Higher-Form Symmetries and their Spontaneous Breaking

by

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Abstract

In this thesis we review aspects of higher-form symmetries by primarily studying the free $U(1)$ gauge theory and its one-form symmetries. At first we discuss differential forms and some necessary aspects of differential geometry. We continue by examining different formulations of Maxwell's theory, derive the equations of motion in the differential form notation, and touch on the topic of electromagnetic duality. We proceed by formulation the usual notion of ordinary symmetries in terms of operators and extending this notion to higher-form symmetries by considering higher dimensional charged objects and currents. Finally, we review essential topics related to spontaneous symmetry breaking and conclude that the photon can be viewed as the Goldstone boson of a spontaneously broken one-form symmetry of Maxwell's theory.

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Contents

Chapter 1

Introduction

Few ideas have proven to be as fruitful and far-reaching in regards to our understanding of nature as the notion of symmetries in physics. They pervade our understanding of the standard model of particle physics in the form of Yang-Mills theory and, under the guise of diffeomorphisms, that of general relativity. Perhaps of equal importance, if not of even greater eminence, is the concept of spontaneous symmetry breaking. The reason for this is that whenever a continuous global symmetry is spontaneously broken there must exist gapless excitations of the system. In particle physics, this means particles of zero mass. This is a result owed to Nambu [1] and Goldstone [2], who investigated this more thoroughly, at the beginning of the 1960s.

In recent years, a particularly elegant generalization of our usual notion of ordinary symmetries has been realized by Gaiotto, Kapustin, Seiberg & Willet [3]. These are know as *generalised global symmetries* or *higher-form symmetries* and are symmetries whose charged objects have dimension $p > 0$. Pivotally, these symmetries can also be spontaneously broken, implying the existence of Goldstone bosons.

This work aims to explore this new type of symmetries, with a particular focus on the free field Maxwell theory and its $U(1)$ one-form symmetry. The ultimate goal is to verify the conclusion of [3], namely that the photon is the Goldstone boson of a broken one-form symmetry.

The remainder of this thesis is structured as follows: in Section 2 we begin by reviewing notions in differential geometry that allow us to make the rest of our statements independent of coordinates. This is done by viewing vectors and duals (and more generally tensors) as maps from the respective tangent and cotangent spaces to $\mathbb{R}$, and realizing that what we often refer to as tensors in physics are tensor components after a choice of basis. We then discuss the different formulations of Maxwell's theory in Section 3, starting from the usual formulation in terms of electric $\mathbf{E}$ and and magnetic $\mathbf{B}$ fields and then to the covariant theory, in terms of the field strength tensor $F_{\mu\nu}$, which properly transforms with respect to the full Lorentz group. In 3.3 we formulate the theory in terms of differential forms and derive the equations of motion while also discussing the dual formulation of electromagnetism, showing that the duality transformation is exact in the source-free case. This proves to be useful for analysing the action of one of the higher-form symmetry in the context of electromagnetism.

In Section 4 we consider at first ordinary symmetries and how we can understand them in terms of operators. This language allows us to easily generalize our usual notions of symmetry to that of higher-form symmetries by extending our view of currents and charged objects by allowing them to be higher dimensional. In 4.2 we introduce the notion of Wilson loops (and 't Hooft loops) which are gauge-invariant objects charged under the one-form symmetry. We then proceed by considering the one-form symmetry in the source-free Maxwell's theory and conclude this section by discussing some general aspects of higher-form symmetries. Finally, in Section 5 we review the notion of spontaneous symmetry breaking in the context of ordinary symmetries, with a particular focus on the linear σ model, in analogy to which we base most of our discussion of the spontaneous breaking of the one-form symmetry in electromagnetism. Ultimately, this leads to our interpretation of the photon as the Goldstone boson of a spontaneously broken one-form symmetry.[1]

[1]Note that this is different from the suggestion of [4] that the photon is a Goldstone boson of a broken Lorentz symmetry.

Chapter 2

Differential Forms

Differential forms are extremely useful when working with electromagnetism, or any gauge theory for that matter, in curved spaces by allowing us to work independent of coordinates. They are a natural way of expressing and generalizing the idea of integration. While we will be primarily working with Minkowski spacetime, we review these notions in order to simplify our subsequent notation and make some generalizations more plain to be seen.

This section largely follows [5] and [6] in order to introduce the needed ideas and establish the convention used throughout this work.

2.1 Tensors and differential forms

To start off, consider the usual notion of *vectors*. Rather than "stretching" along the manifold, we associate them to a single point p and they belong to the corresponding *tangent space* T_p, at that point. We can introduce a set of basis vectors $\hat{e}_\mu$, which span T_p, such that any abstract vector V could be written as a linear combination

$$V = V^\mu \hat{e}_\mu, \tag{2.1}$$

with V^μ the components of the vector.[1] Oftentimes in physics, when talking about a vector, one simply refers to its components V^μ (often called *contravariant vectors*) and suppresses the explicit basis vectors. However, the real vector is an abstract geometrical entity while the components are just the coefficients of the basis vectors [5]. In the study of manifolds, especially curved ones, a natural choice of basis for T_p is the *coordinate basis* comprised of the partial derivative operators $\hat{e}_\mu = \partial_\mu$ [5]. From this point of view, the transformation law under a change of coordinates $x^\mu \to x^{\mu'}$ readily follows. By virtue of the chain rule, the new basis vectors are

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu, \tag{2.2}$$

[1]Throughout this thesis, the Einstein summation convention is adopted with Greek indices running from $\mu = 0, 1, 2, 3$ and for Latin ones $i = 1, 2, 3$.

and by demanding that the vector $V = V^\mu \partial_\mu$ remain unchanged by a change of basis we have that

$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'} \partial_{\mu'} \\ &= V^{\mu'} \frac{\partial x^\mu}{\partial x_{\mu'}} \partial_\mu. \end{aligned} \tag{2.3}$$

Since the inverse of $\partial x^\mu / \partial x^{\mu'}$ is $\partial x^{\mu'} / \partial x^\mu$, it implies that

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x_\mu} V^\mu. \tag{2.4}$$

A closely associated notion is that of *dual vectors* which belong to a *cotangent space* (or *dual vector space*) denoted T_p^*, spanned by a set of basis dual vectors $\hat{\theta}^\mu$ for which we demand

$$\hat{\theta}^\mu \hat{e}_\nu = \delta^\mu_\nu. \tag{2.5}$$

Thus, an arbitrary dual can be written as $\omega = \omega_\mu \hat{\theta}^\mu$. Once again, when talking about dual vectors one often refers only to the components ω_μ (sometimes called *covariant vectors*). We view the dual space as the set of all linear maps from the vector space T_p to the real numbers $\mathbb{R}$, i.e. $\omega : T_p \to \mathbb{R}$.[2] This is perhaps made more clear by considering the action of a dual on a vector:

$$\omega(V) = \omega_\mu \hat{\theta}^\mu V^\nu \hat{e}_\nu = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu \in \mathbb{R}. \tag{2.6}$$

The natural basis for the cotangent space is the gradient of the coordinates $\hat{\theta}^\mu = \mathrm{d}x^\mu$ for which (2.5) implies [5]

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x_\nu} = \delta^\mu_\nu. \tag{2.7}$$

In a similar vein to the case of vectors on arbitrary manifolds, we can consider a change of basis $x^\mu \to x^{\mu'}$ for which

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \mathrm{d}x^\mu, \tag{2.8}$$

and asking that the dual $\omega = \omega_\mu \mathrm{d}x^\mu$ be invariant under a coordinate transformation implies that the components transform as

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu. \tag{2.9}$$

By virtue of the definitions introduced above, one can consider the generalization of vectors and dual vectors to that of *tensors*. We view a tensor T of rank $(k,\, l)$ as a multilinear map from k dual vectors and l vectors to $\mathbb{R}$:

$$T : T_p^{*(1)} \times ... \times T_p^{*(k)} \times T_p^{(1)} \times ... \times T_p^{(l)} \to \mathbb{R}, \tag{2.10}$$

where $\times$ is the usual cartesian product. In this language, a vector is identified as a $(1,\, 0)$ tensor while a dual is a $(0,\, 1)$ tensor. A new operation can be defined, the

[2] Similarly, one can view vectors as maps from T_p^* to $\mathbb{R}$, i.e. $V : T_p^* \to \mathbb{R}$.

tensor product $\otimes$, which produces a $(k+m,\, l+n)$ tensor $T \otimes S$, where T is a $(k,\, l)$ tensor and S is a $(m,\, n)$ one:

$$\begin{aligned} T \otimes S(\omega^{(1)}, ..., \omega^{(k+m)}, V^{(1)}, ..., V^{(l+n)}) = T(\omega^{(1)}, ..., \omega^{(k)}, V^{(1)}, ..., V^{(l)}) \\ \times S(\omega^{(1)}, ..., \omega^{(m)}, V^{(1)}, ..., V^{(n)}). \end{aligned} \tag{2.11}$$

We can now introduce the notion of a *differential form*. A differential p-form (or simply a p-form) is a completely antisymmetric $(0,\, p)$ tensor. The space of all p-forms over a manifold M is $\Omega^p(M)$. Note that if the manifold is d-dimensional, there are no p-forms with $p > d$ by virtue of the antisymmetry [5]. In this language, a dual vector is seen as a 1-form.

It is useful to consider an example which illustrates the use of differential forms. A gauge field A_μ can be integrated along the worldline γ of a charged particle, thus writing the integral $\int_\gamma A_\mu \mathrm{d}x^\mu$. A way of approaching this is by parametrizing the curve γ in terms of a function, with parameter λ, to spacetime $x^\mu(\lambda)$ [6]. This would suggest that the integral would now become $\int_b^a \mathrm{d}\lambda \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} A_\mu$. Nevertheless, the answer is independent of the parametrization, so that the integral is simply written as $\int_\gamma A_\mu \mathrm{d}x^\mu$. Differential forms take this notion further, with the integral becoming simply $\int_\gamma A$ where the object A, defined as $A = A_\mu \mathrm{d}x^\mu$, is a 1-form. It contains information about both the field A_μ which is to be integrated and about the measure of the integration $\mathrm{d}x^\mu$.

