UNIVERSITY OF GRONINGEN

APPLIED MATHEMATICS BACHELOR PROJECT

# Switched nonlinear DAEs in electrical circuit theory

*Author:*
Darina Kuzmenko

*Supervisor:*
1st: Prof. Stephan Trenn
2nd: Dr. Henk van Waarde

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# *Abstract*

Inconsistent initial conditions in RLC circuits with switches or faulty components can cause Dirac impulses in the solution. These electrical impulses can damage certain parts of the circuit, and thus warrant a study into the unique solvability of the circuit. However, the presence of nonlinear elements in such a circuit make this challenging. In this paper, the uniqueness and solvability of nonlinear RLC circuits is investigated through the framework of nonlinear switched DAEs. A theory developed by Kausar and Trenn in their 2017 paper *"Impulses in structured nonlinear switched DAEs"* is discussed and the extent to which it can be applied to DAE models of RLC circuits is investogated.

# *Acknowledgements*

I would like to thank my supervisor Stephan Trenn for his endless patience and valuable guidance on this project. I would also like to thank my friends for their encouragement and commiseration on long study nights, without which I may have gone slightly mad.

# Contents

# List of abbreviations

| | |
|---|---|
| **DAE** | **D**ifferential **A**lgebraic **E**quation |
| **ITP** | **I**nitial **T**rajectory **P**roblem |
| **QWF** | **Q**uasi-**W**eierstrass **F**orm |
| **RLC** | **R**esistor **(L)**Inductor **C**apacitor |
| **KVL** | **K**irchhoff's **V**oltage **L**aws |
| **KCL** | **K**irchhoff's **C**urrent **L**aws |

# Chapter 1

# Introduction

Electrical circuits are intensely prevalent in modern daily life. Even the most minute and common household devices often rely on skillfully assembled arrangements of wires and electrical elements. However, even in those minute devices, electrical circuits can get very intricate. To avoid accidentally damaging an entire electrical system due to an oversight or unforeseen circumstance, it would be best to mathematically solve the system first (as mathematical failure is famously less devastating than electrical failure). Finding this solution usually involves running a numerical method, and the bigger the system, the more time and energy the computer needs to come up with an answer. In order to avoid wasting time and resources on an unsolvable problem, it would be best to verify existence and uniqueness of solutions beforehand.

Alas, it becomes quickly apparent that this is much easier said than done. In particular, circuits with electric switches or unexpected electrical faults (which can act as "unwanted" switches) may experience so-called Dirac impulses, which, in combination with non-linear electrical components, entirely complicate the process of deriving a (unique) solvability result.

In their 2017 paper "Impulses in structured nonlinear switched DAEs" [8], authors Kausar and Trenn devised a set of conditions that, when met, ensure unique solvability of switched nonlinear DAEs. This result was developed in the context of water networks. The aim of this paper is to investigate if this result is applicable to switched nonlinear DAE models of nonlinear RLC circuits.

Chapter 2 provides a comprehensive introduction to switched differential-algebraic equations (DAEs). Chapter 3 introduces the necessary background information on RLC circuits and demonstrate how to translate a nonlinear RLC circuit into a nonlinear DAE system. The latter is heavily based on the research done by Riaza in his paper "DAEs in Circuit Modelling: A survey" [11] and Engelaar in his Bachelor Project "Controllability of RLC electrical circuits with ideal components" [6].

Chapter 4 contains the uniqueness and solvability result for nonlinear switched DAEs developed by Kausar and Trenn, complete with detailed explanations and theoretical background. Finally, in Chapter 5 this result is applied to nonlonear DAE systems of nonlinear RLC circuits. Specific examples are examined and potential limitations in the research framework are discussed.

# Chapter 2

# Switched DAEs

*Differential-algebraic equations* (*DAEs*) (sometimes referred to as *semistate* systems) are systems comprised of both differential and algebraic equations. The presence of the latter is what distinguishes these systems from ODEs (which, by definition, consist exclusively of differential equations). DAEs are found to be particularly useful in the fields of circuit theory and mechanics [12].

One particularly useful class of DAEs is the class of so-called *switched* (or *hybrid*) *DAEs*. These allow us to combine the discrete and continuous dynamics of a system in one mathematical model. Switched *nonlinear* DAEs allow us even more flexibility in modelling more complex systems.

This chapter presents an introduction to switched nonlinear DAE theory. Section 2.1 is dedicatd to the structure of these systems, complete with relevant notation. Section 2.3 examines the solution space of switched nonlinear DAEs, with a particular focus on solutions containing Dirac impulses.

## 2.1 Anatomy of switched DAEs

This paper deals with nonlinear switched nonlinear DAEs, i.e. systems of the form

$$\boxed{E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + g_{\sigma(t)}(x(t)) + f(t)} \tag{2.1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is the state vector, $E_p, A_p \in \mathbb{R}^{n\times n}$ are the system matrices, $g_p : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinearity, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the inhomogeneity and $\sigma(t)$ is the switching signal, given as follows

**Definition 1** *The (time-dependent) switching signal is the function*

$$\begin{aligned} \sigma : [0,\infty) &\to \mathcal{P} \\ t &\mapsto \sigma(t) = p \quad \forall t \in [t_p, t_{p+1}) \end{aligned}$$

*where $\mathcal{P} = \{1, 2, ..., m\}$ is the index set and $\sigma$ is a right-continuous piecewise-constant function. [4]*

Such a function has finitely many discontinuities on every bounded time interval of $[0,\infty)$ and takes a constant value between any two consecutive switching times.

The role of the switching signal is to specify the active subsystem of (2.1) at any given time $t$. Each subsystem takes the form of

$$E_p\dot{x}(t) = A_p x(t) + g_p(x(t)) + f \tag{2.2}$$

where $p \in \mathcal{P}$ and $E_p, A_p \in \mathbb{R}^{n\times n}$ are constant. In other words, (2.1) is a time-varying nonlinear DAE whose system matrices $A_{\sigma(t)}, E_{\sigma(t)}$ are piecewise-constant.

This switched structure of the switched nonlinear DAEs is what will later allow us to represent a complete circuit with multiple branches and switches as one mathematical system.

### 2.1.1 A note on notation

To improve readability, the time dependency of the state vector $x(t)$ and the switching signal $\sigma(t)$ will typically be omitted:

$$E_\sigma \dot{x} = A_\sigma x + g_\sigma(x) + f$$

### 2.1.2 A note on classification

Based on the switching signal, switched DAEs can be classified into

- Time-dependent ($\sigma(t)$),
- State-dependent ($\sigma(x(t))$),
- Both time- and state-dependent (i.e. multiple switching signals involved).

In time-dependent DAEs, the switch from one active subsystem to another can happen irrespective of the behaviour of the mathematical system. An example would be a electric switch that breaks of completes a circuit. These are the systems that will be considered in this paper.

In state-dependent DAEs, the switch from one active subsystem to another is induced by the behaviour of the system itself. As an example, consider any circuit with an electrical fuse. A fuse is designed to melt (and hence break the circuit) when the current running through it exceeds some specified threshold. Here, the fuse acts as a state-dependent switch, where part of the state $x(t)$ is the current running through the fuse-branch. The complexity of a such systems puts them outside of the scope of our paper.

