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Symmetric Group and Double Copy Building Blocks of Scattering Amplitudes

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Abstract

The double copy construction was originally derived as a consequence of colour-kinematics duality in Yang-Mills theory. Since then, the concept was extended to hold for many other theories. In this work, notions related to scattering amplitudes are reviewed, and the idea of double copy is introduced. The goal of this study is to find representations which the building blocks of the double copy follow when transforming under the symmetric group. In order to achieve this, a Mathematica script was developed and is attached to this thesis. Results were obtained for scatterings of up to seven particles. Moreover, the method is general and extendable to higher point. Possible applications of the results in scalar kinematics are also indicated.

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Contents

1 Introduction

In early 20th century, two fundamental theories of the universe were born - general relativity and quantum mechanics. In the following decades, the latter developed into quantum field theory and branched off into more specific theories describing various parts of nature [1].

Nowadays, the most successful quantum theory is the standard model of particle physics which describes, among other things, the strong, weak, and electromagnetic interactions between particles. It gives accurate predictions which are tested and pushed to their limits by many experiments, such as those performed at the Large Hadron Collider at CERN. Although the standard model celebrated many great successes, it is incomplete. It cannot explain many mysteries of the universe such as the matter-antimatter asymmetry, dark matter and dark energy, and the origin of the universe. It also lacks any connection to gravity [2].

On the other hand, general relativity is all about gravity. When it was invented, it revolutionised the way we think about the gravitational pull. Instead of gravity being a force between two massive bodies in Euclidean space, it is a consequence of the curvature of spacetime which is caused by the contents of the universe [3]. This realisation had many consequences such as the anticipation of the existence of gravitational waves, which were recently detected [4]. General relativity provides excellent results at cosmic scales, however it fails at short distances where quantum effects of gravity would be important.

In the majority of scenarios, effects of either of these two theories can be neglected. For example when investigating the scattering of electrons, the gravitational effects are many ten-fold weaker than the electroweak force. Similarly, when observing motions of celestial bodies, the quantum forces on average cancel out over such long distances or they lack a sufficient range entirely.

There are however situations when both effects become relevant, such as near the centers of black holes or during the very early universe. The description of such environments requires a joint theory. This is not an easy task as combining these two theories together, into one testable, unified, self-consistent framework is one of the greatest open problems in theoretical physics [1].

According to the authors of [1], achieving this goal requires brand new approaches and symmetries. One such way is through the double copy perspective discussed in this work, which interprets gravitons as "gluons squared". This relation is possible due to a duality between colour and kinematic numerators which together make up a scattering amplitude of the Yang-Mills theory, underlying a large part of the standard model [1], [5], [6].

Similar dualities were found in other theories and so although the initial claim focused on Yang-Mills theory and perturbative general relativity, soon it was extended to hold for a large web of theories. This raises a few questions - how far does the double copy description reach? Are there general rules guiding the numerators? In this work, we attempt to solidify arguments leading to some of the results in article [7] and extending them. We aim to answer the questions

What representations govern the transformations of double-copy numerators under the symmetric group? Can we find a general method for obtaining them?

In order to be able to answer this research question, we must first understand the context and significance of the double copy construction.

We begin by stating notation which will be used in the rest of this work. Afterwards, in chapter 3, Lie groups and algebras are introduced, focusing on concepts important for the Yang-Mills theory. This is followed by a review of the symmetric group, its group-theoretical properties and representations. Furthermore, the notation and main features of Young tableaux, which are used to express representations of the symmetric group, are also stated here.

Chapter 4 is devoted to the two theories introduced earlier - Yang-Mills theory and general relativity. The concepts of Lie algebra are used here to define the field strength tensor of Yang-Mills theory and subsequently the Lagrangian. The Yang-Mills part of this chapter is finished off with a discussion on gauge symmetries. In the section about general relativity, the Einstein-Hilbert action is introduced, followed by Einstein equations. Those are yet to be generally solved, as of now only special solutions exist. A particular result is that of perturbative gravity which is introduced thereafter, together with diffeomorphism invariance.

Colour-kinematics duality is a statement about scattering amplitudes. Those will be covered from a traditional point of view in chapter 5. The Feynman approach of tackling their calculation is explained, as well as an example for a particular four-particle scattering.

In chapter 6, the notion of soft limit and Adler zero are introduced. Two useful results following from soft limits of photons and gravitons are derived. The infrared behaviour is also introduced as way of classifying scalar theories, in particular focusing on those with an enhanced soft limit.

Finally, the colour-kinematics duality and double copy are described in more details in chapter 7. After an introduction of the core ideas, an example about the scattering of four gluons demonstrates the main properties of this framework, such as the generalised gauge invariance. This chapter is finished off with an outlook beyond Yang-Mills theory and general relativity, discussing examples of other theories which have numerators suitable for the double copy construction.

At last, chapter 8 describes our research in this field - a search for a representation of the symmetric group. First, a strategy of how the research was conducted is outlined, and a complementary code is attached in appendix B. Afterwards, the results are presented, and lastly, suggestions for future research are made.

The work is closed off by a few concluding remarks about a potential value of research in the double copy construction among (not only) scalar theories.

2 Note on Notation

Before diving in, let us first introduce notation and conventions which will be used in the rest of this work.

2.1 Natural Units

There are three fundamental constants of nature relevant for the studied subject. They are the speed of light c, reduced Planck's constant $\hbar$, and the gravitational (or Newton's) constant G_N.

Following the example of other scientists in the field of (not only) particle physics and quantum field theory [8], [9], we will be working in the natural units,

$$c = 1 \quad \text{and} \quad \hbar = 1.$$

So say goodbye to these two as you will not be seeing them again beyond this section.

Although general relativity (GR) is briefly touched upon here and researchers focused on GR sometimes prefer to use a system which sets $G_N = 1$, instead of fixing $\hbar$ [3], it will not be done here. This is because the discussed subject is significantly more closely related to particle physics, and general relativity is covered only marginally.

It is discouraged to set all three constants to one, as that would remove the three related dimensions (time, length, mass) entirely and erase the opportunity to perform dimensional analysis [3].

In this natural system, the units of mass, energy, time, and length are related - some of them coincide directly while others are inverted. In particular,

$$[\text{energy}] = [\text{mass}] = [\text{time}]^{-1} = [\text{length}]^{-1}.$$

2.2 Einstein Summation

The majority of this work uses the Einstein summation convention. This means that whenever there is a repeated index, once as a superscript and once as a subscript, this index is implicitly summer over [10]. So for example,

$$x_i y^i \equiv \sum_i x_i y^i.$$

The sum always runs over all possible values of the index. Greek indices are reserved for denoting the four spacetime dimensions. They run from 0 to 3, where 0 denotes time and the positive integers denote space.

When differentiating with respect to a lower index, it counts as an upper index and vice versa. So,

$$\frac{\partial y^i}{\partial x^i} \equiv \sum_i \frac{\partial y^i}{\partial x^i}.$$

Furthermore, when differentiating with respect to spacetime coordinates, the following short-hand notation is used

$$\frac{\partial y^\mu}{\partial x^\mu} \equiv \partial_\mu y^\mu,$$

for spacetime coordinates x^μ. Notice that in ∂_μ the index is at an appropriate place for the summation. So such derivative signs have the indices in the opposite place than the differentiating coordinate did [10].

2.3 Spacetime Metric

In general,[1] upper and lower indices do not have the same meaning. This is because they transform differently (lower-indexed vectors are covariant while upper-indexed are contravariant). However, it is possible to switch between upper and lower indices by the use of an appropriate metric [9].

For flat four-dimensional (Minkowski) spacetime (which is assumed in all parts of this work unless specified otherwise), the following metric is used

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Once again, since the topic belongs to the field of particle physics, the mostly-negative version of the metric is considered.

In order to switch between the position of indices, the following relationship can be used $x^\mu = n^{\mu\nu} x_\nu$ [9].

[1] This does not hold for e.g. structure constants which will be encountered later in this work. This is because the metric for them is simply the identity (in an appropriate choice of basis).

3 Groups

In this chapter Lie groups and Lie algebras are introduced. These concepts are important in order to understand the essence of Yang-Mills theory in chapter 4.1. Afterwards, the focus is on the symmetric group which lies in the center of the research described in section 8. First, its basic concepts are reviewed, then representations and characters are described in general, until finally Young tableaux and representations of the symmetric group are discussed.

Before diving in, let us quickly review the definition of a group. A group is a set with an associative operation under which the set is closed. Furthermore, the set should contain an inverse of every element of the set as well as the identity element (with respect to the given group operation).

As an introductory example, let us demonstrate that the set of integers is a group under addition. It is closed because adding two integers yields an integer. The identity element is 0 (adding 0 to anything has no effect) which belongs to this set. Inverse of an integer n under addition is $-n$, which is also an integer. Lastly, addition is associative, and so we can conclude that it is indeed a group.

3.1 Lie Groups

Lie groups depend continuously on a number of local parameters $\alpha_1, ..., \alpha_n$. This number, n, determines the dimension of the Lie group. Each element of the group, g can be seen as a function of the n parameters;[2] $g = g(\alpha_1, .., \alpha_n) = g(\alpha)$. The parametrization will be chosen so that the identity element $e = g(0, 0, .., 0)$. Such parametrization always exists, for more details see [11], [12]. For instance, the special orthogonal group $SO(2)$, which consists of rotations in a plane, depends continuously on a single parameter, θ, which can be interpreted as the angle of the rotation.

Special Unitary Groups $SU(n)$ are examples of Lie groups which are relevant for particle physics. This is because strong force has roots in the $SU(3)$ group and the weak force in $SU(2)$ [4]. What this means in more detail will be covered in chapter 4.1.

Special unitary groups are defined as $n \times n$ unitary matrices with determinant equal to 1. Unitarity here refers to $U^\dagger U = \mathbb{1}$ for Hermitian conjugate $\dagger$ and $n \times n$ identity matrix $\mathbb{1}$. The determinant requirement decrease the number of degrees of freedom by one from the maximum possible, leading to dimension of the group being $n^2 - 1$ [13]. Only $SU(1)$ is an Abelian group, higher dimensional special unitary matrices are generally non-Abelian.

[2] However the parameters are only local so if we move too far away we may need to use a different parametrization. This becomes relevant for e.g. the unitary group $U(2)$ which consists of two disconnected components that have different parametrizations.

Representations. Elements of groups can be expressed as matrices via the notion of representation. In general, a representation R of group G is defined as the homomorphism (group-operation preserving map) from G most commonly to a general linear group $\mathrm{GL}(S)$ on space S which has a linear structure.

As our interest lies in the application of groups to physics, rather than in the algebraic structures themselves, for the rest of this work we will consider representations mapping to the group of $d \times d$ dimensional invertible matrices. We call this space $\mathrm{GL}(d, K)$ on a carrier space K.

As an example, let us consider a representation of the group $SO(2)$. Using our intuition about the physical meaning of its elements (rotations in a plane), we can write down its defining representation R to the group $\mathrm{GL}(2, \mathbb{R})$. For an element g of $SO(2)$,

$$R : g(\theta) \mapsto \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

A representation is reducible if there exists a subspace of K which remains invariant under the homomorphism. In the case of mapping into the group of invertible matrices, this translates into an existence of a basis in which the matrices $R(g)$ are block upper triangular for all $g \in G$. If this is not possible, or equivalently no invariant subspace exists, the representation is called irreducible [12].

The previously presented representation R of $SO(2)$ is irreducible. This can be seen for example by considering the absence of invariant subspaces in its carrier space which is the two-dimensional plane. By rotating about an axis perpendicular to the plane, any vector can be rotated out of a subspace which previously contained it.

However, a representation R' of the same group into the general linear group of a three-dimensional real space, $\mathrm{GL}(3, \mathbb{R})$, given by

$$R' : g(\theta) \mapsto \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is reducible. This can be seen in two ways. Firstly, the matrix can be partition into blocks such that it is block upper triangular. Secondly, when applying rotations in a plane, the vectors pointing along the axis about which we rotate will never leave this line as they are unaffected by the transformations. This means that there exists an invariant subspace (namely the axis about which we rotate), hence the representation must be reducible.

3.2 Lie Algebra

A Lie algebra $\mathfrak{g}$ is a vector space with Lie product $[T^a, T^b]$ which is defined for all $T^a, T^b \in \mathfrak{g}$ and is

1. closed: $[T^a, T^b] \in \mathfrak{g}$, for all $T^a, T^b \in \mathfrak{g}$
2. distributive: $[\alpha T^a + \beta T^b, T^c] = \alpha[T^a, T^c] + \beta[T^b, T^c]$ for all $T^a, T^b, T^c \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{C}$
3. antisymmetric: $[T^a, T^b] = -[T^b, T^a]$, for all $T^a, T^b \in \mathfrak{g}$

and satisfies the Jacobi identity: $[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$ for all $T^a, T^b, T^c \in \mathfrak{g}$. [12]

In this work, the only Lie product considered is the commutator of matrices $[T^a, T^b] = T^aT^b - T^bT^a$, however different products exist on more abstract spaces, see for example [14]. Furthermore note that the Lie product is sometimes also called a *Lie bracket* or a *two-bracket* [6], [14].

How does this vector space relate to the Lie groups mentioned above? Suppose a faithful d-dimensional representation R of a Lie group G, elements of which are determined by n parameters α, i.e. $g = g(\alpha)$. As discussed above, it is always possible to choose the parametrisation such that $g(\alpha = 0) = e$, and so $\mathbb{1} = R(e) = R(g(0)) = R(0)$, where 0 is the n-dimensional zero vector and $\mathbb{1}$, the d-dimensional identity matrix.

In order to obtain the elements of Lie Algebra, T^a, also called the generators of the Lie group, we Taylor expand and linearize $R(\alpha)$ around the origin, $\alpha = 0$. For small values of α we obtain

$$R(\alpha) \approx R(0) + \alpha_a \frac{\partial R(\alpha)}{\partial \alpha_a} \mid_{\alpha=0} = \mathbb{1} + i\alpha_a \left(-i \frac{\partial R(\alpha)}{\partial \alpha_a} \mid_{\alpha=0} \right) =: \mathbb{1} + i\alpha_a T^a.$$

Therefore we define T^a as $-i\frac{\partial R(\alpha)}{\partial \alpha_a} \mid_{\alpha=0}$ for a running from 1 to n, labelling the parameters α_a. Notice that $R(\alpha)$ is a $d \times d$ matrix and hence so is T^a. In principle, one could include more indices, labelling the elements of the matrix, then one would obtain $[T^a]^i_j$, however this is rarely done. Hence unless relevant, the two further indices will be kept implicit [12], [15].

Lastly, note that the number of the generators is equal to the number of the parameters defining group G, hence if G is an n-dimensional group, the associated Lie Algebra $\mathfrak{g}$ also has n dimensions.

3.2.1 Structure Constants

We can view the generators $T^1, ..., T^n$ as a basis for the Lie Algebra $\mathfrak{g}$, and so any element of $\mathfrak{g}$ can be expressed as a linear combination of the generators [12]. By the closeness

property of the Lie product (point 1 in the definition above), $[T^a, T^b] \in \mathfrak{g}$, so in particular, the commutator of two elements can be expressed as a combination of the basis elements

$$[T^a, T^b] = i f^{ab}{}_c \, T^c. \tag{1}$$

The coefficients in this linear combination are given by the structure constants f^{abc}, which, as tensors, are antisymmetric in all its indices [15]. Note that for the generators and structure constants, there is no difference between subscripts and superscripts (unlike in the case of, for example, position vectors in four-dimensional spacetime x^μ), however their order matters.

Equation 1 uniquely determines the Lie product of all elements of a particular Lie Algebra. Furthermore note that although we defined the structure constants using generators in representation R, they are independent of the representation and depend only on the structure of the group operation [13].

Furthermore, the Jacobi identity can be also rewritten in terms of the structure constants as

$$f_{bc}{}^e f_{ea}{}^d + f_{ca}{}^e f_{eb}{}^d + f_{ab}{}^e f_{ec}{}^d = 0. \tag{2}$$

This result can be obtained as follows. We begin from the Jacobi identity given in the definition of Lie Algebra, and substitute the Lie products using the basis decomposition as per equation 1,

$$\begin{aligned}
0 &= [T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] \\
&= [T^a, i f^{bc}{}_e T^e] + [T^b, i f^{ca}{}_e T^e] + [T^c, i f^{ab}{}_e T^e] \\
&= i f^{bc}{}_e [T^a, T^e] + i f^{ca}{}_e [T^b, T^e] + i f^{ab}{}_e [T^c, T^e] \\
&= i f^{bc}{}_e i f^{ae}{}_d T^d + i f^{ca}{}_e i f^{be}{}_d T^d + i f^{ab}{}_e i f^{ce}{}_d T^d \\
&= -f^{bc}{}_e f^{ae}{}_d T^d - f^{ca}{}_e f^{be}{}_d T^d - f^{ab}{}_e f^{ce}{}_d T^d \\
&= \left(f^{bc}{}_e f^{ea}{}_d + f^{ca}{}_e f^{eb}{}_d + f^{ab}{}_e f^{ec}{}_d\right) T^d.
\end{aligned}$$

As this result should hold for all elements of the Lie algebra, in particular for the basis elements such as T^d, and for non-trivial groups there is always a non-zero basis element, we obtain that the expression in parentheses must be 0, yielding equation 2.

3.2.2 Adjoint Representation

The adjoint representation is of particular importance and interest because gluons, the mediators of the strong force and the central elements of Yang-Mills theory, transform following this representation [6].

The core observation when establishing the adjoint representation is that a Lie algebra is, by definition, a vector space, which means that it can serve as a carrier space K for representations [12].

The adjoint representation of n-dimensional Lie algebra leads to $n \times n$ matrices

$$[R_a]^c_b = -i f_{ab}{}^c.$$

One can verify that these objects again form a Lie algebra by checking the properties listed in the definition [13]. Not only do they form a Lie algebra but it is the same one that we started with. In other words, they have the same structure constants. One can check this by working out the commutator,

$$\begin{aligned}
[R_a, R_b]_c^d &= [R_a R_b - R_b R_a]_c^d = [R_a]_c^e [R_b]_e^d - [R_b]_c^e [R_a]_e^d \\
&= (-if_{ac}{}^e)(-if_{be}{}^d) - (-if_{bc}{}^e)(-if_{ae}{}^d) \\
&= -f_{ac}{}^e f_{be}{}^d + f_{bc}{}^e f_{ae}{}^d = -f_{ca}{}^e f_{eb}{}^d - f_{bc}{}^e f_{ea}{}^d = f_{ab}{}^e f_{ec}{}^d \quad \text{by equation 2} \\
&= -if_{ab}{}^e if_{ec}{}^d = if_{ab}{}^e [R_e]_c^d,
\end{aligned}$$

which exactly agrees with equation 1.

