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UNIFYING THE DESCRIPTION OF FREE SYSTEMS WITH LIE GROUP ANALYSIS

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Abstract

This paper unifies rigid bodies and fluid flow through Lie group analysis. By exploiting similarities in the symmetries of their Lagrangian, the study reveals that their behavior can be seen as geodesic motion on their respective configuration spaces when an appropriate metric is chosen. Using the Euler-Poincaré reduction theorem, the Euler-Arnold equations for rigid bodies and fluids are derived. Additionally, different systems described by Lie group analysis with various groups and metrics are mentioned, demonstrating the framework's versatility. The behavior of the Hopf/inviscid Burgers' equation and the Euler ideal flow equation is derived, confirming their adherence to the Lie group analysis framework.

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Contents

1 Mathematical introduction

Lie group analysis, a branch of mathematics utilizing Lie groups, offers valuable insights into the analysis of various systems. In this thesis, I use Lie group analysis and build upon Arnold's work [1] to present a comprehensive mathematical framework that unifies the description of different free physical systems. The objectives of this thesis are as follows: (1) to introduce the Euler-Arnold equation, (2) to establish the equivalence between this new framework and the existing descriptions of these physical systems, and (3) to determine the applicability of Lie group analysis to different types of systems.

The thesis starts of by introducing certain mathematical concepts. First are *groups and manifolds* which are fundamental mathematical tools used to model a wide range of physical systems. Manifolds describe the space of states in which a system can exist, while groups capture the symmetries exhibited by physical systems. On a manifold, *a tangent vector* represents the rate of change of a curve. For instance, in a fluid, the velocity of a particle corresponds to a tangent vector on the container that confines the fluid. The velocities of all particles collectively give rise to the velocity field. The length of such a tangent vector can then be measured using a *metric*. The metric is used to convert a tangent vector (a velocity) into a number (a speed). It can also be used to find an inner product between two tangent vectors, this is a measure for how much the two vectors point in the same direction. Finally, by integration, the metric can be used to find the length of a curve on the manifold. Curves that extremize the length are called *geodesics*. They are usually seen when describing free physical systems. The paths of free particles, light rays, or fluids can all be seen as geodesics.

To mathematically rigorously describe geodesics one requires the *connection*. The connection connects nearby tangent spaces to each other and in doing so allows us to find the acceleration of a curve. Previously, the velocity vectors at different points in time were also tangent vectors to different parts of the manifold so they could not be directly compared. The connection allows for such comparisons. The geodesic equation states that there should be no acceleration.

The specific systems in this thesis all have the additional property that their configuration is a Lie group. This means that they are a group *and* a manifold. In some sense, this means that the manifold that forms the configuration space looks the same from any point. This is the case for example for a particle in empty space, but not for a particle inside a finitely sized container.

In this thesis, we will discuss in detail how Lie group analysis can be used to arrive at two first order equations which together describe the behavior of a rigid body. The rigid body's configuration space consists of translations and rotations of the body in N-dimensional space. This is the Lie group $\mathrm{SE}(N)$. We will find a metric so that the equation of motion of the rigid body is exactly a geodesic on $\mathrm{SE}(N)$ by exploiting the similarity between geodesics and the least action principle.

The metric that we will obtain will necessarily be left-invariant, as a left-multiplication on $\mathrm{SE}(N)$ is equivalent to change the coordinates in which we view the system. Therefore, it cannot change the energy. This symmetry allows for the use of Noether's theorem which will reduce our system to the two first order equations. Namely, a first order equation that states that the momentum is preserved when brought back to the Lie algebra by right-multiplication, and another equation stating that the derivative of the position is the velocity.

We will generalize this to the Euler-Poincaré reduction theorem, which applies to all geodesics under a left-invariant metric. We will also construct an alternative equation which describes the change in velocity rather than momentum. Here, velocity refers to the derivative of the position whereas momentum refers to the dual of the velocity. Thereafter, this Euler-Poincaré reduction theorem will be applied to several fluid systems. Fluids have a right- rather than left-invariant metric which corresponds to a relabeling of the particles, however we will provide a slight modification to the Euler-Poincaré reduction theorem so that it can still be applied.

2 Physical introduction

Free systems are physical systems that are not affected by any external potential and whose Lagrangian depends entirely solely on time-derivatives of the generalized coordinates. The behavior of these systems can be simpler than that of systems that do experience a potential, but nevertheless is far from uninteresting. A rigid body is one example of a system that can display many complicated types of behavior. The intermediate axis (or tennis racket theorem) comes to mind as an example of interesting, non-intuitive behavior displayed by a freely rotating rigid body. Fluids too are a free system (when gravity is absent or neglected).

Both fluids and rigid bodies have been studied by Euler who has found two Euler's equations. Euler's equation for a rigid body is derived based on the conservation of angular momentum and describes the time evolution of a rigid body's angular momentum when the rigid body rotates around a fixed point with some external torque applied. Euler's equation for a fluid describe an ideal fluid which means that the fluid is incompressible and inviscid. Notably, Euler's equations for both rigid bodies and fluids do include an external forcing term which means the systems described by Euler were not necessarily free.

Specifically when these external force terms are set to zero, the systems are free and interestingly the different physical systems can be described through an almost equivalent mathematical equation. This specific equation is obtained by simplifying the physical description from an embedding of the physical object in 3-dimensional space into a description of the system as a single point in the configuration space of the object. For example, in the case of a 3-dimensional rigid body rotating about a fixed point, the configuration space is the space of rotations SO(3). Then a reformulation of Lagrangian mechanics is made on this configuration space so that the equation of motion of the physical system becomes a geodesic on the configuration space.

That geodesic can be simplified to a first order using a method called the Euler-Poincaré reduction which is similar to how the Euler-Lagrange equation is obtained. The resulting first order equation is called the Euler-Arnold equation and it describes the behavior of any free physical system. Although the Euler-Arnold equation can be easily derived for a very small group of systems, it turns out that many more systems (mainly simplifications or extensions of fluid dynamics) can be seen as the Euler-Arnold equation on some manifold under some metric. This is interesting and hints at fundamental similarities between these different systems.

This thesis will end by covering the different fluid systems of which it is known that they can be described using Lie group analysis. The Lie groups and metrics that are used for this will be explained and for several examples, the methods shown in the earlier sections will be applied to derive equations such as the Hopf/inviscid Burger's equation and the Euler Ideal flow equation.

Additionally, the relation between the Lie group analysis and the conserved quantities of several systems will be discussed. In particular, the conservation of momentum and angular momentum for a rigid body rotating about its center of mass and the vorticity equation, which states that the vorticity (a generalization of the curl) of a fluid moves with the fluid as if it was frozen in.

3 Manifolds

This section contains superficial rehearsal of the definitions of a manifold. For more detailed and intuitive explanations refer to [2].

A manifold is a mathematical object that can describe various things. It can represent an object embedded in Euclidean space, like a shape, or it can describe a group of symmetries, like the different ways you can rotate a ball. When we say it's "continuous," we mean that it can be described using a set of continuously changing parameters [2].

To define a finite-dimensional manifold, we say that it is a set called M that looks locally like Euclidean space. This means that we can divide it into smaller parts, each of which can be mapped to a piece of Euclidean space using a set of rules called an atlas. The atlas consists of pairs of sets U_i and maps ϕ_i that convert points in U_i to points in Euclidean space. The sets U_i cover the entire manifold, and the maps ϕ_i can smoothly transition between each other.

Definition 3.1. A n-dimensional smooth manifold is a set M which is locally Euclidean. That means that there is an atlas $\mathcal{A} = \big\{(U_i, \phi_i) : i \in I\big\}$ for some indexing set I. So that the following conditions hold.

- U_i is an open subset of M for all $i \in I$;
- ϕ_i is an injective map from U_i to $\mathbb{R}^n$ (or to the upper half plane $\mathcal{H}^n$ in case of a manifold with boundary);
- the U_i form an open cover of M;
- $\phi_i \circ \phi_j^{-1}$ is in C^∞ for all $i, j \in I$.

Not all manifolds can however be parametrized by a finite number of parameters. If this is not possible, the object in question may still be an infinite dimensional manifold. An infinite dimensional manifold is a space that is locally diffeomorphic to a Banach space or a Fréchet space which can be seen as the infinite dimensional analogues to Euclidean space. Infinite dimensional manifolds are generally much nastier to work with. The description of fluids does require specifically Fréchet manifolds however we will not go into much detail here [3].

3.1 Vector fields

Vector fields are important in physics and have various applications. For example, electric and magnetic fields, as well as the gravitational field in Newton's theory, are all examples of vector fields. In more complex theories, we also encounter tensor fields, which are a generalization of vector fields. But let's start by understanding what a vector field is, which requires us to first define a vector.

A vector is a special object that belongs to something called the tangent space to a manifold (denoted as TM where M is the manifold). If we imagine a manifold embedded in a larger space like $\mathbb{R}^n$, the tangent space at a point can be thought of as a flat plane that touches the manifold at that point. It's like a sheet of paper lying flat on a curved surface. But instead of thinking about it visually, we often think of the tangent space as a space that contains derivations along curves. These derivations must have certain properties namely linearity and obeying the product rule. On a functional v from $C^1(M, \mathbb{R})$ the conditions for v being a functional are

$$v(\lambda f + g) = \lambda v(f) + v(g) \quad \text{and} \quad v(fg) = fv(g) + gv(f),$$

where $f, g \in C^1(M, \mathbb{R})$ are continuous functions from the manifold to the real numbers and λ is a real number. There are several complications to this definition such as the fact that the elements do not *really* act on functions in $C^1(M, \mathbb{R})$ but rather on equivalence classes that are the same sufficiently close to a certain point. We say that such functions have the same 'germ'. For more details and an explanation of what a 'germ' is, see [2].

Using this definition, we can think of a tangent vector as something that acts on a function and outputs a number. We do this by looking at a curve $\gamma : [a, b] \to M$ and taking its derivative. The derivative of a curve tells us how the function changes as we move along the curve. We can write this as

$$\dot{\gamma}(\tau)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau} f(\gamma(t)).$$

When we have a coordinate system φ (recall the definition of a chart on a manifold) centered on a point $p := \varphi^{-1}(0)$, which gives us a way to assign coordinates to points on our manifold, we can define paths that behave like straight lines in our chart. The 'straight' line γ_i is given by

$$\gamma_i(t) = \phi^{-1}(0, \ldots, 0, t, 0),$$

where the nonzero parameter is on the ith position. The derivative of such a path at a certain point is written as $\dot{\gamma}_i(0) = \left.\frac{\partial}{\partial x^i}\right|_p$. These $\left.\frac{\partial}{\partial x^i}\right|_p$ form a basis for the tangent space at that point p [2].

Now that we understand what a tangent space is, we can talk about vector fields. A vector field is a map that assigns to every point in the manifold a tangent vector to the manifold at that specific point. This can be made rigorous as follows

Definition 3.2. A vector field associates to every point on a manifold a vector in the tangent space. It is defined by a map $X : M \to TM$ so that $\pi \circ X = \mathrm{id}$, where π is the projection operator that maps a tangent vector $(p, v) \in T_pM \subset TM$ to its footpoint, p. The vector field is smooth if the map X is smooth.

Vector fields are essential in the study of fluid dynamics. The velocity field of a fluid - its velocity vector at every point - is a vector field as well. This velocity field is essentially the derivative of the flow function, Φ_t which maps the position of a particle at time 0 to the position of that particle at time t.

3.1.1 The pushforward

There is one extremely important concept related to vectors that may not be absent in this section. The pushforward of a function is a way of letting a function from a manifold M to itself $f \in C^\infty(M, M)$ act on vectors which are elements of TM. Conceptually, the pushforward takes a vector at a point and moves it along with all the other points to a new position on the manifold. If the neighbourhood around the vector's footpoint is rotated, so is the vector and if that neighbourhood is stretched, sheared, or contracted, those transformations are also applied to the vector.

The pushforward of a function $f \in C^\infty(M, M)$ is denoted as either $\mathrm{d}f$ or as f_*. When it is applied to a vector $v_p \in T_pM$ (which then successively should be applied to a function $g \in C^1(M, \mathbb{R})$). The defining equality is

$$f_*(v)(g) := \mathrm{d}f(v)(g) := v(g \circ f). \tag{1}$$

The footpoint of the new vector will then be $f(p)$. This definition can be extended to take a function from one manifold to another $f \in C^\infty(M, N)$ which will turn a vector $v \in TM$ into a new vector in TN, using the same equality from 1 but now with a function $g \in C^1(N, \mathbb{R})$.

3.2 Index notation

Before we delve into alternating k-form which is the topic of the next chapter, it is important to explain what the meaning is of the position of the different indices. In physics, we often write a vector to be $\mathbf{v}$ as v^i. In mathematics, such notation is considered sacriligious. The v^i refer to the *components* of the vector v and the full vector is given by

$$\mathbf{x} = x^i \frac{\partial}{\partial x^i}.$$

Here the index i is summed over using the Einstein summation convention. Whenever a single index appears twice, once as a superscript and once as a subscript, it should be summed over. For example for two vectors $\mathbf{v}$ and $\mathbf{u}$, their Euclidean inner product is given by

$$\langle \mathbf{v}, \mathbf{u} \rangle = v^i \delta_{ij} u^j.$$

where δ_{ij} equals 1 if $i = j$ and zero otherwise. We could also define an object that has only a subscript. If we make the object f_i then it can be 'contracted' with a vector v^i to obtain

$$f_i v^i = \sum_i f_i v^i.$$

As said before, a mathematician would not accept this notation since the v^i and the f_i have a hanging, uncontracted index. To make the above statement rigorous, we would write $v = v^i \frac{\partial}{\partial x^i}$ and $f = f_i \mathrm{d}x^i$ where $\mathrm{d}x^i$ is the linear function defined by $\mathrm{d}x^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j$. We could then state

$$f(v) = f_i \mathrm{d}x^i \left(v^j \frac{\partial}{\partial x^j} \right) = f_i v^j \mathrm{d}x^i \left(\frac{\partial}{\partial x^j} \right) = f_i v^j \delta^i_j = f_i v^i,$$

which amounts to the exact same result. In general, one can convert from a physicists index notation to a mathematicians index notation by assuming that all uncontracted indices should be contracted with either $\frac{\partial}{\partial x^i}$ or $\mathrm{d}x^i$, unless this index only appears on one side of the equation.