## 2.2 Initial trajectory problem (ITP)

An analogue to the probably more familiar initial value problem (IVP), the *initial trajectory problem (ITP)* is defined as follows

**Definition 2** *An initial trajectory problem (ITP) is defined as*

$$x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)} \tag{2.3}$$
$$(E\dot{x})_{[t_0,\infty)} = (Ax + g(x) + f)_{[t_0,-\infty)} \tag{2.4}$$

where $x^0 : \mathbb{R} \to \mathbb{R}^n$ is some given initial trajectory.

## 2.3 Distributional solutions

As will be shown in Section 4.1, the solution to a switched DAE problem could include the Dirac impulse.

**Definition 3** *The Dirac impulse $\delta_t$ at time $t \in \mathbb{R}$ is given by*

$$\begin{aligned} \delta_t :\ & \mathcal{C}_0^\infty \to \mathbb{R}, \\ & f \mapsto \delta_t(f) = f(t) \end{aligned}$$

[13]This definition makes use of the space of test functions $\mathcal{C}_0^\infty$. To define it, first consider the space of *smooth functions* $\mathcal{C}^\infty$ defined as

**Definition 4**

$$\mathcal{C}_0^\infty = \{f : \mathbb{R} \to \mathbb{R} \mid f \textit{ is arbitrarily often differentiable}\} \tag{2.5}$$

**[2009]** and the *support of* $f \in \mathcal{C}^\infty$ defined as

**Definition 5**

$$\textit{supp } f = \textit{cl } \{x \in \mathbb{R} \mid f(x) \neq 0\} \tag{2.6}$$

[13]

Then, the space of *test functions* $\mathcal{C}_0^\infty$ is defined as

**Definition 6**

$$\mathcal{C}_0^\infty = \{f \in \mathcal{C}^\infty \mid \textit{supp } f \textit{ is bounded}\} \tag{2.7}$$

[13]

Moreover, in a later definition, the space of *piecewise-smooth functions* will be referenced.

**Definition 7**

$$\mathcal{C}_{pw}^\infty = \left\{ \alpha = \sum_{i \in \mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}\alpha_i \;\middle|\; \begin{array}{l} (\alpha_i)_{i\in\mathbb{Z}} \in (\mathcal{C}^{\infty\mathbb{Z}}), \\ \{t_i \in \mathbb{R} \mid i \in \mathbb{Z}\} \textit{ locally finite} \\ \textit{with } t_i < t_{i+1}, i \in \mathbb{Z} \end{array} \right\} \tag{2.8}$$

[13]

The Dirac impulse is a distribution; hence, the solution space will need to be enlarged from the space of classical solutions to the space of *classical distributions*, defined as follows

**Definition 8** *The space of classical distributions is given by*

$$\mathbb{D} = \{D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous}\} \tag{2.9}$$

[13] Notably, any locally integrable function $f : \mathbb{R} \to \mathbb{R}$ induces a distribution via

$$f_{\mathbb{D}}(g) = \int_{\mathbb{R}} fg$$

However, it turns out that the whole space of distributions $\mathbb{D}$ is not a suitable (or entirely necessary) solution space for switched DAEs. Namely, in general distributions cannot be evaluated at a certain time (which becomes a problem when a solution needs to be defined at exactly the time of switching from one active subsystem to another). To combat this, one can consider switched DAEs as DAEs valid on certain intervals. Thus, a special subclass of distributions needs to be considered such that it can be defined on restricted intervals. Moreover, the solution space needs to contains the Dirac impulse and it's derivatives (exactly why will be discussed in Section 4.1). [10] [13]

These reasons give rise to the space of *piecewise-smooth distributions* $\mathbb{D}_{pw\mathcal{C}^\infty}$:

**Definition 9** *[2009] The space of piecewise-smooth distributions is given by*

$$\mathbb{D}_{pw\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{\tau \in T} D_\tau \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{pw},\ T \subseteq \mathbb{R} \text{ is discrete}, \\ \forall \tau \in T : D_\tau \in span\{\delta_\tau, \delta'_\tau, \delta''_\tau, ...\} \end{array} \right\}$$

This space clearly contains the Dirac impulse and it's derivatives. Moreover, a piecewise-smooth distribution can be evaluated at any time $t \in \mathbb{R}$ in the following three ways:

$$\begin{cases} D(t^+) = f(t^+), \\ D(t^-) = f(t^-), \\ D[t] = \begin{cases} D_t, & t \in T \\ 0, & t \notin T \end{cases} \end{cases}$$

[8]where $f(t^\pm)$ is the right/left limit of the function $f$ at time $t$. [8]

# Chapter 3

# RLC Circuits

In this paper, RLC circuits will be considered. These comprise of resistors (R), inductors (L), and capacitors (C), as well as voltage and current sources. Despite their relative simplicity, the real-life applications of RLC circuits are far from limited [1], and entirely justify a study into (unique) solvability.

DAE systems are a convenient way of modelling RLC circuits as they allow to combine the algebraic Kirchhoff and Ohms Laws and the differential component laws in one system. Expanding to nonlinear DAEs allows for nonlinear electrical elements, and expanding to switched nonlinear DAEs allows for time-dependent electric switches.

There are several ways to approach modelling circuits as DAEs. In his 2013 paper titled "DAEs in Circuit Modelling: A survey" [11], Riaza provides a comprehensive overview of strategies for a wide variety of circuit modelling problems. From this text, the concepts most closely related to RLC circuits and branch-oriented modelling were skillfully synthesized by University of Groningen student Maico Engelaar in his 2019 Bachelor Thesis, "Controllability of RLC electrical circuits with ideal components" [6].

This chapter will present a brief summary of the findings from Engelaar's paper, supplementing with additional theory where necessary. Section 3.1 presents a concise introduction to graph theory. Sections 3.2 and 3.3 discuss Kirchhoff Laws and component laws respectively. Finally, Section 3.4 presents a practical framework for modelling RLC circuits as DAEs. Curious readers are strongly encouraged to consult Riaza's and Engelaar's works to get a more thorough understanding of the structure.

## 3.1 Graph theory

In mathematics, graph theory refers to the study of *graphs*, or mathematical structures used to represent relationships between objects. For reasons that will become apparent, this paper focuses on the subset of *directed graphs*. [5]

**Definition 10** *A directed graph $G = (V, E, f)$ is a triple consisting of two sets and a map. The elements of $V$ are the nodes, the elements of $E$ are the edges. The map*

$$f : E \to V \times V$$
$$f(e) = (f_1(e), f_2(e))$$

*assigns to every edge $e$ an initial node $f_1(e)$ and a terminal node $f_2(e)$. The edge $e$ is said to be directed from $f_1(e)$ to $f_2(e)$.*

One may already see how a mathematical graph can be seen as a convenient analogue for an electrical circuit. The nodes are the points where one or more active elements in a circuit meet. The edges correspond to the branch of a circuit. *(NOTE: The number of branches corresponds to the number of active elements in a circuit)*. The direction of the branch, once identified, remains consistent.

In Figure 3.1 (below), there are 5 branches and 4 nodes. The direction is set by the current flowing out of the voltage source.