The adjoint representation can be also defined in a more rigorous way as the map

$$\begin{array}{rccl} \mathrm{ad}_X & : & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & & Y & \longmapsto & \mathrm{ad}_X(Y) = [X, Y] \end{array} \tag{3}$$

Using a particular set of generators, it can be written as $\mathrm{ad}_{T_a}(T_b) = [T_a, T_b] = if_{ab}{}^c\, T_c$ [13]. Since the representation is a linear map, it can also be written as a matrix acting on the generators: $[\mathrm{ad}_{T_a}]_b^c\, T_c = if_{ab}{}^c\, T_c$. By taking inner product with another generator and using the orthogonality of the generators[3], we find

$$\begin{gathered}
[\mathrm{ad}_{T_a}]_b^c\, T_c T^d = if_{ab}{}^c\, T_c T^d \implies [\mathrm{ad}_{T_a}]_b^c\, \alpha\delta_c^d = if_{ab}{}^c\, \alpha\delta_c^d \implies [\mathrm{ad}_{T_a}]_b^d = if_{ab}{}^d \\
\implies [\mathrm{ad}_{T_a}]_b^c = if_{ab}{}^c \implies [\mathrm{ad}_{T_a}]_b^c = -if_{ab}{}^c = [T_a]_b^c,
\end{gathered}$$

leading to the definition above.

The adjoint representation of the Lie algebra induces adjoint representation of the group, by using the exponential map. More details can be found on page 6 of [13].

$$\begin{array}{rccl} \mathrm{Ad} & : & G & \longrightarrow & GL(\mathfrak{g}) \\ & & g & \longmapsto & \mathrm{Ad}_{T_a}(g) = g\, T_a\, g^{-1} \end{array} \tag{4}$$

3.3 The Symmetric Group

In general, the symmetric group S_Σ is the group of all bijections (injective and surjective maps) from the non-empty set Σ to itself, with a map composition as its group operation [16].

Although the symmetric group is defined on any non-empty set, its most common use is for the set of n integers, ranging from 1 to n. In such case it is viewed as a group of permutations of n integers and is denoted as S_n. Although there are multiple ways of writing down the elements of the group, we will stick to cycle notation. Example of such can be $(1\ 3\ 2)(4\ 7) \in S_8$, meaning $1 \mapsto 3 \mapsto 2 \mapsto 1$ and $4 \mapsto 7 \mapsto 4$, and the omitted elements (in this case 5, 6, and 8) are mapped onto themselves. Each bracket is called

[3] they need not be normalised and we will avoid making such assumption as this is not the case in all conventions, see for example [15]

a cycle. Although the integers can appear multiple times in various cycles of one group element, each cycle can contain each number at most once. For more details, see chapter IV.2 of lecture notes [16].

Let us briefly review the most notable properties of the symmetric group S_n. Proofs and details can be found in [11], [16].

1. The order (size) of the symmetric group is $n!$, which can be seen by thinking of all the ways one can permute n objects.

2. Cycles which do not have any integers in common are called disjoint and commute.

3. The order of each cycle is equal to its length l, that being the number of integers it contains.

4. Each group element (permutation) can be uniquely written as a composition of pairwise disjoint cycles.

5. The cycle type of a permutation σ is the list of lengths of disjoint cycles that make up σ, denoted by $\{l_1, l_2, ..., l_k\}$. The order does not matter as disjoint cycles commute, but by convention we write the lengths in non-increasing order, i.e. $l_1 \geq l_2 \geq ... \geq l_k$.

6. Each cycle of length m can be written as a composition of $m - 1$ transpositions (cycles of length 2).

7. The sign of a permutation is defined as $(-1)^k$, where k is the number of transposition in the decomposition. If the sign of a group element is $+1$, it is called even, otherwise it is odd.

8. The set of all even elements forms a subgroup of S_n called the alternating group A_n.

9. The order of the alternating group is $n!/2$.

10. Any finite group of order n is isomorphic to, that means can be seen as, a subgroup of S_n.

3.3.1 Conjugation

Conjugation of an element σ by element ρ means performing $\rho\sigma\rho^{-1}$, where ρ^{-1} denotes the inverse of ρ. For the symmetric group, this actually translates to $(\rho(i_1)\ \rho(i_2)\ ...\ \rho(i_k))$, supposing $\sigma = (i_1\ i_2\ ...i_k)$ for integers i_j [16].

Two elements σ_1 and σ_2 of the symmetric group are conjugate if there exists $\rho \in S_n$ such that $\sigma_1 = \rho\sigma_2\rho^{-1}$. A conjugacy class is a set containing all elements which are conjugate to each other,

$$C_\sigma = \{\alpha \in S_n \mid \exists \rho \in S_n \ \text{ s.t. } \ \rho\sigma\rho^{-1} = \alpha\}.$$

In S_n, the conjugacy classes are simple to find as they correspond to the cycle types defined above [16]. This means that all permutations of the same cycle type are conjugate,

while permutations with different cycle types can never be in the same conjugacy class. Due to this result, the number of conjugacy classes in the symmetric group S_n is equal to the number of partitions of integer n. There is no closed form solution to this problem, but the values for the first few integers are displayed in Table 1 [17].

integer	1	2	3	4	5	6	7	...
partitions	1	2	3	5	7	11	15	...

Table 1: Number of partitions of integers one to seven, adapted from [17]. The number of partitions of n is equal to the cycle types in the symmetric group S_n.

3.3.2 Representations and Characters

The notion of conjugation and conjugacy classes is important and extends beyond the symmetric group[4]. Especially useful result comes from representation theory, which states that for finite groups, the number of nonequivalent irreducible representations (irreps) of a group is equal to the number of conjugacy classes [11].

Irreducible representations are interesting because they serve as building blocks for all other representations of the group. Once the irreps have been found, any other representation can be written as their linear combination. However finding the irreps and decomposing other representations is not a straightforward task; at least not with the tools introduced so far.

A very useful concept for finding irreps, decomposing representations and noticing equivalence among them, are *characters*. Character $\chi^R(g)$ is a mapping which takes in a group element g and maps it onto a trace of the representing matrix $R(g)$ [18]. Often the name refers to merely the output of this mapping - the trace - rather than the map, similarly as is the case for representations.

Here is a list of a few useful properties of characters of finite groups, for details and proofs see [11], [18].

1. Character of the unit element is equal to the dimension of the representation. This is because it is always mapped onto the identity matrix.

2. Equivalent representations (representations related by a mere basis transformation) share the same characters.

3. Elements of the same conjugacy class have the same characters.

An inner product can be defined on the characters as

$$\langle \chi^R, \chi^S \rangle = \frac{1}{\#G} \sum_{g \in G} \chi^R(g) \chi^S(g)^* = \frac{1}{\#G} \sum_i (\#K_i) \chi^R(k_i) \chi^S(k_i)^*, \tag{5}$$

[4]definitions introduced above hold for all groups, except for the parts explicitly talking about cycles and cycle types

where R and S are representation of group G with elements g, which are in conjugacy classes K_i. The size of a set is denoted as # and k_i is an arbitrary element from conjugacy class K_i [18]. Notice that the latter equality comes from the property that the characters of conjugate elements are identical.

Characters of irreducible representations are orthonormal, meaning that $\langle \chi^R, \chi^S \rangle = \delta^{RS}$ [11]. This makes it possible to use the character inner product to decompose representations. Namely, by taking inner product of the representation with each irrep, the number of times the irrep occurs in the decomposition can be found. Meaning that if

$$\langle \chi^R, \chi^{D^j} \rangle = m_j$$

for irrep D^j, representation R can be written as

$$R = \bigoplus_j m_j \, D^j. \tag{6}$$

Everything which was discussed in this section holds for all finite groups G, not only the symmetric group.

3.3.3 Young Tableaux and Diagrams

Young tableaux are multi-purpose combinatorial objects, which might remotely remind one of the classic game Tetris. They are shapes formed from n squares of same sizes. The squares are filled with integers from 1 to n. The tableau as a whole follows certain rules and has a specific, context-dependent meaning.

Young diagrams follow the same rules as Young tableaux but consist typically of empty boxes [19]. In this work, the filling of the tableaux is not of relevance. Therefore, although strictly speaking tableaux and diagrams are not the same, the terms will be used interchangeably unless explicitly stated otherwise.

Although Young tableaux are used in this work to express representations of the symmetric group, the objects themselves are not our main focus, hence they will be introduced only briefly and for more details about the underlying mathematics we refer you to other sources, such as [12], [19]–[21].

A Young diagram is "*a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing*" by a definition from Y. Zhao [19]. Weakly decreasing means that the number of boxes in each subsequent row either decreases or remains the same.

If a diagram satisfies this definition, it is called legitimate [12]. Non-legitimate diagrams will not be considered further but Figure 1 displays a few examples of both allowed and not allowed shapes to provide some illustration for the abstract definition.

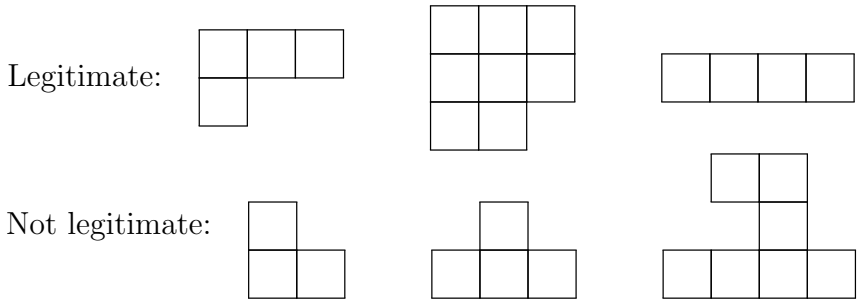
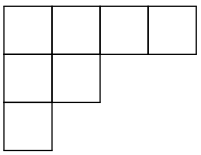


Figure 1: Examples of legitimate and not legitimate Young diagrams.

A Young diagram made up of n boxes can be also described as a partition of n. That is, a sequence $(\lambda_1, \lambda_2, ..., \lambda_k)$ of positive integers λ_i which sum up to n and satisfy $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$. A corresponding Young diagram has λ_i boxes on the ith row and, by definition, all are left-aligned. So for example $(4, 2, 1)$ looks like



Notice the similarity between this description of Young tableaux and cycle types, which determine the conjugacy classes of S_n as we saw above.

Young tableaux can be grouped into so-called tabloids. These are equivalence classes based on the tableaux having the same integers in each row. Using the tabloids, we can define a permutation module M^λ. This is simply a vector space spanned by all the possible tabloids for a given λ [19].

The permutation module is interesting because it can serve as a carrier space for the defining representation of the symmetric group, and because it is needed when talking about Specht modules later. However, that is all we need from permutation modules for our purposes. For an extensive discussion of filled tableaux and the permutation module see [19], [20].

The defining representation really captures the full scope of the group, but our interest lies in irreducible representations and it can be shown that the defining representation is generally not irreducible. However, a representation mapping the elements of S_n into a particular sub-module of the permutation module, called the Specht module S^λ, is irreducible [19], [20]. In fact, the Specht modules form a complete set of carrier spaces for irreducible representations of the symmetric group over the complex numbers [19]. This one-to-one correspondence makes them a very useful tool when working with representations of S_n.

Using the tableaux to express the representations leads to new ways of finding irreducible representations of S_n [20] and their characters [21]. The characters, useful for the decomposition of representations, can be found using the Frobenius formula, which

is detailed in section 4 of work by M. Niggli [21]. However, performing calculations using this formula directly is not necessary as J. Gibson created an accessible calculator for characters of Specht modules, as well as their products up to S_{30} which is sufficient for our purposes [22].

Young diagrams can be used to express symmetry/antisymmetry under exchange of the permuted integers. Rows of Young diagrams signalise symmetry while columns point towards antisymmetry [13].

Lastly, Young diagrams have the same dimensions as the corresponding representations. The dimension of a diagram consisting of n boxes can be found using the hook formula [12], [21]

$$\text{dimension} = \frac{n!}{\prod_{i=1}^{n}(\text{length of hook } i)}. \tag{7}$$

The number of hooks is the same as the number of boxes as there is exactly one hook centered on each box. The length of a hook, centered on box i, can be found by summing the number of boxes directly to the right of box i with the number of boxes directly below box i plus one for box i itself. The product of hook lengths is also called the hook factor [12].

Example: Finding the dimension of a Young diagram

Consider Young diagram (4,2,1) as above. It consists of seven boxes. In order to find its dimension, we first need to find the hook length for each hook, i.e. box. To do that, the number of boxes to the right of and below each box are counted, adding one for the box itself. The hook lengths for each box (hook) are shown in the right-most diagram in Figure 2.

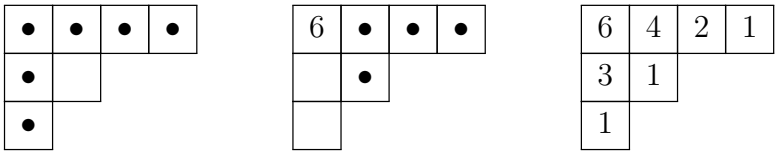


Figure 2: Finding hook lengths of a Young tableau. First, a hook is drawn centered on the first box (left). Then, a hook length is written in that box and a second hook is drawn (center). By repeating this process for all boxes, hook lengths of all hooks can be found (right).

In the left and middle diagrams of Figure 2, it is illustrated where the name "hook" length comes from and shows the simple idea behind the long explanation. For each box, a hook is drawn (going to the right and down) and its length is counted in the units of box lengths. In this figure, each unit is symbolised by a black dot.

First, the hook length of the longest hook (centered on the box in top left corner) is found. A hook is drawn centered on this box, as shown on the left panel of Figure 2. The number of black spots is six, hence a six is written down in that box, marking the hook length. Next, a hook centered on a box to the right of the first box is drawn in the

middle diagram. This hook has length four, hence a four is written down in that box. This process is repeated for all the remaining boxes, leading to the result in the last panel of Figure 2.

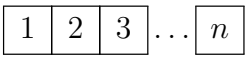
Finally, the hook formula can be used to determine the dimension

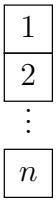
$$\dim = \frac{7!}{6 \cdot 4 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 7 \cdot 5 = 35.$$

3.3.4 Irreducible Representations of S_n

Let us consider a few particular representations of the symmetric group.

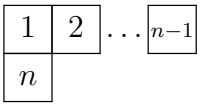
There are two one-dimensional representations - the trivial representation and the alternating representation. The former sends all elements to the identity, the latter maps each permutation to either +1 or -1 depending on its sign. They can be represented by the following Young diagrams (note that the numbers here merely count the boxes but have no other meaning)





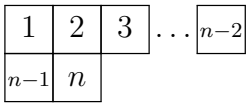
Trivial irrep $\sim$ (n) Alternating irrep $\sim$ (1,1,...,1)

The defining, also called permutation representation was already mentioned before and is represented by permutation modules related to objects called tabloids but not Young tableaux themselves because the representation is not irreducible. However, it can be decomposed into the trivial and a representation called the "standard representation" which are irreducible. The trivial representation was given above already, and the standard representation has a Young diagram of the form [23]



Standard irrep $\sim (n-1, 1)$

The last irreducible representation that will be explicitly mentioned here has partition $(n-2, 2)$[5] and Young diagram



This representation describes the transformations of Mandelstam variables under a permutation of particles in n-point scatterings. Mandelstam variables will be introduced later, but in short they are squares of sums of momenta in particle interactions.

[5]Note that it is only possible for $n \geq 4$.

4 Two Great Theories

Most will agree that two of the greatest theories in high-energy physics of the past century are the standard model of particle physics and general relativity. While the standard model gives an accurate description of particle interactions, it is incomplete because it does not include e.g. gravitational effects. On the other hand, general relativity, which is all about gravity, fails to describe quantum effects [4].

In this chapter, the basics of these two theories will be discussed, so that some similarities and differences become apparent. The standard model is represented here by $SU(3) \otimes SU(2) \otimes U(1)$ Yang-Mills theory which provides results for all the main interactions [4]. The fermionic part of the standard model will not be discussed as it goes beyond this work's scope and relevance. The reason for focusing on Yang-Mills theory is its central role in the discovery of colour-kinematics duality, discussed in chapter 7.

4.1 Yang-Mills Theory

Yang-Mills theory is a gauge theory describing interacting quantum fields with spin 1. It is very relevant for particle physics as it underlies the widely accepted and thoroughly tested Standard Model. Yang-Mills provides results for strong, weak, and in the non-interacting limit also for the electromagnetic force [4]. Though for the latter, quantum electrodynamics are used more commonly. As this is a quantum field theory, the force carrier particles, such as gluons, are viewed as excitations of underlying (in this case gauge) fields. For gluons and photons, the fields are massless.

The mathematical framework of the theory is based on Lie groups and algebras. If one wants to build a particular Yang-Mills theory, one should begin with a Lie group G [15]. As discussed in chapter 3, the group has an associated Lie Algebra $\mathfrak{g}$. By choosing a particular representation, one can obtain the generators T^a as described earlier. For example, when writing a theory for gluons of the strong force, the adjoint representation of special unitary group $SU(3)$ is used [6].

Next, let A^a_μ be vector fields, corresponding to the generators T^a. We work with vector fields because Yang-Mills theory is used for describing spin-1 particles. The number of generators is equal to the number of dimensions and hence, so is the number of introduced fields [15]. For special unitary groups $SU(N)$, the dimension is $N^2 - 1$, hence there are $3^2 - 1 = 8$ types of gluons of the strong force, distinguished by their colours [13].

By contracting each gauge field with a corresponding generator, a gauge potential A_μ is obtained[6]

$$A_\mu = \sum_a A^a_\mu T^a. \tag{8}$$

Similarly as in the case of electromagnetism, the gauge potential can be used to construct a field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{9}$$

[6]Note that from a purely mathematical standpoint, the gauge potential is not a vector, although it looks like one. A_μ has two implicit indices which it has inherited from T^a.

where g is the Yang-Mills coupling constant. Although it is a constant with respect to spacetime, it depends on various parameters such as the energy scale, see [24]–[26].

Notice that the field strength is very similar to the one for electromagnetism. The only difference is the presence of the commutator term. This commutator is responsible for the self-interaction of the fields. In the case of Abelian theory, which occurs when the underlying Lie group is Abelian, the commutator vanishes for all components of the potential. This leads to a lack of interaction between the gauge particles. An example occurs in the electromagnetism - as a gauge theory, it is characterised by the $U(1)$ group, which is Abelian, hence photons do not interact with one another [4], [15].

Returning to the field strength $F_{\mu\nu}$, it can be used to build the action S, or the Lagrangian density $\mathcal{L}$ [15]. Similar to the electromagnetic action,

$$S = -\frac{1}{2}\int \mathrm{d}^4x \,\mathrm{Tr}\big(F^{\mu\nu}F_{\mu\nu}\big), \tag{10}$$

where Tr suggests to take the trace of the resulting matrix - one must not forget about the uncontracted indices which were kept implicit since the introduction of the generators T^a. Note that the constants multiplying the trace vary - while some sources include the coupling constant in the definition of $F_{\mu\nu}$, others add it only when building the action. A similar story goes for factors of half which can be compensated for by the normalisation of the generators for example. [6], [15]

Gauge Symmetry refers to the invariance of the action under a gauge transformation. Together, the gauge transformations, which are defined as maps from spacetime to the Lie group G that do not change the action, form the gauge group. Although it is very closely related to the group G, the two are not identical [15].

