Bachelor Research Project Physics

Probing Axion Inflation Coupled to U(1) Gauge Fields

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Abstract

This bachelor thesis investigates axion-driven inflation coupled to a generic U(1) gauge field. The intrinsic shift symmetry inherent to axions forbids or highly suppresses possible UV corrections, ensuring a substantial degree of flatness in the inflationary potential over an extended duration. The interplay between the axion and the gauge field significantly affects the dynamics of the axion field, deviating from a homogeneous behavior. Backreaction effects, such as the sourcing of inflaton perturbations through the inverse decay of gauge field fluctuations and contributions to the universe energy density, are analyzed. Curvature perturbations arising from the coupling and their impact on the power spectrum at the superhorizon limit are derived and discussed. The resulting power spectrum is found to exhibit the expected mild scale dependence characteristic of slow-roll inflation. By imposing COBE normalization to the power spectrum, the mutual interaction between the axion and gauge field is highlighted. Specifically, it is found how for $\xi \gg 4$, where the parameter ξ is linked to the coupling and growth of gauge field fluctuations, the gauge field contribution to the spectrum becomes dominant and completely surpasses the vacuum fluctuations. Further research directions are proposed: the replication of results can involve considering full scalar and metric perturbations. Additionally, investigating tensor perturbations may provide insights into plausible gravitational wave production. Motivated by recent Cosmic Microwave Background (CMB) observations, exploring non-Gaussianity effects through higher order correlation functions is also recommended.

Contents

1 Introduction

Cosmological inflation is a fascinating theory of physics that was first introduced to solve some crucial open problem about the Universe that the theory of the Big Bang could not directly answer. Inflation is a period of time after the Big Bang and before recombination, characterised by a rapid quasi-exponential expansion of space. Introducing this inflationary period leads to a possible explanation to the apparent uniformity of the Cosmic Microwave Background (CMB), whereby patches of sky that have never been in causal contact are, nevertheless, measured to be in thermal equilibrium. This is known as the horizon problem, and a solution to this puzzle is able to explain also other important cosmological issues such as the flatness problem or the monopole problem [1]. By means of inflation's exponential expansion, distinct regions of the universe become causally connected as the particle horizon expands more rapidly than the Hubble radius, the characteristic scale of the universe. Consequently, particles re-enter the horizon and gain sufficient time to attain thermal equilibrium [2].

A theory of inflation frees the universe from the fine tuning problem, as now generic initial condition can lead to the solution of the problems mentioned above. Moreover, during the inflationary period, quantum fluctuations appear and undergo significant amplification. These perturbations get streched to cosmic significance, thereby affecting and becoming imprinted on the CMB signatures, with reminiscents that are still observable today. As famously known, quantum fluctuations are interpreted as the ancestral source of what are now large-scale structures in the universe [3].

Nevertheless, there still exist various open questions about the theory of inflation: first of all, the exact form of the so called inflaton field that characterised the inflationary epoch is still unknown. Indeed, a successful inflationary period must effectively be driven by the vacuum energy of a suitable field, allowing for the conversion of its potential energy into exponential expansion of spacetime. Additionally, the inflationary potential is required to be considerably flat for a prolonged period of time: this requirement ensures that the potential energy of the inflaton field dominates over other energy density components, serving as the driving force for inflation. Eventually, the inflationary phase concludes with a process known as reheating, during which the inflaton field transfers its stored energy to other forms. This energy transfer can occur through mechanisms such as decay into numerous particles in the Standard Model. Reheating marks the effective initiation of the Big Bang epoch [4].

It then follows that studies of inflation become crucially dependent on the exact nature of the inflaton field. The first proposal was presented by Guth in the 80's, where he believed that the Higgs field was the inflaton field [5]. As of current research, this latter hypothesis is not considered as valid as more probable hypothesis such as inflation driven by an axion field [6]. Axions are fields that posses shift symmetry, i.e. the action remains invariant when the axion field is shifted by a constant. This symmetry is crucial to ensure that quantum corrections to the so called slow roll parameters, essentials to quantify the degree to which the inflaton potential is flat and thus able to effectively drive the inflationary period, are suppressed or at least limited enough such that UV physics corrections to the inflationary epoch become negligible [7].

In conclusion, the main goal of this research project is to investigate cosmological signatures of axionic-driven inflation, in which an axion field is coupled to a generic U(1) gauge field. The thesis is structured as follows: firstly, a concise overview of modern cosmology and the standard slow-roll inflationary paradigm is provided in section 2. Following this, section 3 delves into the primary objective of this thesis, starting with presenting and analyzing the action of the system under consideration. The relevant equations of motion are then derived and discussed, with particular attention on studying the dynamic interplay between the axion and the gauge field. This is subject of subsections 3.2 to 3.5. Finally, in subsection 3.6 an explicit expression for the late time power spectrum is presented and therefore, the plausibility of this type of axionic-driven inflation is physically and mathematically argued through phenomenological and observation-based arguments. Lengthy or less pertinent calculations are provided in the appendix (A), while a supplementary Mathematica code to the thesis can be found in the appendix (B).

2 Mathematical Description of Inflation

The concept of cosmological inflation was first introduced by A. Guth in the 1980 in his famous article "Inflationary universe: A possible solution to the horizon and flatness problems" [5]. As the title of the article suggests, inflation was presented as a possible resolution to overcome two important problems arising from Big Bang cosmology: the horizon and the flatness problem. While this section will primarily concentrate on the former issue, it is essential to begin with a concise overview of the current understanding in standard cosmology.

2.1 FRW Metric

Standard cosmology, more widely known as the Big Bang model, relies on the Friedmann-Robertson-Walker (FRW) spacetime metric [8]. This metric is derived from the observation-driven assumption that the Universe is effectively homogenous and isotropic on large scales, the so called Cosmological Principle. In this regard, a compelling example is given by the cosmic microwave background (CMB) temperature uniformity, with anisotropies in the order of one part in 10^5 [9]. As the CMB can be interpreted as a relic of the thermal radiation that was permeating the Universe at the time of last scattering (at a redshift z of about $z \approx 1100$ [10]), this measured uniformity indicates that the Universe was highly isotropic and homogeneous. With these assumptions, in natural units $c = \hbar = 1$ the FRW metric takes the form [11]

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\,\mathrm{d}t^2 + a^2\left(t\right)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right), \tag{1}$$

where the metric $g_{\mu\nu}$ is expressed in comoving coordinates (t, r, θ, ϕ)[1]. The term $a\left(t\right)$ is the cosmic scale factor characterising the time dependence of the spatial components of the metric. As such, knowledge of the scale factor $a\left(t\right)$ is crucial to understand and describe the evolution of the universe. Lastly, the curvature of space is associated to the curvature parameter k, which can take values of $k = -1, 0, 1$, describing respectively a negatively curved, flat or positively curved universe.

A ubiquitous quantity in cosmology is the Hubble parameter H defined as

$$H \equiv \frac{\dot{a}}{a}, \tag{2}$$

which effectively is the change of $a\left(t\right)$ per unit of scale factor. The Hubble parameter sets a characteristic length scale d_H of the universe via the relation

$$d_H = H^{-1}. \tag{3}$$

d_H is known as the Hubble radius or Hubble length. For an observer at the center of a sphere with radius d_H, objects outside this sphere will appear to recede at a speed faster than the speed of light [3]. As it will be explained further, the time evolution of the comoving Hubble radius $(aH)^{-1}$ plays a crucial role in inflation.

[1] Comoving coordinates refer to a coordinate system in which an observer is static with respect to the relative expansion of space. As an example, the comoving distance between two objects with no peculiar velocities remains fixed.

2.2 Conformal Time

The FRW metric can be recast in a pseudo-Minkowski form by introducing conformal time τ defined as

$$\mathrm{d}\tau \equiv \frac{\mathrm{d}t}{a\left(t\right)}. \tag{4}$$

Conformal time may be interpreted as a clock that slows down alongside the expansion of the universe [1]. In this thesis, derivatives with respect to cosmic time are denoted by an overdot, e.g. $\frac{\mathrm{d}a}{\mathrm{d}t} \equiv \dot{a}$, whereas derivatives with respect to conformal time by an apostrophe, e.g. $\frac{\mathrm{d}a}{\mathrm{d}\tau} \equiv a'$. Furthermore, the Hubble parameter H has a corresponding term $\mathcal{H}$ in conformal time:

$$\mathcal{H} \equiv \frac{a'}{a}. \tag{5}$$

Conformal time τ can then be substituted into (1) to obtain

$$\mathrm{d}s^2 = a^2\left(\tau\right)\left[-\,\mathrm{d}\tau^2 + \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right)\right]. \tag{6}$$

According to observations, our universe is considerably flat [12]. As such, $k = 1$ can be substituted into (6) to yield

$$\mathrm{d}s^2 = a^2\left(\tau\right)\left[-\,\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega\right], \tag{7}$$

where the typical substitution $\mathrm{d}\Omega \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ was employed. We can now observe how the FRW metric factorizes as a product of a static Minkowski metric times the time dependent scale factor $a\left(\tau\right)$. In matrix notation, the metric (7) takes the diagonal form

$$\mathbf{g} = \begin{pmatrix} -a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}, \tag{8}$$

with its inverse metric $g^{\mu\nu}$ obtained by inverting (8):

$$\mathbf{g}^{-1} = \begin{pmatrix} -1/a^2 & 0 & 0 & 0 \\ 0 & 1/a^2 & 0 & 0 \\ 0 & 0 & 1/a^2 & 0 \\ 0 & 0 & 0 & 1/a^2 \end{pmatrix}. \tag{9}$$

2.2.1 Particle Horizon

Considering an isotropic universe, light propagates radially [1], by which $\mathrm{d}\Omega = 0$. Additionally, light follows a null geodesic $\mathrm{d}s^2 = 0$. Substituting these into (7) yields

$$\mathrm{d}r^2 = \mathrm{d}\tau^2\,. \tag{10}$$

Integrating (10) leads to

$$r\left(\tau\right) = \pm\tau + C, \tag{11}$$

where C is an arbitrary constant. From (11), it follows that light, travelling between two events denoted by initial time t_1 and final time t_2, covers a distance of

$$\Delta r \equiv \tau_2 - \tau_1 = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{a\,(t)}. \tag{12}$$

By the cosmic speed limit, this is also the maximum distance any other particle can travel between those times. The causal evolution of two events is therefore defined by the amount of conformal time between them. As such, the comoving particle horizon is defined as

$$\Delta\tau_{max} \equiv \tau - \tau_0 = \int_0^t \frac{\mathrm{d}t}{a\,(t)} = \tau(t) - \tau(0), \tag{13}$$

where, by convention, $t_1 = 0$ is set such that $a\,(t_1 = 0) \equiv 0$ corresponds to the Big Bang singularity. From (4), it can be inferred how $t_1 = 0$ does not necessarily imply $\tau_1 = 0$, and furthermore, we will see how this disagreement is crucial in inflation, where the initial Big Bang singularity is actually pushed back in conformal time to $\tau_1 = -\infty$.

It is common practice to rewrite the integral of (13) in terms of the comoving Hubble radius $(aH)^{-1}$ as

$$\tau = \int \frac{1}{a} \cdot \frac{\dot{a}\,\mathrm{d}t}{a \cdot \frac{\dot{a}}{a}} = \int \frac{\mathrm{d}\ln a}{aH}. \tag{14}$$

Equation 14 shows how the behaviour of the comoving Hubble radius is strictly connected to the particle horizon: if $(aH)^{-1}$ increases, then τ grows in a similar fashion. In order to mathematically obtain the evolution of $(aH)^{-1}$, we need to solve for the universe dynamics governed by $a\,(t)$. This is briefly developed in the following subsection.

2.3 Universe Dynamics

Consider the Einstein field equations

$$G_{\mu\nu} = 8\pi G\,T_{\mu\nu}. \tag{15}$$

The universe dynamics for a perfect fluid with energy density ρ and pressure p is obtained by solving the 00 Einstein field equation together with the trace of (15). Detailed derivations are copious in literature (e.g. see Ref [13]), therefore only the final results is presented for a flat universe with $k = 0$ [2]:

$$H^2 = \frac{\rho}{3}, \tag{16}$$

$$\dot{H} + H^2 = -\frac{1}{6}\left(\rho + 3p\right). \tag{17}$$

Equations (16) and (17) are famously known as the Friedmann equations, and their solutions provide the time evolution of the scale factor $a\,(t)$, which as argued in the previos sect As an example related to the previous subsection, solutions to (16) and (17) yield [1]

$$(aH)^{-1} \propto a^{\frac{1}{2}\left(1+3\frac{p}{\rho}\right)}. \tag{18}$$

[2]Here units are chosen such that $8\pi G = 1$

For conventional matter sources, the Strong Energy Condition (SEC) imposes that $1 + 3\frac{p}{\rho} > 0$ [14]. It then follows from (18) that the comoving Hubble radius is a monotonically increasing function, and so is τ, which can be verified by a straightforward integration of (14)

$$\tau = \int a^{\frac{1}{2}\left(1+3\frac{p}{\rho}\right)} \, \mathrm{d}\ln a = \int a^{\frac{1}{2}\left(-1+3\frac{p}{\rho}\right)} \, \mathrm{d}a \,, \tag{19}$$

$$\tau \propto \frac{2}{1+3\frac{p}{\rho}} a^{\frac{1}{2}\left(1+3\frac{p}{\rho}\right)} . \tag{20}$$

This shows that for sources satisfying the SEC, the particle horizon is a monotonically increasing function of the scale factor as well, and therefore the initial Big Bang singularity is conventionally set at $\tau(0) = 0$, with vanishing contributions to the conformal time coming from the lower boundary of integration. This sets a limit on the extension of the comoving particle horizon (13), effectively making it a finite quantity, with troublesome consequences which we are about to explore in the following sections.

2.4 CMB Observations and the Horizon Problem

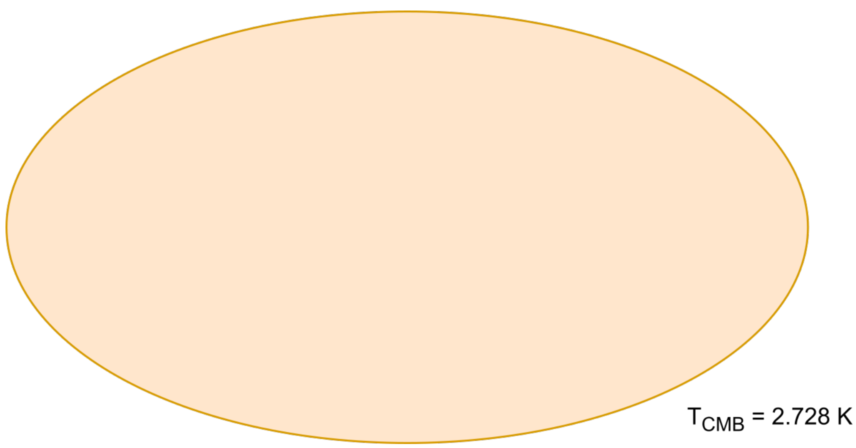

Figure 1: Self produced heat sky map of the CMB radiation at a temperature of $T = 2.728\,\mathrm{K}$. A clear uniformity is displayed at this temperature as the monopole contribution to the spectrum is dominant. Figure adapted from [15].

We have seen in the previous section that conformal time plays a crucial role in defining the causal structure of the universe, as it defines the maximum distance at which two separated particles could ever transmit and receive information. Particles at a distance greater that τ could have never been in causal contact, since light simply did not have time to travel between them during a period equal to the age of the universe [16]. In particular, it can be calculated that patches of the sky that are separated by angles larger than 2 degrees have never been in causal contact during the history of the universe [17]. Nonetheless, although they are lacking a direct causal connection, these regions exhibit an unexpected thermal equilibrium, a phenomenon which is known as the horizon problem. Indeed, observations from the COBE and WMAP experiments have provided valuable insights into the temperature spectrum of the CMB at different angular separations. These experiments have found remarkable uniformities in the CMB even for angular scales larger than 2 degrees [18]. A clear visual display of this issue can be found in figure 1, where the heat sky map of the CMB recorded by the COBE Differential Microwave Radiometers (DMR) experiment is graphically reproduced [19]. The resolution of this puzzle, along with other cosmological issues such as the flatness problem and the monopole problem is one of the key motivations for studying the theory of inflation [20] [21].

2.4.1 The Horizon Problem and the Inflationary Solution

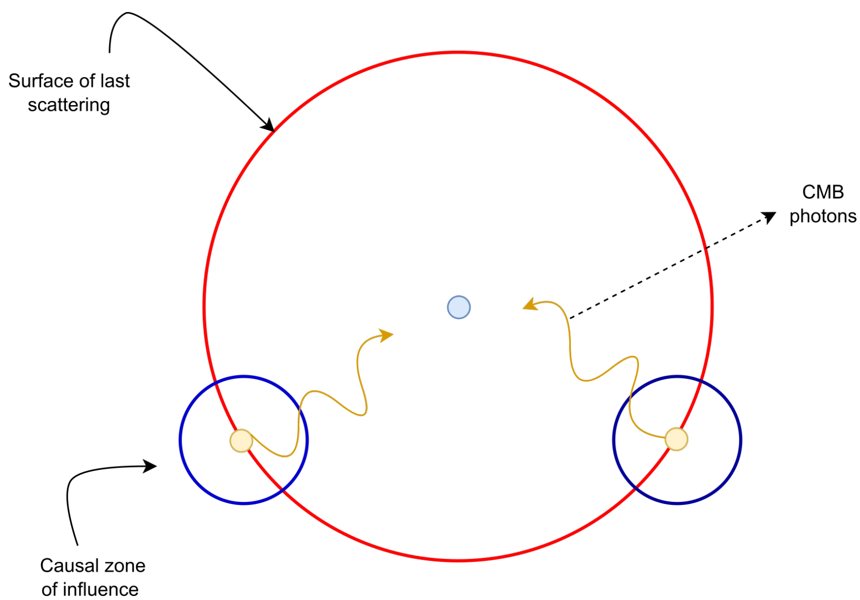

Figure 2: The figure illustrates the horizon problem in cosmology. Two distinct patches of the sky, denoted by the yellow spots, were causally independent during the epoch of recombination. The CMB photons emitted from these regions eventually reached the observer (depicted as the blue spot at the center of the figure). Remarkably, despite the lack of direct causal interaction, the CMB photons from both patches exhibit the same temperature when measured by the observer. This is known as the horizon problem in cosmology. Adapted from [1].

Figure 2 gives a visual explanation of the horizon problem: patches of sky that were causally disconnected at the time of recombination emitted CMB photons which are detected today to be in thermal equilibrium. This surprising phenomenon cannot be just a coincidence, as at the time of recombination the sky consisted of more than 10^5 causally disconnected regions [22]. It appears that there is not enough conformal time from the Big Bang singularity until the epoch of recombination to explain the apparent uniformity of the CMB temperature spectrum. As stated in section 2.3, the particle horizon or equivalent conformal time is a monotonically increasing function of the scale factor. A possible solution to the horizon problem is, then, simply to postulate a period in which the particle horizon is a decreasing function of time, thereby allowing patches of sky to share information and approach thermal equilibrium. Mathematically, this is equivalent to saying

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0, \tag{21}$$

from which using (18) implies

$$\frac{1}{2}\left(1+3\frac{p}{\rho}\right) < 0 \quad \Rightarrow \quad \frac{p}{\rho} < -\frac{1}{3}. \tag{22}$$

We can thus see that a period of decreasing comoving horizon may be sustained by a fluid violating the SEC. Consequently, (20) yields a negative conformal time due to the negative coefficient $\frac{2}{1+3\frac{p}{\rho}} < 0$

$$\tau \propto \frac{2}{1+3\frac{p}{\rho}} a^{\frac{1}{2}\left(1+3\frac{p}{\rho}\right)}, \tag{23}$$

and as such the Big Bang singularity is actually displaced to

$$\tau(0) = -\infty. \tag{24}$$

The latter equation (24) plays a crucial role in the context of inflation: a significantly greater amount of conformal time is generated between the Big Bang singularity and the epoch of recombination.[3] This extended duration allows patches of sky to have interacted and become causally connected in the far past, thereby addressing the fundamental issue arising from the horizon problem. Figure 3 provides a visual description of the solution to the horizon problem as given by the theory of inflation.