Another example is that of the field strength $F_{\mu\nu}$, an antisymmetric rank 2 tensor. The 2-form F can be defined as:

$$F = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{2.12}$$

where the *wedge product* $\wedge$ is an antisymmetrized tensor product, meaning:

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu. \tag{2.13}$$

The extra factor of $1/2$ appearing in (2.12) is due to the fact that any given term in the expression above, like $\mathrm{d}x^0 \otimes \mathrm{d}x^1$, appears twice. The reason why a differential form is necessarily antisymmetric is due to the fact that a surface element is composed of each independent coordinate. As an example, on a plane, the area element is given by $\mathrm{d}x\, \mathrm{d}y$; meanwhile, integrating $\mathrm{d}x\, \mathrm{d}x$ would be meaningless.

Evidently, differential forms play an important role in the integration on manifolds so it useful to make this idea more precise. From this perspective, an integral over an n-dimensional manifold Σ is understood as a map from an n-form field ω to $\mathbb{R}$ [5]:

$$\int_\Sigma : \omega \to \mathbb{R}. \tag{2.14}$$

As an example of this, consider a one dimensional integral which can be written as $\int \omega(x) \mathrm{d}x$ with $\omega(x)$ denoting the component function and the integration measure $\mathrm{d}x$ now properly understood as a differential form. In order to see this, consider the usual role of the volume element $\mathrm{d}\mu$ (on a three-dimensional manifold, for simplicity), its role is to assign to each (infinitesimal) region an (infinitesimal) real number, namely its volume [5]. In other words, the volume element is a map from 3 vectors (which define a region) to the real numbers $\mathrm{d}\mu(V, U, W) \in \mathbb{R}$. Because of this, the volume element is a $(0, n)$ tensor (on a n-dimensional manifold) which is also antisymmetric since under the interchange of two vectors we get the same volume, but opposite sign, while in the case that two vectors are collinear, the volume vanishes. Thus, the volume element in n dimensions is a n-form.

Generally, a p-form can be written down as:

$$\omega_p = \frac{1}{p!}\omega_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}, \tag{2.15}$$

with $\omega_{\mu_1 \ldots \mu_p}$ the components of a rank p antisymmetric tensor.[3] The wedge product of the forms is viewed as a signed sum over permutations π,

$$\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} = \sum_{\pi \in S_p} \mathrm{sgn}(\pi)\, \mathrm{d}x^{\pi(\mu_1)} \otimes \ldots \otimes \mathrm{d}x^{\pi(\mu_p)}, \tag{2.16}$$

where π is a permutation of the p indices and $\mathrm{sgn}(\pi)$ is its sign (1 if even, -1 if odd). The following example illustrates the basic idea:

$$\begin{aligned} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \mathrm{d}x \otimes \mathrm{d}y \otimes \mathrm{d}z - \mathrm{d}x \otimes \mathrm{d}z \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}x \otimes \mathrm{d}y \\ - \mathrm{d}z \otimes \mathrm{d}y \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}z \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}x \otimes \mathrm{d}z. \end{aligned} \tag{2.17}$$

Finally, the order of the wedge product between a p-form ω_p and a q-form η_q can be changed by being careful with the signs

$$\omega_p \wedge \eta_q = (-1)^{pq} \eta_q \wedge \omega_p. \tag{2.18}$$

2.2 Exterior derivative

The *exterior derivative* is an operator which maps p-forms to $(p+1)$-forms. At first, consider its action on a 0-form (a usual function $f(x)$):

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\, \mathrm{d}x^\mu = \partial_\mu f\, \mathrm{d}x^\mu. \tag{2.19}$$

Generally, the exterior derivative maps a p-form ω to a $(p+1)$-form

$$\mathrm{d}\omega = \frac{\partial \omega_{\mu_1 \ldots \mu_p}}{\partial x^\mu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}. \tag{2.20}$$

In other words, the exterior derivative of a p-form, $\mathrm{d}\omega$, is the antisymmetrized derivative whose components are the derivatives of the components of ω [6].

Of particular interest to this text is the action of the exterior derivative on the 1-form $A = A_\mu\, \mathrm{d}x^\mu$:

$$\mathrm{d}A = \frac{\partial A_\nu}{\partial x^\mu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \partial_\mu A_\nu\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \partial_\nu A_\mu\, \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu = F. \tag{2.21}$$

[3] The case of a 0-form is the degenerate one and corresponds to no differentials $\mathrm{d}x$. We identify these as regular function $f(x)$.

In components this is simply $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the field strength of a vector field.

The exterior derivative obeys a modified version of the product rule due to its antisymmetric nature, for a p-form ω_p and a q-form η_q

$$\mathrm{d}(\omega_p \wedge \eta_q) = (\mathrm{d}\omega_p) \wedge \eta_q + (-1)^p \omega_p \wedge (\mathrm{d}\eta_q). \tag{2.22}$$

Another important property of exterior differentiation is that for any p-form ω_p, $\mathrm{d}^2 = 0$, i.e.

$$\mathrm{d}(\mathrm{d}\omega_p) = 0. \tag{2.23}$$

This is a consequence of the definition of d and the fact that partial derivatives (acting on anything) commute $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ [5]. If $\mathrm{d}\omega_p = 0$ we refer to it as a *closed* form. In subsequent sections we will identify these to be *conserved currents*. Furthermore, if there exists a $(p-1)$-form η_{p-1} such that $\mathrm{d}\eta_{p-1} = \omega_p$, we refer to ω_p as an *exact* form. Notice that by virtue of (2.23) all exact forms are also closed.[4] From these forms one can define a new vector space called the *de Rahm cohomology* class $H^p(M)$, comprised of all closed forms modulo the exact ones on a given manifold M. Interestingly, the dimensionality of $H^p(M)$ depends only on the topology of M and thus allows us to directly extract information about the manifold itself [5]. Nevertheless, in this report we typically only consider Minkowski space as a manifold.

The form notation is powerful because it allows us to make statements independent of coordinates which is particularly appealing when studying curved spaces. Additionally, the usual notions of calculus take very simple forms, as a specific example consider the generalized Stoke's theorem. For a p-dimensional manifold M, bounded by ∂M, and a $(p-1)$-form ω_{p-1}

$$\int_{\partial M} \omega_{p-1} = \int_M \mathrm{d}\omega_{p-1}. \tag{2.24}$$

2.3 Hodge duality

The final operation needed on differential forms is *Hodge duality*. The *Hodge star* $\star$, on a d-dimensional manifold, takes a p-form to a $(d-p)$-form. In terms of components, the operation is essentially contracting a tensor with the *Levi-Civita tensor*[5] with d raised indices [6]. In terms of the differentials appearing in a given form, the Hodge star substitutes the ones already present with all the others [6]. For instance, in flat 3-dimensional space, $\mathrm{d}x$ is replaced by $\mathrm{d}y \wedge \mathrm{d}z$.

[4] This statement is not as straight forward in physics. The fact that $F = \mathrm{d}A$ leads us to expect that $\mathrm{d}F = 0$. However, this can be violated if the gauge field is ill-defined at a singular point (e.g. the core of a magnetic monopole). See chapter 2.2 of [6] for a more detailed discussion.

[5] Note that we distinguish between the Levi-Civita *symbol* $\tilde{\epsilon}_{\mu_1 \ldots \mu_d}$, defined in the usual manner (i.e. $\tilde{\epsilon}_{01\ldots(d-1)} = +1$), and the Levi-Civita *tensor* $\epsilon_{\mu_1 \ldots \mu_d}$, related to the former via the metric: $\sqrt{|g|}\tilde{\epsilon}_{\mu_1 \ldots \mu_d} = \epsilon_{\mu_1 \ldots \mu_d}$. This difference arises when considering curved spaces and the two can be treated as equal when dealing with flat spaces.

We define the action of the Hodge star on a p-form (on a d-dimensional manifold with metric $g_{\mu\nu}$) as

$$\star(\mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_p}) = \frac{1}{(d-p)!}\epsilon_{\nu_1...\nu_d} g^{\mu_1\nu_1}...g^{\mu_p\nu_p}\mathrm{d}x^{\nu_{p+1}}...\mathrm{d}x^{\nu_d}. \tag{2.25}$$

It is common to refer to $\star\omega_p$ as the (Hodge) dual of ω_p. Acting with $\star$ twice returns the original form up to a minus sign:

$$\star(\star\omega_p) = (-1)^{p(d-p)+s}\omega_p, \tag{2.26}$$

where s is the number of minus signs in the metric signature ($s = 1$ for a Lorentzian metric).[6]

[6]Depending on convention, one can take $s = 3$ but since both are odd there is no difference.

Chapter 3

Electromagnetism and Duality

In this chapter we review some of the formulations of the laws of electromagnetism, starting from the classical vector one and building up to the differential form notation. Additionally, we look at the duality transformation of the Maxwell theory which maps electric and magnetic charges to each other.

3.1 The classical theory

Classical electrodynamics is described by the usual Maxwell equations with a charge density ρ and current density $\mathbf{J}$ [5]:

$$\nabla \cdot \mathbf{E} = \rho \tag{3.1}$$

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \tag{3.1a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.1c}$$

with $\mathbf{E}$ and $\mathbf{B}$ the usual electric and magnetic fields. If a system lacks sources (i.e. $\rho = 0$ and $\mathbf{J} = 0$) Maxwell's equations take the symmetric form

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{3.2}$$

They are invariant under a *duality transformation* $(\mathbf{E}, \mathbf{B}) \to (\mathbf{B}, -\mathbf{E})$. Equations (3.1b) and (3.1c) imply the existence of the scalar ϕ and vector $\mathbf{A}$ potentials defined by [7]:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{3.3}$$

However, the equations above do not determine the potential uniquely as the fields $\mathbf{E}$ and $\mathbf{B}$ remain unchanged under a transformation

$$\phi \to \phi' = \phi + \frac{\partial f}{\partial t}, \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla f, \tag{3.4}$$

with $f(t, \mathbf{x})$ an arbitrary function. We view (3.4) as a *gauge transformation* (corresponding to a *gauge symmetry*) and, generally, require that the observables of a theory are invariant under such transformations (are *gauge invariant*).

