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Modular curves of genus one and their associated period lattices

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Abstract

In this thesis we construct modular elliptic curves associated to modular curves $X_0(N)$ of genus one. We introduce the notion of a Riemann surface and we study modular curves in this context. We study holomorphic differentials on modular curves and relate these to the set of cusp forms. We write down generators of $\mathcal{S}_2(\Gamma_0(N))$ which are realized as $\eta-$products and use these to calculate the period lattices associated to $X_0(N)$. The theory of elliptic functions is developed and used to construct an elliptic curve from a period lattice. Finally, we put this theory together to compute the elliptic curves which are isomorphic to $X_0(N)$.

Contents

Introduction

Fermat's Last Theorem is one of the most monumental results of twentieth-century mathematics. It states that $a^n + b^n = c^n$ has no nontrivial integer solutions for $n \geq 3$. In 1990, Ribet proved Fermat's Last Theorem (in [31]) assuming that every so-called semistable elliptic curve over $\mathbb{Q}$ is modular. In 1995 Wiles proved Ribet's assumption [38, Theorem 0.4] proving the 350 year old theorem. This is only a part of the full picture however; as conjectured by Taniyama and Shimura in 1957, every elliptic curve over $\mathbb{Q}$ is modular. In 2001, the collaborative efforts of Breuil, Conrad, Diamond, and Taylor proved this conjecture in [6, Theorem A].

In this thesis we explore what it means for an elliptic curve to be modular. To do this, we require a Riemann surface called the modular curve $X_0(N)$ where N is a positive integer. On this Riemann surface, objects along which we can integrate are defined. By integrating along these elements, one finds a canonical way to construct an elliptic curve. The case where a modular curve has genus one immensely simplifies this process compared to higher genera. And in this case, the elliptic curve we obtain is isomorphic to $X_0(N)$. In this thesis we rigorously carry out the process of constructing an elliptic curve in this way and develop the necessary theory to do so.

In the first section we develop the theory of Riemann surfaces which will serve as the framework surrounding the objects of study. In the next two sections we take a closer look at modular curves and holomorphic differentials. In the final two sections we develop the theory and a way to construct a (modular) elliptic curve from a modular curve. For $X_0(N)$ of genus one, we implement this method and write down the elliptic curves isomorphic to it. It turns out that these elliptic curves are defined over $\mathbb{Q}$, the reason why this is true is not treated in this thesis as we mainly take a complex analytical approach.

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1 Riemann surfaces

A Riemann surface is a topological space that looks like an open set of the complex plane around each point. In this section we define Riemann surfaces and the basic notions that relate to these objects. As a main source we use Miranda's book [28]. In the sections afterwards we go through two examples of Riemann surfaces mainly using [27] and [13].

1.1 Definitions and morphisms

Much like a 2−dimensional real manifold, the idea of a Riemann surface is a topological space that locally looks like an open set of the complex plane. Contrary to a real manifold, a Riemann surface has additional structure, namely the structure from the complex plane. To define this more precisely, we require charts.

Definition 1.1. [28, Definition II.1.1] Let X be a topological space. A *chart* for X is a homeomorphism $\varphi\colon U \to V$ where $U \subset X$ and $V \subset \mathbb{C}$ are open. The chart φ is *centered* around $x \in U$ if $\varphi(x) = 0$. ▲

A chart $\varphi : U \to V$ gives coordinates on an open subset of a topological space. As an example, consider the following.