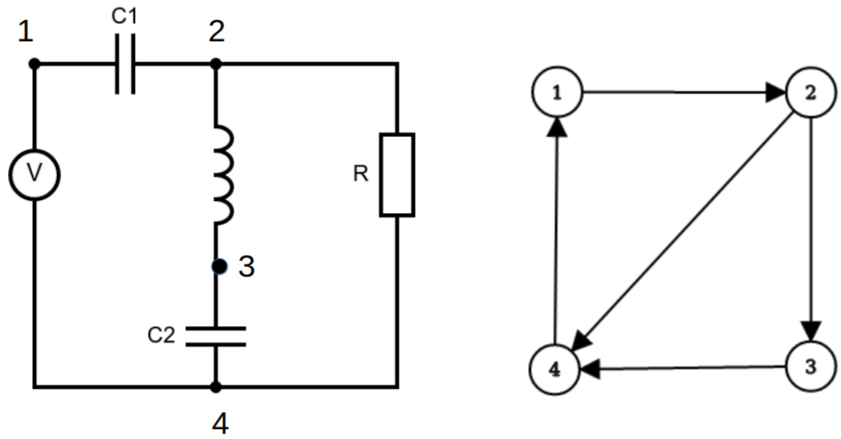


FIGURE 3.1: An electrical circuit on the left and its corresponding graph on the right. The nodes are numbered.

The final relevant component is a *loop*.

**Definition 11** *A path is a nonempty graph $P = (V_P, E_P)$ of the form*

$$V_P = (x_0, x_1, ..., x_k), \qquad E_P = (x_0x_1, x_1x_2, ..., x_{k-1}x_k)$$

*where all the nodes $x_i$ are distinct. The vertices $x_0$ and $x_k$ are the ends of the path.*

**Definition 12** *A loop is a path with $x_0 = x_k$*

A loop in an electrical circuit is similarly defined as a closed path that does not "pass through" the same node more than once. In Figure (3.1), there are three loops.

## 3.2 Kirchhoff's laws

In 1845, German physicist Gustav Kirchhoff introduced two equalities regarding the current and voltage in an electrical circuit.

**Kirchhoff's Current Law (KCL)** dictates that the sum of the currents leaving the node is equal to the sum of the currents entering the node.

**Kirchhoff's Voltage Law (KVL)** dictates that the sum of the voltages along the branches of any closed loop is equal to zero.

These laws are what will constitute the algebraic equations in our DAE.

## 3.3 RLC components

The three components of an RLC circuit (save the voltage and current sources) are the resistor, inductor, and capacitor. The relationship between the current and voltage within each *ideal* element are described below. These are easily generalized into their nonlinear counterparts. The symbols used for each element in circuit diagrams are given in Figure 3.2.

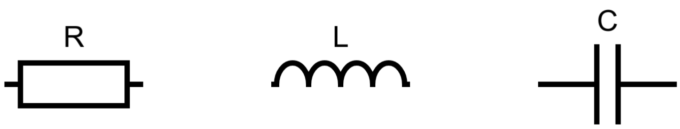


FIGURE 3.2: (From left to right) symbols for resistor, capacitor and inductor.

### 3.3.1 Resistor

Resistors help regulate the flow of current in a circuit. They can be used to, for example, prevent a too strong current from frying other elements in the circuit.

The relationship between the voltage and current within an ideal resistor was described by German physicist and mathematician Georg Simon Ohm in 1827 in a statement known as Ohm's Law:

$$v_R = Ri_R \tag{3.1}$$

Here, the constant $R$ is referred to as the *resistance* and is measured in Ohms ($\Omega$).

### 3.3.2 Inductor

An inductor stores electrical energy in the form of a magnetic field. The relationship between the voltage and current within an ideal inductor is described by the following relation:

$$\frac{d}{dt} i_L = \frac{1}{L} v_L \tag{3.2}$$

Here, the constant $L$ is referred to as the *inductance* and is measured in Henry ($H$).

### 3.3.3 Capacitor

A capacitor stores electrical energy in the form of an electrical field. The relationship between the voltage and current within an ideal capacitor is described by the following relation:

$$\frac{d}{dt} v_C = \frac{1}{C} i_C \tag{3.3}$$

Here, the constant $C$ is referred to as the *capacitance* and is measured in farads ($F$).

*NOTE: Readers may be more familiar with the induction and capacitance constants being on the left-hand side of the equations. The form used in this paper is, obviously, equivalent, and more useful to us when introducing nonlinearity into the circuit system.*

*NOTE: The above equations hold for ideal linear electrical elements. In generalizing to nonlinear electrical elements, replace the right-hand side of the equation with the corresponding nonlinear relation. E.g. for a resistor, $v_R = Ri_R$ becomes $v_R = R(i_R)$.*

### 3.3.4 Voltage and current sources

Voltage and current sources generate a voltage and current respectively. These are interpreted in the system as input variables $u(t)$. For structural reasons explained later, they are also included in the state vector $x(t)$.

## 3.4 RLC into DAE

This section details the process for translating an RLC circuit into a DAE system as it was summarized in Engelaar's paper [6] (with a minor modification regarding the voltage and current sources). Again, this framework does not explicitly include nonlinear elements, but the modification required to do so is very straightforward.

**Step 1:** Count the nodes, branches, loops and elements in your circuit.

$$\begin{cases} n = \text{number of nodes}, \\ b = \text{number of branches}, \\ m = \text{number of loops}, \\ r = \text{number of resistors}, \\ \ell = \text{number of inductors}, \\ c = \text{number of capacitors}. \\ vs = \text{number of voltage sources}, \\ cs = \text{number of current sources} \end{cases} \tag{3.4}$$

**Step 2:** Combine all of the current and voltage measurements into a state vector:

$$x = \begin{bmatrix} i_R \\ i_L \\ i_C \\ i_{VS} \\ i_{CS} \\ v_R \\ v_L \\ v_C \\ v_{VS} \\ v_{CS} \end{bmatrix} \in \mathbb{R}^{2b} \tag{3.5}$$

The subscript indicates the active element to which the current or voltage measurements pertain. If there are multiple elements of the same type (e.g. two resistors), make sure to keep their order consistent throughout the calculations.

**Step 3:** Construct the so-called All-node incidence (ANI) matrix $A_0$ and the All-loop (AL) matrix $B_0$. Each row of the ANI corresponds to an equation of the KCL, and each row of the AL corresponds to an equation of the KVL.

The ANI matrix is defined by $A_0 = \{a_{jk}\} \in \mathbb{R}^{m \times n}$

$$a_{jk} = \begin{cases} 1 & \text{if branch k leaves node j} \\ -1 & \text{if branch k enters node j} \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

The AL matrix is defined by $B_0 = b_{lk} \in \mathbb{R}^{b \times n}$ where

$$b_{jk} = \begin{cases} 1 & \text{if branch k is in loop j and has the same orientation} \\ -1 & \text{if branch k is in loop j and has the same orientation} \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

**Step 3:** For reasons more thoroughly elaborated on in Engelaar's Bachelor Thesis [6], $A_0$ and $B_0$ are not full-rank matrices. They may be reduced into such for the purposes of making future calculations more compact. For the ANI matrix $A_0$, it suffices to remove an arbitrary row, which results in a full-rank *incidence matrix* $\tilde{A}$.