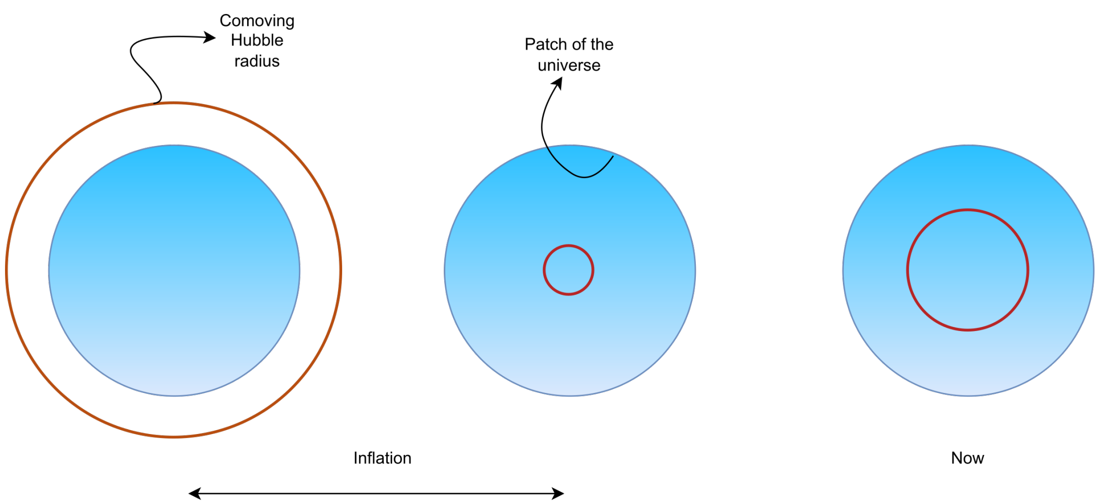

Figure 3: Solution to the horizon problem given by the introduction of an inflationary period: the Big Bang singularity is shifted arbitrarily far back to $\tau(0) = -\infty$, thereby allowing for a phase in which the comoving Hubble radius was decreasing. Consequently, patches of sky had time to interact and share information in order to reach thermal equilibrium. Adapted from [23].

2.5 Slow Roll Parameters

It was described how inflation postulates that a phase of decreasing Hubble radius can provide a satisfactory solution to the horizon problem. It is crucial that this inflationary epoch lasts long enough, thereby permitting all regions of the universe sufficient time to attain thermal equilibrium and thereby address the horizon problem successfully. Consequently, it is worth

[3] Furthermore, the ending and reheating phase of inflation happens now at $\tau = 0$. See Ref [1].

deriving two well-known important quantities that describe the effectiveness of the inflationary period: the relation (21) can be expanded to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = \frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon) < 0 \quad \text{where} \quad \epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{25}$$

Equivalently,

$$\epsilon < 1. \tag{26}$$

The latter expression tells us that during inflation we expect the Hubble rate to be slowly changing, as well as we require ϵ to remain smaller than one "for a sufficient number of Hubble times" [17]:

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon}. \tag{27}$$

The two parameters ϵ and η introduced above characterize successful inflation whenever

$$\{\epsilon, |\eta|\} \ll 1. \tag{28}$$

2.6 UV Sensitivity of Slow Roll Parameters and the η Problem

The slow roll parameters quantify the degree of flatness of the inflationary potential, which is required to drive and sustain the inflationary period for a sufficient amount of time, in order to address correctly cosmological issues such as the horizon problem explained in the previous sections. On the other hand, it is reasonable to assume that quantum and gravity corrections in UV physics could introduce fluctuations in the flatness of the potential, therefore affecting the minuteness of the slow roll parameters and eventually spoiling inflation. In order to quantify this, let's consider a typical slow roll inflation action in the presence of a generic scalar field ϕ minimally coupled to gravity

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_p}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{29}$$

where $\mathcal{R}$ is the Ricci scalar, $\sqrt{-g}$ is the determinant of the metric $g_{\mu\nu}$, and $V(\phi)$ is a generic inflaton potential. It is straightforward to derive the equation of motion for the inflaton and the 00 Einstein equation obtained by varying with respect to the metric [17], respectively

$$\ddot{\phi} + 3H\dot{\phi} = -V', \tag{30}$$

$$H^2 = \frac{1}{3M_p}\left[\frac{1}{2}\dot{\phi}^2 + V\right], \tag{31}$$

where ϕ is considered a perfect fluid obeying the equation of state [24]

$$\frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{32}$$

From equations (30) and (31) an expression for the slow roll parameter ϵ can be derived as

$$\epsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_p^2 H^2}. \tag{33}$$

As discussed in section 2.5, $\epsilon \ll 1$ for successful inflation. Using (33) leads to the slow roll condition $\frac{1}{2}\dot{\phi}^2 \ll V$, from which we can infer that the kinetic energy of the scalar field is considerably smaller than its potential energy. Consequently, (30) and (31) simplify to

$$3H\dot{\phi} \approx -V', \tag{34}$$

$$H^2 \approx \frac{V}{3M_p^2}, \tag{35}$$

where the $\ddot{\phi}$ term in the inflaton equation of motion (30) was also neglected, given that $\dot{\phi}$ is already slowly varying. Substituting (34) and (35) into (33) yields

$$\epsilon \equiv \epsilon_v \approx \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2. \tag{36}$$

Additionally, in literature it is also usually considered the parameter

$$\eta_v \equiv M_p^2 \frac{V''}{V}, \tag{37}$$

which effectively assesses the degree of curvature of the potential: ϵ_v and η_v are also referred to as potential slow roll parameters [25] [26]. Slow roll inflation is then assured for $\{\epsilon_v, |\eta_v|\} \ll 1$ [17].

Quantum corrections may modify the inflaton potential by introducing higher order terms. For instance, let's consider a simple example where the inflaton potential receives quantum corrections of the form:

$$\Delta V(\phi) = \Lambda^4 f\left(\frac{\phi}{\Lambda}\right), \tag{38}$$

where $\Lambda > H$ represents the characteristic energy scale associated with the UV physics and $f\left(\frac{\phi}{\Lambda}\right)$ is an arbitrary dimensionless function [4]. Consequently, the potential $V(\phi)$ is modified to

$$V(\phi) \to V(\phi) + \Delta V(\phi) = V(\phi) + \Lambda^4 f\left(\frac{\phi}{\Lambda}\right), \tag{39}$$

and the η_v parameter changes by

$$\eta_v = \frac{V''(\phi)}{V(\phi)} + \frac{\Lambda^2}{V(\phi)} f''\left(\frac{\phi}{\Lambda}\right) \Rightarrow \Delta\eta_v \approx \frac{\Lambda^2}{H^2} > 1, \tag{40}$$

Changes in the eta parameter are of order $\mathcal{O}(1)$! It then follows that the duration of inflation is considerably reduced. This is known as the eta problem [27].

3 Axion-Driven Inflation

As introduced in the previous section, inflationary slow roll parameters are highly sensitive to UV-physics. As shown in [7], higher n-dimensional terms of the type $\phi^n M_p^{4-n}$ can also induce

[4]Notice how if Λ were smaller than H, then the quantum corrections would be highly suppressed and have negligible impact on the potential.

order $\mathcal{O}(1)$ corrections to the inflationary potential, which may considerably reduce the duration of the inflationary epoch and hence its effectiveness. Therefore, in order to avoid fine tuning arguments that claim to suppress these higher order terms because of special initial conditions, a possible solution comes from considering a shift symmetric invariant action, which is the only symmetry that does not allow these dangerous corrections [7]. With shift symmetry we refer to a transformation of the type $\phi \to \phi + \text{constant}$ that leaves the action invariant. Fields ϕ embedding this symmetry are called *Axions*.

There exists a simple logical argument to understand why axions forbid quantum corrections to the inflationary potential: let's denote the axion field as ϕ, and its potential energy as $V(\phi)$. The shift symmetry of the axion field implies that the potential energy $V(\phi)$ remains invariant under the transformation $\phi \to \phi + C \Rightarrow V(\phi) = V(\phi + C)$, where C is a constant. UV corrections to the potential energy can be expressed as a power series expansion:

$$V(\phi) = V_0(\phi) + \sum_i \Delta V_i(\phi), \tag{41}$$

where $V_0(\phi)$ represents the background potential energy, and $\Delta V_i(\phi)$ are higher order corrections. Since the potential energy $V(\phi)$ is invariant under the transformation $\phi \to \phi + C$, the quantum corrections should also respect this symmetry. This means that each term in the power series expansion should have the same shift symmetry:

$$\Delta V_i(\phi) = \Delta V_i(\phi + C). \tag{42}$$

However, since the quantum corrections are meant to break the shift symmetry, then the effective potential $V(\phi)$ would no longer be shift symmetric, leading to a contradiction. Therefore, axions, which possess a robust shift symmetry, forbid quantum corrections to the inflationary potential.

In this thesis, we consider axion driven inflation coupled to a generic gauge field A_μ via the operator $c\phi F\tilde{F}$ where c is a coefficient. Our analysis thus begins by presenting the action of the system and describing its constituent terms.

3.1 Action of the System

The model in consideration has the following action:

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_p}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\alpha}{4f}\phi\,\tilde{F}^{\mu\nu}F_{\mu\nu}\right], \tag{43}$$

where ϕ is the axion inflaton field, $\mathcal{R}$ the Ricci scalar, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the field strength, and $\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\eta^{\mu\nu\alpha\beta}F_{\alpha\beta}$ the field strength dual, with $\eta^{\mu\nu\alpha\beta}, \eta^{0123} \equiv 1$ the Levi-Civita tensor. The dimensionless parameter α is expected to be of order unity, whereas the coefficient f, known as the axion decay constant, sets the scale of the interaction [28]. As described in Ref [7], f can be interpreted as a parameter quantifying the least non-negligible coupling, such as the interaction between the axion field and the gauge field via terms of the form $c\phi F\tilde{F}$. It is expected that $f < M_p$, since physics above the Planck scale M_p may introduce shift-symmetry breaking effects, thus vanishing the effort of studying axion driven inflation [29][30].

3.2 Obtaining the Gauge Field Equations of Motion

As a first step, we derive the equations of motion for the gauge field A_μ, in order to understand how the created "gauge quanta" are produced and affected by the background inflaton field ϕ. As such, both metric and inflaton perturbations can be neglected for the current section. The usual Einstein summation convention is adopted, by which Greek indexes, such as μ, imply summation over all spacetime coordinates $\mu = \{0, 1, 2, 3\} = \{\tau, x, y, z\}$ while Latin indexes, such as j, refer to spatial coordinates only $j = \{1, 2, 3\} = \{x, y, z\}$. The complete derivation can be found in the appendix A.1. The equation of motion for the gauge field, then, take the form [31]

$$\partial_\sigma \left(\sqrt{-g}F^{\sigma\rho}\right) + \frac{\alpha}{2f}\partial_\sigma \left(\phi\, \eta^{\sigma\rho\mu\nu} F_{\mu\nu}\right) = 0. \tag{44}$$

Equation (44) can be simplified by means of the Bianchi identity [32][33]

$$\partial_\sigma \left(\eta^{\sigma\rho\mu\nu} F_{\mu\nu}\right) = 0, \tag{45}$$

which has also been verified computationally through the code provided in the appendix B. Expanding (44) by using the derivative product rule and substituting for (45) yields

$$\partial_\sigma \left(\sqrt{-g}F^{\sigma\rho}\right) + \frac{\alpha}{2f}\eta^{\sigma\rho\mu\nu} F_{\mu\nu}\partial_\sigma\phi = 0. \tag{46}$$

In solving equation (46), the Coulomb Gauge $A_0 = \partial_i A_i = 0$ is adopted. The solution is then given by

$$\partial_0^2 A_i - \partial_j^2 A_i - \frac{\alpha}{f}\phi' \epsilon_{ijk}\partial_j A_k = 0, \tag{47}$$

where ϵ_{ijk} is the Levi-Civita tensor in three dimensions. Equation (47) takes the vector form

$$\vec{A}'' - \nabla^2 \vec{A} - \frac{\alpha}{f}\phi' \vec{\nabla} \times \vec{A} = 0, \tag{48}$$

where the identity $\left(\vec{\nabla} \times \vec{A}\right)_i = \epsilon_{ijk}\partial_j A_k$ was used.

3.2.1 Converting to Fourier Space

As commonly done in QFT, in order to solve (48), $\vec{A}(\tau, \mathbf{x})$ [5] is promoted to an operator and then decompose it in its Fourier modes as

$$\vec{A}(\tau, \mathbf{x}) = \sum_{\lambda=\pm} \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[\vec{\epsilon}_\lambda(\mathbf{k}) A_\lambda(\tau, \mathbf{k}) a_\lambda(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \vec{\epsilon^*}_\lambda(\mathbf{k}) A^*_\lambda(\tau, \mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{49}$$

where A_λ are the Fourier modes and $a_\lambda, a^\dagger_\lambda$ respectively the usual annihilation and creation operators obeying the canonical commutation relations

$$\left[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')\right] = \delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{50}$$

In (49) a sum is performed over the two polarization states[6] $\lambda = \pm$ represented by the circular polarization vectors $\vec{\epsilon}_\pm$ obeying [28]

[5] We remark that, to avoid excessive cluttered expressions, we interchangeably use both the arrow $\vec{v}$ and the boldface notation $\mathbf{v}$ to denote vectors.

[6] Exactly as in classical Electrodynamics [34]

$$\vec{k} \cdot \vec{\epsilon}_{\pm}(\vec{k}) = 0, \tag{51}$$

$$\vec{k} \times \vec{\epsilon}_{\pm}(\vec{k}) = \mp i k \vec{\epsilon}_{\pm}, \tag{52}$$

$$\vec{\epsilon}_{\pm}(-\vec{k}) = \vec{\epsilon}_{\pm}(\vec{k})^{*}, \tag{53}$$

$$\vec{\epsilon}_{\lambda}(\vec{k})^{*} \cdot \vec{\epsilon}_{\lambda'}(\vec{k}) = \delta_{\lambda\lambda'}. \tag{54}$$

By means of decomposition (49) and (51) to (54), equation (48) can be converted to Fourier space. The exact calculations can be found in A.2. As such, it is found that the gauge field modes satisfy equations

$$\left(\partial_{\tau}^{2} + k^{2} \pm \frac{2k\xi}{\tau}\right) A_{\pm}(\tau, k) = 0, \qquad \xi \equiv \frac{\alpha\dot{\phi}}{2fH}, \tag{55}$$

where the vector notation on the k argument is dropped as the partial differential equation involves only the magnitude of the momentum vectors $\vec{k}$. We will see how the parameter ξ will play an important role in characterizing the inflationary period and the interplay between the axion and the gauge field.

3.2.2 Solution of the Gauge Field Modes Equations of Motion

A first look at equation (55) shows that one of the mode functions $A_{\pm}$ undergoes rapid growth of fluctuations, also known as tachyonic instability. This can be seen clearer if (55) is rewritten in terms of $k\tau$ (for negative conformal time $\tau < 0$):

$$\left(\partial_{k\tau}^{2} + 1 \pm \frac{2\xi}{k\tau}\right) A_{\pm}(\tau, k) = 0, \tag{56}$$

which resembles the standard form of a harmonic oscillator equation with a time dependent oscillating frequency $\omega^2 = 1 \pm \frac{2\xi}{k\tau}$. As such, instabilities will grow when $\omega^2 \leq 0$. The convention in which $\dot{\phi} > 0$ can be adopted, such that the parameter ξ, which can be effectively treated as a constant during inflation, since both $\dot{\phi}$ and H are slowly varying, remains positive. Consequently, the mode A_{+} is the one selected to exhibit tachyonic instability when

$$\omega^2 = 1 + \frac{2\xi}{k\tau} \leq 0 \quad \Rightarrow \quad k\tau \geq -2\xi, \tag{57}$$

that is, equivalently, the growth occurs in the vicinity of horizon crossing for the given mode. By inspection of (56), it then follows that production of gauge field fluctuations for the remaining mode A_{-} can be neglected.

Equation (55) allows for an analytical solution in terms of hypergeometric functions as

$$A_{\pm}(\tau, k) = e^{-ik\tau}\tau \left[c_0 \, {}_1F_1\left(1 \pm i\xi, 2, 2ik\tau\right) + c_1 \, U\left(1 \pm i\xi, 2, 2ik\tau\right)\right], \tag{58}$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function, U is the confluent hypergeometric function and c_0, c_1 are constants defined by initial conditions. The solution (58) was confirmed computationally, see Appendix B. In the current case, it is required that the gauge field satisfies the Bunch-Davies vacuum [35][36]

$$\lim_{\tau \to -\infty} A_{\pm}(\tau, k) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{59}$$

The exact form of the solution with the appropriate c_0 and c_1 is rather complicated and can be found in the code provided in appendix B. On the other hand, in order to account for a more tractable analysis, solutions to (55) may be recast and presented in terms of Coulomb functions[7] G_0 and F_0, which are themselves a combination of hypergeometric functions [38], in the form

$$A_+(\tau, k) = \frac{1}{\sqrt{2k}} \left[G_0(\xi, -k\tau) + iF_0(\xi, -k\tau) \right]. \tag{60}$$

Although (60) is an exact solution, it is worth applying some approximations to reduce it to a more practical form. Firstly, from (56) it is possible to infer that A_+ modes in the range $-k\tau \gg 2\xi$ will not develop substantial growth and they will remain in their vacuum state. Additionally, phenomenologically it is expected for the paramater ξ to satisfy $\xi \geq \mathcal{O}(1)$, in order for the oscillation frequency (57) to deviate sufficiently from $\omega \approx 1$ and to allow for a meaningful exponential growth amplitude of fluctuations[8]. Therefore the aim is to find a reasonable approximation in the regime

$$k\tau \gg -2\xi, \quad e^{\gamma\xi} \gg 1, \tag{61}$$

where γ is an arbitrary parameter of order one. As argued in [28], solutions to (56) in the region (61) are well approximated by the expression

$$A_+(\tau, k) \approx \sqrt{\frac{-2\tau}{\pi}} e^{\pi\xi} K_1\left(2\sqrt{-2\xi k\tau}\right), \tag{62}$$

where $K_1(z)$ is the modified Bessel function of the second kind.

Finally, it is stressed that since $k\tau \geq -2\xi$ is required to develop substantial gauge field fluctuation, an asymptotic approximation and the large argument of expression (62) may be employed to obtain (for further details, refer to the appendix A.3)

$$A_+(\tau, k) = \sqrt{\frac{1}{2k}} \left(\frac{-k\tau}{2\xi}\right)^{1/4} e^{\pi\xi - 2\sqrt{-2\xi k\tau}}. \tag{63}$$

The expression (63) are plotted in figure 4 as a function of conformal time, for a range of values of the parameter ξ. It can be graphically confirmed how the gauge field instabilities are built towards the end of inflation, as $\tau \to 0$. Physically, this can be interpreted as the final period by which the perturbations are stretched to cosmic significance and thus are about to exit the horizon. Indeed, as expected, for $k\tau \ll -2\xi$ the gauge field modes remain in their vacuum state and exhibit an oscillatory motion. Furthermore, it can be observed how the parameter ξ affects the rate at which the gauge field fluctuations start to grow: this is evident from the exponential factor $e^{\pi\xi}$ in the expression (63). To further validate the findings presented in this section, figure 5 provides a visual comparison between the exact solution (60) of the gauge field equation of motion and the approximate solution (63): the two functions exhibit a high degree of agreement, indicating that (63) can be confidently employed in all subsequent analyses with considerable accuracy.

[7] Coulomb functions are solutions to the Coulomb equation $\frac{d^2W}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right] W = 0$ [37]. In our case, (56) is exactly the Coulomb equation with $L = 0$, $\rho = k\tau$ and $\eta = \pm\xi$.

[8] Here it is simply anticipated that given the similarity of (56) to a harmonic oscillator, then the unstable models will likely display a growing exponential behavior.

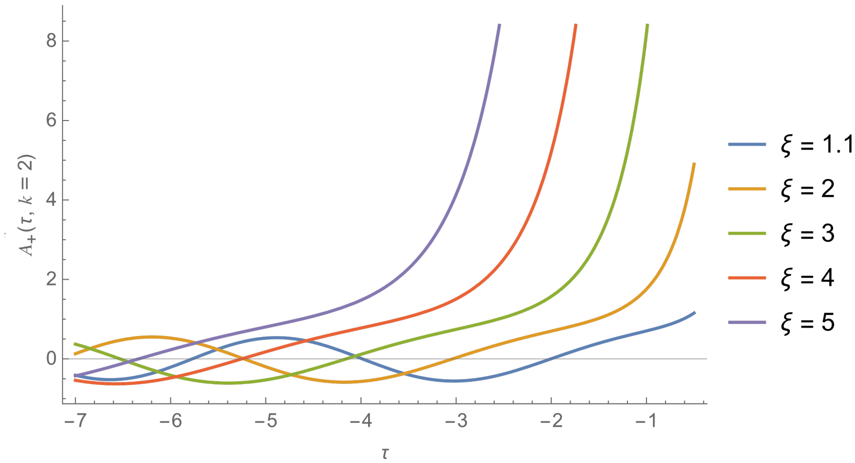

Figure 4: Plot of the gauge field modes A_+ as in expression (63) as a function of conformal time, for five different values of the parameter ξ. The plot reveals a significant increase in the amplitude of fluctuations towards the end of the inflationary period $\tau \to 0$. Moreover, the rate of production of these fluctuations becomes more rapid as the value of ξ increases.