Example 1.2. [28, Example I.1.13] The 2−sphere

$$\mathbb{S}^2 = \{(x, y, w) \in \mathbb{R}^3 : x^2 + y^2 + w^2 = 1\} \subset \mathbb{R}^3$$

is a connected and compact topological space. Identify $\mathbb{C}$ with the plane $\{(x, y, 0)\}$ where $z = x + iy$. The map $\varphi_1 : \mathbb{S}^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$ defined by

$$\varphi_1(x, y, w) = \frac{x}{1-w} + i\frac{y}{1-w}$$

is a chart of $\mathbb{S}^2$. It has continuous inverse

$$\varphi_1^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

◆

Suppose that we have two charts $\varphi_1\colon U_1 \to V_1$ and $\varphi_2\colon U_2 \to V_2$ such that $U_1 \cap U_2 \neq \varnothing$. In this situation we get to choose which chart we use. It is desirable that the choice of chart here does not affect definitions. To ensure this, we require charts to satisfy the following definition.

Definition 1.3. [28, Definition I.1.6] Let X be a topological space. The charts $\varphi_1\colon U_1 \to V_1$ and $\varphi_2\colon U_2 \to V_2$ are *compatible* if the *transition function*

$$\varphi_2\varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a holomorphic map of open subsets in $\mathbb{C}$ whenever U_1 and U_2 have nonempty intersection. ▲

An interpretation of the transition function $\varphi_2\varphi_1^{-1}$ is as a change of coordinates between two local coordinates. By the inverse function theorem [9, Proposition III.2.20] the map $\varphi_1\varphi_2^{-1}$ is also holomorphic. A Riemann surface is covered in such charts so that we can take coordinates at every point.

Definition 1.4. [28, Definition I.1.18] A *Riemann surface* is a connected topological space X which is Hausdorff and second countable together with a set of pairwise compatible charts. These charts are such that around every point $x \in X$ there is a chart $\varphi\colon U \to V$ where U is an open neighborhood of x and $V \subset \mathbb{C}$ is an open subset of the complex plane. ▲

Example 1.5. Any open subset $U \subset \mathbb{C}$ along with the global chart $\mathrm{id}\colon U \to U$ is a Riemann surface. ◆

Example 1.6. [28, Example I.1.20] The 2−sphere $\mathbb{S}^2$ as in Example 1.2 is a Riemann surface. The sphere $\mathbb{S}^2$ is Hausdorff and second countable. In Example 1.2 we defined the chart $\varphi_1\colon \mathbb{S}^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$. This chart does not include the point $(0, 0, 1)$. Define the chart $\varphi_2\colon \mathbb{S}^2 \setminus \{(0, 0, -1)\} \to \mathbb{C}$ by

$$\varphi_2(x, y, w) = \frac{x}{1+w} - i\frac{y}{1+w}$$

with inverse

$$\varphi_2^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right).$$

A simple (but tedious) computation shows that the transition map $\varphi_2\varphi_1^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ maps z to $1/z$ which is holomorphic on its domain. Interpret the point $(0, 0, 1)$ on $\mathbb{S}^2$ as the point at ∞. The chart φ_1 maps $\mathbb{S}^2 \setminus \{\infty\}$ to $\mathbb{C}$, this allows us to consider this Riemann surface as the complex plane with the point ∞ added to it. The Riemann surface constructed in this way is usually referred to as the *Riemann sphere* and is denoted by $\mathbb{C}_\infty$. ♦

Using the coordinates around every point of a Riemann surface we can define a notion of holomorphic functions on these objects.

Definition 1.7. [28, Definition II.1.1] Let X be a Riemann surface. Let x be a point in X and let $\varphi\colon U \to V$ be a chart about x. We say that $f\colon X \to \mathbb{C}$ is *holomorphic* at x if $f\varphi^{-1}\colon \varphi(U) \to \mathbb{C}$ is holomorphic at $\varphi(x)$. We call f *holomorphic* if it is holomorphic at every point $x \in X$. ▲

We have to be careful that this definition does not depend on the coordinates that we take. Thankfully, compatibility of the charts of X ensures choice independence. To see this, let $\varphi_1 \colon U_1 \to V_1$ and $\varphi_2 \colon U_2 \to V_2$ be charts of a point $x \in X$. Suppose that $f\varphi_1^{-1}\colon X \to \mathbb{C}$ is holomorphic at $\varphi_1(x)$. Then

$$f\varphi_2^{-1} = (f\varphi_1^{-1})(\varphi_1\varphi_2^{-1}).$$

Both $f\varphi_1^{-1}$ and $\varphi_1\varphi_2^{-1}$ are holomorphic at $\varphi_1(x)$ and $\varphi_2(x)$ respectively, the former by assumption and the latter by compatibility. This shows that Definition 1.7 is independent of the chosen local coordinates.