Finally, the difference between subscripts and superscripts is how they change under a coordinate transformation. If I were to do a change of basis from some coordinates $(x_1, \ldots, x_n)$ to the coordinates $(y_1, \ldots y_n)$ where a coordinate should be seen as a function of the point described by the coordinate. Then at some point the basis of the tangent space transforms as follows

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial y^j}{\partial x^i} \left.\frac{\partial}{\partial y^j}\right|_p . \tag{2}$$

However, a vector should not change, so if we take the same vector $\mathbf{v}$ expressed both in the coordinates x as v_x^i and in the coordinates y as v_y^j then

$$v_x^i \frac{\partial}{\partial x^i} = v_x^i \frac{\partial y^j}{\partial x^i} \left.\frac{\partial}{\partial y^j}\right|_p = v_y^i \left.\frac{\partial}{\partial y^j}\right|_p$$

so

$$\frac{\partial y^j}{\partial x^i} v_x^i = v_y^j,$$

which has the term $\frac{\partial y^j}{\partial x^i}$ on the opposite side as Equation 2 does. This is the meaning of the subscript or superscript, the way in which the different components will transform under a change of coordinates.

3.3 Alternating k-forms

A k-form is a mathematical object that generalizes certain concepts related to vector fields. Let's start with the simplest type, which is a 1-form. A 1-form is an element from the dual space of smooth vector fields, denoted as T^*M. In other words, it is a smooth section of the dual tangent bundle T^*M that associates each point on the manifold with a 1-form α_p at that point.

$$\begin{array}{rcccl} \alpha & : & M & \longrightarrow & T^*M \\ & & p & \longmapsto & \alpha_p \in T_p^*M \end{array}$$

The set of all 1-forms on a manifold is written as $\Omega^1(M)$, and it has a basis consisting of n 1-forms: $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$. These 1-forms are defined in terms of coordinate differentials to have the property

$$\mathrm{d}x^i \left(\frac{\partial}{\partial x^j} \right) = \delta_j^i,$$

which we already briefly encountered in the previous section.

We can use the wedge product to define higher-degree forms. For k-forms $\alpha_1, \ldots, \alpha_k$ and vectors $v_1, \ldots, v_k$ at a point p, the wedge product $\alpha_1 \wedge \cdots \wedge \alpha_k$ is defined as the determinant of a matrix involving evaluations of the 1-forms on the vectors [2]. The wedge product is associative, linear, and anticommutative. A k-form is a form that takes k inputs, and the space of k-forms is denoted as $\Omega^k(M)$ [2].

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)_p(v_1, \ldots, v_k) = \det \begin{pmatrix} \alpha_1(v_1) & \cdots & \alpha_1(v_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(v_1) & \cdots & \alpha_k(v_k) \end{pmatrix},$$

There are several important operations on the space of k-forms. The first is the exterior derivative denoted by d which sends a k-form to a $(k+1)$-form. For a k-form $u = u_{a_1 \ldots a_k} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_k}$, the exterior derivative is

$$\mathrm{d}u = \frac{\partial u_{a_1 \ldots a_k}}{\partial x^i} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_k} \wedge \mathrm{d}x^i.$$

It can also be defined in coordinate-free notation by taking Stokes' theorem as the definition of the exterior derivative. Usually, Stokes' theorem states that for a k-form $\omega \in \Omega^k(M)$ and for a subset $V \subset M$ which is $(k+1)$-dimensional,

$$\int_V \mathrm{d}\omega = \int_{\partial V} \omega,$$

where ∂ denotes the boundary operator and the integral is defined in coordinates as

$$\int_{\partial V} \omega := \int_{\partial V} \omega_{a_1 \ldots a_k} \mathrm{d}x^{a_1} \cdots \mathrm{d}x^{a_k}.$$

The second operation sends a k-form to a $(k-1)$-form. This operation is called the interior product and functions by taking a vector field v and injecting it into the first slot of the k-form α.

$$(i_v \alpha)(u_1, \ldots, u_{k-1}) = \alpha(v, u_1, \ldots, u_{k-1}).$$

Lastly, there is the Lie derivative, denoted as $\mathcal{L}_X$, which involves pulling a k-form along a given vector field X. It is defined using the smooth curve $g_t(x)$ in the space of smooth diffeomorphisms on the manifold which crosses through the identity at $t = 0$.

$$g_0(x) = x.$$

The Lie derivative can be seen as the rate of change of the pulled-back form $g_t^* \omega$ with respect to time t. When the derivative of the curve g_t at $t = 0$ is given by the vector field[1] X, the Lie derivative is defined as

$$\mathcal{L}_X \omega := \frac{\partial}{\partial t}(g_t^* \omega)\bigg|_{t=0},$$

where $g_t^* \omega(v_1, \ldots, v_k) := \omega(g_t(v_1), \ldots, g_t(v_k))$.

$$\frac{\partial}{\partial t}(g_t^* \omega) = \mathcal{L}_X \omega,$$

[1] In Section 9 it will be shown that this derivative is indeed a vector field.

where $g_t^*\omega(v_1, \ldots, v_k) := \omega(g_t(v_1), \ldots, g_t(v_k))$.

This means that one can see the Lie derivative as the object which pulls a k-form along a certain vector field. By integrating the definition of the Lie derivative, the expression is obtained

$$g_t^*\omega = \int_0^t \mathcal{L}_X \omega \mathrm{d}t + \omega.$$

When working with the Lie derivative, Cartan's magic formula is often used as it is a useful result which states that [2]

$$\mathcal{L}_X = i_X \mathrm{d} + \mathrm{d} i_X.$$

3.4 Riemannian manifolds

A Riemannian manifold is a manifold together with a metric. A metric should be seen as a way to measure length of a curve on a manifold. In a Hilbert space we can use the inner-product to compute the distance[2] $d(x, y)$ between two points x and y. The formula for computing this distance is [4]

$$d(x - y) = \sqrt{\langle x - y, x - y \rangle}.$$

When considering the simple case of Euclidean space $\mathbb{R}^n$, the inner product between two vectors $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{u} = (u_1, \ldots, u_n)$ is given by

$$\langle \mathbf{v}, \mathbf{u} \rangle = \sum_{i=1}^{n} v_i u_i.$$

The distance between the points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ is

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

which is exactly what would be expected from Pythagoras' theorem.

To compute the length of a non-straight curve, you simply sum the infinitesimal distances that make up the curve. For a curve $\gamma : [0, 1] \to H$ where H is a Hilbert space,

$$L(\gamma) = \lim_{n \to \infty} \sum_{i=0}^{n-1} d\left(\gamma\left(\frac{i}{n}\right), \gamma\left(\frac{i+1}{n}\right)\right).$$

When n becomes sufficiently large, this successive distance may be approximated using the derivative of γ at t, $\dot{\gamma}_t$, which is an element of the tangent space of H. In this case, because a Hilbert space is already a vector space, we may identify the tangent space and

[2] This concept of distance is also called metric however it is a different metric than the one we are interested in.

the Hilbert space itself $T_{\gamma(t)}H \cong H$. The inner product then carries over to the tangent space so the length of the curve is

$$L(\gamma) = \int_0^1 \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} \mathrm{d}t.$$

In general, however, a manifold is not a vector space, so an inner product cannot always be defined. On the other hand, the tangent spaces will always be vector spaces. So, we would like to define an inner product for every tangent space to our manifold. This is called a metric.

Definition 3.3. A metric g is a map which associates to every point x of the manifold M a symmetric, positive definite bilinear map on the tangent space T_xM. The inner product between two elements $u, v \in T_xM$ is written as

$$\langle u, v \rangle = g_x(u, v).$$

It is worth noting that not all measures of distance are induced by an inner product and as such not all measures of length can be described with a metric [5].

Theorem 3.4. A norm $\|\cdot\|$ on a vector space V is induced by an inner product $\sqrt{\langle \cdot, \cdot \rangle}$ if and only if the norm obeys the parallelogram identity: for all $u, v \in V$

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 .$$

In this case, the inner product is given by

$$\langle u, v \rangle = \frac{1}{4} \left(\|u + v\|^2 - \|u - v\|^2 \right) .$$

Technically there is an even more general notion of distance (the one that is confusingly named a metric but is *not* the metric we have been discussing). This one is not of interest to us since we should want our length measure to satisfy something called 'absolute homogeneity' $d(0, sv) = |s| \, d(0, v)$ for a real number s and a vector v so that a reparametrization of a curve will not change its length.

Fortunately, both the parallelogram identity and this 'absolute homogeneity' are satisfied by the Euclidean norm

$$\|\mathbf{v}\|^2 = \mathbf{v}^T \mathbf{v},$$

which is induced by the inner product

$$\langle u, v \rangle = u^T v.$$

It also holds for the norm on a vector space X when that norm on the vector space is induced by an inner product on the carrier space

$$\|X\|^2 = \int_M \langle X(x), X(x) \rangle \mathrm{d}x,$$

as can be checked easily using linearity of the integral.

An example of a norm that cannot described as an inner product is the ∞-norm. This norm on $\mathbb{R}^n$ is defined by

$$\left\| (v_1, \ldots, v_n)^T \right\| = \max_i |v_i|$$

3.5 The Connection

The connection is the mathematical formalism that allows us to treat geodesics on arbitrary manifolds. One can see a connection as a generalization of the derivative operator. Although the concept of a connection is defined on all bundles, we will apply it only to the tangent bundle with a projection operation $\pi : TM \to M$ that sends vectors to their footpoint. The connection in this case can be seen loosely as differentiating one vector field along another vector field. In this manner, we 'connect' the tangent spaces that were initially unrelated because they belonged to different points.

Since the tangent bundle itself is a manifold, we can define the tangent bundle of the tangent bundle, TTM. This space contains derivations of tangent vectors. Through some constructions, whose details do not matter here, it is possible to define the subspace HTM which consists of derivations on the tangent bundle in the direction of the manifold [6]. Rigorously this means that it is a space orthogonal to the space of derivations in the direction of tangent vectors, this would be called the vertical subspace VTM. Figure 1 gives a visual aid to understand this.

The ultimate result is that a connection on the tangent space of a real manifold can be defined as a linear operator

$$\boldsymbol{\nabla} : \Gamma(TM) \to \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM, TM)) \cong T^*M \otimes_{\mathbb{R}} TM,$$

that additionally satisfies a product rule

$$\boldsymbol{\nabla}_X fs = \mathrm{d}f(X)s + f\boldsymbol{\nabla}_X s$$

for all $f \in C^\infty(M, \mathbb{R})$, $s \in \Gamma(TM)$, and $X \in \mathfrak{X}(M)$. Here $\Gamma(\Omega)$ denotes the space of smooth sections on a bundle Ω.

The Levi-Cevita connection is one particularly special connection which 'preserves' the metric. It has the property that

$$\boldsymbol{\nabla}_X \langle Y, Z\rangle = \langle \boldsymbol{\nabla}_X Y, Z\rangle = \langle Y \boldsymbol{\nabla}_X Z\rangle$$

for all $X, Y, Z \in \Gamma(TM)$.

3.6 Geodesics

Geodesics are length minimizing[3] curves between points. An easy example of a geodesic is in Euclidean space, denoted as $(\mathbb{R}^n, \delta)$ where δ is the metric. The metric δ means that

$$\left\langle u^i \frac{\partial}{\partial x^i}, v^j \frac{\partial}{\partial x^j} \right\rangle_\delta = u^i \delta_{ij} v^j = \sum_i u^i v^i.$$

In Euclidean space, the geodesics are straight lines from one point to another. The statement of Newton's law can be reformulated to state that in the absence of any forces, the path taken by a particle is a Euclidean geodesic.

[3]However only locally. When considering geodesics between points that are not sufficiently near eachother on that geodesic, the curve need only extreme the length.