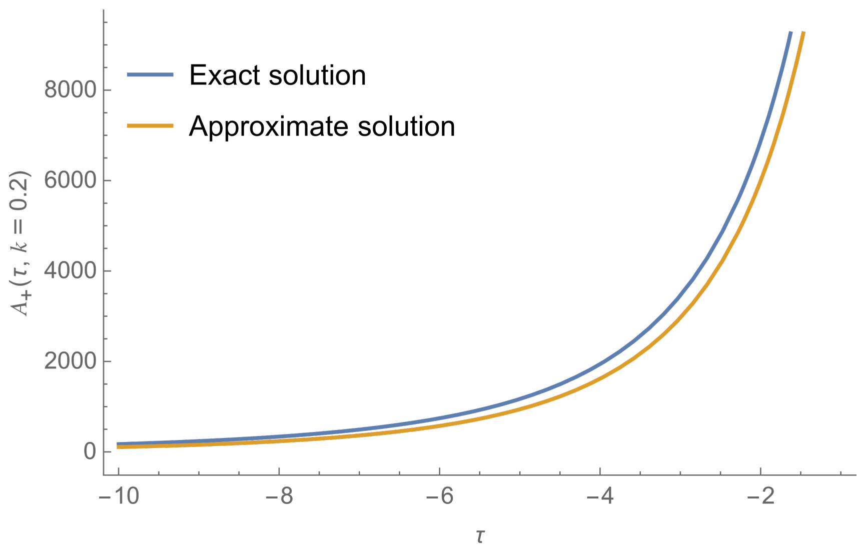

Figure 5: Comparison between the exact solutions (60) versus the approximate solution (63). As it can be inferred, the two agree to a high degree of accuracy in the regime (61) where the approximation was effectively derived. The parameter ξ is set to $\xi = 4$.

3.3 Derivation Equations of Motion for Inflaton

In order to study the backreaction effects that the gauge field perturbations have on the inflationary epoch, the equation of motion of the inflaton field needs to be derived. This is the subject of the next sections. Additionally, the 00 Einstein equation is computed, as the gauge field perturbations may also affect the homogeneous background evolution given by the corresponding Friedmann equation.

3.3.1 Inflaton Equation of Motion

Equation (43) is varied with respect to ϕ. In particular,

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\sqrt{-g}\left(\frac{\alpha}{4f}\phi\, \tilde{F}^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\right), \tag{64}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} = -\frac{1}{2}\sqrt{-g}g^{\sigma\rho}\left(\delta^\mu_\sigma \partial_\rho \phi + \delta^\mu_\rho \partial_\sigma \phi\right) = \tag{65}$$

$$= -\frac{1}{2}\sqrt{-g}\left(g^{\mu\rho}\partial_\rho \phi + g^{\sigma\mu}\partial_\sigma \phi\right) = \tag{66}$$

$$-\frac{1}{2}\sqrt{-g}\left(\partial^\mu \phi + \partial^\mu \phi\right) = -\sqrt{-g}\partial^\mu \phi, \tag{67}$$

where from (66) to (67) the symmetric property of the FRW metric $g^{\mu\nu} = g^{\nu\mu}$ was used, which can also be inferred from its matrix form (8). Given the result of (64) and (67), the Euler-Lagrange equation for the axion field ϕ becomes

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right) = 0, \tag{68}$$

$$-\sqrt{-g}\left(\frac{\alpha}{4f}\phi\, \tilde{F}^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\right) - \partial_\mu\left(\sqrt{-g}\partial^\mu \phi\right) = 0. \tag{69}$$

The first term in (69) contains the backreaction of the gauge field on the dynamics of the axion, whereas the second term introduces a Hubble friction contribution to the equation of motion due to time dependence of the FRW metric, specifically as in the usual slow-roll homogeneous inflation [39]. Further simplification of (69) leads to

$$-a^4\left(\frac{\alpha}{4f}\phi\, \tilde{F}^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\right) - \left(-2a\, a'\phi' + a^2\left(-\phi'' + \nabla^2\phi\right)\right) = 0, \tag{70}$$

$$\phi'' - \nabla^2\phi - 2\frac{a'}{a}\phi' + a^2\frac{\mathrm{d}V}{\mathrm{d}\phi} = a^2\frac{\alpha}{4f}\phi\, \tilde{F}^{\mu\nu}F_{\mu\nu}, \tag{71}$$

where all the details of the derivation can be found in the appendix A.4

In order to simplify the analysis of equation (71), the tensorial formulation of the gauge field term is converted to the physical fields, which by convenience are denoted "Electric" and "Magnetic" fields, although they do not necessarily bear any resemblance to the corresponding fields as in Standard Model physics[9]. It is thus proposed [31] [40]

$$E_i \equiv -\frac{1}{a^2}A_i{}', \quad B_i \equiv \frac{1}{a^2}\epsilon_{ijk}\partial_j A_k, \tag{72}$$

or equivalently, in vector notation

$$\vec{E} \equiv -\frac{1}{a^2}\vec{A}', \quad \vec{B} \equiv \frac{1}{a^2}\nabla\times\vec{A}. \tag{73}$$

[9] As an example, the expectation value of the dot product between the electric and magnetic field vectors $\langle\vec{E}\cdot\vec{B}\rangle_{\text{Electromagnetism}}$ is zero in QED. On the other hand, in section 3.4 we will find an explicit expression for $\langle\vec{E}\cdot\vec{B}\rangle$ assuming the U(1) gauge field of our system.

Consequently, (71) is rewritten as

$$\phi'' - \nabla^2\phi - 2\mathcal{H}\phi' + a^2\frac{\mathrm{d}V}{\mathrm{d}\phi} = a^2\frac{\alpha}{f}\vec{E}\cdot\vec{B}, \tag{74}$$

which has been verified computationally through the code provided in appendix B.

3.3.2 00 Einstein Equation

The 00 Einstein equation is obtained by varying the action with respect to the 00 component of the metric [41]. The derivation is performed through the code provided in the appendix B and leads to a modified Friedmann equation with an additional energy density contribution from the gauge field:

$$\mathcal{H}^2 = \frac{1}{3M_p^2}\left[\frac{1}{2}\phi'^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + a^2V + \frac{a^2}{2}\left(\vec{E}^2 + \vec{B}^2\right)\right]. \tag{75}$$

3.4 Backreaction of Gauge Field on the Homogeneous Inflaton Dynamics

The homogeneous inflation provided by the axion ϕ sources a growth of perturbations for A_μ in its respective equation of motion (55) through the term ξ dependent on $\dot{\phi}$. In section 3.3.1, the equation of motion for the axion field ϕ coupled to the gauge field A_μ through the term $\phi\tilde{F}^{\mu\nu}F_{\mu\nu}$ in the action (43) was derived. This equation captures the dynamics of the axion field and its interaction with the gauge field. Therefore, it follows similarly that the evolution of ϕ is affected by the production of the gauge field fluctuations, which backreact on the homogeneous background evolution through the last term of (74). In order to allow for a more tractable study of these backreaction effects, a mean field approximation of (74) and (75) is employed, by which inflaton perturbations are ignored and the gauge field fluctuations are averaged out. Mathematically, this leads to

$$\phi'' - 2\mathcal{H}\phi' + a^2\frac{\mathrm{d}V}{\mathrm{d}\phi} \approx a^2\frac{\alpha}{f}\langle\vec{E}\cdot\vec{B}\rangle, \tag{76}$$

$$\mathcal{H}^2 \approx \frac{1}{3M_p^2}\left[\frac{1}{2}\phi'^2 + a^2V + \frac{a^2}{2}\langle\vec{E}^2 + \vec{B}^2\rangle\right], \tag{77}$$

where the terms inside angle brackets are expectation value outlining the backreaction of the gauge field on the homogeneous dynamics of the axion during inflation [28]. The presence of these backreaction effects allows for several physical considerations to be made: firstly, from (76) it can be observed how the gauge field fluctuations effectively dissipate kinetic energy ϕ' from the axion in order to source their exponential growth. Secondly, the gauge field also serves as an additional energy density term in the Friedmann equation (77). Consequently, the usual inflationary period may be affected due to the different energy contributions and thus the inflaton potential may not be dominant. It is then crucial to find explicit expressions for both expectation values and impose slow roll conditions. The procedure is as follows:

Both spatial averages can be evaluated by referring to the Fourier mode decomposition (49), with the only difference that the ladder operator quantisation is now neglected as the work is performed in a classical picture[10]. As such, it is found that

[10] The same results for the spatial averages in (76) and (77) are obtained by considering the full quantum Fourier mode decomposition (49) if the angle brackets are considered as effectively defining vacuum expectation values.

$$\vec{A}(\tau, \mathbf{k}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \vec{\epsilon}(\mathbf{k}) A(\tau, k) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{78}$$

$$\vec{E} = -\frac{1}{a^2} \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \vec{\epsilon}(\mathbf{k}) A'(\tau, k) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{79}$$

$$\vec{B} = \frac{1}{a^2} \int \frac{\mathrm{d}^3 q}{(2\pi)^{3/2}} q\, \vec{\epsilon}(\mathbf{q}) A(\tau, q) e^{i\mathbf{q}\cdot\mathbf{x}}, \tag{80}$$

where the subscripts $\lambda = \pm$ is dropped since it is now implicit that only the A_+ polarization is considered, as it was found in section 3.2.2 that the A_- mode does not produce sensible fluctuations and thus it can be neglected. Inserting (78) into the expectation values in (76) leads to

$$\begin{aligned}
\langle \vec{E} \cdot \vec{B} \rangle &= -\frac{1}{a^4} \int \frac{\mathrm{d}^3 x\, \mathrm{d}^3 k\, \mathrm{d}^3 q}{(2\pi)^{9/2}} q\, \vec{\epsilon}(\mathbf{k}) \cdot \vec{\epsilon}(\mathbf{q}) A'(\tau, k) A(\tau, q) e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}} = && (81) \\
&= -\frac{1}{a^4} \int \frac{\mathrm{d}^3 k\, \mathrm{d}^3 q}{(2\pi)^3} \delta^{(3)}(\mathbf{k} + \mathbf{q})\, q\, \vec{\epsilon}(\mathbf{k}) \cdot \vec{\epsilon}(\mathbf{q}) A'(\tau, k) A(\tau, q) = && (82) \\
&= -\frac{1}{a^4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} k\, A'(\tau, k) A(\tau, k) = && (83) \\
&= -\frac{1}{a^4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} k\, \frac{\mathrm{d}}{\mathrm{d}\tau} |A|^2, && (84)
\end{aligned}$$

where the normalization (54) of the polarization vectors was used. Similarly, for the spatial average in (77) it is obtained (see appendix A.5)

$$\langle \vec{E}^2 + \vec{B}^2 \rangle = \frac{1}{a^4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\left| A' \right|^2 + k^2 |A|^2 \right]. \tag{85}$$

Unsurprisingly, direct evaluation of (84) and (85) by substitution of the exact mode solutions (60) is impractical as the integrals are divergent. On the other hand, the approximations (63) can be employed inside the integrals (84) and (85), where the integration regions can eb expanded from $k = 0$ to $k = \infty$. This latter approximation is argued to be accurate as it is discussed in 3.2.2 how the A_+ modes do not produce perturbations and oscillate along their vacuum state for $-k\tau \ll -2\xi$. Consequently the regions contribute negligibly to the integrals and can be evaluated explicitly as

$$\langle \vec{E} \cdot \vec{B} \rangle = -\frac{135 H^4 e^{2\pi\xi}}{65536 \pi^2 \xi^4} \approx -2.1 \cdot 10^{-4} \frac{H^4 e^{2\pi\xi}}{\xi^4}, \tag{86}$$

$$\langle \vec{E}^2 + \vec{B}^2 \rangle = \frac{63 H^4 e^{2\pi\xi} \left(4\xi^2 + 5\right)}{262144 \pi^2 \xi^5} \approx 2.4 \cdot 10^{-5} \frac{H^4 e^{2\pi\xi} \left(4\xi^2 + 5\right)}{\xi^5}. \tag{87}$$

Slow roll inflation is then reasonable when the source term $a^2 \frac{\alpha}{f} \langle \vec{E} \cdot \vec{B} \rangle$ in (76) is much smaller than the variation of the inflaton potential (such that its degree of flatness is not considerably altered). Mathematically, this can be expressed as

$$\left|\frac{\mathrm{d}V}{\mathrm{d}\phi}\right| \gg \left|a^2\frac{\alpha}{f}\langle\vec{E}\cdot\vec{B}\rangle\right| \quad \Rightarrow \quad \frac{H^2}{|\phi'|} \ll 69\xi^{3/2}e^{-\pi\xi}, \tag{88}$$

where the definition of ξ and the slow roll approximation (34) was used. An additional constraint comes from assuming that the inflaton potential remains dominant and effectively drives inflation. The gauge field energy density produced by the growth of perturbations in (77) must satisfy

$$V \gg \frac{1}{2}\langle\vec{E}^2+\vec{B}^2\rangle \quad \Rightarrow \quad \frac{H}{M_p} \ll 124\xi^{3/2}e^{-\pi\xi}, \tag{89}$$

where for simplicity factors of $\frac{1}{\xi^2}$ are neglected since $\mathcal{O}(\xi) \geq 1$. Both constraints (88) and (89) must be satisfied in order for inflation coupled to the U(1) gauge field to be successful.

3.5 Inflaton Perturbations

The axion field ϕ is now considered as a function of both space and time, that is

$$\phi = \phi(\tau) + \delta\phi(\tau, \mathbf{x}), \tag{90}$$

and study how its perturbations $\delta\phi(t, \mathbf{x})$ evolve. This can be performed by first substituting (90) into (74):

$$(\phi(\tau)+\delta\phi(\tau,\mathbf{x}))''-\nabla^2\delta\phi(\tau,\mathbf{x})-2\mathcal{H}(\phi(\tau)+\delta\phi(\tau,\mathbf{x}))'+a^2\frac{\mathrm{d}V}{\mathrm{d}\left(\phi(\tau)+\delta\phi(\tau,\mathbf{x})\right)} = a^2\frac{\alpha}{f}\vec{E}\cdot\vec{B}. \tag{91}$$

Expanding to linear order the potential in the last equation yields

$$\frac{\mathrm{d}V}{\mathrm{d}\left(\phi(\tau)+\delta\phi(\tau,\mathbf{x})\right)} \approx \frac{\mathrm{d}V}{\mathrm{d}\phi(\tau)} + \frac{\mathrm{d}^2V}{\mathrm{d}\phi(\tau)^2}\delta\phi(\tau,\mathbf{x}) = \tag{92}$$

$$= V(\phi)' + V''(\phi)\delta\phi(\tau,\mathbf{x}). \tag{93}$$

Plugging (93) into (91) leads to

$$\left[\phi(\tau)'' - 2\mathcal{H}(\phi(\tau)+a^2V'\right] + \phi(\tau,\mathbf{x})'' - \nabla^2\delta\phi(\tau,\mathbf{x}) + 2\mathcal{H}\delta\phi(\tau,\mathbf{x}) + a^2V''(\phi)\delta\phi(\tau,\mathbf{x}) = a^2\frac{\alpha}{f}\vec{E}\cdot\vec{B}. \tag{94}$$

The terms in square brackets are exactly equation (76), hence this leads to the equation of motion for the axion perturbations in the form

$$\delta\phi(\tau,\mathbf{x})'' - \nabla^2\delta\phi(\tau,\mathbf{x}) + 2\mathcal{H}\delta\phi(\tau,\mathbf{x}) + a^2V''(\phi)\delta\phi(\tau,\mathbf{x}) = a^2\frac{\alpha}{f}\left(\vec{E}\cdot\vec{B} - \langle\vec{E}\cdot\vec{B}\rangle\right). \tag{95}$$

3.5.1 Solution to the Equation of Motion for Inflaton Perturbations

Equation (95) is a partial linear inhomogeneous differential equation. As such, its general solution can be expressed as a sum of the homogeneous solution plus a particular solution:

$$\delta\phi(\tau,\mathbf{x}) = \delta\phi_{\text{homogeneous}}(\tau,\mathbf{x}) + \delta\phi_{\text{particular}}(\tau,\mathbf{x}). \tag{96}$$

Physically, the homogeneous solution corresponds to the classic vacuum inflaton perturbations, which are extensively studied in literature, e.g. see [17] [42]. On the other hand, the inhomogeneous term can be interpreted as sourcing inflaton perturbations via inverse decay contributions of the form $\delta A + \delta A \to \delta\phi$ [43]. These inverse fluctuations are worth studying, as their evolution may radically affect usual slow-roll inflation. As an example, Ref [44] derived how the inverse decay term actually dominates over the vacuum perturbations in the range $f \leq 10^{-2} M_p$.

With this premise, it is proceeded to solve equation (95) in a similar way as in section 3.2.1, by firstly performing a mode expansion of the axion perturbations and thus converting the equation to Fourier space:

$$\delta\phi(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \frac{Q_{\mathbf{k}}(\tau)}{a(\tau)} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{97}$$

where the artificial extraction of the scale factor $a(\tau)$ from the mode function $Q_{\mathbf{k}}(\tau)$ will become clear further in the derivation. The following identities follow:

$$\delta\phi(\tau,\mathbf{x})'' = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left[\frac{Q_{\mathbf{k}}''a - a'Q_{\mathbf{k}}'}{a^2} - \frac{(a''Q_{\mathbf{k}} + a'Q'_{\mathbf{k}}) - 2a'^2 aQ_{\mathbf{k}}}{a^4} \right], \tag{98}$$

$$\nabla^2\delta\phi(\tau,\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} k^2 \frac{Q_{\mathbf{k}}(\tau)}{a(\tau)} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{99}$$

Substituting (98) and (99) into (95) leads to

$$\begin{aligned} &\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left[\frac{Q_{\mathbf{k}}''}{a} - \frac{a'}{a^2}Q'_{\mathbf{k}} - \frac{a''}{a^2}Q_{\mathbf{k}} - \frac{a'Q'_{\mathbf{k}}}{a^2} + 2\frac{(a')^2 Q_{\mathbf{k}}}{a^3} + 2\frac{a'}{a^2}Q'_{\mathbf{k}} - 2\frac{(a')^2 Q_{\mathbf{k}}}{a^3} + \frac{k^2}{a} + \frac{a^2V''}{a} \right] e^{i\mathbf{k}\cdot\mathbf{x}} = \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left[\frac{Q_{\mathbf{k}}''}{a} - \frac{a''}{a^2}Q_{\mathbf{k}} + \frac{k^2}{a}Q_{\mathbf{k}} + \frac{a^2V''}{a}Q_{\mathbf{k}} \right] e^{i\mathbf{k}\cdot\mathbf{x}} = a^2\frac{\alpha}{f}\left(\vec{E}\cdot\vec{B} - \langle \vec{E}\cdot\vec{B}\rangle \right). \end{aligned} \tag{100}$$

The $\langle\vec{E}\cdot\vec{B}\rangle$ term in the last equation can be neglected, since it does not depend on the momentum modes $\vec{k}$ as the expectation value integrates over all momentum space. Therefore, the final expression is reached by converting the source term to Fourier space as well:

$$\left[\partial_\tau^2 - \frac{a''}{a} + k^2 + a^2V'' \right] Q_{\mathbf{k}} = \mathcal{J}_{\mathbf{k}}(\tau), \tag{101}$$

where $\mathcal{J}_{\mathbf{k}}(\tau)$ is the source term in momentum space defined as

$$\mathcal{J}_{\mathbf{k}}(\tau) = a^3\frac{\alpha}{f} \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}} \left[\vec{E}\cdot\vec{B} \right](\tau,\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}. \tag{102}$$

Some readers may recognize equation (101) as a resemblance to the inhomogeneous Mukhanov-Sasaki equation, which arises naturally in slow-roll inflation when perturbing the minimally coupled action of the inflaton through considering metric and scalar perturbations [45]. As such, solutions to (101), which are about to be derived and discussed, will be in an analogous form as the canonical Mukhanov-Sasaki equation. Equation (101) can be solved similarly as (95), by splitting between the homogeneous and particular solutions

$$Q_{\mathbf{k}}(\tau) = Q_{\mathbf{k}}^{\text{homogeneous}}(\tau) + Q_{\mathbf{k}}^{\text{particular}}(\tau). \tag{103}$$

As usually done when solving for inflaton vacuum fluctuations, an operator expansion of the homogeneous term is employed [17]

$$Q_{\mathbf{k}}^{\text{homogeneous}}(\tau) = b\left(\mathbf{k}\right) Q_k(\tau) + b^\dagger\left(-\mathbf{k}\right) Q_k^*(\tau), \tag{104}$$

where $Q_k(\tau)$ and $Q_k^*(\tau)$ are linearly independent solutions of the homogeneous equation. The corresponding axion raising/lowering operators obey the common canonical commutation relation

$$\left[b\left(\mathbf{k}\right), b^\dagger\left(\mathbf{k}'\right)\right] = \delta^{(3)}\left(\mathbf{k} - \mathbf{k}'\right), \tag{105}$$

and since the axion ϕ and the gauge field A_μ are a priori independent, their respective ladder operators are consequently statistically independent. Thus, they commute:

$$\left[b\left(\mathbf{k}\right), a_\lambda\left(\mathbf{k}'\right)\right] = \left[b\left(\mathbf{k}\right), a_\lambda^\dagger\left(\mathbf{k}'\right)\right] = 0. \tag{106}$$

This latter property is crucial as the commutativity between the ladder operators for the homogeneous and particular solution can be exploited to argue that correlation functions involving cross terms such as $\langle Q_{\mathbf{k}}^{\text{homogeneous}} Q_{\mathbf{k'}}^{\text{particular}}\rangle$ vanish, implying that the curvature perturbations can be expressed as the sum of two independent terms

$$\zeta = \zeta^{\text{homogeneous}} + \zeta^{\text{particular}}. \tag{107}$$

This will be the subject of the following sections. But firstly, we take some time to derive the standard expression for the homogeneous solution to (101).