The potential formulation has the advantage of identically satisfying (3.1b) and (3.1c)

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \nabla \cdot (\nabla \times \mathbf{A}) = 0 \\ \nabla \times \mathbf{E} &= -\nabla \times \nabla\phi - \nabla \times \frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \tag{3.5}$$

3.2 The covariant theory

Rather than working with two 3-vectors, **E** and **B**, which do not transform under the full Lorentz group, we can introduce the *electromagnetic field strength tensor*, a (0, 2) antisymmetric tensor $F_{\mu\nu}$, defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}, \tag{3.6}$$

where E_i and B_i are the components of the electric and magnetic fields [5]. Observe that the field components can be identified from the field strength as $E^i = F^{0i}$ and $\tilde{\epsilon}^{ijk}B_k = F^{ij}$.

We can also perform a duality transformation, but now in terms of the dual tensor

$$G_{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix}, \tag{3.7}$$

which in index notation can be written as $F_{\mu\nu} \to G_{\mu\nu} = \frac{1}{2}\tilde{\epsilon}_{\mu\nu\sigma\rho}F^{\sigma\rho}$. As such, the entries of the two tensors are

$$F^{0i} = E^i \quad F^{ij} = \tilde{\epsilon}^{ijk}B_k \quad , \quad G^{0i} = B^i \quad G^{ij} = -\tilde{\epsilon}^{ijk}E_k. \tag{3.8}$$

The four source-free equations of motion (3.2) can be reduced to two, in terms of the field strength and its dual respectively:

$$\partial_\mu F^{\mu\nu} = 0 \quad \partial_\mu G^{\mu\nu} = 0. \tag{3.9}$$

By introducing the *current 4-vector* $J^\mu = (\rho, \mathbf{J})$, the classical Maxwell's equations can be written in component notation as

$$\begin{aligned} \partial_i E^i &= J^0 \\ \tilde{\epsilon}^{ijk}\partial_j B_k &= J^i + \partial_0 E^i \\ \partial_i B^i &= 0 \\ \tilde{\epsilon}^{ijk}\partial_j E_k &= -\partial_0 B^i. \end{aligned} \tag{3.10}$$

The first two equations, in terms of $F_{\mu\nu}$, take the form

$$\begin{aligned} \partial_j F^{ij} - \partial_0 F^{0i} &= J^i \\ \partial_i F^{0i} &= J^0, \end{aligned} \tag{3.11}$$

while the others can be written down as $\partial_{[\sigma} F_{\mu\nu]} = 0$.[1] Finally, by using the antisymmetry of $F_{\mu\nu}$, Maxwell's equations become

$$\partial_\nu F^{\mu\nu} = J^\mu, \quad \partial_{[\sigma} F_{\mu\nu]} = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0. \tag{3.12}$$

Thus, the original four equations are replaced by two which transform as tensors, meaning that if they are satisfied in some inertial frame, they will remain true in any frame transformed through a Lorentz transformation.

The field strength can also be express in terms of the *four-vector potential* $A_\mu = (\phi, \mathbf{A})$ as

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu, \tag{3.13}$$

which is identical to (3.3). Additionally, the gauge transformation takes the simpler form

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu f, \tag{3.14}$$

under which it is easy to see that the field strength is invariant

$$\begin{aligned} F_{\mu\nu} \to F'_{\mu\nu} &= \partial_\nu A'_\mu - \partial_\mu A'_\nu \\ &= \partial_\nu A_\mu + \partial_\nu \partial_\mu f - \partial_\mu A_\nu - \partial_\mu \partial_\nu f \\ &= \partial_\nu A_\mu - \partial_\mu A_\nu = F_{\mu\nu}. \end{aligned} \tag{3.15}$$

Finally, we can write the Lagrangian density for the free electromagnetic field as

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}, \tag{3.16}$$

with e the coupling constant. It behaves correctly under Lorentz and gauge transformations and can be used to find the usual equations of motion through the usual variational procedure of finding the Euler-Lagrange equations where A_μ is the dynamical variable.

3.3 The geometric theory

Recall equation (2.21), we interpret A_μ, the 4-vector potential, to be the components of the differential 1-form $A = A_\mu \mathrm{d}x^\mu$ such that we can succinctly relate the potential to the field strength through the exterior derivative:

$$F = \mathrm{d}A, \tag{3.17}$$

[1] For some arbitrary tensor with n indices $T_{\mu_1 \ldots \mu_n}$ we denote its antisymmetrization by $T_{[\mu_1 \ldots \mu_n]} = \frac{1}{n!}(T_{\mu_1 \ldots \mu_n} +$ alternating sum over permutation of the indices) [5].

which we recall to take the familiar form in terms of components $(\mathrm{d}A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$. Concurrently, the Bianchi identity for the electromagnetic field, namely $\partial_{[\sigma} F_{\mu\nu]} = 0$, is simply understood as the closure of the two-form F

$$\mathrm{d}F = 0. \tag{3.18}$$

In regards to the other equation in (3.12), it is interesting to consider the Euler-Lagrange machinery in the differential form notation in order to derive it. The standard electromagnetic action, in the presence of electric charges (alternatively, we can interpret this as coupling the theory to a current) becomes:

$$\begin{aligned} S &= \int \mathrm{d}^4x \left(-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \right) \\ &= \int_X \left(-\frac{1}{2e^2} F \wedge \star F - A \wedge \star J \right), \end{aligned} \tag{3.19}$$

where X is the full space-time manifold and J is the electric current 1-form whose components are J_μ [6]. Consider a variation of the potential of the form

$$A \to A' = A + \delta A, \tag{3.20}$$

where δA is *any* arbitrary variation. Using (3.17), we obtain

$$\begin{aligned} \delta S &= \delta \int_X \left(-\frac{1}{2e^2} \mathrm{d}A \wedge \star(\mathrm{d}A) - A \wedge \star J \right) \\ &= \int_X \left(-\frac{1}{2e^2} \left[\mathrm{d}(\delta A) \wedge \star \mathrm{d}A + \mathrm{d}A \wedge \star \mathrm{d}(\delta A) \right] - \delta A \wedge \star J \right). \end{aligned} \tag{3.21}$$

We can now use the fact that for 2 differential forms $\omega, \lambda \in \Omega^p(X)$ will obey[2]

$$\omega \wedge \star\lambda = \lambda \wedge \star\omega, \tag{3.22}$$

to obtain $\mathrm{d}A \wedge \star\mathrm{d}(\delta A) = \mathrm{d}(\delta A) \wedge \star\mathrm{d}A$, which implies

$$\delta S = \int_X \left(-\frac{1}{e^2} [\mathrm{d}(\delta A) \wedge \star \mathrm{d}A] - \delta A \wedge \star J \right). \tag{3.23}$$

We also make use of the product rule (2.22), with $p = 1$ here, to get

$$\begin{aligned} \mathrm{d}(\delta A \wedge \star \mathrm{d}A) &= \mathrm{d}\delta A \wedge \star \mathrm{d}A - \delta A \wedge \mathrm{d}\star\mathrm{d}A \\ \Rightarrow \mathrm{d}\delta A \wedge \star \mathrm{d}A &= \mathrm{d}(\delta A \wedge \star \mathrm{d}A) + \delta A \wedge \mathrm{d}\star\mathrm{d}A. \end{aligned} \tag{3.24}$$

Thus, the variation of the action becomes

$$\delta S = \frac{1}{e^2} \left[\int_X \mathrm{d}(\delta A \wedge \star \mathrm{d}A) + \int_X (\delta A \wedge \mathrm{d}\star\mathrm{d}A) \right] - \int_X (\delta A \wedge \star J). \tag{3.25}$$

[2] In general we view this as the scalar product on the space $\Omega^p(X)$ which can be written as $\langle \lambda, \omega \rangle = \lambda \wedge \star\omega \in \mathbb{R}$.

Using Stoke's theorem (2.24) the first integral can be converted to a surface one and we assume that the fields fall off sufficiently fast at infinity/at the boundary such that the boundary term will also vanish as a result, thus obtaining

$$\begin{aligned} 0 = \delta S &= \frac{1}{e^2}\int_{\partial X}(\delta A \wedge \star \mathrm{d}A) + \int_X \left(\frac{1}{e^2}[\delta A \wedge \mathrm{d}\star\mathrm{d}A] - \delta A \wedge \star J\right) \\ &= \int_X \left(\frac{1}{e^2}[\delta A \wedge \mathrm{d}\star F] - \delta A \wedge \star J\right) \end{aligned} \tag{3.26}$$

Since this must hold for any variation δA we find that $\frac{1}{e^2}\mathrm{d}\star F = \star J$. We have thus arrived at the very compact form of the Maxwell equations in terms of differential forms[3]

$$\frac{1}{2\pi}\mathrm{d}F = 0 \quad \frac{1}{e^2}\mathrm{d}\star F = \star J. \tag{3.27}$$

The vacuum Maxwell's equations, meaning $J_\mu = 0$, are invariant under the duality transformation

$$F \to \star F \quad \star F \to -F = \star(\star F), \tag{3.28}$$

where it is straightforward to notice from the discussion around (3.8) that the components of $\star F$ are nothing else than $G_{\mu\nu}$. We can thus define a dual field strength

$$G = \frac{2\pi}{e^2}\star F. \tag{3.29}$$

By assuming that we are in the vacuum $\mathrm{d}\star F = 0 \Rightarrow \mathrm{d}G = 0$, meaning that we can introduce (locally) a "dual" gauge field $\tilde{A}$ such that

$$G = \mathrm{d}\tilde{A}. \tag{3.30}$$

It might be tempting to think of $\tilde{A}$ as a 1-form field analogously to A but this is generally not the case by virtue of the $\star$ operator. Depending on the dimensionality of your spacetime manifold X, $d = \dim(X)$, G will be a $(d-2)$-form implying that $\tilde{A}$ is generally a $(d-3)$-form.

