2017. ISBN: 978-0-9885572-1-5.

[Alu09] P. Aluffi. *Algebra: Chapter 0.* American Mathematical Society, 1st edition, 2009. ISBN: 978-0-821-84781-7.

[Arm83] M. A. Armstrong. *Basic Topology.* Springer New York, NY, 1st edition, 1983. ISBN: 978-0-387-90839-7.

[Bae05] J. C. Baez. An Introduction to n-Categories. *Springer, Berlin, Heidelberg*, 2005. https://doi.org/10.1007/BFb0026978.

[Bar14] K. Barnum. The Axiom of Choice and its Implications. *The University of Chicago*, 2014. Link: https://math.uchicago.edu/~may/REU2014/REUPapers/Barnum.pdf.

[CE56] H. Cartan and S. Eilenberg. *Homological Algebra.* Princeton University Press, 1st edition, 1956. ISBN:978-0691049915.

[DF04] D. S. Dummit and R. M. Foote. *Abstract Algebra.* John Wiley and Sons, 3rd edition, 2004. ISBN: 978-0471433347.

[Eil60] S. Eilenberg. Abstract Description of some Basic Functors. *The Journal of the Indian Mathematical Society*, 24(1-2):231–234, 1960. https://www.informaticsjournals.com/index.php/jims/article/view/16908.

[EM45] S. Eilenberg and S. MacLane. General Theory of Natural Equivalences. *Transactions of the American Mathematical Society*, 58(2):231–294, 1945. https://doi.org/10.2307/1990284.

[Fre64] P. Freyd. *Abelian Categories, an Introduction to the Theory of Functors.* Harper & Row, 1st edition, 1964. ISBN:978-0-598-55334-8.

[Fre66] P. Freyd. Stable Homotopy. *Springer, Berlin, Heidelberg*, 1966. https://doi.org/10.1007/978-3-642-99902-4_5.

[Har77] R. Hartshorne. *Algebraic Geometry.* Springer New York, NY, 1st edition, 1977. ISBN: 978-0-387-90244-9.

[Hat01] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 1st edition, 2001. ISBN: 978-0521795401.

[Hov09] M. Hovey. The Eilenberg-Watts Theorem in Homotopical Algebra. *arXiv: Algebraic Topology*, 2009. https://doi.org/10.48550/arXiv.0910.3842.

[HS97] P.J. Hilton and U. Stammbach. *A Course in Homological Algebra.* Springer New York, NY, 2nd edition, 1997. ISBN: 978-0-387-94823-2.

[Jec07] T. Jech. *Set Theory.* Springer New York, NY, 3rd edition, 2007. ISBN: 978-3-540-44761-0.

[Loa00] C. F. Van Loan. The Ubiquitous Kronecker Product. *Journal of Computational and Applied Mathematics*, 123(1):85–100, 2000. https://doi.org/10.1016/S0377-0427(00)00393-9.

[LOT17] H. Lenstra, F. Oort, and J. Top. *Algebraic Structures*. University of Groningen, 2017.

[Mac98] S. MacLane. *Categories for the Working Mathematician*. Springer New York, NY, 2st edition, 1998. ISBN: 978-1-4757-4721-8.

[Mes07] S. Messick. Limits In Category Theory. *University of Chicago*, 2007. https://math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALAPP/Messick.pdf.

[Mik20] R. Mike. An Introduction to Sheaf Cohomology in Algebraic Geometry. *University of Colorado Boulder, Department of Mathematics*, 2020. https://scholar.colorado.edu/concern/undergraduate_honors_theses/w0892c024.

[nLa23] nLab. Adjoint Functor. https://ncatlab.org/nlab/show/adjoint+functor#basic_properties, March 2023. .

[NS16] A. Nyman and S. P. Smith. A Generalization of Watts's Theorem: Right Exact Functors on Module Categories. *Communications in Algebra*, 44(7):3160–3170, 2016. https://doi.org/10.1080/00927872.2015.1065873.