3.5.2 Solution of the Homogeneous Equation

The homogeneous part of the axion perturbation equation of motion (101) can be recast into the form

$$\left[\partial_\tau^2 + \left(k^2 - \frac{n^2 - \frac{1}{4}}{\tau^2}\right)\right] Q_k(\tau) = 0, \tag{108}$$

where we refer to appendix A.6 for a detailed derivation.

Equation (108) allows for an analytical solution in terms of Bessel functions of the first and second kind:

$$Q_k(\tau) = \sqrt{\tau}\left[c_1 J_n(k\tau) + c_2 Y_n(k\tau)\right], \tag{109}$$

although in literature they are most frequently presented in terms of the equivalent Hankel functions of the first and second kind as

$$Q_k(\tau) = \sqrt{-\tau}\left[\alpha H_n^{(1)}(-k\tau) + \beta H_n^{(2)}(-k\tau)\right], \tag{110}$$

where c_1, c_2, α, β are all constants determined by initial conditions. In (110), a $-i$ term was extracted from the initial condition constants to convert the Hankel function from a negative to a positive argument (since τ is negative during inflation) using the property

$$J_n(k\tau) = -iJ_n(-k\tau) \quad \text{for} \quad n \approx \frac{3}{2} \quad \text{and} \quad \tau \leq 0, \tag{111}$$

which was verified computationally. This leads an equivalent solution to (108) which is usually presented in literature in order to conventionally select only the Hankel function of the first kind, e.g see Refs [1] and [46]. It is stressed that these are all arbitrary normalization procedures, and the final results (including the power spectrum) will not be affected.

In order to avoid the vacuum ambiguity, impose Bunch-Davies vacuum can be imposed

$$\lim_{\tau \to -\infty} Q_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{112}$$

Asymptotic approximations of the Hankel functions of the first and second kind are [47]

$$\lim_{\tau \to -\infty} H^{(1)}_{n\approx 3/2}(k\tau) = i\sqrt{\frac{2}{\pi}} \frac{1}{k\tau} e^{ik\tau}, \quad \lim_{\tau \to -\infty} H^{(2)}_{n\approx 3/2}(k\tau) = i\sqrt{\frac{2}{\pi}} \frac{1}{k\tau} e^{-ik\tau}, \tag{113}$$

where a physically unobservable phase factor of i was kept to conventionally make the mode functions real in the limit $-k\tau \to 0$[11] [28]. Thus, (110) takes the asymptotic form

$$\lim_{\tau \to -\infty} Q_k(\tau) = \sqrt{\frac{2}{\pi}} \left[\alpha \sqrt{\frac{1}{k}} e^{-ik\tau} + \beta \sqrt{\frac{1}{k}} e^{ik\tau} \right]. \tag{114}$$

Matching (114) with (112) leads to

$$\alpha = i\frac{\sqrt{\pi}}{2}, \tag{115}$$

and consequently, the modes functions that serve as the homogeneous solutions to the axion perturbation equation are

$$Q_k(\tau) = i\frac{\sqrt{\pi}}{2} \sqrt{-\tau} H^{(1)}_n(-k\tau). \tag{116}$$

3.5.3 Particular Solution

Any particular solution to the axion perturbation equation of motion (101) can be expressed by employing the Green's function method. It is thus solved

$$\left[\partial_\tau^2 + \left(k^2 - \frac{n^2 - \frac{1}{4}}{\tau^2} \right) \right] G_k(\tau, \tau') = \delta\left(\tau - \tau'\right). \tag{117}$$

The normalized solution to (117) is therefore (see Ref [28])

$$G_k(\tau, \tau') = i\Theta\left(\tau - \tau'\right) \left[Q_k(\tau) Q_k^*(\tau') - Q_k^*(\tau) Q_k(\tau') \right], \tag{118}$$

such that a particular solution to (101) can be expressed as

[11] Here $\lim_{-\tau \to 0} H^{(1)}_{3/2} = -\frac{i\sqrt{\frac{2}{\pi}}}{(k\tau)^{3/2}}$ is used.

$$Q_{\mathbf{k}}^{\text{particular}}(\tau) = \int_{-\infty}^{0} \mathrm{d}\tau' \, G_k(\tau, \tau') \mathcal{J}_{\mathbf{k}}(\tau), \tag{119}$$

where as usual during inflation conformal time runs over the negative real axis.

3.6 Derivation of the Power Spectrum

3.6.1 Vacuum 2-point Correlator

In the previous section the mode functions describing the axion perturbations during the inflationary epoch were derived. The focus now shifts to obtaining an expression for the power spectrum of curvature perturbations $\zeta(\tau, \vec{x})$, which are strictly connected to the inflaton perturbations by the approximate relation [48]

$$\zeta(\tau, \mathbf{x}) = -\frac{H}{\dot{\phi}} \delta\phi(\tau, \mathbf{x}). \tag{120}$$

Derivation of expression (120) is beyond the scope of this thesis. Nevertheless, it is a standard relation extensively mentioned in literature, e.g. see [49]. Curvature perturbations are also known as adiabatic, as they equally affect all relative changes in any observable scalar quantities, i.e. $\frac{\delta \mathcal{X}}{\dot{\mathcal{X}}}$ [50]. As a relevant example to this thesis, observations led to considering CMB anisotropies fluctuations, which arise from energy density perturbations on the surface of last scattering, as utterly adiabatic [51].

Firstly, the curvature perturbations $\zeta(\tau, \vec{x})$ are decomposed into their Fourier modes

$$\zeta(\tau, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \zeta_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{121}$$

Comparing (121) with (97), and using the relation (120) it is found that

$$\zeta_{\mathbf{k}} = -\frac{H}{\dot{\phi}} \frac{Q_{\mathbf{k}}}{a}. \tag{122}$$

As argued in section 3.5.1, the curvature perturbations take the form

$$\zeta_{\mathbf{k}} = \zeta_{\mathbf{k}}^{\text{homogeneous}} + \zeta_{\mathbf{k}}^{\text{particular}}, \tag{123}$$

implying a correlation function $\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k'}} \rangle \equiv \langle 0 | \, \zeta_{\mathbf{k}} \zeta_{\mathbf{k'}} \, | 0 \rangle$ of the type

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k'}} \rangle = \langle \zeta_{\mathbf{k}}^{\text{homogeneous}} \zeta_{\mathbf{k'}}^{\text{homogeneous}} \rangle + \langle \zeta_{\mathbf{k}}^{\text{particular}} \zeta_{\mathbf{k'}}^{\text{particular}} \rangle, \tag{124}$$

where again cross terms vanish by the same reasoning as in section 3.5.1. The homogeneous correlation function can be evaluated as follows: the homogeneous modes are replaced with their operator expansion (104) such that

$$\langle \zeta_{\mathbf{k}}^{\text{homogeneous}} \zeta_{\mathbf{k'}}^{\text{homogeneous}} \rangle = \frac{H^2}{\dot{\phi}^2 a^2} \cdot \langle \Big(b\left(\mathbf{k}\right) Q_k(\tau) + b^\dagger\left(-\mathbf{k}\right) Q_k^*(\tau) \Big) \Big(b\left(\mathbf{k}'\right) Q_{k'}(\tau) + b^\dagger\left(-\mathbf{k}'\right) Q_{k'}^*(\tau) \Big) \rangle = \tag{125}$$

$$= \frac{H^2}{\dot{\phi}^2 a^2} \Big(\langle b\left(\mathbf{k}\right) Q_k(\tau) b\left(\mathbf{k}'\right) Q_{k'}(\tau) \rangle + \langle b\left(\mathbf{k}\right) Q_k(\tau) b^\dagger\left(-\mathbf{k}'\right) Q_{k'}^*(\tau) \rangle + \tag{126}$$

$$\langle b^\dagger\left(-\mathbf{k}\right) Q_k^*(\tau) b\left(\mathbf{k}\right) Q_k(\tau) \rangle + \langle b^\dagger\left(-\mathbf{k}\right) Q_k^*(\tau) b^\dagger\left(-\mathbf{k}'\right) Q_{k'}^*(\tau) \rangle \Big) =$$

$$= \frac{H^2}{\dot{\phi}^2 a^2} Q_k(\tau) Q_{k'}^*(\tau) \langle b\left(\mathbf{k}\right) b^\dagger\left(-\mathbf{k}'\right) \rangle, \tag{127}$$

where to pass from (126) to (127) the standard vacuum identities

$$b\left|0\right\rangle = 0, \quad \left\langle 0\right| b^\dagger = 0, \tag{128}$$

are used. Equation (127) can be further simplified by means of the canonical commutation relation for the axion ladder operators (105) as

$$\langle \zeta_{\mathbf{k}}^{\text{homogeneous}} \zeta_{\mathbf{k'}}^{\text{homogeneous}} \rangle = \frac{H^2}{\dot{\phi}^2 a^2} Q_k(\tau) Q_{k'}^*(\tau) \Big\langle \left[b\left(\mathbf{k}\right), b^\dagger\left(-\mathbf{k}'\right) \right] \Big\rangle = \tag{129}$$

$$= \frac{H^2}{\dot{\phi}^2 a^2} Q_k(\tau) Q_k^*(\tau) \delta^3(\mathbf{k} - \mathbf{k}') = \tag{130}$$

$$= \frac{H^2}{\dot{\phi}^2 a^2} \left|Q_k(\tau)\right|^2 \delta^3(\mathbf{k} - \mathbf{k}'). \tag{131}$$

The modulus squared of the axion perturbation mode functions in (131) is obtained directly from (116):

$$\left|Q_k(\tau)\right|^2 = \frac{\pi}{4}(-\tau) \left| H_n^{(1)}(-k\tau) \right|^2. \tag{132}$$

The latter equation can be simplified further by considering a key feature of curvature perturbations. Indeed, they posses the remarkable property by which their time evolution is stopped on superhorizon scales. This can be interpreted as a subsequent effect from locality: as they get stretched to horizon scale, the fluctuations become causally disconnected from the region of space they originated, and as such they cannot be altered by local physics [52] [53]. Most importantly, the features observed in the cosmic microwave background (CMB) radiation are imprinted by curvature fluctuations created during the inflationary period and subsequently frozen [54]. Consequently, since the interest lies in curvature perturbations well beyond the horizon, the superhorizon limit $\frac{k}{aH} \approx -k\tau \ll 1$ of the Hankel function can be considered [55]

$$\lim_{\tau \to 0} H_n^{(1)} = \frac{i}{\pi} \Gamma(n) \left(\frac{-k\tau}{2} \right)^{-n}, \tag{133}$$

which then gives

$$|Q_k(\tau)|^2 = -\frac{1}{4\pi}\tau\,\Gamma^2(n)\left(\frac{-k\tau}{2}\right)^{-2n} = \tag{134}$$

$$\frac{2^{2n}}{4\pi}\Gamma^2(n)\frac{\tau^{1-2n}}{k^{2n}} = \frac{2^{2n}}{4\pi}\Gamma^2(n)\frac{\tau^{1-2n}}{k^3}k^{3-2n} = \tag{135}$$

$$\approx \frac{1}{2}\left(\frac{1}{aH}\right)^{-2}\frac{1}{k^3}\left(\frac{1}{aH}\right)^{3-2n} = \tag{136}$$

$$= \frac{1}{2}a^2H^2\frac{1}{k^3}\left(\frac{k}{aH}\right)^{3-2n} = \tag{137}$$

$$= \frac{1}{2}a^2H^2\frac{1}{k^3}\left(\frac{k}{aH}\right)^{n_s-1}, \tag{138}$$

where the numerical and gamma terms are evaluated at exactly $n = 3/2$ in the amplitude (as this is expected not to change dramatically the scale invariance of the spectrum) and the spectral index n_s defined as $n_s - 1 \equiv 3 - 2n$ is introduced. The latter is exploited to measure deviation of the power spectrum from scale invariance, i.e. when $n_s - 1 \neq 0$. In conclusion, the 2-point correlator function for the vacuum fluctuations has the standard expression

$$\langle\zeta_{\mathbf{k}}^{\text{homogeneous}}\zeta_{\mathbf{k'}}^{\text{homogeneous}}\rangle = \frac{H^2}{\dot{\phi}^2a^2}\frac{1}{2}a^2H^2\frac{1}{k^3}\left(\frac{k}{aH}\right)^{n_s-1} = \tag{139}$$

$$= \frac{H^4}{2\dot{\phi}^2}\frac{1}{k^3}\left(\frac{k}{aH}\right)^{n_s-1}\delta^3(\mathbf{k}-\mathbf{k}') = \tag{140}$$

$$= \frac{2\pi}{k^3}\,\mathcal{P}\left(\frac{k}{aH}\right)^{n_s-1}\delta^3(\mathbf{k}-\mathbf{k}'), \tag{141}$$

where the term $\mathcal{P}$ is defined as

$$\mathcal{P}^{1/2} \equiv \frac{H^2}{2\pi\left|\dot{\phi}\right|}. \tag{142}$$

3.6.2 Particular 2-point Correlator

The correlation function for the particular solution is independent from the vacuum contribution and is expressed in terms of the Green's function found in section 3.5.3 as

$$\langle\zeta_{\mathbf{k}}^{\text{particular}}\zeta_{\mathbf{k'}}^{\text{particular}}\rangle = \frac{H^2}{\dot{\phi}^2}\int \mathrm{d}\tau'\,\mathrm{d}\tau''\,\frac{1}{a(\tau)^2}G_k(\tau,\tau')G_{k'}(\tau,\tau'')\langle\mathcal{J}_{\mathbf{k}}(\tau')\,\mathcal{J}_{\mathbf{k}}(\tau'')\rangle. \tag{143}$$

By substituting for the Green's function (118) and after some algebra

$$\langle\zeta_{\mathbf{k}}^{\text{particular}}\zeta_{\mathbf{k'}}^{\text{particular}}\rangle = \frac{H^2}{\dot{\phi}^2}\frac{4}{a(\tau)^2}Q_k^2(\tau)\int \mathrm{d}\tau'\,\mathrm{d}\tau''\;\mathrm{Im}\left[Q_k(\tau')\right]\,\mathrm{Im}\left[Q_k(\tau'')\right]\,\langle\mathcal{J}_{\mathbf{k}}(\tau')\,\mathcal{J}_{\mathbf{k}}(\tau'')\rangle, \tag{144}$$

where it was invoked the property by which the 2-point correlator exhibits a non-zero value solely when the magnitudes of the momenta involved are identical ($|\mathbf{k}| = |\mathbf{k}'|$), similarly to the vacuum case. All algebraic simplifications are confirmed with the code provided in the appendix B. Taking into consideration our interest in the superhorizon power spectrum of the modes, the mode function outside the integral in (144) can be replaced with the mode (138) leading to

$$\langle\zeta_{\mathbf{k}}^{\text{particular}}\zeta_{\mathbf{k'}}^{\text{particular}}\rangle = \frac{2H^4}{\dot{\phi}^2}\frac{1}{k^3}\left(\frac{k}{aH}\right)^{n_s-1}\int \mathrm{d}\tau'\,\mathrm{d}\tau''\ \mathrm{Im}\left[Q_k(\tau')\right]\mathrm{Im}\left[Q_k(\tau'')\right]\langle\mathcal{J}_{\mathbf{k}}(\tau')\,\mathcal{J}_{\mathbf{k}}(\tau'')\rangle. \tag{145}$$

The source 2-point correlator $\langle\mathcal{J}_{\mathbf{k}}(\tau')\,\mathcal{J}_{\mathbf{k'}}(\tau'')\rangle$ can be evaluated explicitly, although the calculation is rather lengthy and involved. Therefore, only the final result is presented, and the reader may refer to Refs [28] and [44] for further details. As such, the correlator for the source term takes the rather convoluted form

$$\langle\mathcal{J}_{\mathbf{k}}(\tau')\,\mathcal{J}_{\mathbf{k}}(\tau'')\rangle = \frac{\alpha^2\delta^{(3)}(\mathbf{k}+\mathbf{k'})}{8f^2a(\tau')a(\tau'')}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\left[1+\frac{|\mathbf{q}|^2-\mathbf{q}\cdot\mathbf{k}}{|\mathbf{q}||\mathbf{k}-\mathbf{q}|}^2\right]\mathcal{A}\left[\tau',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right]\mathcal{A}^*\left[\tau'',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right], \tag{146}$$

where

$$\mathcal{A}\left[\tau',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right] \equiv |\mathbf{q}|A'_+(\tau',|\mathbf{q}-\mathbf{k}|)A_+(\tau',|\mathbf{q}|)+|\mathbf{q}-\mathbf{k}|A'_+(\tau',|\mathbf{q}|)A_+(\tau',|\mathbf{q}-\mathbf{k}|). \tag{147}$$

Equation (146) can now be inserted into the correlator (145):

$$\begin{aligned}\langle\zeta_{\mathbf{k}}^{\text{particular}}\zeta_{\mathbf{k'}}^{\text{particular}}\rangle &= \frac{H^6}{2\dot{\phi}^2}\frac{1}{k^3}\left(\frac{k}{aH}\right)^{n_s-1}\frac{\alpha^2}{f^2}\,\delta^3(\mathbf{k}+\mathbf{k'})\int\frac{\mathrm{d}^3q}{(2\pi)^3}\left[1+\frac{|\mathbf{q}|^2-\mathbf{q}\cdot\mathbf{k}}{|\mathbf{q}||\mathbf{k}-\mathbf{q}|}^2\right]\times \\ &\int \mathrm{d}\tau'\,\mathrm{d}\tau''\ (-\tau')(-\tau'')\mathrm{Im}\left[Q_k(\tau')\right]\mathrm{Im}\left[Q_k(\tau'')\right]\mathcal{A}\left[\tau',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right]\mathcal{A}^*\left[\tau'',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right],\end{aligned} \tag{148}$$

where, as previously done, the de Sitter approximation $\tau \approx -\frac{1}{aH}$ was used. The time integrals in (148) can be rewritten as

$$\begin{aligned}&\int \mathrm{d}\tau'\,\mathrm{d}\tau''\ (-\tau')(-\tau'')\mathrm{Im}\left[Q_k(\tau')\right]\mathrm{Im}\left[Q_k(\tau'')\right]\mathcal{A}\left[\tau',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right]\mathcal{A}^*\left[\tau'',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right] = \\ &\qquad\int \mathrm{d}\tau'\,(-\tau')^2\left|\mathrm{Im}\left[Q_k(\tau')\right]\mathcal{A}\left[\tau',|\mathbf{q}|,|\mathbf{q}-\mathbf{k}|\right]\right|^2.\end{aligned} \tag{149}$$

The $\mathbf{k}$ dependence of the integrals in (148) can be overcome by converting to the dimensionless integration variable $\bar{q} \equiv \frac{q}{|\mathbf{k}|}$ and substituting for the gauge field mode functions (63) as well as the axion mode functions (116). After some algebraic simplifications [28],

$$\begin{aligned}\langle\zeta_{\mathbf{k}}^{\text{particular}}\zeta_{\mathbf{k'}}^{\text{particular}}\rangle &= \frac{H^6e^{4\pi\xi}}{2^8\pi^2\dot{\phi}^2}\frac{1}{k^3}\left(\frac{k}{aH}\right)^{n_s-1}\frac{\alpha^2}{f^2}\,\delta^3(\mathbf{k}+\mathbf{k'})\times \\ &\int\mathrm{d}\bar{q}\left[1+\frac{\bar{\mathbf{q}}^2-\bar{\mathbf{q}}\cdot\hat{k}}{\bar{\mathbf{q}}\left|\hat{k}-\bar{\mathbf{q}}\right|}\right]|\bar{\mathbf{q}}|^{1/2}\left|\bar{\mathbf{q}}-\hat{k}\right|^{1/2}\left[|\bar{\mathbf{q}}|^{1/2}+\left|\bar{\mathbf{q}}-\hat{k}\right|^{1/2}\right]^2\times \\ &\left(\sqrt{\frac{\pi}{2}}\int_{-k\tau}^{\infty}\mathrm{d}x\,x^{3/2}\mathrm{Re}\left[H_{3/2}^{(1)}(x)\right]e^{-z\sqrt{x}}\right)^2.\end{aligned} \tag{150}$$

Following the same convention adopted in [28], the particular 2-point correlator is rewritten as

$$\langle \zeta_{\mathbf{k}}^{\text{particular}} \zeta_{\mathbf{k'}}^{\text{particular}} \rangle = \frac{2\pi}{k^3} \left(\frac{k}{aH} \right)^{n_s - 1} \mathcal{P}^2 \chi(\xi) e^{4\pi\xi} \delta^3(\mathbf{k} + \mathbf{k}'), \tag{151}$$

where as in section (3.2.1), $\xi \equiv \frac{\alpha\dot{\phi}}{2fH}$ and the ξ dependent function $\chi(\xi)$ is defined as

$$\chi(\xi) \equiv \frac{\xi^2}{8\pi} \int \mathrm{d}\bar{q} \left[1 + \frac{\bar{\mathbf{q}}^2 - \bar{\mathbf{q}} \cdot \hat{k}}{\bar{\mathbf{q}} \left| \hat{k} - \bar{\mathbf{q}} \right|} \right] |\bar{\mathbf{q}}|^{1/2} \left| \bar{\mathbf{q}} - \hat{k} \right|^{1/2} \left[|\bar{\mathbf{q}}|^{1/2} + \left| \bar{\mathbf{q}} - \hat{k} \right|^{1/2} \right]^2 \times$$
$$\left(\sqrt{\frac{\pi}{2}} \int_{-k\tau}^{\infty} \mathrm{d}x \, x^{3/2} \mathrm{Re} \left[H_{3/2}^{(1)}(x) \right] e^{-z\sqrt{x}} \right)^2. \tag{152}$$

In the spirit of the cosmological principle, the power spectrum is expected to be directionally independent [54]. As such, a reference scale direction $\hat{k} = (1, 0, 0)$ is taken in order to solve (152) numerically through the code provided in the appendix B. A plot of $\chi(\xi)$ is shown in figure 6. It can be inferred how the amplitude of the function decreases rapidly as the parameter ξ increases, although the exponential factor in the correlator strongly counterbalance this steep decline. As such, the amplitude of (151) could grow substantially for higher values of ξ. This gives another physical reason to phenomenologically exclude too large values of the parameter ξ, and thus it is reasonable to expect axionic-inflation to be happening around $\mathcal{O}(\xi) \approx 1$.