We can obtain a dual description of the Maxwell theory from the dual field strength by writing down an action of the form

$$S = \int_X \left(-\frac{1}{2\tilde{e}^2} G \wedge \star G\right), \tag{3.31}$$

where the dual coupling constant $\tilde{e} = \frac{2\pi}{e}$ [8]. Interestingly, the coupling constant for the dual theory is inversely proportional to the original one, meaning that our electric description, which is weakly coupled, goes to a strongly coupled theory in the magnetic description! This comes from Dirac's considerations [9] of magnetic monopoles, realizing that their fundamental charge must be inversely proportional

[3]We choose to explicitly include the factors of $1/e^2$ and $1/2\pi$ in order to account for the convention on flux quantization in terms of $\mathbb{Z}$ numbers. This will become apparent in the subsequent sections.

to the fundamental electric charge. This is an example of S-duality (S referring to strong coupling) which is shown, rather trivially, to be exact for the free Maxwell theory. The duality plays an important role in supersymmetric theories involving non-Abelian Yang-Mils fields and showing this duality becomes a much more involved endeavour [10].

Chapter 4

Generalised Symmetries

In recent years, a powerful generalization of our notion of global symmetries has been understood. These symmetries are often called higher symmetries, p-form symmetries, gauge-like symmetries or generalized global symmetries [3], [11]. In essence, this generalization consists in considering charges that are carried by p-dimensional excitations, created by p-dimensional operators in spacetime. It is thus useful to begin by formulating our usual notions of ordinary symmetries in terms of operators.

4.1 Ordinary symmetries

In Quantum Field Theory (QFT), we associate a continuous ordinary (0-form) symmetry with a conserved current J_μ satisfying $\partial_\mu J^\mu = 0$. In terms of differential forms, we understand the usual current as a 1-form $J = J_\mu \mathrm{d}x^\mu$ and the conservation law is equivalent to the closure condition [6]

$$\mathrm{d}\star J = 0. \tag{4.1}$$

Typically, one integrates the conserved current over a codimension-1[1] submanifold of spacetime $\Sigma \subset X$, representing space, but it is more general to think of it as a $(d-1)$-space separating spacetime to two regions. Hence, we find the charge as

$$Q(\Sigma) = \int_\Sigma \star J. \tag{4.2}$$

We can construct a type of operators which implement the action of the symmetry within a limited region of spacetime called *symmetry operators* [3]. They act on *charged operators*, of charge q, corresponding to some operator insertion $\mathcal{O}(x)$ that is charged under the symmetry. The former are defined by exponentiating the integrated (conserved) current and a coefficient α:

$$U(\Sigma, \alpha) := \exp\left(\mathrm{i}\alpha \int_\Sigma \star J\right). \tag{4.3}$$

[1] We refer to the codimension of a submanifold as the number of dimensions transverse to the submanifold, i.e. a codimension-p manifold in a d-dimensional spacetime is $(d-p)$-dimensional.

From these, we can see the action of the symmetry operators on the charged operators as

$$U(\Sigma,\alpha)\mathcal{O}(x)U^{\dagger}(\Sigma,\alpha) = D(g)\mathcal{O}(x), \tag{4.4}$$

where $D(g)$ is a representation of the symmetry group G corresponding to the symmetry transformation. For $U(1)$ symmetries (the focus of this work), the representation is one-dimensional and the coefficient α, labeling the symmetry operator, corresponds to an element $e^{\mathrm{i}\alpha} \in U(1)$, a parameter of the symmetry transformation.[2] In other words, the phase α is defined mod 2π. Hence, we expect that for the $U(1)$ case the charged operator will transform as

$$U(\Sigma,\alpha)\mathcal{O}(x)U^{\dagger}(\Sigma,\alpha) = \mathrm{e}^{\mathrm{i}\alpha}\mathcal{O}(x). \tag{4.5}$$

The symmetry operators are *topological*, meaning that the surface Σ can be deformed arbitrarily without changing the answer, as long as it does not cross any charged operators [3]. In order to see this, suppose that the operator is deformed to a different surface, i.e. $\Sigma \to \Sigma'$. This deformation sweeps out a region M such that, if no operators were crossed, $\partial M = \Sigma - \Sigma'$. This means that the operators are related as

$$\begin{aligned} U(\Sigma',\alpha) &= \exp\left(\mathrm{i}\alpha \int_{\Sigma-\Sigma'} \star J\right) U(\Sigma,\alpha) \\ &= \exp\left(\mathrm{i}\alpha \int_{M} \mathrm{d}\star J\right) U(\Sigma,\alpha) = U(\Sigma,\alpha), \end{aligned} \tag{4.6}$$

where the second line is by virtue of Stoke's theorem (2.24) and current conservation $\mathrm{d}\star J = 0$. Conversely, if a charged object is present inside M, $U(\Sigma',\alpha)$ and $U(\Sigma,\alpha)$ will differ by precisely its contribution [6].

Formulating symmetries in terms of operators is a powerful approach since it gives us an intrinsic description of the symmetry which is valid even when there is no Lagrangian or when there are multiple Lagrangians (e.g. in duality) [3]. Alas, this also means that the action on the fundamental fields might not be clear or, as we will see for the case of the Maxwell theory, one might need to consider the dual description in order to understand the symmetry. Perhaps surprisingly, this formalism also applies for the the case of discrete symmetries (like $\mathbb{Z}_N$) for which a Noether current does not exist. One of the simplest examples of this is that of a theory with a $\mathbb{Z}_2$ symmetry which acts on the field as $U\phi U^{\dagger} \to -\phi$, implemented by the unitary operator U. Nevertheless, in this thesis we are only interested in continuous global symmetries, hence Lie groups, and refer to [3] for a deeper discussion on higher-form discrete symmetries.

[2] We aim to emphasize this point in order to differentiate between a $U(1)$ symmetry (and hence an underlying $U(1)$ gauge theory) and a $\mathbb{R}$ one since the groups are *locally* the same and both have the Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$. Nevertheless, they are *globally* different and a $U(1)$ gauge theory is one which admits quantized charges and magnetic monopoles. On the other hand, $\mathbb{R}$ gauge theory does not have quantized charge and forbids magnetic monopoles. We choose not to study this in detail by deriving these statements explicitly, but rather refer the interested reader to [6].

4.2 Wilson loops

Before continuing our discussion to higher-form symmetries (i.e. $p > 0$), it is useful to consider what are the charged *objects* under it. For a 0-form symmetry, they have been pointlike (0-dimensional), local operators. Similarly, a 1-form global symmetry acts on 1-dimensional operators, the charged objects are now strings rather than point particles. Additionally, since we are only interested in the global symmetry rather than the gauge one, we also require these operators to be gauge invariant. In gauge theory, these are the *Wilson loops* associated with loops γ in spacetime of charge q [6] [3]

$$W(\gamma) := \exp\left(\mathrm{i}\oint_\gamma A_\mu \mathrm{d}x^\mu\right) = \exp\left(\mathrm{i}\oint_\gamma A\right). \tag{4.7}$$

Their gauge invariance is straightforward to see by considering the action of the gauge transformation $A \to A + \mathrm{d}\lambda$ on $W(\gamma)$

$$\begin{aligned} W(\gamma) \to W'(\gamma) &= \exp\left(\mathrm{i}\oint_\gamma (A + \mathrm{d}\lambda)\right) = \exp\left(\mathrm{i}\oint_\gamma \mathrm{d}\lambda\right) W(\gamma) \\ &= W(\gamma), \end{aligned} \tag{4.8}$$

since $\oint_\gamma \mathrm{d}\lambda = 0$, by using Stoke's theorem (2.24) and the fact that γ is a closed loop. We understand $W(\gamma)$ to serve as an electric probe of our theory.

We can also look at Wilson loops in the dual description, in terms of $\tilde{A}$, now identifying them as *'t Hooft loops*

$$H(\gamma) := \exp\left(\mathrm{i}\oint_\gamma \tilde{A}\right), \tag{4.9}$$

which we view as a magnetic probe of the theory [3].

It is worth noting that the Wilson loop has a pleasantly physical interpretation, particularly in the electromagnetic case. Generally, if we take a vector along a closed path $\mathcal{C}$ then $W(\mathcal{C})$ tells us how it will differ from its starting value, in mathematics this refers to the notion of holonomy [12]. In general relativity, we understand this in terms of the parallel transport around a manifold, related to the Christoffel connection $\Gamma^\sigma_{\mu\nu}$. Finally, in Maxwell's theory, we can think of transporting a particle of charge e along $\mathcal{C}$. When returning to the initial position, the particle picks up a phase $\psi \to \mathrm{e}^{\mathrm{i}e\alpha}\psi$ with $\alpha = \oint_\mathcal{C} A$ [12]. In essence, this is nothing else than the Aharonov-Bohm effect.

4.3 Higher-form symmetries

Having introduced the notion of symmetries in terms of operators for ordinary symmetries, the generalization to p-form symmetries readily follows. We generalize the conservation law (4.1) to $\mathrm{d}\star J_{(p+1)} = 0$, with $J_{(p+1)}$ a $(p+1)$-form current

[3] Generally, the definition of $W(\gamma)$ requires *path-ordering* $\mathcal{P}$ along γ and working with the trace, however this is unnecessary for the Abelian case.

rather than simply a 1-form. The symmetry operators $U(\Sigma, \alpha)$ are now supported on codimension-$(p+1)$ manifolds, while the objects charged under these are p-dimensional. For the 1-form case, this corresponds to Wilson loops. More generally, when dealing with higher-form symmetries we consider higher dimensional "Wilson objects" which correspond to surfaces or p-volumes [3].

Before formulating a general statement regarding the action of higher-form symmetries, we start by analysing the source-free electromagnetic theory.