For the AL matrix $B_0$, one may use any method to select $b - n + 1$ linearly independent rows from $B_0$ to construct the *loop matrix* $\tilde{B}$. (Electrical circuit theory supports this! The number of independent loops in a circuit (i.e. loops that contain at least one branch that is not part of any other independent loop) is indeed equal to the number of branches minus the number of nodes plus one. [1])

**Step 4:** Combine the Kirchhoff Laws, component laws, and voltage and current sources into matrices $E \in \mathbb{R}^{2b \times 2b}$ and $A \in \mathbb{R}^{2b \times 2b}$

$E =$

<table>
<tr><th></th><th>r</th><th>$\ell$</th><th>c</th><th>vs</th><th>cs</th><th>r</th><th>$\ell$</th><th>c</th><th>vs</th><th>cs</th></tr>
<tr><td>b+r</td><td colspan="10">0</td></tr>
<tr><td>$\ell$</td><td>0</td><td>I</td><td colspan="8">0</td></tr>
<tr><td>c</td><td colspan="7">0</td><td>I</td><td colspan="2">0</td></tr>
<tr><td>vs+cs</td><td colspan="10">0</td></tr>
</table>

$A =$

<table>
<tr><th></th><th>r</th><th>$\ell$</th><th>c</th><th>vs</th><th>cs</th><th>r</th><th>$\ell$</th><th>c</th><th>vs</th><th>cs</th></tr>
<tr><td>n-1</td><td colspan="5">$\tilde{A}$</td><td colspan="5">0</td></tr>
<tr><td>b-n+1</td><td colspan="5">0</td><td colspan="5">$\tilde{B}$</td></tr>
<tr><td>r</td><td>-R</td><td colspan="4">0</td><td>I</td><td colspan="4">0</td></tr>
<tr><td>$\ell$</td><td colspan="6">0</td><td>$1/L$</td><td colspan="3">0</td></tr>
<tr><td>c</td><td colspan="2">0</td><td>$1/C$</td><td colspan="7">0</td></tr>
<tr><td>vs</td><td colspan="8">0</td><td>I</td><td>0</td></tr>
<tr><td>cs</td><td colspan="4">0</td><td>I</td><td colspan="5">0</td></tr>
</table>

**Step 5:** The voltage and current sources are accounted for in the form of the input vector $u$:

$$u = \begin{bmatrix} u_{VS} & u_{CS} \end{bmatrix} \in \mathbb{R}^{vs+cs}$$

where $u_{VS}$ are the input voltage sources and $v_{CS}$ are the input current sources.

The input matrix is then defined as

$B =$

| | vs+cs |
|---|---|
| b+r+c+$\ell$ | 0 |
| vs+cs | -I |

**Step 6:** Steps 4-5 combine into the final form

$$E\dot{x} = Ax + Bu \tag{3.8}$$

*NOTE: The code for this construction is included in Appendix B.*

**Step 7:** If there is a nonlinear active element in the circuit, replace the relevant constant in matrix $A$ with zero and add on the nonlinearity as a column vector $g(x)$.

### 3.4.1 A note on input

In this paper, a special case of the system $E\dot{x} = Ax + Bu$ will be considered, specifically one where the input $Bu$ is treated as (part of the) inhomogeneity $f$.

Another alternative would be to assume a constant voltage and current source, i.e.

$$\frac{d}{dt} i_{CS} = \frac{d}{dt} v_{VS} = 0$$

This would require a few more modifications to the construction of $E$ and $A$ themselves.

## 3.5 Example

Consider the following circuit

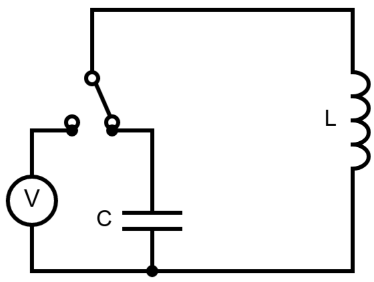


FIGURE 3.3: Example circuit with one capacitor, one inductor, one voltage source and one switch

The capacitance $C$, inductance $L$ and generated voltage $V$ are indicated on the Figure 3.3.

Define the switching signal $\sigma(t)$ as

$$\sigma(t) = \begin{cases} 1 & \text{switch is on the voltage source} \\ 2 & \text{switch is on the capacitor} \end{cases}$$

Then, the two subsystems of the complete switched SDAE are

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E_1} \begin{bmatrix} \dot{i}_L \\ \dot{i}_{VS} \\ \dot{v}_L \\ \dot{v}_{VS} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1/L & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A_1} \begin{bmatrix} i_L \\ i_{VS} \\ v_L \\ v_{VS} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} v_{VS}$$

and

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{E_2} \begin{bmatrix} \dot{i}_L \\ \dot{i}_C \\ \dot{v}_L \\ \dot{v}_C \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1/L & 0 \\ 0 & 1/C & 0 & 0 \end{bmatrix}}_{A_2} \begin{bmatrix} i_L \\ i_C \\ v_L \\ v_C \end{bmatrix}$$

If, for example, the capacitor were nonlinear $C(i_C)$, subsystem $(E_2, A_2)$ would contain a nonlinear term

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{E_2} \begin{bmatrix} \dot{i}_L \\ \dot{i}_C \\ \dot{v}_L \\ \dot{v}_C \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1/L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_2} \begin{bmatrix} i_L \\ i_C \\ v_L \\ v_C \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ C(i_C) \end{bmatrix}}_{g_2(x)}$$

# Chapter 4

# Nonlinear switched DAEs: existence and uniqueness of solutions

As mentioned in the introduction, being able to determine the (unique) solvability of a system is hugely benefitial in saving time and resources. However, in the context of nonlinear switched DAEs, this proves to be no easy feat, which is shown in Section 4.1. In their 2017 paper "Impulses in structures nonlinear switched DAEs" [8], authors Kausar and Trenn present a set of conditions under which existence and uniqueness of solution of a switched nonlinear DAE can be guaranteed. This result is presented in Section 4.3. The remainder of the chapter is dedicated to the thorough dissection of their theorem.

## 4.1 Problem

The main challenge in studying the solvability of nonlinear switched DAEs (2.1) is the nonlinear evaluation of the potentially impulsive parts of the solution. Recall that, by definition, the space of piecewise-smooth distributions $\mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$ (i.e. our solution space) is linear. Hence, if there was a Dirac impulse in the solution $x$, the nonlinear evaluation $g_\sigma(x)$ would take us out of that solution space.

To illustrate, consider a linear resistor

$$v_R = Ri_R$$

A Dirac impulse in $i_R$ would translate directly to a Dirac impulse in $v_R$, just scaled by the constant $R$. Now, consider the following instead a nonlinear resistor

$$v_R = R(i_R) = cos(i_R),$$

and imagine again a Dirac impulse in $i_R$. How would this translate in $v_R$?

So when can the Dirac impulse show up in our solution? Well, it can become an issue when so-called *inconsistent initial value* are present. For example, consider the circuit in Figure 4.1 (below).