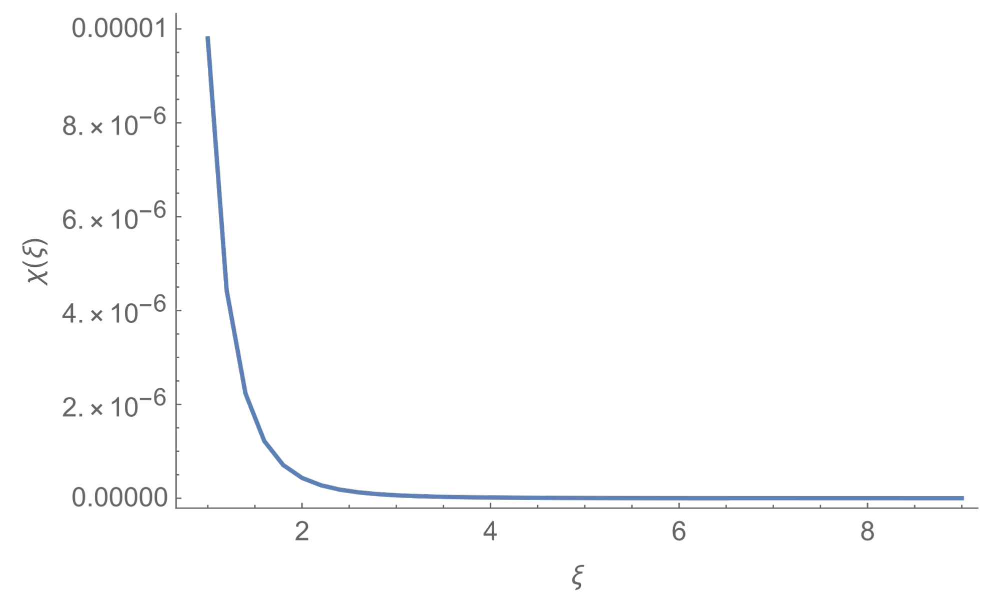

Figure 6: The dimensionless function χ from equation (152) as a function of ξ.

3.6.3 The Cumulative Power Spectrum

Expressions for both the vacuum and particular correlation functions, respectively (141) and (151), can now finally be related to the power spectrum by the well known equation [3]

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = P(k) \frac{2\pi}{k^3} \delta^{(3)}(\mathbf{k} + \mathbf{k}'). \tag{153}$$

It is now clearer why it was decided to cast the vacuum and particular correlators into the respective forms (141) and (151), since it can now be effortlessly inferred that the power spectrums P_{vacuum} and $P_{\text{particular}}$ are

$$P_{\text{vacuum}} = \mathcal{P}\left(\frac{k}{aH}\right)^{n_s-1}, \tag{154}$$

$$P_{\text{particular}} = \mathcal{P}^2\left(\frac{k}{aH}\right)^{n_s-1}\chi(\xi)e^{4\pi\xi}. \tag{155}$$

Physically, by inspection of (155), it can be observed how the particular power spectrum is strongly dependent on the parameter ξ, which effectively quantifies the backreaction of the gauge field on the axionic inflationary evolution. In particular, the gauge field mode functions A_+ in the representation given by (63) were employed, which were obtained in the limit $e^{4\pi\xi} \gg 1$. It is, thus, reasonable to anticipate that the exponential term in (155) will dominate the power spectrum as ξ increases, while being in a subdominant position with respect to the vacuum contribution as $\xi \to 0$ due to the relative small magnitude of the function $\chi(\xi)$ (see section 3.6.2.) Consequently, the duration of the inflationary period can be significantly affected by these dynamics.

In order to delve deeper into this aspect, an expression for the cumulative power spectrum is derived, which encompasses the combined effects of both the vacuum and particular contributions. Through this analysis, the interplay between these two components can be examined, studying how they interact and evolve as a function of the parameter ξ. Firstly, it was discussed previously that the vacuum and particular contributions are statistically independent, as the corresponding solutions to the equation of motions can be expanded in terms of commuting ladder operators. Consequently, (154) and (155) can be substituted into (153) with $P(k) = P_{\text{vacuum}} + P_{\text{particular}}$ to obtain an expression for the early power spectrum

$$P(k) = \left(\frac{k}{aH}\right)^{n_s-1}\left[\mathcal{P} + \mathcal{P}^2\chi(\xi)e^{4\pi\xi}\right]. \tag{156}$$

It is often convenient to parametrize the power spectrum with respect to a pivot scale k_0 usually set by experimental capabilities and observations [56]. A common adopted pivot is the Wilkinson Microwave Anisotropy Probe (WMAP) scale $k_0 = 0.002\,\text{Mpc}^{-1}$ [21] [28]. Following this convention finally yields

$$P(k) = \left(\frac{k}{k_0}\right)^{n_s-1}\left[\mathcal{P} + \mathcal{P}^2\chi(\xi)e^{4\pi\xi}\right], \qquad k_0 = 0.002\text{Mpc}^{-1}. \tag{157}$$

Interestingly, both the vacuum and particular contributions to the power spectrum have the same mild scale dependence as expected for typical slow roll inflation. Several insights can be derived from this observation: firstly, it is noted that the mathematical reason that led to the scale invariance of the particular power spectrum was the possibility to rewrite the integrals of $\chi(\xi)$ in the dimensionless variable $\bar{q} = \frac{q}{|\mathbf{k}|}$. Consequently, the only scale dependence came from the same vacuum solutions $|Q_k(\tau)|^2$. This procedure was employed due to the usage of the approximate gauge field mode solutions into the source correlator (146). These were derived within the regime $k\tau \gg -2\xi$, specifically near the conclusion of the inflationary period. At this stage, scalar fluctuations are anticipated to have undergone substantial stretching to cosmic scales in a reasonably uniform manner, leading to a more gradual evolution of the power spectrum during late times. This is similar to mapping the degree of smoothness of the surface of an ocean: on small enough scales, smaller than the characteristic wavelength of a wave, the spectrum appears smooth. As the scale is increased, uniformities arise and can be detected in the power spectrum. On the other hand, on large enough scales (bigger than a typical wave wavelength), the spectrum can appear fairly even again [57]. It is then reasonable to predict deviation from the scale invariance for the particular power spectrum as the exact solutions (60) to the gauge field equation of motion are considered in evaluating the 2-point correlator (150). Furthermore, it is remarked how the particular power spectrum is amplified by the exponential factor $e^{4\pi\xi}$, which given large enough values of ξ could eventually overtake the standard vacuum fluctuations, therefore spoiling inflation. This can be better observed from figure 7, where the standard and particular contributions to the late time power spectrum (157) are plotted and normalized according to the COBE normalization $P(k) \approx 25 \cdot 10^{-10}$ [58] [28]. It can be clearly inferred how the gauge field fluctuations completely surpass the vacuum contribution in the range $\xi \gtrapprox 4$, from where the latter must be dramatically reduced to ensure the spectrum normalization is maintained.

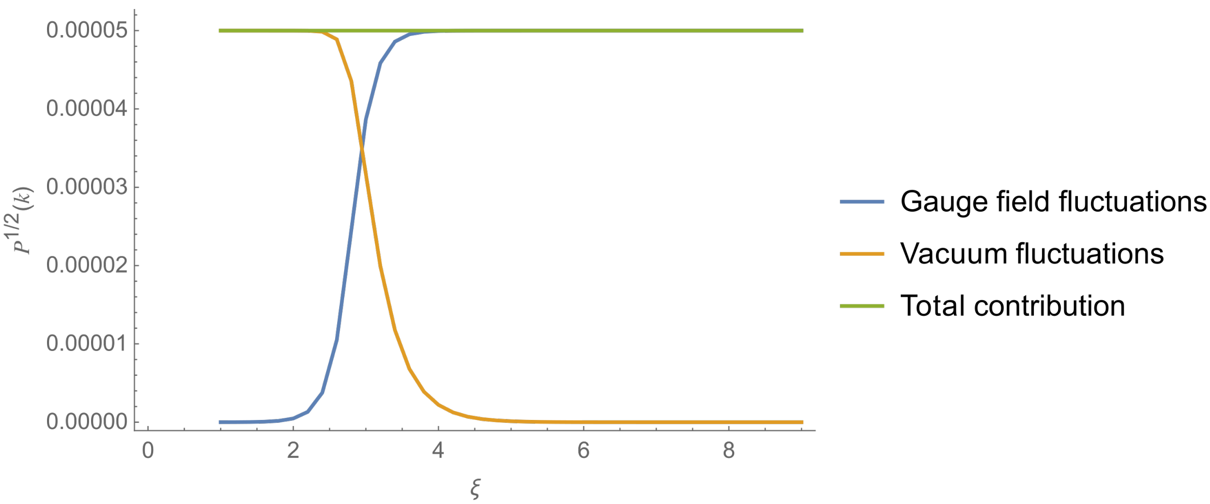

Figure 7: The COBE normalized power spectrum derived from expression (157). The behavior is closely dependent on the parameter ξ: as its value increases, the gauge field contribution dramatically overcomes the vacuum solution, and thus an exponential decrease of the latter has to be inserted in order to retain the spectrum at the observed normalized value of $\mathcal{P}_{COBE} \approx 5 \cdot 10^{-5}$.

4 Conclusions

Axion-driven inflation was extensively investigated within the framework of U(1) gauge field coupling. The interplay between the axion and the gauge field was discovered to possess notable effects on the dynamics of the axion field, influencing the standard homogeneous behavior. This was achieved through several backreaction effects including the sourcing of inflaton perturbations via inverse decay of gauge field fluctuations and contributions to the overall energy density of the universe.

This thesis focused on analyzing the curvature perturbations arising from this coupling and their impact on the power spectrum at the superhorizon limit. It was derived that the power spectrum exhibited the characteristic mild scale dependence expected in usual slow-roll inflation. Additionally, the synergy between the axion and gauge field was clearly observed from applying COBE normalization to the power spectrum, where it was discovered that in the region $\xi \gg 4$, involving the parameter ξ strictly connected to the coupling and growth of the gauge field fluctuations, the source contribution dominated the spectrum by completely surpassing the vacuum perturbations.

Several possible further paths of research are available on this subject: firstly, reproduction of the results obtained in this thesis could be performed by considering full scalar and metric perturbations. Furthermore, tensor perturbations may also be considered, with possible results that would lead to production of gravitational waves during axionic-driven inflation. Additionally, motivated by fairly recent CMB observations [59], non-Gaussianity effects in the spectrum could be worth investigating by computing higher order correlation functions, similarly to the work accomplished by Refs [28] and [60].

In conclusion, axions demonstrate promising characteristics as potential drivers of an inflationary period that shaped the observed universe. Their intrinsic shift symmetry is crucial to forbid potential UV corrections to Lagrangian and thus maintain a considerable degree of flatness of the inflationary potential for a prolonged period of time. The coupling of axions to U(1) gauge fields leads to a remarkable interplay between these components, resulting in substantial fluctuations that manifest in the late-time power spectrum.

5 References

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A Detailed Calculations and Derivations

A.1 Derivation Gauge Field Equation of Motion

The Coulomb Gauge $A_0 = 0$ is applied and thus the action (43) is varied by means of the following Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A_\rho} - \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\sigma A_\rho \right)} \right) = 0, \tag{158}$$

where $\mathcal{L}$ is the Lagrangian of the system. Additionally, some common known identities in gauge field theory are exploited, such as

$$F^{\mu\nu} F_{\mu\nu} = \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) = 2 \left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu \right) \tag{159}$$

$$F_{\alpha\beta} F_{\mu\nu} = \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha \right) \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) = \partial_\alpha A_\beta \partial_\mu A_\nu - \partial_\alpha A_\beta \partial_\nu A_\mu - \partial_\beta A_\alpha \partial_\mu A_\nu + \partial_\beta A_\alpha \partial_\nu A_\mu, \tag{160}$$

together with

$$\frac{\partial}{\partial \left(\partial_\mu A_\nu \right)} \left(\partial^\alpha A^\beta \partial_\beta A_\alpha \right) = 2 \partial^\nu A^\mu \tag{161}$$

$$\frac{\partial}{\partial \left(\partial_\mu A_\nu \right)} \left(\partial^\alpha A^\beta \partial_\alpha A_\beta \right) = 2 \partial^\mu A^\nu . \tag{162}$$

By inspection of (43), it is inferred that the only term depending on the gauge field A_μ or one of its derivatives are

$$\mathcal{L}_A = \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\alpha}{4f} \phi \tilde{F}^{\mu\nu} F_{\mu\nu} \right], \tag{163}$$

$$\begin{aligned} \mathcal{L}_A = \sqrt{-g} \Big[& -\frac{1}{2} \left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu \right) + \\ & - \frac{\alpha}{8f} \phi \frac{\eta^{\mu\nu\alpha\beta}}{\sqrt{-g}} \left(\partial_\alpha A_\beta \partial_\mu A_\nu - \partial_\alpha A_\beta \partial_\nu A_\mu - \partial_\beta A_\alpha \partial_\mu A_\nu + \partial_\beta A_\alpha \partial_\nu A_\mu \right) \Big]. \end{aligned} \tag{164}$$

Therefore, equation (158) can now be applied to (164):

$$\frac{\partial \mathcal{L}}{\partial A_\rho} = 0, \tag{165}$$

as there are no terms depending explicitly on A_ρ. Furthermore,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \left(\partial_\sigma A_\rho \right)} &= \sqrt{-g} \Big[- \partial^\sigma A^\rho + \partial^\rho A^\sigma - \\ & \frac{\alpha}{8f} \phi \frac{\eta^{\mu\nu\alpha\beta}}{\sqrt{-g}} \Big(\delta^\sigma_\alpha \delta^\rho_\beta \partial_\mu A_\nu + \partial_\alpha A_\beta \delta^\sigma_\mu \delta^\rho_\nu - \delta^\sigma_\alpha \delta^\rho_\beta \partial_\nu A_\mu - \partial_\alpha A_\beta \delta^\sigma_\nu \delta^\rho_\mu - \delta^\sigma_\beta \delta^\rho_\alpha \partial_\mu A_\nu - \partial_\beta A_\alpha \delta^\sigma_\mu \delta^\rho_\nu + \delta^\sigma_\beta \delta^\rho_\alpha \partial_\nu A_\mu + \partial_\beta A_\alpha \delta^\sigma_\nu \delta^\rho_\mu \Big) \Big], \\ \frac{\partial \mathcal{L}}{\partial \left(\partial_\sigma A_\rho \right)} &= \sqrt{-g} \Big[- \partial^\sigma A^\rho + \partial^\rho A^\sigma - \\ & \frac{\alpha}{8f} \frac{\phi}{\sqrt{-g}} \Big(\eta^{\mu\nu\sigma\rho} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) + \eta^{\sigma\rho\alpha\beta} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha \right) + \eta^{\rho\sigma\alpha\beta} \left(\partial_\beta A_\alpha - \partial_\alpha A_\beta \right) + \eta^{\mu\nu\rho\sigma} \left(\partial_\nu A_\mu - \partial_\mu A_\nu \right) \Big) \Big]. \end{aligned} \tag{166}$$

Equation (166) can be further simplified by recalling that the Levi-Civita tensor $\eta^{\mu\nu\alpha\beta}$ is totally anti-symmetric, hence a minus sign is obtained upon index swapping procedures. As such, an appropriate swap of indexes of the last four terms in (166) shows that these terms are indeed the same. Thus, one is left with

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\sigma A_\rho\right)} = \sqrt{-g}\left[\partial^\rho A^\sigma - \partial^\sigma A^\rho - \frac{\alpha}{8f}\frac{\phi}{\sqrt{-g}}4\,\eta^{\sigma\rho\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right], \tag{167}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\sigma A_\rho\right)} = \sqrt{-g}F^{\sigma\rho} - \frac{\alpha}{2f}\phi\,\eta^{\sigma\rho\mu\nu}F_{\mu\nu} \tag{168}$$

where from (167) to (168) the definition of field tensor $F_{\mu\nu}$ was substituted for. Therefore, the equation of motion for the gauge field are given by the solutions to [31]

$$\partial_\sigma\left(\sqrt{-g}F^{\sigma\rho}\right) + \frac{\alpha}{2f}\partial_\sigma\left(\phi\,\eta^{\sigma\rho\mu\nu}F_{\mu\nu}\right) = 0. \tag{169}$$

A.2 Converting the Guage Field Equation of Motion to Fourier Space

Firstly,

$$\vec{A}'' = \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left[\vec{\epsilon}_\lambda(\mathbf{k})A_\lambda''(\tau,\mathbf{k})a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \vec{\epsilon^*}_\lambda(\mathbf{k})A_\lambda^{*\prime\prime}(\tau,\mathbf{k})a_\lambda^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{170}$$

Similarly, the Laplacian of (49) is straightforwardly given by

$$\nabla^2\vec{A} = -\sum_{\lambda=\pm}\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}k^2\left[\vec{\epsilon}_\lambda(\mathbf{k})A_\lambda(\tau,\mathbf{k})a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \vec{\epsilon^*}_\lambda(\mathbf{k})A_\lambda^*(\tau,\mathbf{k})a_\lambda^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{171}$$

whereas the curl of $\vec{A}$ requires further computations:

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ A_x & A_y & A_z\end{vmatrix} = \tag{172}$$

$$= \hat{x}\left(\partial_y A_z - \partial_z A_y\right) - \hat{y}\left(\partial_x A_z - \partial_z A_x\right) + \hat{z}\left(\partial_x A_y - \partial_y A_x\right), \tag{173}$$

such that

$$\vec{\nabla}\times\vec{A} = \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left[\vec{\nabla}\times\vec{\epsilon}_\lambda(\mathbf{k})A_\lambda(\tau,\mathbf{k})a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \vec{\nabla}\times\vec{\epsilon^*}_\lambda(\mathbf{k})A_\lambda^*(\tau,\mathbf{k})a_\lambda^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{174}$$

$$\vec{\nabla}\times\vec{A} = \sum_{\lambda=\pm}\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\Bigg[i\begin{pmatrix}k_y\epsilon_z(\vec{k}) - k_z\epsilon_y(\vec{k})\\ k_z\epsilon_x(\vec{k}) - k_x\epsilon_z(\vec{k})\\ k_x\epsilon_y(\vec{k}) - k_y\epsilon_x(\vec{k})\end{pmatrix}A_\lambda(\tau,\mathbf{k})a_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \tag{175}$$

$$-\,i\begin{pmatrix}k_y\epsilon_z(\vec{-k}) - k_z\epsilon_y(\vec{-k})\\ k_z\epsilon_x(\vec{-k}) - k_x\epsilon_z(\vec{-k})\\ k_x\epsilon_y(\vec{-k}) - k_y\epsilon_x(\vec{-k})\end{pmatrix}A_\lambda^*(\tau,\mathbf{k})a_\lambda^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\Bigg], \tag{176}$$

where the λ subscript on the polarization vectors is for ease of notation. Condition (52) is then applied to obtain

$$\vec{\nabla} \times \vec{A} = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \Big[i(-ik)\vec{\epsilon}_+(\mathbf{k}) A_+(\tau, \mathbf{k}) a_+(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - i(-ik)\vec{\epsilon}_+(-\mathbf{k}) A_+^*(\tau, \mathbf{k}) a_+^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + \\ i(-ik)\vec{\epsilon}_-(\mathbf{k}) A_-(\tau, \mathbf{k}) a_-(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - i(-ik)\vec{\epsilon}_-(-\mathbf{k}) A_-^*(\tau, \mathbf{k}) a_-^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \Big], \tag{177}$$

$$\vec{\nabla} \times \vec{A} = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \Big[k\vec{\epsilon}_+(\mathbf{k}) A_+(\tau, \mathbf{k}) a_+(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - k\vec{\epsilon}_+(-\mathbf{k}) A_+^*(\tau, \mathbf{k}) a_+^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + \\ -k\vec{\epsilon}_-(\mathbf{k}) A_-(\tau, \mathbf{k}) a_-(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + k\vec{\epsilon}_-(-\mathbf{k}) A_-^*(\tau, \mathbf{k}) a_-^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \Big]. \tag{178}$$