4.3.1 Free Maxwell theory

The free field Maxwell theory is described (in 4 dimensions) by

$$S = \int_X \left(-\frac{1}{2e^2} F \wedge \star F \right). \tag{4.10}$$

We interpret the two equations of motion (3.27) in terms of currents

$$J_e = \frac{1}{e^2} F \qquad J_m = \frac{1}{2\pi} \star F, \tag{4.11}$$

with both F and $\star F$ 2-forms, conserved in the sense that both satisfy $\mathrm{d}\star J = 0$.

For concreteness, we take the manifold Σ, on which the symmetry operator is supported, to be the 2-sphere S^2. The associated conserved charges (from which we construct symmetry operators) are then

$$Q_e^{(1)} = \int_{S^2} \star J_e = \frac{1}{e^2} \int_{S^2} \star F \quad \text{and} \quad Q_m^{(1)} = \int_{S^2} \star J_m = \frac{1}{2\pi} \int_{S^2} F. \tag{4.12}$$

The two are nothing else than the electric flux and magnetic flux through S^2 respectively [3]. Equivalently, they measure charge (electric or magnetic) enclosed by S^2. We also see here the reason for our choice of normalization, owed to the "Dirac quantization" condition, such that $Q_e^{(1)} \in \mathbb{Z}$ and $Q_m^{(1)} \in \mathbb{Z}$ [12].

The electric symmetry

We follow the procedure outlined in the previous sections to construct the symmetry operator

$$U(S^2, \alpha) = \exp\left(\frac{\mathrm{i}\alpha}{e^2} \int_{S^2} \star F \right). \tag{4.13}$$

The charged object under this symmetry will be the Wilson loop (of some charge $q \in \mathbb{Z}$), with the symmetry transformation being

$$W_q(\gamma) \to \mathrm{e}^{\mathrm{i}q\alpha} W_q(\gamma). \tag{4.14}$$

In order to see this, take the equal-time commutation relation

$$[E_i(t, \mathbf{x}), A_j(t, \mathbf{y})] = -\mathrm{i}e^2 \delta_{ij} \delta(\mathbf{x} - \mathbf{y}), \tag{4.15}$$

and consider the case where both the Wilson loop and the symmetry operator are completely located inside the spatial manifold [13]. By doing this, we obtain[4]

$$\left[Q_e^{(1)}(S^2), \oint_\gamma A_i \mathrm{d}x^i\right] = -\mathrm{i}\int_{S^2} \mathrm{d}S^i \oint_\gamma \mathrm{d}x^j \delta_{ij}\delta(\mathbf{x}-\mathbf{y}) = -\mathrm{i}I(S^2,\gamma), \tag{4.16}$$

where $I(S^2,\gamma)\in\mathbb{Z}$ is the signed intersection number of S^2 and γ in space.

We can now verify that the action of the symmetry of the Wilson loop is[5]

$$\begin{aligned} U(S^2,\alpha)W_q(\gamma)U^\dagger(S^2,\alpha) &= \mathrm{e}^{\mathrm{i}\alpha Q_e^{(1)}(S^2)}\mathrm{e}^{\mathrm{i}q\oint_\gamma A}\mathrm{e}^{-\mathrm{i}\alpha Q_e^{(1)}(S^2)} \\ &= \mathrm{e}^{\mathrm{i}\alpha q I(S^2,\gamma)}W_q(\gamma), \end{aligned} \tag{4.17}$$

which is precisely (4.14), under the assumption that the charged operator and symmetry operator intersect once.

In terms of gauge field, the symmetry corresponds to a translation

$$A \to A + \lambda, \tag{4.18}$$

with λ a flat/closed 1-form ($\mathrm{d}\lambda = 0$) that is *not* exact [14]. This requirement is owed to the fact that we are interested in global symmetries (under which the Wilson loop picks up a phase) rather than gauge symmetries of the form $A \to A + \mathrm{d}\eta$ (under which the Wilson loop is invariant). Recall the discussion at the end of Section 2.2, this is a statement that the transformation is classified by the first cohomology group of the spacetime manifold X, i.e. $\lambda \in H^1(X)$.

The magnetic symmetry

We turn now to the second conserved current $J_m = \frac{1}{2\pi}\star F$, from which we construct the symmetry operator

$$U(S^2,\alpha) = \exp\left(\frac{\mathrm{i}\alpha}{2\pi}\int_{S^2} F\right). \tag{4.19}$$

The charged objects under the symmetry are the 't Hooft loops $H_m(\gamma)$, but the action of the symmetry is more subtle and difficult to see in the electric description.

Nevertheless, we can work with the electromagnetic dual and the "magnetic" gauge field $G = \mathrm{d}\tilde{A}$, with $G \propto \star F$, in order to check the action of the symmetry. From this perspective, we understand the conserved current to be $J_m = \frac{1}{\tilde{e}^2}G$ and the charged object will be $H(\gamma)$ rather than $W(\gamma)$.

The 't Hooft loop (of charge $m\in\mathbb{Z}$) will also transform as a $U(1)$ charged object $H_m(\gamma)\to \mathrm{e}^{\mathrm{i}m\alpha}H_m(\gamma)$. We can check this in the dual frame in an analogous manner to the discussion for the electric symmetry[6]

$$\begin{aligned} U(S^2,\alpha)H_m(\gamma)U^\dagger(S^2,\alpha) &= \mathrm{e}^{\mathrm{i}\alpha Q_m^{(1)}(S^2)}\mathrm{e}^{\mathrm{i}m\oint_\gamma \tilde{A}}\mathrm{e}^{-\mathrm{i}\alpha Q_m^{(1)}(S^2)} \\ &= \mathrm{e}^{\mathrm{i}\alpha m I(S^2,\gamma)}H_m(\gamma), \end{aligned} \tag{4.20}$$

[4] Since we take S^2 to only extend in the spatial directions, we find that $Q_e^{(1)}(S^2) = \frac{1}{e^2}\int_{S^2}\tilde{\epsilon}_{\mu\nu\rho\sigma}F^{\mu\nu}\mathrm{d}S^{\rho\sigma} = \frac{1}{e^2}\int_{S^2}\mathbf{E}\cdot\mathrm{d}\mathbf{S}$.

[5] In doing this, we are tacitly making use of the BCH theorem in the form $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{[A,B]}\mathrm{e}^B\mathrm{e}^A$.

[6] Similarly to the electric case, because S^2 is a pure spatial slice, we find $Q_m^{(1)}(S^2) = \frac{1}{2\pi}\int_{S^2}F_{\mu\nu}\mathrm{d}S^{\mu\nu} = \frac{1}{2\pi}\int_{S^2}\mathbf{B}\cdot\mathrm{d}\mathbf{S}$.

with the commutator (4.15), now understood in terms of $\tilde{A}$ and the magnetic field B_i, i.e.

$$[B_i(t,\mathbf{x}), \tilde{A}_j(t,\mathbf{y})] = -\mathrm{i}\tilde{e}^2\delta_{ij}\delta(\mathbf{x}-\mathbf{y}). \tag{4.21}$$

Similarly, we view this as a translation of the dual (magnetic) gauge field $\tilde{A} \to \tilde{A} + \eta$, with η a flat form [3].

Therefore, the free Maxwell theory has $U(1)_e^{(1)} \times U(1)_m^{(1)}$ global (one-form) symmetry under which the Wilson and 't Hooft loops pick up a phase.[7]

4.3.2 General discussion

We discuss now some general aspects of higher-form symmetries which make them different from ordinary symmetries.

In general, p-form symmetries (with $p \geq 1$) must be Abelian. Two symmetry operators will fuse according to the corresponding group law

$$U(\Sigma^{(d-p-1)}, \alpha)U(\Sigma^{(d-p-1)}, \beta) = U(\Sigma^{(d-p-1)}, \zeta), \tag{4.22}$$

with $\zeta = \alpha\beta$ in the group G [3]. In the $p = 0$ case, $\Sigma^{(d-p-1)}$ is of codimension-1, so we can think inserting the two operators at different times, say $t \pm \epsilon$, and we make sense of the ordering of operators through time-ordering so the operators might not commute. Because of this, the symmetry group can be non-Abelian. For $p > 0$, there is no such ordering as the manifold at $t + \epsilon$ can be smoothly deformed to $t - \epsilon$ [3]. Therefore, the two operators must commute and the group must be Abelian.[8]

We also address one of the assumptions made when computing the commutator (4.16), namely that both S^2 and γ are within the spatial manifold and they intersect only at points in space. Generically, S^2 and γ can be placed in spacetime and the action of the symmetry is through linking $\ell(S^2, \gamma)$ [14]. Thus, for a p-form symmetry, we view the action of a symmetry operator, supported on a codimension-$(p+1)$ manifold Σ, on the charged operator, supported on a p-dimensional manifold Γ, to be[9]

$$U(\Sigma, \alpha)\mathcal{O}_\Gamma U^\dagger(\Sigma, \alpha) = \mathrm{e}^{\mathrm{i}\alpha q \ell(\Gamma,\Sigma)}\mathcal{O}_\Gamma. \tag{4.23}$$

It is useful to consider an illustrative example which shows how the symmetry is implemented for the case of a 1-form symmetry in 3 dimensional spacetime, see Figure 4.1 for this. Recall that one of the reasons why we took the Wilson loops to

[7]The electric symmetry remains a 1-form one for arbitrary spacetimes of dimension d while the magnetic one is generally identified as a $(d-3)$-form symmetry $U(1)_m^{(d-3)}$. Note that in saying this, we ignore that, depending on d or p, there exists the possibility of adding a Chern-Simons term to the action (which is not invariant under the transformation $A \to A + \lambda$ [14]) or a θ term.

[8]In saying this, we assume trivial topology. If the theory is placed on a manifold with a more complicated topology, the symmetry can become non-Abelian. We refer to [3] for a discussion on this topic.

[9]We show this for the specific case of a $U(1)$ global symmetry but would generally refer to a representation $D(g)$ of our symmetry group as in (4.4).

