University of Groningen

Master Thesis in Physics

# Classifications of Galilean and Carrollian Spacetimes with and without indices

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**Abstract**

Non-Lorentzian geometries have resurfaced recently as a topic of interest in various fields, including non-relativistic string theory [1, 2], cosmology [3, 4], and condensed matter physics [5, 6]. Two specific non-Lorentzian geometries have been of particular interest: Galilean and Carrollian ones. Recently, the spacetimes whose structure is dictated by these groups were classified for particles [7] and Strings [8]. Following two approaches, inspired by the mathematics and physics literatures on the topic, we will extend this classification to cover generic $p$-brane foliations, and provide the geometric interpretation corresponding to the classification. Both methods will employ an approach of finding the possible forms of *intrinsic torsion* that the spacetimes may possess, as well as a mathematical tool called the *Spencer differential*. We find five potential classes of intrinsic torsion and five corresponding constraints on the geometry. Finally, we will derive the corresponding theories of gravity, and demonstrate that it matters whether a limit to a theory is taken from a second or first-order formulation of general relativity.

# Contents

# Chapter 1

# Introduction

Non-Lorentzian geometries have been brought up in recent work [7–10] as objects of interest in both mathematics and physics. In physics, non-Lorentzian geometries typically surface as various limits of general relativity (GR). Afterwards, applications have also been found in unrelated sub-fields of physics, ranging from condensed matter physics to gravitational waves research. Because some non-Lorentzian geometries are found as limits of general relativity, they are also important for understanding the universe - they probe various corners of a well-accepted theory. For an overview of non-Lorentzian geometry in general, and some applications to non-relativistic and ultra-local gravity, the reader is directed to a recent review article [11], and citations therein.

This work studies the Galilei and Carroll spacetimes in particular. Both of these spacetimes arise as a limit of general relativity, with the former arising as the $c \to \infty$ limit, while the latter results from the $c \to 0$ limit. Although both of these geometries are exotic, many potential applications have been found for both, especially recently. Both Galilean and Carrollian spacetimes have seen use as target spaces in string theory, for non-relativistic strings [1] and for Carrollian strings [2] respectively. Applications of the symmetries that generate these spacetimes have been studied in various contexts. A prime example of this is the fact that black hole horizons have been shown to possess Carroll symmetries [12, 13]. Interest in Carroll geometry has also surfaced in condensed matter theory [5, 6], fluid dynamics [14], and conformal field theory [15].

Galilei spacetimes are of particular interest because they are, in some sense, a first step towards Newton-Cartan (NC) gravity. NC gravity was originally studied by Cartan [16, 17], who wrote classical Newtonian gravity in terms of differential geometry instead of forces. Galilei spacetimes are separated from NC spacetimes by the fact that the Galilean structure group lacks a central extension that corresponds to the conservation of mass in NC gravity. This relation will be further discussed in section 3. A large part of the interest in NC gravity is driven by the goal of improving the understanding of post-Newtonian corrections to conventional Newtonian gravity. This can then be used to improve the modeling of gravitational wave sources, since NC gravity can be applied to strong field regimes of the theory, unlike general relativity. However, in this work, no additional geometric ingredients to the usual Galilei spacetime will be added. Although this has been done in the literature for particles, strings [8], and in some cases for other extended objects [18, 19]. In addition, applications of Galilean symmetry have surfaced in hydrodynamics [14].

It is well-known that in general relativity it is possible to select a unique metric-compatible and torsion-free connection - the Levi-Civita connection. In Newton-Cartan gravity, as well as most other theories of gravity that arise through limiting procedures of GR, this is no longer the case. In fact, some torsion always remains irrespective of the choice of connection. This remaining torsion is called *intrinsic*. In the cases treated here, the intrinsic torsion can take on various values, with each value enforcing a different constraint on the geometry of the spacetime itself. Because of this relation between intrinsic torsion and geometric constraints, different spacetime geometries can be classified based on the values that the intrinsic torsion can take in those geometries, and the resulting geometric constraints.

In [7] Galilean and Carrollian spacetimes, among others, were classified for particles by their intrinsic torsion in a systematic manner. The extension of this method to string and p-brane spacetimes is of physical interest for describing the non-relativistic theories of the related extended objects, because a priori, different extended objects couple to fundamentally different target geometries. Moreover, due to the analysis methodology employed in [7], together with a formal duality that is introduced in this work, the consideration of Galilean and Carrollian spacetimes can be simply incorporated into the same procedure. The formal duality between the non-relativistic and ultra-local geometries can be realized by exchanging the particle geometry of one for the domain-wall geometry of the other, together with exchanging the time direction for spatial ones. The latter step corresponds to changing the direction that boosts act - whether they send temporal directions to spatial ones, or vice versa. For generic $p$-branes, this operation generalizes to exchanging a $p$-brane with a $(D-p-2)$-brane, together with the exchange of dimensions as outlined previously.

In the end, we derive Galilean theories of gravity where each case of the classification is realized. We discuss three interesting details of these theories. First, we show that the second-order formalism of Galilean gravity, which is reached as a direct limit of the second-order formalism of general relativity, is different to that found by solving for all spin-connection components in the first-order formalism of Galilei gravity. Second, we discuss different ways of dealing with a divergent term that is found in the limit of the second-order formulation of GR. Finally, we show that electric Carroll gravity, a theory characterized by arising in a particular limit of GR, is in fact not unique, and that this specific limit can be taken in three different ways.

The goal of this work is to extend the classification of particle Galilean and Carrollian spacetimes in [7] to also cover the cases of Galilean and Carrollian $p$-branes. Moreover, this procedure will be done in a second way, inspired by [8]. The former method corresponds to using the language of Cartan, and the latter to using that of tensor fields. Although throughout the work (and especially in the final section) we often refer explicitly to only Galilean cases, through the aforementioned duality the results can be extended to include Carrollian ones.

This work is structured as follows: section 2 recalls the theory of general relativity, and introduces three different formalisms of the theory, the metric formalism, the formalism of non-coordinate bases, and finally the Cartan formalism. The first one will serve to introduce some basic concepts of GR, while the other two will be used throughout the text. In section 3 the Lie groups of relevance to non-relativistic gravity, their algebras, and related curvatures are derived from the Poincaré group. Section 4 sets the physical expectations for our results by discussing Galilean gravity as limits of general relativity for particles. In section 5 the classification of p-brane Galilean spacetimes is carried out in two different languages, and section 6 discusses the relevant realizations of these spacetimes as theories of gravity. Finally, conclusions and outlook are presented.

## 1.1 On Notation

Before we begin, some general notes on the notation used throughout the paper are given. Because we attempt to bridge the gap between the mathematics and physics perspectives on the classification, notation may be varied. We make our best attempt to point out any inconsistencies that might occur, as well as to link the two notations with each other.

Throughout the text, several different kinds of indices are used, with the type of index generally related to the type of object in the following manner:

- Greek indices $\mu, \nu, etc.$ correspond to objects that are "curved", i.e. relate to the base manifold $M$
- Capital Latin indices with "hats" $\hat{A}, \hat{B}, etc.$ are called "flat" or "internal" indices, and refer to things that live on the tangent bundle (or in the tangent space). These indices range over all dimensions of the tangent space $A = \{0, \ldots, D-1\}$,, or equivalently any space or bundle that can be reached via isomorphism from the tangent one,
- Lower case Latin indices $i, j, etc.$ correspond to the spatial (or "transversal") components of the flat indices, i.e. for particles $\hat{I} = \{0, i\}$, where $i = \{1, ..., D-1\}$, while for general $p$-branes we have $\hat{I} = \{0, \ldots, p, i\}$, with $i = \{p+1, \ldots, D-1\}$,
- Capital Latin indices $I, J, etc.$ correspond to the longitudinal components of the tangent bundle, i.e. for particles only a single index, $I = 0$, or for $p$-branes $\hat{I} = \{I, p, \ldots, D-1\}$, with $I = \{0, \ldots, p\}$.

Capital ${E_\mu}^{\hat{A}}$ is generally used for the vielbein in the general relativity case, while lower case $e_\mu^a$ is used for the spatial vielbein in the non-relativistic case.

# Chapter 2

# Formulations of General Relativity

Like most fundamental theories, general relativity can be described using multiple different, but equivalent formulations. This section will give a basic overview of the most common formulation, the *metric formulation*, and then introduce two other formulations that will be used extensively in this work, the *formulation of non-coordinate bases*, and the *Cartan Formulation*. We will also discuss the difference between the usual metric *second-order* formalism, and the *Palatini*, or *first-order* formalism of general relativity.

Why do we want to study different formulations of general relativity? Richard Feynman said in his nobel lecture in 1965:

> There is always another way to say the same thing that doesn't look at all like the way you said it before. I don't know what the reason for this is. I think it is somehow a representation of the simplicity of nature? Perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing.
>
> *Richard Feynman, 1965*

Of course, Feynman was talking about electrodynamics, rather than general relativity. To him, the fact that many different representations of the theory exist was a hint signifying the fundamentality of the theory. We can make the same argument for GR. Of course, the different formalisms can suggest different paths to further study - they're essentially different viewpoints, offering different tools to tackle problems. What might seem obvious in one viewpoint might be completely hidden in another.

## 2.1 The Metric Formalism

The most common way to formulate general relativity is by starting with the metric tensor $g_{\mu\nu}$ as the sole dynamical object of the theory. It is useful here to recall that the Christoffel symbols are given in terms of the metric by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{2.1}$$

To construct an action from the metric, we need to find a covariant scalar quantity that can be interpreted as the Lagrangian density of the theory. This cannot be done using only the metric and it's first derivatives (in the form of the Christoffel symbols), but rather we have to introduce the Riemann curvature tensor

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_{[\mu}\Gamma^{\sigma}{}_{\rho\nu]} + \Gamma^{\lambda}{}_{\rho[\nu}\Gamma^{\sigma}{}_{|\lambda|\mu]}, \tag{2.2}$$

where the square brackets denote anti-symmetrization, i.e.

$$\Gamma^{\lambda}{}_{\rho[\nu}\Gamma^{\sigma}{}_{|\lambda|\mu]} = \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\sigma}{}_{\lambda\mu} - \Gamma^{\lambda}{}_{\rho\mu}\Gamma^{\sigma}{}_{\lambda\nu}. \tag{2.3}$$

The Riemann curvature is often contracted in two ways, corresponding to the Ricci tensor and scalar

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}. \tag{2.4}$$

The Ricci scalar turns out to be the simplest covariant scalar in Riemannian geometry, and we can proceed to write down the (Einstein-)Hilbert action (presented here with zero cosmological constant)

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int d^dx\sqrt{-g}R \tag{2.5}$$

where $\kappa = 8\pi G_N$, with $G_N$ Newton's gravitational constant, and $g$ is the metric determinant. It should be noted that $R \sim \partial\Gamma \sim \partial^2 g$, and thus our action is written in terms of up to second derivatives in the metric.

The variation of (2.5) yields

$$\begin{aligned}
\delta S &= \frac{1}{2\kappa}\int d^dx\left(\sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \delta(\sqrt{-g})R\right) && (2.6)\\
&= \frac{1}{2\kappa}\int d^dx\left(\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}R\delta g^{\mu\nu}\right) && (2.7)\\
&= \frac{1}{2\kappa}\int d^dx\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}. && (2.8)
\end{aligned}$$

Requiring that this variation equals zero yields the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{2.9}$$

In the case of non-zero cosmological constant, the same process can be repeated identically.

In order to couple matter to general relativity, we can introduce an arbitrary extra term $S_M$ in the action (2.5). The variation of this term can then be considered separately from the variation of the rest of the action, creating an additional term in the equations of the motion (2.9)

$$\frac{\delta S_M}{\delta g_{\mu\nu}} := T^{\mu\nu}, \tag{2.10}$$

where $T^{\mu\nu}$ is called the stress-energy tensor.

Note that now the constant $\frac{1}{2\kappa}\sqrt{-g}$ has to be retained in the equation of motion, but we can move this to the right hand side as well, making the equation of motion in the presence of matter

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{2\kappa}{\sqrt{-g}}T^{\mu\nu}. \tag{2.11}$$

Often it is convenient to absorb the constant $-\frac{2\kappa}{\sqrt{-g}}$ into the definition of $T^{\mu\nu}$.

## 2.2 The Palatini Formalism

In metric formalism, the metric is taken to be the only dynamical field. In the *Palatini* (or sometimes *first order*) formalism this assumption is changed, and the metric $g_{\mu\nu}$ and the affine connection ${\Gamma^\rho}_{\mu\nu}$ are taken to be independent of each other. Moreover, the connection is assumed to be *torsion free*, meaning

$$ {\Gamma^\rho}_{\mu\nu} = {\Gamma^\rho}_{\nu\mu}. \tag{2.12} $$

The action then remains the same as in (2.5), with the connection entering through the Ricci scalar $R(g,\Gamma) := g^{\mu\nu}R_{\mu\nu}(\Gamma)$.

$$ S_{\mathrm{P}} = \frac{1}{2\kappa}\int d^dx\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma). \tag{2.13} $$

Varying this action with respect to $g_{\mu\nu}$ still yields the Einstein equation (2.9), while varying with respect to ${\Gamma^\rho}_{\mu\nu}$ one finds

$$ \frac{\delta S}{\delta\Gamma^\rho_{\mu\nu}} = \sqrt{-g}g^{\mu\nu}\nabla_{[\rho}{\Gamma^\rho}_{\nu]\mu} = 0. \tag{2.14} $$

Integrating this by parts, we find that

$$ 0 = -\nabla_{[\rho}\left(\sqrt{-g}g^{\mu\nu}\right){\Gamma^\rho}_{\nu]\mu}, \tag{2.15} $$

which we can, in turn, write out and subsequently simplify the coefficient of ${\Gamma^\rho}_{\nu\mu}$ by contracting with $g_{\mu\nu}$

$$ \begin{aligned} 0 &= -g_{\mu\nu}\left[(\nabla_\rho\sqrt{-g})g^{\mu\nu} + \sqrt{-g}(\nabla_\rho g^{\mu\nu})\right] &(2.16)\\ &= \sqrt{-g}\left[g_{\mu\nu}g^{\lambda\mu}g^{\sigma\nu}(\nabla_\rho g_{\lambda\sigma}) - \frac{1}{2}g_{\mu\nu}g^{\mu\nu}g^{\lambda\sigma}(\nabla_\rho g_{\lambda\sigma})\right] &(2.17)\\ &= \frac{1}{2}\sqrt{-g}g^{\sigma\lambda}(\nabla_\rho g_{\sigma\lambda}) &(2.18)\\ &= (\nabla_\rho\sqrt{-g}). &(2.19) \end{aligned} $$

Plugging this back in to equation (2.16), we find

$$ \sqrt{-g}\nabla_\rho g^{\mu\nu} = 0, \tag{2.20} $$

which is nothing but the statement that the connection $\Gamma$ is metric compatible.

## 2.3 Non-Coordinate Bases

Although the metric formulation of GR is the most popular one, it has a glaring weakness - spinors cannot be coupled to this formalism. Since spinors form much of the basis for modern physics, the source of interest in formulations of general relativity that they can couple to is evident. Among other things, coupling to spinor fields is one of the advantages of formulating GR in terms of a non-coordinate basis. For a treatment of this formalism, see for example [20, app. J].

Even though the formalism of non-coordinate bases does not introduce much of the mathematical structure that the Cartan formalism does, it is still unequivocally powerful. In fact, in order to work with non-coordinate bases, it is not necessary to introduce any bundles in addition to the usual tangent and co-tangent bundles $TM$ and $T^*M$.

The conventional basis for the tangent bundle $TM$ of a manifold $M$ is given by the *local* coordinate derivatives

$$\left\{ \frac{\partial}{\partial x^\mu} := \partial_\mu \right\}. \tag{2.21}$$

However, as is evident from some basic examples, this basis is usually not orthonormal. However, since $TM$ is a vector bundle, it is possible to select a different, orthonormal basis

$$\{E_{\hat{A}}\}, \tag{2.22}$$

where we require that $E_{\hat{A}}$ are orthonormal in the sense that

$$g(E_{\hat{A}}, E_{\hat{B}}) = \eta_{\hat{A}\hat{B}}. \tag{2.23}$$

We can then write the original basis of coordinate derivatives in terms of the $E_{\hat{A}}$ as

$$\partial_\mu = E_\mu{}^{\hat{A}} E_{\hat{A}}. \tag{2.24}$$

The coefficient $E_\mu^{\hat{A}}$ is called the *vielbein*[1] or tetrad. We can relate the metric to the vielbeine by

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} g(E_{\hat{A}}, E_{\hat{B}}) = E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \eta_{\hat{A}\hat{B}}. \tag{2.25}$$

Finally, we wish to introduce the spin connection. The derivative of a vector $v$ in $TM$ is given by

$$\nabla v = \nabla(v^{\hat{A}} E_{\hat{A}}) \tag{2.26}$$
$$= (dv^{\hat{A}}) E_{\hat{A}} + v^{\hat{A}} \nabla E_{\hat{A}}. \tag{2.27}$$

Defining the connection coefficients $\Omega_{\hat{A}}{}^{\hat{B}}$ by

$$\nabla E_{\hat{B}} = \Omega^{\hat{A}}{}_{\hat{B}} E_{\hat{A}}, \tag{2.28}$$

and substituting into (2.27) we find

$$\nabla(v^{\hat{B}} E_{\hat{B}}) = (\partial v^{\hat{B}}) E_{\hat{B}} + v^{\hat{B}} \Omega^{\hat{A}}{}_{\hat{B}} E_{\hat{A}} \tag{2.29}$$
$$\implies \nabla v^{\hat{A}} = dv^{\hat{A}} + \Omega^{\hat{A}}{}_{\hat{B}} v^{\hat{B}}. \tag{2.30}$$

Since a change of basis should not change any results, we can compare this expression with that in terms of the coordinate basis, where the corresponding derivative is given by

$$\nabla_\rho(v^\mu \partial_\mu) = (\partial_\rho v^\mu + \Gamma_{\nu\rho}{}^\mu v^\nu) \partial_\mu \tag{2.31}$$
$$= (\partial_\rho v^\mu + \Gamma_{\nu\rho}{}^\mu v^\nu) E_\mu{}^{\hat{A}} E_{\hat{A}}. \tag{2.32}$$

[1]The name changes in different numbers of dimensions: in 4 dimensions $E_\mu^{\hat{A}}$ is called the vierbein, in 2 the zweibein, and so on.

This should equal our previous expression

$$\nabla(v^\mu E_\mu{}^{\hat{A}} E_{\hat{A}}) = \left[\partial(v^\mu E_\mu{}^{\hat{A}}) + \Omega^{\hat{A}}{}_{\hat{B}} v^\mu E_\mu{}^{\hat{B}}\right] E_{\hat{A}}. \tag{2.33}$$

Comparing the two expressions gives us an expression for the Christoffel symbols in terms of the connection coefficients $\Omega^{\hat{A}}{}_{\hat{B}}$:

$$\Gamma_{\nu\rho}{}^\mu E_\mu{}^{\hat{A}} = \partial_\rho E_\nu{}^{\hat{A}} + \Omega_\rho{}^{\hat{A}}{}_{\hat{B}} E_\nu{}^{\hat{B}}, \tag{2.34}$$

or using the inverse vielbein $E^\mu{}_{\hat{A}}$, defined by

$$E^\mu{}_{\hat{A}} E_\mu{}^{\hat{B}} = \delta^{\hat{B}}_{\hat{A}} \qquad \text{and} \qquad E^\mu{}_{\hat{A}} E^{\hat{A}}{}_\nu = \delta^\mu_\nu, \tag{2.35}$$

we can write the Christoffel symbols as

$$\Gamma_{\nu\rho}{}^\mu = E^\mu{}_{\hat{A}} \partial_\rho E_\nu{}^{\hat{A}} + E^\mu{}_{\hat{A}} \Omega_\rho{}^{\hat{A}}{}_{\hat{B}} E_\nu{}^{\hat{B}}. \tag{2.36}$$

To write the action (2.5) in terms of the frame fields $E_\mu{}^{\hat{A}}$, we take the conversion $g_{\mu\nu} = E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \eta_{\hat{A}\hat{B}}$ from equation (2.25). Then $g$ becomes

$$\det(g_{\mu\nu}) = \det(E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \eta_{\hat{A}\hat{B}}) = -\det(E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}}) := -E^2.$$

By substitution, (2.5) is then given by

$$S = \frac{1}{2\kappa} \int d^d x E R \tag{2.37}$$

$$= \frac{1}{2\kappa} \int d^d x E E^\mu{}_{\hat{A}} E_{\hat{B}}{}^\nu R_{\mu\nu}{}^{\hat{A}\hat{B}}, \tag{2.38}$$

where $R_{\mu\nu}{}^{\hat{A}\hat{B}}$ is the Riemann curvature tensor, given in terms of the spin-connection $\Omega_\mu{}^{\hat{A}\hat{B}}$ by

$$R_{\mu\nu}{}^{\hat{A}\hat{B}} = 2\partial_{[\mu} \Omega_{\nu]}{}^{\hat{A}\hat{B}} - 2\Omega_{[\mu}{}^{\hat{B}\hat{C}} \Omega_{\nu]}{}^{\hat{A}}{}_{\hat{C}}. \tag{2.39}$$

The action (4.1) is invariant under general coordinate transformations, as well as local Lorentz transformations [9].

We can vary the action (2.38) with respect to the two separate dynamical fields $E_\mu{}^{\hat{A}}$ and $\Omega_\mu{}^{\hat{A}\hat{B}}$:

$$\frac{\delta \mathcal{S}}{\delta E^\mu{}_{\hat{A}}} = \frac{E}{\kappa} \left( R_{\mu\hat{B}}{}^{\hat{A}\hat{B}} - \frac{1}{2} E_\mu{}^{\hat{A}} R_{\hat{B}\hat{C}}{}^{\hat{B}\hat{C}} \right) \tag{2.40}$$

$$\frac{\delta \mathcal{S}}{\delta \left( \partial_\rho E^\mu{}_{\hat{A}} \right)} = 0 \tag{2.41}$$

$$\frac{\delta \mathcal{S}}{\delta \Omega_\mu{}^{\hat{A}\hat{B}}} = \frac{2E}{\kappa} \left( E^{[\mu}{}_{\hat{A}} E^{\rho]}{}_{\hat{C}} \Omega_\rho{}^{\hat{C}}{}_{\hat{B}} \right) \tag{2.42}$$

$$\frac{\delta \mathcal{S}}{\delta \left( \partial_\rho \Omega_\sigma{}^{\hat{A}\hat{B}} \right)} = \frac{E}{2\kappa} E^{[\rho}{}_{\hat{A}} E^{\sigma]}{}_{\hat{B}} \tag{2.43}$$

From this we find the equations of motion to be

$$R_{\mu\hat{B}}{}^{\hat{A}\hat{B}} - \frac{1}{2}E_{\mu}{}^{\hat{A}}R = 0 \tag{2.44}$$

and

$$\begin{aligned} 0 = 2\left(E^{[\rho}{}_{\hat{A}}E^{\nu]}{}_{\hat{B}}E^{\mu}{}_{\hat{C}} + \frac{1}{2}E^{\nu}{}_{\hat{A}}E^{\rho}{}_{\hat{C}}E^{\mu}{}_{\hat{B}}\right)\partial_{[\mu}E_{\nu]}{}^{\hat{C}} - \\ - 2E^{[\mu}{}_{\hat{C}}E^{\rho]}{}_{[\hat{A}}\Omega_{\mu}{}^{\hat{C}}{}_{\hat{B}]} \end{aligned} \tag{2.45}$$

We can write (2.45) as

$$R_{\hat{C}[\hat{A}}{}^{\hat{C}}E^{\mu}{}_{\hat{B}]} + \frac{1}{2}E^{\mu}{}_{\hat{C}}R_{\hat{A}\hat{B}}{}^{\hat{C}} = 0. \tag{2.46}$$

It is clear that (2.44) is the Einstein field equation in terms of the frame fields. The interpretation of (2.46) is not immediately clear, but we can use it to find

$$R_{\mu\nu}{}^{\hat{A}} = 0. \tag{2.47}$$

This equation can be used to solve for the spin-connection $\Omega_{\mu}{}^{AB}$ in terms of the vielbein $E_{\mu}{}^{A}$. The solution is given by

$$\Omega_{\mu}{}^{\hat{A}\hat{B}} = -2E^{\rho[\hat{A}}\partial_{[\mu}E_{\rho]}{}^{\hat{B}]} + E_{\mu\hat{C}}E^{\rho\hat{A}}E^{\nu\hat{B}}\partial_{[\rho}E_{\nu]}{}^{\hat{C}}. \tag{2.48}$$

If we introduce a matter term $S_M$ to the Lagrangian this changes since the right-hand sides will gain a current term,

$$T_{\mu}{}^{\hat{A}} = \frac{\kappa}{E}\frac{\delta S_M}{\delta E^{\mu}{}_{\hat{A}}} \tag{2.49}$$

for (2.44), and

$$J^{\mu}{}_{\hat{A}\hat{B}} = \frac{\kappa}{E}\frac{\delta S_M}{\delta\Omega_{\mu}{}^{\hat{A}\hat{B}}} \tag{2.50}$$

for (2.46). The solution for $R_{\mu\nu}{}^{\hat{A}}$ in equation (2.47) in this case will be proportional to the current $J^{\mu}{}_{\hat{A}\hat{B}}$. The full details of this can be found in [9].