Expressions in Fourier space for all the terms in (48) are now obtained. Finally, substituting (170), (171) and (178) into (48) yields

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \Bigg[\left(A_+{''}(\tau, \mathbf{k}) + k^2 A_+(\tau, \mathbf{k}) - \frac{\alpha}{f}\phi' k A_+{''}(\tau, \mathbf{k}) \right) \vec{\epsilon_+}(\mathbf{k}) a_+(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \\ \left(A_+^*{''}(\tau, \mathbf{k}) + k^2 A_+^*(\tau, \mathbf{k}) + \frac{\alpha}{f}\phi' k A_+^*(\tau, \mathbf{k}) \right) \vec{\epsilon_+^*}(\mathbf{k}) a_+^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + \\ \left(A_-{''}(\tau, \mathbf{k}) + k^2 A_-(\tau, \mathbf{k}) + \frac{\alpha}{f}\phi' k A_-(\tau, \mathbf{k}) \right) \vec{\epsilon_-}(\mathbf{k}) a_-(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + \\ \left(A_-^*{''}(\tau, \mathbf{k}) + k^2 A_-^*(\tau, \mathbf{k}) - \frac{\alpha}{f}\phi' k A_-^*(\tau, \mathbf{k}) \right) \vec{\epsilon_-^*}(\mathbf{k}) a_-^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \Bigg] = 0. \tag{179}$$

Equation (179) implies that each term within round brackets in is set to zero. In particular,

$$A_+{''}(\tau, \mathbf{k}) + k^2 A_+(\tau, \mathbf{k}) - \frac{\alpha}{f}\phi' k A_+{''}(\tau, \mathbf{k}) = 0, \tag{180}$$

$$A_-{''}(\tau, \mathbf{k}) + k^2 A_-(\tau, \mathbf{k}) + \frac{\alpha}{f}\phi' k A_-(\tau, \mathbf{k}) = 0. \tag{181}$$

Equations (180) and (181) can be further simplified by converting ϕ' to cosmic time

$$\phi' = \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \dot{\phi}\, a. \tag{182}$$

During inflation, the de Sitter approximation of the scale factor $a(\text{inflation}) \approx -\frac{1}{H\tau}$ can be used [31][1] such that (180) and (181) take the form

$$A_+{''}(\tau, \mathbf{k}) + k^2 A_+(\tau, \mathbf{k}) - \frac{\alpha}{f}\dot{\phi}\left(-\frac{1}{H\tau}\right) k A_+(\tau, \mathbf{k}) = 0, \tag{183}$$

$$A_-{''}(\tau, \mathbf{k}) + k^2 A_-(\tau, \mathbf{k}) + \frac{\alpha}{f}\dot{\phi}\left(-\frac{1}{H\tau}\right) k A_-(\tau, \mathbf{k}) = 0. \tag{184}$$

By denoting $\xi \equiv \frac{\alpha\dot{\phi}}{2fH}$, ultimately it is arrived at

$$A_+''(\tau, \mathbf{k}) + k^2 A_+(\tau, \mathbf{k}) + \frac{2k\xi}{\tau} A_+(\tau, \mathbf{k}) = 0 \tag{185}$$

$$A_-''(\tau, \mathbf{k}) + k^2 A_-(\tau, \mathbf{k}) - \frac{2k\xi}{\tau} A_-(\tau, \mathbf{k}) = 0 \tag{186}$$

which can be rewritten for ease of notation into one equation as

$$\left(\partial_\tau^2 + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(\tau, k) = 0, \tag{187}$$

where the vector notation on the k argument is dropped as the differential equation involves only the magnitude of the momentum vectors $\vec{k}$.

A.3 Large Argument Asymptotic of A_+ Modes

The large argument asymptotic behavior of the expression

$$\sqrt{\frac{-2\tau}{\pi}} e^{\pi\xi} K_1\left(2\sqrt{-2\xi k\tau}\right) \tag{188}$$

involving the modified Bessel function of the second kind can be obtained by employing the large argument asymptotic behavior of $K_1(z)$, using the following asymptotic expansion [61]:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8z)^2} + \cdots\right), \quad \text{as } z \to \infty. \tag{189}$$

Applying this asymptotic expansion to (188) yields

$$\begin{aligned}\sqrt{\frac{-2\tau}{\pi}} e^{\pi\xi} K_1\left(2\sqrt{-2\xi k\tau}\right) \sim \sqrt{\frac{-2\tau}{\pi}} e^{\pi\xi} \sqrt{\frac{\pi}{4\sqrt{-2\xi k\tau}}} e^{-2\sqrt{-2\xi k\tau}} \\ \times \left(1 + \frac{3}{8\sqrt{-2\xi k\tau}} + \frac{3 \cdot 7}{2!(8\sqrt{-2\xi k\tau})^2} + \cdots\right),\end{aligned} \tag{190}$$

as $k\tau \to -\infty$. By keeping terms up to first order and after some algebra, it is arrived at

$$\sqrt{\frac{-2\tau}{\pi}} e^{\pi\xi} K_1\left(2\sqrt{-2\xi k\tau}\right) \approx \frac{1}{\sqrt{2k}} \left(\frac{-k\tau}{2\xi}\right)^{1/4} e^{\pi\xi - 2\sqrt{-2\xi k\tau}}. \tag{191}$$

A.4 Derivation of the Axion Equation of Motion

The aim is to simplify equation (69). As such, $\sqrt{-g}$ is evaluated with the help of (8):

$$\sqrt{-g} = \sqrt{-\det(\mathbf{g})}, \tag{192}$$

$$\sqrt{-g} = \sqrt{-\left(-a^8(\tau)\right)} = a^4(\tau). \tag{193}$$

The second term of (69) can then be expanded as follows:

$$\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) = \partial_0\left(\sqrt{-g}\partial^0\phi\right) + \partial_i\left(\sqrt{-g}\partial^i\phi\right), \tag{194}$$

$$= \partial_0\left(a^4 g^{0\nu}\partial_\nu\phi\right) + a^4 g^{ij}\partial_i\partial_j\phi, \tag{195}$$

$$= \partial_0\left(a^4 g^{00}\partial_0\phi\right) + a^4 g^{ij}\partial_i\partial_j\phi, \tag{196}$$

where the symmetric property of the FRW metric was used. From (9), it is inferred that $g^{00} = -\frac{1}{a^2}$ and $|g^{\mu\nu}| = \frac{1}{a^2}$ such that

$$\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) = \partial_0\left(a^4\left(-\frac{1}{a^2}\right)\partial_0\phi\right) + a^4\left(\frac{1}{a^2}\right)\partial_i\partial_i\phi = \tag{197}$$

$$= \partial_0\left(-a^2\partial_0\phi\right) + a^2\nabla^2\phi \tag{198}$$

$$= -2a\,a'\phi' + a^2\left(-\phi'' + \nabla^2\phi\right). \tag{199}$$

Substituting (199) into (69) finally leads to

$$-a^4\left(\frac{\alpha}{4f}\phi\,\tilde{F}^{\mu\nu}F_{\mu\nu} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\right) - \left(-2a\,a'\phi' + a^2\left(-\phi'' + \nabla^2\phi\right)\right) = 0, \tag{200}$$

$$\phi'' - \nabla^2\phi - 2\frac{a'}{a}\phi' + a^2\frac{\mathrm{d}V}{\mathrm{d}\phi} = a^2\frac{\alpha}{4f}\phi\,\tilde{F}^{\mu\nu}F_{\mu\nu}. \tag{201}$$

A.5 Electric and Magnetic Fields Expectation Values

The aim is to derive an expression for

$$\langle\vec{E}^2 + \vec{B}^2\rangle = \langle\vec{E}^2\rangle + \langle\vec{B}^2\rangle, \tag{202}$$

since the integration is linear in $\vec{E}^2$ and $\vec{B}^2$. As such, it is found

$$\langle\vec{E}^2\rangle = \frac{1}{a^4}\int\frac{\mathrm{d}^3x\,\mathrm{d}^3k\,\mathrm{d}^3q}{(2\pi)^{9/2}}\vec{\epsilon}(\vec{k})\cdot\vec{\epsilon}(\vec{q})A'(\tau,k)A'(\tau,q)e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = \tag{203}$$

$$= \frac{1}{a^4}\int\frac{\mathrm{d}^3k\,\mathrm{d}^3q}{(2\pi)^3}\delta^{(3)}(\vec{k}+\vec{q})\vec{\epsilon}(\vec{k})\cdot\vec{\epsilon}(\vec{q})A'(\tau,k)A'(\tau,q) = \tag{204}$$

$$= \frac{1}{a^4}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\vec{\epsilon}(\vec{k})\cdot\vec{\epsilon}(-\vec{k})A'(\tau,k)A'(\tau,k) = \tag{205}$$

$$= \frac{1}{a^4}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left|A'(\tau,k)\right|^2, \tag{206}$$

where properties (53) and (54) are used. Similarly for the magnetic field

$$\langle\vec{B}^2\rangle = \frac{1}{a^4}\int\frac{\mathrm{d}^3x\,\mathrm{d}^3k\,\mathrm{d}^3q}{(2\pi)^{9/2}}q\,k\vec{\epsilon}(\vec{k})\cdot\vec{\epsilon}(\vec{q})A(\tau,k)A(\tau,q)e^{i(\vec{k}+\vec{q})\cdot\vec{x}} = \tag{207}$$

$$= \frac{1}{a^4}\int\frac{\mathrm{d}^3k\,\mathrm{d}^3q}{(2\pi)^3}\delta^{(3)}(\vec{k}+\vec{q})q\,k\vec{\epsilon}(\vec{k})\cdot\vec{\epsilon}(\vec{q})A(\tau,k)A(\tau,q) = \tag{208}$$

$$= \frac{1}{a^4}\int\frac{\mathrm{d}^3q}{(2\pi)^3}q^2|A(\tau,q)|^2. \tag{209}$$

The expectation value of the energy density is then obtained by summing expressions (206) and (209):

$$\langle \vec{E}^2 + \vec{B}^2 \rangle = \frac{1}{a^4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\left| A'(\tau, k) \right|^2 + k^2 |A(\tau, k)|^2 \right]. \tag{210}$$

A.6 Derivation of Mukhanov-Sasaki Equation

For notational convenience, the double derivative of the potential in (95) is renamed as a mass term $m^2 \equiv V''$. Secondly, the definition of the slope and curvature slow-roll parameters ϵ and η is recalled as[12] [62]

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \tag{211}$$

$$\eta \equiv \frac{1}{3} \frac{V''}{H^2}, \quad \Rightarrow \quad V'' = 3\eta H^2, \tag{212}$$

as well as the well known relation

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1 - \epsilon) H^2, \tag{213}$$

which can be converted to conformal time as follows:

$$\ddot{a} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}a}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\mathrm{d}t} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(a' \cdot \frac{1}{a} \right) = \tag{214}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{a'}{a} \right) \frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{a''}{a^2} - \frac{(a')^2}{a^3}, \tag{215}$$

such that

$$\frac{\ddot{a}}{a} = (1 - \epsilon) H^2 = \frac{1}{a} \left(\frac{a''}{a^2} - \frac{(a')^2}{a^3} \right) = \frac{a''}{a^3} - \frac{(a')^2}{a^4}. \tag{216}$$

Expression (216) can be inverted to obtain

$$\frac{a''}{a} = a^2 (1 - \epsilon) H^2 + \left(\frac{a'}{a} \right) = \tag{217}$$

$$= (aH)^2 (1 - \epsilon) + \mathcal{H}^2, \tag{218}$$

but,

$$\mathcal{H} = \frac{a'}{a} = \frac{1}{a} \dot{a} \frac{\mathrm{d}t}{\mathrm{d}\tau} = \dot{a}, \tag{219}$$

hence,

$$\frac{a''}{a} = (aH)^2 (2 - \epsilon). \tag{220}$$

[12] We remark that units are chosen in order to set the Planck mass to $M_p = 1$.

This whole bookkeeping was meant to simplify the $-\frac{a''}{a}+a^2m^2$ term in (101) to

$$a^2m^2-\frac{a''}{a}=3\left(aH\right)^2\eta-\left(aH\right)^2\left(2-\epsilon\right)= \tag{221}$$

$$=\left(aH\right)^2\left[3\eta-2+\epsilon\right]. \tag{222}$$

The first order approximation $aH\approx-\frac{1}{\tau}\left(1+\epsilon\right)$ can now be utilized (see equation (2.4.89) of [17]), which is valid during the inflationary epoch $\{\epsilon,\mid\eta\mid\}\ll 1$, into the last expression (222) to yield

$$\frac{1}{\tau^2}\left(1+\epsilon\right)^2\left[3\eta-2+\epsilon\right]\approx\frac{1}{\tau^2}\left(1+2\epsilon\right)\left(3\eta-2+\epsilon\right)= \tag{223}$$

$$=-\frac{1}{\tau^2}\left(2+3\left(\epsilon+\eta\right)\right)= \tag{224}$$

$$=-\frac{1}{\tau^2}\left(n^2-\frac{1}{4}\right), \tag{225}$$

where $n\equiv\frac{3}{2}+\epsilon+\eta$. Finally, by substituting (225) into (101), the homogeneous part to the simil Mukhanov-Sasaki equation takes the form

$$\left[\partial_\tau^2+\left(k^2-\frac{n^2-\frac{1}{4}}{\tau^2}\right)\right]Q_k(\tau)=0. \tag{226}$$

B Mathematica Supplementary Code

In this appendix the Mathematica code used as supplement to the calculations and computations presented in this thesis is outlined.

Probing Axion Inflation Coupled to U(1) Gauge Fields

Gianmarco Morbelli s4513932

Code supplement to the respective bachelor thesis

NB: Ricci package is required to run this notebook. Use the following line to update to the relevant directory where the packages is located.
I am considerably grateful to Martino Michelotti for the help provided in shaping this code.

```
<< "Your directory" (* Set the path to Ricci.m *)

LastIndex = 3;   (* 3+1 dimensions, indices from 0 to 3 *)

Coordinate[0] = τ;
Coordinate[1] = x;
Coordinate[2] = y;
Coordinate[3] = z;

 (* Insert the non-zero metric components *)

MetricTensor[0, 0] = -n[τ]^2;
MetricTensor[1, 1] = a[τ]^2;
MetricTensor[2, 2] = a[τ]^2;
MetricTensor[3, 3] = a[τ]^2;

Do[Ad[mu], {mu, 0, 3}]  (* U(1) gauge fields *)

Ad[0] = A0[τ, x, y, z];
Ad[1] = A1[τ, x, y, z];
Ad[2] = A2[τ, x, y, z];
Ad[3] = A3[τ, x, y, z];

Do[Fdd[mu, nu] = D[Ad[nu], Coordinate[mu]] - D[Ad[mu], Coordinate[nu]],
 {mu, 0, 3}, {nu, 0, 3}]

Do[Fuu[mu, nu] = Sum[InverseMetric[mu, al] × InverseMetric[nu, be] × Fdd[al, be],
   {al, 0, 3}, {be, 0, 3}], {mu, 0, 3}, {nu, 0, 3}]
```

Check Bianchi identity for the gauge field

In[18]:=
```
Sum[D[Signature[{σ, ρ, μ, ν}] × Fdd[μ, ν], Coordinate[σ]],
  {σ, 0, 3}, {μ, 0, 3}, {ν, 0, 3}]
```

Out[18]=
```
0
```

It works!

Equation of motion for the gauge field A_μ

In[19]:=
```
infla = χ[τ];

(* Now we write the action *)


sg = Simplify[Sqrt[-DetMetric], {n[τ] > 0, a[τ] > 0}];


az = Expand[sg ((Mp^2 / 2) ScalarCurvature -
        (1 / 4) Sum[Fuu[mu, nu] × Fdd[mu, nu], {mu, 0, 3}, {nu, 0, 3}] -
        (1 / 2) (Sum[InverseMetric[mu, nu] × D[infla, Coordinate[mu]] ×
            D[infla, Coordinate[nu]], {mu, 0, 3}, {nu, 0, 3}]) - V[infla]) +
     (λ / (8 f)) infla * Sum[Signature[{mu, nu, al, be}] × Fdd[mu, nu] × Fdd[al, be],
       {mu, 0, 3}, {nu, 0, 3}, {al, 0, 3}, {be, 0, 3}]];
```

In[22]:=
```
eqA = Block[{ν = 1}, Expand[D[az, Ad[ν]] -
       Sum[D[D[az, D[Ad[ν], Coordinate[μ]]], Coordinate[μ]], {μ, 0, 3}]] /.
     {A0 → 0, n'[τ] → a'[τ], n[τ] → a[τ]}] // FullSimplify
```

Out[22]=

$$A1^{(0,0,0,2)}[\tau, x, y, z] + \frac{\lambda\, \chi'[\tau]\, \left(A2^{(0,0,0,1)}[\tau, x, y, z] - A3^{(0,0,1,0)}[\tau, x, y, z]\right)}{f} + A1^{(0,0,2,0)}[\tau, x, y, z] - A3^{(0,1,0,1)}[\tau, x, y, z] - A2^{(0,1,1,0)}[\tau, x, y, z] - A1^{(2,0,0,0)}[\tau, x, y, z]$$

In Coulomb gauge

In[23]:=
```
eqA1 = A1^(0,0,0,2)[τ, x, y, z] + (λ χ'[τ] (A2^(0,0,0,1)[τ, x, y, z] - A3^(0,0,1,0)[τ, x, y, z]))/f +
   A1^(0,0,2,0)[τ, x, y, z] + A1^(0,2,0,0)[τ, x, y, z] - A1^(2,0,0,0)[τ, x, y, z];
eqA2 = A2^(0,0,0,2)[τ, x, y, z] + A2^(0,0,2,0)[τ, x, y, z] +
   (λ χ'[τ] (-A1^(0,0,0,1)[τ, x, y, z] + A3^(0,1,0,0)[τ, x, y, z]))/f +
   A2^(0,2,0,0)[τ, x, y, z] - A2^(2,0,0,0)[τ, x, y, z];
eqA3 = A3^(0,0,0,2)[τ, x, y, z] + A3^(0,0,2,0)[τ, x, y, z] +
   (λ χ'[τ] (A1^(0,0,1,0)[τ, x, y, z] - A2^(0,1,0,0)[τ, x, y, z]))/f +
   A3^(0,2,0,0)[τ, x, y, z] - A3^(2,0,0,0)[τ, x, y, z];
```

Inflaton equation of motion and the 00 Einstein equation

Now we include also inflaton perturbations

In[26]:=
```
Clear[infla]
infla = χ[τ, x, y, z];

az2 = Expand[sg ((Mp^2 / 2) ScalarCurvature -
       (1 / 4) Sum[Fuu[mu, nu] × Fdd[mu, nu], {mu, 0, 3}, {nu, 0, 3}] -
       (1 / 2) (Sum[InverseMetric[mu, nu] × D[infla, Coordinate[mu]] ×
           D[infla, Coordinate[nu]], {mu, 0, 3}, {nu, 0, 3}]) - V[infla]) +
     (λ / (8 f)) infla * Sum[Signature[{mu, nu, al, be}] × Fdd[mu, nu] × Fdd[al, be],
       {mu, 0, 3}, {nu, 0, 3}, {al, 0, 3}, {be, 0, 3}]];
eqinf = Simplify[
  Expand[(1 / a[τ] ^2) ((Sum[D[D[az2, D[infla, Coordinate[μ]]], Coordinate[μ]],
          {μ, 0, 3}] - D[az2, infla]))] /.
   {n'[τ] → a'[τ], n[τ] → a[τ], A0 → 0}, Assumptions → a[τ] ≠ 0]
```

Out[29]=
$$a[\tau]^2\, V'[\chi[\tau,x,y,z]] - \chi^{(0,0,0,2)}[\tau,x,y,z] - \chi^{(0,0,2,0)}[\tau,x,y,z] - \chi^{(0,2,0,0)}[\tau,x,y,z] + \frac{1}{f\, a[\tau]^2} \lambda \big(A2^{(0,0,0,1)}[\tau,x,y,z]\, A1^{(1,0,0,0)}[\tau,x,y,z] - A3^{(0,0,1,0)}[\tau,x,y,z]\, A1^{(1,0,0,0)}[\tau,x,y,z] - A1^{(0,0,0,1)}[\tau,x,y,z]\, A2^{(1,0,0,0)}[\tau,x,y,z] + A3^{(0,1,0,0)}[\tau,x,y,z]\, A2^{(1,0,0,0)}[\tau,x,y,z] + A1^{(0,0,1,0)}[\tau,x,y,z]\, A3^{(1,0,0,0)}[\tau,x,y,z] - A2^{(0,1,0,0)}[\tau,x,y,z]\, A3^{(1,0,0,0)}[\tau,x,y,z]\big) + \frac{2\, a'[\tau]\, \chi^{(1,0,0,0)}[\tau,x,y,z]}{a[\tau]} + \chi^{(2,0,0,0)}[\tau,x,y,z]$$