The morphisms of Riemann surfaces, that is, the maps that preserve the structure of a Riemann surface, are defined as follows.

Definition 1.8. [28, Definition II.3.1] Let X and Y be Riemann surfaces. A mapping $\Psi\colon X \to Y$ is *holomorphic* at x if there exist charts $\varphi_X : U_X \to V_X$ and $\varphi_Y : U_Y \to V_Y$ of x and $\Psi(x)$ respectively such that $\varphi_Y \Psi \varphi_X^{-1}$ is holomorphic at $\varphi_X(x)$. We say that Ψ is a *holomorphic map of Riemann surfaces* (or just *holomorphic*) if it is holomorphic at every point $x \in X$. ▲

Definition 1.9. [28, Definition II.3.6] Let X and Y be Riemann surfaces. A mapping $\Psi\colon X \to Y$ is an *isomorphism of Riemann surfaces* (or just isomorphism) if it is a bijective holomorphic mapping such that $\Psi^{-1}\colon Y \to X$ is holomorphic. If an isomorphism $\Psi\colon X \to Y$ exists then we say that X and Y are *isomorphic* and we write $X \cong Y$. ▲

Again, by a similar argument to the above, these definitions do not depend on the choice of chart due to compatibility. Holomorphic maps between Riemann surfaces are generalizations of holomorphic maps between open subsets of $\mathbb{C}$. Most results that hold for the latter carry over to the former. The ones that are important for us are summarized here.

Proposition 1.10. [28, Proposition II.3.11] Let X be a compact Riemann surface and $\Psi\colon X \to Y$ be a non-constant holomorphic map. Then Y is compact and Ψ is surjective. ⬣

Proposition 1.11. [28, Proposition II.3.12] Let X and Y be compact Riemann surfaces and $\Psi\colon X \to Y$ a holomorphic map. Then for every $y \in Y$ the preimage $\Psi^{-1}(y)$ is a nonempty finite subset of X. ⬣

In the next two sections we study two examples of Riemann surfaces that are significant for our purposes. The first example are complex tori which turn out to be elliptic curves over $\mathbb{C}$. When studying isomorphic complex tori one encounters a group action on

$$\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}.$$

The orbits of this group action are a Riemann surface called a modular curve, these serve as our second example.

1.2 Complex Tori

A complex torus is a quotient of the complex plane by a $\mathbb{Z}-$module of rank 2. We use [27] as a main source to define these concepts more precisely.

Definition 1.12. [27, Section III.1] Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. A *lattice* Λ is a $\mathbb{Z}-$module generated by ω_1 and ω_2. ▲

A lattice Λ which is generated by ω_1 and ω_2 has the form

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

Without loss of generality we can order ω_1 and ω_2 such that $\operatorname{Im}(\omega_1/\omega_2) > 0$. This is possible if and only if ω_1 and ω_2 are linearly independent over $\mathbb{R}$. Let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ be elements of the lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and write

$$\begin{aligned}\tilde{\omega}_1 &= a\omega_1 + b\omega_2\\ \tilde{\omega}_2 &= c\omega_1 + d\omega_2\end{aligned}$$