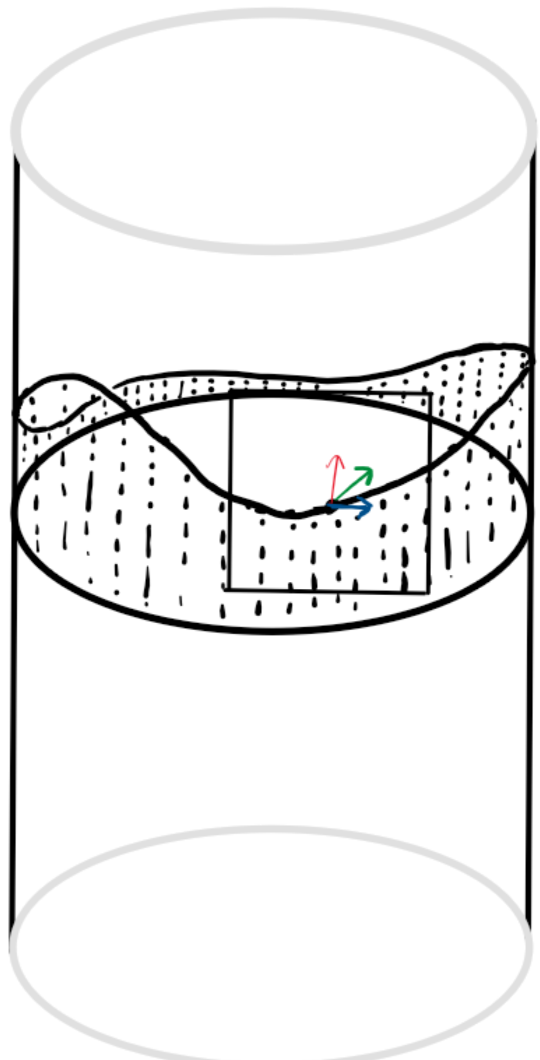

Figure 1: The tangent space to a circle can be visualized as a cylinder with the tangent vectors pointing in the vertical direction. The black curve is a section of TM, where the dotted lines show how each vector projects down to the circle. In the square we see a small part of TTM. This part contains three vectors, the red vector points vertically and represents a derivation purely in the direction of a tangent vector. The green vector points somewhat in the direction of the manifold, but still points somewhat in the direction of the tangent vectors as well - it is not orthogonal to VTM. Lastly, the blue vector points perfectly horizontally so it is an element of the horizontal subspace. The acceleration of a curve should be a vector pointing in the same direction of the blue curve.

Different metrics can lead to different paths. For example, the behavior of light can be seen as a geodesic by rescaling the euclidean metric by $1/v$ where v is the speed of light inside the medium.

General relativity uses geodesics to great effect to describe particles' gravitational interactions. For this purpose, the $3+1$ dimensional space-time is introduced. Movement of the particles affects the metric as described by Einstein's field equations creating a time-dependent metric. When there are no massive bodies present, the metric of special relativity is accurate. It is given by

$$\eta = \mathrm{diag}(-1, +1, +1, +1).$$

Although geodesics are generally used as paths that minimize distance, it turns out to be (almost) equivalent to consider paths extremising the energy [7]. In a space with a Riemannian metric g, the path followed by a geodesic $\gamma : [a, b] \to G$ satisfies the equation

$$\ddot{\gamma}^i(t) = -{\Gamma^i}_{jk}\dot{\gamma}^j\dot{\gamma}^k, \quad \text{where} \quad \Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

are the Christoffel symbols. This equation is called the geodesic equation.

Example 3.5. We will find the equation for a geodesic in the special case of hyperbolic space parametrized as an open upper half plane. The metric on hyperbolic n-space is

$$g_{ij}(x_1, \dots, x_n) = \frac{\delta_{ij}}{x_n}.$$

This means that

$$\frac{\partial g_{ij}}{\partial x^k} = \delta_{ij}\delta_{kn} \cdot -\frac{1}{x_n^2} \quad \text{and} \quad \Gamma_{ijk} = -\frac{1}{2x_n^2}\left(\delta_{ij}\delta_{kn} + \delta_{ki}\delta_{jn} - \delta_{jk}\delta_{in}\right).$$

The equation of motion is given by

$$\ddot{\gamma}^i = \frac{1}{2x_n^2}\left(\dot{\gamma}^i\dot{\gamma}^n + \dot{\gamma}^n\dot{\gamma}^i - (\dot{\gamma})^2\delta_{in}\right).$$

Importantly, this means that for $i \neq n$, $\ddot{\gamma} = \frac{1}{x_n^2}\dot{\gamma}^i\dot{\gamma}^n$ which, with the initial condition $\dot{\gamma}^i = 0$ will give that γ^i is constant for all time. This means that (in our particular choice of basis), geodesics on hyperbolic n-space are always contained within an affine[4] plane. For the sake of simplicity, take a parametrisation so that that affine plane is exactly the plane spanned by γ^1 and γ^n.

The equations of motion are then

$$\ddot{\gamma}^1 = \frac{1}{x_n^2}\dot{\gamma}^1\dot{\gamma}^n \quad \text{and} \quad \ddot{\gamma}^n = \frac{1}{2x_n^2}\left((\dot{\gamma}^n)^2 - (\dot{\gamma}^1)^2\right).$$

[4] They are affine because they need not pass through the origin. A plane is closed under addition of its vectors whereas an affine plane is not. Affine planes are always parralel to some true plane.

Interestingly enough, the solution to this equation is always the unique circle contained in the affine plane, with center on the $x_n = 0$ hyperplane and which matches the initial conditions. In the case where the initial $\dot{\gamma}$ points solely in the x_n direction, it will be a line rather than a circle [7].

It is possible to write the geodesic equation in a basis-independent way. This is what the concept of a connection is used for. The geodesic equation can be written

$$\boldsymbol{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0,$$

where $\boldsymbol{\nabla}$ is the Levi-Civita connection [6].

4 Lie groups

Definition 4.1. A Lie group G is a set which simultaneously has a group structure and a smooth manifold structure[2], [8], [9]. Additionally, the maps

$$L_h : g \mapsto hg \quad \text{and} \quad R_h : g \mapsto gh$$

should be smooth. For a reminder of what a group is, see [10]–[12].

Lie groups are often used for describing symmetries, such as the Poincaré group or the Euclidean group. These Lie groups describe respectively the symmetries of Minkowski and Euclidean space.

Example 4.2. The orthogonal group $\mathrm{SO}(N)$ consists of rotations in N-dimensional space. It is a group under composition and can be seen as a manifold through the use of Euler angles. In three dimensions, this means to decompose a rotation into three successive rotations along comoving axis. See Figure 2 on page 17. In higher dimensions, rotations are no longer about an axis, but rather within a plane. This leads to an $N(N-1)/2$ dimensional manifold [13].

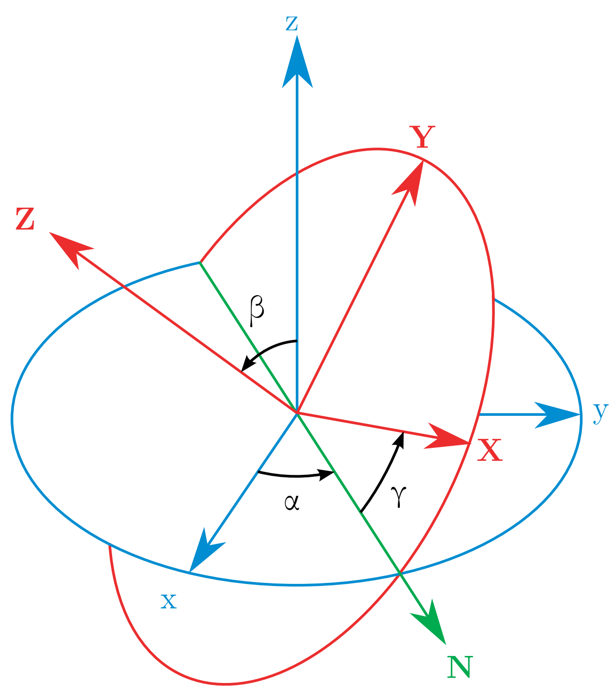

Figure 2: The Euler angles first rotate by an angle α about the z-axis, then by an angle β about the N-axis (which is where the first rotation moved the x-axis, finally by an angle γ about the Z-axis (which is where the previous rotations moved the z-axis to). In the figure, the lowercase letters indicate axis in the space frame whereas the capital letters indicate axis in the body frame.

4.1 Lie algebra

Whereas the Lie group contains finite-sized symmetries, the Lie algebra contains the infinitesimal symmetries. For example, the Lie group $SO(3)$ may contain a rotation by 30 degrees around the z-axis whereas the Lie algebra $\mathfrak{so}(3)$ only contains rotations by an infinitesimal angle. These infinitesimal symmetries are understood as the tangent space to the Lie group at the identity element. The Lie algebra is often written in the Fraktur font.

The Lie algebra forms a group under the operation

$$\begin{array}{rccl} [\cdot,\cdot] & : & \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & & (\xi,\omega) & \longmapsto & [\xi,\omega] =: \operatorname{ad}_\xi \omega \end{array}$$

This operation is called the commutator and can be defined as

$$[\xi,\omega] = \tilde{\xi}\circ\tilde{\omega} - \tilde{\omega}\circ\tilde{\xi}\Big|_e ,$$

where $\tilde{\xi}$ and $\tilde{\omega}$ are the vector fields generated by ξ and ω using left-invariance. Although this definition explains the name commutator, it does not elucidate why we should call the commutator by this weird name "ad" (which is short for adjoint).

To explain how we got to the ad operator, we will need to build up slowly. First we define the conjugation operator $A_g : G \to G, x \mapsto gxg^{-1}$ and from there we can get to the ad by first definining the Ad operator. Importantly, the conjugation operator sends the identity element to itself. Thus its pushforward defines a map from $\mathfrak{g}$ to $\mathfrak{g}$. This pushfoward is the operator called $\operatorname{Ad}_g := A_{g*}$. However, we can now keep the input fixed and vary the parameter g. Doing this, we can again define a pushforward, this time the pushforward we obtain is ad which is a linear operator on the lie algebra which depends lineary on its parameter. The map $\Phi : \mathfrak{g} \to \mathrm{GL}(\mathfrak{g}), \xi \mapsto \operatorname{ad}_\xi$ is called the adjoint representation of the Lie algebra.

Theorem 4.3. The commutator satisfies the following properties [1], [2]

i) anticommutativity $[\xi,\omega] = -[\omega,\xi]$;

ii) bilinearity, $[a\xi + b\omega, \eta] = a[\xi,\eta] + b[\omega,\eta]$ (this is equivalent to bilinearity only if anticommutativity is already known);

iii) Jacobi identity $[\xi,[\omega,\eta]] + [\omega,[\eta,\xi]] + [\eta,[\xi,\omega]] = 0$.

Besides the adjoint representation, we also have the coadjoint representation which is given by the map $\Theta : \mathfrak{g} \to \mathrm{GL}(\mathfrak{g}^*), \xi \mapsto \operatorname{ad}^*_\xi$, which is defined by the fact that it satisfies the following relation. Let $\xi, \eta \in \mathfrak{g}$ and $a \in \mathfrak{g}^*$. Then

$$a(\operatorname{ad}_\xi \eta) = (\operatorname{ad}^*_\xi a)(\eta).$$

This representation is particularly useful because many times in physics, velocities are represented by elements of the Lie algebra whereas momenta are elements of the dual to the Lie algebra. Thus, it will often occur that we need to apply some sort of adjoint action to the dual of the Lie algebra.

Example 4.4. The Lie Algebra of $\mathrm{SO}(N)$ consists of antisymmetric matrices. One can see this by differentiating the expression

$$R(t)^T R(t) = I$$

at $t = 0$, for a path $R : [-\epsilon, \epsilon] \to \mathrm{SO}(N)$ with $R(0) = I$. The derivative is found with a product rule and equals

$$\dot{R}^T(t)R(t) + R^T(t)\dot{R}(t)\bigg|_{t=0} = \dot{R}^T(t) + \dot{R}(t)\bigg|_{t=0} = 0.$$

This is indeed exactly the expression satisfied by an antisymmetric matrix. In the specific case of SO(3) we can even find precisely what the matrix will look like. For a rotation about the unit vector $\hat{u} = (u_x, u_y, u_z)$ by an angle φ the matrix is given by Rodgrigues' rotation formula [14]

$$R = I + \sin(\varphi)\begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} + \sin^2(\varphi/2)\begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}^2.$$

For the Lie algebra we care only about *infinitesimal* rotations so the $\sin^2(\varphi/2)$ will vanish. This gives us not only that the Lie algebra consists of antisymmetric matrices but also that the Lie algebra can be represented as

$$\mathbb{R} \otimes S^2 \cong \mathbb{R}^3.$$

If this notation is unfamiliar, the S^2 represents the unit sphere which contains all unit vectors and the $\mathbb{R}$ represents a scaling factor essentially for how rapidly space is rotating. We need to use the tensor product ($\otimes$) rather than the Cartesian product ($\times$) specifically because elements of the form $(0, \hat{u})$ and $(0, \hat{v})$ should be the same (when the object is not rotating, it does not matter about what axis this rotation is happening).

Using the matrix representation we can find explicity the commutator. Denote the matrix elements of the Lie algebra induced by the vectors $\mathbf{u}$ and $\mathbf{v}$ as U and V. Note that these need no longer be unit vectors as their length now represents the angular speed.

$$\begin{aligned}
[U, V] &= \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} -u_z v_z - u_y v_y & -u_z v_x & u_z v_x \\ u_x v_y & -u_z v_z - u_x v_x & u_z v_y \\ u_x v_z & u_y v_z & -u_y v_y - u_x v_x \end{pmatrix} \\
&\quad - \begin{pmatrix} -v_z u_z - v_y u_y & -v_z u_x & v_z u_x \\ v_x u_y & -v_z u_z - v_x u_x & v_z u_y \\ v_x u_z & v_y u_z & -v_y u_y - v_x u_x \end{pmatrix}
\end{aligned}$$

$$
= \begin{pmatrix} 0 & v_z u_x - u_z v_x & u_z v_x - v_z u_x \\ u_x v_y - v_x u_y & 0 & u_z v_y - v_z u_y \\ u_x v_z - v_x u_z & u_y v_z - v_y u_z & 0 \end{pmatrix}
$$

which is in fact exactly the element in the matrix representation of the Lie algebra that corresponds to the element $\mathbf{u} \times \mathbf{v}$ in the vector representation. Thus, the adjoint action on $\mathfrak{so}(3)$ in its vector representation can be written explicitly

$$
\mathrm{ad}_{\mathbf{u}} \mathbf{v} = \mathbf{u} \times \mathbf{v}.
$$

From this we can immediately obtain also the coadjoint ad^* which should satisfy

$$
\mathbf{u} \cdot \mathrm{ad}_{\mathbf{v}} \mathbf{w} = \mathrm{ad}^*_{\mathbf{v}} \mathbf{u} \cdot \mathbf{w}
$$

which is satisfied specifically by $\mathrm{ad}^*_{\mathbf{v}}(\cdot) = -\mathbf{v} \times (\cdot)$ since we know from their geometric interpretations as a parallelapiped that

$$
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}
$$

4.2 The exp map

In general, the exponential function shows up when solving an equation of the sort

$$
\dot{x} = kx;
$$

the solution to which is

$$
x(t) = x(0)e^{kt}.
$$

Similarly, on a manifold, we denote the solution to the equation $\dot{\gamma} = X(\gamma(t))$ as $e^{tX}\gamma(0)$. This means that e^{tX} is equal to the flow of vector field Φ_t. In the literature, you may find the flow of a single point referred to as an integral curve of the vector field X.