The 00 Einstein equation is obtained by varying the action with respect to the metric

In[30]:=
```
eq0 = FullSimplify[(1 / (2 a[τ]))
    Expand[D[az2, n[τ]] - D[D[az2, n'[τ], τ]] /. {n'[τ] → a'[τ], n[τ] → a[τ]}],
   Assumptions → a[τ] ≠ 0]
```

Out[30]=

$$-\frac{1}{4\,a[\tau]^2}\Big(2\,a[\tau]^4\,V[\chi[\tau,x,y,z]]-6\,Mp^2\,a'[\tau]^2+\left(A2^{(0,0,0,1)}[\tau,x,y,z]-A3^{(0,0,1,0)}[\tau,x,y,z]\right)^2+\left(A1^{(0,0,1,0)}[\tau,x,y,z]-A2^{(0,1,0,0)}[\tau,x,y,z]\right)^2+\left(A1^{(0,0,0,1)}[\tau,x,y,z]-A3^{(0,1,0,0)}[\tau,x,y,z]\right)^2+\left(A0^{(0,1,0,0)}[\tau,x,y,z]-A1^{(1,0,0,0)}[\tau,x,y,z]\right)^2+\left(A0^{(0,0,1,0)}[\tau,x,y,z]-A2^{(1,0,0,0)}[\tau,x,y,z]\right)^2+\left(A0^{(0,0,0,1)}[\tau,x,y,z]-A3^{(1,0,0,0)}[\tau,x,y,z]\right)^2+a[\tau]^2\left(\chi^{(0,0,0,1)}[\tau,x,y,z]^2+\chi^{(0,0,1,0)}[\tau,x,y,z]^2+\chi^{(0,1,0,0)}[\tau,x,y,z]^2+\chi^{(1,0,0,0)}[\tau,x,y,z]^2\right)\Big)$$

In[31]:=
```
eq0 /. {A0 → 0, χ → (χ[#] &)}
```

Out[31]=

$$-\frac{1}{4\,a[\tau]^2}\Big(2\,a[\tau]^4\,V[\chi[\tau]]-6\,Mp^2\,a'[\tau]^2+a[\tau]^2\,\chi'[\tau]^2+\left(A2^{(0,0,0,1)}[\tau,x,y,z]-A3^{(0,0,1,0)}[\tau,x,y,z]\right)^2+\left(A1^{(0,0,1,0)}[\tau,x,y,z]-A2^{(0,1,0,0)}[\tau,x,y,z]\right)^2+\left(A1^{(0,0,0,1)}[\tau,x,y,z]-A3^{(0,1,0,0)}[\tau,x,y,z]\right)^2+A1^{(1,0,0,0)}[\tau,x,y,z]^2+A2^{(1,0,0,0)}[\tau,x,y,z]^2+A3^{(1,0,0,0)}[\tau,x,y,z]^2\Big)$$

We convert to physical electric and magnetic field

In[ ]:=
```
electricf =
  - (1 / a[τ] ^2) {A1^(1,0,0,0)[τ, x, y, z], A2^(1,0,0,0)[τ, x, y, z], A3^(1,0,0,0)[τ, x, y, z]};
magneticf =
  1 / a[τ] ^2 Curl[{A1[τ, x, y, z], A2[τ, x, y, z], A3[τ, x, y, z]}, {x, y, z}];
```

In[ ]:=
```
electricf.magneticf /. a[τ] → 1
```

Out[ ]=

$$-\Big(\left(-A2^{(0,0,0,1)}[\tau,x,y,z]+A3^{(0,0,1,0)}[\tau,x,y,z]\right)A1^{(1,0,0,0)}[\tau,x,y,z]\Big)-\left(A1^{(0,0,0,1)}[\tau,x,y,z]-A3^{(0,1,0,0)}[\tau,x,y,z]\right)A2^{(1,0,0,0)}[\tau,x,y,z]-\left(-A1^{(0,0,1,0)}[\tau,x,y,z]+A2^{(0,1,0,0)}[\tau,x,y,z]\right)A3^{(1,0,0,0)}[\tau,x,y,z]$$

We check whether the two expression in the equation of motion for phi and the converted one with electric and magnetic field coincide (for simplicity we set a to 1).

In[ ]:=
```
Simplify[Expand[(a[τ] ^4) electricf.magneticf] ==
  (A2^(0,0,0,1)[τ, x, y, z] A1^(1,0,0,0)[τ, x, y, z] -
    A3^(0,0,1,0)[τ, x, y, z] A1^(1,0,0,0)[τ, x, y, z] -
    A1^(0,0,0,1)[τ, x, y, z] A2^(1,0,0,0)[τ, x, y, z] + A3^(0,1,0,0)[τ, x, y, z]
     A2^(1,0,0,0)[τ, x, y, z] + A1^(0,0,1,0)[τ, x, y, z] A3^(1,0,0,0)[τ, x, y, z] -
    A2^(0,1,0,0)[τ, x, y, z] A3^(1,0,0,0)[τ, x, y, z]), Assumptions → a[τ] == 1]
```

Out[ ]=

True

```
In[•]:= right = (A2^(0,0,0,1)[τ, x, y, z] - A3^(0,0,1,0)[τ, x, y, z])^2 +
          (A1^(0,0,1,0)[τ, x, y, z] - A2^(0,1,0,0)[τ, x, y, z])^2 +
          (A1^(0,0,0,1)[τ, x, y, z] - A3^(0,1,0,0)[τ, x, y, z])^2 +
          A1^(1,0,0,0)[τ, x, y, z]^2 + A2^(1,0,0,0)[τ, x, y, z]^2 + A3^(1,0,0,0)[τ, x, y, z]^2;
        left = electricf.electricf + magneticf.magneticf /. a[τ] → 1
Out[•]=
```

$$\left(-A2^{(0,0,0,1)}[\tau, x, y, z] + A3^{(0,0,1,0)}[\tau, x, y, z]\right)^2 + \left(-A1^{(0,0,1,0)}[\tau, x, y, z] + A2^{(0,1,0,0)}[\tau, x, y, z]\right)^2 + \left(A1^{(0,0,0,1)}[\tau, x, y, z] - A3^{(0,1,0,0)}[\tau, x, y, z]\right)^2 + A1^{(1,0,0,0)}[\tau, x, y, z]^2 + A2^{(1,0,0,0)}[\tau, x, y, z]^2 + A3^{(1,0,0,0)}[\tau, x, y, z]^2$$

We confirm again that the right equation of motion is obtained by comparing the gauge field terms with their physical field expressions

```
In[•]:= Simplify[left == right]
Out[•]=
        True
```

Let’s try to define A as a vector. We start with considering plus polarization

```
In[•]:= kvec = {k1, k2, k3};
        r = {x, y, z};
        ϵv = {ϵ1[k1, k2, k3], ϵ2[k1, k2, k3], ϵ3[k1, k2, k3]};
        ϵvh = {ϵ1h[k1, k2, k3], ϵ2h[k1, k2, k3], ϵ3h[k1, k2, k3]};
        Cross[kvec, ϵv] == I Sqrt[kvec.kvec] ϵv;
```

Convert to Fourier Space

```
In[•]:= A1[τ_, x_, y_, z_] :=
         av[k1, k2, k3] * Av[τ, k1, k2, k3] Exp[I (k1 x + k2 y + k3 z)] * ϵ1[k1, k2, k3]
        A2[τ_, x_, y_, z_] :=
         av[k1, k2, k3] * Av[τ, k1, k2, k3] Exp[I (k1 x + k2 y + k3 z)] * ϵ2[k1, k2, k3]
        A3[τ_, x_, y_, z_] :=
         av[k1, k2, k3] * Av[τ, k1, k2, k3] Exp[I (k1 x + k2 y + k3 z)] * ϵ3[k1, k2, k3]
```

Normalization and properties of polarization vectors

```
In[•]:= crosspola = Thread[Cross[kvec, ϵv] → I Sqrt[kvec.kvec] ϵv];

In[•]:= eqmotionA =
         FullSimplify[FullSimplify[Expand[eqA1 == 0 /. {A1 → (A1[#1, #2, #3, #4] &),
               A2 → (A2[#1, #2, #3, #4] &), A3 → (A3[#1, #2, #3, #4] &)}],
            {f ≠ 0, λ ≠ 0, av[k1, k2, k3] ≠ 0, k > 0}] /. Thread[
            Cross[kvec, ϵv] → I Sqrt[kvec.kvec] ϵv] /.
          (kvec.kvec) → k^2, {k > 0, ϵ1[k1, k2, k3] ≠ 0}]
Out[•]=
```

$$k\,A\nu[\tau, k1, k2, k3]\,(f\,k - \lambda\,\chi'[\tau]) + f\,A\nu^{(2,0,0,0)}[\tau, k1, k2, k3] == 0$$

We now switch back to cosmic time for the inflaton term and make the approximation that during inflation the scale factor $a \approx -\frac{1}{H\,\tau}$

In[◦]:=
```
approxeqA = ExpandAll@FullSimplify[eqmotionA[[1]] / f, f ≠ 0] /. χ'[τ] λ / f → -2 ξ / τ
```
Out[◦]=

$$k^2\,\text{A}\nu[\tau, \text{k1}, \text{k2}, \text{k3}] + \frac{2\,k\,\xi\,\text{A}\nu[\tau, \text{k1}, \text{k2}, \text{k3}]}{\tau} + \text{A}\nu^{(2,0,0,0)}[\tau, \text{k1}, \text{k2}, \text{k3}]$$

Solutions of the gauge field equations of motion

Momentum vector k only appears in magnitude, hence we can set the argument k_1, k_2, k_3 to just k.

In[◦]:=
```
equationforA = approxeqA /. Aν → (Aν[#1, k] &)
```
Out[◦]=

$$k^2\,\text{A}\nu[\tau, k] + \frac{2\,k\,\xi\,\text{A}\nu[\tau, k]}{\tau} + \text{A}\nu^{(2,0)}[\tau, k]$$

In[◦]:=
```
approxequationAminus =
  k^2 Aν[τ, k1, k2, k3] - (2 k ξ Aν[τ, k1, k2, k3])/τ + Aν^(2,0,0,0)[τ, k1, k2, k3];
```

In[◦]:=
```
equationforAminus = approxequationAminus /. Aν → (Aν[#1, k] &)
```
Out[◦]=

$$k^2\,\text{A}\nu[\tau, k] - \frac{2\,k\,\xi\,\text{A}\nu[\tau, k]}{\tau} + \text{A}\nu^{(2,0)}[\tau, k]$$

In[◦]:=
```
initialcondition = Asymptotic[Aν[τ, k], τ → -Infinity] == E^(-I k τ) / Sqrt[2 k];
```

In[◦]:=
```
solutionA =
 FullSimplify@DSolve[{equationforA == 0}, Aν[τ, k], τ, Assumptions → k > 0]
```
Out[◦]=

$$\left\{\left\{\text{A}\nu[\tau, k] \to e^{-i k \tau}\,\tau\,\left(c_2\,\text{Hypergeometric1F1}[1 + i\,\xi, 2, 2\,i\,k\,\tau] + c_1\,\text{HypergeometricU}[1 + i\,\xi, 2, 2\,i\,k\,\tau]\right)\right\}\right\}$$

In[◦]:=
```
solutionAminus =
 FullSimplify@DSolve[{equationforAminus == 0}, Aν[τ, k], τ, Assumptions → k > 0]
```
Out[◦]=

$$\left\{\left\{\text{A}\nu[\tau, k] \to e^{-i k \tau}\,\tau\,\left(c_2\,\text{Hypergeometric1F1}[1 - i\,\xi, 2, 2\,i\,k\,\tau] + c_1\,\text{HypergeometricU}[1 - i\,\xi, 2, 2\,i\,k\,\tau]\right)\right\}\right\}$$

In[ ]:=
```
asymp = Asymptotic[ⅇ^(-ⅈ k τ) τ
    (ℂ2 Hypergeometric1F1[1 + ⅈ ξ, 2, 2 ⅈ k τ] + ℂ1 HypergeometricU[1 + ⅈ ξ, 2, 2 ⅈ k τ]),
   τ → -Infinity, Assumptions → ξ > 0]
```

Out[ ]=

$$e^{-i k \tau} (i k \tau)^{-1-i\xi} \left(-\frac{2^{-2-i\xi}(1+i\xi)\,\xi\, c_1}{k} - \frac{2^{-4-i\xi}\xi(-i+\xi)^2(-2i+\xi)\,c_1}{k^2\tau} + 2^{-1-i\xi}\tau\, c_1\right) + e^{-i k \tau}(-i k \tau)^{-1-i\xi} \left(-\frac{2^{-2-i\xi}(1+i\xi)\,\xi\, c_2}{k\,\text{Gamma}[1-i\xi]} - \frac{2^{-4-i\xi}\xi(-i+\xi)^2(-2i+\xi)\,c_2}{k^2\tau\,\text{Gamma}[1-i\xi]} + \frac{2^{-1-i\xi}\tau\, c_2}{\text{Gamma}[1-i\xi]}\right) + e^{i k \tau}(i k \tau)^{-1+i\xi}\left(-\frac{2^{-2+i\xi}(1-i\xi)\,\xi\, c_2}{k\,\text{Gamma}[1+i\xi]} - \frac{2^{-4+i\xi}\xi(i+\xi)^2(2i+\xi)\,c_2}{k^2\tau\,\text{Gamma}[1+i\xi]} + \frac{2^{-1+i\xi}\tau\, c_2}{\text{Gamma}[1+i\xi]}\right)$$

In[ ]:=
```
c1c2 = FullSimplify[Solve[{asymp == E^(-I k τ) / Sqrt[2 k],
     D[asymp, τ] == - I k E^(-I k τ) / Sqrt[2 k] }, {ℂ1, ℂ2}], {k > 0, ξ > 0}]
```

Out[ ]=

$$\Bigg\{\Bigg\{c_1 \to \Bigg(2^{\frac{5}{2}-i\xi} k^{5/2}\tau^2(-i k \tau)^{-i\xi}\Bigg(-i\, 2^{2i\xi}\left(k^2\tau^2\right)^{i\xi}\left(\xi\left(-2+3i\xi+\xi^2\right)^2 + 2k\xi^2(i+\xi)^2\tau - 8ik^2\xi^2\tau^2 - 16k^3\tau^3\right) + \frac{e^{-2ik\tau}\xi^2\left((-2+\xi(-3i+\xi))^2 + 4ik(-i+\xi)^2\tau - 8k^2\tau^2\right)\text{Gamma}[i\xi]}{\text{Gamma}[1-i\xi]}\Bigg)\Bigg) \Big/ \left(\xi^3\left(1+\xi^2\right)^2\left(4+\xi^2\right) + k\xi^2\left(1+\xi^2\right)\left(12-23\xi^2+\xi^4\right)\tau - 8k^2\xi\left(4-17\xi^2+3\xi^4\right)\tau^2 + 64k^5\tau^5\right),$$

$$c_2 \to \left(2^{\frac{5}{2}-i\xi} e^{-2ik\tau} k^{5/2}\xi^2\tau^2(ik\tau)^{-i\xi}\left((-2+\xi(-3i+\xi))^2 + 4ik(-i+\xi)^2\tau - 8k^2\tau^2\right)\text{Gamma}[i\xi]\right) \Big/ \left(\xi^3\left(1+\xi^2\right)^2\left(4+\xi^2\right) + k\xi^2\left(1+\xi^2\right)\left(12-23\xi^2+\xi^4\right)\tau - 8k^2\xi\left(4-17\xi^2+3\xi^4\right)\tau^2 + 64k^5\tau^5\right)\Bigg\}\Bigg\}$$

The final solution given by Mathematica is rather complicated

In[ ]:=
```
finasol = FullSimplify@solutionA /. c1c2
```

Out[ ]=

$$
\begin{aligned}
&\Big\{\Big\{\Big\{\mathrm{A}\nu[\tau,\mathrm{k}] \to e^{-i\,k\,\tau}\,\tau \\
&\quad\Bigg(\bigg(2^{\frac{5}{2}-i\,\xi}\,e^{-2\,i\,k\,\tau}\,k^{5/2}\,\xi^2\,\tau^2\,(i\,k\,\tau)^{-i\,\xi}\,\left((-2+\xi\,(-3\,i+\xi))^2+4\,i\,k\,(-i+\xi)^2\,\tau-8\,k^2\,\tau^2\right) \\
&\qquad\quad \mathrm{Gamma}[i\,\xi]\,\mathrm{Hypergeometric1F1}[1+i\,\xi,2,2\,i\,k\,\tau]\bigg)\Big/ \\
&\qquad\left(\xi^3\left(1+\xi^2\right)^2\left(4+\xi^2\right)+k\,\xi^2\left(1+\xi^2\right)\left(12-23\,\xi^2+\xi^4\right)\tau-8\,k^2\,\xi\left(4-17\,\xi^2+3\,\xi^4\right)\tau^2+\right. \\
&\qquad\quad\left.64\,k^5\,\tau^5\right)+\Bigg(2^{\frac{5}{2}-i\,\xi}\,k^{5/2}\,\tau^2\,(-i\,k\,\tau)^{-i\,\xi}\,\Bigg(-i\,2^{2\,i\,\xi}\left(k^2\,\tau^2\right)^{i\,\xi} \\
&\qquad\qquad\left(\xi\left(-2+3\,i\,\xi+\xi^2\right)^2+2\,k\,\xi^2\,(i+\xi)^2\,\tau-8\,i\,k^2\,\xi^2\,\tau^2-16\,k^3\,\tau^3\right)+ \\
&\qquad\qquad\frac{e^{-2\,i\,k\,\tau}\,\xi^2\left((-2+\xi\,(-3\,i+\xi))^2+4\,i\,k\,(-i+\xi)^2\,\tau-8\,k^2\,\tau^2\right)\mathrm{Gamma}[i\,\xi]}{\mathrm{Gamma}[1-i\,\xi]}\Bigg) \\
&\qquad\quad\mathrm{HypergeometricU}[1+i\,\xi,2,2\,i\,k\,\tau]\Bigg)\Big/\left(\xi^3\left(1+\xi^2\right)^2\left(4+\xi^2\right)+\right. \\
&\qquad\quad\left.k\,\xi^2\left(1+\xi^2\right)\left(12-23\,\xi^2+\xi^4\right)\tau-8\,k^2\,\xi\left(4-17\,\xi^2+3\,\xi^4\right)\tau^2+64\,k^5\,\tau^5\right)\Bigg)\Big\}\Big\}\Big\}
\end{aligned}
$$

But it effectively reduces to Bunch-Davies Vacuum

In[ ]:=
```
FullSimplify@Asymptotic[finasol[[1, 1, 1, 2]], τ → -Infinity]
```

Out[ ]=

$$\frac{e^{-i\,k\,\tau}}{\sqrt{2}\,\sqrt{k}}$$

Solutions can be recast in terms of Coulomb functions

In[ ]:=
```
modeAplus[τ_, k_] := 1 / Sqrt[2 k] (CoulombG[0, ξ, -k τ] + I CoulombF[0, ξ, -k τ])
```

In[ ]:=
```
modeAminus[τ_, k_] := 1 / Sqrt[2 k] (CoulombG[0, -ξ, -k τ] + I CoulombF[0, -ξ, -k τ])
```

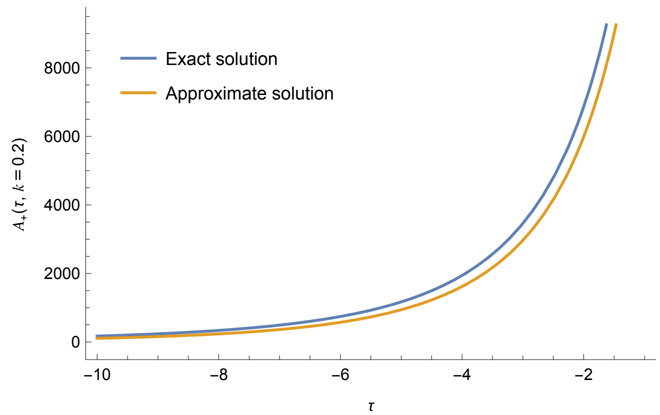
Whereas the approximate function as derived in the thesis are given by

In[ ]:=
```
approximatefuncfrompaper[τ] :=
  1 / Sqrt[2 k] (-k τ / (2 ξ)) ^ (1 / 4) E ^ (Pi ξ - 2 * Sqrt[-2 ξ k τ]);
```

We now show that fluctuations for mode A_- are negligible, and that perturbations are enhanced almost only for $-k\,\tau \ll 2\,\xi$ and for $e^{\pi\xi} \gg 1$.