for integers $a, b, c, d \in \mathbb{Z}$. Equivalently,

$$\begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \tag{1.1}$$

Write $(a, b; c, d)$ for the 2×2 matrix in (1.1). If $(a, b; c, d)$ is invertible over $\mathbb{Z}$ then we can write every element in Λ as a $\mathbb{Z}-$linear combination of $\tilde{\omega}_1$ and $\tilde{\omega}_2$. In this case, $\Lambda = \mathbb{Z}\tilde{\omega}_1 \oplus \mathbb{Z}\tilde{\omega}_2$. The coefficients of $(a, b; c, d)$ are in $\mathbb{Z}$, we can therefore invert $(a, b; c, d)$ if and only if $ad - bc = \pm 1 \in \mathbb{Z}^\times$. We still require $\operatorname{Im}(\tilde{\omega}_1/\tilde{\omega}_2) > 0$, from which we obtain,

$$\operatorname{Im}\left(\frac{\tilde{\omega}_1}{\tilde{\omega}_2}\right) = \operatorname{Im}\left(\frac{a(\omega_1/\omega_2) + b}{c(\omega_1/\omega_2) + d}\right) = \frac{\operatorname{Im}(ac|\omega_1/\omega_2|^2 + ad(\omega_1/\omega_2) + bc(\overline{\omega_1/\omega_2}) + bd)}{|c(\omega_1/\omega_2) + d|^2} = \frac{(ad - bc)\operatorname{Im}(\omega_1/\omega_2)}{|c(\omega_1/\omega_2) + d|^2}.$$

The last expression is greater than 0 if and only if $ad - bc > 0$ and thus equal to 1.

Definition 1.13. The *special linear group over* $\mathbb{Z}$ is the group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathbb{Z}^{2\times 2} \colon ad - bc = 1\right\}$$

with matrix multiplication as group law and identity matrix $I = (1, 0; 0, 1)$ as identity element. ▲

Summarizing the argument above, we obtain the following.

Proposition 1.14. [27, Proposition III.1.1] Let $\omega_1, \omega_2, \tilde{\omega}_1, \tilde{\omega}_2 \in \mathbb{C}$ such that $\operatorname{Im}(\omega_1/\omega_2) > 0$ and $\operatorname{Im}(\tilde{\omega}_1/\tilde{\omega}_2) > 0$. If there is a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$A\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix},$$

then the lattices $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\mathbb{Z}\tilde{\omega}_1 \oplus \mathbb{Z}\tilde{\omega}_2$ are equal. ⬣

The set $\mathrm{SL}_2(\mathbb{Z})$ induces an equivalence relation on the set of pairs $(\omega_1, \omega_2) \in \mathbb{C}$ satisfying $\operatorname{Im}(\omega_1/\omega_2) > 0$. Define $(\omega_1, \omega_2) \sim (\tilde{\omega}_1, \tilde{\omega}_2)$ when there is some $A \in \mathrm{SL}_2(\mathbb{Z})$ satisfying the hypotheses of Proposition 1.14. By projecting onto this equivalence relation, we obtain a bijection

$$\{\text{Lattices } \Lambda \subset \mathbb{C}\} \leftrightarrow \{(\omega_1, \omega_2) \in \mathbb{C} \colon \operatorname{Im}(\omega_1/\omega_2) > 0\}/\sim.$$

Let $\Lambda \subset \mathbb{C}$ be a lattice. The lattice Λ is a subgroup of the additive group $\mathbb{C}$. Therefore the quotient $\mathbb{C}/\Lambda$ is a group. Let $\pi \colon \mathbb{C} \to \mathbb{C}/\Lambda$ be the projection homomorphism $z \mapsto z + \Lambda$. Using this homomorphism we define a topology on $\mathbb{C}/\Lambda$. Define a set $U \subset \mathbb{C}/\Lambda$ to be open whenever $\pi^{-1}(U)$ is open in $\mathbb{C}$. In this topology, the map π is continuous. Additionally, π is an open map as for open $U \subset \mathbb{C}$ we have

$$\pi^{-1}(\pi(U)) = \bigcup_{\omega\in\Lambda} U + \omega$$

where $U + \omega$ is the image of the homeomorphism $z \mapsto z + \omega$ and is therefore open, thus $\pi(U)$ is open.