Gauge transformations lead to the field strength tensor being conjugated by an element from the Lie group G. The action depends only on the trace of the field strength, which is cyclically invariant, meaning that the trace of a conjugated object is same as the original. Therefore the action is indeed invariant.

Although this type of invariance is called "symmetry", it is not the usual symmetry that we are used to. Rotating an equilateral triangle by 120° around an axis piercing through its center keeps the triangle same from the outside, but if we label the corners, it is clearly in a different state. This is not the case for the gauge symmetry. States which are equivalent under gauge transformations are completely identical. It is not a "physical" symmetry of the system, it is merely a redundancy in our mathematical description [5], [8]. This leads to the conclusion that all physical results must be invariant under gauge transformations.

One might wonder, why do we use a mathematical description which contains redundancies that we need to remove subsequently by demanding gauge invariance? It is due to the convenience and intuition behind the action. The current phrasing of the theory makes unitarity, locality and Lorentz invariance more apparent than if we were to use an alternative, less redundant framework [5], [15].

4.2 General Relativity

General relativity is one of the most successful theories of today. It was invented by Albert Einstein in 1915 and more than hundred years later, new results, confirming the theory's correctness, are being found. The predicted gravitational waves were discovered only recently, gravitational lensing, which was observed for the first time in the second half of the last century, is still frequently used in research today, and on daily basis we take advantage of the precise GPS. All of these are results of general relativity [27], [28].

This theory revolutionised the way we perceive gravity and spacetime. While in the previous (Newtonian) approach, gravity was merely a force between two massive bodies, general relativity views the gravitational force as a consequence of the deformation of spacetime itself. Spacetime is now no longer static but instead dynamically connected to the bodies in the Universe. While mass bends the spacetime, the curvature of spacetime influences the way mass moves. Keep in mind that mass and energy are equivalent, another result of Einstein, and so the curved spacetime changes also the trajectories of, for example, massless photons.

As briefly mentioned above, in general relativity (GR), gravity is very closely related to spacetime. Similarly as for particle physics or electromagnetism, a field theory approach will be used. This means that our primary object will be the gravitational field. In GR, it is identified with the metric $g_{\mu\nu}$ of the spacetime, which mathematically is a four dimensional Lorentzian manifold. This means that one of the metric's diagonal entries is negative. Linking gravity and spacetime results in a dynamical, time-evolving metric [3].

In a classical approach, a theory begins with a Lagrangian or an action, and afterwards one can find the equations of motion and solve them. For now, we will do no different. The Einstein-Hilbert action for an empty universe is given by

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \sqrt{-g} R, \tag{11}$$

where G_N is the gravitational constant, expressing the strength of gravity-to-matter coupling, $g = \det\big(g_{\mu\nu}\big)$ which is negative due to one of the negative diagonal entries, and R is the Ricci scalar, equal to metric-contracted Ricci tensor, i.e. $g^{\mu\nu}R_{\mu\nu}$. Ricci tensor is symmetric and encapsulates information about the curvature of the manifold - spacetime. If desired, it is possible to also include cosmological constant Λ or a matter term [3], [6].

From the Lagrangian density, we can obtain the equations of motion, called the Einstein equations, which read

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{12}$$

for an empty space, or

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{13}$$

in the case of Universe with cosmological constant Λ and matter. Information about their distribution is contained in the energy-momentum tensor $T_{\mu\nu}$ [6].

General solution to the Einstein equation has not been found yet, however many special-case solutions exist such as Friedmann-Robertson-Walker for cosmological scales

or Schwarzschild for black holes. Of particular interest to us are the perturbative solutions which are valid in a nearly flat space for weakly coupled gravity. Non-perturbative states such as black holes are exponentially suppressed in scattering processes due to the weak coupling [6].

In an almost flat spacetime, one can write the gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric introduced in section 2.3, and $h_{\mu\nu}$ is a small perturbation to the flat space. This ansatz can be substituted into the action and the Einstein equation, and expanded up to a linear order in $h_{\mu\nu}$ around $\eta_{\mu\nu}$. Terms higher order in $h_{\mu\nu}$ can be discarded because $h_{\mu\nu}$ is assumed to be small. The interpretation of the linearised theory is that now we no longer view gravity as a dynamical part of spacetime, but rather as a symmetric, massless, spin-2 field $h_{\mu\nu}$, which we will call the graviton field, propagating through static Minkowski space [3], [6].

The obtained field is not too different from the Yang-Mills gauge fields encountered in chapter 4.1. It is an interacting field with symmetries and redundancies (discussed below). It mediates a force and after a quantization, particles (gravitons) can be obtained [6].

Diffeomorphism Invariance. Similar to earlier gauge symmetries, both graviton ($h_{\mu\nu}$) and gravitational ($g_{\mu\nu}$) fields are diffeomorphism invariant. A diffeomorphism is a smooth map between two manifolds which has a smooth inverse. In the cases related to this work, these maps can be usually seen as changes of coordinate systems. In other words, general relativity states that a change of coordinates should not influence the result, just like the choice of gauge should not in Yang-Mills [3].

At linear order, which is relevant when using perturbations, the diffeomorphism transformation should change $h_{\mu\nu}$ into $h_{\mu\nu} + \partial_\mu\theta_\nu + \partial_\nu\theta_\mu$, as then the action remains invariant. Although the transformation only has this specific form, θ_μ can be an arbitrary compactly supported function of spacetime. This yields a large symmetry group [5].

Although diffeomorphisms are often referred to as symmetries of the system, they are closer to redundancies than real symmetries. As was the case for gauge symmetries, states which are related by a diffeomorphism invariance must be viewed as identical, not merely equivalent. Otherwise, one would arrive at non-sensical meanings [3], [5]. Think back to change of coordinates - merely because one describes the system in a different basis, the reality should not be any different.

5 Scattering Amplitudes

A scattering amplitude in particle physics is the overlap between initial state i and final state f,

$$\mathcal{A} = \langle f | \, S \, | i \rangle = \lim_{t_\pm \to \pm\infty} \langle f | \, U(t_+, t_-) \, | i \rangle \, ,$$

where $U(t, t_0)$ is the unitary time evolution operator which takes states from time t_0 to time t: $\big|\psi(t)\big\rangle = U(t, t_0) \big|\psi(t_0)\big\rangle$, and S is the so-called S-matrix, an abbreviation for a scattering matrix [8].

The main significance of the scattering amplitude is that it is related to a probability of transition from state i to state f. In particular,

$$P \propto |\mathcal{A}|^2.$$

This relation leads to Fermi's golden rule for transition probability per unit time (for details see chapter 3.6.1 in [8])

$$\dot{P}_{i \to f} \propto | \, \langle f | \, H_{int} \, | i \rangle \, |^2,$$

for interaction Hamiltonian H_{int}.

Although the probability is an observable in quantum mechanical sense, it is not very "experiment-friendly". Instead, decay rates and cross-sections are measured. This makes the amplitude no less useful as its square is also proportional to either of these.

Decay rate is measured for example when a state is observed and the number of decays per unit time is counted. The decaying state can be anything from a particle changing energy-level to decaying into a group of products. The decay rate expresses the ratio between the probability for the state i to transition to state f to the probability of the states i and f to remain unchanged [9], [29].

Cross-section is measured mostly in collider experiments, be it for fixed-target collisions or when colliding two beams of particles. The cross-section expresses the ratio between scattering events and the incoming flux per unit area. The square of the amplitude is proportional to differential cross section which is an infinitesimal element in a distribution over a solid angle [2], [9], [29].

5.1 The Feynman Approach

Before diving into colour-kinematics duality, which is the main focus of this work and introduces a new way of constructing amplitudes, let us examine how scattering amplitudes are "usually" built. This will serve to provide not only a general review of amplitudes but also prepare us to appreciate the beauty of the later-discussed duality approach.

The usual approach uses Feynman diagrams and then a few rules to write down the algebraic expressions. In Feynman diagrams, lines are used to symbolise particles. Different types of lines are often used to mark different particles - for example gluons are usually drawn as curly lines, photons as wavy lines and fermions as solid lines. The conventions

are however not universal as for example [8] uses in chapter 3 solid lines for nucleons and dotted lines for mesons.

In the example below, solid lines are used for scalars, in particular for the scalar field ϕ. Black lines symbolise external particles satisfying the on-shell conditions, and grey lines are used for exchange particles.

5.1.1 The Lagrangian

Scattering amplitude calculation begins with a Lagrangian as that is central to quantum field theories. For the purpose of explicit explanation of the rules, we will pick a particular Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\ \partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4. \tag{14}$$

First thing one can notice is that this is a scalar field Lagrangian. Scalars, ϕ, were chosen for two reasons. Firstly, they are a simple example as there is no need for considering spin and associated polarisation of the field, and so the explanation can be more to the point. Secondly, amplitudes and Lagrangians of scalar fields will come back later in this work when discussing the infrared behaviour of theories in chapter 6, and when looking into the future of this research at the end of this thesis.

The first term represents the kinetic energy and the gradient potential [8], the second term introduces consequences due to the mass m of the field, and the last two terms are responsible for interactions. More specifically, they for example dictate what kind of vertices are allowed in Feynman diagrams. For this Lagrangian there are three-ϕ and four-ϕ vertices allowed [8], [9], [30].

The constants λ_i are the couplings, describing the strength of the interactions. They are assumed to be small, relatively to an appropriate power of the energy scale at which the scattering occurs [8]. This assumption is crucial for the method of Feynman diagrams, as treating the interaction as a small perturbation simplifies calculation by providing a ground for neglecting higher order terms. This negligence is essential because calculating a scattering amplitude requires summing over all possible Feynman diagrams with the same in- and out-going particles. Unfortunately, there is infinitely many of those. Although there is a limited number of possible tree diagrams, one can always add an extra loop, creating a new, more complicated diagram, adding an extra term to the sum. However, if the coupling constant is small, the higher-order diagrams, meaning diagrams with larger number of vertices, are strongly suppressed. In other words, their contribution to the scattering amplitude is small, and so they can be neglected as infinite precision is rarely required.

Under this assumption of weak coupling, only tree-level diagrams will be considered in the example below as those are the lowest order in λ_3 and λ_4 (for four-point amplitude) and hence contribute the largest part to the resulting scattering amplitude [30]. Furthermore, let us assume that λ_4 is of similar order of magnitude as λ_3 (after rescaling by appropriate energy scale to dimensionally allow for comparison), so that the two types of interaction are comparable.

5.1.2 Feynman Rules

Using the arguments above, one can draw the relevant Feynman diagrams, so now all that is needed are the rules for writing down the mathematical expressions based on the diagrams.

We begin by associating momentum p_i to each line (particle), both internal and external, the latter obeying the on-shell condition $p^2 = m^2$ and energy conservation. Note that unless explicitly stated otherwise, "momentum" always refers to the four-momentum, and so p^2 is the contraction $p^\mu p_\mu$.

Next, for each vertex, one writes down

$$(-ig)(2\pi)^4 \delta^{(4)}\left(\sum p_i\right), \tag{15}$$

where g is the coupling constant relevant for a given vertex, $\delta^{(4)}(...)$ is the four-dimensional Dirac delta function and in its argument one can find sum over all the momenta involved in the vertex, assuring momentum conservation [8]. This way of expressing momentum conservation assumes all momenta are incoming. If a different convention is used and some momenta are exiting, they are included in the summation with a minus sign.

The coupling constants and momentum conservation terms are multiplied by a propagator for each internal line, which has the form

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}. \tag{16}$$

Here p_μ and m are respectively the momentum and mass of the exchange (propagating) particle [8]. The ϵ in the denominator is present merely to prepare for a contour integration around poles which are likely to arise in scattering amplitudes [30].

Lastly, if needed, we divide by the symmetry factor which is equal to the number of ways of interchanging particles without changing the diagram. As this is merely a constant factor, not important for the example below, we will skip further explanation. More details can be found on page 93 of a book by Peskin and Schröder [9].

These rules can be applied to each permissible Feynman diagram for a desired situation[7]. Subsequently summing over the resulting expressions yields the total scattering amplitude [29].

A possible way of interpreting the rules is that the first factor, equation 15, gives the amplitude for emission or absorption of particles in a given vertex, while the propagator (eq. 16) sums over all possible momenta which the exchange particle could carry. These "probability" functions for each vertex and propagator are multiplied as they are treated as independent probabilities, leading to the total probability for the process which is displayed in the given diagram [9].

[7] The "situation" depends on the number and type of in- and out-going particles and their momenta, and possibly other properties.

5.2 Four-particle Scattering

After discussing the Feynman rules for calculating the scattering amplitudes, it is time to let them shine in practice. In this section a four-point tree-level scattering amplitude for a scalar field with Lagrangian density as given in equation 14 will be derived.

In this process, four external particles appear, labelled a, b, c and d. Some of these are in-going, others out-going. Whether one views particles as entering or exiting is a matter of preference and convention. In this example, we will view the situations as two incoming particles a, b and two outgoing particles c, d, as it is the most intuitive case. However, frequently all-entering convention is used in literature. The motivation for such an approach comes largely from not needing to be concerned with the signs of momenta, in particular in momentum conservation as seen earlier in equation 15.

The Lagrangian in question permits three-point and four-point vertices, but not any higher order. As we will consider only tree-level diagrams (i.e. those which do not contain any loops), two types of diagrams are possible - exchange (Figure 4) and contact (Figure 3) diagrams [30]. Exchange diagrams include an exchange particle, which means that a propagator will be present in the expression. As loops are not considered, this mediator can propagate only between two three-particle vertices. On the other hand, contact diagrams do not require any such constructions as all four particles interact at one point, the four-particle vertex. This diagram corresponds to a delta-function interaction potential [8].

5.2.1 Contact Diagrams

We begin with the contact diagrams as their lack of propagator makes the use of Feynman rules simpler.

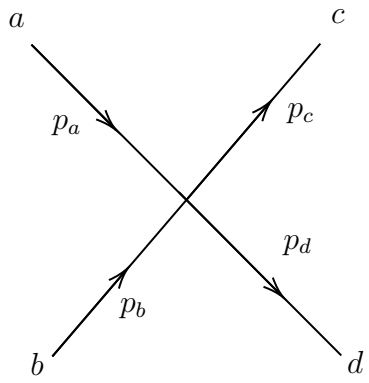


Figure 3: Contact diagram for four interacting particles

Following the Feynman rules from section 5.1.2, the expression corresponding to the diagram in Figure 3 is

$$\mathcal{A}^{(1)}_{contact} = -i\frac{\lambda_4}{4!}$$

However, there are in total 4! diagrams of this form as we can freely permute the

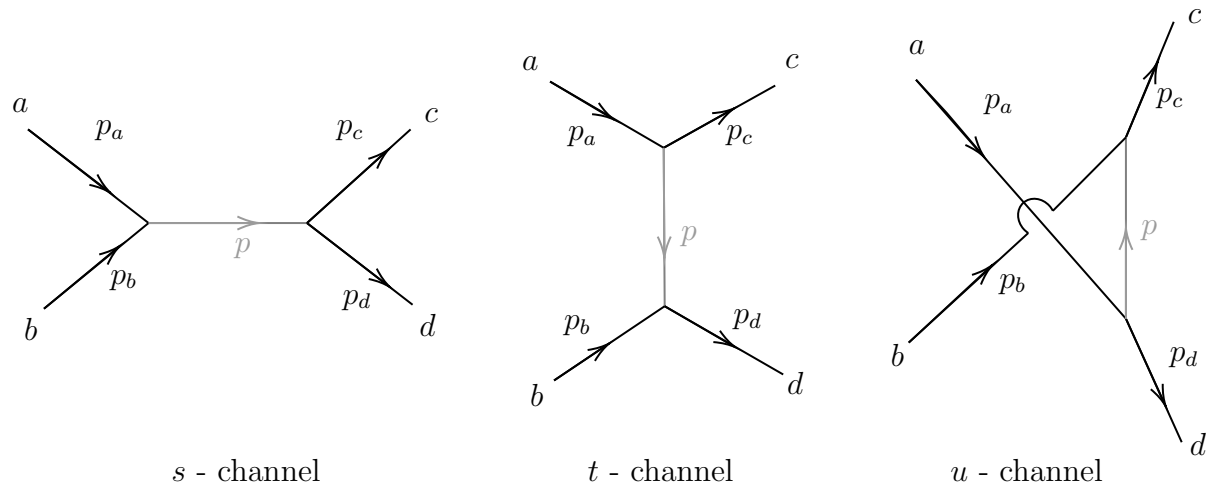
labelling of the legs, and so to get the total contact amplitude, one must multiply this result by 4! [8], yielding

$$\mathcal{A}_{contact} = -i\lambda_4. \tag{17}$$

Note that here the term assuring the conservation of momentum on external legs, $(2\pi)^4\delta^{(4)}(p_a + p_b - p_c - p_d)$, was omitted as it is not commonly written in the amplitude expression, although it is present in the S-matrix [8], [29]. It can be left implicit in the amplitude because this conservation of momentum always holds and is identical for all Feynman diagrams of particular scattering, hence it is trivial to add it later if needed.

5.2.2 Exchange Diagrams

There are three possible tree-level exchange diagrams for a scattering of four particles, shown in Figure 4. Each contains only ϕ^3 vertices, so the coupling is λ_3 in this case.



s - channel t - channel u - channel

Figure 4: Exchange diagrams for four interacting particles

Using the Feynman rules, the scattering amplitudes for each diagram can be obtained. We discuss this in detail for the first diagram and the other two follow by similar arguments.

The momenta in Figure 4 are already labelled, and so one begins with the vertex rules. The diagram has two vertices, each with a coupling strength of $\lambda_3/3!$, so we obtain

$$\left(-i\frac{\lambda_3}{3!}\right)^2 (2\pi)^8\delta^{(4)}(p_a + p_b - p)\delta^{(4)}(p - p_c - p_d).$$

There is a single propagator which contributes

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}$$

to the amplitude.