## 2.4 The Cartan Formalism

Over the course of the early 20th century, Cartan developed a new language for differential geometry - that of principal bundles. While general relativity is often still taught using the original notation that Einstein used to develop the theory, Cartan's formulation allows for the use of the powerful tools that differential geometry offers. In many contexts, these tools can prove incredibly useful.

Physically speaking, why be interested in such a theory? Einstein gives us the answer - he is quoted saying "There is much reason to be attracted to a theory with no space and no time. But nobody has any idea how to build it up."[21, p. 787] It could be said that the Cartan formalism (or Einstein-Cartan theory) is precisely this theory, since it explains, using geometric tools, why a metric exists at all. When developing his formalism, Cartan's main innovation was to write general relativity in terms of a reduction of the frame bundle, rather than in terms of tensor calculus.

This section introduces the concepts that will be employed in the ultimate classification of Galilean and Carrollian spacetimes in section 5. We'd like to direct readers interested only in the classification to that section, or [7], where $G$-structures are discussed in this context for particle spacetimes.

The Cartan formalism of general relativity is partially formulated in terms of some of the same objects as the formalism of non-coordinate bases discussed in the previous section. However, while the previous section does discuss the vielbeine, it doesn't introduce any bundles beyond the "usual" tangent and cotangent ones. This section formalizes the methods of the previous section by building extra mathematical structure on the base manifold $M$. Let us then first relate the two formalisms by relating the two ways they discuss frames.

Let $M$ be a smooth manifold, with $p \in M$ some point in it. Since the tangent space to $M$ at $p$, $T_pM$, is a vector space, we can then establish a basis on it by considering a *frame*

**Definition 2.4.1.** A *frame* $u$ at $p$ is an isomorphism

$$u : \mathbb{R}^n \to T_pM. \tag{2.51}$$

We can indeed use this to define a basis for $T_pM$, by taking the image $u(\mathbf{e}_I)$ of the basis vectors $\mathbf{e}_I$ of $\mathbb{R}^n$ under $u$. The set of all frames at a particular point is denoted by $F_pM$, and called the *frame space* at $p$. Since different bases in $\mathbb{R}^n$ are related by $GL(n, \mathbb{R})$-transformations, the elements of $F_pM$, because they are frames, are as well. In the subsequent discussion (and indeed throughout much of this work) we will use $\mathbb{V}$ as notation for $\mathbb{R}^n$, rather than for an arbitrary vector space.

Analogously to the tangent bundle $TM$, we define the *frame bundle* $FM$ as the disjoint union of frame spaces over all points in $M$, i.e.

$$FM = \bigsqcup_{p\in M} T_pM. \tag{2.52}$$

We then define

**Definition 2.4.2.** Let $s : M \supset U \to FM$ be a local section of the frame bundle. Then a local moving frame or *inverse vielbein*[2] ${E^\mu}_{\hat{A}}$ is given by

$$E_{\hat{A}} = E^\mu_{\hat{A}}\partial_\mu = s(p)(\mathbf{e}_{\hat{A}}). \tag{2.53}$$

Note that for two overlapping coordinate charts $U, V$ on $M$, the vielbeine on $U$ and $V$ are related on their overlap by $GL(n, \mathbb{R})$-transformations. Let $G$ be some subgroup of $GL(n, \mathbb{R})$. In fact, we usually take $G$ to be the (defining) representation of some group on $\mathbb{V}$. Here it is possible to exploit Cartan's idea - we restrict the frames we consider to a subset of those in $FM$, specifically to those related on overlaps by $G$-transformations instead of $GL(n, \mathbb{R})$ ones.

[2]In [7] this is called the *vielbein*. However here we adopt the physics convention of calling ${E^\mu}_{\hat{A}}$ the inverse vielbein, and its dual ${E_\mu}^{\hat{A}}$ the vielbein.

**Definition 2.4.3.** Let $G$ be a subgroup of $GL(n, \mathbb{R})$. At a point $p \in M$, the restriction of frames to those related by $G$-transformations gives a subset $P_p \subset F_pM$. The disjoint union $P = \bigsqcup_{p \in M} P_p$ is a subset of $FM$, and the original $\mathrm{GL}(n, \mathbb{R})$-bundle is reduced to a principal $G$-bundle $P$. This reduction is called a *$G$-structure*.

We can project down from the reduced frame bundle $P$ to the base manifold $M$ in a canonical way, by defining the map

$$\begin{aligned} \pi &: P \to M \\ P_p &\ni u \mapsto p. \end{aligned} \tag{2.54}$$

We will denote the *$G$-structure* as the reduced frame bundle $P$, together with the projection $\pi$, and the base manifold $M$, or more concisely as $P \xrightarrow{\pi} M$.

In order to relate this discussion to that of the section on non-coordinate bases, we want to express the vielbein in the Cartan formalism. This is possible by considering the tangent space $T_uP$ to $P$ at some particular frame $u \in P$ (points in $P$ being frames), since elements of this tangent space describe the way that the frame changes as we move around the manifold, which is precisely the purpose of the vielbein of the previous section. Similarly to $\pi$, we can canonically project down from $T_uP$ to $T_pM$ by the push-forward of $\pi$

$$\begin{aligned} d\pi &: T_uP \to T_pM \\ X_u &\mapsto Y_p, \end{aligned} \tag{2.55}$$

where $Y_p$ is some tangent vector to $M$ at $p$. From here, it is possible to find the vector in $\mathbb{R}^n$ corresponding to $Y_p$ (relative to the frame $u$) by inverting $u$. The composition of $u^{-1}$ and $d\pi$ then lets us send tangent vectors to $P$ to vectors in $\mathbb{R}^n$. This defines an $\mathbb{R}^n$-valued one-form $\theta$ on $P$ which we call the *solder form* associated to the $G$-structure. Explicitly $\theta$ is given by

$$\theta_u(X_u) = u^{-1}(d\pi X_u). \tag{2.56}$$

In terms of the components $\theta^{\hat{A}}$ of $\theta$, we can write the vielbein as

$$E^{\hat{A}} = E^{\hat{A}}_{\mu} dx^{\mu} \tag{2.57}$$

$$= s^*(\theta^{\hat{A}}) \tag{2.58}$$

$$= \theta^{\hat{A}}_{s(p)}(ds(p)(v)) \tag{2.59}$$

where $v \in T_pM$. In the previous expression, the local section $s$ restricts our solder form to some local coordinate chart $U \subset M$. Thus the vielbein is precisely the local expression for the solder form. Since it is typical in physics to always work within some local chart, it is usual to only discuss this local expression, as in the previous section. Following this way of thinking, in the Cartan formalism the vielbein is often introduced as an isomorphism

$$E : T_xM \to \mathbb{V}. \tag{2.60}$$

From this we simply recover the same object as before, but without worrying about global properties as much. In fact this way of considering things is in thinking closer to that of the subsection on non-coordinate frames.

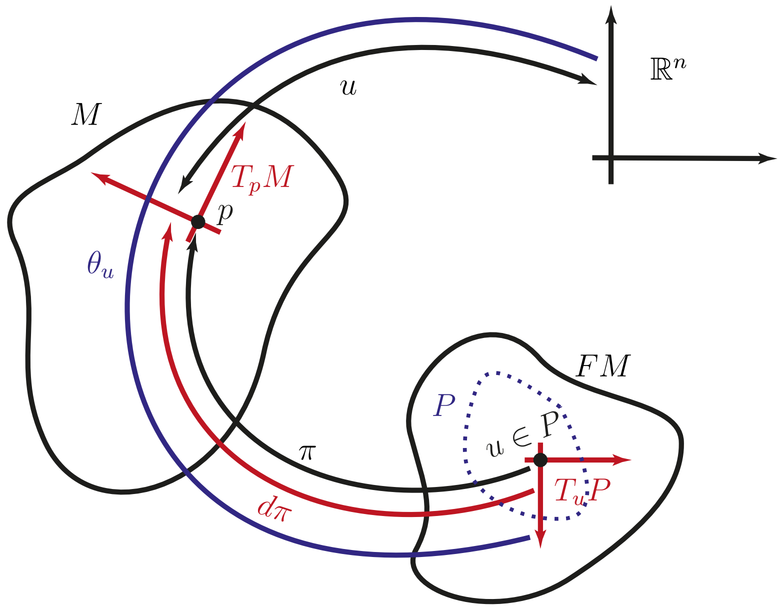


Figure 2.1: Relations between the spaces $M$, $FM$, $\mathbb{R}^n$, via the maps $u, \pi$, and $\theta_u$.

The way that the different spaces we have introduced are related is portrayed in figure 2.1.

Note that with respect to the basis $e_{\hat{A}}$ of $\mathbb{V}$, $\theta^{\hat{A}}$ define the canonical dual basis, since $s^*(\theta^{\hat{A}})(X_{\hat{B}}) = \delta^{\hat{A}}_{\hat{B}}$ for a local frame $s = (X_1, ..., X_n)$, or equivalently

$$E^{\mu}{}_{\hat{A}} E_{\mu}{}^{\hat{B}} = \delta^J_I \qquad\qquad E^{\mu}{}_{\hat{A}} E_{\nu}{}^{\hat{A}} = \delta^{\mu}_{\nu}, \tag{2.61}$$

which is precisely the same expression as in the formalism of non-coordinate bases.

### 2.4.1 The Spin-connection

So far, we have introduced the reduction of the frame bundle $P$. By definition, $P$ is a vector bundle over $M$, with each fibre $\mathbb{V} = \mathbb{R}^n$. We are free to split the dimension $n = p + q$, with $p$ and $q$ such that the signature of the space (and by extension the metric $\langle \cdot, \cdot \rangle_\eta$ or $\eta_{\hat{A}\hat{B}}$) match our purposes. Additionally, we defined the soldering form $\theta : P \to \mathbb{V}$ which, in a sense, solders the additional bundle $P$ to the tangent space $TM$, and by extension to $\mathbb{V}$.

Next, we want to introduce the *spin-connection*. A spin-connection $\omega$, commonly referred to as the *Ehresmann connection* in mathematical texts, is a one-form taking values in the Lie algebra $\mathfrak{g}$ associated to $G$. Because the groups of interest should be formulated as a subgroup of $GL(\mathbb{V})$, we will most often directly consider $\omega$ to take values in the representation of $\mathfrak{g}$ on $\mathbb{R}^n$, $d\rho$.

We can define the spin-connection as follows. We start with a *local*, one-form valued, metric connection $\omega_{\hat{A}}{}^{\hat{B}}$ on the bundle $P$. The covariant derivative with respect to the connection, $\nabla$[3] is given analogously to equation (2.30) by

$$\nabla v^{\hat{A}} = dv^{\hat{A}} + \omega^{\hat{A}}{}_{\hat{B}} v^{\hat{B}}. \tag{2.62}$$

Since the connection is metric,

$$\nabla \eta^{\hat{A}\hat{B}} = \omega^{\hat{A}}{}_{\hat{C}} \eta^{\hat{C}\hat{B}} + \omega^{\hat{B}}{}_{\hat{C}} \eta^{\hat{A}\hat{C}} = 0. \tag{2.63}$$

This, upon inspection, reveals that in fact the connection $\omega^{\hat{A}}{}_{\hat{B}}$ takes values in the Lie algebra $\mathfrak{g}$. For example, in the case where $\eta^{\hat{A}\hat{B}}$ is the Minkowski metric, $\mathfrak{g}$ is the Lorentz algebra, and since the Lorentz algebra is given by

$$\left\{ X \in \mathrm{GL}(4, \mathbb{R}) \mid \eta X + X^T \eta = 0 \right\}, \tag{2.64}$$

which is precisely the condition we have recovered.

Equivalently (and more commonly in the physics literature), we can start with the knowledge that the spin-connection should take values in the Lie algebra. We then introduce the structure group connection $\Omega$. Mathematically, we write this in terms of the pullback of the vielbein $E^*$ and the spin-connection as

$$\Omega := E^* \omega. \tag{2.65}$$

This can be expanded in terms of a local co-basis $dx^\mu$ as

$$\Omega = \Omega_\mu dx^\mu, \tag{2.66}$$

where $\Omega_\mu$ is the object that is usually called the structure group connection in physics texts. We can write $\Omega_\mu$ down in terms of the generators of (which also form a basis for) the Lie algebra. In the case of general relativity, we introduce a basis $J_{\hat{A}\hat{B}} = -J_{\hat{B}\hat{A}}$ for the Poincaré algebra. We then define the spin-connection $\omega_\mu{}^{\hat{A}\hat{B}}$ as the coefficients that yield the structure group connection, i.e.

$$\Omega_\mu = \omega_\mu{}^{\hat{A}\hat{B}} J_{\hat{A}\hat{B}}. \tag{2.67}$$

Since the Lorentz generators $J_{\hat{A}\hat{B}}$ are anti-symmetric, i.e. $J_{\hat{A}\hat{B}} = -J_{\hat{B}\hat{A}}$, the spin-connection has to cancel this symmetry. Then, for any anti-symmetric generator we have a corresponding anti-symmetric spin-connection $\omega_\mu{}^{\hat{A}\hat{B}} = -\omega_\mu{}^{\hat{B}\hat{A}}$.

How does the Cartan formalism relate to the usual Metric one? In terms of the soldering form, we can write the metric in the way introduced for non-coordinate bases in equation (2.25). The other important (and possibly dynamic) variable in GR is the affine connection. We can relate it to the spin-connection in a simple way. First, we introduce the *torsion*, given by

$$T^{\hat{A}} := \nabla \theta^{\hat{A}} = d\theta^{\hat{A}} + \omega^{\hat{A}}{}_{\hat{B}} \theta^{\hat{B}}, \tag{2.68}$$

or in index-free notation

$$\Theta := d\theta + \omega \wedge \theta. \tag{2.69}$$

We may require that the torsion of our spin-connection $\omega^{\hat{A}}{}_{\hat{B}}$ be zero, which restricts our chosen spin-connection. In fact this restriction is maximal, and the requirement $T^{\hat{A}} = 0$ uniquely selects the spin-connection to equal the affine connection [22].

[3] Notation in differential geometry is famously varied. This is often denoted $d^\omega$

# Chapter 3

# Relevant Lie Groups and Their Algebras

Having discussed general relativity, the natural group theoretical starting point is the Poincaré group. Since we wish to develop theories that arise in various limits of GR, we want to study the relationship between the Poincaré group and the groups of symmetries of those theories. Some relationship should exist, since the symmetries themselves should result through a limit of the symmetries of GR. However, a simple limiting procedure does not accomplish this, and we need to do a little more work.

In the example of the Galilei group, it is well known that in Newtonian gravity time is universal. In general relativity this is famously not the case. Therefore we need to somehow separate symmetries related to time from the other symmetries of GR. The desired effect can be accomplished by scaling some components of the generators of the Poincaré group with a *dimensionless contraction parameter* and then performing a limit with respect to that contraction parameter. Such a process is called an İnönü-Wigner contraction. In a simple example, suppose we have a group with generators $X_1, X_2$, and $X_3$, and the commutation relations between them

$$[X_i, X_j] = X_k, \qquad i, j, k \in 1, 2, 3 \tag{3.1}$$

We can take some contraction parameter $\lambda$, and redefine the generators with factors of $\lambda$. For instance, let

$$Y_1 = \lambda X_1, \qquad Y_2 = \lambda X_2, \qquad Y_3 = X_3. \tag{3.2}$$

The commutation relation with $i = 1, j = 2, k = 3$ becomes

$$[\frac{1}{\lambda}Y_1, \frac{1}{\lambda}Y_2] = Y_3 \tag{3.3}$$

$$\implies [Y_1, Y_2] = \lambda^2 Y_3, \tag{3.4}$$

while the commutation relations for any other combination remain the same. We can then perform a limit, for instance with $\lambda \to 0$, to recover a different group structure.

Let us first discuss the Poincaré group, and then the Carroll and Galilei groups as contractions. In the final part of this section, we will discuss the generalization of the Galilei and Carroll groups for the extended objects of interest.

## 3.1 Poincaré Group and Algebra

The Poincaré group and algebra are particularly well studied in theoretical physics, since they form the backbone of Einstein's special theory of relativity. Specifically, the Poincaré group is the group of isometries of Minkowski spacetime, i.e. the Lorentz group, together with spacetime translations. We can describe the Poincaré group via the basis of generators; $P_{\hat{A}}$ and $J_{\hat{A}\hat{B}}$ for spacetime translations and Lorentz transformations respectively. These generators obey the following non-zero commutation relations

$$[P_{\hat{A}}, J_{\hat{B}\hat{C}}] = 2\eta_{\hat{A}[\hat{C}}P_{\hat{B}]}, \tag{3.5}$$

$$[J_{\hat{A}\hat{B}}, J_{\hat{C}\hat{D}}] = 4\eta_{[\hat{A}[\hat{D}}J_{\hat{C}]\hat{B}]}, \tag{3.6}$$

with all other commutators being zero. In the defining matrix representation, the group of dimension $n$ is given by matrices of the form

$$G_P = \left\{ \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} \;\middle|\; A \in O(1, n-2), \mathbf{v} \in \mathbb{R}^{n-1} \right\}, \tag{3.7}$$

with the corresponding algebra

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & \mathbf{v} \\ 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{so}(1, n-2), \mathbf{v} \in \mathbb{R}^{n-1} \right\}. \tag{3.8}$$

Here, the matrices $A$ form the Lorentz group, and the vectors $\mathbf{v}$ describe spacetime translations. As already seen in the previous section, we can write the structure group connection $\Omega_\mu$ in terms of the generators $P_{\hat{A}}$ and $J_{\hat{A}\hat{B}}$ as

$$\Omega_\mu = \Omega_\mu{}^{\hat{A}\hat{B}} J_{\hat{A}\hat{B}}, \tag{3.9}$$

where the field $\Omega_\mu{}^{AB}$ is the gauge fields associated to the generators $J_{\hat{A}\hat{B}}$. When studying general relativity, we already saw that the gauge fields are instrumental for describing the theory. This pattern will repeat later when we are studying the gravitational theories in sections 4 and 6. In addition, the corresponding gauge fields of the $p$-brane Galilei and Carroll groups are a key ingredient for the classification in section 5. A decomposition similar to that in equation (3.9) will be done for the case of every group.

Finally, we want to introduce the curvature associated to an algebra. The curvature is described by a Lie algebra valued 2-form, usually denoted by $R$ in physics texts, and somewhat confusingly by $\Omega$ in some mathematics ones. Here we will stick to calling the curvature $R$. It is given by

$$R = d\omega + \frac{1}{2}\omega \wedge \omega, \tag{3.10}$$

where $\omega$ is the spin-connection. We can write the formula (3.10) in indices as the already familiar quantity

$$R_{\mu\nu}{}^{\hat{A}\hat{B}} = 2\partial_{[\mu}\omega_{\nu]}{}^{\hat{A}\hat{B}} - 2\omega_{[\mu}{}^{\hat{B}\hat{C}}\omega_{\nu]}{}^{\hat{A}}{}_{\hat{C}}. \tag{3.11}$$

In cases where more than one generator is required to describe the algebra - like in the following subsections - a different curvature can be associated with each generator.

Readers with a keen eye might have already spotted that there is a curvature missing. This would be that constructed from translations, given by

$$T_{\mu\nu}{}^{\hat{A}} = 2\partial_{[\mu}E_{\nu]}{}^{\hat{A}} - 2\omega_{[\mu}{}^{\hat{A}}{}_{\hat{B}}E_{\nu]}{}^{\hat{B}}, \tag{3.12}$$

where the vielbein $E_\mu{}^{\hat{A}}$ has been interpreted as the gauge field associated to translations. However, rather than looking like a curvature, equation (3.12) looks like a torsion 2-form, defined by

$$\Theta = dE + \omega \wedge E. \tag{3.13}$$

In the following sections on various groups and algebras, we will include these terms as well, labeling them as torsion tensors[1].

## 3.2 Carroll Group and Algebra

As mentioned previously, both the Carroll, and the Galilei algebras can be obtained as İnönü-Wigner contractions of the Poincaré algebra. In particular, we want to scale the relevant generators of the algebra with powers of a dimensionless contraction parameter[2] $c$. It should be emphasized that the contraction parameter should be dimensionless. This is reconciled with a limit in a physical sense by interpreting $c$ as a ratio of the physical speed of light to some other reference speed, in comparison to which the speed of light becomes large. To match the future separation of the vielbein $E_\mu^A$ into a time and space components $\tau_\mu$ and $e_\mu^A$, we split, and then re-scale the generators of the Poincaré algebra in the following way:

$$\begin{aligned} P_0 &= cH, & P_a &= P_a, \\ J_{0a} &= cG_a, & J_{ab} &= J_{ab}, \end{aligned} \tag{3.14}$$

where $H$ generates time-translations, and $G_a$ generate boosts. The brackets of the Poincaré group, when the limit $c \to \infty$ is taken, become

$$[P_a, G_b] = \delta_{ab}H, \qquad [P_c, J_{ab}] = -2\delta_{c[a}P_{b]} \tag{3.15}$$

$$[G_c, J_{ab}] = -2\delta_{c[a}G_{b]}, \qquad [J_{ab}, J_{cd}] = 4\delta_{[a[d]}J_{c]b]} \tag{3.16}$$

Similarly to what was presented for the Poincaré group, we can write down the Carroll in its defining matrix representation as

$$G_C = \left\{ \begin{pmatrix} 1 & \mathbf{v}^T \\ \mathbf{0} & A \end{pmatrix} \middle| \ \mathbf{v} \in \mathbb{R}^{n-1}, A \in O(n-1) \right\}, \tag{3.17}$$

with the corresponding algebra

$$\mathfrak{c} = \left\{ \begin{pmatrix} 0 & \mathbf{v}^T \\ \mathbf{0} & A \end{pmatrix} \middle| \ \mathbf{v} \in \mathbb{R}^{n-1}, A \in \mathfrak{so}(n-1) \right\}. \tag{3.18}$$

[1]In super-gravity literature it is not uncommon for these to also be labeled as curvatures.

[2]In literature it is quite common to find the contraction parameter called $\omega$ instead, to separate it from the physical speed of light. Here we elect to not do this since $\omega$ is also used to denote spin-connections.

In equation (3.17), the matrices $A$ are interpreted as the transversal rotations, while $\mathbf{v}$ form the Carroll boost component of the group. The structure group connection for the Carroll algebra can be written as

$$\Omega_\mu = \omega_\mu{}^{ab} J_{ab} + \omega_\mu{}^{a} G_a. \tag{3.19}$$

Finally, the curvatures are given by

$$\begin{aligned} T_{\mu\nu} &= 2\partial_{[\mu}\tau_{\nu]} - 2\omega_{[\mu}{}^{a} e_{\nu]a}, && (3.20)\\ T_{\mu\nu}{}^{a} &= 2\partial_{[\mu}e_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab} e_{\nu]b}, && (3.21)\\ R_{\mu\nu}{}^{a}(G) &= 2\partial_{[\mu}\omega_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab}\omega_{\nu]b}, && (3.22)\\ R_{\mu\nu}{}^{ab}(J) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{bc}\omega_{\nu]}{}^{a}{}_{c}, && (3.23) \end{aligned}$$

where the first two torsion tensors are those associated to time translations and spatial translations respectively.