Definition 4.5. For any vector field $X : M \to TM$, the integral curve is defined to be a curve $\gamma : [a, b] \to M$ which satisfies the property that $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in [a, b]$. Notice that although these curves can be defined for any vector field, the vector field by our definition can only depend on time if there is an explicit time coordinate in our manifold. In this case we would want the vector field to have a constant component in the time-direction.

In the case of a Lie group, this map from vector fields to functions can be turned into a map from elements of the Lie algebra to elements of the Lie group. To do this, we must first introduce the concept of left (and right) invariant vector fields.

Definition 4.6. A vector field is left (respectively right) invariant if $X(gh) = L_{g*}X(h)$ (respectively $X(hg) = R_{g*}X(h)$).

Using the previous definition, we can reconstruct the entire vector field if even one vector is given. In particular, we can construct an entire vector field using just an element from the Lie algebra. For example, on the Lie group $\mathbb{R}^n$ the vector field generated by $(0,\xi) \in \mathfrak{g}$ is $X : g \mapsto (g,\xi)$.

Definition 4.7. The exponential of an element X in the Lie algebra is defined to be the exponential of the vector field generated using X by the condition of left-invariance.

Theorem 4.8. For every element X of the Lie algebra $\mathfrak{g}$, there exists an element γ of the Lie group G so that

$$e^X = L_\gamma.$$

Proof. Denote by $\tilde{X}$ the vector field generated from X using left-invariance and let $h \in \mathfrak{g}$. So

$$e^X h = e^{\tilde{X}} h = \Phi_1(h),$$

where Φ_t is the flow of $\tilde{X}$. Let $\chi(t) := R_{h^{-1}}\Phi_t(h)$ so that $\chi(0) = 0$. The equation satisfied by $\dot{\chi}$ is

$$\begin{aligned}
\dot{\chi} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(R_{h^{-1}}\Phi_t(h)\big) \\
&= R_{h^{-1}*}\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi_t(h)\big) \\
&= R_{h^{-1}*}\tilde{X}(\Phi_t(h)) \\
&= \tilde{X}(R_{h^{-1}}\Phi_t(h)) \\
&= \tilde{X}(\chi).
\end{aligned}$$

This differential equation is independent of h. So we may choose $\gamma := \chi(1)$ to obtain that indeed $e^X h = R_h\gamma = L_\gamma h$ for all $h \in G$. □

5 The classical approach to a rigid body

In classical mechanics, the motion of a rigid body can be described by the Euler equations. These equations govern the rotational motion of the body and are derived based on Newton's laws of motion and the principles of rotational dynamics. In this derivation, we assume a rigid body with a fixed axis of rotation.

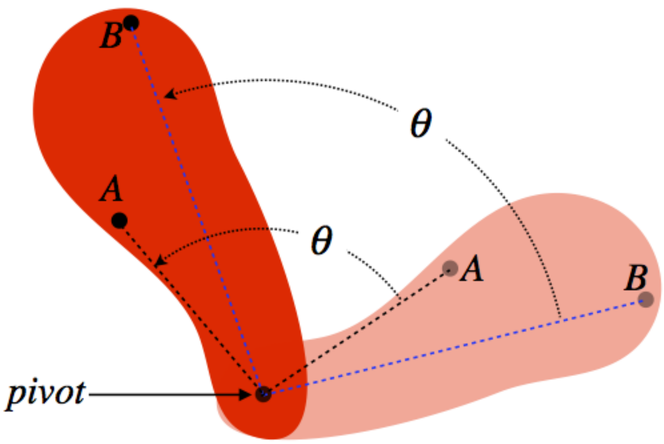

Figure 3: A 2-dimensional rigid body rotates about a fixed point, the pivot. The configuration can be indicated by a parameter θ [15].

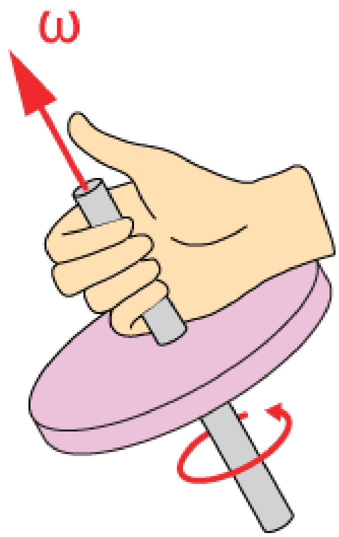

Figure 4: In 3 dimensions, a rigid body's rotation is defined by the angular velocity vector. The angular velocity is turned into a rotation with the right-hand rule [16].

Consider a 3-dimensional rigid body rotating about a fixed axis passing through the origin. Let's denote the density function of the rigid body by ρ and the angular velocity vector of the body which points in the direction around which the object is rotating ω. It has magnitude $\|\omega\| := \frac{\mathrm{d}}{\mathrm{d}t}\theta$. See Figure 4.

The velocity of any one part of mass in the body is $\mathbf{v} = \omega \times \mathbf{r}$. This gives an angular momentum

$$\begin{aligned}\mathbf{L} &= \int \mathbf{r} \times \mathbf{v} \mathrm{d}m \\ &= \iiint \rho(\mathbf{r})\mathbf{r} \times (\omega \times \mathbf{r}) \mathrm{d}V\end{aligned}$$

If we write this in a basis, $\mathbf{r} = (r_x, r_y, r_z)$, and $\omega = (\omega_x, \omega_y, \omega_z)$, this integrand becomes

$$\mathbf{r} \times (\omega \times \mathbf{r}) = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \times \begin{pmatrix} r_z\omega_y - r_y\omega_z \\ r_x\omega_z - r_z\omega_x \\ r_y\omega_x - r_x\omega_y \end{pmatrix}$$

$$= \begin{pmatrix} -r_y r_x \omega_y + r_y^2 \omega_x + r_z^2 \omega_x - r_z r_x \omega_z \\ -r_z r_y \omega_z + r_z^2 \omega_y + r_x^2 \omega_y - r_x r_y \omega_x \\ -r_x r_z \omega_x + r_x^2 \omega_z + r_y^2 \omega_z - r_y r_z \omega_y \end{pmatrix}$$

$$= \begin{pmatrix} r_y^2 + r_z^2 & -r_y r_x & -r_z r_x \\ -r_x r_y & r_x^2 + r_z^2 & -r_z r_y \\ -r_x r_z & -r_y r_z & r_x^2 + r_y^2 \end{pmatrix} \omega.$$

For this reason, we can use the inertia tensor

$$\mathbf{I} = \iiint \rho(\mathbf{r}) \begin{pmatrix} r_y^2 + r_z^2 & -r_y r_x & -r_z r_x \\ -r_x r_y & r_x^2 + r_z^2 & -r_z r_y \\ -r_x r_z & -r_y r_z & r_x^2 + r_y^2 \end{pmatrix} \mathrm{d}V,$$

which then has the property that

$$\mathbf{L} = \mathbf{I}\omega.$$

The way we currently wrote this, the inertia tensor appears to be a matrix, however, this is not really true. To be entirely precise, we should distinguish momenta from velocities, as a velocity is a vector whereas a momentum is a covector (in indices this means that velocity has a superscript whereas momentum has a subscript). In order for the moment of inertia *tensor* to send a vector to a covector, it must be a $(0, 2)$-tensor[5], not a $(1, 1)$-tensor like a matrix is. This distinction will become important during the Lie group analysis, but for now it suffices to treat both as regular vectors.

According to Newton's second law for rotational motion, the net torque acting on the body is equal to the rate of change of angular momentum. Mathematically, this can be written as:

$$\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{3}$$

where τ is the total torque acting on the body, and $\mathbf{L}$ is the angular momentum of the body. In the case of a free body, this external force must be equal to zero.

In the case of a free system, τ must be zero so the equation of motion becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\omega) = \mathbf{I}\dot{\omega} = 0,$$

where $\dot{\omega}$ represent the time derivatives of $\dot{\omega}$.

However, this will not hold. As the object rotates, the density function will change, so, the notation $\rho(\mathbf{r})$ has been incorrect from the start. Instead, we should have been writing $\rho(\mathbf{r}, t)$. To resolve this issue, we observe the rigid body from a rotating reference frame, which is rotating with the body. We could instead have chosen to let $\mathbf{I}$ vary in time to obtain

$$\dot{\mathbf{I}}\omega + \mathbf{I}\dot{\omega} = 0,$$

[5] Or $(2, 0)$-tensor depending on what notation you prefer to follow. The point is, it turns a vector into a covector, not a vector into a vector.

but this would only complicate matters further.

In the rotating reference frame we must deal with the issue that the basis unit vectors are no longer constant in time. Instead, a basis vector $\hat{u}$ rotates at a rate

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = \omega \times \hat{u}.$$

Writing $\mathbf{L} = \mathbf{L}_x\hat{x} + \mathbf{L}_y\hat{y} + \mathbf{L}_z\hat{z}$, where the components and unit vectors are given in the rotating frame, the product rule lets one find the derivative of $\mathbf{L}$.

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{L}_x\hat{x} + \mathbf{L}_y\hat{y} + \mathbf{L}_z\hat{z}\right) \\
&= \dot{\mathbf{L}}_x\hat{x} + \dot{\mathbf{L}}_y\hat{y} + \dot{\mathbf{L}}_z\hat{z} + \mathbf{L}_x(\omega \times \hat{x}) + \mathbf{L}_y(\omega \times \hat{y}) + \mathbf{L}_z(\omega \times \hat{z}) \\
&= \dot{\mathbf{L}}_{\mathrm{rot}} + \omega \times \mathbf{L},
\end{aligned}$$

where $\dot{\mathbf{L}}_{\mathrm{rot}}$ refers to the change of the decomposition in the rotating reference frame. As stated before, in the absence of external forces, this derivative should vanish and as such, in the rotating frame, we obtain that

$$\mathbf{I}\dot{\omega} + \omega \times (\mathbf{I}\omega) = 0.$$

Solving this equation will tell us everything that we care to know about the motion of the rigid body.

6 The classical approach to an inviscid fluid

To describe a classical fluid system, there are two equations that may always be used [17]. These are the continuity equation and the Navier-Stokes equation, which are derived from respectively conservation of mass and conservation of momentum. Let's start with deriving the continuity equation.

Consider a small volume element V within the fluid. The rate of change of mass within this volume element is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho \mathrm{d}V$$

where ρ is the fluid density. Mass is conserved so the change in mass must be equal to the net mass flow into or out of the volume element through its boundaries. This can be expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho \mathrm{d}V = -\oint\!\!\!\oint_{\partial V} \rho \mathbf{v}\cdot\hat{n}\mathrm{d}A$$

where $\mathbf{v}$ is the velocity vector of the fluid and $\hat{n}$ is the outward unit normal vector of the surface element $\mathrm{d}A$. The negative sign accounts for the flow out of the volume element. The time derivative can be moved into the integral as long as the density as a function of time and position is sufficiently smooth (this would not be the case for example at the surface of a river where the fluid's density suddenly drops to zero). By applying the divergence theorem, the right-hand side can be simplified too.

$$\iiint_V \frac{\partial\rho}{\partial t}\mathrm{d}V = -\iiint_V \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\mathrm{d}V$$

The only way for this equality to hold with any volume V is if the two integrands are the same. Thus,

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\mathbf{v}).$$

This is the continuity equation [17]. Next, let's consider the conservation of momentum. It shows that at any moment, the change in mass at a specific point must be exactly equal to the amount of mass moving away from or into that point. For the Navier-Stokes equation we use the conservation of momentum. The rate of change of momentum within a volume element V is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\mathbf{v}\mathrm{d}V$$

As with matter, momentum must also be conserved, so the change in momentum of the matter inside our volume is given by the amount of momentum crossing through the boundaries of the volume.