Multiplying together the vertex factors and the propagator yields

$$\begin{aligned}\mathcal{A}_s &= \left(-i\frac{\lambda_3}{3!}\right)^2 (2\pi)^8 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}\, \delta^{(4)}(p_a + p_b - p)\delta^{(4)}(p - p_c - p_d) \\ &= \left(-i\frac{\lambda_3}{3!}\right)^2 (2\pi)^4 \frac{i}{(p_a + p_b)^2 - m^2 + i\epsilon} \delta^{(4)}(p_a + p_b - p_c - p_d).\end{aligned}$$

Finally, same as for the contact amplitude, we leave the conservation of momentum, $(2\pi)^4\delta^{(4)}(p_a + p_b - p_c - p_d)$, implicit, arriving at

$$\mathcal{A}_s = \left(-i\frac{\lambda_3}{3!}\right)^2 \frac{i}{(p_a + p_b)^2 - m^2}.$$

Via a similar process for the other two diagrams, we find

$$\begin{aligned}\mathcal{A}_t &= \left(-i\frac{\lambda_3}{3!}\right)^2 \frac{i}{(p_a - p_c)^2 - m^2} \\ \mathcal{A}_u &= \left(-i\frac{\lambda_3}{3!}\right)^2 \frac{i}{(p_a - p_d)^2 - m^2}.\end{aligned}$$

In the labelling of the amplitudes for the three diagrams, letters s, t, and u were used, as that is a common way to reference these three interaction channels. The names originate from the so-called Mandelstam variables, which are defined as follows

$$\begin{aligned}s &= (p_a + p_b)^2 = (p_c + p_d)^2 \\ t &= (p_a - p_c)^2 = (p_d - p_b)^2 \\ u &= (p_a - p_d)^2 = (p_b - p_c)^2.\end{aligned}$$

Note that in the all-entering convention of external particles, there are only plus-signs in this definition.

The Mandelstam variables were established largely out of convenience due to their frequent occurrence in scattering amplitude expressions. Subsequently, types of diagrams were labelled by them in a way that the exchange particle (the propagator) in a given diagram has a square of its momentum equal to the respective Mandelstam variable [8]. Notice that this is indeed the case for the expressions derived here, and so we can rewrite the amplitudes using the Mandelstam variables as

$$\begin{aligned}\mathcal{A}_s &= \left(-i\frac{\lambda_3}{3!}\right)^2 \frac{i}{s - m^2}, \\ \mathcal{A}_t &= \left(-i\frac{\lambda_3}{3!}\right)^2 \frac{i}{t - m^2}, \\ \mathcal{A}_u &= \left(-i\frac{\lambda_3}{3!}\right)^2 \frac{i}{u - m^2}.\end{aligned}$$

Finally, to obtain the complete exchange amplitude, one must sum the contributions from all diagrams

$$\mathcal{A}_{exchange} = i\left(-i\frac{\lambda_3}{3!}\right)^2 \left(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2}\right). \tag{18}$$

5.2.3 Combining Diagrams

Now that the amplitudes for each discussed diagram were found, it is time to add the amplitudes from all four diagrams together to obtain the tree-level scattering amplitude for four particle scattering

$$\mathcal{A}_{tree} = \mathcal{A}_{contact} + \mathcal{A}_{exchange} = -i\lambda_4 + i\left(-i\frac{\lambda_3}{3!}\right)^2\left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right),$$

leading to the result

$$\mathcal{A}_{tree} = -i\left[\lambda_4 + \left(\frac{\lambda_3}{3!}\right)^2\left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right)\right]. \tag{19}$$

For very small λ_3, λ_4, this result serves as a reasonable approximation of the total four-particle scattering amplitude.

6 Infrared Behaviour

In this chapter, the infrared behaviour of general relativity, Yang-Mills and scalar theories will be discussed. That is, what happens at low energies. It will be investigated via the soft limit. Before diving in, let us begin with a bit of motivation first.

When considering massless, interacting spin-1 field theories, there is essentially only a single renormalisable choice, which is the Yang-Mills theory discussed in chapter 4.1. For spin-2, the story is similar except that now to obtain at least some non-trivial theory, the renormalisation requirement must be dropped and we end up with gravity characterised by Einstein-Hilbert action [5].

In contrast, consider scalars. Although they are simpler to deal with due to their lack of spin, there are many permitted theories, and distinguishing them requires determining many parameters. This differs significantly from Yang-Mills theory, which needs only one parameter (the coupling strength g) after being given a group. Part of this problem has a root in the lack of constraints put on scalars. Unlike fields with spin, scalar fields are not anyhow restricted by the Little group[8] [5].

Therefore, in order to reduce the number of investigated scalar theories and choose some "interesting" ones, one could for example require a specific behaviour in the soft limit. In the rest of this section, the soft limit will be discussed, as well as possible interesting properties and theories possessing those.

Note that this section uses the all-incoming particles convention of labelling directions of momenta.

6.1 Soft Limit

Choose an external particle k with momentum p_k and consider transformation

$$p_k \to \epsilon p_k.$$

The soft limit of a scattering amplitude occurs when ϵ goes to 0. In such case, we talk about "taking particle k to be soft". Physically, this means considering the chosen particle to contribute only negligible momentum to the scattering process, effectively decreasing the number of external particles from n to $n-1$ [30].

To characterise the infrared behaviour of a theory, let us introduce the soft degree σ. It is defined as the leading power of momentum p_k when taken soft. That is, the smallest power which has a non-zero contribution to the scattering amplitude,

$$\lim_{\epsilon \to 0} \mathcal{A}(p_1, ..., \epsilon p_k, ..., p_n) \propto p_k^\sigma.$$

In all usual situations, σ is an integer.[9] As an example, for local gauge theories and gravity, $\sigma = -1$ [5]. One can notice that $\sigma = -1$ means that the amplitude goes to infinity in the soft limit.

[8] The Little group is a subset of Lorentz transformations which leave the momentum of a particle invariant but not necessarily its polarisation (spin orientation)

[9] It could be a non-integer in certain special cases such as for a particular scattering of phonons in solids, detailed in [31]. However, such situations are rare and not entirely relevant for the discussion here.

6.2 Consequences for Electromagnetism and Gravity

Later in this chapter, we will look at cases where an amplitude goes to zero in the soft limit. However, as can be apparent from the soft degrees, this is not the case for gauge theories and gravity. Instead, up to the leading order in soft momentum, an amplitude for $n-1$ particle scattering is obtained, multiplied by a so-called Weinberg soft factor. Those were found by Weinberg in the 1960's [32], [33] for photons and gravitons. As we will see in this section, they lead to universality of graviton coupling strength and a conservation of electric charge [5].

6.2.1 Photons

The first result comes from quantum electrodynamics, which can be seen as a massless Yang-Mills theory with underlying Lie group $U(1)$. The excitations of the quantum fields are then called photons. Since $U(1)$ is an Abelian group, the photons cannot interact with themselves, and so any non-trivial scatterings must also include some charged particles to which the photons couple.

Suppose an interaction of n external particles, some of which are charged, while others are photons. Now let us take one of the photons soft, i.e. its momentum $p^{(\gamma)}$ goes to zero. Photons are massless, hence making its momentum vanish leads to also vanishing energy and so the photon effectively stops existing. Based on Weinberg's work [33], the amplitude

$$\mathcal{A}_n \to \varepsilon_\mu S^\mu \mathcal{A}_{n-1} \text{ as } \ p^{(\gamma)} \to 0,$$

where ε_μ is the polarisation vector of the soft photon and S^μ the Weinberg soft factor for photons,

$$S^\mu = \sum_{i=1}^{n-1} q_i \frac{p_i^\mu}{p_i^\nu p_\nu^{(\gamma)}}.$$

The index i runs over all hard (not soft) particles involved in the interaction, which are connected to the process via vertices with coupling constants q_i. Note that the coupling for electromagnetic interaction is equal to the electric charge [5].

The soft limit should be invariant under gauge transformations, which means that it should not change upon the replacement $\varepsilon_\mu \to \varepsilon_\mu + \alpha p_\mu^{(\gamma)}$ for arbitrary scalar α. Applying this transformation yields

$$\lim_{p_{(\gamma)}\to 0} \mathcal{A}_n = \varepsilon_\mu S^\mu \mathcal{A}_{n-1} \to \varepsilon_\mu S^\mu \mathcal{A}_{n-1} + \alpha p_\mu^{(\gamma)} S^\mu \mathcal{A}_{n-1}.$$

This means that the soft limit is invariant if and only if the second term vanishes,

$$\alpha p_\mu^{(\gamma)} S^\mu \mathcal{A}_{n-1} = 0.$$

Substituting the Weinberg soft factor leads to

$$0 = \alpha p_\mu^{(\gamma)} \sum_{i=1}^{n-1} q_i \frac{p_i^\mu}{p_i^\nu p_\nu^{(\gamma)}} \ \mathcal{A}_{n-1} = \alpha \sum_{i=1}^{n-1} q_i \frac{p_\mu^{(\gamma)} p_i^\mu}{p_i^\nu p_\nu^{(\gamma)}} \ \mathcal{A}_{n-1} = \alpha \sum_{i=1}^{n-1} q_i \ \mathcal{A}_{n-1}.$$

As this relation holds for arbitrary α, we obtain

$$\sum_{i=1}^{n-1} q_i \, \mathcal{A}_{n-1} = 0 \quad \Longrightarrow \quad \sum_{i=1}^{n-1} q_i = 0 \;\text{ or }\; \mathcal{A}_{n-1} = 0.$$

This means that if the sum of q_i's is non-zero, the amplitude of the process must vanish, hence it cannot happen. Remember that the couplings q_i are equal to the electric charge of the particles, and so we can conclude that the gauge invariance requirement implies conservation of electric charge.

6.2.2 Gravitons

A similar process can be repeated for gravitons from perturbative general relativity. In this case the Weinberg soft factor is a tensor given by

$$S^{\mu\nu} = \sum_{i=1}^{n-1} \kappa_i \frac{p_i^\mu p_i^\nu}{p_i^\rho p_\rho^{(g)}},$$

with same conventions as in the photon case, except that now the coupling constants are κ_i's [5]. One way for justifying that it must be a tensor in this case, is reasoning with the polarisation of the graviton. As mentioned in chapter 4.2, gravitons are spin-2 particles and so their polarisation has two indices. The matters because the soft limit for graviton scattering is essentially identical to the one for photons, containing contraction of the polarisation with the Weinberg factor [5]

$$\mathcal{A}_n \to \varepsilon_{\mu\nu} S^{\mu\nu} \mathcal{A}_{n-1} \text{ as } \; p_{(g)} \to 0.$$

The derivation now follows a similar, though slightly more complicated, reasoning as for the photon case. As mentioned earlier, gravitons are invariant under diffeomorphism transformations, in place of gauge transformations relevant for photons. In the polarisation formalism, a diffeomorphism transformation translates to $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \alpha_\mu p_\nu^{(g)} + \alpha_\nu p_\mu^{(g)}$ for an arbitrary vector α_μ [5].

Applying this transformation to the amplitude in soft limit leads to

$$\varepsilon_{\mu\nu} S^{\mu\nu} \mathcal{A}_{n-1} \to \varepsilon_{\mu\nu} S^{\mu\nu} \mathcal{A}_{n-1} + \alpha_\mu p_\nu^{(g)} S^{\mu\nu} \mathcal{A}_{n-1} + \alpha_\nu p_\mu^{(g)} S^{\mu\nu} \mathcal{A}_{n-1}.$$

In order to have a diffeomorphism invariance, the second and the last terms must vanish

$$\begin{aligned}
0 &= \alpha_\mu p_\nu^{(g)} S^{\mu\nu} \mathcal{A}_{n-1} + \alpha_\nu p_\mu^{(g)} S^{\mu\nu} \mathcal{A}_{n-1} \\
&= \alpha_\mu p_\nu^{(g)} \sum_{i=1}^{n-1} \kappa_i \frac{p_i^\mu p_i^\nu}{p_i^\rho p_\rho^{(g)}} \mathcal{A}_{n-1} + \alpha_\nu p_\mu^{(g)} \sum_{i=1}^{n-1} \kappa_i \frac{p_i^\mu p_i^\nu}{p_i^\rho p_\rho^{(g)}} \mathcal{A}_{n-1} \\
&= \sum_{i=1}^{n-1} \left(\kappa_i \alpha_\mu \frac{p_\nu^{(g)} p_i^\mu p_i^\nu}{p_i^\rho p_\rho^{(g)}} + \kappa_i \alpha_\nu \frac{p_\mu^{(g)} p_i^\mu p_i^\nu}{p_i^\rho p_\rho^{(g)}} \right) \mathcal{A}_{n-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} \left(\kappa_i \alpha_\mu p_i^\mu \frac{p_\nu^{(g)} p_i^\nu}{p_i^\rho p_\rho^{(g)}} + \kappa_i \alpha_\nu p_i^\nu \frac{p_\mu^{(g)} p_i^\mu}{p_i^\rho p_\rho^{(g)}} \right) \mathcal{A}_{n-1} \\
&= \sum_{i=1}^{n-1} \left(\kappa_i \alpha_\mu p_i^\mu + \kappa_i \alpha_\nu p_i^\nu \right) \mathcal{A}_{n-1} = \sum_{i=1}^{n-1} \left(\kappa_i \alpha_\mu p_i^\mu + \kappa_i \alpha_\mu p_i^\mu \right) \mathcal{A}_{n-1} \\
&= 2 \sum_{i=1}^{n-1} \left(\kappa_i \alpha_\mu p_i^\mu \right) \mathcal{A}_{n-1}.
\end{aligned}
$$

On the last row, the ν index was relabelled to μ because it is summed over anyways and there is no μ in that particular term. This relation should hold for all α_μ, so we can rewrite this as

$$
\sum_{i=1}^{n-1} \left(\kappa_i p_i^\mu \right) \mathcal{A}_{n-1} = 0 \quad \Longrightarrow \quad \sum_{i=1}^{n-1} \kappa_i p_i^\mu = 0 \ \text{ or } \ \mathcal{A}_{n-1} = 0.
$$

Therefore we conclude that in order to have a non-vanishing amplitude, the first condition must be satisfied. However, by the conservation of momentum (which holds for all scatterings of on-shell particles with non-zero amplitude),

$$
\sum_{i=1}^{n-1} p_i^\mu = 0.
$$

As a consequence, the only way to satisfy the diffeomorphism invariance for all combinations of momenta which obey momentum conservation without requiring a special configuration of the particles, is for all κ_i's to be the same [5]. In such case, the condition reduces to momentum conservation which is assumed to be satisfied.

Since κ is the graviton coupling constant (related to Newton's constant G_N), we can say that gravitons couple with same ease to all particles. In other words, the strength of gravity is universal.

6.3 Adler Zero

Alder zero is a phenomenon occurring in all derivatively-coupled scalar theories. Such a theory has a Lagrangian containing only derivatives of the field $\partial\phi$ but not the field on its own. As an example can serve the following Lagrangian which will be discussed in more detail later

$$
\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{\lambda_4}{4!}(\partial\phi)^4 + \frac{\lambda_6}{6!}(\partial\phi)^6 + ... \tag{20}
$$

In general, the amplitude can have terms proportional to $1/p$, p^0, p, p^2, etc. for momentum p which will be taken soft,

$$
\mathcal{A} \sim c_- \frac{1}{p} + c_0 + c_1 p + c_2 p^2 + ...
$$

However, amplitudes possessing Adler zero, have vanishing coefficients up to and including c_0. Therefore the leading order term in the soft limit is linear in momentum, so the soft degree is $\sigma = 1$ and the amplitude goes to zero in the soft limit. Hence the name.

This concept extends further, to theories with larger number of vanishing coefficients and higher soft degrees. In those cases, we talk about generalised Adler zero [7].

Adler zero occurs for all derivatively-coupled Lagrangians, but it can also appear in other Lagrangians. Such cases are more interesting as for derivatively-coupled theories the statement is fairly straightforward because they trivially possess shift symmetries[10] leading to the Adler zero.

An example of theory with Alder zero which does not have a purely derivatively-coupled fields is the non-linear σ model for soft pions [5].

Non-linear σ model is an effective field theory related to particular symmetry breaking. Effective field theories are a way of simply representing dynamics in low energy limits. This can be done by using only the parts of the description which are relevant at low energies, and building a full field theory around them.

The σ model is useful for understanding a very low energy limit of quantum chromodynamics[11] by moving away from quark and gluon interactions to pions which are exchanged between nucleons. It has a Lagrangian containing terms of the form

$$\mathcal{L} \sim (\partial_\mu \pi)^2 + \pi^2 (\partial_\mu \pi)^2 + ...$$

for scalar fields π [34].

Unlike the Lagrangian in equation 20, this one includes also non-derivatively coupled fields and so the shift symmetry cannot be immediately assumed. Yet it realises an Adler zero. This is due to cancellations among Feynman diagrams of different shapes which comes from internal non-linear symmetry [4].

6.4 Special Scalar Theories

We started this chapter by introducing the issue of too many permissible scalar theories which we would like to restrict in some way. A possible first step to more constraints is requiring an Adler zero or a higher soft degree.

The infrared behaviour of theories depends on the number of derivatives in the interaction term of the Lagrangian. Consider the Lagrangian from equation 20 which contains only even powers of single derivatives of the field. This Lagrangian belongs to a general Nambu Goldstone Boson [5]. Since the fields in this Lagrangian are derivately-coupled, we are guaranteed to get an Adler zero. Therefore requiring a normal Adler zero (equivalent to $\sigma = 1$), does not bear any fruits and we are still left with infinitely many theories.

[10] Shift symmetry means that the quantity (in this case the Lagrangian) does not change upon translation of the field. For derivative-coupling this is fairly obvious as any constant shift will disappear when taking the derivative.

[11] theory dealing with the strong force and particles susceptible to it, i.e. those possessing colour quantum number

However, when $\sigma = 2$ is required, only a single theory is permissible, the Dirac-Born-Infeld (DBI) theory [5]. The fact that this is possible in the first place is an unexpected result as the number of derivatives is equal to the number of fields in the Lagrangian, and so it seems to allow only for $\sigma = 1$. Nonetheless, similarly as in the case of non-linear σ model, contributions from different diagrams can cancel out upon the requirement of the second soft degree. The only situation how such cancellation can happen is when all the parameters are related and we are left with a single degree of freedom.

Upon continuing further and looking for $\sigma = 3$ theory with Lagrangian from equation 20, only the free theory is found with all parameters (couplings λ_i's) equal to zero [5].

Besides requiring (generalised) Adler zero, a different approach to constraining the theories can be taken. The truly exceptional theories have a so-called *enhanced soft limit*, which means that their soft degree is higher than what one would expect from the number of derivatives in their respective Lagrangians [30]. One can recall that such theories were already encountered earlier in this chapter.

For $\sigma = 1$, such theory is the non-linear σ-model discussed in chapter 6.3. For $\sigma = 2$, the DBI theory mentioned just now is obtained. Although there was no interacting theory with third soft degree for the Lagrangian from equation 20, if a different Lagrangian is chosen, a unique $\sigma = 3$ theory can be found, the Special Galileon [5].

Due to the too-high values of σ (relative to the derivatives in their Lagrangians), all three theories are highly constrained, each having only a single parameter which determines their coupling [30]. Similarly as explained for the DBI theory above, any further restrictions would make these theories trivial. In this sense they are similar to Yang-Mills theory and gravity, which makes them an attractive research topic [5]. Furthermore, these theories also realise additional symmetries (another similarity to Yang-Mills theory and general relativity), which permit this special behaviour in the first place, and can be interpreted as Goldstone modes [7].