## 3.3 Galilei Group and Algebra

The appropriate rescalings of the Poincaré algebra for the Galilean case are

$$\begin{aligned} P_0 &= c^{-1}H, \quad & P_a &= P_a,\\ J_{0a} &= cG_a, \quad & J_{ab} &= J_{ab}. \end{aligned} \tag{3.24}$$

Comparing with the corresponding scalings for the Carroll algebra (3.14), we see that the factor $c$ for the time-translations in the Carroll case has been exchanged to a factor $c^{-1}$. This is to be expected, since we tend to discuss the Galilei limit as the $c \to \infty$ limit of general relativity, and the Carroll limit the $c \to 0$ limit.

The exchange in the power of the scaling factor transforms the usual Lorentzian lightcone in the opposite way to the Carroll case; instead of closing up to become a line, it opens up to form a (hyper)plane of equal universal time. This difference is illustrated in figure 3.1. These hyperplanes foliate the entire manifold, as one would expect for a conventional Newtonian view of the universe.

Performing the $c \to \infty$ limit, we find the following (usual) commutation relations

$$[J_{bc}, P_a] = 2\delta_{a[c}P_{b]}, \qquad [J_{ab}, J_{cd}] = 4\delta_{[a[d}J_{c]b]}, \tag{3.25}$$

as well as

$$[J_{ab}, G_c] = 2\delta_{c[a}G_{b]}, \qquad [H, G_a] = P_a. \tag{3.26}$$

The second commutation relation of equation (3.26) points to the fact that under Galilean boosts, time is sent to space, but space does not get sent to time.

In the defining matrix representation, we can write the Galilei group $G_G$ as the matrices that preserve the Euclidean metric on the spatial leaves of equal universal time, which form $O(n-1)$, together with boosts from time to space, which can be represented either as a vector in $\mathbb{R}^{n-1}$, or a map

$$\mathbf{v} : \mathbb{R} \to \mathbb{R}^{n-1}, \tag{3.27}$$

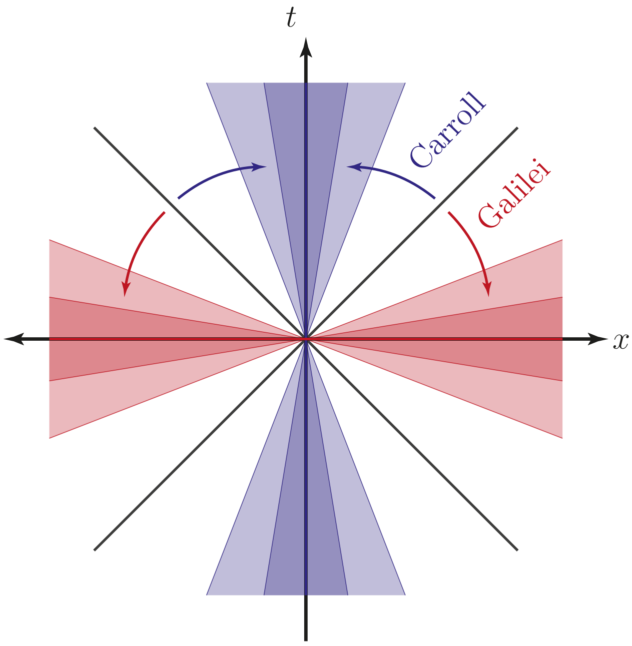


Figure 3.1: The effect of the limiting procedure leading to Carrollian and Galilean spacetimes on the usual GR lightcones. For the Carrollian case, the lightcone closes up to a "time-like" line, while for the Galilean case it opens up to a purely spacial hyperplane.

where $\mathbb{R}$ represents the 0-component of the spacetime, i.e. the time component.

Thus the representation of the Galilei group is given by

$$G_G = \left\{ \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & A \end{pmatrix} \;\middle|\; \mathbf{v} \in \mathbb{R}^{n-1}, A \in O(n-1) \right\}, \tag{3.28}$$

with the corresponding algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{v} & A \end{pmatrix} \;\middle|\; \mathbf{v} \in \mathbb{R}^{n-1}, A \in \mathfrak{so}(n-1) \right\}. \tag{3.29}$$

The structure group connection $\Omega_\mu$ can be written as

$$\Omega_\mu = \omega_\mu{}^{ab} J_{ab} + \omega_\mu{}^a G_a. \tag{3.30}$$

Finally, the curvatures associated to the Galilei group are

$$\begin{aligned} T_{\mu\nu} &= 2\partial_{[\mu}\tau_{\nu]}, & (3.31)\\ T_{\mu\nu}{}^a &= 2\partial_{[\mu}e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^a\tau_{\nu]}, & (3.32)\\ R_{\mu\nu}{}^a(G) &= 2\partial_{[\mu}\omega_{\nu]}{}^a - 2\omega_{[\mu}{}^a{}_b\omega_{\nu]}{}^b, & (3.33)\\ R_{\mu\nu}{}^{ab}(J) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^a{}_c\omega_{\nu]}{}^{cb}. & (3.34) \end{aligned}$$

Upon comparison between the curvatures of the Galilei group and the Carroll group, it is apparent that the boosts act a different way between the two groups.

Next, we will present the generalization of the last two groups for $p$-branes.

## 3.4 $p$-Brane Galilei and Carroll Groups

The groups that were previously discussed are no longer sufficient when considering extended physical objects, such as strings or membranes. This is a result of the fact that the additional dimensions to which these objects extend into carry different kinds of symmetries than the other "regular" spatial dimensions. Conventionally, we term the direction of extension together with time *"longitudinal"*, while calling the usual spacial dimensions *"transversal"*. Because of the inclusion of time as a longitudinal dimension, and the boosts (in the Galilei case) take longitudinal dimensions to transversal ones.

Although the Galilei and Carroll groups are different, we will define them here in one go by utilizing characteristic fields. In terms of these tensor fields, the two groups are separated by signature, with the signature of one component being Minkowski, and the other Euclidean. The groups are then separated by an exchange of which component has which signature, together with a change in the direction that the boosts act. When left signature agnostic, the characteristic tensor fields define both groups congruently, with one assignment of signatures resulting in the Galilei group, and the other in the Carroll group. This is in line with the rest of the work - we can characterize the geometries generated by both of these groups at the same time precisely because in what follows, we will choose to work without reference to signature.

The Galilei (and equivalently Carroll) group is the subgroup of the general linear group, that leaves invariant the non-degenerate tensor fields $\delta$ and $\eta$, with

$$\delta \in \odot^2 W, \quad \eta \in \odot^2 \operatorname{Ann} W, \tag{3.35}$$

where $\operatorname{Ann}W$ denotes the annihilator of $W$, and $\odot$ is the symmetric product. We have also associated the group to the vector space $\mathbb{V} := \mathbb{R}^n$ that it is to act on. $\mathbb{V}$ can be broken down to two components (this step will be elaborated on in section 5)

$$\mathbb{V} = V \oplus W. \tag{3.36}$$

In the case of the Galilei group, $V$ would be the longitudinal, and $W$ the transversal component of $\mathbb{V}$, however, since we wish to not fix our signature, for now we will leave these subspaces arbitrary. The elements $g \in \mathrm{GL}(\mathbb{V})$ that preserve $\eta$ and $\delta$ are those that comprise $\mathrm{SO}(\eta)$ and $\mathrm{SO}(\delta)$ respectively. In addition to these components, we want one-way boosts that send, in the Galilei case, longitudinal components to transverse ones. In the Carroll case, boosts should act the opposite way. Thus the remaining subgroup of $\mathrm{GL}(\mathbb{V})$ is

$$G = \left\{ g = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in \mathrm{GL}(\mathbb{V}) \;\middle|\; \begin{array}{c} A \in \mathrm{SO}(\eta) \\ C \in \mathrm{SO}(\delta) \\ B : V \to W \end{array} \right\}. \tag{3.37}$$

Where, for the Galilei group, the tensor field $\eta$ would carry signature $(1, D-1)$, while $\delta$ would be of Euclidean signature. For the Carroll group, this is of course vice versa.

As is hinted by (3.37), we can also write the group as a semi-direct product

$$(\mathrm{SO}(1, p-1) \times \mathrm{SO}(D-p-2)) \ltimes \mathbb{R}^{(p+1)(D-p-1)}. \tag{3.38}$$

For another possible description of the group, we can describe both groups by picking a basis of generators. We pick a generator for each component of the group, that is

$$\{J_{ab}, L_{AB}, G_{aA}\}\,, \tag{3.39}$$

where, in the Galilean case, $J_{ab} = -J_{ba}$ generates the transversal rotations, $L_{AB} = -L_{BA}$ the longitudinal Lorentz transformations, and $G_{aA}$ the Galilean boosts. The structure group connection $\Omega_\mu$ can be written in terms of the generators and spin-connection parameters $\omega$ as

$$\Omega_\mu = {\omega_\mu}^{ab} J_{ab} + {\omega_\mu}^{AB} L_{AB} + {\omega_\mu}^{aA} G_{aA}. \tag{3.40}$$

The parameters $\omega$ are the spin-connections associated with the various types of transformations.

The curvatures in the Galilei and Carroll cases are simply generalized to $p$-branes from those presented in the sections of the particle groups. For instance, in the Galilean case we find

$$\begin{aligned}
{T_{\mu\nu}}^A &= 2\partial_{[\mu}{\tau_{\nu]}}^A - 2{\omega_{[\mu}}^A{}_B{\tau_{\nu]}}^B, && (3.41)\\
{T_{\mu\nu}}^a &= 2\partial_{[\mu}{e_{\nu]}}^a - 2{\omega_{[\mu}}^a{}_b{e_{\nu]}}^b - 2{\omega_{[\mu}}^a{}_A{\tau_{\nu]}}^A, && (3.42)\\
{R_{\mu\nu}}^{aA}(G) &= 2\partial_{[\mu}{\omega_{\nu]}}^{aA} - 2{\omega_{[\mu}}^a{}_b{\omega_{\nu]}}^{bA}, && (3.43)\\
{R_{\mu\nu}}^{ab}(J) &= 2\partial_{[\mu}{\omega_{\nu]}}^{ab} - 2{\omega_{[\mu}}^a{}_c{\omega_{\nu]}}^{cb}. && (3.44)
\end{aligned}$$

In sections 5 and 6, the torsion tensors ${T_{\mu\nu}}^A$ and ${T_{\mu\nu}}^a$ will rise to play a central role.

# Chapter 4

# Non-relativistic Gravity

Non-relativistic gravity theories usually occur as different limits of general relativity, with perhaps the most well-known one of these theories being Newton-Cartan (NC) gravity. Many other examples exist, although most of them are exotic theories. Although NC gravity won't be discussed here, it is still relevant to mention, since it arises as the resulting gravity from spacetimes with *Bargmann* symmetries, with the Bargmann group being the centrally extended Galilei group. Therefore, although Galilei gravity is exotic, we can move from it towards NC gravity, since the only missing component is the field $m_\mu$ that results from the central extension. In order to arrive at NC gravity from general relativity, the mass conservation field $m_\mu$ has to either be added "by hand", or via central extension, rather than organically appearing in the limit-taking process. Without this extra ingredient, Galilei gravity is the massless representation of NC gravity, and in some sense the pure theory resulting from the $c \to \infty$ limit of GR. The same caveat discussed in the previous section regarding the dimensionality of $c$ will apply throughout this section.

The non-relativistic limit of general relativity can be taken via a few different methods. Namely, the limit can be taken at the level of the equations of motion, or at the level of the action. Additionally, instead of a pure limit-taking approach, we can consider an expansion of GR in powers of the dimensionless parameter $c$, where in the end lower-order terms are neglected. Although the expansion approach is different from the others, it is well known that it results in the same theory [23], and won't be treated here.

## 4.1 Galilei Gravity For Particles

### 4.1.1 As a Limit

For the purpose of taking the non-relativistic limit, we wish to recall the action of general relativity in the formalism of non-coordinate bases. As introduced in section 2 equation (2.38), the action in vacuum is given by

$$S = \frac{1}{2\kappa} \int d^d x E E^\mu{}_{\hat{A}} E^\nu{}_{\hat{B}} R_{\mu\nu}{}^{\hat{A}\hat{B}}, \tag{4.1}$$

where we recall that ${R_{\mu\nu}}^{\hat{A}\hat{B}}$ is the Riemann curvature tensor, given in terms of the spin-connection ${\Omega_\mu}^{\hat{A}\hat{B}}$ by

$$ {R_{\mu\nu}}^{\hat{A}\hat{B}} = 2\partial_{[\mu}{\Omega_{\nu]}}^{\hat{A}\hat{B}} - 2{\Omega_{[\mu}}^{\hat{B}\hat{C}}{\Omega_{\nu]}}^{\hat{A}}{}_{\hat{C}}. \tag{4.2} $$

### 4.1.2 Galilei Action

To pass onto an action for Galilean gravity, we need to scale the relativistic dynamical fields of our theory by factors of the contraction parameter $c$. Physically, it is possible to interpret $c$ as the speed of light. More properly, $c$ would be the ratio of the physical speed of light with some reference speed, in comparison to which it grows large. After the rescaling, we will then proceed to take the limit $c \to \infty$.

Since in GR space and time are unified, in order to reach a theory with a difference between these dimensions we have to break up the vielbein ${E_\mu}^{\hat{A}}$ into two parts. The zero component of the vielbein becomes the time-like vielbein $\tau_\mu := {E_\mu}^0$, while the rest of the dimensions comprise the spatial vielbein $e^a_\mu := {E_\mu}^a$, with $a$ ranging from 1 to $D-1$.

We then introduce the following scaling in terms of the contraction parameter $c$:

$$ \begin{aligned} {E_\mu}^0 &= c\tau_\mu, & {E_\mu}^a &= {e_\mu}^a, \\ {\Omega_\mu}^{0a} &= \frac{1}{c}{\omega_\mu}^a, & {\Omega_\mu}^{ab} &= {\omega_\mu}^{ab}, \end{aligned} \tag{4.3} $$

with ${\Omega_\mu}^{\hat{A}\hat{B}}$ the spin-connection of the relativistic Poincaré transformations, and ${\omega_\mu}^{ab}$ and ${\omega_\mu}^a$ those of the corresponding non-relativistic transformations (spatial rotations and Galilean boosts respectively). Additionally, we perform a rescaling of the constant $\kappa$ to

$$ \kappa \to c\bar{\kappa}, \tag{4.4} $$

where the factor $c$ is due to a rescaling of the Newtonian gravitational constant. Depending on the approach we take to reach the non-relativistic theory, this scaling could be omitted (it is just an overall scaling of the action after all). In particular, scaling $\kappa$ is not necessary in the expansion approach, as it only considers the leading terms, regardless of their power of $c$.

In accordance with the Galilei group introduced in the previous section, the vielbeine transform as

$$ \delta\tau_\mu = \lambda\tau_\mu \qquad \delta{e_\mu}^a = \lambda^a{}_b{e_\mu}^b + \lambda^a\tau_\mu. \tag{4.5} $$

The inverse vielbeine $\tau^\mu$ and $e^\mu{}_a$ are defined by the relations

$$ \tau_\mu e^\mu{}_a = 0, \qquad \tau^\mu{e_\mu}^a = 0, \tag{4.6} $$

$$ \tau_\mu\tau^\mu = 1, \qquad {e_\mu}^a e^\mu{}_b = \delta^a_b, \tag{4.7} $$

$$ {e_\mu}^a e^\nu{}_a = \delta^\nu_\mu - \tau^\nu\tau_\mu. \tag{4.8} $$

At this point, in physics literature it is conventional to write down the so-called "vielbein postulates"

$$ \begin{aligned} 0 &= \partial_\mu\tau_\nu - {\Gamma_{\mu\nu}}^\rho\tau_\rho \\ 0 &= \partial_\mu{e_\nu}^a + {\omega_\mu}^a{}_b{e_\nu}^b - {\omega_\mu}^a\tau_\nu - {\Gamma_{\mu\nu}}^\rho{e_\rho}^a. \end{aligned} \tag{4.9} $$

These can be anti-symmetrized to find

$$T_{\mu\nu}{}^{0} = 2\partial_{[\mu}\tau_{\nu]} \tag{4.10}$$
$$T_{\mu\nu}{}^{a} = 2\partial_{[\mu}e_{\nu]}{}^{a} + 2\omega_{[\mu}{}^{a}{}_{b}e_{\nu]}{}^{b} - 2\omega_{[\mu}{}^{a}\tau_{\nu]}, \tag{4.11}$$

where $T_{\mu\nu}{}^{0}$ and $T_{\mu\nu}{}^{a}$ the longitudinal and transversal projections of the torsion

$$T_{\mu\nu}{}^{\rho} = 2\Gamma_{[\mu\nu]}{}^{\rho}. \tag{4.12}$$

It should be noted that the statements (4.10) and (4.11) always hold - together they comprise the statement that the affine connection $\nabla$ with components $\Gamma_{\mu\nu}{}^{\rho}$ and the connection $\tilde{\nabla}$ with components $\omega_{\mu}{}^{\hat{A}\hat{B}}$ coincide.

With the decompositions and scalings introduced in equation (4.3), the action (4.1) becomes

$$S = \frac{1}{2\bar{\kappa}}\int d^dxe\left[e^{\mu}{}_{a}e^{\nu}{}_{b}\left(2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{bc}\omega_{\nu]}{}^{a}{}_{c}\right) + \frac{2}{c^2}\tau^{\mu}e^{\nu}{}_{a}\left(2\partial_{[\mu}\omega_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{c}\omega_{\nu]}{}^{a}{}_{c}\right)\right], \tag{4.13}$$

where $e$ is the determinant over both vielbeine, i.e. $e = \det(\tau, e)$. Taking the limit $c \to \infty$, and recognizing $2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{bc}\omega_{\nu]}{}^{a}{}_{c}$ as the curvature associated to spatial rotations, $R_{\mu\nu}{}^{ab}(J)$, we have

$$\lim_{c\to\infty} S = \frac{1}{2\bar{\kappa}}\int d^dxee^{\mu}{}_{a}e^{\nu}{}_{b}\left(2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{bc}\omega_{\nu]}{}^{a}{}_{c}\right) \tag{4.14}$$
$$= \frac{1}{2\bar{\kappa}}\int d^dxee^{\mu}{}_{a}e^{\nu}{}_{b}R_{\mu\nu}{}^{ab}(J). \tag{4.15}$$

From this action, we can derive the dynamics of the system by the principle of least action. For convenience, we define $\mathcal{L} = ee^{\mu}_{a}e^{\nu}_{b}R^{ab}_{\mu\nu}(J) := eR(J)$ . The variations of $S$ with respect to the different dynamical fields give:

$$\frac{\delta\mathcal{L}}{\delta\tau_{\mu}} = e\left(\tau^{\mu}R(J) + 2e^{\mu}{}_{a}\tau^{\rho}R_{\rho b}{}^{ab}\right), \qquad \frac{\delta\mathcal{L}}{\delta\partial_{\mu}\tau_{\nu}} = 0; \tag{4.16}$$
$$\frac{\delta\mathcal{L}}{\delta e_{\mu}{}^{a}} = e\left(e^{\mu}{}_{c}R(J) + 2e^{\mu}{}_{a}e^{\rho}{}_{c}R_{\rho b}{}^{ab}\right), \qquad \frac{\delta\mathcal{L}}{\delta\partial_{\mu}e_{\nu}{}^{a}} = 0; \tag{4.17}$$
$$\frac{\delta\mathcal{L}}{\delta\omega_{\mu}{}^{a}} = 0, \qquad \frac{\delta\mathcal{L}}{\delta\partial_{\mu}\omega_{\nu}{}^{a}} = 0; \tag{4.18}$$
$$\frac{\delta\mathcal{L}}{\delta\omega_{\mu}{}^{ab}} = 4e\left(e^{[\rho}{}_{[a}e^{\mu]}{}_{c}\omega_{\rho}{}^{c}{}_{b]}\right), \qquad \frac{\delta\mathcal{L}}{\delta\partial_{\mu}\omega_{\nu}{}^{ab}} = 2ee^{[\mu}{}_{a}e^{\nu]}{}_{b}. \tag{4.19}$$

Using the Euler-Lagrange equation, we then find the equations of motion

$$\tau^{\mu}R(J) + 2e^{\mu}{}_{a}\tau^{\rho}R_{\rho b}{}^{ab} = 0 \tag{4.20}$$
$$e^{\mu}{}_{c}R(J) + 2e^{\mu}{}_{a}e^{\rho}{}_{c}R_{\rho b}{}^{ab} = 0 \tag{4.21}$$
$$4ee_{[a}{}^{[\mu}e_{c}{}^{\nu]}\omega_{\mu}{}^{c}{}_{b]} - 2\partial_{\mu}\left(ee^{[\mu}{}_{a}e^{\nu]}{}_{b}\right) = 0 \tag{4.22}$$

Expanding out the second term of equation (4.22), we find

$$2\partial_\mu \left(e e^{[\mu}{}_a e^{\nu]}{}_b\right) = 2\left(\partial_\mu e\right) e^{[\mu}{}_a e^{\nu]}{}_b + 2e\partial_\mu \left(e^{[\mu}{}_a e^{\nu]}{}_b\right) \tag{4.23}$$

$$= 2e\left(\tau^\rho \partial_\mu \tau_\rho + e_c{}^\rho \partial_\mu e_\rho{}^c\right) e^{[\mu}{}_a e^{\nu]}{}_b + 2e\partial_\mu \left(e^{[\mu}{}_a e^{\nu]}{}_b\right), \tag{4.24}$$

with the second term of (4.24) in turn being

$$2e\partial_\mu \left(e^{[\mu}{}_a e^{\nu]}{}_b\right) = -2e\Big(\tau^{[\mu} e^{\nu]}{}_b e^\rho{}_a (\partial_\mu \tau_\rho) + \tau^{[\nu} e^{\mu]}{}_a e^\rho{}_b (\partial_\mu \tau_\rho) \tag{4.25}$$

$$+ e^{[\mu}{}_c e^{\nu]}{}_b e^\rho{}_a (\partial_\mu e_\rho{}^c) + e^{[\nu}{}_c e^{\mu]}{}_a e^\rho{}_b (\partial_\mu e_\rho{}^c)\Big) \tag{4.26}$$

Plugging this back in to equation (4.24), and regrouping terms, the final equation of motion (4.22) is then

$$\begin{gathered} 4e^{[\mu}{}_{[a} e^{\nu]}{}_c \omega_\mu{}^c{}_{b]} + 2\tau^\nu e^\mu{}_a e^\rho{}_b \partial_{[\mu}\tau_{\rho]} + 4\tau^\mu e^\nu{}_{[a} e^\rho{}_{b]} \partial_{[\mu}\tau_{\rho]} + \\ + 2e^\nu{}_c e^\mu{}_b e^\rho{}_a \partial_{[\rho} e_{\mu]}{}^c + 4e^\mu{}_c e^\nu{}_{[a} e^\rho{}_{b]} \partial_{[\mu} e_{\rho]}{}^c = 0. \end{gathered} \tag{4.27}$$

The equations of motion (4.20), (4.21) and (4.27), can be refined slightly. Immediately, combining the first two equations of motion, (4.20) and (4.21), we find

$$0 = \tau_\nu \tau^\mu R + 2e^\mu{}_a \tau_\nu \tau^\rho R_{\rho b}{}^{ab} + e_\nu{}^c e^\mu{}_c R + 2e_\nu{}^c e^\mu{}_a e^\rho{}_c R_{\rho b}{}^{ab} \tag{4.28}$$

$$= 3e^\mu{}_a R_{\nu b}{}^{ab}, \tag{4.29}$$

from which we see that

$$R_{\nu b}{}^{ab} = 0. \tag{4.30}$$

Since this statement only concerns the curvature $R_{\mu b}{}^{ab}(J)$, we interpret this directly as an equation of motion.

For $D > 2$ we can also derive other restrictions on the geometry by taking projections of equation (4.27).