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\mathbf{v}\mathrm{d}V = -\oint\!\!\!\oint_{\partial V}(\rho\mathbf{v})(\mathbf{v}\cdot\hat{n})\mathrm{d}A + \oint\!\!\!\oint_{\partial V}\mathbf{f}\mathrm{d}A$$

where $\mathbf{f}$ is the force pushing on the volume from the outside and all other variables are defined as before. One could add a term for forces acting on the particles inside the volume, but since such forces are absent in the free systems considered in this work, they are also not included in the previous equation. Specifically for an inviscid fluid (which is what we will be discussing in the rest of this thesis too), the force term is of a particularly easy form.

$$\mathbf{f} = -P\hat{n},$$

where P is the pressure and $\hat{n}$ the normal vector to the surface. In case the pressure in the fluid is constant, this would result in no net force on the volume as the volume is compressed equally from all sides. The divergence theorem can be used to write this as a volume integral[6]

$$\oiint -P\hat{n}\mathrm{d}V = -\iiint \boldsymbol{\nabla} P\mathrm{d}A.$$

To proceed any further, we must make the somewhat dubious step

$$\oiint_{\partial V} (\rho\mathbf{v})(\mathbf{v}\cdot\hat{n})\mathrm{d}A = \iiint_V \boldsymbol{\nabla}\cdot(\rho\mathbf{v}\mathbf{v})\mathrm{d}V.$$

This step is dubious since the product $\mathbf{vv}$ is not a defined object and neither is its divergence. To make such a step mathematically rigorous, we would need to resort to the tensor product however that is beyond what I intend to cover in this thesis [2]. Instead, we may pretend that the product is a matrix $\mathbf{vv} := \mathbf{v}\mathbf{v}^T$. This allows for an expansion in coordinates. The ith component of the vector $\boldsymbol{\nabla}\cdot(\rho\mathbf{vv})$ is given by

$$\big(\boldsymbol{\nabla}\cdot(\rho\mathbf{vv})\big)_i = \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\rho v_j v_i\right),$$

where n is the dimension of the fluid flow (for physically realizable systems, this will 1, 2, or 3) and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. We can now write the equation as a single volume integral

$$\iiint_V \left(\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}\cdot\rho\mathbf{v}\mathbf{v}^T + \boldsymbol{\nabla}P\right)\mathrm{d}V = 0$$

Similar to the continuity equation, this integral can be equal to zero for all volumes V only if the integrand is identically zero.

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}\mathbf{v}^T) = -\boldsymbol{\nabla}P.$$

This is the Navier-Stokes equation for an inviscid fluid [17]. Now it is important to note that the two equations we found - the continuity equation and the Navier-Stokes equation - are not sufficient to solve for the number of variables that we have. The Navier-Stokes equation is a vector equation so if we consider a fluid in n-dimensions, we have $n+1$

[6] This might look like a generalization of the gradient theorem but really it is an application of the divergence theorem with a vector field $P\mathbf{c}$ for any constant vector $\mathbf{c}$.

equations in $n + 2$ unknowns. These unknowns are the velocity of the fluid, the density of the fluid, and the pressure. To resolve this, assumptions are often made about a fluid. One could assume that the fluid is incompressible which would give a constraint that the velocity field remains divergence free which can be used to determine the pressure term [17].

7 Lie group analysis of a rigid body

The tools described before can be used to model a rigid body. The Euclidean group, $\mathrm{SE}(N)$, describes all transformations that can be done that will keep the structure of the rigid body intact. That is, it can be moved and it can be rotated. This motivates us to describe the time-evolution of the rigid body as paths within the Euclidean group. As we will later find, these paths are, in fact, geodesics under some special inner product [18].

Although cases with $N > 3$ cannot be realized in the real world - space is three-dimensional after all - we will describe the motion of an N-dimensional rigid body [19]. This more general approach will reveal more of the underlying structure and is simultaneously more useful as a preparation for our later description of ideal fluids [18].

The group structure of $\mathrm{SE}(N)$ similar to that of the affine group except that instead of any invertible matrix M, we only allow rotation matrices from the special orthogonal group. The multiplication rule is

$$(R_2, \mathbf{s}_2)(R_1, \mathbf{s}_1) = (R_2 R_1, R_2 \mathbf{s}_1 + \mathbf{s}_2) \tag{4}$$

A rigid body can be seen as a density function $\rho \in \Omega^n(\mathbb{R}^n)$ on $\mathbb{R}^N$ which represents the mass-distribution of the rigid body in space. For mathematical convenience, a density function should be used with a compact support. Since our rigid body lives in $\mathbb{R}^N$ the condition of compact support means nothing more than that the rigid body is finite in size.

Let $\mathbf{r}(t)$ be the position vector at time t of the infinitesimal volume that at time 0 was at $\mathbf{r}_0$, then the kinetic energy of the rigid body is given by

$$K(t) = \frac{1}{2} \int \left\| \dot{\mathbf{r}}(t) \right\|^2 \rho.$$

The variable $\mathbf{r}_0$ does not appear explicit but one should consider both $\mathbf{r}(t)$ and ρ to be functions of $\mathbf{r}_0$ which is the variable over which we integrate.

The kinetic energy equation before is essentially just an integral version of the well known equation $K(t) = \frac{1}{2} m \dot{\mathbf{r}}^2$. As described before, the configuration space of the rigid body is $\mathrm{SE}(N)$ so $\mathbf{r}(t) = g(t)\mathbf{r}_0$ for some path $g(t) \in \mathrm{SE}(N)$. Specifically since the body is *rigid*, the curve $g(t)$ is the same for all $\mathbf{r}$ regardless of which position $\mathbf{r}_0$ they started off at. In fact, since $\mathbf{r}_0$ is a constant in time, $\dot{\mathbf{r}}(t) = \dot{g}(t)\mathbf{r}_0$. Therefore,

$$K(t) = \frac{1}{2} \int \left\| \dot{g}(t)\mathbf{r}(0) \right\|^2 \rho.$$

Now, using that the action of an element $(R, \mathbf{s})\mathbf{v} = R\mathbf{v} + \mathbf{s}$, we proceed to find

$$K(t) = \frac{1}{2} \int \left\| \dot{R}\mathbf{r}_0 + \dot{\mathbf{s}} \right\|^2 \rho = \frac{1}{2} \int \left[\left\| \dot{R}\mathbf{r}_0 \right\|^2 + 2\langle \dot{R}\mathbf{r}_0, \dot{\mathbf{s}} \rangle + \|\dot{\mathbf{s}}\|^2 \right] \rho,$$

where we have for now made time-dependence implicit.

Using the standard inner product on $\mathbb{R}^n$, $\langle \mathbf{v}, \mathbf{u}\rangle = \mathbf{v}^T\mathbf{u} = \operatorname{Tr}\left(\mathbf{u}\mathbf{v}^T\right)$, and that the Lie algebra of $\mathrm{SO}(N)$, $\mathfrak{se}(N) \ni \dot{R}$ consists of antisymmetric matrices,

$$\left\|\dot{R}\mathbf{r}_0\right\|^2 = -\operatorname{Tr}\left(\dot{R}\mathbf{r}_0\mathbf{r}_0^T\dot{R}\right),$$

and using linearity of the inner product,

$$\int \langle \dot{R}\mathbf{r}_0, \dot{\mathbf{s}}\rangle \rho = \left\langle \dot{R}\int \mathbf{r}_0\rho, \dot{\mathbf{s}}\right\rangle.$$

Here, it can be seen that $\int \mathbf{r}_0\rho$ is the object's center of mass, scaled by the total mass. By choosing the initial coordinates in such a way that the center of mass coincides with the origin, it can be made so that this inner product term vanishes. This coordinate system is called the center of mass (COM) frame. Note that the center of mass is not necessarily the origin of our frame at any time other than 0.

The remaining kinetic energy consists of two terms, one from the momentum, $\frac{1}{2}\|\dot{\mathbf{s}}\|^2 M$ and one term from the angular momentum, $-\frac{1}{2}\operatorname{Tr}\left(\dot{R}J\dot{R}\right)$ where $M = \int \rho$ and $J = \int \mathbf{r}_0\mathbf{r}_0^T\rho$.

As is usual at this point, we define an inertia operator $A : \mathfrak{se}(N) \to \mathfrak{se}^*(N)$. The inertia operator must have the property $A((\Omega, \mathbf{v}))((\Omega, \mathbf{v})) = K(t)$ where $(\Omega, \mathbf{v}) = L_{(R,\mathbf{s})^{-1}*}(\dot{R}, \dot{\mathbf{s}})$. This is well-defined, since the metric on $\mathbb{R}^N$ is invariant under the action of $\mathrm{SE}(N)$.

The fact that our energy can be so neatly divided into two parts motivates us to partition this operator into two operators as well. Namely, A_{L} and A_{P} which are respectively given by

$$A_{\mathrm{L}}((\Omega, \mathbf{v}))((\Omega, \mathbf{v})) = -\frac{1}{2}\operatorname{Tr}(\Omega J\Omega) \quad \text{and} \quad A_{\mathrm{P}}((\Omega, \mathbf{v}))((\Omega, \mathbf{v})) = \frac{1}{2}M\|\mathbf{v}\|^2.$$

What do we get for all this work? A metric on $\mathrm{SE}(N)$. We define the inner product

$$\begin{aligned}
\langle(\dot{R}_1, \dot{\mathbf{s}}_1), (\dot{R}_2, \dot{\mathbf{s}}_2)\rangle_A &:= A_g((\dot{R}_1, \dot{\mathbf{s}}_1))((\dot{R}_1, \dot{\mathbf{s}}_1)) \\
&:= \left(L^*_{g^{-1}}AL_{g^{-1}*}\right)((\dot{R}_1, \dot{\mathbf{s}}_1))((\dot{R}_1, \dot{\mathbf{s}}_1)) \\
&= A(L_{g^{-1}*}(\dot{R}_1, \dot{\mathbf{s}}_1))(L_{g^{-1}*}(\dot{R}_1, \dot{\mathbf{s}}_1)) \\
&= A_{\mathrm{L}}(L_{g^{-1}*}(\dot{R}_1, 0))(L_{g^{-1}*}(\dot{R}_2, \dot{\mathbf{s}}_2)) + A_{\mathrm{P}}(L_{g^{-1}*}(0, \dot{\mathbf{s}}_1))(L_{g^{-1}*}(\dot{R}_2, \dot{\mathbf{s}}_2)) \\
&=: A_{g\mathrm{L}}((\dot{R}_1, 0))((\dot{R}_2, \dot{\mathbf{s}}_2)) + A_{g\mathrm{P}}((0, \dot{\mathbf{s}}_1))((\dot{R}_2, \dot{\mathbf{s}}_2))
\end{aligned}$$

The splitting of this metric in second to last step is allowed because the pushforward of left-multiplication does not mix translation and rotation components as can be verified by a straightforward linearisation of Equation 4.

$$L_{(R_1,\mathbf{s}_1)*}(\dot{R}, \dot{\mathbf{s}}) = (R_1\dot{R}, R_1\dot{s})$$

The beauty of this metric is the information it contains about the rigid body. The energy[7] of a path under this metric is the same as the action of the physical object using

[7] energy is the integral of the square of the norm $E = \int \|\dot{x}\|^2 \,\mathrm{d}t$

the Lagrangian $\mathcal{L} = K$. Thus, the geodesic $g(t)$ obtained in $\mathrm{SE}(N)$ will describe exactly the time-evolution that the rigid body will undergo.

We could alternatively have chosen to use right multiplication when we extended our inner-product with right multiplication but this would not have given us the same result. Since left multiplication happens after everything else, it takes the velocity vectors with it while rotating the body. Right multiplication would rotate the body *without* taking the velocity vector with it. This is a problem since rotating the body can change the moment of inertia and if we do not rotate the angular velocity in the same way as we rotated the body, the energy can change due to this changed moment of inertia.

Fortunately, the metric we created is invariant under left multiplication. Through Noether's Theorem [18], [20], this invariance creates a conserved current of the form

$$J(g, \dot{g}) = R_g^* \left.\frac{\partial \mathcal{L}}{\partial \dot{g}}\right|_g = 2R_g^* \langle \dot{g}, \cdot \rangle = 2R_g^* A_g(\dot{g}) := 2m_s = \text{constant}. \tag{5}$$

The subscript 's' denotes that this is the momentum with respect to space. We can also define a momentum with respect to the rigid body's own coordinates. In this case we would obtain $m_b := L_g^* A_g(\dot{g}) = \mathrm{Ad}_g^* m_s$. Since m_s is constant, the time derivative of m_b is then given by [21],

$$\dot{m_b} = \mathrm{ad}_{\omega_b}^* m_b, \tag{6}$$

with $\omega_b := L_{g^{-1}*}\dot{g}$. This equation is known as the Euler-Arnold equation [21] and is one of the most fundamental and valuable results of the are of mathematics that concerns itself with Lie group analysis.

Although it might seem that we lost something by going from m_s to m_b - our equation certainly looks more complicated, this was always the case and the complexity of the equation was simply made more evident. This is because in terms of just the A operator,

$$m_b = A(L_{g^{-1}*}\dot{g}) \quad \text{whereas} \quad m_s = \mathrm{Ad}_g^* A(L_{g^{-1}*}\dot{g}).$$

Thus, the complexity that is visible in Equation 6 was not absent in Equation 5, it was merely well hidden.

The previous construction is not limited to the Lie group $\mathrm{SE}(N)$ and can be done for any Lie group equipped with a left-invariant inner product. In fact, one can considers the group of all rotations about a fixed point $\mathrm{SO}(3)$ and make the identifications with the objects from the previous section, $m_b = \mathbf{I}\omega$ and $\omega_b = \omega$. Then, the explicit form of the coadjoint operator on $\mathfrak{so}(3)$, which is negative the cross product, does indeed yield the same equation as previously found

$$\dot{m_b} = -\omega_b \times m_b.$$

It is possible to write this equation in terms of ω_b without explicit dependence on m_b [1]. This will require the creation of a bilinear form precisely for this purpose.