7 Colour-Kinematics Duality and Double Copy

Yang-Mills theory was discussed from the viewpoint of action and we saw that the gauge symmetries introduce a large amount of redundancy in the description. Although this redundancy makes some properties clearer, it obscures others and makes computations of scattering amplitudes more complicated. This is because redundancy in the description leads to redundant Feynman diagrams, and computing redundant Feynman diagrams means that unnecessary calculations are being done [5].

An alternative way of dealing with scattering amplitudes is by taking advantage of the colour-kinematics duality, discovered by Bern, Carrasco, and Johansson, hence also known as BCJ duality [35]. They found that the kinematic structures and colour factors which make up scattering amplitudes of Yang-Mills theory obey the same algebraic relations. In particular, there is a kinematic analog of the Jacobi identity (equation 2), which holds for colour essentially by definition. As a consequence, the colour and kinematic factors can be mutually interchanged in the S-matrix, that determines the scattering amplitudes [5], [30].

7.1 Double Copy

A close consequence of the colour-kinematics (CK) duality is the double copy (DC) structure of theories. Namely, the replacement described above leads to new relations between existing theories, or even a generation of new theories. As a primary example, consider Yang-Mills amplitude for the strong force. By replacing colour for kinematics, a graviton scattering amplitude can be obtained [5], [7]. After this replacement, the factors are "doubled up", hence the name *double copy* [30].

This particular relation of theories immediately brings two great results - connection between gravity and gauge theories, and simplification of calculations for gravitons.

First of all, finding a common framework for gauge theories and gravity is one of the most daunting challenges of modern physics. Gauge theory, describing the electromagnetic, strong and weak forces, and general relativity, giving results about gravity, are two most used, generally recognised theories of the past and present century [7]. Although they have some similarities, mentioned in chapters 4.1 and 4.2, they are largely distinct and incompatible. Connecting the two would mean bridging the gap between scales, forces, and geometries. The authors of a review about the CK duality with applications [1] believe that such a unification is possible only by finding and taking advantage of new symmetries. Colour-kinematics duality could be one of those. Particularly, it brings a new view on gravity as a double copy of the strong force.

Secondly, complicated graviton amplitudes can be simplified by exploiting the double copy. One of the differences between gluon and graviton scattering is the number of particles which can meet in one point. For gluons, each vertex of Feynman diagrams can only have three or four particles interacting, and multi-gluon scatterings are mediated by propagators which are not directly visible to an outside observer. On the other hand, an arbitrary number of gravitons can interact in each vertex, which leads to great complications in the calculations of graviton amplitudes [1].

This is where the practicality of BCJ duality comes in. When trying to compute the graviton scattering amplitude, the duality gives us the opportunity to instead write down the gluon amplitude, replace colour factors for kinematics and after taking care of appropriate coupling constants and normalisation, a result for graviton scattering is obtained [1], [30].

7.2 Closer Look at Amplitudes

After the quick-paced overview, it is time to slow down and take a look at the colour-kinematics duality in more detail.

In its original form, the BCJ duality states that a representation can be found such that the perturbative expansion of tree-level Yang-Mills amplitude can be rewritten into a sum of cubic diagrams within which the kinematic numerators n_i have the same symmetry and algebraic structure as the corresponding colour factors c_i. Later this statement was generalised to hold for all gauge theories and more. It was proven to hold at tree-level, and is conjectured and thoroughly checked, though not proven to hold, at loop-level [1], [7].

Thanks to this result, we can write the BCJ scattering amplitude $\mathcal{A}$ as

$$\mathcal{A} = \sum_{i \in \{\text{cubic diagrams}\}} \frac{c_i n_i}{d_i}, \tag{21}$$

where d_i is the product of squares of the propagator momenta, and for all triplets i, j, k the following statements hold [6],

$$c_i + c_j + c_k = 0 \iff n_i + n_j + n_k = 0 \tag{22}$$

$$c_i = -c_j \iff n_i = -n_j. \tag{23}$$

Notice that the only diagrams summed over here are those containing only trivalent vertices. This is another difference from the Feynman approach where quartics are included for Yang-Mills. For the graviton amplitude, the standard approach goes even further, to higher-point diagrams, and so the simplification coming from CK duality is even larger. The number of nonequivalent cubic diagrams for n-point scattering is $(2n-5)!!$ [7].

The colour factors are combinations of the structure constants which contain information related to the gauge group of a particular Yang-Mills theory. For four-point, they are defined to be products of two structure constants, $c_i = f_{ab}{}^e f_{ec}{}^d$ [5]. Note that literature differs in convention as some sources also include a factor of -2 in the definition [1].

While the colour factors carry information about the colour and the group structure of the interacting fields, making it fairly obvious that the Jacobi identity $c_i + c_j + c_k = 0$ holds, it is less obviously the case for the kinematic numerators. These contain information about the momenta p^μ and polarization ε^μ of the particles. The exact form of the numerators heavily depends on the particular amplitude in question. Below is an example of what one such numerator looks like for an s-channel scattering of four massless spin-1 particles [1]

$$
\begin{aligned}
n_s = &-\frac{1}{2}\left[(\varepsilon_1\cdot\varepsilon_2)p_1^\mu + 2(\varepsilon_1\cdot p_2)\varepsilon_2^\mu - (\varepsilon_2\cdot\varepsilon_1)p_2^\mu - 2(\varepsilon_2\cdot p_1)\varepsilon_1^\mu\right] \\
&\cdot\left[(\varepsilon_3\cdot\varepsilon_4)p_{3\mu} + 2(\varepsilon_3\cdot p_4)\varepsilon_{4\mu} - (\varepsilon_4\cdot\varepsilon_3)p_{4\mu} - 2(\varepsilon_4\cdot p_3)\varepsilon_{3\mu}\right] \\
&-\frac{s}{2}\left[(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4) - (\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\right].
\end{aligned}
$$

In section 7.1, a replacement of kinematics for colour was mentioned to obtain the graviton amplitude:

$$
\mathcal{A}_{grav} = \kappa \sum_{i\in\{\text{cubic}\}} \frac{n_i^2}{d_i},
$$

where κ contains the gravitational coupling constant. The kinematic numerators do not necessarily need to be identical, it is also possible to replace c_i by $\tilde{n}_i$ [5]. Any such replacement, leading to a double copy theory, is possible precisely due to CK duality [1]. That is, because the kinematic numerators satisfy the same identities as the colour factors, that being equations 22 and 23.

KLT. This method of obtaining the result yields the Kawai-Lewellen-Tye (KLT) amplitude for graviton scattering. It was derived from the low-energy limit of string theory, and stood in the foundations of CK duality, so this should not come as a surprise [30].

Kawai, Lewellen and Tye noticed a similarity in scattering amplitudes of Yang-Mills theory and general relativity in the framework of string theory already in 1986 [36]. They discovered that at tree-level, a gravity amplitude can be written as a product of two Yang-Mills amplitudes, which lead to the so-called KLT relations. Afterwards the BCJ trio generalised it based on the underlying similarity in Jacobi relations for kinematics and colour, leading to CK duality [4].

7.3 Four-gluon Scattering

In this section, a scattering of four gluons will be considered. Similarly as in an earlier example 5.2, a convention with two entering and two-exiting particles will be deployed.

Let us write the amplitude as per equation 21, following the BCJ. In order to do this, we begin by identifying all possible tree-level cubic diagrams for four-gluon scattering. These happen to topologically correspond to the exchange diagrams from section 5.2.2. Their Feynman versions are shown in Figure 4. However, for CK duality, often different type of diagrams - BCJ diagrams - are used [30]. For four-gluon scattering, the only tree-level shape is the so-called half-ladder with four legs, the labelling of which can be done in three non-equivalent ways. These three options are shown in Figure 5. Note that at higher-point, starting from six-particle scattering, other shape, called the snowflake, also becomes possible. For more details on six-point interactions see [30].

There are three tree-level cubic diagrams, and so it is possible to write down the scattering amplitude as

$$
\mathcal{A}_4 = \frac{c_s n_s}{d_s} + \frac{c_t n_t}{d_t} + \frac{c_u n_u}{d_u}.
$$

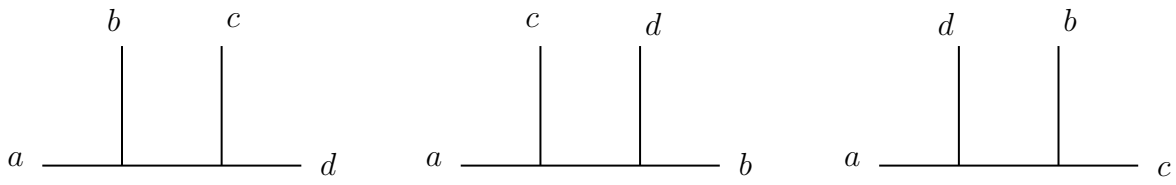


Figure 5: BCJ diagrams for four interacting particles

Note that the naming convention of the three channels from section 5.2 is used, even though the three corresponding summands are not identical as will be discussed below.

The denominators d_i's are the products of the propagators and the diagrams involved here each have only one propagator with a conveniently named momentum, the amplitude can be written as

$$\mathcal{A}_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}. \tag{24}$$

As a reminder, the Mandelstam variables s, t, and u are defined to be

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_c + p_d)^2, \\ t &= (p_a - p_c)^2 = (p_d - p_b)^2, \\ u &= (p_a - p_d)^2 = (p_b - p_c)^2. \end{aligned}$$

7.3.1 Colour Factors

Now, let us consider what the colour factors are in this case. Earlier, we saw that each colour factor is defined as a product of two structure constants. In particular [5], [6],

$$\begin{aligned} c_s &= f_{ab}{}^e f_{ecd} \\ c_t &= f_{ac}{}^e f_{edb} \\ c_u &= f_{ad}{}^e f_{ebc}. \end{aligned}$$

Notice that the structure constants involved in the colour factor of each channel correspond to the vertices of the particular diagram. For example, in the s-channel, the first vertex has particles a, b, and e interacting, and the second vertex includes particles e, c, and d, therefore c_s is made up of structure constants f_{abe} and f_{ecd}.

As a side-note, the Jacobi identity for the colour factors can be checked. As there are only three diagrams involved, there is only one Jacobi identity of the form shown in equation 22, which is the sum of the three colour factors,

$$c_s + c_t + c_u = f_{ab}{}^e f_{ecd} + f_{ac}{}^e f_{edb} + f_{ad}{}^e f_{ebc} = 0 \tag{25}$$

by equation 2. This yields a third way of writing down the Jacobi identity, expressed in a yet different quantity.

7.3.2 Four-vertex Problem

In the perturbative expansion, the scattering amplitude also has a contribution from a four-vertex as gluons can have both three- and four-point interactions, similarly to the scalar example from section 5.2. A Feynman diagram of such vertex is depicted on Figure 3. Notice that this diagram has no propagator involved, and so the corresponding term in the scattering amplitude will have no poles, which can also be seen in the case of a scalar field in example 5.2.

At first sight, this may appear as an issue because all terms in expression 24 have non-constant denominators depending on the momenta of the propagators, while the term coming from the contact diagram does not. However, this seeming problem can be solved by multiplying the contact term by 1 of the form s/s.[12] Afterwards, by tweaking the numerator, it can be included in one of the terms we already have. As any Mandelstam can be used, this way of rewriting the amplitude is not unique, giving us even more freedom in choosing a representation which would satisfy BCJ identities (equations 22, 23). This additional freedom contributes to what is called a *generalised gauge invariance* [1], [5], [6].

7.3.3 Generalised Gauge Invariance

Gauge invariance is a crucial property of all gauge theories. If the result is not invariant under a gauge transformation of the field, it cannot be physical. The generalised gauge invariance is an extension of this concept, requiring invariance under a transformation of the numerator, rather than the field itself. This is checked by replacing the polarisation ε_4 by the momentum p_4 in the expression of the kinematic numerators. If a quantity is gauge invariant, it should vanish upon this replacement [1]. Alternatively, ε_4 could be replaced by $\varepsilon_4 + p_4$ and a quantity would be invariant if it does not change after this transformation.

When applying the first transformation to the four-gluon kinematic numerators, they do not vanish. Instead they become

$$\begin{aligned} n_s &\to s\ \alpha(\varepsilon, p) \\ n_t &\to t\ \alpha(\varepsilon, p) \\ n_u &\to u\ \alpha(\varepsilon, u), \end{aligned}$$

where α is a non-zero function of the polarisation and momenta [5]. This implies that the kinematic numerators are not physical observables. This is similar to Feynman diagrams which are individually also not gauge invariant, only the total scattering amplitude is [1].

To see whether this will also be the case for the BCJ amplitude under the generalised gauge invariance, the numerator transformations are substituted into equation 24. Notice that the colour factors and the denominators remain constant as they do not depend on the polarisation ε_4. So the amplitude becomes

$$\mathcal{A} \to \frac{c_s s\alpha}{s} + \frac{c_t t\alpha}{t} + \frac{c_u u\alpha}{u} = c_s\alpha + c_t\alpha + c_u\alpha = \alpha(c_s + c_t + c_u) = 0,$$

[12] Any other Mandelstam variable can be used as well.

where the last equality brings the transformation of the amplitude to zero by Jacobi identity, equation 25.

In the discussed case, $\alpha(\varepsilon, p)$ is a particular function that is obtained by making the replacement, however one can notice that there is no special property of α that is used here, merely that it is same for all three channels. Therefore the scattering amplitude is invariant under any transformation of the numerators which leads to

$$\begin{aligned} n_s &\to n_s + s\ \beta(\varepsilon, p), \\ n_t &\to n_t + t\ \beta(\varepsilon, p), \\ n_u &\to n_u + u\ \beta(\varepsilon, p). \end{aligned}$$

These transformations form a set which gives rise to the generalised gauge invariance [6].

7.4 Beyond Yang-Mills and Gravity

As was mentioned in the introduction, although the double copy construction originates from connecting Yang-Mills theory and general relativity, it was found to hold for a broad web of theories. A schematic overview of DC theories can be found in [1] and [30], however a complete systematisation of all permissible double copies is yet to be made.

An example of another BCJ-satisfying theory, perhaps the most straight forward extension, is obtained by starting from Yang-Mills theory but instead of replacing colour for kinematics, one replaces kinematics for colour. This way a double-coloured scalar[13] is obtained, corresponding to bi-adjoint scalar theory [5], [7].

The focus was for now on colour-kinematics duality and double copy following from that, and although this was the original result, it was found that the double copy holds also for other types of numerators. Instead of using colour and kinematic factors with polarisation as is the case for Yang-Mills, it is possible to use kinematics without polarisation (in other words, those of scalar fields) or completely different quantities occurring in scattering amplitudes which satisfy the appropriate relations. This significantly increases the scope of the web of double copy theories.

As an example, other theories constructable by the double copy are the Non-linear σ model, Dirac-Born-Infeld theory, and Special Galileon - theories with enhanced soft limit which were discussed in chapter 6. These can be found by using flavour factors and kinematic numerators without polarisation [7].

[13] It must be a scalar theory because polarisation, present in non-scalar theories, can only be included in the kinematic factors.

8 BCJ Representation

In the previous chapter, it was mentioned that any numerator satisfying the appropriate relations can serve as a building block for the double copy. The goal of this chapter is to find a representation of the symmetric group in the carrier space of BCJ numerators. These are the numerators which can, in principle, be used in the double copy construction of a theory. This is the case if they satisfy particular constraint equations which are detailed in the first part of this chapter.

8.1 Constraint Equations

The BCJ scattering amplitude can be written as a product of two numerators divided by propagators and summed over all trivalent diagrams, as shown in equation 21. The numerators depend on the n external particles which we label $1, 2, 3, ..., n$ for a particular diagram, or in general $a, b, c, d, ..$ (which take on integer values from the earlier list), and so for a general double-copy theory the amplitude can be written as

$$A_n = \sum_{\text{trivalent diagrams}} \frac{N_{abcd...}\,\tilde{N}_{abcd...}}{D_{abcd...}}. \tag{26}$$

The numerator factors in equation 26 can be viewed as functions of the indices and are restricted by constraint equations. For factors of n-point scattering, there are in general $n-1$ forms of constraint equations. In particular, there is the reverse order condition which requires

$$N_{abcd...} = (-1)^n N_{...dcba} \tag{27}$$

and there are $n-2$ forms of Jacobi-like[14] constraints (which will be called BCJ constraints from now on) which assert

$$-N_{abcd...} = N_{bacd...} = N_{c[ab]d...} = N_{d[[ab]c]...} = ... \,. \tag{28}$$

These constraint equations were taken from an article [7]. Note that listed here are the *forms* of equations but these equations hold for all permutations of the indices $1, 2, ..., n$ which are substituted for $a, b, c, ...$. For the rest of this work, whenever we label or count the equations, we are really working just with the forms of equations. There are in reality $n!$ equations of each form, which are not all independent.

The condition about reverting order is fairly straightforward, but the BCJ equations will benefit from a bit of clarification.

Firstly, a note on notation; the square brackets mark a commutator, which would typically mean $[a, b] = ab - ba$. However, a difference of subscripts on one N does not

[14] The name comes from the similarity of the first equations to the Jacobi equations satisfied by colour factors which inherit their structure from the underlying Lie algebra.

make much sense. Instead, the idea behind this notation is that the difference propagates up to become a difference between the numerators themselves. Symbolically, this means

$$N_{c[ab]d} = N_{cabd} - N_{cbad}.$$

Similarly, nesting of commutators leads to for example

$$N_{d[[ab]c]e} = N_{d[ab]ce} - N_{dc[ab]e} = N_{dabce} - N_{dbace} - N_{dcabe} + N_{dcbae}.$$

Secondly, one can notice that there is an indication that the string of the constraints continues, to more complicated expressions with more nested commutators. As mentioned above already, the number of BCJ constraints is $n-2$, and so one must consider only the first $n-2$ equalities of the long sequence. By counting the equalities in equation 28, one can easily reach the conclusion that these suffice for five-point interactions, however new conditions are needed for higher points. Those can be found by writing down a nested commutator of the first m indices, then placing the $(m+1)$th index all the way to the left, and keeping the rest on the right. The next equations are hence

$$-N_{abcdef\ldots} = N_{e[[[ab]c]d]f\ldots} = N_{f[[[[ab]c]d]e]\ldots} = \ldots .$$

Equations 27 and 28 relate various numerators together, causing them to depend on each other, therefore decreasing the number of linearly independent $N_{abcd\ldots}$'s from the original $n!$.

8.2 Basis for the BCJ Numerators

In order to work more concretely and make it easier to obtain a particular representation later in section 8.3, we will now set up a basis for our carrier space - the space of BCJ numerators for scattering of n particles.