First, taking a longitudinal projection, we find that only the second term contributes, giving us the geometric constraint

$$0 = e^\mu{}_{[b} e^\rho{}_{a]} \partial_\mu \tau_\rho \tag{4.31}$$

$$= e^\rho{}_b e^\mu{}_a T_{\rho\mu} \tag{4.32}$$

$$= T_{ba}(H), \tag{4.33}$$

where we interpret the condition $T_{ab} = 0$ as a constraint on the geometry instead of an equation of motion, due to the presence of a torsion tensor $T$ instead of a curvature.

The transversal projection along $e_\nu{}^a$ yields

$$0 = -2(D-3)\left(e^\mu{}_c \omega_\mu{}^c{}_b + e^\mu{}_b e^\rho{}_c (\partial_{[\mu} e_{\rho]}{}^c)\right) + 2(D-2)\left(\tau^\mu e^\rho{}_b (\partial_{[\mu}\tau_{\rho]})\right) \tag{4.34}$$

$$= -2(D-3)T_{bc}{}^c + 2(D-2)T_{0b}, \tag{4.35}$$

which we can simply rearrange to the form

$$T_{0b} = \frac{D-3}{D-2} T_{bc}{}^c. \tag{4.36}$$

Finally, we can project along ${e_\nu}^d$ to find

$$
\begin{aligned}
0 &= 2{e^\mu}_{[a}{\omega_\mu}^d{}_{b]} - 2{e^\mu}_a{e^\rho}_b\partial_{[\mu}{e_{\rho]}}^d - \delta^d_{[b}\left({e^\mu}_c{\omega_\mu}^c{}_{a]} - 4{e^\rho}_{a]}{e^\mu}_c\partial_{[\mu}{e_{\rho]}}^c\right) - 4\delta^d_{[b}{e^\rho}_{a]}\tau^\mu\partial_{[\mu}\tau_{\rho]} && (4.37)\\
&= {T_{ba}}^d + 2\delta^d_{[b}{T_{a]c}}^c - 2\delta^d_{[b}T_{|0|a]}, && (4.38)
\end{aligned}
$$

which, using equation (4.36), can be written as

$$
\implies {T_{ba}}^d = -\frac{2}{D-2}\delta^d_{[b}{T_{a]c}}^c \tag{4.39}
$$

Therefore, the equations of motion (4.20)-(4.21) can equivalently be written as

$$
\begin{aligned}
{R_{\nu b}}^{ab} &= 0 && (4.40)\\
T_{ab} &= 0 && (4.41)\\
T_{0b} &= \frac{D-3}{D-2}{T_{bc}}^c && (4.42)\\
{T_{ab}}^d &= -\frac{2}{D-2}\delta^d_{[a}{T_{b]c}}^c && (4.43)
\end{aligned}
$$

What are the interpretations of these equations? Equations (4.42) and (4.43) both involve the spin connection ${\omega_\mu}^{ab}$ and can be used to find an expression for the spin connection in terms of the vielbeine. However, since in both equations the longitudinal components of the spin connection are projected away, they cannot be solved for, and only the component ${\omega_c}^{ab}$ can be determined. The expression for this component, with all indices written down, is

$$
\omega_{cab} = {e^\rho}_{[a}{e^\mu}_{b]}\partial_\rho e_{\nu c} + {e^\rho}_{[a}{e^\nu}_{c]}\partial_\rho e_{\nu b} + {e^\rho}_{[c}{e^\nu}_{b]}\partial_\rho e_{\nu a} + \frac{4}{D-3}\delta_{c[b}{e^\rho}_{a]}\tau^\nu\partial_{[\rho}\tau_{\nu]}, \tag{4.44}
$$

where $\omega_{cab}$ is explicitly anti-symmetric in the last two indices $a$ and $b$. In (super-)gravity literature, equations that are used to solve for spin connection (components) are referred to as *conventional constraints* [24, 25]. This will also be the language used throughout section 5.

On the other hand, equations (4.40) and (4.41) are purely constraints on the geometry. In fact, we can identify (4.41) as the *twistless torsion* constraint that has been found in relation to Galilei and Newton-Cartan gravity [9, 26]. We can write the constraint in two different ways

$$
{T_{ab}}^0 = 0 \iff \tau \wedge d\tau = 0. \tag{4.45}
$$

From the second formulation it is more clear that this is in fact a condition on integrability. Namely, the base manifold is foliated by hyperplanes of equal universal time. This means that all observers can agree on the ordering of events, and in particular on what events are simultaneous, thus generating slices of simultaneous events on the manifold. Note that since the totality of time-like torsion ${T_{\mu\nu}}^0$ has not been set to zero, it is still possible for there to remain some effects that do not usually appear in non-relativistic theories, such as time dilation, regardless of the aforementioned foliation.

# Chapter 5

# $p$-brane Galilei Geometry

In this section, it is our desire to classify the different $p$-brane spacetimes that result from Galilei and Carrol $G$-structures. Because this classification can be done in terms of the torsion of the space, in order to complete the classification we need to simply determine the different possible torsions that the geometry can have. Since we are free to pick, or change our connection, the specific interest lies in those parts of the torsion that do not depend on the choice of connection. This type of torsion we call *intrinsic*. There are two similar approaches to classifying the intrinsic torsion - it can be done in the Cartan language of principal bundles, or in what resembles the formalism of non-coordinate bases. Although these two "languages" describe largely the same objects, the former approach allows for a more generalizable, systematic classification, while the latter is faster, and more convenient for concrete computations. We will refer to the former approach as the "mathematical perspective", and the latter as the "physics perspective", based on the literature that inspired each approach.

Let us first make some general remarks, and recall some concepts from section 2. First, as a tensor, we write the torsion as $T_{ab}{}^{A}$, with anti-symmetry in the indices $a$ and $b$. This hints at the space that the torsion (when expressed on the manifold) is an element of,

$$T^{\nabla} \in \hom(\wedge^2 TM, TM) \cong \hom(\wedge^2 \mathbb{V}, \mathbb{V}). \tag{5.1}$$

Where we would like to remind the reader that we are using $\mathbb{V}$ as notation for $\mathbb{R}^n$. Here, we will proceed to break $\mathbb{V}$ down to two components

$$\mathbb{V} = V \oplus W. \tag{5.2}$$

In the case of Galilei spacetimes, $V$ would be termed the "longitudinal" component of $\mathbb{V}$, and $W$ as the "transverse" one. Note that here we do not yet fix the interpretation of $V$ or $W$ at all; this is to enable everything that follows to work for both Galilei and Carroll $G$-structures. It is important to emphasize that in (5.2) we are making an implicit choice of $V$, which is not "natural", i.e. our choice is made arbitrarily. This choice can be avoided altogether, and this is the approach adopted in the mathematics part of a paper that is to appear on Arxiv. However, since we are interested in the physical characterization of spacetimes, we will proceed with a choice of longitudinal subspace $V$. Note that we then work in an explicitly basis dependent way.

With remarks out of the way, let us proceed with the classification, first from the perspective of mathematics, and then in that of physics.

## 5.1 From the Perspective of Mathematics

Let us establish bases for the two components of $\mathbb{V}$; $V = \langle \pi_A \rangle$, and $W = \langle \pi_a \rangle$, where by $\langle \pi \rangle$ we denote the span of $\pi$. In what follows, we will adopt the physics convention and have lowercase Latin indices $a, b$ range over the transversal component of $\mathbb{V}$, and uppercase ones $A, B$ to range over the longitudinal one.

### 5.1.1 Torsion and the Spencer Differential

Recall (see section 2.4) that the torsion of a manifold can be described in two different ways, as a two-form on the sub-bundle $P$ of the frame bundle, with values in the Lie algebra $\mathfrak{g}$, given by

$$\Theta = d\theta + \omega \wedge \theta, \tag{5.3}$$

or as a tensor on the manifold $M$

$$T^{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{5.4}$$

Not much attention was paid to the torsion in section 2, because in GR it is always to pick a connection with no torsion. However, this is no longer the case for other kinds of gravitational theories, in particular in the theories of interest here.

Let us define the contorsion $\kappa$ to be a difference between two connections $\omega$ and $\omega'$,

$$\kappa = \omega' - \omega. \tag{5.5}$$

Looking at the corresponding difference in torsion, we find that the first term drops out, giving the difference between the torsions of the connections

$$\Theta - \Theta' = \kappa \wedge \theta. \tag{5.6}$$

Since this object should be of the same kind as $\Theta$, we can conclude that $\Theta - \Theta' \in \Omega^2(P, \mathbb{V})$. Plugging in two vector fields $X$ and $Y$ on $P$, $\Theta - \Theta'$ reads

$$(\Theta - \Theta')(X, Y) = \kappa(X)\theta(Y) - \kappa(Y)\theta(X) \tag{5.7}$$
$$= (\partial\kappa)(X, Y). \tag{5.8}$$

In order to pass from $\kappa$ to $\Theta - \Theta'$, we have defined a map $\partial$,

$$\partial : \Omega^1(P, \mathfrak{g}) \to \Omega^2(P, \mathbb{V}) \tag{5.9}$$
$$\kappa \mapsto \partial\kappa = \kappa \wedge \theta. \tag{5.10}$$

The map $\partial$ in fact descents from a linear map, to which we will interchangeably refer to as $\partial$ as well. This map is a special case of a *Spencer differential*, and is defined by

$$\begin{aligned} \partial : \mathrm{Hom}(\mathbb{V}, \mathfrak{g}) &\to \mathrm{Hom}(\wedge^2\mathbb{V}, \mathbb{V}) \\ (\partial\kappa)(v, w) &= \kappa(v)w - \kappa(w)v, \end{aligned} \tag{5.11}$$

which now sends a map $\kappa : \mathbb{V} \to \mathfrak{g}$ to $\partial\kappa : \wedge^2\mathbb{V} \to \mathbb{V}$. In other words, $\partial$ sends a contorsion $\kappa$ to the associated difference in torsion. Since the contorsion is defined by $\kappa = \omega - \omega'$, we can describe the Spencer differential as sending a difference in connections to a difference in their torsions

$$\text{“}\partial : \nabla - \nabla' \to T^{\nabla} - T^{\nabla'}\text{”}. \tag{5.12}$$

The Spencer differential (5.11) can also be understood as a composite map

$$\mathbb{V}^* \otimes \mathfrak{g} \xrightarrow{\mathrm{id}_{\mathbb{V}^*}\otimes\rho} \mathbb{V}^* \otimes \mathbb{V}^* \otimes \mathbb{V} \xrightarrow{\wedge\otimes\mathrm{id}_{\mathbb{V}}} \wedge^2\mathbb{V}^* \otimes \mathbb{V}. \tag{5.13}$$

In the above, the first step assigns to the Lie algebra a representation

$$\rho : \mathfrak{g} \to \mathbb{V}^* \otimes \mathbb{V}, \tag{5.14}$$

and the second on anti-symmetrizes the result in the two copies of $\mathbb{V}^*$. Employing an isomorphism $\mathrm{Hom}(\mathbb{V}, \mathfrak{g}) \cong \mathfrak{g} \otimes \mathbb{V}^*$, we can associate to the Spencer differential the exact sequence

$$0 \to \ker\partial \to \mathfrak{g} \otimes \mathbb{V}^* \xrightarrow{\partial} \mathbb{V} \otimes \wedge^2\mathbb{V}^* \to \operatorname{coker}\partial \to 0 \tag{5.15}$$

Figure 5.1 equivalently illustrates this exact sequence, with the leftmost space the domain of the map $\partial$, highlighting the kernel of the map. The middle space is the codomain, together with the image of $\partial$. Finally, the rightmost space presents the cokernel of $\partial$, and seeks to highlight the fact that the cokernel is a coset space, with elements being equivalence classes up to the image of the Spencer differential. Elements of the spaces involved in the sequence have the following significances:

1. $\ker\partial$: the difference in connection does not result in a difference in torsion.
2. $\operatorname{im}\partial$: torsions which result purely from a difference in connection. In physics (especially in the field of super-gravity), torsions of this type are referred to as *conventional* tensors, which can be turned into constraints by enforcing these torsions to become zero. The equations resulting from the definition of the torsions when this constraint is enforced can then be used to solve for spin-connections in terms of the vielbeine.
3. $\operatorname{coker}\partial$: the torsions which cannot result from differences in connection. The torsions that make up the cokernel are termed *intrinsic*, and can be used to classify spacetimes associated to $G$, due to their independence of the choice of connection.

Here it is good to point out that by a pure coincidence in the Galilean (and Carrollian) case the spaces $\ker\partial$ and $\operatorname{coker}\partial$ are isomorphic.

Let us now proceed to find the structure of the cokernel of $\partial$. We can begin by inspecting the transformations of each element of $\mathbb{V}^* \otimes \mathfrak{g}$ under the Spencer map. Since the Galilei and Carroll Lie algebras can be described in terms of the generators $L_{ab}, L_{\alpha\beta}$, and $B_{a\alpha}$, of $SO(V)$, $SO(W)$, and the appropriate boosts, respectively, we can focus our attention on these basis elements to study the structure of the spacetimes the algebras generate. First, under the aforementioned representation $\rho$, the basis elements of $\mathfrak{g}$ are given by

$$\rho(L_{ab}) = \pi_{[b} \otimes P_{a]} \tag{5.16}$$

$$\rho(L_{AB}) = \pi_{[B} \otimes P_{A]} \tag{5.17}$$

$$\rho(B_{aA}) = -\pi_A \otimes P_b. \tag{5.18}$$

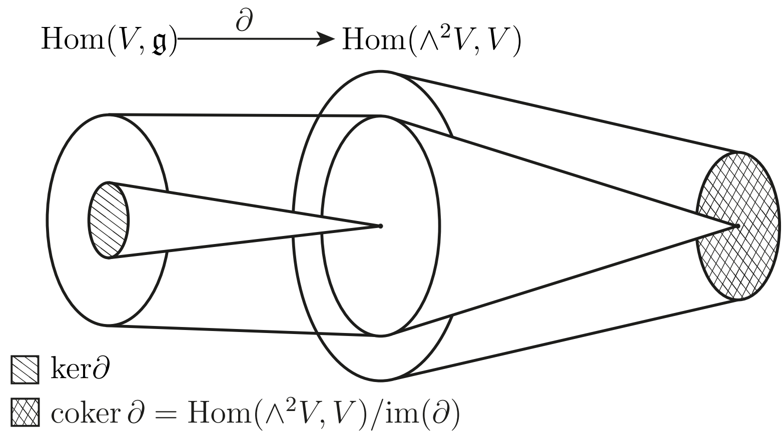


Figure 5.1: Spaces under $\partial$. The final step seeks to illustrate the fact that the cokernel is a coset space.

We can then find the transformations of elements in $\mathbb{V}^* \otimes \mathfrak{g}$ under $\partial$. These are

$$\begin{aligned}
\partial(\pi^c \otimes L_{ab}) &= \pi^c \wedge \pi_{[b} \otimes P_{a]} \\
\partial(\pi^c \otimes L_{AB}) &= \pi^c \wedge \pi_{[B} \otimes P_{A]} \\
\partial(\pi^c \otimes B_{aA}) &= -\pi^c \wedge \pi_A \otimes P_a,
\end{aligned} \tag{5.19}$$

as well as the same ones, but with the transverse index $c$ replaced by a longitudinal one, $C$:

$$\begin{aligned}
\partial(\pi^C \otimes L_{ab}) &= \pi^C \wedge \pi_{[b} \otimes P_{a]} \\
\partial(\pi^C \otimes L_{AB}) &= \pi^C \wedge \pi_{[B} \otimes P_{A]} \\
\partial(\pi^C \otimes B_{aA}) &= -\pi^C \wedge \pi_A \otimes P_a.
\end{aligned} \tag{5.20}$$

The kernel of $\partial$ can then be systematically determined

**Theorem 5.1.1.** *The kernel of $\partial$ is given by $\langle \pi_{(B} \otimes B_{|a|A)}, \pi_{[a} \otimes B_{b]A} + \pi_A \otimes L_{ab} \rangle$*

*Proof.* The kernel of $\partial$ is given by those combinations of elements of equations (5.19) and (5.20) that are sent to zero under the map $\partial$. We already examined the image of the basis elements of $\mathbb{V}^* \otimes \mathfrak{g}$ under this map. We can then write a generic vector $v$ in $\mathbb{V}^* \otimes \mathfrak{g}$ in terms of the basis as[1]

$$\begin{aligned}
v = \frac{1}{2} v^{cab}(\pi_c \otimes L_{ab}) + \frac{1}{2} v^{cAB}(\pi_c \otimes L_{AB}) + v^{caA}(\pi_c \otimes B_{aA}) + \\
+ \frac{1}{2} v^{Cab}(\pi_C \otimes L_{ab}) + \frac{1}{2} v^{CAB}(\pi_C L_{AB}) + v^{CaA}(\pi_C \otimes B_{aA}),
\end{aligned} \tag{5.21}$$

where all the coefficients $v^{\hat{A}bc}$ and $v^{\hat{A}BC}$ of the $L_{ab}$ and $L_{AB}$ terms are anti-symmetric in the last two indices.

[1] Here, for clarity we write all indices down. The proper positioning of the indices on the basis would be of the form $\pi^{\hat{A}} \otimes T_{\hat{B}\hat{C}}$ and correspondingly $\pi^{\hat{A}} \wedge \pi_{\hat{C}} \otimes P_{\hat{B}}$. The index positioning of the coefficients $v$ should be the opposite.

In order to find the composition of the kernel, we examine the image of the vector (5.21) under $\partial$, and then set it to zero. We then wish to solve the equation

$$\begin{aligned}\partial v = v^{cab}(\pi_c \wedge \pi_b \otimes P_a) + v^{cAB}(\pi_c \wedge \pi_B \otimes P_A) - v^{caA}(\pi_c \wedge \pi_A \otimes P_a) + \\ + v^{Cab}(\pi_C \wedge \pi_b \otimes P_a) + v^{CAB}(\pi_C \wedge \pi_B \otimes P_A) - v^{CaA}(\pi_C \wedge \pi_A \otimes P_a) = 0,\end{aligned} \tag{5.22}$$

where we have exploited the aforementioned anti-symmetry of the coefficients to get rid of the anti-symmetrizations present in equations (5.19) and (5.20).

The above terms can be grouped to reduce the number of terms by one. The remaining terms are

1. $v^{cab}(\pi_c \wedge \pi_b \otimes P_a)$,
2. $v^{cAB}(\pi_c \wedge \pi_B \otimes P_A)$,
3. $(v^{Cab} + v^{baC})\pi_C \wedge \pi_b \otimes P_a$,
4. $v^{CAB}(\pi_C \wedge \pi_B \otimes P_A)$,
5. $v^{CaA}(\pi_C \wedge \pi_A \otimes P_a)$.

We can then set each term to zero separately.

1. $v^{cab}(\pi_c \wedge \pi_b \otimes P_a) = 0$ implies, by the anti-symmetry of $\pi_c \wedge \pi_b$, that $v^{cab} = v^{bac}$. However, using the anti-symmetry of $v^{cab}$ in the last two indices, this sets all components of $v^{cab}$ to zero,
2. $v^{cAB}(\pi_c \wedge \pi_B \otimes P_A) = 0$ implies $v^{cAB} = 0$,
3. $(v^{Cab} + v^{baC})\pi_C \wedge \pi_b \otimes P_a = 0$ implies $-v^{Cab} = v^{baC} \implies v^{Cba} = v^{baC}$,
4. $v^{CAB}(\pi_C \wedge \pi_B \otimes P_A) = 0$ implies $v^{CAB} = 0$ by the same argument as in case 1,
5. $v^{CaA}(\pi_C \wedge \pi_A \otimes P_a) = 0$ implies $v^{CaA} = v^{AaC}$ due to the anti-symmetry in $\pi_C \wedge \pi_A$.

Thus the components in cases 1, 2, and 4 are trivially zero. The kernel is then given by the two remaining cases. In fact, we can be slightly more strict - the condition of case 3 can be broken down due to the anti-symmetry of $v^{Cba}$

$$v^{C(ba)} = 0 \implies v^{(ba)C} = 0 \tag{5.23}$$

trivially. Thus the non-trivial solution is given by

$$v^{[ba]C} = v^{Cba}. \tag{5.24}$$

We can translate this to a condition on the basis, instead of the coefficients. The resulting structure of the kernel is given by

$$v = \frac{1}{2}v^{CaA}(\pi_{(C|} \otimes B_{a|A)}) + \frac{1}{2}v^{Cab}(\pi_C \otimes L_{ab} + \pi_{[b} \otimes B_{a]C}), \tag{5.25}$$

which, as required, shows that the kernel is given by

$$\ker \partial = \langle \pi_{(C|} \otimes B_{a|A)}, \pi_C \otimes L_{ab} + \pi_{[b} \otimes B_{a]C} \rangle. \tag{5.26}$$

■

We can now move on to find the cokernel. It is given by

**Theorem 5.1.2.** *The cokernel of $\partial$ is given by $\langle[\pi^a \wedge \pi^b \otimes P_C], [\pi^a \wedge \pi_{(B} \otimes P_{C)}]\rangle$, where $[\pi]$ denotes the equivalence class of $\pi$*

*Proof.* We know from equation (5.22) and the subsequent analysis that the image of $\mathbb{V}^* \otimes \mathfrak{g}$ under $\partial$ is given by

$$\partial v = v^{cab}(\pi_c \wedge \pi_b \otimes P_a) + v^{cAB}(\pi_c \wedge \pi_B \otimes P_A) + (v^{Cab} + v^{baC})\pi_C \wedge \pi_b \otimes P_a + \tag{5.27}$$

$$+ v^{CAB}(\pi_C \wedge \pi_B \otimes P_A) - v^{CaA}(\pi_C \wedge \pi_A \otimes P_a) = 0. \tag{5.28}$$

We then want to inspect which components of the space $\wedge^2\mathbb{V}^* \otimes \mathbb{V}$ are not included in the above expression. We can write a generic vector $\xi$ in the space $\wedge^2\mathbb{V}^* \otimes \mathbb{V}$ in terms of a basis as

$$\xi = \frac{1}{2}\xi^{abc}(\pi_a \wedge \pi_b \otimes P_c) + \frac{1}{2}\xi^{abC}(\pi_a \wedge \pi_b \otimes P_C) + \frac{1}{2}\xi^{ABc}(\pi_A \wedge \pi_B \otimes P_c) + \tag{5.29}$$

$$+ \frac{1}{2}\xi^{ABC}(\pi_A \wedge \pi_B \otimes P_C) + \xi^{aBc}(\pi_a \wedge \pi_B \otimes P_c) + \xi^{aBC}(\pi_a \wedge \pi_B \otimes P_C). \tag{5.30}$$

We now wish to find which components form the quotient space $\frac{\wedge^2\mathbb{V}^* \otimes \mathbb{V}}{\operatorname{im}\partial}$. We can consider this quotient by basis element to see which quotients remain nontrivial.

1. $\frac{1}{2}\xi^{abc}(\pi_a \wedge \pi_b \otimes P_c)/v^{abc}(\pi_a \wedge \pi_b \otimes P_c)$. This component is trivial.
2. $\frac{1}{2}\xi^{abC}(\pi_a \wedge \pi_b \otimes P_C)$. We see that there is no corresponding term in the vector $\partial v$. Thus this term as a whole lands in $\operatorname{coker}\partial$.
3. $\frac{1}{2}\xi^{ABc}(\pi_A \wedge \pi_B \otimes P_c)/v^{AcB}(\pi_A \wedge \pi_B \otimes P_c)$. Neither of the coefficients has any (anti-)symmetry, and thus this component of the quotient is trivial.
4. $\frac{1}{2}\xi^{ABC}(\pi_A \wedge \pi_B \otimes P_C)/v^{ABC}(\pi_A \wedge \pi_B \otimes P_C)$. The coefficient $v^{ABC}$ is anti-symmetric in the last two indices, but we can simply set $\xi^{A(BC)} = 0$, and equate the rest of the coefficient components. This implies that the component is trivial.
5. $\xi^{aBc}(\pi_a \wedge \pi_B \otimes P_c)/(v^{Cab} + v^{baC})\pi_C \wedge \pi_b \otimes P_a$. Since $v^{baC}$ has no symmetries, this coefficient can be used to account for the non-symmetric components of $\xi^{aBc}$, and thus this component is also trivial.
6. $\xi^{aBC}(\pi_a \wedge \pi_B \otimes P_C)/v^{aBC}(\pi_a \wedge \pi_B \otimes P_C)$. Since $v^{aBC} = -v^{aCB}$, the symmetric component of $\xi^{aBC}$, $\xi^{a(BC)}$ remains independent and is thus in the cokernel.