Theorem 7.1. The time evolution of the angular velocity obeys the equation

$$\frac{\mathrm{d}\omega_b}{\mathrm{d}t} = B(\omega_b, \omega_b),$$

where B is a bilinear operator defined by the equality

$$\langle a, [b, c] \rangle = \langle B(a, b), c \rangle.$$

Proof. For any vector a in the Lie algebra, we have

$$\begin{aligned}
\left\langle \frac{\mathrm{d}\omega_b}{\mathrm{d}t}, a \right\rangle &= \left(A \frac{\mathrm{d}\omega_b}{\mathrm{d}t}, c \right) \\
&= \left(\frac{\mathrm{d}m_b}{\mathrm{d}t}, c \right) \\
&= \left(\operatorname{ad}^*_{\omega_b} m_b, c \right) \\
&= \left(m_b, \operatorname{ad}_{\omega_b} c \right) \\
&= \left(m_b, [\omega_b, c] \right) \\
&= \left\langle \omega_b, [\omega_b, a] \right\rangle \\
&= \left\langle B(\omega_b, \omega_b), a \right\rangle .
\end{aligned}$$

Which holds for any a in the Lie algebra, so

$$\frac{\mathrm{d}\omega_b}{\mathrm{d}t} = B(\omega_b, \omega_b).$$

□

7.1 Preserved Quantities

From physics, we know of the existence of two constants of motion. They are the linear momentum $\mathbf{P}$, and the angular momentum $\mathbf{L}$ which is a bivector.

$$\mathbf{P} := \int \left(\dot{R}\mathbf{r}_0 + \dot{\mathbf{s}} \right) \mathrm{d}\mu, \quad \text{and} \quad \mathbf{L} := \int (R\mathbf{r}_0 + \mathbf{s}) \wedge \left(\dot{R}\mathbf{r}_0 + \dot{\mathbf{s}} \right) \mathrm{d}\mu.$$

In the COM frame, they simplify to

$$\mathbf{P} := \int \dot{\mathbf{s}} \mathrm{d}\mu = M\dot{\mathbf{s}}, \quad \text{and} \quad \mathbf{L} := \int \left(R\mathbf{r}_0 \wedge \dot{R}\mathbf{r}_0 + \mathbf{s} \wedge \dot{\mathbf{s}} \right) \mathrm{d}\mu.$$

The second angular momentum term is the term that represents angular momentum of a particle moving in an (affine) line that does not cross through the origin. We will assume that $\mathbf{s}(0) = \mathbf{0}$ so that $\mathbf{s}(t) = \int_0^t \dot{\mathbf{s}}(\tau)\mathrm{d}\tau = \dot{\mathbf{s}}t$. The anti-symmetric wedge product $\mathbf{s} \wedge \dot{\mathbf{s}} = \dot{\mathbf{s}}t \wedge \dot{\mathbf{s}}$ will therefore vanish.

It is interesting to see how these can both be derived directly form the Euler-Arnold equation. As stated before, the inertia operator can be separated in two distinct parts relating to linear and angular momenta, respectively

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(R_g^* A_{g\mathrm{L}}((\dot{R}, 0)) + R_g^* A_{g\mathrm{P}}((0, \dot{\mathbf{s}})) \right) = 0.$$

These are conservation of angular momentum and linear momentum. We can proceed further by using the statement that the pullback of right-multiplication does not mix translations into rotations (although it does mix rotations into the translations!). Thus, when we apply this operator to a pure rotation vector $(\Omega_2, 0)$, we will find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R_g^*\left(A_{g\mathrm{L}}((\dot{R},0)) + A_{g\mathrm{P}}((0,\dot{\mathbf{s}}))\right)(0,\mathbf{v})\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(A_{g\mathrm{P}}((0,\dot{\mathbf{s}}))(R_g(0,\mathbf{v}))\right) = 0.$$

This holds for any $\mathbf{v} \in \mathfrak{g}$, and $\ker A_{g\mathrm{P}}((0,\dot{\mathbf{s}})) \supset \left\{(\dot{R},\dot{s}) : \dot{R} = 0\right\}$, so it must be the case that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R_g^* A_{g\mathrm{P}}((0,\dot{\mathbf{s}}))\right) = 0 \quad \text{and separately} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(R_g^* A_{g\mathrm{L}}((\dot{R},0))\right) = 0.$$

These two equations are respectively equal to conservation of momentum and conservation of angular momentum.

7.2 The mathematical essence of this analysis

From a mathematical point, many of the constructions above are more complicated than necessary. The decomposition of our group into a rotation and translation part provides additional physical intuition, but does not aid mathematical understanding. For this reason, I provide here a very brief overview of exactly what steps were taken without any relation to their physical meaning.

We began with a configuration space that formed a Lie group G. On the Lie algebra of this Lie group, $\mathfrak{g}$, we defined an operator $A : \mathfrak{g} \to \mathfrak{g}^*$ which was linear, positive definite, and symmetric. For $X, Y \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$,

1. $A(\lambda X + Y) = \lambda A(X) + A(Y)$,
2. $A(X)(X) \geq 0$,
3. $A(X)(Y) = A(Y)(X)$.

This operator provides an inner product on the Lie algebra by

$$\langle X, Y \rangle_A = A(X)(Y).$$

We then extended this inner product through left-invariance to obtain a metric on the entire Lie group. We chose left-invariance because we knew from the physical interpretation of our system that the metric should be left-invariant.

Theorem 7.2. *When extending an inner product on the Lie algebra $\mathfrak{g}$ of the Lie algebra G through left-invariance, we obtain a metric on G.*

$$\langle X, Y \rangle_{A_g} := A_g(X)(Y) := L_{g^{-1}}^* A L_{g^{-1}*}(x)(Y) = A(L_{g^{-1}*}X)(L_{g^{-1}*}Y)$$

for all $g \in G$ and $X, Y \in T_gG$.

Proof.

(i) The operator is linear in every tangent space. For $g \in G$, $X, Y, Z \in T_gG$, and $\lambda \in \mathbb{R}$,

$$\begin{aligned} A_g(\lambda X + Y) &= L^*_{g^{-1}} A(L_{g^{-1}*}(\lambda X + Y)) \\ &= L^*_{g^{-1}} A(\lambda L_{g^{-1}*} X + L_{g^{-1}*} Y) \\ &= \lambda L^*_{g^{-1}} A(L_{g^{-1}*} X) + L^*_{g^{-1}} A(L_{g^{-1}*} Y) \\ &= \lambda A_g(X) + A_g(Y). \end{aligned}$$

(ii) The operator is symmetric in every tangent space. For $g \in G$ and $X, Y \in T_gG$

$$A_g(X)(Y) = A(L_{g^{-1}*}X)(L_{g^{-1}*}Y) = A(L_{g^{-1}*}Y)(L_{g^{-1}*}X) = A_g(Y)(X).$$

(iii) The operator is positive definite (since we already showed linearity and symmetry, we only have to show positive semidefiniteness)

$$A_g(X)(X) = A(L_{g^{-1}*}X)(L_{g^{-1}*}X) \geq 0.$$

□

We then applied Noether's theorem to find that the equation of motion for a geodesic $g(t)$ may be split into two first order equation.

$$\xi = L_{\gamma^{-1}*}\dot{\gamma} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial A(\xi)(\xi)}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial A(\xi)(\xi)}{\partial \xi}$$

Symmetry and linearity of the A operator gives that $\frac{\partial A(\xi)(\xi)}{\partial \xi} = 2A(\xi)$ so that these equations may instead be written in terms of $\eta := A(\xi)$ as

$$\eta = L^*_{\gamma^{-1}} A(\dot{\gamma}) \quad \text{and} \quad \frac{\mathrm{d}\eta}{\mathrm{d}t} = \mathrm{ad}^*_{A^{-1}\eta}\,\eta.$$

Finally, we defined the bilinear operator B by the equality

$$\langle [a, b], c\rangle_A = \langle B(c, a), b\rangle_A$$

for all $a, b, c \in \mathfrak{g}$. Using Theorem 7.1 we then found that the Geodesic equation may be written as the pair of equations

$$\xi = L_{\gamma^{-1}*}\dot{\gamma}, \quad \text{and} \quad \frac{\mathrm{d}\xi}{\mathrm{d}t} = B(\xi, \xi).$$

Instead of using Noether's theorem, we could have also used the more specific Euler-Poincaré reduction theorem. The Euler-Poincaré reduction theorem has the advantage that it brings us immediately to the Euler-Arnold equation and is more readily generalized to infinite dimensional manifolds.

Theorem 7.3 (Euler-Poincaré Reduction Theorem (for a Matrix Lie group))**.** For any left-invariant Lagrangian L whose restriction to the Lie algebra is given by ℓ. Then for a path $\gamma(t) \in C^\infty([a,b], M)$ with $\xi(t) := L_{\gamma(t)^{-1}*}\dot{\gamma}(t)$ and M a Matrix Lie group, the following are equivalent [22], [23]

1. $\gamma(t)$ satisfies the Euler–Lagrange equations for L on G;
2. $\gamma(t)$ obeys the variational principle $\delta \int_a^b L(\dot{\gamma})\mathrm{d}t = 0$ for all variations with fixed endpoints;
3. $\xi(t)$ obeys the variational principle $\delta \int_a^b \ell(\xi)\mathrm{d}t = 0$ for variations of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$ with $\eta \in C^\infty([a,b], \mathfrak{g})$ and $\eta(a) = \eta(b) = 0$.
4. $\xi(t)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}$ where ℓ is the restriction of the Lagrangian to the Lie algebra.

Proof. The equivalence of the first and the second is immediate as this is how the Euler-Lagrange equations are obtained.

The second to the third is quite technical. Since the Lagrangian is left-invariant,

$$\int_a^b L(\dot{\gamma})\mathrm{d}t = \int_a^b \ell(\xi)\mathrm{d}t$$

The proof then boils down to the claim that all variations of γ with fixed endpoints correspond to variations of ξ of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$. If we take a smooth family of curves $\gamma(t, \varepsilon)$ and let $\xi = L_{\gamma^{-1}*}\frac{\partial \gamma}{\partial t}$ and $\eta = L_{\gamma^{-1}*}\frac{\partial \gamma}{\partial \varepsilon}$, then we would need

Lemma 7.4. For $\xi = L_{\gamma^{-1}*}\frac{\partial \gamma}{\partial t}$ and $\eta = L_{\gamma^{-1}*}\frac{\partial \gamma}{\partial \varepsilon}$ with $\gamma(t, \varepsilon)$ some smooth family of curves with fixed endpoints on a matrix Lie group.

$$\frac{\partial \xi}{\partial \varepsilon} = \frac{\partial \eta}{\partial t} + [\xi, \eta],$$

This lemma will be proved right after the current proof is complete. The equivalence of the third and fourth follows through integration by parts

$$\begin{aligned}
\delta \int_a^b \ell(\xi)\mathrm{d}t &= \delta \int_a^b \frac{\partial \ell}{\partial \xi}\mathrm{d}t \\
&= \delta \int_a^b \frac{\partial \ell}{\partial \xi}\left(\dot{\eta} + [\xi, \eta]\right)\mathrm{d}t \\
&= \delta \int_a^b -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \xi}\eta + \mathrm{ad}^*_\xi\left(\frac{\partial \ell}{\partial \xi}\right)\eta\mathrm{d}t = 0.
\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \xi}\eta = \mathrm{ad}^*_\xi\left(\frac{\partial \ell}{\partial \xi}\right).$$

□

Proof of Lemma 7.4. In the case of a Matrix Lie group such as $\mathrm{SE}(N)$ or $\mathrm{SO}(N)$, this can be derived using the product rule

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon}\left(g^{-1}\frac{\partial g}{\partial t}\right) &= -g^{-1}\frac{\partial g}{\partial \varepsilon}g^{-1}\frac{\partial g}{\partial t} + g^{-1}\frac{\partial^2 g}{\partial \varepsilon \partial t} \\
&= -\eta\xi + \xi\eta - \xi\eta + g^{-1}\frac{\partial^2 g}{\partial \varepsilon \partial t} \\
&= [\xi, \eta] + \frac{\partial \eta}{\partial t}
\end{aligned}$$

□

8 Lie group analysis of other physical systems

It turns out that the rigid body is far from being the only physical system that can be described through a Lie group analysis. One notable example is the flow of an ideal(incompressible and inviscid) fluid in a manifold M which is described by geodesics in the space of volume preserving diffeomorphisms of that manifold to itself, $\mathrm{Diff}_\mu(M)$. This example will be discussed in more detail in the next section. In [1] and in Table 1, many more systems are given.

Table 1: This table, adapted from [1] shows a list of relevant physical equations that may be derived as a geodesic equation, along with the Lie group and metric that can be used to derive the equation.

Lie group	metric	Euler-Arnold equation
$\mathrm{SO}(N)$	$A(\Omega)(\Omega)$	Euler top (a rigid body with a fixed point)
$\mathrm{SE}(N)$	$A((\Omega,\mathbf{v}))((\Omega,\mathbf{v}))$	rigid body moving in a body in a fluid
$\mathrm{Diff}(S^1)$	L^2	Hopf/inviscid Burgers' equation
$\mathrm{Diff}(S^1)$	$\dot{H}^{1/2}$	Constantin–Lax–Majda-type equation
Virasoro	L^2	Korteweg de Vries equation
Virasoro	H^1	Camassa–Holm equation
Virasoro	$\dot{H}^1$	Hunter–Saxton (or Dym) equation
$\mathrm{Diff}_\mu(M)$	L^2	Euler ideal fluid
$\mathrm{Diff}_\mu(M)$	H^1	averaged Euler flow
$\mathrm{Sympp}_\omega(M)$	L^2	symplectic fluid
$\mathrm{Diff}(M)$	L^2	EPDiff equation
$\mathrm{Diff}_\mu(M) \rtimes \mathrm{Vect}_\mu(M)$	$L^2 \oplus L^2$	Magnetohydrodynamics
$C^\infty(S^1, \mathrm{SO}(3))$	H^{-1}	Heisenberg magnetic chain

The analysis of these systems is simplified by the fact that Theorem 7.3 from the analysis of the rigid body does in fact hold for *all* systems with a left-invariant metric. The proof is readily generalized using a result from [24]

Theorem 8.1 (Euler-Poincaré reduction theorem)**.** Consider a left-invariant Lagrangian L whose restriction to the Lie algebra is given by ℓ. Then for a path $\gamma(t) \in C^\infty([a,b], M)$ with $\xi(t) := L_{\gamma(t)^{-1}*}\dot{\gamma}(t)$ and M any Lie group, the following are equivalent [22], [23]

1. $\gamma(t)$ satisfies the Euler-Lagrange equations for L on G;
2. $\xi(t)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial \ell}{\partial \xi}$.