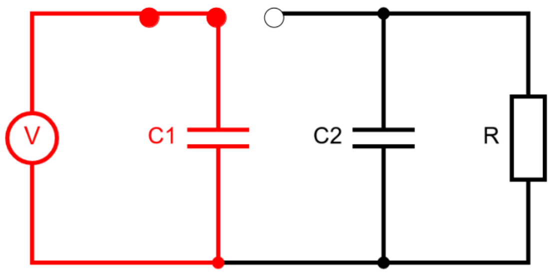


FIGURE 4.1: Circuit with one resistor, two capacitors and one voltage source

This circuit has two subcircuits that can be switched between electronically. When the switch is flipped to the left (as it currently is in Figure), the active circuit includes the voltage source and the capacitor C1. Say the capacitor is charged to voltage E. When the switch is flipped to the right, it effectively "adds" another capacitor and resistor into the active circuit. At the moment that the switch is flipped to the right, the left capacitor is supplying a voltage of E volts into node 1, and the right capacitor is supplying no voltage into node 1. But this is a loop, and by Kirchhoff's Laws, the sum of all voltages in a loop is equal to zero. Hence, the problem has an inconsistent initial value at the time of switching from left to right. [15] [2]

This inconsistency is equalized instantaneously by a Dirac impulse. Hence, the presence of inconsistent initial conditions is conducive to Dirac impulses in the solution. Consequently, when looking at a switched nonlinear DAE of this form (2.1), a (unique) solution cannot always be guaranteed.

## 4.2 Existence and uniqueness of solution: informal reasoning

In their 2017 paper "Impulses in structured nonlinear switched DAEs", Kausar and Trenn bypass the problem descrived in the previous section by assuming that the nonlinearity $g_\sigma(x)$ is sparse in such a way that it "overlooks" the possibly impulsive parts of $x$. Following this, they show that under some additional conditions, the original DAE (2.1) can be "split up" into an equivalent form comprised of three DAEs, for each of which one can verify existence and uniqueness of solutions using preexisting theory.

## 4.3 Existence and uniqueness of solution: formal result

The formal existence and uniqueness result for nonlinear DAEs derived by authors Kausar and Trenn is as follows.

**Theorem 1** *For $\omega \in [0, \infty)$, consider the local nonlinear ITP*

$$\begin{cases} x_{(-\infty,0)} = x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\omega)} = (Ax + g(x) + f)_{[0,\omega)} \end{cases} \tag{4.1}$$

*with initial trajectory $x^0 \in \mathbb{D}^n_{pwC^\infty}$. If the following assumptions hold:*

*(R): $(E, A)$ is regular,*

*(F): The inhomogeneity $f$ is induced by a piecewise-smooth function $\overline{f} : \mathbb{R} \to \mathbb{R}^n$, i.e. $f = \overline{f}_{\mathbb{D}}$,*

*(S): $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and picewise-smooth,*

*(G): $\exists \overline{g} : \mathbb{R}^{m_g} \to \mathbb{R}^{n_g}$ $\exists \mathcal{M} \in \mathbb{R}^{m_g \times n}$ $\exists \mathcal{N} \in \mathbb{R}^{n \times n_g}$ $\forall \xi \in \mathbb{R}^n$ such that $g(\xi) = \mathcal{N}\overline{g}(\mathcal{M}\xi)$,*

*(M): $\mathcal{M}E^{imp} = 0$,*

*(N): $im\, \mathcal{N} \subseteq im\, E$,*

*Then there exists $\omega > 0$ such that the local nonlinear ITP (4.1) has a unique solution $x \in \mathbb{D}^n_{pwC^\infty}$ on $(-\infty, \omega)$.*

*NOTE: A particularly cautious reader may not feel entirely convinced by the final statement. After all, $\omega$ may turn out to be very small, thus severely (or entirely) limiting the real-life applicability of the theorem. This is, unfortunately, the case for many results regarding existence and uniqueness of solutions of systems of differential equations. A possible way to bypass this issue is to extend the solution by re-using the initial value. This, of course, breaks down if our solution is undefined at $\omega$ or is constantly increasing. In this case the problem has what is known as a* maximal solution.

Following this, they formulated a corollary that extends the above result to switched nonlinear DAEs in the form (2.1).

**Corollary 1** *Consider the switched DAE (2.1) such that $\forall p$ the subsystem satisfies the conditions stated in Theorem 1. Then, for any initial trajectory $x^0 \in \mathbb{D}^n_{pwC^\infty}$ on $(-\infty, 0)$, there exists a unique distributional solution $x \in \mathbb{D}^n_{pwC^\infty}$ of (2.1) defined on $(\omega, \infty)$ for some finite $\omega \in \mathbb{R}$.*

## 4.4 Explanation

The purpose of this section is to convince the reader of the relevance and necessity of each of the six conditions listed in 4.1. For a rigorous and very comprehensive proof, the reader is welcome to consult page 3184 in the original paper by Kausar and Trenn [8].

Recall that Section 4.3 outlined two main assumptions for unique solvability:

1. The nonlinearity $g_\sigma$ is sparse in such a way that it "overlooks" the potentially impulsive parts of $x$,
2. The DAE (2.1) can be split up into three uniquely solvable systems.

Each condition in Theorem 1 works to translate these assumptions into a mathematically rigorous form.

### 4.4.1 QWF

Before any explanation is to proceed, several formal definitions and concepts must be introduced.

The first condition **(R)** in Theorem 1 is regularity of matrix pair $(E, A)$.

**Definition 13** *A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is said to be regular if $det(s \cdot E - A)$ is not the zero polynomial.*

A consequence of regularity is that the matrix pair can necessarily be transformed into the *quasi-Weierstrass form (QWF)*.

**Proposition 1** *A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular iff there exist invertible transformation matrices $S, T \in \mathbb{R}^n$ that put $(E, A)$ into quasi-Weierstrass form (QWF)*

$$(SET, SAT) = \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{4.2}$$

*where $N \in \mathbb{R}^{n_2\times n_2}$ is nilpotent and $J \in \mathbb{R}^{n_1\times n_1}$ with $n_1 = n - n_2$ is some matrix.*

In [3] it was shown that a convenient method to obtain the transformation matrices $S, T$ is through the Wong sequences.

**Theorem 2** *Let $(E, A)$ be a regular matrix pair. Define the Wong sequences as*

$$\begin{aligned} \mathcal{V}^0 &= \mathbb{R}^n, \quad \mathcal{V}^{i+1} = A^{-1}(E\mathcal{V}^i), \\ \mathcal{W}^0 &= 0, \quad \mathcal{W}^{i+1} = E^{-1}(A\mathcal{W}^i), \end{aligned}$$

*where*

$$\begin{aligned} M^{-1}(N) &= \{x \in \mathbb{R}^n \mid Mx \in N\} = \textit{pre-image of } N \subseteq \mathbb{R}^n \textit{ under } M, \\ N(M) &= \{y = Mx \in \mathbb{R}^n \mid x \in N\} = \textit{image of } M \subseteq \mathbb{R}^n \textit{ under } N. \end{aligned}$$

*These sequences converge after the same number of finite steps. Denote the limits with $\mathcal{V}^*$ and $\mathcal{W}^*$. Choose full rank matrices $V$ and $W$ such that*

$$\begin{aligned} im\ V &= \mathcal{V}^* \\ im\ W &= \mathcal{W}^* \end{aligned}$$

*Then, the transform matrices*

$$\begin{aligned} T &= \begin{bmatrix} V & W \end{bmatrix} \\ S &= \begin{bmatrix} EV & AW \end{bmatrix}^{-1} \end{aligned}$$

*are invertible and put $(E, A)$ into quasi-Weierstrass form.*

*NOTE: A MATLAB code for finding the QWF of a matrix pair $(E, A)$ was derived by Trenn in his 2009 PhD Thesis "Distributional Differential Algebraic Equations" [13]. This code was used in calculations in Chapter 4 of this paper and is included in full is Appendix B for convenience of reference.*