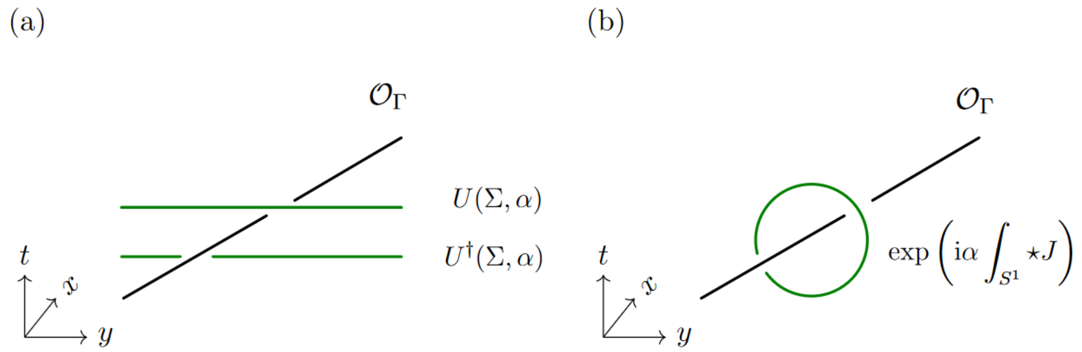

Figure 4.1: The 1-form symmetry action on line operator $\mathcal{O}_\Gamma$ in 2+1 spacetime, demonstrated in two ways. In (a), we look at a product of operators parallel to the xy plane, evaluated at times t for $\mathcal{O}_\Gamma$ and $t \pm \epsilon$ for $U(\Sigma,\alpha)$ and $U^\dagger(\Sigma,\alpha)$ respectively. We evaluate the whole expression at time t to be $U(\Sigma,\alpha)\mathcal{O}_\Gamma U^\dagger(\Sigma,\alpha)$. We view this as a linking between Σ and Γ in spacetime (corresponding $\ell(\Sigma,\Gamma) = 1$ in this scenario), by using the topological invariance of $\int_\Sigma \star J$ to smoothly deform (a) into (b) [15]. Figure adapted from [16].

be the charged objects under the 1-form symmetry of the Maxwell theory was due to their gauge invariance. Generally, we can consider any 1-dimensional charged operator which intersects/links with Σ in an unavoidable way (viz. Γ and Σ in Fig. 4.1, the symmetry operator cannot be smoothly deformed past the charged operator such that they would become disjoint, meaning $\mathrm{U}(\Sigma,\alpha)\mathcal{O}_\Gamma U^\dagger(\Sigma,\alpha) \neq \mathcal{O}_\Gamma$).[10] Additionally, since we are working with a 3-dimensional theory, the symmetry operator, which is still codimension-2, is now 1-dimensional. This means that we can take it to be supported on the circle S^1 such that we could more clearly picture the linking between the two manifolds Γ and Σ.

[10]Meaning that, in fact, we can also consider lines supported on the boundary (or at infinity) to also be charged under such a symmetry. In saying this, we aim to mention that Wilson *lines* with endpoints on the boundary are also gauge-invariant but this requires a careful treatment of the boundary conditions which is why we chose, for simplicity, to focus on Wilson *loops* for our discussion of the Maxwell theory. It is worth noting that these operators are closely related to the notion of asymptotic symmetry for which we understand the shift to be by a p-form that has support at infinity. See [14] for a treatment of Wilson lines and boundary effects.

Chapter 5

Spontaneous Symmetry Breaking

We have seen that our understanding of ordinary symmetries can be generalized by considering charges carried by p-dimensional operators (and hence extending our notion of currents to $(p+1)$-form ones). As with ordinary symmetries, higher-form symmetries can be spontaneously broken which leads to the existence of Nambu-Goldstone (NG) bosons [3]. In this framework, we understand the photon to be the NG boson of the spontaneously broken 1-form symmetry of the Maxwell theory!

In this chapter we only discuss the salient points behind the idea of symmetry breaking before specializing to the 1-form Maxwell case. For a more thorough introduction, the reader is advised to consult [17] or [18].

5.1 Discrete symmetries

We start in the same vein as [19] and consider the simple physical example of a thin, circular rod sitting vertically on a table pushed down along its length with a force $\vec{F}$. If $\vec{F}$ is small, the rod remains unbent and retains its invariance under rotations about its axis. If $\vec{F}$ exceed some critical value $\vec{F}_{\text{crit}}$, it bends in a plane chosen at random. The new ground state of the rod is no longer symmetric under rotations. While the rod bends in only one of the many degenerate configurations, the others can be reached through a rotation. This example aims to to show the context in which symmetry breaking occurs: once a parameter goes beyond a critical value ($\vec{F}_{\text{crit}}$ here), the system becomes unstable and the new ground state is now degenerate and no longer invariant under the symmetry [19].

In order to illustrate Spontaneous Symmetry Breaking (SSB) in a relativistic system, we consider the scalar ϕ^4 theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{5.1}$$

where m and λ are temperature dependent [17]. We take that for a certain critical temperature $T > T_C$: $m^2 > 0$ while for $T < T_C$: $m^2 < 0$ [17]. For $T > T_C$, the potential $V = -\mathcal{L}_{\text{int}} = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$ has a minimum at $\phi = 0$.

For $T < T_C$, in our Lagrangian, we make the change $m^2 \to -m^2$ (such that m^2 is still positive) and obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{5.2}$$

In this scenario, our potential V has a local maximum at $\phi = 0$ and is unstable. On the other hand, the other minima can be found from

$$\frac{\partial V}{\partial\phi} = m^2\phi - \frac{\lambda}{6}\phi^3 = 0 \Rightarrow \phi = \pm\sqrt{\frac{6m^2}{\lambda}} = \pm v. \tag{5.3}$$

We can expand our field around one of the minima, $\phi = \sqrt{\frac{6m^2}{\lambda}} + \tilde{\phi}$, with the Lagrangian taking the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{3m^4}{\lambda} - m^2\tilde{\phi}^2 - \sqrt{\frac{\lambda}{6}}m\tilde{\phi}^3 - \frac{\lambda}{4!}\tilde{\phi}^4. \tag{5.4}$$

The initial Lagrangian (5.2) had the $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. Once we expand around the new minimum, we say that the symmetry is spontaneously broken since it looks like (5.4) is no longer invariant under the symmetry $\tilde{\phi} \to -\tilde{\phi}$, it is "spoiled" by the $\tilde{\phi}^3$ term. However, the symmetry is still there but is "hidden", it is now realized nonlinearly as $\tilde{\phi} \to -\tilde{\phi} - 2v$ [17].

Generally, we identify SSB from the *Vacuum Expectation Value* (VEV) of the appropriate *order parameter* (in the example from above, this is simply the field ϕ). We identified two minima, so there must be two different vacua: $|\Omega_+\rangle$ such that $\langle\Omega_+|\phi|\Omega_+\rangle = \sqrt{6m^2/\lambda}$ and $|\Omega_-\rangle$ with $\langle\Omega_-|\phi|\Omega_-\rangle = -\sqrt{6m^2/\lambda}$. The $\mathbb{Z}_2$ symmetry took $\phi \to -\phi$ so it must also take $|\Omega_+\rangle \to |\Omega_-\rangle$ and while the two vacua are equivalent, the system has to choose one or the other [17].

5.2 Continuous symmetries

Continuous global symmetries which, by Noether's theorem, are associated to a conserved current $\partial_\mu J^\mu = 0$ are particularly interesting in a SSB framework. This is owed to *Goldstone's theorem* which we state without proof[1]: *The spontaneous breaking of a continuous global symmetry implies the existence of massless bosons* [2]. For ordinary symmetries, we can create a 1-particle NG boson state from the vacuum by acting with the symmetry current corresponding to the broken symmetry

$$\langle\Omega|J_\mu(x)|p\rangle = p_\mu e^{\mathrm{i}p\cdot x}f, \tag{5.5}$$

with f a constant [6].

To get an intuitive understanding of why this is true, consider the *linear sigma model*, with ϕ now a complex scalar field, as outlined in [17]. The Lagrangian is given by

$$\mathcal{L} = (\partial_\mu\phi^\star)(\partial_\mu\phi) + m^2\phi\phi^\star - \frac{\lambda}{4}\phi^2\phi^{\star 2}. \tag{5.6}$$

[1]For a proof the reader is advised to consult [18] for ordinary symmetries or [14] for the case of higher-form symmetries.

The symmetry in this case is $U(1)$ and acts as $\phi(x) \to e^{i\alpha}\phi(x)$. The potential $V = -m^2|\phi|^2 + \frac{\lambda}{4}|\phi|^4$ is once more unstable for $\phi = \phi^\star = 0$ and instead we find the minimum to be

$$\frac{\partial V}{\partial \phi} = -2m^2|\phi| + \lambda|\phi|^3 = 0 \Rightarrow |\phi|^2 = \frac{2m^2}{\lambda}. \tag{5.7}$$

We interpret this to mean that there are an infinite number of equivalent vacua $|\Omega_\theta\rangle$ such that $\langle\Omega_\theta|\phi|\Omega_\theta\rangle = \sqrt{\frac{2m^2}{\lambda}}\mathrm{e}^{\mathrm{i}\theta}$.

To expand around the vacuum, we parameterize $\phi(x)$ in terms of two real fields $\sigma(x)$ and $\pi(x)$ [17]

$$\phi(x) = \left(\sqrt{\frac{2m^2}{\lambda}} + \frac{1}{\sqrt{2}}\sigma(x)\right)\mathrm{e}^{\mathrm{i}\frac{\pi(x)}{F_\pi}}, \tag{5.8}$$

with F_π a real number. In terms of the new fields, the Lagrangian takes the form

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \left(\sqrt{\frac{2m^2}{\lambda}} + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{1}{F_\pi^2}(\partial_\mu\pi)^2 \\ - \left(-\frac{m^4}{\lambda} + m^2\sigma^2 + \frac{1}{2}m\sqrt{\lambda}\sigma^3 + \frac{1}{16}\lambda\sigma^4\right).\end{aligned} \tag{5.9}$$

This Lagrangian describes two particles, a massless one π and a massive one σ. We visualize this in terms of the excitation of the Mexican hat (wine bottle) potential shown in Figure 5.1. The massive σ field is visualized as radial excitations for which displacements against the restoring force of the potential will cost energy. This is typically associated with the Higgs boson. Meanwhile, the π field corresponds to excitations along the equipotential minimum (the degenerate vacua) and the associated particle is massless, i.e. the NG boson [18]!