Proof. The proof is extremely similar to the proof of the Euler-Poincaré reduction theorem for a matrix Lie group. To generalize the proof to non-matrix Lie groups, we must prove Lemma 7.4 in the case of an arbitrary Lie group. For this, we use the shorthand notation provided by the logarithmic derivative. The left logarithmic derivative is given by [23]

$$\delta^l \gamma = L_{\gamma^{-1}(t)*}\mathrm{d}\gamma,$$

which relates the the derivative of a logarithm

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\bigl(f(t)\bigr) = \frac{1}{f(t)}f'(t).$$

In case γ is a function of t and ε, this means

$$\delta^l\gamma = \xi\mathrm{d}t + \eta\mathrm{d}\varepsilon,$$

where η and ξ are defined the same as in Theorem 7.3. Now, it is the case that [23], [24]

$$\mathrm{d}\delta^l f + \left[\delta^l f, \delta^l f\right]^\wedge = 0 \tag{7}$$

for all $f \in C^\infty([a,b], M)$. Here $[\cdot,\cdot]^\wedge$ denotes the exterior product induced by the commutator. It is defined by

$$[\alpha, \beta]^\wedge(u,v) = \frac{1}{2}\Bigl(\bigl[\alpha(u), \beta(v)\bigr] - \bigl[\alpha(v), \beta(u)\bigr]\Bigr),$$

for $u, v \in \mathfrak{g}$ and $\alpha, \beta \in \Omega^1(U)$ with U some open set containing the identity element.

If we now apply this vanishing 2-form from equation 7 to the basis elements $(\frac{\partial}{\partial t}, \frac{\partial}{\partial\varepsilon})$, we obtain

$$\begin{aligned} 0 &= \left[\left(\frac{\partial\xi}{\partial\varepsilon} - \frac{\partial\eta}{\partial t}\right)\mathrm{d}t\wedge\mathrm{d}\varepsilon + [\xi\mathrm{d}t + \eta\mathrm{d}\varepsilon, \xi\mathrm{d}t + \eta\mathrm{d}\varepsilon]^\wedge\right]\left(\frac{\partial}{\partial\varepsilon}, \frac{\partial}{\partial t}\right) \\ &= \frac{\partial\xi}{\partial\varepsilon} - \frac{\partial\eta}{\partial t} + \frac{1}{2}\left([\eta,\xi] - [\xi,\eta]\right), \end{aligned}$$

which is indeed equivalent to

$$\frac{\partial\xi}{\partial\varepsilon} = \frac{\partial\eta}{\partial t} + [\xi,\eta].$$

□

9 Lie groups for the analysis of fluids

When analysing fluid flows, we will be in all but a few cases dealing with groups of diffeomorphisms. When the flow is contained within a manifold M, the group of diffeomorphisms of that manifold $\mathrm{Diff}(M)$ is defined as follows

$$\left\{F \in C^\infty(M, M) : F \text{is a bijection and } F^{-1} \in C^\infty(M, M)\right\},$$

with composition as the group rule. The diffeomorphisms should also be continuously connected to the identity element, so that for all $F \in \mathrm{Diff}(M)$ there is some smooth flow on the manifold $g(t, x)$ and some $T \in \mathbb{R}$ so that $g(0, x) = x$ and $g(T, x) = F(x)$. For example, if we take diffeomorphisms of $(0, 1) \subset \mathbb{R}^n$, the diffeomorhism $x \mapsto 1 - x$ is not included.

Within this diffeomporphism group, there are two important subgroups. The subgroup of volume-preserving diffeomorphisms and the subgroup of symplectomorphisms. The subgroup of volume preserving diffeomorphisms is written $\mathrm{Diff}_\mu(M)$ where μ is the volume form that is preserved. This group has the additional condition that

$$|\det \mathrm{d}F_x| = 1,$$

where $\det \mathrm{d}F_x$ is defined to be the volume of a unit parallelapiped after it has been transformed by the map $\mathrm{d}F_x$. That is, take some set of vectors $e_1, \ldots, e_n$ so that $\mu(e_1, \ldots, e_n) \neq 0$, then

$$\det(\mathrm{d}F_x) = \frac{\mu(\mathrm{d}F_x(e_1), \ldots, \mathrm{d}F_x(e_n))}{\mu(e_1, \ldots, e_n)}.$$

The choice of e_i does not matter here since in an n-dimensional vector space, any n linearly independent vectors can be written as a linear combination of any other n linearly independent vectors.

The Lie algebra of the diffeomorphism group consists of smooth vector fields on M, $\mathfrak{X}(M)$. This may be seen to hold because for any path $g(t, \cdot) \in \mathrm{Diff}(M)$ with $g(0, x) = x$, $\frac{\partial}{\partial t}\Big|_{t=0} g(t, x) \in T_{g(0,x)}M = T_xM$. The condition that $g(t, \cdot) \in C^\infty(M, M)$ is what gives us the additional constraint that vector fields in the Lie algebra must be smooth [1].

When restricting to the volume-preserving diffeomorphisms, the Lie algebra will be restricted to those vector fields whose divergence vanishes. In case the manifold has a boundary, the vector fields should be parallel to that boundary [1].

The subgroup of symplectomorphisms is the subgroup that preserves a symplectic form $\omega \in \Omega^2(M)$.

9.1 The Virasoro group

The Virasoro group is the 'central universal extension' of $\mathrm{Diff}(S^1)$. It can be written as a semidirect product

$$\mathrm{Vir} = \mathrm{Diff}(S^1) \rtimes \mathbb{R}.$$

The composition rule is given by

$$(f,a)\circ(g,b)=\left(f\circ g, a+b+\int_{s^1}\log\bigl(f(g(x))\bigr)'\mathrm{d}\left(\log g'(x)\right)\right),$$

where the logarithm is defined as the inverse of the exp map and we use an identification between the Lie algebra of S^1 and the real numbers. The choice of which identification allows one to add an arbitrary constant into the equation but we will use the identification such that $\ker(\exp)=\{2\pi k : k\in\mathbb{Z}\}$.

The Lie algebra of the virasoro group is then a semidirect product between the Lie algebra os $\mathrm{Diff}(S^1)$ and $\mathbb{R}$. The Lie algebra of $\mathrm{Diff}(S^1)$ is the vector fields on S^1 so

$$\mathfrak{vir}=\mathrm{Vect}(S^1)\oplus\mathbb{R},$$

with the commutation operation

$$\left[(f\frac{\partial}{\partial x},a),(g\frac{\partial}{\partial x},b)\right]=\left(\left(f'\cdot g-g'\cdot f\right)\frac{\partial}{\partial x},\int_{S^1}f'\cdot g''\mathrm{d}x\right),$$

where $\cdot$ denotes pointwise multiplication.

9.2 The commutator on a diffeomorphism group

The commutator on the lie algebra of vector fields is taken to be negative the Poisson bracket [1]. This can be written in coordinates as

$$[X,Y]_i=-\{X,Y\}_i=-\sum_j\left(X_j\frac{\partial Y_i}{\partial x^j}-Y_j\frac{\partial x_i}{\partial x^j}\right)=\sum_j\left(Y_j\frac{\partial x_i}{\partial x^j}-X_j\frac{\partial Y_i}{\partial x^j}\right).$$

In particular the case of a one-dimensional fluid flow can be described by the group $\mathrm{Diff}(S^1)$. In this case the commutator simplifies to be

$$[X,Y]=Y\frac{\partial X}{\partial y}-X\frac{\partial Y}{\partial x}.$$

Although the commutator is important, the Euler-Arnold equation is written in terms of ad^*, not ad. The expression for ad^* can be found by first finding the expression of Ad. The right-invariance of the metric gives reason to identify all the tangent spaces of the metric with vector fields. For $X\in T_x\,\mathrm{Diff}(M)$ and $Y\in T_y\,\mathrm{Diff}(M)$, we write

$$X\sim Y \iff \exists g\in\mathrm{Diff}(M): X=R_{g*}Y.$$

This allows us to associate a unique element of $\mathrm{Vect}(M)$ to every element in $T\,\mathrm{Diff}(M)$. In addition, we obtain that when we define equivalence in this way, the Ad operator becomes

$$\mathrm{Ad}_g^*\,\eta\sim L_g^*\eta.$$

This means that the ad operator is given by

$$\mathrm{ad}_\xi^*\,\eta\sim\mathcal{L}_\xi\eta,$$

where $\mathcal{L}_\xi$ is the Lie derivative.

10 Metrics for the analysis of fluids

There are several possible metrics on the space of diffeomorphisms. They all differ from the metric used for the analysis of a rigid body in one critical way. Instead of being left-invariant, they are right invariant. The difference between left- and right-invariance is essentially whether the operation is applied before or after the time-evolution of the system. Right-invariance, can then be interpreted as a relabeling of the particles in the system done before the system's time-evolution takes place. See Figure 5.

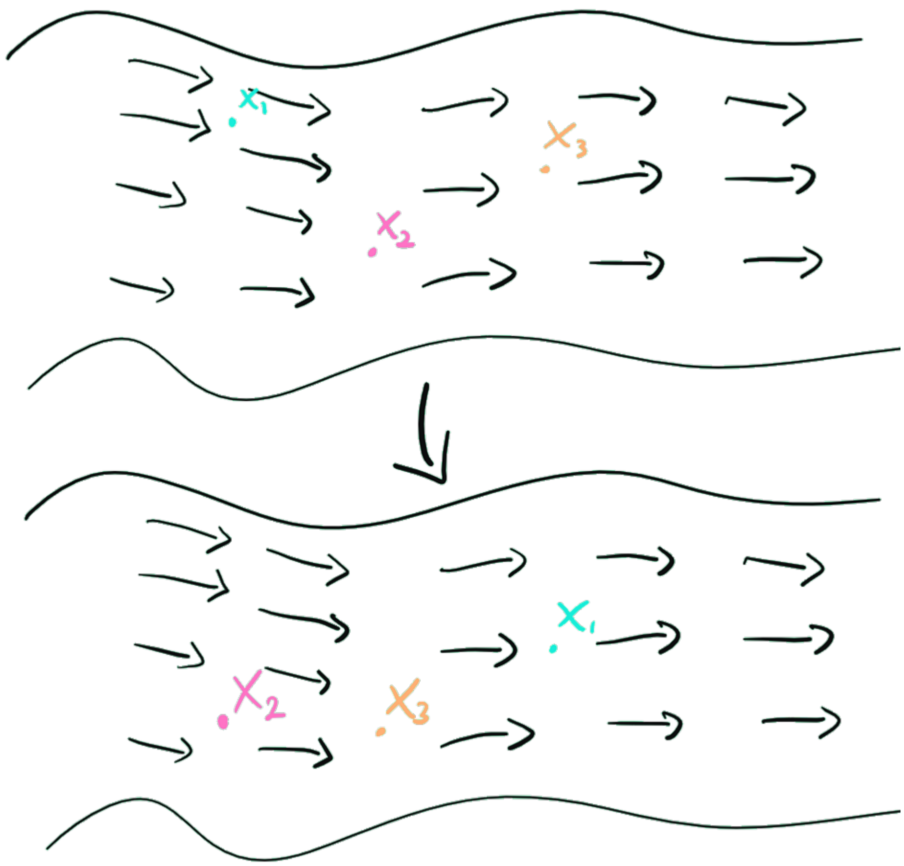

Figure 5: The action of a diffeomorphism permutes the position of all points (particles) in the fluid. When done through right multiplication i.e. before the fluid flows, this amounts to a mere relabeling of particles which does not change the Lagrangian of the system. The same is not true in case of left multiplication since the pushfoward of left-multiplication will also move around velocity vectors and possibly stretch or contract them.

One could mistakenly presume that the fact that these groups are right and not left-invariant means Theorem 8.1 is no longer useful. Fortunately, a simple change in definition of the group operator can turn a right-invariant metric into a left-invariant metric. Instead of endowing the manifold with $\circ : (g, h) \mapsto g \circ h$, we can use the operator $* : (g, h) \mapsto h \circ g$. Under the operator $*$, the manifold remains a Lie group and as such all theorems for left-invariant metrices can be applied in the right-invariant case except for the need to interchange left and right multiplications in the definitions of all terms.

The exchange of left and right multiplication has one crucial effect on the Euler-Arnold equation. This is that the adjoint operator Ad_g under the $\circ$ operator becomes the $\mathrm{Ad}_{g^{-1}}$ under the $*$ operator. In the Euler-Arnold equation this means that for a right-invariant metric we have

$$\eta = L^*_{\gamma^{-1}}\dot{\gamma}, \quad \frac{\mathrm{d}\eta}{\mathrm{d}t} = -\,\mathrm{ad}^*_{A^{-1}\eta}\,\eta,$$

and correspondingly

$$\xi = L^*_{\gamma^{-1}}\dot{\gamma}, \quad \frac{\mathrm{d}\xi}{\mathrm{d}t} = -B(\xi,\xi),$$

with B defined the same as in Theorem 7.1. The L^2 metric is perhaps the most intuitive metric to place on a diffeomorphism group. As discussed before, the Lie algebra of the (volume-preserving) diffeomorphism group consists of (divergence-free) vector fields. One can define the inner product on such a vector field by the equation

$$\langle V, V\rangle = \int_M (V\cdot V)\mu.$$

For a review of how such integrals on manifolds are defined, see [2]. This metric encapsulates in a very tangible way the energy stored in the fluid flow. The volume form μ may be treated as the density at a point whilst $V\cdot V$ is the square velocity of the flow at that point.