Then the components of $\xi$ which remain in the quotient are

$$\frac{1}{2}\xi^{abC}(\pi_a \wedge \pi_b \otimes P_C) + \frac{1}{2}\xi^{a(BC)}(\pi_a \wedge \pi_B \otimes P_C). \tag{5.31}$$

Transferring the symmetrization onto the basis element, we find that the cokernel of the Spencer differential is given by the span of the equivalence classes (mod $\operatorname{im}\partial$)

$$\langle[\pi^a \wedge \pi^b \otimes P_C], [\pi^a \wedge \pi_{(B} \otimes P_{C)}]\rangle, \tag{5.32}$$

as required. ∎

Thus, in the case where $G$ is the p-brane Galilei group, we find that $\operatorname{coker}\partial$ breaks down to the following components

$$\operatorname{coker}\partial = \langle [\pi^a \wedge \pi^b \otimes P_C], [\pi^a \wedge \pi_{(B} \otimes P_{C)}] \rangle, \tag{5.33}$$

where the intrinsic torsion could reside in either the entirety of the cokernel, the symmetric $G$-submodule $\mathcal{T}_1 = [\pi^a \wedge \pi_{(b} \otimes P_{\gamma)}]$, the traceless part $\mathcal{T}_1^0$ of $\mathcal{T}_1$, in the trace $\mathcal{T}_1^{\text{tr}}$, or be zero. It is not possible for the intrinsic torsion to be only in the component $\mathcal{T}_2$, since under boosts

$$\mathcal{T}_2 = [\pi^a \wedge \pi^b \otimes P_C] \xrightarrow{B_{Aa}} \mathcal{T}_1, \tag{5.34}$$

meaning that $\mathcal{T}_2$ is not a $G$-submodule. The remaining $G$-submodules form a chain

$$0 \subset \langle \mathcal{T}_1^{\text{tr}} \rangle, \langle \mathcal{T}_1^0 \rangle \subset \langle \mathcal{T}_1 \rangle \subset \operatorname{coker}\partial, \tag{5.35}$$

of spaces which the intrinsic torsion can be an element of. This corresponds to the Hasse diagram portrayed in figure 5.2. In the figure, the arrows indicate the action of the boosts $B_{Aa}$ on the submodules.

$$\begin{tikzcd}
 & & \langle \mathcal{T}_1^{\text{tr}} \rangle \arrow[dr] & \\
\operatorname{coker}\partial \arrow[r] & \langle \mathcal{T}_1 \rangle \arrow[ur] \arrow[dr] & & 0 \\
 & & \langle \mathcal{T}_1^0 \rangle \arrow[ur] &
\end{tikzcd}$$

Figure 5.2: A Hasse diagram portraying the relations between the different $G$-submodules of $\operatorname{coker}\partial$. The arrows indicate the action of boosts on the submodules.

In terms of constraints on the torsion tensor $T_{\mu\nu}{}^{\hat{A}}$, as is typical in physics, we can also write the above options as the following[2]

1. For $T^\nabla \in \operatorname{coker}\partial$ the intrinsic torsion is not constrained.
2. For $T^\nabla \in \langle \mathcal{T}_1 \rangle$, $T_{ab}{}^A = 0$.
3. For $T^\nabla \in \langle \mathcal{T}_1^{\text{tr}} \rangle$, the constraint of 2 holds, together with $T_a{}^{(AB)} - \frac{1}{2}\eta^{AB} T_{aC}{}^C = 0$, i.e. $T_{ab}{}^A$ has only a trace component
4. For $T^\nabla \in \langle \mathcal{T}_1^0 \rangle$, the constraint of 2 also holds, together with $T_{aA}{}^A = 0$ and $T_a{}^{AB} = T_a{}^{BA}$, i.e. $T_{ab}{}^A$ is symmetric and traceless.
5. For $T^\nabla = 0$, we find $T_{\mu\nu}{}^{\hat{A}} = 0$.

In summary, we find the following

**Theorem 5.1.3.** *The intrinsic torsion $T^\nabla$ of a p-brane Galilean geometry can reside in either* $\operatorname{coker}\partial$, $\mathcal{T}_1$, $\mathcal{T}_1^0$, $\mathcal{T}_1^{tr}$, *or be* $0$. *These options have the following interpretations*

1. *$T^\nabla \in \operatorname{coker}\partial$. The intrinsic torsion is generic. For $Y \in \Gamma(E)$, and $X \in \mathscr{X}(M)$, we have $\nabla_X Y \in \Gamma(E)$,*
2. *$T^\nabla \in \langle \mathcal{T}_1 \rangle$. The underlying manifold $M$ is foliated by integrable submanifolds of $E$,*

[2]Here we fully adopt the physics conventions on indices: greek indices range over all curved values, capital indices with hats range over all tangent-space values, while the others remain the same.

*3. $T^\nabla \in \langle \mathcal{T}_1^{tr} \rangle$. $\mathcal{L}_X \eta = c(X)\eta$ for any $X \in \Gamma(E)$, with $c(X) = \frac{1}{2}h(\xi, X)$ i.e. transversal vectors are $\eta$-homothetic,*

*4. $T^\nabla \in \langle \mathcal{T}_1^0 \rangle$. The top form $\Omega$ is closed,*

*5. $T^\nabla = 0$. Then for all $X \in \Gamma(E)$, $\mathcal{L}_X \eta = 0$, i.e. any transversal $X$ is $\eta$-killing.*

*Proof.* 1. Since $\nabla$ is adapted, $\nabla \eta = 0$, and

$$0 = (\nabla_X \eta)(Y, Z) = X\eta(Y, Z) - \eta(\nabla_X Y, Z) - \eta(Y, \nabla_X Z) \tag{5.36}$$

for any vector fields $X, Y, Z \in \mathscr{X}(M)$. By letting $Y \in \Gamma(E)$ and letting $X$ and $Z$ be arbitrary vector fields, we obtain the desired relation.

2. First, note that if $[X, Y] \in \Gamma(E)$ for all $X, Y \in \Gamma(E)$, then Frobenius' theorem implies that $M$ is foliated by integrable submanifolds of $E$. It then remains to be shown that $[X, Y] \in \Gamma(E)$.

Let $X, Y \in \Gamma(E)$. Then

$$0 = T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{5.37}$$

Note that the first two terms of the right-most expression are in $\Gamma(E)$ by part 1. of theorem 5.1.3, and that then $[X, Y]$ has to be in $\Gamma(E)$ as well.

3. We compute

$$\begin{aligned} (\mathcal{L}_X \eta)(Y, Z) &= X\eta(Y, Z) - \eta([X, Y], Z) - \eta(Y, [X, Z]) &(5.38)\\ &= \eta(T^\nabla(X, Y), Z) + \eta(Y, T^\nabla(X, Z)), &(5.39) \end{aligned}$$

where we can, for both terms, write

$$\begin{aligned} \eta(Y, T^\nabla(X, Z)) = \eta(Y, T^\nabla(X, Z)) &- \frac{1}{2}\eta(\varepsilon_C, T^\nabla(X, \varepsilon^C))\eta(Y, Z) \\ &+ \frac{1}{2}\eta(\varepsilon_C, T^\nabla(X, \varepsilon^C))\eta(Y, Z). \end{aligned} \tag{5.40}$$

Since $T_a^{AB} = \eta(\varepsilon^A, T^\nabla(X_a, \varepsilon^B))$, and for $T^\nabla \in \langle \mathcal{T}_1^{\mathrm{tr}} \rangle$

$$T_a{}^{(AB)} - \frac{1}{2}\eta^{AB} T_{aC}{}^C = 0, \tag{5.41}$$

we conclude that the first two terms of equation (5.40) equal to zero, and thus

$$\begin{aligned} (\mathcal{L}_X \eta)(Y, Z) &= \eta(\varepsilon_C, T^\nabla(X, \varepsilon^C))\eta(Y, Z) &(5.42)\\ &= c(X)\eta(Y, Z). &(5.43) \end{aligned}$$

To find the form of $c(X)$, we note that the map $h^\flat : \Gamma(E) \to \Gamma(E^*)$, given by $h^\flat(X_1)(X_2) = h(X_1, X_2)$ is an isomorphism, and since $c(X)$ is linear, it is an element of $\Gamma(E^*)$. Then by the non-degeneracy of $h$, $c(X) = \frac{1}{2}h(\xi, X)$, for some unique $\xi \in \Gamma(E)$.

4. Note that we can write $\Theta^\alpha$ in two different ways, namely

$$\Theta^A = d\theta^A + \omega^A{}_B \wedge \theta^B, \tag{5.44}$$

and

$$\Theta^A = T_{aB}{}^A \theta^a \wedge \theta^B + \frac{1}{2} T_{BC}{}^A \theta^B \wedge \theta^C. \tag{5.45}$$

This allows us to write $d\theta^A$ as

$$\begin{aligned} d\theta^A &= T_{aB}{}^A \theta^a \wedge \theta^B - \omega^A{}_B \wedge \theta^B + \frac{1}{2} T_{BC}{}^A \theta^B \wedge \theta^C & (5.46) \\ &= (T_{aB}{}^A \theta^a - \omega^A{}_B - \frac{1}{2} T_{BC}{}^A \theta^C) \wedge \theta^B. & (5.47) \end{aligned}$$

Now, taking the differential of the top form $\Omega$, defined by

$$\Omega = \frac{1}{(p+1)!} \theta^{A_0} \wedge \cdots \wedge \theta^{A_p}. \tag{5.48}$$

Using the form found for $d\theta^A$ in equation (5.47), we find

$$\begin{aligned} d\Omega &= \frac{1}{(p+1)!} (p+1) d\theta^{A_0} \wedge \cdots \wedge \theta^{A_p} & (5.49) \\ &= \frac{1}{p!} (T_{aB}{}^{A_0} \theta^a - \omega^{A_0}{}_B - \frac{1}{2} T_{BC}{}^{A_0} \theta^C) \wedge \theta^B \wedge \cdots \wedge \theta^{A_p}. & (5.50) \end{aligned}$$

Recognizing the final factor as $\delta^B_{A_0}\Omega$ gives

$$\begin{aligned} d\Omega &= \frac{1}{p!} \delta^B_{A_0} (T_{aB}{}^{A_0} \theta^a - \omega^{A_0}{}_B - \frac{1}{2} T_{BC}{}^{A_0} \theta^C) \wedge \Omega & (5.51) \\ &= (T_{aB}{}^B \theta^a - \omega^B{}_B) \wedge \Omega, & (5.52) \end{aligned}$$

which is zero, since $\omega^A{}_B$ is anti-symmetric, and $T_{aA}{}^A = 0$.

5. ( $\Longrightarrow$ ) Let $X \in \Gamma(E)$. Then

$$\begin{aligned} (\mathcal{L}_X \eta)(Y, Z) &= X\eta(Y, Z) - \eta([X, Y], Z) - \eta(Y, [X, Z]) & (5.53) \\ &= \eta(\nabla_X Y - [X, Y], Z) + \eta(Y, \nabla_X Z - [X, Z]) & (5.54) \\ &= \eta(T^\nabla(X, Y), Z) + \eta(Y, T^\nabla(X, Z)), & (5.55) \end{aligned}$$

which is clearly zero when $T^\nabla$ is zero.

( $\Longleftarrow$ ) Let $\mathcal{L}_X \eta = 0$ for any $X \in \Gamma(E)$. Then $T^\nabla(X, -) \in SO(V) \oplus W$, both of which are in the image of $\partial$, and thus there exists a connection $\nabla'$ such that $T^{\nabla'} = 0$. ■

## 5.2 From the Perspective of Physics

In the "physics language", the main focus is on determining constraints on the torsion, in contrast to determining the sub-representations in which the torsion could reside. In the end the procedure ends up being very analogous to that of the last part, although this similarity is well disguised by the fundamental difference in thinking, and as a result in approach. The procedure of this section will be very closely mirror that used in section 4, i.e.

1. Find an expression for the torsion in terms of the vielbeine and spin-connections.
2. Determine what happens to which components of the spin-connections as we apply the Spencer map to them. This provides us with the components that constitute $\ker \partial$, and those that are completely dependent on torsion, in other words, $\operatorname{im} \partial$.
3. Pick out those components of the torsion that do not contain any spin-connection components. These constitute $\operatorname{coker} \partial$.
4. Derive the geometric constraints that result from constraining each components occurring in $\operatorname{coker} \partial$ to be zero.

So, we begin by breaking the relativistic vielbein into $p+1$ longitudinal and $D-p-1$ transverse components by

$$E_\mu{}^A = c\tau_\mu{}^A, \qquad E_\mu{}^a = e_\mu{}^a, \tag{5.56}$$

where we have broken down the relativistic flat index $\hat{A}$ into $\hat{A} = (A, a)$, with $A = 0, \ldots, p-1$, and $a = p, \ldots, D-1$. Under the Galilei group, the new non-relativistic vielbeine transform in the expected way as

$$\delta\tau_\mu{}^A = \lambda^A{}_B \tau_\mu{}^B \qquad \delta e_\mu{}^a = \lambda^a{}_b e_\mu{}^b + \lambda^a{}_B \tau_\mu{}^B, \tag{5.57}$$

with $\lambda_{AB}$, $\lambda_{ab}$, and $\lambda_{aA}$ the transformation parameters of the longitudinal Lorentz transformations, transversal rotations, and Galilean boosts respectively.

It is good to again emphasize that although we will usually explicitly refer only to the Galilei case, any results can be formally related to the Carrollian one by an exchange of the signatures of the conserned metrics, as well as exchanging $p$-branes for $(D-p-2)$-branes. The exchange of metrics can be seen as a way of reinterpreting "where the time resides" in the final spacetime. This is in an exact correspondence to an exchange of the longitudinal and transversal directions. Formally we could write an exchange

$$\left(\tau_\mu{}^A, e_\mu{}^a\right) \longleftrightarrow \left(e_\mu{}^b, \tau_\mu{}^B\right). \tag{5.58}$$

A corresponding exchange of longitudinal and transversal directions can be seen in the exchange of the capital indices with new lowercase ones. The ranges of the dimensions have also been switched, i.e. range $A$ = range $b$, and range $a$ = range $B$.

The inverses of the vielbeine can be introduced through the equations

$$\tau^\mu{}_A \tau_\mu{}^B = \delta^B_A \qquad e_\mu{}^a e^\mu{}_b = \delta^a_b \tag{5.59}$$

$$\tau^\mu{}_A e_\mu{}^a = 0 \qquad e^\mu{}_a \tau_\mu{}^A = 0 \tag{5.60}$$

$$\tau_\mu{}^A \tau^\nu{}_A + e_\mu{}^a e^\nu{}_a = \delta^\nu_\mu. \tag{5.61}$$

The transformations of these inverses are given by

$$\delta\tau^\mu{}_A = \lambda_A{}^B \tau^\mu{}_B + \lambda^a{}_A e^\mu{}_a \qquad \delta e^\mu{}_a = \lambda_a{}^b e^\mu{}_b. \tag{5.62}$$

It may be noted that the inverses transform, similarly to the particle case, the "opposite" way to the vielbeine in terms of Galilean boosts.

Recalling from section 2 that the connection $\Omega_\mu$ takes values in the lie algebra $\mathfrak{g}$ of the structure group $G$, we can write it in terms of the generators of $\mathfrak{g}$ as

$$\Omega_\mu = \omega_\mu{}^{AB} J_{AB} + \omega_\mu{}^{ab} J_{ab} + \omega_\mu{}^{aA} J_{aA}, \tag{5.63}$$

where $J_{AB}$, $J_{ab}$, and $J_{aA}$ generate longitudinal Lorentz transformations, transversal rotations, and $p$-brane Galilean boosts, respectively. The "vielbein postulates" now take the form

$$0 = \partial_\mu \tau_\nu{}^A - \omega_\mu{}^A{}_B \tau_\nu{}^B - \Gamma_{\mu\nu}{}^\rho \tau_\rho{}^A, \tag{5.64}$$
$$0 = \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\mu{}^a{}_A \tau_\nu{}^A - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a. \tag{5.65}$$

These postulates are, again, nothing but a statement of the equality of the connections associated to $\omega$ and $\Gamma$. As before, we anti-symmetrize the equations to find two expressions in terms of the longitudinal component $T_{\mu\nu}{}^A$ and transversal component $T_{\mu\nu}{}^a$ of the torsion tensor $T_{\mu\nu}{}^{\hat{A}}$

$$\begin{aligned} T_{\mu\nu}{}^A &= 2\partial_{[\mu}\tau_{\nu]}{}^A - 2\omega_{[\mu}{}^A{}_B \tau_{\nu]}{}^B, \\ T_{\mu\nu}{}^a &= 2\partial_{[\mu} e_{\nu]}{}^a - 2\omega_{[\mu}{}^a{}_b e_{\nu]}{}^b - 2\omega_{[\mu}{}^a{}_A \tau_{\nu]}{}^A. \end{aligned} \tag{5.66}$$

It is possible to connect this to the discussion in the previous section by absorbing the curl terms into the definitions of the torsion components, i.e.

$$\widetilde{T}_{\mu\nu}{}^A := T_{\mu\nu}{}^A - 2\partial_{[\mu}\tau_{\nu]}{}^A \tag{5.67}$$
$$\widetilde{T}_{\mu\nu}{}^a := T_{\mu\nu}{}^a - 2\partial_{[\mu} e_{\nu]}{}^a. \tag{5.68}$$

The equations (5.66) then become

$$\widetilde{T}_{\mu\nu}{}^A = -2\omega_{[\mu}{}^A{}_B \tau_{\nu]}{}^B, \tag{5.69}$$
$$\widetilde{T}_{\mu\nu}{}^a = -2\omega_{[\mu}{}^a{}_b e_{\nu]}{}^b - 2\omega_{[\mu}{}^a{}_A \tau_{\nu]}{}^A. \tag{5.70}$$

Despite the difference in notation, it is easy to make the identifications

$$2\omega_{[\mu}{}^A{}_B \tau_{\nu]}{}^B \longleftrightarrow (\pi^C + \pi^c) \wedge \pi_{[B} \otimes P^{A]}, \tag{5.71}$$
$$2\omega_{[\mu}{}^a{}_b e_{\nu]}{}^b \longleftrightarrow (\pi^C + \pi^c) \wedge \pi_{[b} \otimes P^{a]}, \tag{5.72}$$
$$2\omega_{[\mu}{}^a{}_A \tau_{\nu]}{}^A \longleftrightarrow -(\pi^C + \pi^c) \wedge \pi_A \otimes P^b. \tag{5.73}$$

In the above statements, the opposite positioning of the indices reflects the difference between objects in discussion being elements of the basis, versus the corresponding coefficients. The right-hand side terms correspond to the images of the connection components under the Spencer map $\partial$. As is evident from examining equations (5.19) and (5.20) in the more mathematical notation, the Spencer map anti-symmetrizes a spin-connection $\omega$ with respect to its first and last index. Schematically

$$\partial : \omega_A{}^B{}_C \mapsto 2\omega_{[A}{}^B{}_{C]}. \tag{5.74}$$

It should be noted that the anti-symmetry in the two Lie algebra-valued indices remains. Now, we know that the image of a spin-connection $\partial\omega$ under the Spencer map corresponds to a particular component of the torsion $\widetilde{T}$, and we can begin classifying the different components depending on whether they fall into the image, kernel, or cokernel of $\partial$. As a reminder, we wish to find those components that are in the cokernel of $\partial$, since these constitute the intrinsic torsion of the geometry. The place to start the classification is exactly the same as in the previous subsection - we want to find the image of each spin-connection component under $\partial$. The simplest way to accomplish this is to convert all indices of $\omega$ to flat ones and inspect where they are sent by $\partial$. We find

$$\begin{aligned} \omega_{AB}{}^{C} &\to 2\omega_{[A}{}^{C}{}_{B]}, & \omega_{AB}{}^{c} &\to 2\omega_{[A}{}^{c}{}_{B]}, \\ \omega_{Ab}{}^{C} &\to -\omega_{b}{}^{C}{}_{A}, & \omega_{Ab}{}^{c} &\to \omega_{[A}{}^{c}{}_{b]}, \\ \omega_{ab}{}^{C} &\to 0, & \omega_{ab}{}^{c} &\to 2\omega_{[a}{}^{c}{}_{b]} \\ \omega_{aB}{}^{c} &\to \omega_{[a}{}^{c}{}_{B]}, & \omega_{aB}{}^{C} &\to \omega_{a}{}^{C}{}_{B} \end{aligned} \tag{5.75}$$

The elements of $\ker\partial$ can then be found by setting the right-hand side of (5.75) to zero and considering only those components for which this gives a non-trivial solution, much like was done in the previous part. Writing all indices down for clarity of the $\omega$ structure, the left column yields only the two trivial solutions

$$\omega_{[A|\,C\,|B]} = \omega_{bCA} = 0. \tag{5.76}$$

Note that $\omega_{[A|\,C\,|B]} = 0$ does not yield any non-trivial solutions since it implies that the symmetric part $\omega_{(A|\,C\,|B)}$ also goes to zero by

$$\omega_{(A|\,C\,|B)} = -\omega_{(AB)C} \tag{5.77}$$

$$= -\omega_{C(BA)} = 0. \tag{5.78}$$

where we have used the fact that

$$\omega_{[A|\,B\,|C]} = 0 \implies \omega_{ABC} = \omega_{CBA} \tag{5.79}$$

Using the same argument, the right column yields only one trivial solution,

$$\omega_{[a|\,c\,|b]} = 0 \tag{5.80}$$

and two non-trivial ones

$$\omega_{[A|\,c\,|B]} = 0, \tag{5.81}$$

$$\omega_{Acb} = \omega_{bcA}. \tag{5.82}$$

Since only the anti-symmetric component $\omega_{[A|\,c\,|B]}$ of $\omega_{AcB}$ is set to zero in (5.81), and because $\omega_{AcB}$ is not anti-symmetric in the last two indices, the symmetric part $\omega_{(A|\,c\,|B)}$ cannot be set to zero.

Moreover, equation (5.82) can be broken down to two components. By symmetrizing in $b$ and $c$ on the right-hand side, we find

$$\omega_{(bc)A} = \omega_{Acb} + \omega_{Abc} = 0, \tag{5.83}$$

thus implying that the anti-symmetric components $\omega_{[bc]A}$ constitute the entirety of the left-hand side of (5.82),

$$\omega_{[bc]A} = \omega_{Acb}. \tag{5.84}$$

Note that the last two equations of (5.75) do not yield any new solutions, and can therefore be neglected in this part of the analysis.

We have then found the independent spin-connections to be

$$\omega_{\{A}{}^{a}{}_{B\}},\ \omega_{A}{}^{aA},\ \text{and}\ \omega_{C}{}^{ab} - \omega^{[ab]}{}_{C}, \tag{5.85}$$

where we have broken down the symmetric part $\omega_{(A}{}^{c}{}_{B)}$ into a symmetric traceless part $\omega_{\{A}{}^{c}{}_{B\}}$, and the trace $\omega_{A}{}^{aA}$. These independent spin-connections are precisely those that constitute $\ker\partial$. This result can also be compared to that in the previous part, found in theorem (5.1.1).