As a topological space, $\mathbb{C}/\Lambda$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$, a $1-$holed torus. We therefore fittingly name $\mathbb{C}/\Lambda$ a *complex torus*. If $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ then for $z \in \mathbb{C}$, the connected compact set

$$\overline{D}_z = \{z + w_1\omega_1 + w_2\omega_2 : w_1, w_2 \in [0,1]\} \subset \mathbb{C} \tag{1.2}$$

is a complete set of representatives for $\mathbb{C}/\Lambda$ and hence $\pi(\overline{D}_z) = \mathbb{C}/\Lambda$ incidentally showing that $\mathbb{C}/\Lambda$ is compact and connected. Moreover, the interior D_z of $\overline{D}_z$ does not contain two points which are congruent modulo Λ. For arbitrary $z_0 \in \mathbb{C}$, picking $z = z_0 - \omega_1/2 - \omega_2/2$, we see that $z_0 \in D_z$. We can therefore choose D_z such that it contains any point of our choosing. A set of the form D_z is called a *fundamental domain for* Λ.

For $z_0 \in \mathbb{C}$, the coset $z_0 + \Lambda$ is a set of isolated points of $\mathbb{C}$, therefore there is an open neighborhood $U \subset \mathbb{C}$ of z_0 containing no two elements which are congruent modulo Λ. Take for example a fundamental domain D_z containing z_0 and any open subset $U \subset D_z$ such that $z_0 \in U$. Then $\pi|_U : U \to \pi(U)$ is a bijection, and hence a homeomorphism as π is open and continuous. Let $\varphi\colon \pi(U) \to U$ be the local inverse of π. Then φ satisfies Definition 1.1 and defines a chart for $\mathbb{C}/\Lambda$.

Proposition 1.15. Let $\Lambda \subset \mathbb{C}$ be a lattice. The complex torus $\mathbb{C}/\Lambda$ endowed with the quotient topology is a compact Riemann surface. ⬣

Proof. By the argument above it follows that $\mathbb{C}/\Lambda$ is compact, connected, and around every point in $\mathbb{C}/\Lambda$ there exists a chart. Additionally, $\mathbb{C}/\Lambda$ inherits the second countable and Hausdorff properties from $\mathbb{C}$. What remains to be checked is whether the transition charts are compatible in the sense of Definition 1.3. Let $\varphi_1\colon \pi(U_1) \to U_1$ and $\varphi_2\colon \pi(U_2) \to U_2$ be local inverses of π such that $U_1 \cap U_2 \neq \varnothing$. The transition function $\varphi_1\varphi_2^{-1}\colon \varphi_1(\pi(U_1)\cap\pi(U_2)) \to \varphi_2(\pi(U_1)\cap\pi(U_2))$ sends a point $z \in \varphi_1(\pi(U_1)\cap\pi(U_2))$ to $\pi(z)$ for some $z \in U_1$. We have that $\pi(z) = \pi(u)$ for some $u \in U_2$ hence $z+\omega = u$ for some $\omega \in \Lambda$ and $\varphi_2(\pi(z)) = u = z + \omega$. We obtain that $\varphi_1\varphi_2^{-1}(z) = z + \omega$ for some $\omega \in \Lambda$. It follows that the transition function $\varphi_1\varphi_2^{-1}$ is a holomorphic map of open subsets of $\mathbb{C}$. ∎