The L^2 metric may also be called the (0,2)-Sobolev norm $W^{0,2}$. A Sobolev norm is defined by [4]

$$\|f\|_{W^{k,p}} = \left(\sum_{i=0}^{k}\int_M \left\|\boldsymbol{\nabla}^i f(x)\right\|^p \mathrm{d}x\right)^{1/p}.$$

In the special case where $p = 2$, the norm is induced by a metric which is called H^k. Confusingly, this H^k metric is also called the Sobolev metric. The H^1 metric can for example be used on the Virasoro group to obtain the Camassa Holm equation, and the well known L^2 metric is equal to the H^0 metric [4].

It is also possible to define a so-called degenerate Sobolev metric $\dot{H}^2$. This metric is defined by

$$\|f\|_{\dot{H}^s} = \left(\int_M \left\|\boldsymbol{\nabla}^s f(x)\right\|^2 \mathrm{d}x\right)^{1/2}.$$

In case $0 < s < 1$, this is defined through fractional laplacians $\boldsymbol{\nabla}^s f = (-\Delta)^{s/2} f$, which are themselves defined through the Fourier series [25].

$$\mathcal{F}((-\Delta)^{s/2} f)(\xi) = |\xi|^s\,\mathcal{F}(f)(\xi).$$

10.1 The Hopf or inviscid Burgers' equation

10.1.1 History

Burgers' equation was initially proposed by Henry Bateman in 1915. He introduced the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

with $0 < x < L, \nu \in \mathbb{R}^+$ and $0 < t < T$. The equation was later used by Burgers to mathematically model turbulent fluids. Later on, both Eberhard Hopf and Julian David Cole found a substitution $u(x,t) = -2\nu\frac{\theta_x}{\theta}$ by which Burgers' equation could be simplified into a heat equation in θ [26]. When deriving this equation from a Lie theoretic approach, the viscosity term $\nu\frac{\partial^2 u}{\partial x^2}$ does not appear.

Without the viscosity term, the equation is known as the Hopf equation or the inviscid Burgers' equation. The equation can be solved by the method of characteristics. Provided initial conditions $u(0,x) = f(x)$, the method of characteristics yields the implicit solution

$$u(t,x) = f(x - u(t,x)t).$$

However, this solution may not always be defined for arbitrary time. It is possible for characteristic curves to cross in a process known as shock formation. Figure 6 shows a numeric simulation of the Hopf equation that displays shock formation around $t = 250$.

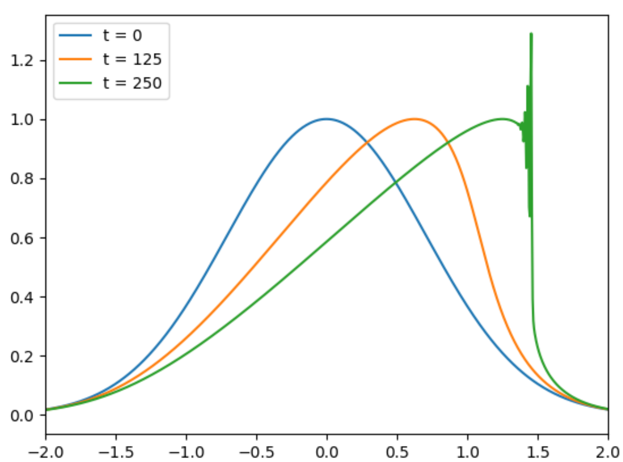

Figure 6: A solution to the inviscid Burgers' equation. The initial condition is $u(0,x) = e^{-x^2}$ and simulation is performed with a timestep of 0.001. At $t = 250$ we see the beginning of a shock wave around $x = 1.5$.

10.1.2 Lie group derivation

There is a very fortunate result that the Lie group of diffeomorphisms from a locally flat manifold to itself is locally flat under the L^2 metric [1], [27]. For this reason, the geodesic equations on such Lie groups are extremely simple. The only requirement is that the acceleration of any curve vanishes [1]

$$\frac{\partial^2 g(t,x)}{\partial t^2} = 0.$$

Let $g(t,x)$ be a geodesic in $\mathrm{Diff}(S^1)$ and let $v(t,x) = \frac{\partial}{\partial t} g(t, g(t,x))$. Then the geodesic equation reads

$$\frac{\partial^2 g(t,x)}{\partial t^2} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0.$$

Had we started from the Euler-Arnold equation, we would have obtained a slightly different (but equivalent) result.

$$\begin{aligned}
\langle a, [b,c]\rangle &= -\langle a, \{b,c\}\rangle \\
&= -\int_{S^1} \left(ab\frac{\partial c}{\partial x} - a\frac{\partial b}{\partial x} c \right) \mu \\
&= -\int_{S^1} \left(-\frac{\partial a}{\partial x} bc - a\frac{\partial b}{\partial x} c - a\frac{\partial b}{\partial x} c \right) \mu \\
&= \int_{S^1} \left(\frac{\partial a}{\partial x} bc + a\frac{\partial b}{\partial x} c + a\frac{\partial b}{\partial x} c \right) \mu \\
&= \int_{S^1} \left(\frac{\partial a}{\partial x} b + a\frac{\partial b}{\partial x} + a\frac{\partial b}{\partial x} \right) c\mu
\end{aligned}$$

so that

$$B(v,v) = \frac{\partial v}{\partial x} v + 2v\frac{\partial v}{\partial x} = 3v\frac{\partial v}{\partial x}.$$

As the metric for a fluid is right-invariant, the equation of motion is then

$$\frac{\partial v}{\partial t} = -3v\frac{\partial v}{\partial x}.$$

This differs by a factor 3 from the original geodesic equation but this difference may be reconciled by rescaling time. Do note that we may rescale time in *this* equation, but we should not simultaneously rescale time in the equation

$$v = \frac{\partial}{\partial t} g(t, g(t,x)).$$

10.2 The Euler equation

10.2.1 History

The Euler equation is one of the oldest equations used in Fluid dynamics. The equations were introduced by Leonhard Euler in the 18th century. Much of Euler's work was inspired by earlier work done by the Bernoullis [28]. Euler's equation describes the behavior of an ideal fluid which means it is both incompressible and inviscid.

10.2.2 Lie group derivation

Like in the derivation of the Hopf equation, we use that $\mathrm{Diff}(M)$ is locally flat under the L^2 metric. In this case, however, we are concerned with the subgroup $\mathrm{Diff}_\mu(M)$. The fact that we are dealing with a subgroup introduces a slight complication. The acceleration $\partial^2_{tt} g$ need no longer vanish! Instead, the acceleration must be orthogonal to the subgroup [1]. This can be interpreted as a constraining force that keeps the geodesic within $\mathrm{Diff}_\mu(M)$. We thus have

$$\langle \partial^2_{tt} g, u \rangle = 0,$$

for all divergence-free vector fields u. More precise information on the curvature of $\mathrm{Diff}_\mu(M)$ can be found in [29].

Lemma 10.1. If for some 1-form $\alpha \in \Omega^1(M)$, $\alpha(u) = 0$ for all divergence free vector fields u on M, then α is the differential of a zero form (scalar field) [1].

Proof. Consider any closed curve γ and let u_ε be supported on a torus T_ε around this curve with radius ε. Denote by Φ the flux of this vector field through any cross-section of the torus. The flux is necessarily constant due to the divergence free-ness of the vector field. As ε goes to zero,

$$\lim_{\varepsilon \to 0} \int_M \alpha(u_\epsilon)\mu = \lim_{\varepsilon \to 0} \int_T \alpha(u) = \Phi \int_\gamma \alpha = 0.$$

Since we are free to choose Φ to have a nonzero value, we obtain that the integral of α vanishes over all closed curves. Thus, α is an exact 1-form. □

Therefore, it must be the case that $\left(\partial^2_{tt} g\right)^\flat$ is an exact 1-form.

$$\left(\frac{\partial^2 g}{\partial t^2}\right)^\flat = -\mathrm{d}P,$$

where $\flat$ is the musical isomorphism[8] between the 1-forms and the vector fields, whose inverse is $\sharp$. Rather neatly then, we may apply the sharp operator to both sides of this

[8] This isomorphism sends a vector v to a covector $\flat v = \langle v, \cdot \rangle$ and may thus be seen to be essentially a contraction with the metric.

equation and use the identity $\boldsymbol{\nabla} = \sharp\mathrm{d}$ to find that $\partial^2_{tt} g$ must be a gradient field $-\boldsymbol{\nabla} P$ [2]. Using the chain rule to work out $\partial^2_{tt} g$, Euler's equation is derived [21]

$$\frac{\partial v}{\partial t} + \boldsymbol{\nabla}_v v = -\boldsymbol{\nabla} P, \qquad \boldsymbol{\nabla} \cdot v = 0,$$

where $\boldsymbol{\nabla}$ is the Levi-Cevita connection. Notice that $\mathbf{P}$ is not constant in time but rather varies as is given by the condition that $\boldsymbol{\nabla} \cdot v$ must remain 0.

10.3 Conserved quantities

The flow of an ideal fluid has several conserved quantities. The number of conserved quantities differs based on the dimension of the manifold. One example of a conserved quantity for an ideal fluid is the vorticity. The vorticity ξ is given by

$$\xi = \mathrm{d}u,$$

where $u = v^\flat$ is the dual vector field associated to the velocity vector field. As discussed in Section 9.2, the time derivative of u is given by

$$\frac{\partial u}{\partial t} = \mathcal{L}_v u.$$

The exterior derivative will commute with the time-derivative for smooth vector fields, so by Cartan's magic formula [2],

$$\frac{\partial \xi}{\partial t} = \mathrm{d}\mathcal{L}_v u = \mathrm{d}\left(i_v \mathrm{d}u + \mathrm{d}i_v u\right) \tag{8}$$

The exterior derivative is nilpotent so the second term will vanish, and in the first term we may replace $\mathrm{d}u$ with ξ.

$$\frac{\partial \xi}{\partial t} = \mathrm{d}i_v \xi, \tag{9}$$

which is in fact equal to the Lie derivative again as the second part of Cartan's magic formula $i_v\mathrm{d}$ will vanish when applied to the vorticity which is an exact 2-form.

$$\frac{\partial \xi}{\partial t} = \mathcal{L}_v \xi. \tag{10}$$

This equation means that the vorticity is essentially frozen into the fluid.

11 Conclusion

In conclusion, the research presented in this thesis provides an interesting new way of viewing the behavior of free physical systems. It provides a comprehensive, albeit brief review of the mathematical concepts required to understand the Euler-Poincaré reduction theorem, the Euler-Arnold equation and the resulting symmetries that can be found.

The method of Lie group analysis allowed for a unification of two different areas of physics - rigid bodies and fluid flow - by exploiting a similarity in their Lagrangian and the symmetries of these systems. Namely, both systems have a Lagrangian that depends solely on the time-derivative of the system's configuration. In the case of the N-dimensional rigid body this configuration space was $\mathrm{SE}(N)$ or $\mathrm{SO}(N)$ when the rigid body had a single fixed point. In the case of fluids that filled a container M, this configuration space was $\mathrm{Diff}(M)$, the space of diffeomorphisms from M to itself.

The fact that these systems had a Lagrangian depending solely on the time derivative of the configuration meant that their Lagrangian could be represented as a quadratic form on the tangent space to these configuration spaces. This quadratic form could then be written as a metric which allowed for the conclusion that - under a certain metric - the equation of motion of the physical systems would be the equation of a geodesic on these configuration spaces.

The symmetries of both systems were discussed. A rigid body had a left-invariant metric which could be understood as a Lagrangian that would not change when one switched to a different reference frame. In contrast, a fluid had a right-invariant metric which could be understood as a Lagrangian that would not change when one gave a different labeling to the particles. Using the Euler-Poincaré theorem, the geodesic equation for a left-invariant metric was reduced from a second order equation to a first order equation and by reversing the order of the group operation, this reduced equation could also be used to describe fluids. In the case of a rigid body with an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ (defined so that $A(\omega)(\xi) = \langle \omega, \xi \rangle$ is the bilinear form induced by the Lagrangian) which followed a path $\gamma : [a, b] \to \mathrm{SE}(N)$ (or $\mathrm{SO}(N)$ respectively), the angular velocity was defined as $\omega(t) := L_{\gamma(t)^{-1}*}\dot{\gamma}(t)$ and the Euler-Arnold equation was found to be

$$\frac{\partial A(\omega)}{\partial t} = \mathrm{ad}^*_{\omega} A(\omega).$$

For a fluid the angular velocity was similarly defined and when the fluid had a velocity field v, the Euler-Arnold equation was found to be

$$\frac{\partial A(v)}{\partial t} = \mathrm{ad}^*_{v} A(v).$$

Finally, several systems that could be described with Lie group analysis using different groups of metric were mentioned and it was explained which metric and group were needed to arrive at each of the different physical equations. Not all these systems were described but in particular the Hopf/inviscid Burgers' equation and the Euler ideal flow equation were, their behavior was derived and shown to indeed follow from the framework provided by Lie group analysis.

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