The remaining spin-connections can be solved for by using the expressions for the torsions $\widetilde{T}_{\mu\nu}{}^{\hat{A}}$, and are therefore labeled dependent. These are

$$\omega_{[A}{}^{c}{}_{B]},\ \omega^{(bc)}{}_{A},\ \omega_{\mu}{}^{AB},\ \text{and}\ \omega_{C}{}^{ab} + \omega^{[ab]}{}_{C}. \tag{5.86}$$

Of special interest in the last section was the cokernel of the Spencer map. This is also what we are working towards here. The cokernel can be found by considering those components of $\widetilde{T}_{\mu\nu}{}^{\hat{A}}$ that do not contain any spin-connection, while the intrinsic torsion itself is the corresponding torsion $T_{\mu\nu}{}^{\hat{A}}$. The obvious first component is $T_{ab}{}^{A}$, since the spin-connection $\omega_{ab}{}^{A}$ drops out under the Spencer map. The second option is the only symmetric component $\omega_{(A}{}^{a}{}_{B)}$, since this is also not present in the equations (5.75). The corresponding torsion component is $T_{a}{}^{(AB)}$, which can again be broken down to the symmetric traceless part $T_{a}{}^{\{AB\}}$, and the trace $T_{a}{}^{A}{}_{A}$. In conclusion, the intrinsic torsion components (or the $G$-submodules of $\operatorname{coker}\partial$) are

$$T_{a}{}^{\{AB\}},\ T_{a}{}^{A}{}_{A},\ \text{and}\ T_{ab}{}^{A}. \tag{5.87}$$

From comparing the structure of the intrinsic torsion components (5.87) with the kernel components (5.85), it is manifest that the cokernel of $\partial$ is isomorphic to its kernel.

The remaining components of $\widetilde{T}_{\mu\nu}{}^{\hat{A}}$ - those still present in the expressions (5.69) and (5.70) - constitute the image $\operatorname{im}\partial$ of the Spencer map. As stated earlier, in the physics literature, the image of $\partial$ constitutes torsion components that are referred to as *conventional constraints*. These components are

$$\widetilde{T}_{a}{}^{[AB]},\ \widetilde{T}_{AB}{}^{C},\ \text{and}\ \widetilde{T}_{\mu\nu}{}^{a} \tag{5.88}$$

As will be important in some special cases of Galilean $p$-brane gravity, the spin-connections in (5.86) are only dependent if we enforce the maximum number of conventional constraints. If some conventional constraints are not applied, these can become an obstruction to solving for some of the spin-connection components (5.86).

Now that the elements of the intrinsic torsion have been derived, we can proceed to classify $p$-brane Galilean geometries by the geometric constraints that result from constraining the intrinsic torsion. However, since we are discussing the matter in terms of constraints, it is important to consider the way boosts act; if we set a components of the intrinsic torsion to zero, then any components into which it transforms under boosts must also be set to zero. Of course, the action of the boosts should correspond to that presented in figure 5.2 in the last section. Moreover, there are two specific cases that need to be considered

$$\begin{tikzcd}
\boxed{T_a{}^{\{AB\}} = 0} \arrow[dr] & & \boxed{T_a{}^A{}_A = 0} \arrow[dl] \\
& \boxed{T_{ab}{}^A = 0} &
\end{tikzcd}$$

Figure 5.4: This Figure indicates the non-vanishing intrinsic torsion components for $p \neq 0$ and $p \neq D-2$. The arrow indicates the direction in which the $p$-brane Galilean boost transformations act. For instance, the boost transformation of $T_a{}^{\{AB\}}$ gives $T_{ab}{}^A$ but not the other way around. Note that the zero intrinsic torsion constraint $T_{\mu\nu}{}^{\hat{A}} = 0$, and the case with no constraint are omitted. A diagram that includes these cases can be found in figure 5.3.

separately - that of Galilean particles ($p = 0$), and Galilean domain-walls ($p = D-2$). The second case corresponds to that of the Carrollian particle under the formal duality discussed earlier. These special cases arise from the fact that for the respective values of $p$, only one longitudinal (resp. transversal) direction remains, and thus any intrinsic torsion components with two different components of that type of direction drop out, while any possible trace components will remain.

It is again important to emphasize two differences between the languages with and without indices. Firstly, while in the previous section we restricted the intrinsic torsion to be an element of a certain submodule of $\operatorname{coker}\partial$, here we make statements about *constraints* on the intrinsic torsion. Therefore the classifications are each the opposites of each other, i.e.

$$T^{\nabla} \in \mathcal{T}_1^{\mathrm{tr}} \iff T_{ab}{}^A = 0 \text{ and } T_a{}^{\{AB\}} = 0, \tag{5.89}$$

in words, if the intrinsic torsion is in the *trace*-submodule of $\operatorname{coker}\partial$, then the *traceless* part of $T_a{}^{AB}$ is zero.

Secondly, it can be noted by comparing figure 5.2 with figure 5.3 that the action of the boosts is labeled in the opposite direction. This is due to the aforementioned difference in discussing action on the elements of the basis, versus on coefficients. An additional implication of this is that in the language without indices, the submodules $\mathcal{T}_1^{\mathrm{tr}}$ and $\mathcal{T}_1^0$ are considered to include in themselves the condition of $\mathcal{T}_1$, whereas when using physics conventions this isn't necessary - it is guaranteed by the boosts.

With these remarks out of the way, we can proceed to discuss the implications of the different intrinsic torsion constraints.

$$\begin{tikzcd}
& \boxed{T_{\mu\nu}{}^A = 0} \arrow[dl] \arrow[dr] & \\
\boxed{T_a{}^{\{AB\}} = 0} \arrow[dr] & & \boxed{T_a{}^A{}_A = 0} \arrow[dl] \\
& \boxed{T_{ab}{}^A = 0} \arrow[d] & \\
& \boxed{\text{No constraint}} &
\end{tikzcd}$$

Figure 5.3: The different possible intrinsic torsion constraints for $p$-brane Galilei spacetimes. Note that here the arrows do not (strictly) denote the action of boosts, but simply which constraints are automatically satisfied by requiring those higher up the chain.

$\mathbf{T_{ab}{}^A = 0}$. This intrinsic torsion constraint implies that the Lie bracket of any two transverse vector fields $X^\mu$ and $Y^\mu$ is transverse. We can write this as

$$\tau_\mu{}^A \left(X^\nu(\partial_\nu Y^\mu) - (\partial_\nu X^\mu)Y^\nu\right) = 0. \tag{5.90}$$

By Frobenius' theorem, this condition further implies that the overall manifold is foliated by transverse submanifolds that are *involutive*.

To prove (5.90) holds, write the derivatives in terms of the connection $\nabla$ as

$$\tau_\mu{}^A \left(X^\nu(\partial_\nu Y^\mu) - (\partial_\nu X^\mu)Y^\nu\right) = \tau_\mu{}^A \left(X^\nu(\nabla_\nu Y^\mu) - (\nabla_\nu X^\mu)Y^\nu + 2X^\rho Y^\sigma \Gamma_{[\rho\sigma]}{}^\mu\right). \tag{5.91}$$

Using integration by parts, as well as the vielbein postulate (5.64) (or the fact that the connection $\nabla$ is metric compatible), we find that the above becomes

$$\begin{aligned}
\tau_\mu{}^A \left(X^\nu(\nabla_\nu Y^\mu) - (\nabla_\nu X^\mu)Y^\nu + 2X^\rho Y^\sigma \Gamma_{[\rho\sigma]}{}^\mu\right) &= 2X^\rho Y^\sigma \tau_\mu{}^A \Gamma_{[\rho\sigma]}{}^\mu &(5.92)\\
&= 2e^\rho{}_a X^a e^\sigma{}_b Y^b \tau_\mu{}^A \Gamma_{[\rho\sigma]}{}^\mu &(5.93)\\
&= 2X^a Y^b T_{ab}{}^A, &(5.94)
\end{aligned}$$

which is zero because of the given intrinsic torsion constraint.

$\mathbf{T_a{}^{\{AB\}} = T_{ab}{}^A = 0}$. In order to make progress toward both of the remaining constraints, we can first consider the overall constraint $T_a{}^{(AB)} = 0$. This translates to

$$T_a{}^{(AB)} = e^\mu{}_a \tau^{\nu(A} \left(\partial_\mu \tau_\nu{}^{B)} - \partial_\nu \tau_\mu{}^{B)}\right) = 0. \tag{5.95}$$

Using the orthogonality relations

$$\tau^\nu{}_B \tau_\nu{}^D = \delta_B^D \qquad \text{and} \qquad e^\mu{}_a \tau_{\mu\nu} = 0 \tag{5.96}$$

(5.95) becomes

$$\tau^\mu{}_A \tau^\nu{}_B \left(e^\lambda{}_a(\partial_\lambda \tau_{\mu\nu}) + 2(\partial_{(\mu} e^\lambda{}_a \tau_{\lambda|\nu)})\right) = \tau^\mu{}_A \tau^\nu{}_B K_{\mu\nu a} := K_{ABa}, \tag{5.97}$$

where we have recognized

$$K_{\mu\nu a} = e^\lambda{}_a \partial_\lambda \tau_{\mu\nu} + 2(\partial_{(\mu} e^\lambda{}_a)\tau_{\lambda\nu)} \tag{5.98}$$

as the Lie derivative $\mathcal{L}_e \tau_{\mu\nu}$ of the longitudinal metric $\tau_{\mu\nu}$ along transverse directions $e^\lambda{}_a$. It can be seen that the constraint (5.97) is not invariant under boosts. Acting on the constraint with the boosts, we find the whole Galilei invariant set of constraints to be

$$K_{ABa} = K_{Aba} = K_{abc} = 0. \tag{5.99}$$

One way to see this is to notice that since Galilean boosts send longitudinal inverse vielbeine to transverse ones each boost will send a capital index to a lowercase one. With all the components of $K_{\mu\nu a}$ covered in equation (5.99), thus the overall constraint can be written as

$$K_{\mu\nu a} = 0. \tag{5.100}$$

Now, to consider the traceless constraint that we are interested in. We can generalize (5.95) by subtracting the trace

$$T_a{}^{\{AB\}} = e^\mu{}_a \tau^{\nu(A} \left(\partial_\mu \tau_\nu{}^{B)} - \partial_\nu \tau_\mu{}^{B)}\right) - \frac{1}{p+1}\eta^{AB}\eta_{CD}T_a{}^{(CD)} = 0. \tag{5.101}$$

In terms of the Lie derivative, this yields an additional term

$$K_{\mu\nu a} = \frac{1}{p+1}(\tau^{\rho\sigma}K_{\rho\sigma a})\tau_{\mu\nu}, \tag{5.102}$$

which, since the Lie derivative of $\tau_{\mu\nu}$ is proportional to $\tau_{\mu\nu}$, is equivalent to the statement that all transversal covectors, i.e. those that can be written in terms of components in the $e^\mu{}_a$ directions, are conformal killing vectors of the metric $\tau_{\mu\nu}$.

$\mathbf{T_a{}^A{}_A = T_{ab}{}^A = 0}$. To preface the results of this constraint, let us define the worldvolume $(p+1)$-form. The worldvolume is the volume swept out by the longitudinal directions of the extended object (for example, the worldsheet in the string ($p = 1$) case). This is a generalization of proper time - the total proper time experienced by a particle is given by the length of it's worldline, and the worldvolume is the analogous quantity for extended objects. The worldvolume form $\Omega$ can be defined as in equation (5.48). Adapting this to the conventions used in this section, we find

$$\Omega_{\mu_0\ldots\mu_p} = \epsilon_{A_0\ldots A_p}\tau_{\mu_0}{}^{A_0}\ldots\tau_{\mu_p}{}^{A_p}, \tag{5.103}$$

where $\epsilon_{A_0\ldots A_p}$ is the fully anti-symmetric Levi-Civita epsilon symbol. As we know from the treatment of the classification without indices, the implication of this constraint should be that the worldvolume is absolute. This can be proven by showing that the form $\Omega$ is closed,

$$d\Omega = 0, \tag{5.104}$$

with $d\Omega$ denoting the exterior derivative of $\Omega$. We can evaluate the external derivative to be

$$d\Omega = (p+1)\epsilon_{A_0\ldots A_p}\left(\partial_{[\rho}\tau_{\mu_0]}{}^{A_0}\right)\tau_{\mu_1}{}^{A_1}\ldots\tau_{\mu_p}{}^{A_p}, \tag{5.105}$$

where we can interpret $\partial_{[\rho}\tau_{\mu_0]}{}^{A_0}$ as some components of the torsion tensor, where the spin-connection term drops out. There are a couple of options for which torsion tensor components $\partial_{[\rho}\tau_{\mu_0]}{}^{A_0}$ could be generically, but since the component $T_{ab}{}^A$ is zero by one of the constraints of this case, and $T_{AB}{}^C$ is set to zero in order to solve for spin-connection components, the only remaining one is $T_{aA}{}^B$. We then write

$$\partial_{[\rho}\tau_{\mu_0]}{}^{A_0} = e_\rho{}^a\tau_{\mu_0}{}^A T_{aA}{}^{A_0}, \tag{5.106}$$

making the total expression

$$\begin{aligned} d\Omega &= (p+1)\epsilon_{A_0\ldots A_p}e_\rho{}^a T_{aA}{}^{A_0}\tau_{\mu_0}{}^A\tau_{\mu_1}{}^{A_1}\ldots\tau_{\mu_p}{}^{A_p} && (5.107)\\ &= (p+1)\epsilon_{A_0\ldots A_p}e_\rho{}^a T_{aA}{}^{A_0}\epsilon^{AA_1\ldots A_p}\Omega && (5.108)\\ &= (p+1)e_\rho{}^a\delta^A_{A_0}T_{aA}{}^{A_0}\Omega, && (5.109)\end{aligned}$$

where the delta function selects the trace component of the torsion tensor, $T_{aA}{}^A$, which by the initial constraint is zero, and thus $d\Omega$ is zero.

In conclusion, the different options for intrinsic torsion constraints imply the following geometric constraints:

**Theorem 5.2.1.**

1. **The intrinsic torsion is unconstrained.** No additional geometric constraints enforced;
2. $\mathbf{T_{ab}{}^{A} = 0}$. $M$ is foliated by transverse submanifolds, which are involutive;
3. $\mathbf{T_a{}^{\{AB\}} = T_{ab}{}^{A} = 0}$. In addition to the geometric constraint of part 2., transversal vectors $e^\mu{}_a$ are conformal Killing vectors with respect to the longitudinal metric $\tau_{\mu\nu}$;
4. $\mathbf{T_a{}^{A}{}_{A} = T_{ab}{}^{A} = 0}$. In addition to part 2., the worldvolume is absolute;
5. $\mathbf{T_{\mu\nu}{}^{A} =}$ . The foliation by transversal submanifolds is involutive, transversal vectors are conformal Killing with respect to $\tau_{\mu\nu}$, and the worldvolume is absolute.

In comparison to theorem 5.1.3, it is clear that despite the difference in notation and conventions, the classification is the same.

As discussed before, $p$-branes with $p = 0$, and $p = D - 2$ are special cases. Although the classification depends only on the group under consideration, these cases are special because for the special dimensions the representation of either the $SO(1,p)$ or $SO(D - P - 1)$ component of the Galilei group becomes reducible.

$\mathbf{p = 0}$. This is the case of the Galilean particle (or equivalently Carrollian domain-wall) geometry. In this case, there is only one choice of longitudinal direction, and as such, only the constraints $T_{ab}{}^{A=0} = T_{ab} = 0$ and $T_{\mu\nu} = 0$ remain. The classification is then

**Theorem 5.2.2.**

1. **The intrinsic torsion is unconstrained.** No additional geometric constraints enforced;
2. $\mathbf{T_{ab} = 0}$. $M$ is foliated by transverse submanifolds, which are involutive. In physics literature, this is often referred to as the *twistless torsional* constraint, or a *hyper-space orthogonal* foliation [9];
3. $\mathbf{T_{\mu\nu} = 0}$. $M$ is foliated by transverse submanifolds, where the submanifolds are slices of equal absolute time.

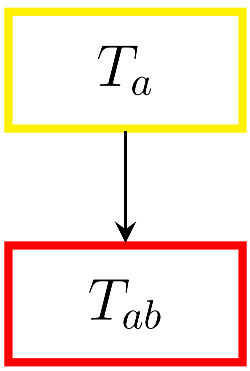


Figure 5.5: This figure indicates the non-zero intrinsic torsion components for $p = 0$ where, with $A = 0$, we have written $T_a{}^0{}_0 = T_a$ and $T_{ab}{}^0 = T_{ab}$

$\mathbf{p = D - 2}$. This case is the opposite of the previous one - the Galilean domain-wall, or Carrollian particle. Only one transverse direction exists, which we can label by $a = z$, or just omit the label altogether. The remaining constraints are $T_z{}^{\{AB\}} = 0$, $T_z{}^{A}{}_{A} = 0$, and $T_z{}^{(AB)} = 0$, with the geometric consequences of each being

**Theorem 5.2.3.**

1. **The intrinsic torsion is unconstrained.** No additional geometric constraints enforced;
2. $\mathbf{T^{\{AB\}} = 0}$. Transverse vectors (those with components only in the unique direction $e^{\mu}_{z} = e^{\mu}$), are conformal Killing vectors with respect to the longitudinal metric $\tau_{\mu\nu}$;
3. $\mathbf{T^{A}{}_{A} = 0}$. The worldvolume is absolute;
4. $\mathbf{T^{(AB)} = 0}$. Both of the above geometric constraints are satisfied.

$$\boxed{T^{\{AB\}}} \qquad\qquad \boxed{T^{A}{}_{A}}$$

Figure 5.6: This figure indicates the non-zero intrinsic torsion components for $p = D - 2$ where, with $a = z$, we have written $T_z{}^{\{AB\}} = T^{\{AB\}}$ and $T_z{}^{A}{}_{A} = T^{A}{}_{A}$.

As an interesting note for future work, there are additional cases where the classifications in theorem 5.2.1 can be refined - in fact any dimension where the representations of either $SO(1,p)$, $SO(D-P-1)$, or both become reducible is such a case. In the string case, where $p = 1$, we can apply the well-known fact that the representation of $SO(1,1)$ is reducible, and refine the classification in terms of light-cone coordinates.

To illustrate the physical relevance of the classification presented here, in the next section we will proceed to show how each case of the classification arises as a limit of general relativity, and thus as a theory of gravity.

# Chapter 6

# $p$-brane Galilei Gravity

This section will generalize the discussion of section 4, and therefore that of [9], to the case of $p$-branes. This procedure gives rise to gravity theories with underlying spacetimes described by $p$-brane Galilei geometries, as introduced in the last section. For $p = 0$, the discussion of this section and section 4 coincide, while for $p = 1$, the discussion here reduces to that presented in [8].

It is again important to emphasize the applicability of the work to both Galilei and Carroll gravity. The duality is the same as discussed in the beginning of section 5.2, i.e. exchanging a $p$-brane for a $(D-p-2)$-brane, together with the exchange of which spacetime component is considered longitudinal, and which transverse. In other words, we exchange the roles of the two kinds of vielbeine, $e$ and $\tau$,

$$\left(\tau_\mu{}^A, e_\mu{}^a\right) \longleftrightarrow \left(e_\mu{}^b, \tau_\mu{}^B\right). \tag{6.1}$$

Again, there are two special cases, which we will call the domain-wall, and the defect-brane gravity. These occur for the cases where $p = D-2$ and $p = D-3$ respectively. First, we discuss the generic $p$-brane case, with $p \leq D-4$.

We again start with the Einstein-Hilbert action

$$S_{\mathrm{EH}} = -\frac{1}{16\pi G_N}\int E E^\mu{}_{\hat{A}} E^\nu{}_{\hat{B}} R_{\mu\nu}{}^{\hat{A}\hat{B}}(\Omega), \tag{6.2}$$

where we have explicitly indicated that the curvature $R_{\mu\nu}{}^{\hat{A}\hat{B}}$ is a function of the relativistic spin-connection $\Omega$. Although this already implies it, for clarity let us state that we are working in the first-order formulation, where $\Omega$ is a priori independent of the vielbein. In (6.2), $E^\mu{}_{\hat{A}}$ is the relativistic vielbein, $E$ is its determinant, and $G_N$ is the Newton constant. In the relativistic case, we define the inverse vielbein the usual way,

$$E_\mu{}^{\hat{A}} E^\mu{}_{\hat{B}} = \delta^{\hat{A}}{}_{\hat{B}}, \qquad\qquad E_\mu{}^{\hat{A}} E^\nu{}_{\hat{B}} = \delta_\mu{}^\nu. \tag{6.3}$$

We can write out the curvature $R_{\mu\nu}{}^{\hat{A}\hat{B}}$ out in terms of the spin-connection $\Omega$ as

$$R_{\mu\nu}{}^{\hat{A}\hat{B}} = 2\partial_{[\mu}\Omega_{\nu]}{}^{\hat{A}\hat{B}} - 2\Omega_{[\mu}{}^{\hat{B}\hat{C}}\Omega_{\nu]}{}^{\hat{A}}{}_{\hat{C}}. \tag{6.4}$$

Following the previously introduced method, we decompose the relativistic index $\hat{A}$ into two parts, $A$ and $a$, where $A$ index the $p+1$ longitudinal components, and $a$ the remaining transverse ones. We then perform the rescalings

$$\begin{aligned} E_\mu{}^A &= c\tau_\mu{}^A, & E_\mu{}^a &= e_\mu{}^a, \\ \Omega_\mu{}^{AB} &= \omega_\mu{}^{AB}, & \Omega_\mu{}^{Aa} &= \frac{1}{c}\omega_\mu{}^{Aa}, \\ \Omega_\mu{}^{ab} &= \omega_\mu{}^{ab}, \end{aligned} \tag{6.5}$$

where $c$ is a dimensionless scaling parameter. After substituting the rescaled definitions into the action (6.2), and exploiting our freedom to perform an overall scaling by some factor of $c$ to make the leading power $c^0$, we find

$$S_{\mathrm{G}} = -\frac{1}{16\pi G}\int e\left\{c^0\left(e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}(J)\right) + c^{-2}\left(\tau^\mu{}_A e^\nu{}_a R_{\mu\nu}{}^{Aa}(G) + \tau^\mu{}_A \tau^\nu{}_B R_{\mu\nu}{}^{AB}(L)\right)\right\}, \tag{6.6}$$

where $G_N = c^{p+1}G$ is the rescaled Newton constant, $R_{\mu\nu}{}^{Aa}(G)$ is the curvature of the Galilean boosts, and $R_{\mu\nu}{}^{AB}$ is the curvature of the longitudinal Lorentz transformations. The leading term is the usual one. Unlike in section 4, the next-to-leading order terms have not been omitted, since the two first subleading terms of order $c^{-2}$ contribute to the two special cases of $p$-brane Galilean gravity. In the case of domain-walls, only one transverse direction remains, and thus the curvature $R_{\mu\nu}{}^{ab}(J)$ goes to zero. This makes the usually subleading curvature terms dominant, thus resulting in the special case. For defect-branes, the special case is a result of the $SO(D-P-1)$-component of the Galilei group becoming Abelian, which results in the anti-symmetrized terms of the associated connections dropping out.

**$p$-branes.** In the generic case, we can directly consider the leading order behavior of (6.6). This yields the action

$$S_{p\text{-brane}} = -\frac{1}{16\pi G}\int e e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}(J), \tag{6.7}$$

where $R_{\mu\nu}{}^{ab}(J)$ is the curvature associated to the transverse rotations,

$$R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{bc}\omega_{\nu]}{}^a{}_c. \tag{6.8}$$

The geometric constraint that results from this action can be determined by finding the spin-connection components that do not occur in the quadratic term of the curvature (6.8), since these are the only terms that can feature a torsion tensor. These spin-connection components instead occur linearly as Lagrange multipliers enforcing a constraint on a particular (corresponding) component of the torsion tensor. These components can be easily determined by performing the suitable decomposition of the spin-connection,

$$\omega_\mu{}^{ab} = \tau_\mu{}^A\omega_A{}^{ab} + e_\mu{}^d\omega_d{}^{ab}. \tag{6.9}$$

When this is inserted into the action (6.7), any terms resulting from expanding the expression

$$2\omega_{[\mu}{}^{bc}\omega_{\nu]}{}^a{}_c = 2(\tau_\mu{}^A\omega_A{}^{bc} + e_\mu{}^d\omega_d{}^{bc})(\tau_\mu{}^A\omega_A{}^b{}_c + e_\mu{}^d\omega_d{}^b{}_c) \tag{6.10}$$

that have a factor $\tau_\mu{}^A$ are cancelled by the transverse vielbeine in front of the curvature in the action (6.7). Thus the only remaining term is $\omega_{[a}{}^{bc}\omega_{b]}{}^a{}_c$. Since $\omega_A{}^{ab}$ no longer occurs in a quadratic term in the action, but only linearly through the first term,

$$2e^\mu{}_a e^\nu{}_b(\partial_{[\mu}\omega_{\nu]}{}^{ab}) = 2e^\mu{}_a e^\nu{}_b\left\{(\partial_{[\mu}\tau_{\nu]}{}^A)\omega_A{}^{ab} + \tau_{[\nu}{}^A(\partial_{\mu]}\,\omega_A{}^{ab})\right\} + 2e^2\partial(e\omega) \tag{6.11}$$

$$= 2T_{ab}{}^A\omega_A{}^{ab} + 2e^2\partial(e\omega), \tag{6.12}$$

where we have omitted the indices on the "irrelevant" term and only kept them for the sake of completeness. We have also used the definition

$$T_{ab}{}^A = e^\mu{}_a e^\nu{}_b T_{\mu\nu}{}^A \tag{6.13}$$

$$= 2e^\mu{}_a e^\nu{}_b \partial_{[\mu}\tau_{\nu]}{}^A. \tag{6.14}$$

From equation (6.12) it is apparent that $\omega_A{}^{ab}$ has become a Lagrange multiplier that enforces the geometric constraint

$$T_{ab}{}^A = 0, \tag{6.15}$$

which corresponds to case 2 in theorem 5.2.1.