NG bosons are closely associated with shift symmetries [17]. To explicitly see this, consider the action of the broken $U(1)$ symmetry ($\phi(x) \to \mathrm{e}^{\mathrm{i}\theta}\phi(x)$) with $\phi(x)$ parameterized in terms $\sigma(x)$ and $\pi(x)$ (5.8). The symmetry is now once more nonlinearly realized as

$$\pi(x) \to \pi(x) + F_\pi\theta, \tag{5.10}$$

leaving $\sigma(x)$ invariant. Additionally, we see that this is also a symmetry of the Lagrangian (5.9). This can be used to place a constraint on the model even if the full theory which is spontaneously broken is not known, namely that *the shift symmetry prohibits a mass term for* $\pi(x)$ [17].

5.3 The Maxwell one-form symmetry

We turn now to the SSB of higher-form symmetries and, for concreteness, consider the breaking of the one-form symmetry of Maxwell's theory (in 4 dimensional spacetime).

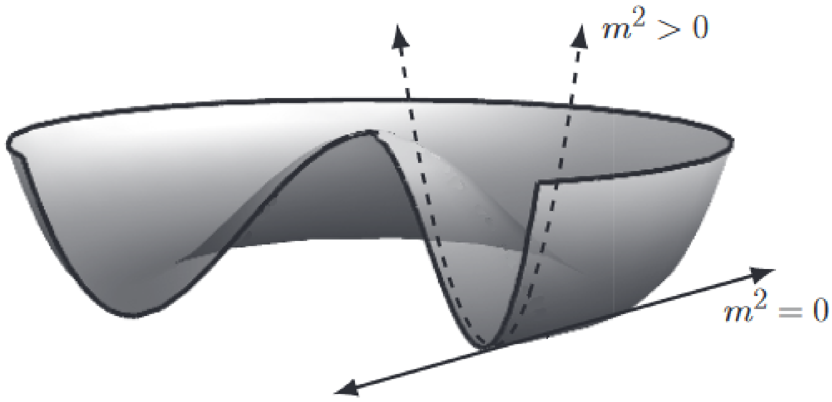

Figure 5.1: The Mexican hat potential. The second derivative of the potential gives the mass squared of the particles. Displacements about the minimum of the potential are decomposed into two modes: a massless one (solid line, the π) and a massive one (dashed line, the σ). Figure adapted from [17].

A key insight is that the appropriate order parameter which distinguishes between different phases of the theory is now the VEV of the Wilson loop $\langle W(\gamma)\rangle$ [3].[2] Generally, $\langle W(\gamma)\rangle$ depends on geometric properties of γ such as either the minimal area bounded by the curve $A(\gamma)$ or its perimeter $L(\gamma)$. For area law scaling, the expectation value of the Wilson loop rapidly goes to zero as the size of the loop increases and we say that the symmetry is unbroken. On the other hand, for a perimeter law (or any scaling milder than this, as will be shortly introduced) we interpret the one-form symmetry to be spontaneously broken. Then, for a large loop γ, we effectively have[3]

$$\begin{aligned} \langle W(\gamma)\rangle \propto \mathrm{e}^{-A(\gamma)} &\Rightarrow \langle W(\gamma)\rangle = 0 \\ \langle W(\gamma)\rangle \propto \mathrm{e}^{-L(\gamma)} &\Rightarrow \langle W(\gamma)\rangle \neq 0. \end{aligned} \tag{5.11}$$

It is useful to consider a simple yet surprisingly illuminating example which shows the relation between the expectation value of the Wilson loop and the potential between two charged particles. Consider the loop γ shown in Figure 5.2, we interpret it as the creation of a static particle-antiparticle pair (viz. the particles can only propagate in time and no kinetic terms can be attributed to them), separated by a distance $r = R$, which propagate forward for a time T after which they annihilate back to the vacuum. We find $\langle W(\gamma)\rangle$, for the given loop, by looking at the euclidean path-integral which, for long times, projects our system to the lowest energy state [12].[4] Before the particles appear, and after they

[2] Equivalently, the VEV of the 't Hooft loop $H(\gamma)$ if working in the dual description.

[3] One could argue that, for large loops ($r \to \infty$ in (5.14) (5.16)), both expectation values will vanish. However, for the perimeter (or milder) scaling, the operator can be redefined by adding a counterterm along the loop such that the expectation value is nonzero for arbitrarily big loops [6], [20]. Conversely, if the scaling is faster than a perimeter law this can not be done [14].

[4] We do not show explicitly this derivation as it falls beyond the scope of this thesis. Nevertheless, we refer to [21] for a wonderful derivation of this and some of the results which follow.

disappear, there is only the vacuum which we take to be the ground state of the system and hence the one with zero energy. However, in the presence of the two sources, the ground state of the system has an energy $V(r)$, corresponding to the inter-particle potential. Thus, one finds [21]

$$\langle W(\gamma)\rangle \propto \lim_{T\to\infty} \mathrm{e}^{-V(r)T}. \tag{5.12}$$

Meaning that we can compute the potential as the following limit

$$V(r) = -\lim_{T\to\infty} \frac{1}{T} \ln\langle W(\gamma)\rangle. \tag{5.13}$$

We aim to emphasize the importance of this result. In the framework of lattice gauge theory, one views the Wilson loop as the product of link variables along a given contour which means that, in principle, the potential between two charged particles can be calculated by using numerical methods [21].[5]

We can discuss now the expectation values of the Wilson loop associated to Figure 5.2. Suppose it has an area law

$$\langle W(\gamma)\rangle \propto \mathrm{e}^{-\sigma T r}, \tag{5.14}$$

by virtue of (5.13), it leads to a potential that grows linearly with the distance

$$V(r) = \sigma r, \tag{5.15}$$

for some σ that has dimensions of energy per length.

If the loop has a perimeter law, the associated potential is constant[6]

$$\langle W(\gamma)\rangle \propto \mathrm{e}^{-\rho(T+r)} \Rightarrow V(r) = \rho. \tag{5.16}$$

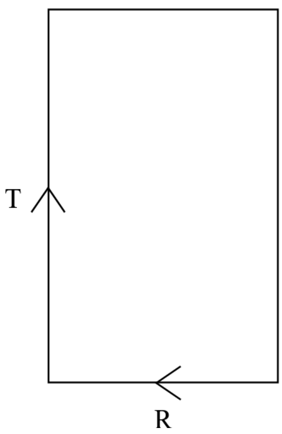

Figure 5.2: Wilson loop describing static particles separated spatially by distance R and propagating for time T.

Finally, we consider a case which we did not mention before, but which is relevant to our discussion of Maxwell's theory. This is the Coulomb behaviour with scale-invariance on the parameters of the loop which decays milder than the perimeter law (and hence also signals a broken phase) [22]. The expectation value depends on the ratios r/T and T/r, for which we find

$$\langle W(\gamma)\rangle \propto \mathrm{e}^{-\alpha\frac{T}{r}-\beta\frac{r}{T}} \Rightarrow V(r) = \frac{\alpha}{r}. \tag{5.17}$$

This is nothing else than a Coulomb-like potential.

[5] We note in passing that a similar procedure can be employed in Quantum Chromodynamics (QCD) in order to test for confinement by finding the interquark potential.

[6] In the context of confinement, we refer to this as a *deconfining* phase since the energetic cost of separating two charges at large distances is finite. Conversely, for the case with $V(r) = \sigma r$, the potential is said to be *confining*.

Having said this, we can turn back to our discussion of Maxwell's theory where, unsurprisingly, two probe particles experience a Coulomb potential $V(r) \propto \frac{1}{r}$. Therefore, our one-form global symmetry is spontaneously broken and the system should have Goldstone bosons. In fact, we now understand this NG boson to be nothing else than the photon [3]! Moreover, we understand the shift by a flat form $A \to A + \lambda$, identified for the one-form symmetry (4.18), to be the generalization of the familiar shift by a constant $\pi \to \pi + c$, discussed in the context of the linear σ-model (5.10) [14].[7]

Recall that, in terms of components, the current corresponding the spontaneously broken symmetry is nothing else than the field strength tensor $J_{\mu\nu} = F_{\mu\nu}$ (4.11). This suggests that we can check whether the photon is indeed the NG boson by acting with the two-form current on the vacuum, and checking its overlap with the one photon state, similarly to (5.5). This is an exercise in Quantum Electrodynamics (QED) which falls beyond the scope of this work, but for which we refer to [14] and [22] for details. In doing so, one obtains a non-zero overlap between the vacuum and the single photon state

$$\langle \Omega | F_{\mu\nu} | \epsilon, p \rangle \propto \epsilon_{[\mu} p_{\nu]} \mathrm{e}^{\mathrm{i}p\cdot x}, \tag{5.18}$$

with ϵ the photon's polarization and p its momentum.[8] Typically, when there is a massless scalar field in a theory, the usual underlying reason is that of Goldstone's theorem. Here, we see that this reasoning can also be applied to explain why the photon is massless, provided that we extend the theorem's validity to higher-form symmetries.

Before concluding this section, we aim to touch on some subtle questions. The keen eyed reader might wonder what is the fate of the magnetic symmetry, after all we identified that Maxwell's theory has $U(1)(1)_e \times U(1)^{(1)}_m$ global one-form symmetry. The answer is that *both*[9] symmetries are spontaneously broken [3], [12]. However, an immediate follow up question arises: To the breaking of which symmetry does the photon correspond? In our discussion of the linear σ model (in analogy to which we look at the one-form symmetry) we had a breaking $G = U(1) \to H = \{\mathbb{1}\}$ and one NG boson. As a consequence, we could naively think that there are two NG bosons (of which one the photon), corresponding to the breaking of the two symmetries. However, due to electromagnetic duality, there is only one NG boson (hence only the photon) corresponding to *either* of the symmetries (i.e. we can give rise to a one photon state by acting on the vacuum with either the electric or the magnetic current [3]).