Next, we investigate the holomorphic maps between two complex tori. Let Λ and Λ' be lattices. Covering space theorists call $\mathbb{C}$ a universal covering space for $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ and $\pi'\colon \mathbb{C} \to \mathbb{C}/\Lambda'$. A result from covering space theory [15, Theorems 5.1, 6.4] states that a continuous map $\Psi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ such that $\Psi(0) = 0$ lifts to a unique continuous map $\tilde{\Psi}\colon \mathbb{C} \to \mathbb{C}$ sending 0 to 0 such that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{\Psi}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda \arrow[r, "\Psi"] & \mathbb{C}/\Lambda' \end{tikzcd} \tag{1.3}$$

commutes. Around every point, locally, π and π' are bijections and commutativity of the above diagram implies $\tilde{\Psi} = \pi'^{-1}\Psi\pi$. If Ψ is a holomorphic map of Riemann surfaces, then by Definition 1.8 the right hand side of this equation is holomorphic hence $\tilde{\Psi}$ is holomorphic. Conversely if $\tilde{\Psi}$ is holomorphic we see that Ψ is holomorphic.

Example 1.16. Let $\alpha \in \mathbb{C}$ define the map $\Psi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ as $z + \Lambda \mapsto \alpha z + \Lambda'$. Such a mapping is well-defined if and only if $\alpha\Lambda \subset \Lambda$. To see this, suppose that $u + \Lambda = z + \Lambda$. Then $u - z \in \Lambda$. We have $\alpha u + \Lambda' = \alpha z + \Lambda'$ if and only if $\alpha(u - z) \in \Lambda'$. In particular, setting $z = 0$ we obtain $\alpha\Lambda \subset \Lambda'$. In this case, the map $\tilde{\Psi}\colon z \mapsto \alpha z$ is holomorphic on $\mathbb{C}$ and is the unique continuous map $\mathbb{C} \to \mathbb{C}$ such that the diagram (1.3) commutes. This shows that Ψ satisfies Definition 1.8 and is a holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$. ♦

Proposition 1.17. [27, Proposition 3.3] Let Λ and Λ' be lattices. Suppose that $\Psi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is holomorphic sending 0 to 0. Then there exists an $\alpha \in \mathbb{C}$ such that $\alpha\Lambda \subset \Lambda'$ and $\Psi\colon z + \Lambda \mapsto \alpha z + \Lambda'$. ⬣

Proof. Let $\tilde{\Psi}\colon \mathbb{C} \to \mathbb{C}$ be the lifting map of Ψ making (1.3) commute and let $\omega \in \Lambda$. Then

$$\pi'(\tilde{\Psi}(z+\omega) - \tilde{\Psi}(z)) = \pi'(\tilde{\Psi}(z+\omega)) - \pi'(\tilde{\Psi}(z)) = \Psi(\pi(z+\omega)) - \Psi(\pi(z)) = \Psi(\pi(z)) - \Psi(\pi(z)) = 0$$

which implies $\tilde{\Psi}(z+\omega) - \tilde{\Psi}(z) \in \Lambda'$. Since $\tilde{\Psi}(z+\omega) - \tilde{\Psi}(z)$ is continuous, it can not jump to different values of Λ' as each point in Λ' is isolated. This forces $z \mapsto \tilde{\Psi}(z+\omega) - \tilde{\Psi}(z)$ to be constant. Its derivative will then be 0. It follows that $\tilde{\Psi}'(z+\omega) = \tilde{\Psi}'(z)$, this means that $\tilde{\Psi}'$ is fully determined on the closure of some fundamental domain $\overline{D}_z$. Such a set is compact and hence $\tilde{\Psi}'$ is bounded on $\overline{D}_z$. Since Ψ' is determined entirely on $\overline{D}_z$, it is bounded on the whole complex plane and hence constant by Liouville's theorem. Write $\tilde{\Psi}'(z) = \alpha$ for $\alpha \in \mathbb{C}$ and take an anti-derivative to obtain $\tilde{\Psi}(z) = \alpha z + \beta$. Plugging in 0 gives $\tilde{\Psi}(0) = \beta = 0$ and

$$\Psi(z+\Lambda) = \Psi(\pi(z)) = \pi'(\tilde{\Psi}(z)) = \alpha z + \Lambda',$$

as required. ■

An immediate consequence of this is that every holomorphic function between complex tori which sends 0 to 0 is a group homomorphism.