**Defect-branes.** For this case, we let $p = D-3$. As mentioned previously, the group $SO(D-P-1)$ of transverse rotations becomes the abelian group $SO(2)$. Thus the term quadratic in spin-connections drops out from the curvature (6.8), leaving us with an action

$$S_{\text{defect-brane}} = \frac{1}{16\pi G}\int ee^\mu{}_a e^\nu{}_b \partial_{[\mu}\omega_{\nu]}{}^{ab}. \tag{6.16}$$

Using the familiar decomposition of the spin-connection $\omega_\mu{}^{ab}$, as well as integration by parts, we find a total derivative, which we drop since it is only a boundary term, as well as the terms

$$\widetilde{\mathcal{L}} = ee^\mu{}_a e^\nu{}_b(\partial_{[\mu}\tau_{\nu]}{}^A)\omega_A{}^{ab} + 2ee^\mu{}_a e^\nu{}_b(\partial_{[\mu}e_{\nu]}{}^c)\omega_c{}^{ab} + 2ee^\mu{}_a e^\nu{}_b e_{[\nu}(\partial_{\mu]}\omega_c{}^{ab}), \tag{6.17}$$

where we can recognize that in the first term $e^\rho{}_b e^\mu{}_a(\partial_{[\mu}\tau_{\rho]}{}^A)$ is the intrinsic torsion tensor component $T_{ab}{}^A$. In the second term, we similarly recognize $e^\mu{}_a e^\nu{}_b(\partial_{[\mu}e_{\nu]}{}^c)$ as the transversal, conventional torsion tensor $T_{ab}{}^c$, which we can ignore using the conventional constraint $T_{\mu\nu}{}^c = 0$. The third term becomes

$$2ee^\mu{}_a e^\nu{}_b e_{[\nu}(\partial_{\mu]}\partial_{\mu]}\omega_c{}^{ab} = 2ee^\mu{}_a\partial_\mu\omega_b{}^{ab} \tag{6.18}$$

$$= -2\partial_\mu(ee^\mu{}_a)\omega_b{}^{ab} \tag{6.19}$$

$$= -2e\left[T_{aA}{}^A + T_{ac}{}^c\right]\omega_b{}^{ab}, \tag{6.20}$$

where, since $T_{ab}{}^c$ has already been set to zero, the latter term disappears. In the above we have used, again, the fact that the derivatives of the inverse vielbeine become

$$\partial_\mu e^\nu{}_a = -\tau^\nu{}_C(\partial_\mu\tau_\rho{}^C)e^\rho{}_a - e^\nu{}_c(\partial_\mu e_\rho{}^c)e^\rho{}_a, \tag{6.21}$$

$$\partial_\mu\tau^\nu{}_A = -\tau^\nu{}_C(\partial_\mu\tau_\rho{}^C)\tau^\rho{}_A - e^\nu{}_c(\partial_\mu e_\rho{}^c)\tau^\rho{}_A, \tag{6.22}$$

and the fact that the derivative of the determinant $e$ is

$$\partial_\mu e = e\left(\tau^\rho{}_A \partial_\mu \tau_\rho{}^A + e^\rho{}_a \partial_\mu e_\rho{}^a\right). \tag{6.23}$$

Equation (6.20) means $\omega_b{}^{ab}$ has also become a Lagrange multiplier, with this component of the spin-connection enforcing the constraint $T_{aA}{}^A = 0$. The full set of geometric constraints resulting from this theory is then

$$T_{ab}{}^A = T_{aA}{}^A = 0, \tag{6.24}$$

corresponding to case 4 of the classification in theorem 5.2.1.

**Domain-walls.** We now let $p = D - 2$. This means that there is only one transverse direction, which we will denote by $z$, or omit completely where this causes no ambiguity, i.e. $e_\mu{}^a = e_\mu$, but $\omega_\mu{}^{aA} = \omega_\mu{}^{zA}$ to preserve the distinction between $\omega_\mu{}^{zA}$ and $\omega_\mu{}^{Az}$. Only having one choice of a transverse direction means that $\omega_\mu{}^{ab} = \omega_\mu{}^{zz} = 0$. Since this is the only spin-connection that is present in the leading term of the action (6.7), the entire leading term disappears. We then consider the action

$$S_{\text{domain-wall}} = -\frac{1}{16\pi G}\int e\left(e^\mu \tau^\nu{}_A R_{\mu\nu}{}^{zA}(G) + \tau^\mu{}_A \tau^\nu{}_B R_{\mu\nu}{}^{AB}(M)\right), \tag{6.25}$$

where we have performed an overall scaling to bring the previously subleading terms to order $c^0$. The curvatures are defined by

$$R_{\mu\nu}{}^{zA}(G) = 2\partial_{[\mu}\omega_{\nu]}{}^{zA} - 2\omega_{[\mu}{}^A{}_B \omega_{\nu]}{}^{Bz}, \tag{6.26}$$

$$R_{\mu\nu}{}^{AB}(L) = 2\partial_{[\mu}\omega_{\nu]}{}^{AB} - 2\omega_{[\mu}{}^A{}_C \omega_{\nu]}{}^{CB}. \tag{6.27}$$

We again decompose the spin-connections to transversal and longitudinal components by

$$\omega_\mu{}^{AB} = \tau_\mu{}^C \omega_C{}^{AB} + e_\mu \omega^{AB}, \tag{6.28}$$

$$\omega_\mu{}^A = \tau_\mu{}^C \widetilde{\omega}_C{}^A + e_\mu \omega^A, \tag{6.29}$$

where we have chosen to distinguish between the spatial component of $\omega_\mu{}^{AB}$ and the longitudinal component of $\omega_\mu{}^{zA} := \omega_\mu{}^A$, by labeling the latter with a tilde, i.e. letting $\omega_C{}^{zA} := \widetilde{\omega}_C{}^A$.

Proceeding with the classification of the resulting geometry, we can use integration by parts on the first term of $R_{\mu\nu}{}^{Az}(G)$ in (6.26). We find that it can be written as

$$\begin{aligned} 2\tau^\mu{}_A e^\nu \partial_{[\mu}\omega_{\nu]}{}^{zA} &= 2ee^\mu \tau^\nu{}_A \partial_{[\mu}\left(\tau_{\nu]}{}^C \widetilde{\omega}_C{}^A + e_{\nu]}\omega^A\right) &(6.30)\\ &= e\bar{\tau}_{zA}{}^C \widetilde{\omega}_C{}^A + e\bar{e}_{zA}\omega^A + e\bar{\tau}_{Bz}{}^B \widetilde{\omega}_A{}^A + e\left[\bar{e}_{Az} + \bar{\tau}_{AB}{}^B\right]\omega^A &(6.31)\\ &= e\bar{\tau}_{zA}{}^C \widetilde{\omega}_C{}^A + e\bar{\tau}_{Bz}{}^B \widetilde{\omega}_A{}^A + e\bar{\tau}_{AB}{}^B \omega^A, &(6.32)\end{aligned}$$

where we have introduced a short-hand notation for the curls of $e$ and $\tau$

$$\bar{\tau}_{\mu\nu}{}^A := 2\partial_{[\mu}\tau_{\nu]}{}^A, \tag{6.33}$$

$$\bar{e}_{\mu\nu} := 2\partial_{[\mu}e_{\nu]}, \tag{6.34}$$

and turning curved indices into flat ones works the usual way, for instance

$$\bar{\tau}_{zA}{}^{B} := e^{\mu}\tau^{\nu}{}_{A}\bar{\tau}_{\mu\nu}{}^{B}. \tag{6.35}$$

By itself, (6.32) does not yet result in intrinsic torsion constraints. Therefore, we proceed to consider the omega squared term of (6.26). This term results in the following two terms

$$2\tau^{\mu}{}_{A}e^{\nu}\omega_{[\mu}{}^{A}{}_{B}\omega_{\nu]}{}^{zB} = -\omega^{A}{}_{B}\widetilde{\omega}_{A}{}^{B} + \omega_{B}{}^{B}{}_{A}\omega^{A}. \tag{6.36}$$

Writing the two terms (6.32) and (6.36) together, the portion of the action (here expressed in the Lagrangian density) resulting purely from the $R_{\mu\nu}{}^{zA}(G)$-term is

$$\begin{aligned}
\mathcal{L}_{\text{domain-wall}}\Big|_{R(L)=0} &= e\bar{\tau}_{zB}{}^{A}\widetilde{\omega}_{A}{}^{B} + e\bar{\tau}_{Bz}{}^{B}\widetilde{\omega}_{A}{}^{A} + e\bar{\tau}_{AB}{}^{B}\omega^{A} - \omega^{A}{}_{B}\widetilde{\omega}_{A}{}^{B} + \omega_{B}{}^{B}{}_{A}\omega^{A} && (6.37)\\
&= e\left[\left(\bar{\tau}_{zB}{}^{A} - \omega^{A}{}_{B}\right)\widetilde{\omega}_{A}{}^{B} + \bar{\tau}_{Bz}{}^{B}\widetilde{\omega}_{A}{}^{A} + \left(\bar{\tau}_{AB}{}^{B} + \omega_{B}{}^{B}{}_{A}\right)\omega^{A}\right] && (6.38)\\
&= e\left[T_{B}{}^{A}\widetilde{\omega}_{A}{}^{B} + T_{B}{}^{B}\widetilde{\omega}_{A}{}^{A} + T_{AB}{}^{B}\omega^{A}\right], && (6.39)
\end{aligned}$$

where $T^{AB} := T_{z}{}^{AB}$.

From (6.39), it seems as if $\widetilde{\omega}_{AB}$ is already a Lagrange multiplier. However, there is one more subtlety to take care of. You may recall from the last section that not every component of $T^{AB}$ can be intrinsic. Namely, only the symmetric part, $T^{(AB)}$ can be intrinsic. In fact, we can already see a glimpse of this in (6.38) - we absorb the spin-connection $\omega^{AB}$ into the torsion tensor. Since only those spin-connection components that don't include any spin-connection may be intrinsic, we need to exclude $\omega^{AB}$ somehow.

We can phrase this as a condition - the sum $\omega^{AB}\widetilde{\omega}_{AB}$ has to be zero. Since $\omega^{AB}$ is anti-symmetric in $A$ and $B$, this condition simply selects the anti-symmetric part of $\widetilde{\omega}_{AB}$, $\widetilde{\omega}_{[AB]}$. The remaining components of $\widetilde{\omega}$ can still take on the role of Lagrange multiplier. These components are

$$\widetilde{\omega}_{(AB)} = \widetilde{\omega}_{\{AB\}} + \frac{1}{p+1}\eta_{AB}\widetilde{\omega}, \tag{6.40}$$

where we have broken down the symmetric portion of $\widetilde{\omega}_{AB}$ to the symmetric traceless $\widetilde{\omega}_{\{AB\}}$ and the trace $\widetilde{\omega} := \widetilde{\omega}_{A}{}^{A}$. The two components $\widetilde{\omega}_{\{AB\}}$ and $\widetilde{\omega}$ enforce the geometric constraints

$$T^{\{AB\}} = T_{A}{}^{A} = 0, \tag{6.41}$$

which amounts to the total symmetric component of $T_{z}{}^{(AB)}$ being zero

$$T^{(AB)} = 0, \tag{6.42}$$

corresponding to case 4 of the classification presented in theorem 5.2.3.

The second terms in (6.32) and (6.36) remain, but can simply be used to solve for $\omega_{A}{}^{A}{}_{B}$.

The $R_{\mu\nu}{}^{AB}(L)$-term of the action (6.25) does not result in any intrinsic torsion constraints, so we will not discuss it further here. This has been somewhat of a trend across the current section - any terms in the action that do not contribute to the classification have been paid no attention to. In the following section we will move through the same process in the second-order formalism of the gravitational theory. This means that the full actions, as written in terms of intrinsic torsion tensors, spin-connections, and conventional torsion tensors are to be found there.

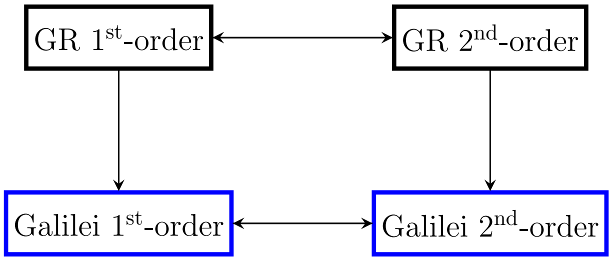


Figure 6.1: Relations between general relativity and galilean gravity theories in their different formalisms. In the first-order formalism the spin-connections $\Omega$ or $\omega$ are independent dynamical objects, while in the second-order formalism they are assumed to be functions of the vielbeine ${\tau_\mu}^A$ and ${e_\mu}^a$.

## 6.1 In the Second-Order Formalism

For the sake of comparison, we now wish to repeat the same process in the second-order formalism of Galilei gravity. Naively it would be expected that the square in figure 6.1 "closes", i.e. that it would not matter whether the $c \to \infty$ limit is taken from the first or second-order formalism of GR.

It turns out that our Naive expectation is shattered upon further consideration - in the second-order formulation of GR, we can solve for all spin-connection components. However, we already saw in section 5 that in Galilean gravity some spin-connection components become independent, and we cannot solve for them. This is, at a fairly fundamental level an obstruction to 6.1 closing.

In order to verify that this is the case, as well as to better understand how the break occurs, we wish to compute the resulting Galilean gravity theories in the second-order formalism through the two paths - solving for all spin connections from the Galilean first-order formalism, and taking the $c \to \infty$ limit of the second-order formalism of general relativity directly.

We first begin by solving for all spin-connections that we can solve for in the first-order formalism of Galilean gravity presented in the previous section. To this end, we will assume the spin-connections, which previously were independent geometric objects, to have some expressions in terms of the vielbeine ${\tau_\mu}^A$ and ${e_\mu}^a$.

### 6.1.1 From First to Second Order Formalism of Galilei gravity

**$p$-branes.** In order to solve for all spin-connections that we can, we proceed from equation (6.12), and expand the remaining term $2e^2\partial(e\omega)$. Restoring the indices, the term becomes

$$2e{e^\mu}_a{e^\nu}_b\partial_{[\mu}\left({e_{\nu]}}^c{\omega_c}^{ab}\right) = 2e{e^\mu}_a{e^\nu}_b\left\{\left(\partial_{[\mu}{e_{\nu]}}^c\right){\omega_c}^{ab} + {e_{[\nu}}^c\left(\partial_{\mu]}{\omega_c}^{ab}\right)\right\} \tag{6.43}$$

$$= 2e{e^\mu}_a{e^\nu}_b\left(\partial_{[\mu}{e_{\nu]}}^c\right){\omega_c}^{ab} + 2e{e^\mu}_a{e^\nu}_b{e_{[\nu}}^c\left(\partial_{\mu]}{\omega_c}^{ab}\right). \tag{6.44}$$

Integrating by parts in the second term, we find

$$\begin{aligned}
2ee^{\mu}{}_{a}e^{\nu}{}_{b}\partial_{[\mu}\left(e_{\nu]}{}^{c}\omega_{c}{}^{ab}\right) &= 2\partial_{[\mu}\left(ee^{\mu}{}_{a}e^{\nu}{}_{b}\right)e_{\nu]}{}^{c}\omega_{c}{}^{ab} && (6.45)\\
&= 2e\left\{-\left[2\tau^{\rho}{}_{A}e^{\mu}{}_{a}\left(\partial_{[\rho}\tau_{\mu]}{}^{A}\right)+2e^{\mu}{}_{a}e^{\rho}{}_{c}\left(\partial_{[\mu}e_{\rho]}{}^{c}\right)\right]\omega_{b}{}^{ab}-\right. && (6.46)\\
&\qquad \left.-\,2e^{\mu}{}_{a}e^{\rho}{}_{b}\left(\partial_{[\mu}e_{\rho]}{}^{c}\right)\omega_{c}{}^{ab}\right\} && (6.47)\\
&= -2e\left\{\left[T_{aA}{}^{A}+\bar{e}_{ac}{}^{c}\right]\omega_{b}{}^{ab}+\bar{e}_{ab}{}^{c}\omega_{c}{}^{ab}\right\}, && (6.48)
\end{aligned}$$

where we have omitted the constants $(D-p-2)$ and $(D-p-3)$ that should occur in front of the $T_{aA}{}^{A}$ and $\bar{e}_{ac}{}^{c}$ term respectively throughout the calculation for clarity.

We can now also find the terms of the action resulting from the $\omega$-squared term. This term is

$$2ee^{\mu}{}_{a}e^{\nu}{}_{b}\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b}. \tag{6.49}$$

Upon applying the usual decomposition of the spin-connection $\omega_{\mu}{}^{ab}$, we find that this term becomes

$$\begin{aligned}
2e^{\mu}{}_{a}e^{\nu}{}_{b}\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b} &= \omega_{a}{}^{ac}\omega_{bc}{}^{b}-\omega_{b}{}^{ac}\omega_{ac}{}^{b} && (6.50)\\
&= -\omega_{ca}{}^{c}\omega_{b}{}^{ab}+\omega_{a}{}^{c}{}_{b}\omega_{c}{}^{ab}, && (6.51)
\end{aligned}$$

where we have omitted the determinant $e$, since it does not contribute to the equation, as well as relabeled indices to reach the final expression.

The overall action, before enforcing any constraints is then given by

$$S=-\frac{1}{16\pi G}\int e\left[\omega_{ca}{}^{c}\omega_{b}{}^{ab}-\omega_{a}{}^{c}{}_{b}\omega_{c}{}^{ab}+T_{ab}{}^{A}\omega_{A}{}^{ab}-2\left(T_{aA}{}^{A}+\bar{e}_{ac}{}^{c}\right)\omega_{b}{}^{ab}+\bar{e}_{ab}{}^{c}\omega_{c}{}^{ab}\right]. \tag{6.52}$$

Grouping terms by spin-connection, we find

$$S=-\frac{1}{16\pi G}\int e\left[\left(\omega_{ca}{}^{c}-2\bar{e}_{ac}{}^{c}-2T_{aA}{}^{A}\right)\omega_{b}{}^{ab}+\left(\bar{e}_{ab}{}^{c}-\omega_{a}{}^{c}{}_{b}\right)\omega_{c}{}^{ab}+T_{ab}{}^{A}\omega_{A}{}^{ab}\right], \tag{6.53}$$

where we can identify

$$T_{ab}{}^{b}=\bar{e}_{ab}{}^{b}-\omega^{b}{}_{ab} \tag{6.54}$$

and

$$T_{ab}{}^{c}=\bar{e}_{ab}{}^{c}-2\delta^{d}_{[a}\omega_{d}{}^{c}{}_{b]}. \tag{6.55}$$

You may notice that the terms as written in (6.53) cannot be directly identified with the torsion tensors (6.54) and (6.55). However, once the equations of motion are taken, we can make the identification.

Upon taking the equations of motion, the equation of motion corresponding to $\omega_{b}{}^{ab}$ makes the coefficient of $\omega_{b}{}^{ab}$ become a conventional constraint in the form of

$$T_{ab}{}^{b}+\frac{D-p-2}{D-p-3}T_{aA}{}^{A}=0, \tag{6.56}$$

where the dimensions result from the trace. Note that the trace $\bar{e}_{ac}{}^{c}$ is not included in the second term of (6.48), since it has been separated to the component occurring in front of $\omega_b{}^{ab}$. The equation of motion for $\omega_c{}^{ab}$ then becomes another conventional constraint,

$$T_{ab}{}^{c} - \text{Trace} = 0. \tag{6.57}$$

The remaining spin-connection terms in the constraints we have derived form precisely the components of $\omega_c{}^{ab}$, and as all other terms in the equations are in terms of the vielbeine, we can write down an expression for $\omega_c{}^{ab}(e,\tau)$. This is given by

$$\omega_c{}^{ab}(e,\tau) = \omega_c{}^{ab}(e) - \frac{2}{D-p-3}\delta_c^{[a}T^{b]}{}_A{}^A, \tag{6.58}$$

where we have separated out the component $\omega_c{}^{ab}(e)$ for reasons that will become clear in the next sub-section. The full expression for $\omega_c{}^{ab}(e)$ is

$$\omega_{cab}(e) = \bar{e}_{ab,c} - \bar{e}_{c[a,b]}. \tag{6.59}$$

Here we have adopted, for clarity, a notation where a comma denotes the separation between two anti-symmetric indices and the remaining distinguished one, for instance, for the spin-connections $\omega_c{}^{ab} = -\omega_c{}^{ba}$, when all indices are brought down, we write

$$\omega_{c,ab}. \tag{6.60}$$

Enforcing the constraints as per the equations of motion, we arrive at the full action

$$S^{\text{2nd order}}_{\text{p-brane}} = -\frac{1}{16\pi G}\int e\left[-\omega_{ca}{}^{c}(e,\tau)\omega_b{}^{ab}(e,\tau) + \omega_a{}^{c}{}_b(e,\tau)\omega_c{}^{ab}(e,\tau) + T_{ab}{}^{A}\omega_A{}^{ab}\right], \tag{6.61}$$

where the dependences of any spin-connections has been explicitly indicated, i.e. $\omega_A{}^{ab}$ is the only independent one present. The switch in the signs between the first two terms of (6.61) and (6.52) is due to the fact that through adding and subtracting these spin-connections we can apply the conventional constraints to set the rest of the coefficients of $\omega_b{}^{ab}$ and $\omega_c{}^{ab}$ to zero.

**Defect-branes.** In the case of defect-branes, we can collect the terms discussed in the previous section and write the action as

$$S^{\text{2nd-order}}_{\text{defect-brane}} = -\frac{1}{16\pi G}\int e\left[T_{ab}{}^{A}\omega_A{}^{ab} - 2T_{aA}{}^{A}\omega_b{}^{ab}\right]. \tag{6.62}$$

However, we see that both of the spin-connections in the action (6.62) occur as Lagrange multipliers of the two constraints of the theory. No spin-connections remain for us to solve for, and therefore the second-order formalism is the same as the first-order one[1] Note that since we do not have the extra terms from the $\omega^2$-term unlike in the last case, we do not find constraints from the $\omega_b{}^{ab}$-term, but rather that it has now become an independent Lagrange multiplier.

[1]In fact, it does not make sense to call the second-order formalism such - we haven't solved for any spin-connections, and therefore the second-order formalism isn't really even second-order.