### 4.4.2 Nonlinearity $g_\sigma$ is sparse

(R): $(E, A)$ is regular,

(G): $\exists \overline{g} : \mathbb{R}^{m_g} \to \mathbb{R}^{n_g}\ \ \exists \mathcal{M} \in \mathbb{R}^{m_g \times n}\ \ \exists \mathcal{N} \in \mathbb{R}^{n \times n_g}\ \ \forall \xi \in \mathbb{R}^n$ such that $g(\xi) = \mathcal{N}\overline{g}(\mathcal{M}\xi)$,

(M): $\mathcal{M}E^{\text{imp}} = 0$

Based on the QWF of $(E, A)$ (condition **(R)**) one defines the *impulse projector*:

$$\Pi^{\text{imp}} = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S \tag{4.3}$$

and the *impulse matrix*

$$E^{\text{imp}} = \Pi^{\text{imp}} E \tag{4.4}$$

The usefullness of this matrix becomes more apparent when compared with the linear analogue of our DAE system. Consider the following linear ITP:

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x}))[0,\infty) &= (Ax + f)_{[0,infty)} \end{aligned}$$

From existing theory [13] [14] one can obtain the explicit solution formula for the linear ITP. Specifically, if $f$ is induced by a piecewise-smooth function, the impulsive part of $x$ is defined as as:

$$x[0] = -\sum_{i=0}^{n-1} (E^{\text{imp}})^i x^0(0^-)\delta^{(i)} - \sum_{i=0}^{n-1} (E^{\text{imp}})^i \sum_{j=0}^{i} f^{(i-j)}(0^+)\delta^{(j)} \tag{4.5}$$

where $\delta^{(i)}$ denotes the $i^{th}$ derivative of the Dirac impulse $\delta$.

$x[0]$ is a linear combination of Diracs and their derivatives, i.e. $x[0] = x_0\delta + x_1\dot{\delta} + x_2\ddot{\delta}$. Then, from 4.5 it is clear that there exist $y_0, y_1, y_2$ such that $x_i = E^{\text{imp}} \cdot y_i$ for $i = 1, 2, 3$. Hence, condition **(M)** would imply that

$$\begin{aligned} \mathcal{M}x[0] &= \mathcal{M}E^{imp} \sum y_i \\ &= 0 \times \sum y_i \quad \textit{condition (M)} \\ &= 0 \end{aligned}$$

If our nonlinearity is indeed in the form **(G)**, then $Mx[0] = 0$ implies that $g$ "overlooks" the potentially impulsive part of $x$ at the time of the inconsistent initial value and can be evaluated even for distributional $x$.

### 4.4.3 DAE can be split up into three uniquely solvable DAEs

(R): $(E, A)$ is regular,

(F): The inhomogeneity $f$ is induced by a piecewise-smooth function $\overline{f} : \mathbb{R} \to \mathbb{R}^n$, i.e. $f = \overline{f}_{\mathbb{D}}$,

(S): $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and picewise-smooth,

(N): $\operatorname{im} \mathcal{N} \subseteq \operatorname{im} E$,

(M): $\mathcal{M}E^{\text{imp}} = 0$

The regularity **(R)** of $(E, A)$ once again allows us to consider the QWF of $(E, A)$. Specifically, it can be shown that there exists a special case of the transformation matrices $S, T$ such that the matrix pair $(E, A)$ assumes the special form

$$(SET, SAT) = \left( \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N_1 & N_2 \end{bmatrix}, \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right) \tag{4.6}$$

$$T^{-1}x = \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}$$

Plugging this into our DAE (2.2),

$$
\begin{aligned}
E\dot{x} &= Ax + g(x) + f \\
ET\begin{bmatrix} \dot{v} \\ \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} &= AT\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix} + g\left(T\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}\right) + f \\
SET\begin{bmatrix} \dot{v} \\ \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} &= SAT\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix} + Sg\left(T\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}\right) + Sf \\
\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N_1 & N_2 \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} &= \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} S^v \\ S_1^w \\ S_2^w \end{bmatrix} g\left(T\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}\right) + \begin{bmatrix} S^v \\ S_1^w \\ S_2^w \end{bmatrix} f \\
&\begin{cases} \dot{v} = Jv + S^v g(x) + S^v f \\ 0 = w_1 + S_1^w g(x) + S_1^w f \\ N_1\dot{w}_1 + N_2 w_2 = w_2 + S_2^w g(x) + S_2^w f \end{cases}
\end{aligned}
$$

Each new equation corresponds to it's own ITP. The remaining conditions **(F)**, **(S)** and **(N)** allow us to use solution theory [10] to verify the unique solvability of each of the three new ITPs. Namely, note that the first condition is just an ODE and can be solved for $v$ accordingly. Then, condition **(N)** implies that $S_1^w g(x) = 0$, making the second equation directly solvable for $w_1$. Lastly, by condition **(M)** it follows that $g(x)$ is independent of $S_2^w$. Hence, the third equation can be solved using known $f$, $w_1$ and $g(x)$ since $\begin{bmatrix} N_1 & N_2 \end{bmatrix}$ is nilpotent [13].

# Chapter 5

# Results

The solution theory of switched nonlinear DAEs in the 2017 paper by Krausar and Trenn was developed in the context of modelling water distribution systems (the main concern was that of the water hammer effect, which occurs when a large quantity of flowing water is suddenly halted and has the potential to cause serious damage to the hydraulic infrastructure) [9]. This section is an investigation into the application of this theorem to RLC circuits.

Section 5.2 considers what happens when the resistor(s) are made nonlinear. Section 5.3 contains further examples to illustrate the complexity of the problem. The remaining sections outline potential caveats related to the structure of the DAE model used and how this may be inhibiting an effective application of Theorem 1.

## 5.1 Note on examples

In order to evaluate the application of Theorem 1 to switched nonlinear RLC circuit models, one does not need to study complete *switched* nonlinear RLC circuit examples. Recall that, as per Corollary 1, the result 1 would need to hold for every subcircuit in a switched system.

## 5.2 Resistors

An immediate consequence of using the DAE framework described in Chapter 3 is that making any resistor nonlinear violates condition $(N)$ of Theorem 1. To illustrate, consider the general structure of a DAE

$$\underbrace{\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&I&0&0&0\\ 0&0&I&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}}_{E}\dot{x} = \underbrace{\begin{bmatrix} \multicolumn{5}{c}{\tilde{A}} &0&0&0&0&0\\ 0&0&0&0&0& \multicolumn{5}{c}{\tilde{B}}\\ 0&0&0&0&0&I&0&0&0&0\\ 0&I&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&I&0&0\\ 0&0&0&0&0&0&0&0&I&0\\ 0&0&0&0&I&0&0&0&0&0 \end{bmatrix}}_{A}x + \underbrace{\begin{bmatrix} 0\\ 0\\ -R(i_R)\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}}_{\mathcal{N}\bar{g}(\mathcal{M}x)} + f$$

(This is by no means a rigorous sketch of the structure, but it works well enough to illustrate the result.)