Additionally, we should note that based on our analysis, the free Maxwell theory is *always* spontaneously broken since we only associate to it the Coulomb

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[7] The generalization for p-form symmetries, with $p > 1$, is then simply understood as the shift of a p-form field A by a flat p-form λ ($\mathrm{d}\lambda = 0$). The transformation is now classified by the p-th cohomology group $\lambda \in H^p(X)$.

[8] We understand the single photon state to be the one created by the appropriate creation operator $|\epsilon, p\rangle = a_\epsilon(p)^\dagger |\Omega\rangle$, with ϵ representing the physical polarizations [22].

[9] By working with the dual theory, the discussion for the magnetic symmetry and the fate of the 't Hooft loop $H(\gamma)$ follows completely analogously to the one described above for the Wilson loop.

potential and respectively the deconfinement of charges. We believe that, while surprising at first, this is simply indicative of an incomplete theory which does not model the existence of a symmetric phase to which phase transitions could happen. Notably, this is contrary to the view of [12] and [20] which see the superconducting phase as the symmetric one, with the photon having an effective mass. For completeness, we offer a few comments on this. Firstly, we do not view the superconducting phase as belonging to the model studied here, i.e. the free field theory (4.10), and argue that, if dealing with superconductors, one has to study the Landau-Ginzburg theory and is no longer in the free field regime (see [17] for a short exposition on this). On the other hand, in a superconductor, the electrons form *Cooper pairs* and the photon has short-range correlations. It is argued by [12] that, in this scenario, the energetic cost of separating two particles (similarly to the case of Figure 5.2) is

$$V(r) = \sigma r, \tag{5.19}$$

with the flux lines between the particles forming collimated tubes. If this were the case, we can see based on our discussion around (5.14) that we would indeed understand this as the symmetric phase of the theory (Wilson loop would have an area law). Nevertheless, we view this in a speculative light and postpone a more complete discussion for future work.

Finally, we mention one aspect in which the breaking of higher-form symmetries is different from that of ordinary symmetries. In the usual scenario, the *explicit* breaking of a spontaneously broken continuous 0-form symmetry gives a mass to the Goldstone boson[10], this typically happens if the symmetry is not exact or if it is gauged [17]. By adding charged matter (either electric or magnetic) to the Maxwell theory we explicitly break the symmetries as the underlying conservation laws are no longer true and take the form

$$\frac{1}{e^2}\mathrm{d}\star F = J_e \quad , \quad \frac{1}{2\pi}\mathrm{d}F = J_m. \tag{5.20}$$

Perhaps concerningly, at a first glance, this might seem to suggest that the photon would gain a mass if charged matter is added to the theory. However, this is not the case. A simple argument for this is that generally, one can only add local operators to the action but only non-local objects are charged under the higher-form symmetry (i.e. $p \geq 1$) [16].[11] Because of this, the pseudo-Goldstone boson of a higher-form symmetry remains massless even if the symmetry is explicitly broken! We consider this aspect beyond the scope of this thesis and refer to [23] for a more detailed discussion.

[10] A concrete example of this is the pion which we understand as the pseudo-Goldstone boson of the chiral Lagrangian of QCD, with only up and down quarks, for which the chiral symmetry (SU(2)×SU(2)) is not exact due to the quark masses [17].

[11] The fact that local objects are uncharged under a higher-form symmetry follows from our discussion at the end of Section 4.3.

Chapter 6

Conclusion

In this thesis we reviewed higher-form symmetries, with an emphasis on Maxwell's theory (free field case), concluding that the photon can be interpreted as the NG boson of a SSB one-form symmetry. This allows us to explain the masslessness of the photon in the framework of SSB, without having to appeal to what we view as the more dire interpretation of the spontaneous breaking of the Lorentz symmetry [4]. This was done by formulating symmetries in the language of operators and extending our notion of currents and charged objects to ones of higher dimensionality. Subsequently, by treating the VEV of the Wilson loop as the appropriate order parameter of the one-form symmetry, we have seen that for the $U(1)$ gauge theory (i.e. electromagnetism, with a Coulomb potential) $\langle W(\gamma)\rangle \neq 0$ and hence signals that the theory is always in a broken phase. By using this information in tandem with Goldstone's theorem, we arrived at our conclusion, namely that the photon can indeed be interpreted as the NG boson of a broken one-form symmetry.

We aim to emphasize that most of our discussion, specifically for the SSB of the one-form symmetry, was done in analogy to the previously existing σ model and it does not offer an intrinsic and complete description (this is further reinforced by the conclusion that the symmetry is always spontaneously broken). Additionally there are still some open questions such as the number of Goldstone modes resulting from the SSB or the existence of a symmetric phase (at least for the free field case), for which (to the best of our knowledge) there does not seem to be an answer in literature.

Finally, our discussion has mostly revolved around the $U(1)$ gauge theory. According to [3], higher-form symmetries also exist in general Yang-Mils theory (with generic gauge group $SU(N)$) and they are associated to the center symmetry ($\mathbb{Z}_N$). It would be thus interesting to analyse the implications of this and the process of constructing symmetry operators for discrete symmetries. Additionally, our discussion of the magnetic symmetry has been rather superficial and it would be helpful to gain a better understanding of the interplay between the electric and magnetic symmetry and the consequences of the two dual descriptions.

References

[1] Y. Nambu and G. Jona-Lasinio, “Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I,” *Phys. Rev.*, vol. 122, pp. 345–358, 1 Apr. 1961. DOI: 10.1103/PhysRev.122.345.

[2] J. Goldstone, “Field theories with Superconductor solutions,” en, *Il Nuovo Cimento*, vol. 19, no. 1, pp. 154–164, Jan. 1961. DOI: 10.1007/BF02812722.

[3] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, *Generalized Global Symmetries*, 2015. arXiv: 1412.5148 [hep-th].

[4] J. Bjorken, *Emergent Gauge Bosons*, 2001. arXiv: hep-th/0111196 [hep-th].

[5] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, 1st ed. Cambridge University Press, Aug. 2019. DOI: 10.1017/9781108770385.

[6] M. Reece, *TASI Lectures: (No) Global Symmetries to Axion Physics*, Apr. 2023. Available: http://arxiv.org/abs/2304.08512.

[7] F. Mandl and G. Shaw, *Quantum field theory*, 2nd ed. Hoboken, N.J: Wiley, 2010, OCLC: ocn460050759.

[8] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, “Dual electromagnetism: Helicity, spin, momentum, and angular momentum,” *New Journal of Physics*, vol. 15, no. 3, p. 033 026, Mar. 2013, arXiv:1208.4523 [hep-th]. Available: http://arxiv.org/abs/1208.4523.

[9] “Quantised singularities in the electromagnetic field,” *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 133, no. 821, pp. 60–72, Sep. 1931. DOI: 10.1098/rspa.1931.0130.

[10] L. Alvarez-Gaumé and S. F. Hassan, “Introduction to S-duality in N = 2 supersymmetric gauge theories (a pedagogical review of the work of seiberg and witten),” *Fortschritte der Physik/Progress of Physics*, vol. 45, no. 3-4, pp. 159–236, 1997. DOI: 10.1002/prop.2190450302.

[11] D. Harlow and H. Ooguri, *Symmetries in quantum field theory and quantum gravity*, arXiv:1810.05338 [gr-qc], Jun. 2019. Available: http://arxiv.org/abs/1810.05338.

[12] D. Tong, *Gauge Theory*. Available: https://www.damtp.cam.ac.uk/user/tong/gaugetheory/gt.pdf, Lecture notes.

[13] Y. Hidaka, Y. Hirono, and R. Yokokura, "Counting Nambu-Goldstone Modes of Higher-Form Global Symmetries," *Phys. Rev. Lett.*, vol. 126, p. 071 601, 7 Feb. 2021. DOI: 10.1103/PhysRevLett.126.071601.

[14] E. Lake, *Higher-form symmetries and spontaneous symmetry breaking*, Feb. 2018. Available: http://arxiv.org/abs/1802.07747.

[15] D. Simmons-Duffin, *Tasi lectures on the conformal bootstrap*, 2016. arXiv: 1602.07982 [hep-th].

[16] R. Thorngren, T. Rakovszky, R. Verresen, and A. Vishwanath, *Higgs Condensates are Symmetry-Protected Topological Phases: II.* $U(1)$ *Gauge Theory and Superconductors*, 2023. arXiv: 2303.08136 [cond-mat.str-el].

[17] M. D. Schwartz, *Quantum field theory and the standard model*. New York: Cambridge University Press, 2014.

[18] T.-P. Cheng and L.-F. Li, *Gauge theory of elementary particle physics* (Oxford science publications). Oxford: New York: Clarendon Press; Oxford University Press, 1984.

[19] L. H. Ryder, *Quantum field theory*, 2nd ed. Cambridge ; New York: Cambridge University Press, 1996.

[20] D. Hofman and N. Iqbal, "Goldstone modes and photonization for higher form symmetries," *SciPost Physics*, vol. 6, no. 1, Jan. 2019. DOI: 10.21468/scipostphys.6.1.006.

[21] H. J. Rothe, *Lattice gauge theories: an introduction* (World Scientific lecture notes in physics v. 74), 3rd ed. Hackensack, N.J: World Scientific, 2005.

[22] P. R. S. Gomes, *An Introduction to Higher-Form Symmetries*, Mar. 2023. Available: http://arxiv.org/abs/2303.01817.

[23] J. McGreevy, "Generalized Symmetries in Condensed Matter," *Annual Review of Condensed Matter Physics*, vol. 14, no. 1, pp. 57–82, Mar. 2023. DOI: 10.1146/annurev-conmatphys-040721-021029.