Our proposal for a basis is the set of elements of the form $N_{1\,\sigma(2)\ldots\sigma(n-1)\,n}$, where the first and last indices are kept fixed across all the basis numerators[15] and $\sigma \in S_\Sigma$ is a permutation of the indices in the middle, which are elements of $\Sigma = \{k \mid k \in \mathbb{N},\ 2 \leq k \leq n-1\}$. Notice that S_Σ is isomorphic to S_{n-2}, so for all purposes it behaves identically to S_{n-2}, except that wherever one would put 1, instead we write $n-1$.

The size of Σ is $n-2$, and so the size of S_Σ is $(n-2)!$ [16]. The numerators are uniquely determined by the order of the subscripts, therefore the size of the to-be-basis is $(n-2)!$, same as the size of the group S_Σ.

The span of this basis will form a carrier space for a BCJ representation, which is sought in the rest of this chapter. Some research looking for these representations was performed earlier by de Neeling, Roest and Veldmeijer [7] who conjectured the dimension of this representations to be $(n-2)!$. Since the dimension of a representation coincides with the dimension of its carrier space, this claim holds if the chosen set is a basis (but

[15] Which numbers/letters/symbols will be fixed in the first and last positions is arbitrary and does not have any influence on the reality, as long as it is consistent across the entire basis. So without loss of generality, we chose the first index to be 1 and the last index to be n as it is the most natural choice.

not vice versa).

Now, we will attempt to argue that this set indeed forms a basis for the numerators satisfying the constraint equations. For the set to be a basis, it must satisfy two conditions - it must span the entire space of numerators (proved in section 8.2.1), and the basis elements must be independent of each other (discussed in section 8.2.2).

8.2.1 Span of the Basis

First, we show that the basis indeed spans the set of all BCJ numerators. This statement will hold if and only if any numerator can be written as a linear combination of the basis elements, i.e.

$$N_{abcd\ldots} = \sum_i k_i N_{1\ \sigma_i(2)\ldots\sigma_i(n-1)\ n}$$

where k_i's are constants and i runs over all the different permutations of the inner $n-2$ indices. As the only relations between the numerators are the reverse order and BCJ constraints, these should be used to form the linear combinations.

A crucial observation for this proof is that in each BCJ equation, there is only one numerator with a at the right-most position relative to the other numerators (as per equation 28), and all other a's are placed more to the left. In particular, when the equations are written in the following form and order

$$\begin{aligned}
\text{1st BCJ constraint:} \quad & -N_{abc\ldots} = N_{bac\ldots} \\
\text{2nd BCJ constraint:} \quad & -N_{abcd\ldots} = N_{c[ab]d\ldots} = N_{cabd\ldots} - N_{cbad\ldots} \\
\text{3rd BCJ constraint:} \quad & -N_{abcde\ldots} = N_{d[[ab]c]e\ldots} = N_{dabce\ldots} - N_{dbace\ldots} - N_{dcabe\ldots} + N_{dcbae\ldots} \\
& \ldots,
\end{aligned}$$

the mth BCJ constraint has a single numerator with a on the $(m+1)$th position and all other a's are on "lower" positions. For example, the second equation has one term with a on the third position and the other terms have a on the first or second positions.

This holds also for all other (more complicated) equations which are not explicitly written here. This can be seen from the way the sequence of the equations is built: in the right-hand side numerator, a is always on the second position and more importantly, it is always nested in the inner-most commutator. When expanding all the commutators, starting from the outside, we always get one term with the commutator more to the front and one with it more to the back. This holds for all commutators, leading to a being on each position, which is in the span of positions taken up by the commutators, exactly once.

The rest of the proof follows by construction. Suppose a general numerator which has n and 1 on arbitrary positions. In order to express it in the basis, n needs to be brought to the end and 1 to the front. First we deal with n, afterwards with 1. For reasons which will become clearer later, it is not handy to operate on both at the same time.

Our basis has n on the last position and so this numerator needs to be written as a linear combination of terms which have n at the end. If n is already at the end, this part can be skipped and one can move to the second part where we deal with 1. If n is not in the last position, the following algorithm with $a = n$ can be applied to move n to the first position in all the terms in the linear combination, and then the reverse order condition can be applied to move it to the end of all the involved numerators. We cannot move n to the end by using BCJ equations because they always leave the last index untouched. Let us see in more detail why this is the case. We know so far that

- for numerators with n indices, there are $n-2$ equations,
- mth equation permutes the first $m+1$ indices,
- more complex equations permute larger number of indices.

Therefore, the last equation for each n permutes the largest number of indices, so that is the only one of concern. This equation is labelled as $m = n-2$, hence it permutes the first $m+1 = n-2+1 = n-1$ indices. This leaves one, the last, index fixed in all the BCJ equations.

Algorithm for moving index a to the first position
Suppose that the index a is in position $i \neq 1$. By observation outlined above, a is exactly once on the $(m+1)$th position in the constraint equation m and all other terms in this equation have a on positions closer to the front. By applying equation $m = i-1$ and rearranging it, the original numerator can be expressed as a linear combination of other numerators which have a closer to the front.

Next, we repeat this process for each of the new numerators separately, i.e. apply equation $m = i-2$ for a numerator with a on the $(i-1)$th position, and equation $m = i-3$ for a on the $(i-2)$th position, and so on. This is repeated over and over on all the new terms. In every iterations, the numerators will have a more and more to the front. Eventually, the first equation will be applied to finalise, bringing the a's to the first position in all the numerators in the linear combination.

Now that all the terms have n at the end, the algorithm is repeated for $a = 1$, leading to having 1 in the first position. Now one might wonder, 1 is at the front but what about n? It was in the last position before applying the algorithm for the second time, but is it still there? The answer is: Yes, it is. This is assured because in all the BCJ equations, the last index is kept fixed (as explained above) and the reverse order equations are not used anymore after bringing n to the end. So, by placing n at the end of the sequence of integers, it is safely "stored" there when the algorithm operates on the first (at most) $n-1$ indices to move 1 to the front.

This concludes the first part of the argument where it was shown that the chosen basis indeed spans the set of all BCJ numerators.

8.2.2 Independence of Basis Elements

After proving that the chosen set is indeed a spanning set of the space of the BCJ numerators, it must also be shown that it is the minimal spanning set. In other words, the elements of this set should be independent of each other.

Multiple attempts were made to directly show the independence of the basis elements, unfortunately a satisfactory proof for arbitrary n was not found. Outlines of the endeavours which resulted in some simplification of the problem are given in appendix A. The missing pieces needed to complete the proofs are highlighted and form assumptions which should be satisfied for the results in the rest of this work to hold.

A persuasive reason to believe that these assumptions hold and the given set is indeed a basis, and so the obtained results are correct, comes from [6]. In chapter 13.1 of this text, the dimension of the space of colour factors of Yang-Mills amplitude is claimed to be $(n-2)!$. A set of particular numerators may have additional constraints imposed on it, further decreasing the dimension of the basis, but it cannot be less restricted (have larger dimension) than the general case considered here. This is because all the conditions from section 8.1 must be satisfied by numerators permitting the double copy. Since the colour factors of Yang-Mills theory form a set of BCJ numerators,[16] the dimension of the general basis considered here should be at least that of the colour factors, i.e. $(n-2)!$.

From previous chapter, we know that the chosen set of $(n-2)!$ elements spans the space of BCJ numerators, therefore the space can be at most $(n-2)!$ dimensional. Combining these two conditions we find that the space must have exactly $(n-2)!$ dimensions and so any set of this many elements which spans the space forms a basis.

8.3 Strategy for Finding the Representation

Next, we would like to find a representation of the symmetric group according to which the BCJ numerators transform. Mathematica script from Appendix B was used to solve the system of equations and perform the calculations. The strategy behind the process is explained here.

In order to make the following of this process easier, comparable steps for less abstract group and representation are mentioned where possible. As this parallel example, a defining representation of a special orthogonal group $SO(2)$ with carrier space $\mathbb{R}^2$, was chosen. This choice was made because this process is simpler to visualise.

Let $N_{abcd\ldots}$ be a general set of numerators which satisfy the reverse order and BCJ constraints. It was shown earlier that not all of the numerators are independent, and so to find a representation of the symmetric group in this carrier space, it is needed to only look at how the basis transforms. Therefore first of all, we choose a basis to be the set of the same form as in section 8.2. In the parallel example of $SO(2)$, this translates to setting up a coordinate system with a basis $\{\hat{x}, \hat{y}\}$.

To each basis numerator, permutations from the symmetric group are applied. This is similar to applying a rotation to the basis vectors of the $\mathbb{R}^2$ plane, as shown in Figure 6. It is the irrep decomposition that is sought here, not directly the representation itself

[16] at least at tree-level

(though it can be reconstructed from the decomposition), and so we only need to find the characters of the representation and afterwards apply equations 5 and 6. As discussed in section 3.3.2, the characters of conjugate elements are identical. This simplifies the process as it is now necessary to only apply one element per conjugacy class to the basis numerators in order to find the characters.

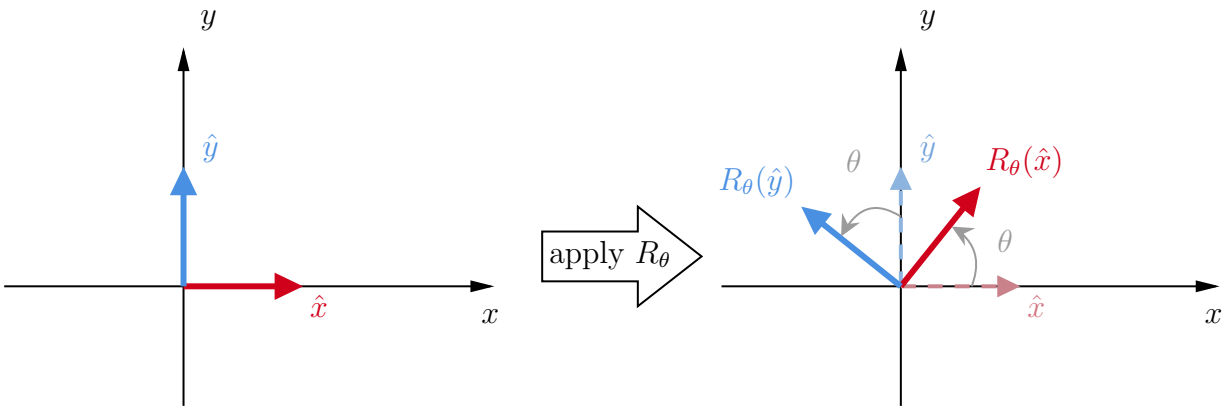


Figure 6: Applying rotation $R_\theta \in SO(2)$ to basis vectors of the space $\mathbb{R}^2$

Applying the permutations to the basis elements maps them to other numerators in the space. By using the reverse order and BCJ constraint equations, the images of the basis elements can be expressed as linear combinations of the old basis. In the parallel example, this corresponds to writing the rotated vectors in the original basis. In the case of rotation by $\pi/6$ counterclockwise, $R_{\pi/6}$, we obtain

$$R_{\pi/6}(\hat{x}) = \frac{\sqrt{3}\hat{x} + \hat{y}}{2} \quad \text{and } R_{\pi/6}(\hat{y}) = \frac{-\hat{x} + \sqrt{3}\hat{y}}{2}.$$

The co-domain of the sought representation should be the general linear group and so we relate the image under the permutation to the basis via a matrix, representing this permutation. This is similar to the case of rotations where $R_{\pi/6}$ can be written as

$$\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}.$$

Similarly, a counterclockwise rotation by a general angle θ, R_θ, is represented by

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

This mapping is performed for one element from each conjugacy class. Once a matrix for each conjugacy class is obtained, the characters are found simply as the traces of these matrices. For the case of the given representation of $SO(2)$, the characters would be $2\cos(\theta)$.

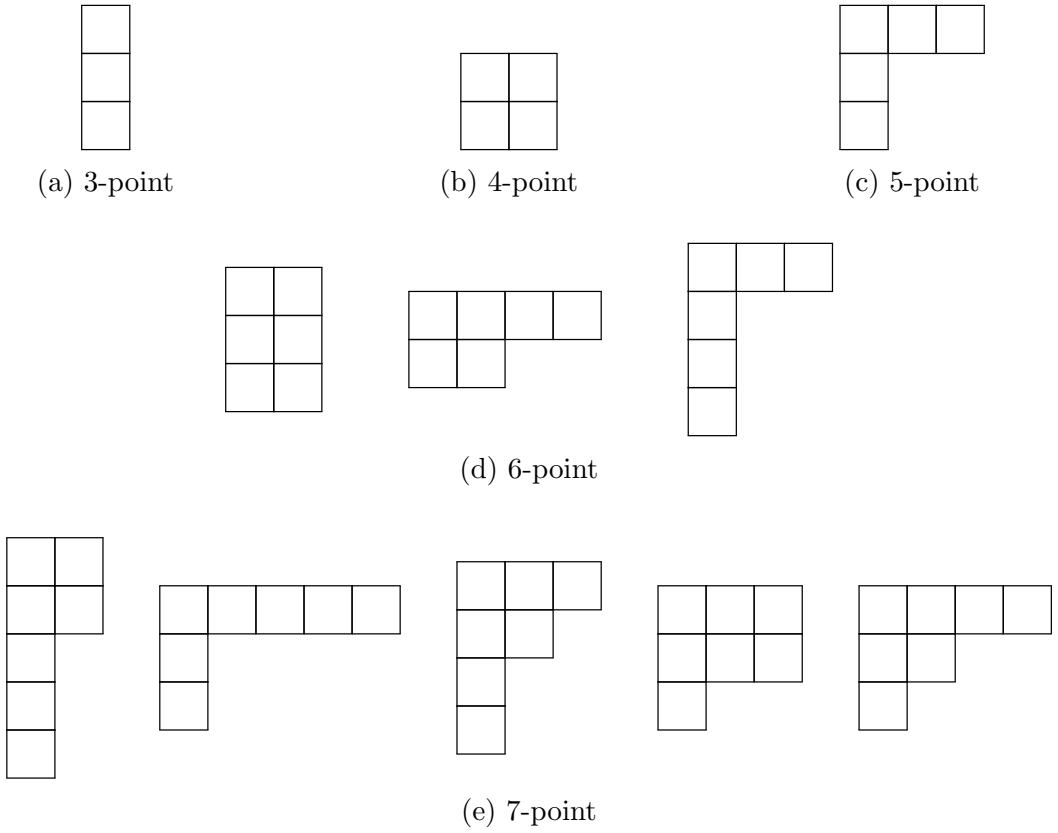
In order to apply equation 5 (which could not be used for $SO(2)$ because this group is not finite) and decompose this representation into irreducible representations, the characters of the irreps of S_n, as well as the number of elements in each conjugacy class must be known. The Frobenius formula can be used to find the characters of the irreps, however this is not the goal of this work and so instead the results of J. Gibson available at [22] were used.

Finally, once decomposed, the irreducible representations can be expressed as Young diagrams.

8.4 Results

Following the procedure described in section 8.3, a Mathematica script, which can be found in Appendix B, was written to find the decomposition of the representations.

The irrep decomposition of the BCJ representations from three-point to seven-point interactions are shown in Figure 7 as Young diagrams. Note that each of these irreps occurs exactly once in the decomposition.



(a) 3-point

(b) 4-point

(c) 5-point

(d) 6-point

(e) 7-point

Figure 7: Irrep decomposition of the BCJ representation for n-point scattering, expressed as Young diagrams. Each irreducible representation (diagram) occurs exactly once in the decomposition for these scatterings.

Columns signal antisymmetry in indices while rows point towards symmetry [13]. Hence in three-point scattering, only fully antisymmetric numerators are possible. Structure constants f_{abc} can serve as examples of such numerators since they gain a minus sign upon permutation in any pair of indices. On the other hand, four-point numerators should have same number of symmetric and antisymmetric pairs of indices. This is satisfied for example by the colour numerators of the form $c_s = N_{abcd} = f_{ab}{}^e f_{ecd}$ introduced in section 7.3. This particular numerator is antisymmetric in indices (a, b) and (c, d), but symmetric in (a, c) and (b, d).

8.5 Scalar Kinematics

The representation of the symmetric group, which describes how the BCJ numerators transform when their indices are permuted, was found. What can be done next with these representations?

The results give a general condition which must be satisfied by numerators of double copy permissible theories. It can be used to not only check validity of numerators, but also to construct the factors and hence find corresponding theories. This was done up to six-point for flavour and scalar kinematics in [7]. This research lead to finding, for instance, theories with enhanced soft limit, discussed in chapter 6.

8.5.1 Powers of Mandelstam

Looking for the numerators or even theories from the representation is not a straightforward task. In this chapter, an outline is given for how to begin such a search in the case of kinematic numerators of scalars, i.e. theories of fields without spin. An example of a Lagrangian for a scalar theory was given in chapter 5.1. For scalars without any underlying (group) structure, this is all that there is and the resulting numerators can be used to directly construct theories. In the case of scalars which have other properties (than momentum) relevant for the scattering amplitude, there is more to choose from but these factors are still valid building blocks. For now, the focus is on numerators consisting purely from momenta.

Momenta can be combined to create Mandelstam variables without loss of relevant information. Mandelstam variables were introduced in chapter 5.2 for four-point interaction, however this concept can be generalised for any n-point. The Mandelstam variables are then defined as

$$s_{ij} = (p_i + p_j)^2, \tag{29}$$

using the all-momenta-incoming convention, and i, j labelling the external particles. The four-point Mandelstam variables can be identified in this convention and notation as

$$s \to s_{12} = s_{34} \quad , \quad t \to s_{13} = s_{24} \quad , \quad u \to s_{23} = s_{14}.$$

As mentioned in section 3.3.4, the Mandelstam variables of n-particle scattering transform following the $(n-2, 2)$ irreducible representation of the symmetric group. Notice that

this representation exists only from four-point onward (i.e. only for S_4 or larger group). This restriction is reasonable as there are no on-shell scatterings of three particles with only momenta, and so one should not expect an option for S_3.

In order to find at which order in Mandelstam there exists a BCJ numerator, the symmetric product of the Mandelstam representation with itself is taken q-times (in other words, we take the qth power of the representation). Upon decomposition of this product into irreducible representations,[17] we look for the irreps making up the BCJ representation. Let k be the number of times the BCJ representation is present, or equivalently the number of times all its irreps (including multiplicities) appear in the decomposition. Then k is the number of BCJ numerators of power q in Mandelstam variables for n-point scattering. This corresponds to power $2q$ in momenta.

Although we are not all the way there yet, this simplifies the problem as it provides information about where to look for the numerators.

8.5.2 Four-point Scalar Kinematic Numerators

In the case of four-particle scattering, the BCJ representation is the so-called *window*, partitioned as (2,2), shown in Figure 7. The Mandelstam representation is by definition $(n-2,2) = (4-2,2) = (2,2)$. The two representations happen to coincide![18] This can also be interpreted as taking the Mandelstam irrep to power 1 and finding the BCJ representation in its "decomposition". Therefore, as a consequence, there is a numerator linear in Mandelstam variables.