[Rie14] E. Riehl. *Categorical Homotopy Theory*. Cambridge University Press, 1st edition, 2014. ISBN: 978-1-107-26145-7.

[Rie16] E. Riehl. *Category Theory in Context*. Dover Publications, 1st edition, 2016. ISBN: 978-0-486-80903-8.

[Rot09] J. J. Rotman. *An Introduction To Homological Algebra*. Springer New York, NY, 2nd edition, 2009. ISBN: 978-0-387-68324-9.

[RV22] E. Riehl and D. Verity. *Elements of ∞-Category Theory*. Cambridge University Press, 1st edition, 2022. ISBN: 9781108936880.

[Ser91] J.-P. Serre. Motifs. *Journées arithmétiques de Luminy*, 198-199-200:333–349, 1991. http://www.numdam.org/item/AST_1991__198-199-200__333_0/.

[Ser23] M. Seri. *Analysis On Manifolds*. University of Groningen, 2023. https://github.com/mseri/AoM/releases.

[Shu08] M. A. Shulman. Set Theory for Category Theory. *arXiv: Category Theory*, 2008. https://doi.org/10.48550/arXiv.0810.1279.

[Tae18] L. Taelman. *Modules and Categories*. University of Amsterdam, 2018. https://staff.fnwi.uva.nl/l.d.j.taelman/ca.pdf.

[Use14] User:YassineMrabet. A wireframe render of a torus with independent cycles in red and blue. https://en.m.wikipedia.org/wiki/File:Simple_torus_with_cycles.svg, March 2014.

[vGLOT17] B. van Geemen, H. Lenstra, F. Oort, and J. Top. *Advanced Algebraic Structures*. University of Groningen, 2017.

[Wat60] C. E. Watts. Intrinsic Characterizations of some Additive Functors. *Proc. Amer. Math. Soc.*, 11:5–8, 1960. https://doi.org/10.1090/S0002-9939-1960-0118757-0.

[Wei94] C. A. Weibel. *An Introduction To Homological Algebra*. Cambridge University Press, 1st edition, 1994. ISBN: 9781139644136.

[Wel07] R. O. Wells. *Differential Analysis on Complex Manifolds*. Springer New York, NY, 3rd edition, 2007. ISBN: 978-0-387-73892-5.

Glossary of Notation

$f : A \to B$	Morphism between two objects. 5
1_A	Identity morphism. 5
$g \circ f$	Composite morphism. 5
▼	Marker for the end of a definition. 5
$\mathsf{Ob}(\mathsf{C})$	Collection of objects. 5
$\mathsf{C}(A, B)$	Collection of morphisms between two objects. 5
$\mathsf{Hom}(A, B)$	Collection of morphisms between two objects. 5, 50, 71
$\dashrightarrow$	Emphasized arrow. 6
(†)	Marker for an important example. 6
Set	Category of sets. 6
Top	Category of topological spaces. 6
Eucl	Category of Euclidean spaces. 6
Man	Category of smooth manifolds. 6
Set_*	Category of pointed sets. 6
Top_*	Category of pointed topological spaces. 6
Eucl_*	Category of pointed Euclidean spaces. 6
Man_*	Category of pointed smooth manifolds. 6
(X, x)	Object of a category with pointed objects. 6
Group	Category of groups. 6
Ring	Category of rings. 6, 105
Field	Category of fields. 6
Monoid	Category of monoids. 6
$\mathbb{Z}$	Set of integers. 6
Mod_R	Category of left R-modules. 7, 43, 73
Ab	Category of abelian groups. 7, 71, 105
Vect_K	Category of K-vector spaces. 7, 73
${}_R\mathsf{Mod}$	Category of right R-modules. 7
Quiver	Category of quivers. 7
C^{op}	Opposite category. 7, 77
$:=$	Is defined as. 7
Mat_R	Category of matrices over R. 7, 73
$\mathsf{B}G$	One-object group category. 7
G^{op}	Opposite group. 7
$(P, \leqslant)$	Poset category. 7, 105
Htpy	Category of topological spaces and homotopy classes. 8
Htpy_*	Category of pointed Category of pointed topological spaces and homotopy classes. 8
$\mathbf{0}$	Empty category. 8
$\mathbf{1}$	Category with one object and one morphism. 8
$\mathbf{2}$	Category with two objects and one non-identity morphism. 8
$\mathbf{n}$	Poset category $(\{1, \ldots, n\}, \leqslant)$. 8
$\mathsf{C} \times \mathsf{D}$	Product category. 8