Comparing matrix $E$ and nonlinearity $g(x) = \mathcal{N}\overline{g}(\mathcal{M}x)$, one observes that

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{E} \qquad \underbrace{\begin{bmatrix} 0 \\ 0 \\ -R(i_R) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathcal{N}\overline{g}(\mathcal{M}x)}$$

In the simplest case, $\mathcal{N} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, and $\operatorname{im}(\mathcal{N}) \subseteq \operatorname{im}(E)$ is clearly not satisfied. It is easy to see that any other choice of $\mathcal{N}$ it is similarly unsatisfactory. Hence, in any scenario with a nonlinear resistor, Theorem 1 cannot be used to ensure (unique) solvability.

## 5.3 Capacitors and Inductors

The rigorous nonlinear DAE structure detailed in Chapter 3 leads us to believe that there may be some compact proof that would allow us to make a concrete judgement on the solvability of some class of RLC circuits. However, so far neither the theoretical nor empirical results supported the existence of any such result.

As an example, consider the following circuit

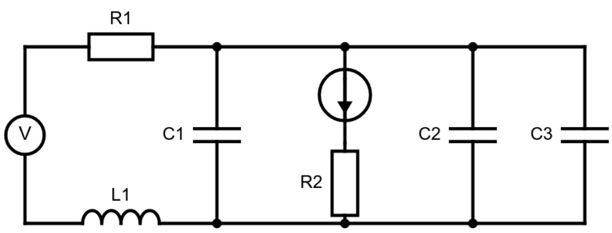


FIGURE 5.1: Circuit with two resistors, three capacitors, one inductor, one voltage source and one current source

If the inductor is the only nonlinear element in the circuit, i.e. the nonlinearity takes the form

$$g(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L(v_L) & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathcal{M}E^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

i.e. condition **(M)** of Theorem 1 is satisfied.

However, if the inductor *and* capacitor 3 to be nonlinear, i.e. the nonlinearity takes the form

$$g(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & C3(i_{C3}) & 0 & 0 & 0 & 0 & L(v_L) & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the calculations show that

$$\mathcal{M}E^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C1 & -C2 & C1+C2, 0, 0], & & \end{bmatrix}$$

and hence the condition **(M)** is not satisfied.

## 5.4 Caveat: regularity

The regularity condition (R) is necessary and sufficient for the existence of the necessary QWF. For smaller circuits (e.g. 2-4 active components), empirical evidence shows the regularity condition holds by default as there is a term independent of the resistance, inductance or capacitance constants in the polynomial $\det(s \cdot E - A)$. However, for more complicated circuits there is a distinct absence of such an independent term, and there is hence a constraint on which (combinations of) circuit elements can be made nonlinear.

Take as an example the circuit from Figure 4.1. Calculations show that
For Figure 5.1, running the code reveals that the regularity result is

$$\begin{aligned} \det(s \cdot E - A) = {} & C1 \cdot C2 \cdot s^2 + C1 \cdot C3 \cdot s^2 + C2 \cdot C3 \cdot s^2 + C1 \cdot C2 \cdot C3 \cdot L1 + \\ & C1 \cdot C2 \cdot L1 \cdot R1 \cdot s + C1 \cdot C3 \cdot L1 \cdot R1 \cdot s + C2 \cdot C3 \cdot L1 \cdot R1 \cdot s \end{aligned}$$

Recall that regularity requires that this polynomial is not equal to the zero polynomial. However, note that there are no terms independent of either the capacitance or inductance. Hence, there is some limitation to the elements which can be made nonlinear, e.g. making capacitor 1 and capacitor 2 nonlinear makes $(E, A)$ irregular, but making capacitor 2 and inductor 1 nonlinear does not affect regularity.

Another related result is that circuits may be divided up into a set of independent loops in different ways [1]. Hence, the loop matrix $\tilde{B}$ may be different for the same circuit, which would in turn affect the regularity result $\det(s \cdot E - A)$.

However, a sort of "loophole" can be employed that will allow any problem to be regular. In examples thus far, to convert a linear element to a nonlinear one, the corresponding constant (resistance $R$, inductance $L$ or conductance $C$) was "removed" from the $A$ matrix and "added" it back into the system in the form of a nonlinear function in $g(x)$. However, one could instead leave matrix $A$ unmodified, and include a term in $g(x)$ such that it cancels the "unwanted" constant in $A$. For example, a nonlinear resistor would result in a nonlinearity $g(x)$ such that

$$g(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R_f(i_R) + R \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Of course, this "loophole" would imply that the DAE matrices $(E, A)$ remain the same regardless of which circuit elements are nonlinear. Moreover, the QWF transformation matrices $S, T$ and, subsequently, the impulse matrix $E^{\text{imp}}$ remain the same throughout.

## 5.5 Caveat: reducing system size

Another way in which it may be possible to bypass the non-regularity of pair of system matrices $(E, A)$ is by reducing the system size. As an example, consider a circuit containing a nonlinear resistor and a linear inductor only. Then, the following equations would be active:

$$\begin{aligned} i_R &= i_L \\ v_R + v_L &= 0 \\ v_L &= \frac{1}{L}\frac{d}{dt} i_L \\ v_R &= R(i_R) \end{aligned}$$

It would be possible to express this system in the following reduced form

$$\frac{1}{L}\frac{d}{dt} i_L = -R(i_L)$$

Similar calculations on irregular systems could reduce the size of the system and potentially bring it into a regular form. Of course, this would then warrant an investigation into exactly how much information about the system would be lost in this way.

# Chapter 6

# Conclusion

In this paper the uniqueness and solvability of switched nonlinear DAEs was considered. Specifically, it was investigated if the uniqueness and solvability result derived by Krausar and Trenn in [8] could be applied to switched nonlinear DAE models of RLC circuits.

In Chapter 1, the notion of switched nonlinear DAES was introduced, with a particular focus being made on the solution space of piecewise-smooth distributions. In Chapter 2, relevant background information on RLC circuits was introduced, and a framework for translating a nonlinear RLC circuit into a nonlinear DAE was introduced.

Chapter 3 discussed the existence and uniqueness result derived by Kausar and Trenn in their paper "Impulses in structured nonlinear switched DAEs" [8]. Lastly, attempts (however successful) at combining the nonlinear RLC circuits were presented in Chapter 4.