**Domain-walls.** In the previous section, we already derived the terms of the domain-wall action resulting from the $R_{\mu\nu}{}^{Az}(G)$-term of the initial action (6.25). These were

$$S_{\text{domain-wall}}\Big|_{R(L)=0} = -\frac{1}{16\pi G}\int e\left[T_B{}^A\widetilde{\omega}_A{}^B + T_B{}^B\widetilde{\omega}_A{}^A + T_{AB}{}^B\omega^A\right]. \tag{6.63}$$

Therefore, we will now focus our attention on the second term, resulting from $R_{\mu\nu}{}^{AB}(L)$, the definition of which has been written out in (6.27). Since the corresponding term in the action (6.25) is preceded by two longitudinal vielbeine, any terms that feature a transverse vielbein can be dropped. What remains is

$$e\tau^\mu{}_A\tau^\nu{}_B R_{\mu\nu}{}^{AB}(L) = e\left[2\bar{\tau}_{BA}{}^C\omega_C{}^{AB} + \bar{e}_{AB}\omega^{AB} + \left(2\bar{e}_{zA} - 2\bar{\tau}_{AC}{}^C\right)\omega_B{}^{AB}\right. \tag{6.64}$$

$$\left. -\omega_A{}^A{}_C\omega_B{}^{CB} + \omega_A{}^B{}_C\omega_B{}^{CA}\right]. \tag{6.65}$$

Grouping everything by the corresponding spin-connections, we write the above as

$$e\tau^\mu{}_A\tau^\nu{}_B R_{\mu\nu}{}^{AB}(L) = e\left[\left(\omega_{BC}{}^A + 2\bar{\tau}_{BC}{}^A\right)\omega_A{}^{BC} + \bar{e}_{AB}\omega^{AB} + \left(2\bar{e}_{zA} - 2\bar{\tau}_{AC}{}^C - \omega_C{}^C{}_A\right)\omega_B{}^{AB}\right]. \tag{6.66}$$

Thus the overall action, prior to enforcing any constraints is

$$S^{\text{2nd-order}}_{\text{domain-wall}} = -\frac{1}{16\pi G}\int e\left[2T_B{}^A\widetilde{\omega}_A{}^B + 2T_B{}^B\widetilde{\omega}_A{}^A + 2T_{AB}{}^B\omega^A + \right. \tag{6.67}$$

$$+\left(\omega_{BC}{}^A + 2\bar{\tau}_{BC}{}^A\right)\omega_A{}^{BC} + \bar{e}_{AB}\omega^{AB} + \tag{6.68}$$

$$\left. +\left(2\bar{e}_{zA} - 2\bar{\tau}_{AC}{}^C - \omega_C{}^C{}_A\right)\omega_B{}^{AB}\right]. \tag{6.69}$$

Similarly to the case of general $p$-branes, upon taking the equations of motion, we find conventional constraints which we can use to solve for spin-connection components. The equations of motion are

$$2\bar{\tau}_{BC}{}^A - 2\omega_{[B}{}^A{}_{C]} = T_{BC}{}^A = 0, \tag{6.70}$$

$$-\bar{\tau}_{AC}{}^C - \omega_C{}^C{}_A + \bar{e}_{zA} - \omega_A = -T_{AB}{}^B + T_{zA}{}^z = 0, \tag{6.71}$$

$$T_{AB}{}^B = 0, \tag{6.72}$$

$$T_{[AB]} = 0, \tag{6.73}$$

$$\bar{e}_{AB} - 2\widetilde{\omega}_{[AB]} = T_{AB}{}^z = 0 \tag{6.74}$$

for $\omega_A{}^{BC}$, $\omega_B{}^{AB}$, $\omega^A$, $\widetilde{\omega}^{[AB]}$, and $\omega^{AB}$ respectively. Summarizing this in the form of conventional constraints on torsion, we have found

$$T_{AB}{}^C = T_{\mu\nu}{}^z = T_{[AB]} = 0. \tag{6.75}$$

Applying the constraints to finding expressions for spin-connection components, we simply write out the corresponding torsion tensor in terms of the curl of $\tau$ or $e$ and the spin-connection components that are included in the definition of torsion. Then we rearrange the equation to express the corresponding spin-connection component. We find, from $T_{[AB]} = 0$

$$\omega_{AB}(\tau, e) = -e^\mu\bar{\tau}_{\mu[A,B]}, \tag{6.76}$$

where we have again used a comma to indicate anti-symmetry in the two indices preceding the comma. ${T_{AB}}^C = 0$ can be used to solve for ${\omega_A}^{BC}$, yielding

$$ {\omega_A}^{BC}(\tau) = -{\bar{\tau}_A}{}^{[B,C]} + \frac{1}{2}\bar{\tau}^{BC,}{}_A\,. \tag{6.77} $$

Next, we can use the fact that

$$ {T_{\mu\nu}}^z = 0 \implies {T_{zA}}^z = 0 \tag{6.78} $$

to write

$$ \omega_A(\tau, e) = \bar{e}_{zA}{}^z = e^\mu \bar{e}_{\mu A}{}^z. \tag{6.79} $$

Finally, using

$$ {T_{\mu\nu}}^z = 0 \implies {T_{AB}}^z = 0, \tag{6.80} $$

we find

$$ \widetilde{\omega}_{[AB]}(\tau, e) = \frac{1}{2}\bar{e}_{AB}. \tag{6.81} $$

So, in conclusion the solutions for conventional spin-connection components in the domain-wall case are

$$ {\omega_A}^{BC}(\tau) = -{\bar{\tau}_A}{}^{[B,C]} + \frac{1}{2}\bar{\tau}^{BC,}{}_A\,, \tag{6.82} $$

$$ \omega_{AB}(\tau, e) = -e^\mu \bar{\tau}_{\mu[A,B]}\,, \tag{6.83} $$

$$ \widetilde{\omega}_{[AB]}(\tau, e) = \frac{1}{2}\bar{e}_{AB}, \tag{6.84} $$

$$ \omega_A(\tau, e) = e^\mu \bar{e}_{\mu A}{}^z. \tag{6.85} $$

Before proceeding to take the limit directly from the second-order formalism of GR, we wish to write down the action (6.67) after enforcing the constraints we have found. The final action is

$$ \begin{aligned} S^{\text{2nd-order}}_{\text{domain-wall}} = -\frac{1}{16\pi G}\int e\,\big[&{\omega_C}^C{}_A{\omega_B}^{AB} + 2\omega_A{\omega_B}^{AB} + {\omega_C}^A{}_B{\omega_A}^{CB} + \\ &+2T_{(BA)}\widetilde{\omega}^{AB} + 2\omega_{AB}\widetilde{\omega}^{[AB]}\big] \end{aligned} \tag{6.86} $$

### 6.1.2 Limit in the second order formalism of GR

In the previous subsection, we proceeded from the first-order formalism of Galilean gravity to the second-order formulation. We can reach the same final theory, Galilean gravity in the second-order formalism, by starting directly from the second-order formalism of general relativity. This is what we will do here. Let us first introduce the second-order formulation of GR briefly.

Starting from the relativistic case, we write the Einstein-Hilbert action as a function of the spin-connection ${\Omega_\mu}^{\hat{A}\hat{B}}$ as

$$ S_{\text{EH}} = -\frac{1}{16\pi G_N}\int E\left({\Omega_{\hat{A}}}^{\hat{B}\hat{C}}{\Omega_{\hat{B}}}^{\hat{A}}{}_{\hat{C}} - {\Omega_{\hat{A}}}^{\hat{A}\hat{C}}{\Omega_{\hat{B}}}^{\hat{B}}{}_{\hat{C}}\right), \tag{6.87} $$

where the spin-connection is dependent on the vielbein $E_{\mu}{}^{\hat{A}}$, and is given by

$$2\partial_{[\mu}E_{\nu]}{}^{\hat{A}} - \Omega_{[\mu}{}^{\hat{A}}{}_{\hat{B}}E_{\nu]}{}^{\hat{B}} = 0. \tag{6.88}$$

The equation (6.88) can be solved by

$$\Omega_{\hat{A}}{}^{\hat{B}\hat{C}} = \frac{1}{2}\bar{E}^{\hat{B}\hat{C}}{}_{\hat{A}} - \bar{E}_{\hat{A}}{}^{[\hat{B}\hat{C}]}, \tag{6.89}$$

where we have defined $\bar{E}_{\mu\nu}{}^{\hat{A}}$ as the curl of the relativistic vielbein

$$\bar{E}_{\mu\nu}{}^{\hat{A}} := 2\partial_{[\mu}E_{\nu]}{}^{\hat{A}}. \tag{6.90}$$

Note that even when indices are raised or lowered, $\bar{E}_{\mu\nu}{}^{\hat{A}}$ is anti-symmetric in its first two indices, as indicated by a comma in what follows.

We can find expressions for the components of the relativistic spin-connections in terms of the non-relativistic ones by simply fixing the spin-connection component in terms of non-relativistic indices $a, A, \ldots$, and then replacing the relativistic vielbeine in $\bar{E}_{\mu\nu}{}^{\hat{A}}$ by the appropriate non-relativistic ones. We find the expressions

$$\Omega_{C,AB} = \frac{1}{c}\omega_{C,AB}(\tau), \tag{6.91}$$

$$\Omega_{C,Ab} = -T_{b(A,C)} - \frac{1}{c^2}\omega_{[C,|b|A]}(\tau, e), \tag{6.92}$$

$$\Omega_{C,ab} = \frac{c}{2}T_{ab,C} + \frac{1}{c}(\omega_{C,ab} + \omega_{[a,b]C})(\tau, e)\,, \tag{6.93}$$

$$\Omega_{c,AB} = \omega_{c,AB}(\tau, e) + \frac{1}{c^2}\omega_{[B|,c|C]}(\tau, e), \tag{6.94}$$

$$\Omega_{c,Ab} = \frac{c}{2}T_{cb,A} - \frac{1}{c}\omega_{(c,b)A}(\tau, e), \tag{6.95}$$

$$\Omega_{c,ab} = \omega_{c,ab}(\tau, e)\,. \tag{6.96}$$

We can express the spin-connections present in these equations in terms of the vielbeine - the expression for $\omega_{c,ab}$ is the same as in equation (6.58), while $(\omega_{C,ab} + \omega_{[a,b]C})$ is given by

$$(\omega_{C,ab} + \omega_{[a,b]C})(\tau, e) = -\bar{e}_{C[a,b]}. \tag{6.97}$$

Switching to the special case of domain-walls ($p = D - 2$), we find that (6.89) reduces to the following expressions

$$\Omega_{C}{}^{AB} = \frac{1}{\omega}\omega_{C}{}^{AB}(\tau)\,, \tag{6.98}$$

$$\Omega_{z}{}^{AB} = \omega_{z}{}^{AB}(\tau, e) + \frac{1}{\omega^2}\omega^{[A}{}_{z}{}^{B]}(\tau, e)\,, \tag{6.99}$$

$$\Omega^{AB} = T^{(AB)} + \frac{1}{\omega^2}\widetilde{\omega}^{[AB]}(\tau, e)\,, \tag{6.100}$$

$$\Omega^{A} = \frac{1}{\omega}\omega^{A}\,, \tag{6.101}$$

where we should recall that

$$\widetilde{\omega}_A{}^B = \omega_A{}^{zB}. \tag{6.102}$$

The solutions for the spin-connection components $\omega_C{}^{AB}(\tau)$, $\omega_z{}^{AB}(\tau, e)$, $\omega^{[A}{}_z{}^{B]}(\tau, e)$, $\widetilde{\omega}^{[AB]}(\tau, e)$, and $\omega^A(\tau, e)$ are the same ones that were found in the previous section, see equation (6.86). Now we can proceed to consider the different cases, similarly to previous sections.

**$p$-branes.** To find the $p$-brane Galilean action, we find the highest-order combination that could arise when the expansions in equations (6.91) - (6.96) are plugged into (6.87), and then take the $c \to \infty$ limit. The highest-order action that arises from this is the one resulting from the square of (6.93), since no other terms of order $c^2$ occur via the combinations available from (6.87). The action, at leading order, is then given by

$$S_{p\text{-brane}} = -\frac{1}{16\pi G} \int \frac{ec^2}{4} T_{ab}{}^A T^{ab}{}_A\,, \tag{6.103}$$

which presents us with a dilemma, since this term is divergent. In general, there are three options, which we can consider:

1. Accept the divergence as the leading term. We may perform an overall scaling of the action to bring this term to order $c^0$, and then take the limit. In literature, the resulting theories are called "electric" Galilean theories. What is interesting about these theories is that they're independent of the spin-connection, i.e. a first-order formalism does not exist for these theories.

2. Use a Hubbard-Stratonovich transformation. In this case, we pacify the divergence by introducing an auxiliary field $\lambda$. We can then write a divergence of the form $c^2X^2$ equivalently as

$$-\frac{1}{c^2}\lambda^2 - 2\lambda X. \tag{6.104}$$

   We can recover the original form by solving for $\lambda$ and substituting the solution in. $\lambda$ also inherits its transformation rule from the field $X$ in the form of the solution

$$\lambda = c^2 X. \tag{6.105}$$

   Once the $c \to \infty$ limit is taken, $\lambda$ becomes a Lagrange multiplier that enforces the constraint $X = 0$. This will result in different term(s) becoming leading in the action and thus yields a different gravitational theory.

3. Cancel the divergence. It is possible to introduce a new $(p+1)$-form field into the theory and tune this field critically such that it precisely cancels the divergent term. Due to transformation rules, the extra field then also contributes at lower orders. This approach, for the $p = 0$ particle and $p = 1$ string cases results in the corresponding Newton-Cartan theory of gravity. This would also be expected for the generic $p$-brane. However, we will not discuss this scenario further in this work.

For generic $p$-branes, the first approach results in the following action

$$S_{\text{electric } p\text{-brane}} = -\frac{1}{16\pi G} \int \frac{e}{4} T_{ab}{}^A T^{ab}{}_A\,, \tag{6.106}$$

where we have performed a rescaling of $G_N$ to $c^{p+3}G$. As mentioned earlier, this action is clearly independent of any spin-connection. The equations of motion yield as a solution any manifold for which $T_{ab}{}^{A}$ is zero. Per the previous section, this is equivalent to the manifold possessing an integrable foliation by transverse submanifolds.

In contrast, the second approach yields the action

$$\begin{aligned}S_{\text{magnetic }p\text{-brane}} = -\frac{1}{16\pi G}\int e\Bigg[&\omega^{a,bc}(e)\omega_{b,ac}(e) - \omega_{b}{}^{ba}(e)\omega_{c}{}^{c}{}_{a}(e) + 2T^{a}{}_{A}{}^{A}\omega_{b}{}^{b}{}_{a}\\ &- T^{a}{}_{A}{}^{A}T_{aB}{}^{B} + T^{a(A,B)}T_{a(A,B)}\\ &+\Big(\lambda^{A,ab} - (\omega^{A,[bc]} + \omega^{[b,c]A})(\tau,e)\Big)T_{ab,A}\Bigg].\end{aligned} \tag{6.107}$$

Upon comparison with the second-order $p$-brane action found in the previous section (6.61), we find that the actions are indeed not the same, with the difference given by

$$\begin{aligned}S_{\text{magnetic }p\text{-brane}} = S^{\text{2nd-order}}_{p\text{-brane}}&\\ -\frac{1}{16\pi G}\int e&\left[T_{a}{}^{\{BC\}}T^{a}{}_{\{BC\}} + \frac{(D-2)}{(p+1)(D-p-3)}T^{a}{}_{A}{}^{A}T_{aB}{}^{B}\right],\end{aligned} \tag{6.108}$$

with the additional difference that the $T_{ab}{}^{A}\omega_{A}{}^{ab}$-term of $S^{\text{2nd-order}}_{p\text{-brane}}$ has been replaced by a rescaled version

$$T_{ab}{}^{A}\left(\omega_{A}{}^{ab}(\tau,e) + \lambda_{A}{}^{ab}\right). \tag{6.109}$$

This presents the remedy to the issue discussed at the start of the section - how is it possible that there are independent spin-connections in the second-order formulation of Galilei gravity, even though clearly all spin-connections were already solved for before the limit was taken? The spin-connection that still occurs in the limit from the first-order formulation of the theory $\omega_{A}{}^{ab}$, can be absorbed via redefinition into the Lagrange multiplier $\lambda_{A}{}^{ab}$.

It should be noted that the situation resulting from the difference between the magnetic $p$-brane and plain second-order $p$-brane actions is quite dire - the original action $S^{\text{2nd-order}}_{p\text{-brane}}$ has an emergent symmetry under an-isotropic dilatations, which the additional terms break. The emergent symmetry can be related to the fact that there is a field, and thus an equation of motion missing from the original $p$-brane action with respect to the relativistic case. This means that the action found in (6.61) should be considered a pseudo-action, rather than a full action of the non-relativistic theory.

**Defect-branes.** Yet again, we note that in the case of defect-branes there are only independent spin-connections occur. This means that in the case of the limit from the second-order formalism, all spin-connections should drop out, since all spin-connection components have already become dependent. Similarly, all torsion tensors that occur have become intrinsic. This means that all terms quadratic in spin-connections drop out from the corresponding actions. For the first method of dealing with the divergence, we recover exactly the same action as in the case of the electric generic $p$-brane, found in equation (6.106). Similarly to the generic case, choosing the second way of dealing with the divergent term, yields an action similar to (6.107), but with all the quadratic terms removed, i.e.

$$S_{\text{magnetic defect-brane}} = -\frac{1}{16\pi G}\int e\Big[T_{ab}{}^{A}\left(\omega_{A}{}^{ab}(e) + \lambda_{A}{}^{ab}\right) + T_{a}{}^{(AB)}T^{a}{}_{(AB)} \tag{6.110}$$

$$+ 2T^{a}{}_{A}{}^{A}\omega_{b}{}^{b}{}_{a} - T^{a}{}_{A}{}^{A}T_{aB}{}^{B}\Big]. \tag{6.111}$$

Since there are only independent spin-connections, no comparison can be made here.

**Domain-walls.** Finally, in the case of domain-walls, identically to our previous treatments of this case, $T_{ab}{}^{A}$ goes to zero identically. This makes the electric theory easy to treat - the divergent $T_{ab}{}^{A}$-term goes to zero, and what remains is

$$S^{(1)}_{\text{electric domain-wall}} = -\frac{1}{16\pi G}\int e\Big[T^{(AB)}T_{(AB)} - T_{A}{}^{A}T_{B}{}^{B}\Big], \tag{6.112}$$

where we have again omitted the singular transverse index $a = z$. The solutions to the equations of motion arising from this action are any geometries where $T^{(AB)} = 0$. Recalling the geometric implications of the classification, we know that those are precisely the geometries where $e^{\mu}$ is a conformal Killing vector with respect to the longitudinal metric $\tau_{\mu\nu}$, and where worldvolume is absolute. Finally, we remark that under the duality (6.1), this theory becomes what is known in the literature as "electric Carroll gravity" [27].

We should note that this electric limit is not unique, since we can apply the Hubbard-Stratonovich transformation to either $T$-squared term separately, i.e.

$$-T_{A}{}^{A}T_{B}{}^{B} \to -2\omega^{-2}\lambda T_{A}{}^{A} + \omega^{-4}\lambda^{2}, \tag{6.113}$$

or

$$T^{(AB)}T_{(AB)} \to 2\omega^{-2}\lambda^{(AB)}T_{(AB)} - \omega^{-4}\lambda^{(AB)}\lambda_{(AB)}. \tag{6.114}$$

These two procedures result in two additional electric domain-wall actions,

$$S^{(2)}_{\text{electric domain-wall}} = -\frac{1}{16\pi G_{NL}}\int eT^{(AB)}T_{(AB)}, \tag{6.115}$$

and

$$S^{(3)}_{\text{electric domain-wall}} = -\frac{1}{16\pi G_{NL}}\int -eT_{A}{}^{A}T_{B}{}^{B}, \tag{6.116}$$

respectively.

Taking the second option for taming the divergence, we tame, this time, $T^{(AB)}$. Thus $\lambda_{(AB)}$ becomes a Lagrange multiplier enforcing the constraint $T^{(AB)} = 0$. The action, in this case, becomes

$$\begin{aligned} S_{\text{magnetic domain-wall}} = -\frac{1}{16\pi G}\int e\,\big[&\omega_{C}{}^{C}{}_{A}\omega_{B}{}^{AB} + 2\omega_{A}\omega_{B}{}^{AB} + \omega_{C}{}^{A}{}_{B}\omega_{A}{}^{CB} + \\ &+2T_{(BA)}\widetilde{\omega}^{AB} + 2\omega_{AB}\widetilde{\omega}^{[AB]}\big]\,, \end{aligned} \tag{6.117}$$

which is identical to the previous action we found for second-order domain-walls after solving for spin-connections in the first-order Galilean theory.

# Chapter 7

# Conclusion

In this work, we presented a classification of Galilean spacetimes for $p$-branes. Additionally, we discussed a duality between Carroll and Galilei spacetimes that allows for this classification to be extended to include Carroll spacetimes. The classification was reached through determining the different types values that can be taken by the intrinsic torsion, and subsequently finding the geometric consequences of each possible intrinsic torsion type. In addition we discussed two special cases that were found in literature previously [7, 9], the Galilean particle and domain-wall. The latter corresponds, via the aforementioned duality, to the classification of the Carrollian particle.

Finally, we derived the gravitational theories that realize the classifications presented in section 5. It was found that every case found in the classification was also realized in a theory of gravity. These theories were discussed in both their first and second-order formulations. Galilean gravity in it's second-order form was reached both as a direct limit of the second-order formalism of general relativity, and through solving for all spin-connections from the first-order Galilean theory. Through this procedure we found that the approach that was taken makes a difference, and may result in a different action being recovered in some cases.

In taking the limit from the second-order formulation of general relativity, we discovered that the leading term was divergent. We then discussed different methods of dealing with this divergent term. The options presented to us were

1. Accept the divergent term, and bring it to order $c^0$ by an overall scaling of the action
2. Eliminate the divergence by applying a Hubbard-Stratonovich transformation. This brings the divergent term from order $c^2$ to $c^0$ via the introduction of an auxiliary field.
3. Introducing an extra field to the theory, and tuning this to a critical value such that it cancels the divergent term. This option is left for future work.

Depending on the approach that was chosen, actions were presented for the different options, termed electric and magnetic limits in the literature.

Finally, we discuss the fact that the electric limit in the domain-wall ($p = D - 2$) is not unique - we can apply a Hubbard-Stratonovich transformation to either of the two resulting quadratic terms, or to both at the same time, yielding three distinct electric limits in this case.

**Outlook** Of course, a lot of work still remains to be done. Here we will highlight some of the most immediate and promising directions for future inquiries to be directed towards.

As we found in section 5, the classification of spacetimes intimately depends on the representation theory of the group under consideration. This provides ample potential for further research since not all special cases are discussed in this work. Some interesting cases that can immediately be thought of are those of non-relativistic strings, where the representations of the longitudinal Lorentz group, $SO(1,1)$ becomes reducible. It is well known that in this case you can express the system in terms of so-called lightcone coordinates [28, 29]. Another interesting special case is that of supermembranes in 11 dimensional supergravity, where the representation of the transversal rotations $SO(8)$ become reducible. Of course, any case where a representation of one of the groups forming the structure group (or indeed the entirety of the structure group) is reducible becomes a special case of the classification. With the tools discussed in this work, these special cases can be explored systematically.

Another further path that is open to explore is that of other structure groups $G$. For instance, the spacetimes resulting from the Aristotelian group for extended objects have thus far not been classified. Some exploration of the geometries resulting from different kinematical Lie groups can be found in [11]. Some groups of interest would be the already mentioned Aristotelian group, with applications to be found in condensed matter physics [6], as well as fluid dynamics [30]. Another Lie algebra of interest is the Bargmann algebra, the centrally extended Galilei algebra. This is significant because the central extension component of the algebra corresponds directly to the additional geometric component that is added in the third option of our approaches to taming a divergence.

Extending the classification presented here is possible also through the third option we presented towards the taming of divergences. Introducing the additional geometric ingredient of a $(p+1)$-form field has been done in some cases in the supergravity literature [31], but the most general $p$-brane case still remains to be done. An additional point of interest in this regard is the fact that using the mathematical framework to this end proves unsuccessful for extended objects. The reason for this is the following: mathematically speaking, the additional geometric field arises as a central extension to the Galilei algebra. This is fine for particles, since in this cases the central extension generator of the algebra is a vector. However, already for strings the generator would need to be a 2-form, which cannot be used to generate a central extension of a group.

Recently, increasing attention has also been directed toward studying Carrollian quantum field theories (CQFTs). Although potentially tangental, employing the concept of duality between Carrollian and Galilean spacetimes introduced here might prove to be of interest in making further progress in this area.

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