As mentioned earlier, columns of the Young diagrams symbolise antisymmetry in indices and rows a symmetry. From the definition of Mandelstam variables (equation 29), it is clear that Mandelstam variables are symmetric in their two indices. However, the BCJ representation requires also antisymmetry in two pairs. This can be achieved by taking a difference of two Mandelstam variables [7], leading to a numerator scale-able by a constant λ

$$N_{abcd} = \lambda(s_{ac} - s_{bc}).$$

For four-point, the on-shell condition asserts that $s_{ij} = s_{kl}$ for all indices, hence the numerator can also be expressed as

$$N_{abcd} = \lambda(s_{bd} - s_{ad}) = \lambda(s_{ac} - s_{ad}) = \lambda(s_{bd} - s_{bc}).$$

We can now see that this numerator is antisymmetric in (a,b) and (c,d) from its first and last expressions, respectively. A concrete example of a numerator of this form is

$$N_{1234} = s_{13} - s_{23} = t - u.$$

The existence of linear numerators is straightforward to see since the two investigated representations are the same. In order to demonstrate the process described above, let us examine whether any numerators quadratic in Mandelstam variables exist. We begin by taking a symmetric product (marked here with $\otimes$) of the Mandelstam representation with itself and decomposing it into irreps. This is performed with the help of [22].

[17] Tool [22] can serve as a useful resource when performing this task.

[18] Note that this is a special case that occurs at four-point and by no means holds for all n.

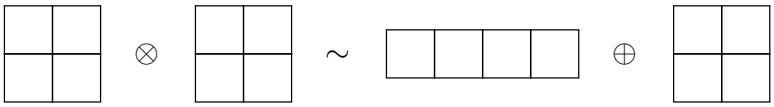


Figure 8: Decomposition of the product of the four-point Mandelstam representation with itself.

As shown in Figure 8, we find that the second power of the Mandelstam representation decomposes into the window and the trivial representation. Since the four-point BCJ representation (the window) is present, we conclude that there exists one set of numerators quadratic in Mandelstam variables which can be used for a double copy constructable amplitude. These numerators are of the form [7]

$$N_{abcd} = \lambda s_{ab}(s_{ac} - s_{bc}).$$

In particular,

$$N_{1234} = s_{12}(s_{13} - s_{23}) = s(t - u).$$

8.6 Possibilities for Improvements and Further Research

The research performed in this work has opened a number of directions for future research, a few of which are outlined in this chapter.

We believe that a goal of research in scalar double copy is to better understand the double copy construction and classify scalar theories, of which there exist many, as discussed in chapter 6. The most natural next step is finding and classifying double-copy-permissible structureless scalars, starting by obtaining information about powers in Mandelstam variables as described in section 8.5, then looking for corresponding numerators. Afterwards, also scalar theories with additional structure can be considered.

In chapter 8.2.2, it was argued why the space of BCJ numerators should be $(n-2)!$ dimensional. However, the argument relies on statements from other sources which were proven to hold only at tree-level. Several attempts were also made to show the independence of the basis elements purely from the constraint equations, however all are left incomplete, requiring additional assumptions.

Therefore a major improvement of this work lies in finishing these efforts, proving the assumptions which they rely on, or inventing an entirely new way of looking at the problem that would lead to a conclusive solution. We believe that the easiest way of reaching the desired result is by showing that the assumptions of Attempt 3 from appendix A always hold.

The decompositions of BCJ representations were found in this work up to and including seven-point. The reason for stopping there were constraints on time and computational power. Although, the process of finding BCJ representations is general and can be repeated for arbitrary n-point interactions, the scaling of this process with increasing n is very steep, both in terms of memory and computing time.

The number of equations is significantly larger than what is necessary to solve the system. It is conjectured that only the reverse order and the most complex BCJ equations are needed but even many of those are superfluous. This conjecture was tested and confirmed to hold up to seven-point. When performing these computations, it was found that providing the computer with fewer equations decreases the amount of memory needed but increases the running time. At higher-point, the memory appears to be a larger concern than the running time, therefore it would be valuable to determine which equations are independent and necessary for finding the solution.

Besides containing more equations than necessary, the Mathematica code in appendix B is far from optimised as that was not the focus of this research. Therefore, further work could be done in making this code run more efficiently, leading to lower demands on computational power.

Lastly, many interesting results related to the double copy structure of spin-0, spin-1, and spin-2 theories exist and were mentioned in this thesis. However, further research could also examine the validity of this construction for fields with half-integer-spin.

9 Conclusion

This work is centered around representations related to the double copy construction. This framework had been derived as a consequence of colour-kinematics duality in Yang-Mills theory, underlying the standard model of particle physics.

The first part of this work focused on providing a broad background in particle physics in order to appreciate the beauty of the double copy. We started off with an introduction to Lie groups and algebras in chapter 3 which helped in understanding Yang-Mills theory later in chapter 4. Next to Yang-Mills theory, a brief tour into general relativity was made, focusing on introducing its perturbative version which allows us to treat it as an effective field theory and gives rise to quantum excitations, gravitons.

Decreasing breadth and delving deeper into scattering amplitudes, we entered chapter 5. The Feynman method of obtaining scattering amplitudes was explained and applied to an example of four-scalar scattering for a particular Lagrangian. Next, the soft limit of amplitudes was used to probe behaviour of both scalar theories and those with spin. Conservation of electric charge and universality of gravity was obtained as a consequence of soft photons and gravitons respectively. In the case of scalars, it was found that the soft limit can be used to discriminate theories and choose special ones.

Eventually, we reached chapter 7, where the colour-kinematics duality was introduced as well as the resulting double copy between Yang-Mills theory and general relativity.

With all the pieces at hand, our research about representations was described. General numerators which can be used for the double copy construction must satisfy particular relations. These were translated into statements about representations which the numerators must follow in their transformations under the symmetric group. This was done by first setting up a basis and afterwards developing and implementing an algorithm which applied permutations from the symmetric group to the basis elements and expressed this process as a set of matrices. By finding the characters, a decomposition into irreducible representations was found.

Results were obtained for up to seven-point, after which the computations became heavy on memory and running time. However, a large part of the value of this work lies in the generality and systematicness of its approach, rather than the particular representations which were found. This makes it more straightforward to continue up to higher n-points. Nevertheless, this method could be further improved. In case one would wish to find representations for scatterings of large numbers of particles, it is advised to first attempt to simplify and optimize the code developed in this work, in order to decrease the requirements on computing power.

The closest next step lies in using the results found in this work to find kinematic numerators of scalar theories. This is the most natural continuation because they depend only on Mandelstam variables, which have a known representation.

Although double copy was first noticed as a relation between Yang-Mills theory and general relativity, it was found to hold for other theories as well. Nowadays, there are entire webs of theories being discovered and systematised in this context.

Scalar theories are of particular interest as there are many realistically allowed ones,

hence there are opportunities for a large number of double copy relations. This could help with a better classification of scalar theories in general, but also in improving our understanding of the physical meaning of the double copy.

The vast space of scalar theories can serve as a sandbox to notice patterns and symmetries in the way the double copy connects them. By interpreting these relations, the meaning behind double copy could be uncovered, perhaps revealing hidden truths about similarities between Yang-Mills theory and general relativity, taking us a step closer to joining these two into a single unified framework.

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Appendix

A Attempts to Prove Independence of Basis Elements

This appendix shows attempts at a direct proof of independence of the numerators in a chosen set $\{N_{1\ \sigma(2)\dots\sigma(n-1)\ n} \mid \sigma \in S_\Sigma, \Sigma = \{k \mid k \in \mathbb{N}, 2 \leq k \leq n-1\}\}$ under the constraint equations. This proof is needed to show that the chosen set is a basis of the space of BCJ numerators. More details can be found in chapter 8.

Attempt 1. This proof relies on the result of the algorithm described in section 8.2.1 to be the unique way of obtaining the linear combinations. This occurs if the set of independent equations contains only

- BCJ equations with the basis elements on the left-hand side,
- BCJ equations with numerators $N_{n\alpha(1)\dots\alpha(n-1)}$ on the left-hand side for $\alpha \in S_{n-1}$,
- reverse order equations relating numerators with n at the first and last position.

The basis numerators are independent if none of them can be written as a linear combination of the other ones. Which means that no basis element $N_{1\rho(2)\dots\rho(n-1)n}$ for any $\rho \in S_\Sigma$ can be written as

$$N_{1\rho(2)\dots\rho(n-1)n} = \sum_i k_i N_{1\sigma_i(2)\dots\sigma_i(n-1)\ n},$$

for permutations $\sigma_i \in S_\Sigma$ such that $\sigma_i \neq \rho$ for all i.

Assume for contradiction that this would be possible. If it is possible, it is via the BCJ equations and it should be possible to find the relations or a way of obtaining them.

Suppose that this connection is possible and in particular uses an equation which we will call $\widetilde{m}$ and which contains the term $N_{1\rho(2)\dots\rho(n-1)n}$ is the numerator on the left-hand side of the equation. This means that n is the last index of all the terms and one can identify $a = 1$. By arguments from section 8.2.1, it is possible to express all the terms with a on position $\widetilde{m}$ or lower as a linear combination of the basis elements by using equations 1 up to $\widetilde{m} - 1$. This way a relation

$$-N_{1\rho(2)\dots\rho(n-1)n} = \sum_i \tilde{k}_i N_{1\sigma_i(2)\dots\sigma_i(n-1)\ n} + N_{\tau(2)\dots\tau(\tilde{m}+1)\ 1\ \tau(\tilde{m}+2)\dots\tau(n-1)n} \tag{30}$$

is obtained for some $\tau \in S_\Sigma$, and 1 in the last term is on the $(\tilde{m}+1)$th position. This looks close to expressing $N_{1\rho(2)\dots\rho(n-1)n}$ as a linear combination of the other basis elements which would disprove the independence. The last missing puzzle piece is expressing $N_{\tau(2)\dots\tau(\tilde{m}+1)\ 1\ \tau(\tilde{m}+2)\dots\tau(n-1)n}$ in the basis.

From the previous section, there is one obvious way of doing this - by using the algorithm. However, the algorithm uses precisely this $\tilde{m}$ equation and leads to exactly this relation and so the other basis elements cancel out and we end up with $-N_{1\rho(2)\dots\rho(n-1)n} = -N_{1\rho(2)\dots\rho(n-1)n}$.

Therefore, should we want to find an expression for $N_{1\rho(2)\dots\rho(n-1)n}$ as a function of the other basis elements, it is necessary to find $N_{\tau(2)\dots\tau(\tilde{m}+1)\ 1\ \tau(\tilde{m}+2)\dots\tau(n-1)n}$ without using equation $\tilde{m}$. Equations 1 to $\tilde{m}-1$ are not going to be of help here (except for perhaps final touches) as they do not change the position of the $(\tilde{m}+1)$th index. Therefore, it is necessary to look at more complex equations, labelled $\tilde{m}+1$ and higher.

If $\tilde{m}$ is the most complex equation (meaning that $\tilde{m}=n-2$), one can immediately reach the conclusion that expressing $N_{\tau(2)\dots\tau(\tilde{m}+1)\ 1\ \tau(\tilde{m}+2)\dots\tau(n-1)n} \equiv N_{\tau(2)\dots\tau(n-1)1n}$ in a different way is not possible as $\tilde{m}$ is the only equation which can move 1 away from position $n-1$[19]. If $\tilde{m}\neq n-2$, more complex equations exist, hence there is still a chance for our initial claim to be false.

As mentioned in the previous section, in each equation (labelled m), $a=1$ is on each position (up to $m+1$) exactly once. So in equation of the form $m'=\tilde{m}+1$ which includes the sought numerator $N_{\tau(2)\dots\tau(\tilde{m}+1)\ 1\ \tau(\tilde{m}+2)\dots\tau(n-1)n}$, all the other terms will have 1 on positions different from $\tilde{m}+1$. Using this new equation we can express $N_{\tau(2)\dots\tau(\tilde{m}+1)\ 1\ \tau(\tilde{m}+2)\dots\tau(n-1)n}$ as a combination of the other terms. We know how to deal with indices more to the front - use less complex equations, which do not include the sough term, to express it in the basis. Substituting the new expression back in equation 30, leads to

$$\begin{aligned}
-N_{1\rho(2)\dots\rho(n-1)n} &= \sum_i \tilde{k}_i N_{1\sigma_i(2)\dots\sigma_i(n-1)\ n} + \sum_i {k'}_i N_{1\sigma_i(2)\dots\sigma_i(n-1)\ n} + \\
&\quad + c\ N_{1\rho(2)\dots\rho(n-1)n} + N_{\tau'(2)\dots\tau'(\tilde{m}+2)\ 1\ \tau'(\tilde{m}+3)\dots\tau'(n-1)n} \\
-(c+1)N_{1\rho(2)\dots\rho(n-1)n} &= \sum_i (\tilde{k}_i + {k'}_i) N_{1\sigma_i(2)\dots\sigma_i(n-1)\ n} + N_{\tau'(2)\dots\tau'(\tilde{m}+2)\ 1\ \tau'(\tilde{m}+3)\dots\tau'(n-1)n},
\end{aligned}$$

which is the same equation as before[20], except now the coefficients are different.

This process can be iterated until the following form is reached

$$-N_{1\rho(2)\dots\rho(n-1)n} = \sum_i k''_i N_{1\sigma_i(2)\dots\sigma_i(n-1)\ n} + N_{\tau''(2)\dots\tau''(n-1)\ 1\ n}. \tag{31}$$

There is a single equation that has a on the one-before-last position. This is the last considered equation $m=n-2$, which was already used to reach this form. Therefore it is not possible to express $N_{1\rho(2)\dots\rho(n-1)n}$ as a linear combination of only the other basis numerators, so the basis elements must be independent.

Attempt 2. An alternative way of solving the problem is by looking at the dimensions. If it can be shown that the dimension of the space of BCJ numerators is $(n-2)!$, then any set of this size which spans the space is its basis. (this is also what is used for the indirect proof provided in Section 8.2.2.

[19] besides the reverse order equation which can bring it to the second position but that would also move n and not lead to the desired result because the only way to get n at the end is by using the reverse order again

[20] Up to a constant multiplying the left-hand side but that does not effect the argument.

The dimension, i.e. the number of independent numerators, can be found as the difference between the total number of numerators and the number of independent equations. The amount of numerators is known to be $n!$, and so the question translates into finding the number of independent equations.

The number of independent reverse order equations is $n!/2$ but this can go down if they are related by BCJ equations. Provided that we want to keep all of these in our set of independent equations, we need to show that the number of independent BCJ constraints is

$$n! - (n-2)! - \frac{n!}{2} = \frac{n!}{2} - (n-2)!$$

Attempt 3. This method uses an assumption similar to the one from Attempt 1 but the proof follows in a more elegant way, using dimension counting from Attempt 2.

Let us assume that all the equations which are not given in the list at the beginning of Attempt 1, can be derived from the listed one. This means that we need at most:

- $(n-2)\cdot(n-2)!$ BCJ equations with the basis on the left-hand side,
- $(n-2)\cdot(n-1)!$ BCJ equations with a numerator starting with n on the left-hand side,
- $(n-1)!$ reverse order equations with a numerator starting with n.

These equations are sufficient for the basis to span the space and we assumed that all other equations can be derived from these. Hence there are at most $(n-2)(n-2)! + (n-2)(n-1)! + (n-1)!$ independent equations.

$$\#\text{all numerators} - \#\text{independent equations} = \#\text{independent numerators}$$

Using the bound mentioned above we get

$$\#\text{all numerators} - \#\text{independent numerators} \leq (n-2)(n-2)! + (n-2)(n-1)! + (n-1)!$$

The total number of numerators is $n!$, so upon rearrangement

$$\begin{aligned}
\#\text{independent numerators} &\geq n! - ((n-2)(n-2)! + (n-2)(n-1)! + (n-1)!) \\
&= n! - (n-1)(n-2)! + (n-2)! - n(n-1)! + 2(n-1)! - (n-1)! \\
&= n! - (n-1)! + (n-2)! - n! + 2(n-1)! - (n-1)! \\
&= (n-2)!
\end{aligned}$$

We obtain that the number of independent numerators is at least $(n-2)!$. However, in section 8.2.1, it was found that the space can be spanned by a set with $(n-2)!$ elements, hence its dimension can be at most $(n-2)!$. These two inequalities together lead to a conclusion that the number of independent numerators, i.e. dimension of the space, must be exactly $(n-2)!$ and since the chosen set spans the space and has the right size, it is

also its basis.

The last attempt appears closest to yielding the desired result in the clearest way. In order to complete the proof, it is necessary to show that indeed all the other BCJ and reverse order equations can be obtained from the ones listed at the beginning.

B Script for Finding BCJ Representations

This section contains Mathematica code used to find decomposition of the BCJ representation into its irreps.

The given version is for seven-point interactions but it is simple to modify for any n-point. The difficulty of performing the changes was further decreased by including comments throughout the code starting with *FOR N-POINT EDIT:* with instructions of what exactly needs to be done and where, to avoid forgetting about changing some parts.