In[ ]:=
```
Block[{k = 0.2, ξ = 4}, Plot[{Evaluate@Re[modeAplus[τ, k]],
    Evaluate[approximatefuncfrompaper[τ]]}, {τ, -10, -1},
  PlotLegends → Placed[{"Exact solution", "Approximate solution"}, {0.25, 0.8}],
  Frame → {{True, False}, {True, False}},
  FrameLabel → {{"A₊(τ, k = 0.2)", None}, {"τ", None}}, FrameTicks → All]]
```

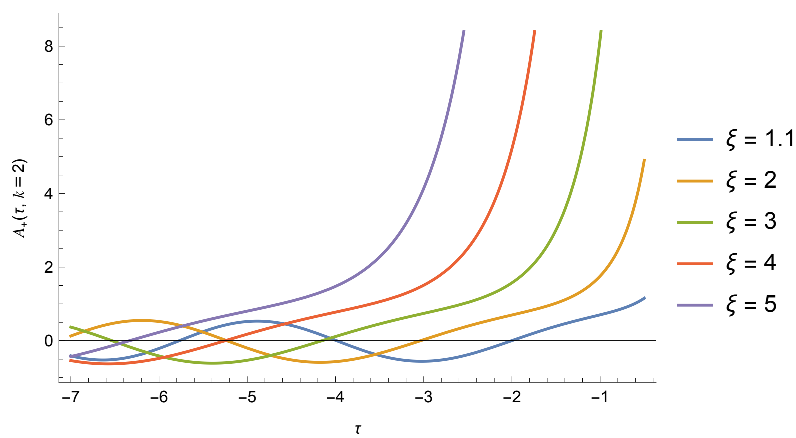
Out[ ]=



In[ ]:=
```
multipleAmodes = Block[{ξ = #, k = 2}, Re[modeAplus[τ, k]]] & /@ {1.1, 2, 3, 4, 5};
```

In[ ]:=
```
Show[Block[{k = 2}, Plot[multipleAmodes, {τ, -7, -0.5}, PlotLegends → {
     "ξ = " <> ToString[#] & /@ {1.1, 2, 3, 4, 5}},
   Frame → {{True, False}, {True, False}},
   FrameLabel → {{"A₊(τ, k = 2)", None}, {"τ", None}}, FrameTicks → All]]]
```

Out[ ]=

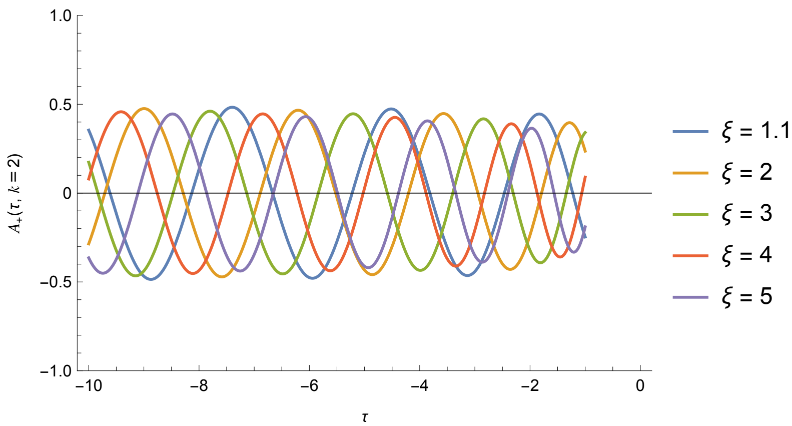


Modes A_- do not develop perturbations and remain in their vacuum state

In[ ]:=
```
multipleAmodesminus =
  Block[{ξ = #, k = 2}, Re[modeAminus[τ, k]]] & /@ {1.1, 2, 3, 4, 5};
```

In[ ]:=
```
Show[Block[{k = 2}, Plot[multipleAmodesminus, {τ, -10, -1}, PlotLegends → {
    "ξ = " <> ToString[#] & /@ {1.1, 2, 3, 4, 5}},
   Frame → {{True, False}, {True, False}},
   FrameLabel → {{"A₊(τ, k = 2)", None}, {"τ", None}},
   FrameTicks → All, PlotRange → {{Automatic, Automatic}, {-1, 1}}]]]
```

Out[ ]=



Expectation values involving $\vec{E}$ and $\vec{B}$.

In[ ]:=
```
approximatemodeAplus[τ, k] =
  1 / Sqrt[2 k] (-k τ / (2 ξ)) ^ (1 / 4) E ^ (Pi ξ - 2 * Sqrt[-2 ξ k τ]);
```

In[ ]:=
```
averageEdotB =
 -1 / (4 Pi ^ 2 a ^ 4) Integrate[k ^ 3 FullSimplify@D[approximatemodeAplus[τ, k] ^ 2, τ],
    {k, 0, Infinity}, Assumptions → {τ < 0, ξ > 0}] //. {τ → 1 / H, a → 1}
```

Out[ ]=

$$-\frac{135\, e^{2\pi\xi}\, H^4}{65\,536\, \pi^2\, \xi^4}$$

In[ ]:=
```
averageE2B2 = FullSimplify@Expand[1 / (4 Pi ^ 2 a ^ 4) Integrate[k ^ 2
       (D[approximatemodeAplus[τ, k], τ] ^ 2 + k ^ 2 approximatemodeAplus[τ, k] ^ 2),
      {k, 0, Infinity}, Assumptions → {τ < 0, ξ > 0}]] //. {τ → 1 / H, a → 1}
```

Out[ ]=

$$\frac{63\, e^{2\pi\xi}\, H^4\, (5 + 4\,\xi^2)}{262\,144\, \pi^2\, \xi^5}$$

Some useful identities

Green's function

In[41]:=
```
g[t1, t2] :=
  I HeavisideTheta[t1 - t2] (q[t1] Conjugate[q[t2]] - Conjugate[q[t1]] q[t2])
```

In[42]:=
```
FullSimplify[Expand[
   (g[t1, t2] /. {t1 → τ, t2 → τ0}) * (g[t1, t2] /. {t1 → τ, t2 → τ1})], q[τ] ∈ Reals]
```

Out[42]=

$4\,\mathrm{HeavisideTheta}[\tau - \tau0,\ \tau - \tau1]\ \mathrm{Im}[q[\tau0]]\ \mathrm{Im}[q[\tau1]]\ q[\tau]^2$

In[43]:=
```
Expand[(g[t1, t2] /. {t1 → τ, t2 → τ0}) * (g[t1, t2] /. {t1 → τ, t2 → τ1})]
```
Out[43]=
```
-Conjugate[q[τ0]] Conjugate[q[τ1]] HeavisideTheta[τ - τ0]
  HeavisideTheta[τ - τ1] q[τ]^2 + Conjugate[q[τ]] Conjugate[q[τ1]]
  HeavisideTheta[τ - τ0] HeavisideTheta[τ - τ1] q[τ] × q[τ0] + Conjugate[q[τ]]
  Conjugate[q[τ0]] HeavisideTheta[τ - τ0] HeavisideTheta[τ - τ1] q[τ] × q[τ1] -
 Conjugate[q[τ]]^2 HeavisideTheta[τ - τ0] HeavisideTheta[τ - τ1] q[τ0] × q[τ1]
```

In[44]:=
```
FullSimplify[-Conjugate[q[τ0]] Conjugate[q[τ1]] +
  Conjugate[q[τ1]] q[τ0] + Conjugate[q[τ0]] q[τ1] - q[τ0] × q[τ1]]
```
Out[44]=
```
4 Im[q[τ0]] Im[q[τ1]]
```

Bessel Function

In[52]:=
```
FullSimplify[BesselJ[3 / 2, k τ] == -I BesselJ[3 / 2, -k τ],
 Assumptions → {τ < 0, k > 0}]
```
Out[52]=
```
True
```

Derivation of power spectrum

Solve for the dimensionless $\chi(\xi)$

In[ ]:=
```
χ3[ξ_] := ξ^2 / (8 Pi) NIntegrate[(1 + ((q1^2 + q2^2 + q3^2) - q1) /
        (Sqrt[q1^2 + q2^2 + q3^2] Sqrt[(1 - q1)^2 + q2^2 + q3^2]))^2 *
    Sqrt[q1^2 + q2^2 + q3^2]^(1 / 2) * Sqrt[(q1 - 1)^2 + q2^2 + q3^2]^(1 / 2) *
    (Sqrt[q1^2 + q2^2 + q3^2]^(1 / 2) + Sqrt[(q1 - 1)^2 + q2^2 + q3^2]^(1 / 2))^2 *
    (Pi / 2) *
    (x^(3 / 2) Re[HankelH1[3 / 2, x]] E^(-(2 Sqrt[2 ξ] (Sqrt[q1^2 + q2^2 + q3^2]^
              (1 / 2) + Sqrt[(q1 - 1)^2 + q2^2 + q3^2]^(1 / 2))) Sqrt[x]))^2,
   {x, 0, Infinity}, {q1, -Infinity, Infinity}, {q2, -Infinity, Infinity},
   {q3, -Infinity, Infinity},
   Method → {"MultidimensionalRule", "Generators" → 9},
   PrecisionGoal → 10, AccuracyGoal → 10]
```

In[ ]:=
```
Table[Evaluate[χ3[ξ]], {ξ, 1, 9, 0.2}]
```
Out[ ]=

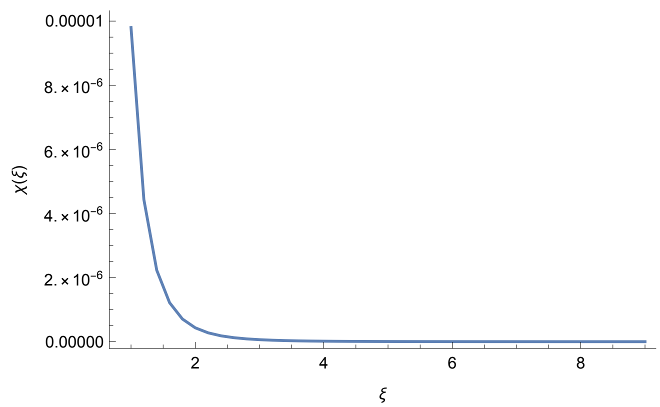
$\{9.79883\times10^{-6}, 4.43487\times10^{-6}, 2.23171\times10^{-6}, 1.21737\times10^{-6}, 7.07661\times10^{-7}, 4.3309\times10^{-7}, 2.76593\times10^{-7}, 1.83084\times10^{-7}, 1.24958\times10^{-7}, 8.75758\times10^{-8}, 6.27965\times10^{-8}, 4.59577\times10^{-8}, 3.42348\times10^{-8}, 2.59059\times10^{-8}, 1.9903\times10^{-8}, 1.54911\times10^{-8}, 1.21918\times10^{-8}, 9.71391\times10^{-9}, 7.7702\times10^{-9}, 6.36605\times10^{-9}, 5.19716\times10^{-9}, 4.29298\times10^{-9}, 3.59795\times10^{-9}, 3.00354\times10^{-9}, 2.52973\times10^{-9}, 2.19165\times10^{-9}, 1.82928\times10^{-9}, 1.56257\times10^{-9}, 1.34321\times10^{-9}, 1.31784\times10^{-9}, 1.15803\times10^{-9}, 1.00607\times10^{-9}, 7.46192\times10^{-10}, 6.58754\times10^{-10}, 6.16365\times10^{-10}, 5.15892\times10^{-10}, 4.60507\times10^{-10}, 4.20034\times10^{-10}, 2.91985\times10^{-10}, 2.54785\times10^{-10}, 3.2253\times10^{-10}\}$

In[ ]:=
```
list = Thread[
   {{9.798825658147578`*^-6, 4.434872816993712`*^-6, 2.2317135077461777`*^-6,
     1.21736601479737`*^-6, 7.076609392820408`*^-7, 4.3309006246320117`*^-7,
     2.7659268739892427`*^-7, 1.8308388319999438`*^-7, 1.2495809954089347`*^-7,
     8.757583296471376`*^-8, 6.279653756484055`*^-8, 4.5957684325696595`*^-8,
     3.423479700469405`*^-8, 2.590586911890827`*^-8, 1.9902990576599183`*^-8,
     1.5491099643824834`*^-8, 1.2191766100672054`*^-8,
     9.713909740489984`*^-9, 7.770202684710494`*^-9, 6.36604781877133`*^-9,
     5.197164341368833`*^-9, 4.292981578349364`*^-9, 3.5979513194070786`*^-9,
     3.003541921815547`*^-9, 2.5297285305690097`*^-9, 2.1916493316087435`*^-9,
     1.8292783327416729`*^-9, 1.562570548262091`*^-9, 1.3432112745491081`*^-9,
     1.317839131499263`*^-9, 1.1580333621141924`*^-9, 1.0060707550758208`*^-9,
     7.461921629772588`*^-10, 6.587535537238211`*^-10,
     6.163646302119666`*^-10, 5.158924010729107`*^-10, 4.605068214417691`*^-10,
     4.2003435008160875`*^-10, 2.9198464661759426`*^-10,
     2.5478508307638025`*^-10, 3.225299118078547`*^-10}, Range[1, 9, 0.2]}];
```

Plot of the $\chi(\xi)$ function.

In[ ]:=
```
ListLinePlot[ReverseSort[#] &/@ list, PlotRange → Full,
 Axes → False, Frame → {{True, False}, {True, False}},
 FrameLabel → {{"χ(ξ)", None}, {"ξ", None}}, FrameTicks → All]
```

Out[ ]=



COBE normalizazion

In[ ]:=
```
pCobe[ξ_] := E^(-4 Pi ξ) / (2 χ3[ξ]) (-1 + Sqrt[1 + 10^(-8) χ3[ξ] E^(4 Pi ξ)])

Table[Evaluate[pCobe[ξ]], {ξ, 1, 9, 0.2}]
```

Out[ ]=

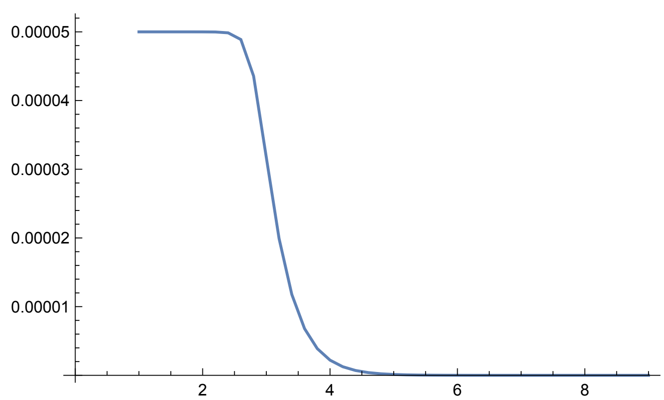
$$\{2.5\times10^{-9}, 2.5\times10^{-9}, 2.5\times10^{-9}, 2.5\times10^{-9}, 2.49997\times10^{-9}, 2.49978\times10^{-9},$$
$$2.49825\times10^{-9}, 2.48582\times10^{-9}, 2.38961\times10^{-9}, 1.89766\times10^{-9}, 1.00488\times10^{-9},$$
$$3.96534\times10^{-10}, 1.38547\times10^{-10}, 4.62072\times10^{-11}, 1.50985\times10^{-11}, 4.88083\times10^{-12},$$
$$1.56689\times10^{-12}, 4.9971\times10^{-13}, 1.59029\times10^{-13}, 5.00055\times10^{-14}, 1.57515\times10^{-14},$$
$$4.9326\times10^{-15}, 1.53348\times10^{-15}, 4.77681\times10^{-16}, 1.48138\times10^{-16}, 4.52969\times10^{-17},$$
$$1.41112\times10^{-17}, 4.34543\times10^{-18}, 1.33392\times10^{-18}, 3.83284\times10^{-19}, 1.1637\times10^{-19},$$
$$3.55334\times10^{-20}, 1.17429\times10^{-20}, 3.55703\times10^{-21}, 1.0466\times10^{-21}, 3.25588\times10^{-22},$$
$$9.80798\times10^{-23}, 2.92284\times10^{-23}, 9.97738\times10^{-24}, 3.0399\times10^{-24}, 7.68972\times10^{-25}\}$$

In[ ]:=
```
pcobelist = {2.499999982605964`*^-9, 2.4999998995534867`*^-9,
   2.49999938967263`*^-9, 2.4999958951388746`*^-9, 2.4999705418952134`*^-9,
   2.4997774683703524`*^-9, 2.498247639000135`*^-9, 2.485822381045987`*^-9,
   2.3896093456512826`*^-9, 1.897664630490458`*^-9, 1.0048762775704315`*^-9,
   3.965336527898024`*^-10, 1.385468271093583`*^-10, 4.620724850371261`*^-11,
   1.509854602602767`*^-11, 4.880830118971101`*^-12, 1.566892420609399`*^-12,
   4.9970955081443`*^-13, 1.5902945173834236`*^-13, 5.0005493279184504`*^-14,
   1.5751498375524562`*^-14, 4.932600964408639`*^-15, 1.5334782061195048`*^-15,
   4.776809818324325`*^-16, 1.4813829011421233`*^-16, 4.5296872835483043`*^-17,
   1.4111179630326825`*^-17, 4.34543099343275`*^-18, 1.3339205685512846`*^-18,
   3.8328373720055126`*^-19, 1.1636986081991456`*^-19, 3.5533357599319686`*^-20,
   1.1742880077705868`*^-20, 3.557033377362587`*^-21, 1.0465983118979531`*^-21,
   3.2558829045515746`*^-22, 9.80798442365184`*^-23, 2.922838745264451`*^-23,
   9.977384533010795`*^-24, 3.0398991634215816`*^-24, 7.689721944486793`*^-25};
```

Now plot P_{cobe} alone

In[ ]:=
```
Show[
 ListLinePlot[ReverseSort[#] & /@ Thread[{Sqrt[pcobelist], Range[1, 9, 0.2]}]]]
```

Out[ ]=

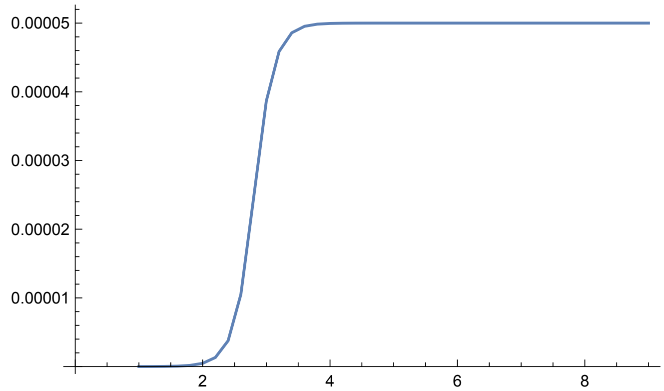


In[ ]:=
```
χlist = {9.798825658147578`*^-6, 4.434872816993712`*^-6,
   2.2317135077461777`*^-6, 1.21736601479737`*^-6, 7.076609392820408`*^-7,
   4.3309006246320117`*^-7, 2.7659268739892427`*^-7, 1.8308388319999438`*^-7,
   1.2495809954089347`*^-7, 8.757583296471376`*^-8, 6.279653756484055`*^-8,
   4.5957684325696595`*^-8, 3.423479700469405`*^-8, 2.590586911890827`*^-8,
   1.9902990576599183`*^-8, 1.5491099643824834`*^-8, 1.2191766100672054`*^-8,
   9.713909740489984`*^-9, 7.770202684710494`*^-9, 6.36604781877133`*^-9,
   5.197164341368833`*^-9, 4.292981578349364`*^-9, 3.5979513194070786`*^-9,
   3.003541921815547`*^-9, 2.5297285305690097`*^-9, 2.1916493316087435`*^-9,
   1.8292783327416729`*^-9, 1.562570548262091`*^-9, 1.3432112745491081`*^-9,
   1.317839131499263`*^-9, 1.1580333621141924`*^-9, 1.0060707550758208`*^-9,
   7.461921629772588`*^-10, 6.587535537238211`*^-10, 6.163646302119666`*^-10,
   5.158924010729107`*^-10, 4.605068214417691`*^-10, 4.2003435008160875`*^-10,
   2.9198464661759426`*^-10, 2.5478508307638025`*^-10, 3.225299118078547`*^-10};
```

In[ ]:=
```
gaugecon = pcobelist^2 * Table[E^(4 Pi ξ), {ξ, 1, 9, 0.2}] * χlist;
```

Plot of gauge quanta contributions

In[ ]:=
```
ListLinePlot[ReverseSort[#] & /@ Thread[{Sqrt[gaugecon], Range[1, 9, 0.2]}]]
```
Out[ ]=



Combined plot of the power spectrum

In[ ]:=
```
sumcontributions = gaugecon + pcobelist;
```

In[ ]:=
```
ListLinePlot[{ReverseSort[#] & /@ Thread[{Sqrt[gaugecon], Range[1, 9, 0.2]}],
  ReverseSort[#] & /@ Thread[{Sqrt[pcobelist], Range[1, 9, 0.2]}],
  ReverseSort[#] & /@ Thread[{Sqrt[sumcontributions], Range[1, 9, 0.2]}]},
 PlotLegends → {"Gauge fluctuations", "Vacuum fluctuations",
   "Total contribution"}, Axes → False, Frame → {{True, False}, {True, False}},
 FrameLabel → {{"P^{1/2}(k)", None}, {"ξ", None}}, FrameTicks → All]
```
Out[ ]=