r/a, $\frac{r}{a}$	Element of the localization. 65
$\mathfrak{p}$	Prime ideal of a ring. 66
$\mathfrak{a} \trianglelefteq R$	Ideal of a ring. 66
$R_{\mathfrak{p}}$	Localization at a prime ideal. 66, 96
$A^{-1}M$	Localization of a module by a multiplicative set. 66
$\hat{f} : A^{-1}M \to A^{-1}N$	Homomorphism induced by the localization functor. 66
$M_{\mathfrak{p}}$	Localization of a module at a prime ideal. 68, 96
$\mathrm{Ann}(a)$	Annihilator of an element of a ring. 68
$\mathfrak{m}$	Maximal ideal of a ring. 69
$\mathrm{coim}\, f$	Coimage of a morphism. 74
$\mathsf{Ab}^{\text{tor-free}}$	Category of torsion-free abelian groups. 78
$(A_\bullet, d_\bullet)$	Chain complex with boundary morphisms. 81
$H_i(A_\bullet)$	i-th homology object of a chain complex. 81
$\mathsf{Ch}(\mathsf{A})$	Category of chain complexes. 81
Δ^n	Standard n-simplex. 84
$[p_0, \ldots, p_n]$	n-simplex defined by $n+1$ points. 84
$C_\bullet(X)$	Simplicial chain complex of a topological space. 84
S^1	Circle space. 85
T	Torus space. 85
$P_\bullet \to A \to 0$	Projective resolution of an object. 86
$0 \to A \to I_\bullet$	Injective resolution of an object. 86
$P_\bullet^A$	Deleted projective resolution. 89
$L_n F$	n-th left derived functor of an additive functor. 90
$\mathrm{Tor}_n^R(M, N)$	n-th Tor functor of two R-modules. 94
$M[p]$	p-torsion submodule. 95
$\mathrm{Tor}(A, B)$	Torsion product of two abelian groups. 96
$\mathrm{Tor}\dim R$	Tor dimension of a ring. 97
$I_\bullet^A$	Deleted injective resolution. 97
$R_n F$	n-th right derived funtor of an additive functor. 97
$\mathrm{Ext}_n^R(M, N)$	n-th Ext functors of two R-modules. 98
Ω^k	Sheaf of smooth differential k-forms. 101
T_X	Poset of open subsets of a topological space. 101, 105
$f^*\omega$	Pullback of a smooth map, applied to a differential form. 101
$H^n_{\mathrm{dR}}(M)$	n-th de Rham cohomology group of a smooth manifold. 101
F_p	Stalk of a sheaf at a point. 101, 105
$\frac{\partial f}{\partial x}$	Partial derivative of a smooth map with respect to the coordinate x. 102
$\underline{\mathbb{R}}$	Sheaf of locally constant real functions. 102
$H^n(X, F)$	n-th sheaf cohomology object. 102, 106
$\Gamma_X : \mathsf{Sheaf}(X) \to \mathsf{Ab}$	Global sections functor. 102, 106
$r_{V,U} : FV \to FU$	Restriction homomorphism. 105
$\mathsf{Sheaf}(X)$	Category of sheaves of abelian groups on a topological space. 105

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