A carefully reader might notice that the code includes unused equations, cycle types, and commented data. These were prepared for an eight-point run, which was however not completed due to steep (more than order of $(n!)^2$) scaling which resulted in lack of time and computing power.

```
(* list of indeces *)
indeces = {a, b, c, d, e, f, g};
(* FOR N-POINT EDIT: include appropriate number of indeces *)

n = Length[indeces] (* n-point interaction *)
N0 = Subscript[N, indeces]; (* define numerator *)


(** find the basis and dependent elements **)

(* generate basis numerators *)
permutingIndeces = indeces[[2 ;; n - 1]];
permForBasis = Permutations[permutingIndeces];
basis = Table[N0 /. {Table[permutingIndeces[[j]] -> permForBasis[[i, j]],
    {j, Length[permutingIndeces]}]}[[1]], {i, 1, (n - 2)!}];

(* generate all numerators *)
permAll = Permutations[indeces];
all = Table[N0 /. {indeces -> permAll[[i]]}, {i, 1, n!}];

(* generate dependent numerators *)
nonBasis = Complement[all, basis];


(** generate all constraits - BCJ and order equations **)

(* reverse order equations *)
equation0 =
  Table[(N0 == ((-1)^n)  Subscript[N, in1]) /. {N0 -> all[[i]],
     in1 -> Reverse[permAll[[i]]]}, {i, 1, Length[all]}];
```

```
(* BCJ equations *)
equation1 = Table[N0 == -Subscript[N, in1]/.{N0 -> all[[i]],
    in1 -> Permute[permAll[[i]]  , Cycles[{{1,2}}]]}, {i, 1, Length[all]}];

equation2 = Table[(-N0 == -Subscript[N, in2] + Subscript[N, in1]) /.
    {N0-> all[[i]], in2 -> Permute[permAll[[i]], Cycles[{{1,3}}]],
    in1 -> Permute[permAll[[i]]  , Cycles[{{1,2,3}}]]},  {i, 1, Length[all]}];

equation3 = Table[ -N0 ==   Subscript[N, in1] - Subscript[N, in2] -
    Subscript[N, in3] + Subscript[N, in4] /. {N0 -> all[[i]],
    in1 -> Permute[permAll[[i]], Cycles[{{1,2,3,4}}]],
    in2 -> Permute[permAll[[i]], Cycles[{{1,3,4}}]] ,
    in3 -> Permute[permAll[[i]], Cycles[{{1,3,2,4}}]],
    in4 -> Permute[permAll[[i]], Cycles[{{1,4},{2,3}}]]} ,{i, 1, Length[all]}];

equation4 = Table[ -N0 ==   Subscript[N, in1] - Subscript[N, in2] -
    Subscript[N, in3] + Subscript[N, in4] - Subscript[N, in5] +
    Subscript[N, in6] + Subscript[N, in7] - Subscript[N, in8]    /.
    {N0 -> all[[i]], in1 -> Permute[permAll[[i]], Cycles[{{1,2,3,4, 5}}]],
    in2 -> Permute[permAll[[i]], Cycles[{{1,3,4, 5}}]] ,
    in3 -> Permute[permAll[[i]], Cycles[{{1,3,2,4, 5}}]],
    in4 -> Permute[permAll[[i]], Cycles[{{1,4, 5},{2,3}}]],
    in5 -> Permute[permAll[[i]], Cycles[{{1,3, 5}, {2,4}}]],
    in6 -> Permute[permAll[[i]], Cycles[{{1,4, 2,3, 5}}]] ,
    in7 -> Permute[permAll[[i]], Cycles[{{1,4,2, 5}}]],
    in8 -> Permute[permAll[[i]], Cycles[{{1,5},{2,4}}]]} ,{i, 1, Length[all]}];

equation5 = Table[ -N0 ==   Subscript[N, in1] - Subscript[N, in2] -
    Subscript[N, in3] + Subscript[N, in4] - Subscript[N, in5] +
    Subscript[N, in6] + Subscript[N, in7] - Subscript[N, in8]  -
    Subscript[N, in9] + Subscript[N, in10] + Subscript[N, in11] -
    Subscript[N, in12] + Subscript[N, in13] - Subscript[N, in14] -
    Subscript[N, in15] + Subscript[N, in16]   /. {N0 -> all[[i]],
    in1 -> Permute[permAll[[i]], Cycles[{{1,2,3,4, 5, 6}}]],
    in2 -> Permute[permAll[[i]], Cycles[{{1,3,4, 5, 6}}]] ,
    in3 -> Permute[permAll[[i]], Cycles[{{1,3,2,4, 5, 6}}]],
    in4 -> Permute[permAll[[i]], Cycles[{{1,4, 5, 6},{2,3}}]],
    in5 -> Permute[permAll[[i]], Cycles[{{1,3, 5, 6}, {2,4}}]],
    in6 -> Permute[permAll[[i]], Cycles[{{1,4, 2,3, 5, 6}}]],
    in7 -> Permute[permAll[[i]], Cycles[{{1,4,2, 5, 6}}]],
    in8 -> Permute[permAll[[i]], Cycles[{{1,5, 6},{2,4}}]],
    in9 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1,2,3,4, 5, 6}}], Cycles[{{2,3,4,5,6}}]]],
    in10 -> Permute[permAll[[i]], PermutationProduct[Cycles[{{1,3,4, 5,6}}],
```

```
    Cycles[{{2,3,4,5,6}}]]],
    in11 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1,3,2,4, 5, 6}}], Cycles[{{2,3,4,5,6}}]]],
    in12 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1,4, 5, 6},{2,3}}], Cycles[{{2,3,4,5,6}}]]],
    in13 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1,3, 5, 6},{2,4}}], Cycles[{{2,3,4,5,6}}]]],
    in14 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1,4, 2,3,5, 6}}], Cycles[{{2,3,4,5,6}}]]],
    in15 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1,4,2, 5,6}}], Cycles[{{2,3,4,5,6}}]]],
    in16 -> Permute[permAll[[i]], PermutationProduct[Cycles[{{1,5,6},{2,4}}],
    Cycles[{{2,3,4,5,6}}]]]} ,{i, 1, Length[all]}];

equation6 = Table[ -N0 ==
     Subscript[N, in1] - Subscript[N, in2] - Subscript[N, in3] +
      Subscript[N, in4] - Subscript[N, in5] + Subscript[N, in6] +
      Subscript[N, in7] - Subscript[N, in8]  - Subscript[N, in9] +
      Subscript[N, in10] + Subscript[N, in11] - Subscript[N, in12] +
      Subscript[N, in13] - Subscript[N, in14] - Subscript[N, in15] +
      Subscript[N, in16] - Subscript[N, in17] + Subscript[N, in18] +
      Subscript[N, in19] - Subscript[N, in20] + Subscript[N, in21] -
      Subscript[N, in22] - Subscript[N, in23] + Subscript[N, in24]  +
      Subscript[N, in25] - Subscript[N, in26] - Subscript[N, in27] +
      Subscript[N, in28] - Subscript[N, in29] + Subscript[N,
      in30 ] +  Subscript[N, in31] - Subscript[N, in32]   /. {N0 ->
      all[[i]],
     in1 -> Permute[permAll[[i]], Cycles[{{1, 2, 3, 4, 5, 6, 7}}]],
     in2 -> Permute[permAll[[i]], Cycles[{{1, 3, 4, 5, 6, 7}}]] ,
     in3 -> Permute[permAll[[i]], Cycles[{{1, 3, 2, 4, 5, 6, 7}}]],
     in4 ->
      Permute[permAll[[i]], Cycles[{{1, 4, 5, 6, 7}, {2, 3}}]],
     in5 -> Permute[permAll[[i]], Cycles[{{1, 3, 5, 6, 7}, {2, 4}}]],
     in6 -> Permute[permAll[[i]], Cycles[{{1, 4, 2, 3, 5, 6, 7}}]] ,
     in7 -> Permute[permAll[[i]], Cycles[{{1, 4, 2, 5, 6, 7}}]],
     in8 -> Permute[permAll[[i]], Cycles[{{1, 5, 6, 7}, {2, 4}}]],
     in9 -> Permute[permAll[[i]],
     PermutationProduct[Cycles[{{1, 2, 3, 4, 5, 6, 7}}],
     Cycles[{{2, 3, 4, 5, 6}}]]],
     in10 -> Permute[permAll[[i]],
     PermutationProduct[Cycles[{{1, 3, 4, 5, 6, 7}}],
     Cycles[{{2, 3, 4, 5, 6}}]]],
     in11 -> Permute[permAll[[i]],
     PermutationProduct[Cycles[{{1, 3, 2, 4, 5, 6, 7}}],
```

```
Cycles[{{2, 3, 4, 5, 6}}]]],
in12 -> Permute[permAll[[i]], PermutationProduct[Cycles[{{1, 4, 5, 6, 7},
{2, 3}}], Cycles[{{2, 3, 4, 5, 6}}]]],
in13 -> Permute[permAll[[i]], PermutationProduct[Cycles[{{1, 3, 5, 6, 7},
{2, 4}}], Cycles[{{2, 3, 4, 5, 6}}]]],
in14 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 4, 2, 3, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6}}]]],
in15 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 4, 2, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6}}]]],
in16 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 5, 6, 7}, {2, 4}}],
Cycles[{{2, 3, 4, 5, 6}}]]],
in17 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 2, 3, 4, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in18 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 3, 4, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in19 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 3, 2, 4, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in20 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 4, 5, 6, 7}, {2, 3}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in21 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 3, 5, 6, 7}, {2, 4}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in22 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 4, 2, 3, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in23 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 4, 2, 3, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in24 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 5, 6, 7}, {2, 4}}],
Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in25 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 2, 3, 4, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]],
in26 -> Permute[permAll[[i]],
PermutationProduct[Cycles[{{1, 3, 4, 5, 6, 7}}],
Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]],
```

```
    in27 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1, 3, 2, 4, 5, 6, 7}}],
    Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]],
    in28 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1, 4, 5, 6, 7}, {2, 3}}],
    Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]],
    in29 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1, 3, 5, 6, 7}, {2, 4}}],
    Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]],
    in30 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1, 4, 2, 3, 5, 6, 7}}],
    Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]],
    in31 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1, 4, 2, 5, 6, 7}}],
    Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]],
    in32 -> Permute[permAll[[i]],
    PermutationProduct[Cycles[{{1, 5, 6, 7}, {2, 4}}],
    Cycles[{{2, 3, 4, 5, 6}}], Cycles[{{2, 3, 4, 5, 6, 7}}]]]} ,
    {i, 1, Length[all]}];

(* FOR N-POINT EDIT: add new constraint equation and
    add it to the list of equations *)
equations = Join[equation0, equation1, equation2, equation3, equation4, equation5];


(** finding solution - expressing dependent tensors in terms of the basis **)

solution = Solve[equationsReduced, nonBasis][[1]];

(** preparing permutations **)

(* write down one permutation for each cycle type (conjugacy class) *)
(* notice that we permute based on "name", not position *)
cycle1 = {a -> a};
cycle2 = {a -> b, b -> a};
cycle22 = {a -> b, b -> a, c -> d, d -> c};
cycle222 = {a -> b, b -> a, c -> d, d -> c, e -> f, f -> e};
cycle2222 = {a -> b, b -> a, c -> d, d -> c, e -> f, f -> e, g -> h, h -> g};
cycle3 = {a -> b, b -> c, c -> a};
cycle32 = {a -> b, b -> c, c -> a, d -> e, e -> d};
cycle322 = {a -> b, b -> c, c -> a, d -> e, e -> d, f -> g, g -> f};
cycle33 = {a -> b, b -> c, c -> a, d -> e, e -> f, f -> d};
cycle332 = {a -> b, b -> c, c -> a, d -> e, e -> f, f -> d, g -> h, h -> g};
cycle4 = {a -> b, b -> c, c -> d, d -> a};
```

```
cycle42 = {a -> b, b -> c, c -> d, d -> a,  e -> f, f -> e};
cycle422 = {a -> b, b -> c, c -> d, d -> a,  e -> f, f -> e,  g -> h, h -> g};
cycle43 = {a -> b, b -> c, c -> d, d -> a,  e -> f, f -> g, g -> e};
cycle44 = {a -> b, b -> c, c -> d, d -> a, e -> f, f -> g, g -> h, h -> e};
cycle5 = {a -> b, b -> c, c -> d, d -> e, e -> a};
cycle52 = {a -> b, b -> c, c -> d, d -> e, e -> a, f -> g, g -> f};
cycle53 = {a -> b, b -> c, c -> d, d -> e, e -> a, f -> g, g -> h, h -> f};
cycle6 = {a -> b, b -> c, c -> d, d -> e, e -> f, f -> a};
cycle62 = {a -> b, b -> c, c -> d, d -> e, e -> f, f -> a, g -> h, h -> g};
cycle7 = {a -> b, b -> c, c -> d, d -> e, e -> f, f -> g, g -> a};
cycle8 = {a -> b, b -> c, c -> d, d -> e, e -> f, f -> g, g -> h, h -> a};

(* FOR N-POINT EDIT: include all cycle types and add them into the cycles list *)


(* merge cycles into a list, make sure to have same order as in,
https://www.jgibson.id.au/articles/characters/ *)

(* ====== 6-point ======= *)
(*
cycles = {cycle1, cycle2, cycle22,cycle222,  cycle3, cycle32,
cycle33, cycle4, cycle42, cycle5,cycle6};
*)

(* ======= 7-point ======== *)

cycles = {cycle1, cycle2, cycle22,cycle222,  cycle3, cycle32, cycle322,
cycle33, cycle4, cycle42, cycle43, cycle5, cycle52, cycle6, cycle7};


(* ======= 8-point ======== *)
(*
cycles = {cycle1, cycle2, cycle22, cycle222, cycle2222,  cycle3, cycle32,
    cycle322, cycle33, cycle332, cycle4, cycle42, cycle422, cycle43, cycle44,
    cycle5, cycle52, cycle53, cycle6, cycle62, cycle7, cycle8};
*)


(** finding characters **)

(* apply elements representing each conjugacy class to each basis element *)
permutedBasis = Table[basis[[i]] /. cycles[[j]], {i,  Length[basis]},
    {j, Length[cycles]}];
```

```
(* use solution to express permuted elements back in the basis *)
image = permutedBasis /. solution ;

(* find matrix representation by "reading off" coefficients from expressing it
in the basis. Equations are linear, so it can be done by differentiation *)
matrices =
  Table[D[image[[i, k]], basis[[j]]], {k, Length[cycles]}, {i,
    Length[basis]}, {j, Length[basis]}];

(* tracing matrices to obtain characters *)
charBCJ = Table[Tr[matrices[[i]]], {i, Length[matrices]}]


(**number of elements in conjugacy classes and characters of irreps **)
(* obtained from https://www.jgibson.id.au/articles/characters/ *)


(* ====== 6-point ======= *)
(*
elementsInCC = {1, 15, 45, 15, 40, 120, 40, 90, 90, 144, 120};

charIrreps = {{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1},
{5, 3, 1,-1, 2, 0,-1, 1,-1, 0,-1},
{9, 3, 1, 3, 0, 0, 0,-1, 1,-1, 0},
{10, 2,-2,-2, 1,-1, 1, 0, 0, 0, 1},
{5, 1, 1,-3,-1, 1, 2,-1,-1, 0, 0},
{16, 0, 0, 0,-2, 0,-2, 0, 0, 1, 0},
{10,-2,-2, 2, 1, 1, 1, 0, 0, 0,-1},
{ 5,-1, 1, 3,-1,-1, 2, 1,-1, 0, 0},
{9,-3, 1,-3 ,0, 0, 0, 1, 1,-1, 0},
{5,-3, 1, 1, 2, 0,-1,-1,-1, 0, 1},
{1,-1, 1,-1, 1,-1, 1,-1, 1, 1,-1}};
*)


(* ========= 7-point ======= *)

elementsInCC =
{1,21,105,105,70,420,210,280,210,630,420,504,504,840,720};

charIrreps = {{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1},
{6,4,2,0,3,1,-1,0,2,0,-1,1,-1,0,-1},
{14,6,2,2,2,0,2,-1,0,0,0,-1,1,-1,0},
{15,5,-1,-3,3,-1,-1,0,1,-1,1,0,0,0,1},
```

```
{14,4,2,0,-1,1,-1,2,-2,0,1,-1,-1,0,0},
{35,5,-1,1,-1,-1,-1,-1,-1,1,-1,0,0,1,0},
{20,0,-4,0,2,0,2,2,0,0,0,0,0,0,-1},
{21,1,1,-3,-3,1,1,0,-1,-1,-1,1,1,0,0},
{21,-1,1,3,-3,-1,1,0,1,-1,1,1,-1,0,0},
{35,-5,-1,-1,-1,1,-1,-1,1,1,1,0,0,-1,0},
{15,-5,-1,3,3,1,-1,0,-1,-1,-1,0,0,0,1},
{14,-4,2,0,-1,-1,-1,2,2,0,-1,-1,1,0,0},
{14,-6,2,-2,2,0,2,-1,0,0,0,-1,-1,1,0},
{6,-4,2,0,3,-1,-1,0,-2,0,1,1,1,0,-1},
{1,-1,1,-1,1,-1,1,1,-1,1,-1,1,-1,-1,1}};


(* ========= 8-point ======= *)
(*
elementsInCC = {1, 28, 210, 420, 105, 112, 1120, 1680, 1120, 1120, 420,
    2520, 1260, 3360, 1260, 1344, 4032, 2688, 3360, 3360, 5760, 5040};

charIrreps = {{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1},
    {7,5,3,1,-1,4,2,0,1,-1,3,1,-1,0,-1,2,0,-1,1,-1,0,-1},
    {20,10,4,2,4,5,1,1,-1,1,2,0,2,-1,0,0,0,0,-1,1,-1,0},
    {21,9,1,-3,-3,6,0,-2,0,0,3,-1,-1,0,1,1,-1,1,0,0,0,1},
    {28,10,4,2,-4,1,1,1,1,1,-2,0,-2,1,0,-2,0,1,-1,-1,0,0},
    {64,16,0,0,0,4,-2,0,-2,-2,0,0,0,0,0,-1,1,-1,0,0,1,0},
    {35,5,-5,-3,3,5,-1,1,2,2,1,-1,1,1,-1,0,0,0,0,0,0,-1},
    {14,4,2,0,6,-1,1,-1,2,-2,-2,0,2,1,2,-1,-1,-1,0,0,0,0},
    {70,10,2,-2,-2,-5,1,-1,1,1,-4,0,0,-1,-2,0,0,0,1,1,0,0},
    {56,4,0,4,8,-4,-2,0,-1,1,0,0,0,0,0,1,-1,1,1,-1,0,0},
    {90,0,-6,0,-6,0,0,0,0,0,0,2,0,0,2,0,0,0,0,0,-1,0},
    {35,-5,-5,3,3,5,1,1,2,-2,-1,-1,-1,-1,-1,0,0,0,0,0,0,1},
    {42,0,2,0,-6,-6,0,2,0,0,0,-2,0,0,2,2,0,-1,0,0,0,0},
    {56,-4,0,-4,8,-4,2,0,-1,-1,0,0,0,0,0,1,1,1,-1,-1,0,0},
    {70,-10,2,2,-2,-5,-1,-1,1,-1,4,0,0,1,-2,0,0,0,-1,1,0,0},
    {64,-16,0,0,0,4,2,0,-2,2,0,0,0,0,0,-1,-1,-1,0,0,1,0},
    {21,-9,1,3,-3,6,0,-2,0,0,-3,-1,1,0,1,1,1,1,0,0,0,-1},
    {14,-4,2,0,6,-1,-1,-1,2,2,2,0,-2,-1,2,-1,1,-1,0,0,0,0},
    {28,-10,4,-2,-4,1,-1,1,1,-1,2,0,2,-1,0,-2,0,1,1,-1,0,0},
    {20,-10,4,-2,4,5,-1,1,-1,-1,-2,0,-2,1,0,0,0,0,1,1,-1,0},
    {7,-5,3,-1,-1,4,-2,0,1,1,-3,1,1,0,-1,2,0,-1,-1,-1,0,1},
    {1,-1,1,-1,1,1,-1,1,1,-1,-1,1,-1,-1,1,1,-1,1,-1,1,1,-1}};
*)


(* FOR N-POINT EDIT: COPY IN DATA ABOUT S_N FOR THE PARTICULAR N *)
```

```
(** Find decomposition into irreps by taking inner product **)

innerProduct =
 Table[(1/Total[elementsInCC])*
   Sum[elementsInCC[[i]]*Conjugate[charBCJ[[i]]]*
     charIrreps[[j, i]], {i, Length[charBCJ]}], {j, 1,
   Length[charIrreps]}]
```