# Appendix A

# RLC into DAE

## A.1 MATLAB code for generating the DAE form of an RLC circuit

```
%%%%%%%%%%%%%%%% Preliminaries %%%%%%%%%%%%%%%%

disp(' ');

n = input("How many elements does your circuit have? ANSWER
   : ");

oldE = zeros(2*n,2*n);
E = sym(oldE);

oldA = zeros(2*n,2*n);
A = sym(oldA);

disp(' ');


%%%%%%%%%%%%%%%% Resistors %%%%%%%%%%%%%%%%

nR = input('How many resistors does your circuit have?
   ANSWER: ');
syms(sym('R',[1,nR]));

disp(' ');

if nR==0
    RES = 0;
else
    disp('Type in the resistance values in order.')
    disp('Please use the following notation (brackets and
       commas')
    disp('included): [R1, R2, ..., ].')
    disp('If the exact values are unknown, use "R1" etc. ')
    RES = input('ANSWER: ');
end

disp(' ');
```

```
%%%%%%%%%%%%%%%%% Capacitors %%%%%%%%%%%%%%%%%

nC = input('How many capacitors does your circuit have?
   ANSWER: ');
syms(sym('C',[1,nC]));

disp(' ');

if nC==0
    CAP = 0;
else
    disp('Type in the capacitance values in order.')
    disp('Please use the following notation (brackets and
       commas')
    disp('included): [C1, C2, ..., ].')
    disp('If the exact values are unknown, use "C1" etc. ')
    CAP = input('ANSWER: ');
end

disp(' ');


%%%%%%%%%%%%%%%%% Inductors %%%%%%%%%%%%%%%%%

nL = input('How many inductors does your circuit have?
   ANSWER: ');
syms(sym('L',[1,nL]));

if nL==0
    IND = 0;
else
    disp('Type in the inductance values in order.')
    disp('Please use the following notation (brackets and
       commas')
    disp('included): [L1, L2, ..., ].')
    disp('If the exact values are unknown, use "L1" etc. ')
    IND = input('ANSWER: ');
end

disp(' ');


%%%%%%%%%%%%%%%%% Voltage sources %%%%%%%%%%%%%%%%%

nVS = input('How many voltage sources does your circuit
   have? ANSWER: ');

disp(' ');
```

```
%%%%%%%%%%%%%%%% Current sources %%%%%%%%%%%%%%%%

nCS = input('How many current sources does your circuit
   have? ANSWER: ');

disp(' ');


%%%%%%%%%%%%%%%%

% At this point, we have two empty matrices,
%     E,
%     A,
% five constants
%     nR -- number of resistors,
%     nCA -- number of capacitors,
%     nL -- number of inductors,
%     nV -- number of voltage sources,
%     nCU -- number of current sources,
% and (max) three arrays
%     RES -- array containing the resistances,
%     CAP -- array containing the capacitances,
%     IND -- array containing the inductances.

%%%%%%%%%%%%%%%%


%%%%%%%%%%%%%%%% Filling in E %%%%%%%%%%%%%%%%

for i=1:nL
    E(n+nR+i,nR+i) = 1;
end

for i=1:nC
    E(n+nR+nL+i,n+nR+nL+i) = 1;
end


%%%%%%%%%%%%%%%% Filling in A %%%%%%%%%%%%%%%%

for i=1:nR
    A(n+i,i) = -RES(i);
    A(n+i,n+i) = 1;
end

for i=1:nL
    A(n+nR+i,n+nR+i) = IND(i);       % Should technically be
       1/IND(i)
end
```

```
for i=1:nC
    A(nR+nL+i,nR+nL+i) = IND(i);      % Should technically
       be 1/CAP(i)
end

for i=1:nVS
    A(n+nR+nC+nL+i,n+nR+nC+nL+i) = 1;
end

for i=1:nCS
    A(n+nR+nC+nL+nVS+i,nR+nC+nL+nVS+i) = 1;
end


%%%%%%%%%%%%%%%% Word of caution %%%%%%%%%%%%%%%%

disp('I will now ask you questions regarding the
   relationship between the branches, nodes, and loops in
   your circuit.');

disp('When answering, please adhere to the following branch
    order: Resistor(s) --> Inductor(s) --> Capacitor(s) -->
    Voltage source(s) --> Current source(s).');

prompt = input('Input [1] to continue. ANSWER: ');

disp(' ');


%%%%%%%%%%% Filling in A: Branches and nodes %%%%%%%%%%%

nodes = input('How many nodes does your circuit have?
   ANSWER: ');

disp(' ');

branches = input('How many branches does your circuit have?
    ANSWER: ');

disp(' ');

ATemp = zeros(nodes,branches);

disp('The next questions all ask "does branch x Enter (-1),
    Leave (1), or Not Connect (0) with node y?');
disp('Answer with -1 (Enter), 1 (Leave) or 0 (Does Not
   Connect).');
disp(' ');

for j=1:nodes
    for k=1:branches
```

```
        X = sprintf('Branch %d and node %d?',k,j);
        disp(X);
        prompt = input('ANSWER (-1/1/0): ');
        ATemp(j,k) = prompt;
    end
end

ANew = ATemp(1:(nodes-1),:); % Arbitrarily deleting one row


%%%%%%%%%%% Filling in A: Branches and loops %%%%%%%%%%%

disp(' ')

loops = input('How many loops does your circuit have?
   ANSWER: ');

disp(' ');

BTemp = zeros(loops,branches);

disp('The next questions all ask "Does branch x have the
   same orientation as loop y?');
disp('Answer with 1 (Same orientation), -1 (Different
   orientation) or 0 (Not in this loop).');
disp(' ');

for j=1:loops
    for k=1:branches
        X = sprintf('Branch %d and loop %d?',k,j);
        disp(X);
        prompt = input('ANSWER (1/-1/0): ');
        BTemp(j,k) = prompt;
    end
end

if rank(BTemp)==1  % Extracting linearly independent rows
    BNew = BTemp;
else
    BNew = licols(transpose(BTemp),1e-10);
end


%%%%%%%%%%% Filling in A: Final compilation %%%%%%%%%%%

A(1:(nodes-1),1:n) = ANew;

A(nodes:n,(n+1):2*n) = transpose(BNew);
```

## A.2 MALAB code for extracting linearly independent subset of matrix rows [7]

```
function [Xsub,idx]=licols(X,tol)

if ~nnz(X) %X has no non-zeros and hence no independent
   columns
    Xsub=[]; idx=[];
     return
end
if nargin<2, tol=1e-10; end
[Q, R, E] = qr(X,0);
if ~isvector(R)
 diagr = abs(diag(R));
else
 diagr = R(1);
end
%Rank estimation
r = find(diagr >= tol*diagr(1), 1, 'last'); %rank
   estimation
idx=sort(E(1:r));
Xsub=X(:,idx);

end
```

# Appendix B

# QWT

*NOTE: These scripts come from "Distributional Differential Algebraic Equations" by Stephan Trenn. [8] They are included here for convenience of reference.*

## B.1 MATLAB code for calculating a basis of the preimage $A^{-1}(\text{im}S)$ for some matrices $A$ and $S$

```
function V=getPreImage(A,S)

    [m1,n1]=size(A);
    [m2,n2]=size(S);

    if m1==m2 || m2==0
        H1=null([A,S]);
        H2=H1(1:n1,:);
        H3=sym(H2);
        V=colspace(H3);
    else
        error('Both matrices must have the same number of
           rows');
    end

end
```

## B.2 MATLAB code for calculating a basis of the space $\mathcal{V}*$

```
function V=getVspace(E,A)

    [m,n]=size(E);

    if (m==n) & size(E)==size(A)
        V=eye(n,n);
        oldsize=n;
        newsize=n;
        finished=0;
        while finished==0;
            symEV=sym(E*V);
            EV=colspace(symEV);
```

```
            V=getPreImage(A,EV);
            oldsize=newsize;
            newsize=rank(V);
            finished=(newsize==oldsize);
        end;
    else
        error('Matrices E and A must be square and of the
           same size');
    end

end
```

## B.3 MATLAB code for calculating a basis of the space $\mathcal{W}*$

```
function W=getWspace(E,A)

    [m,n]=size(A);

    if (m==n) & size(E)==size(A)
        W=zeros(n,1);
        oldsize=n;
        newsize=n;
        finished=0;
        while finished==0;
            symAW=sym(A*W);
            AW=colspace(symAW);
            W=getPreImage(E,AW);
            oldsize=newsize;
            newsize=rank(W);
            finished=(newsize==oldsize);
        end;
    else
        error('Matrices E and A must be square and of the
           same size');
    end

end
```

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