Corollary 1.18. [27, Corollary 3.4] Let Λ and Λ' be lattices. The Riemann surfaces $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic as groups and Riemann surfaces if and only if $\alpha\Lambda = \Lambda'$ for some $\alpha \in \mathbb{C}$. ⬣

For a lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, write $\Lambda = \omega_2(\mathbb{Z}\frac{\omega_1}{\omega_2} \oplus \mathbb{Z})$. By Corollary 1.18, for any lattice Λ there is a lattice of the form $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ with $\tau \in \mathcal{H}$ such that $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda_\tau$. In the next proposition, we classify elliptic tori of the form $\mathbb{C}/\Lambda_\tau$ up to isomorphism.

Proposition 1.19. Let $\tau_1, \tau_2 \in \mathcal{H}$. Define $\Lambda_{\tau_1} = \mathbb{Z}\tau_1 \oplus \mathbb{Z}$ and $\Lambda_{\tau_2} = \mathbb{Z}\tau_2 \oplus \mathbb{Z}$. Then $\mathbb{C}/\Lambda_{\tau_1} \cong \mathbb{C}/\Lambda_{\tau_2}$ if and only if

$$\tau_1 = \frac{a\tau_2 + b}{c\tau_2 + d} \tag{1.4}$$

for some $(a,b;c,d) \in \mathrm{SL}_2(\mathbb{Z})$. ⬣

Proof. Suppose that $\mathbb{C}/\Lambda_{\tau_1} \cong \mathbb{C}/\Lambda_{\tau_2}$. By Corollary 1.18 $\alpha\Lambda_{\tau_1} = \Lambda_{\tau_2}$ for some $\alpha \in \mathbb{C}$. This means that there is some $(a,b;c,d) \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha\tau_1 \\ \alpha \end{pmatrix}.$$

Then

$$\tau_1 = \frac{\alpha\tau_1}{\alpha} = \frac{a\tau_2 + b}{c\tau_2 + d}.$$

Conversely, suppose that τ_1 and τ_2 satisfy (1.4) for some $(a,b;c,d) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\mathbb{C}/\Lambda_{\tau_1} = \mathbb{C}/(\mathbb{Z}\tau_1 \oplus \mathbb{Z}) = \mathbb{C}/\left(\mathbb{Z}\left(\frac{a\tau_2+b}{c\tau_2+d}\right) \oplus \mathbb{Z}\right) \overset{1.18}{\cong} \mathbb{C}/(\mathbb{Z}(a\tau_2+b) \oplus \mathbb{Z}(c\tau_2+d)) \overset{1.14}{=} \mathbb{C}/(\mathbb{Z}\tau_2 \oplus \mathbb{Z}) = \mathbb{C}/\Lambda_{\tau_2}.$$

■

The relation (1.4) between τ_1 and τ_2 defines a group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. This fact is the groundwork of modular curves which is studied in the next section. In Section 4.2 an isomorphism between complex tori and elliptic curves is described, further emphasizing the importance of studying isomorphism classes of complex tori.

1.3 Modular curves

A modular curve is a compact Riemann surface consisting of the set of orbits of a group action of a certain type of subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. These objects are widely studied in algebraic geometry and number theory. Modular curves allows us to study certain types of objects through the lens of Riemann surfaces. In this section we define these objects and show that they are indeed Riemann surfaces. These objects are paramount to us, for this reason we define these objects in quite some detail, mainly using [13, Section 2].

As hinted at in the previous section, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half complex plane $\mathcal{H}$.

Proposition 1.20. The action $\mathrm{SL}_2(\mathbb{Z}) \times \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

is a well-defined group action. ⬣

Proof. Let $\tau \in \mathcal{H}$ and $(a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$. Showing that the action is well-defined amounts to showing that $c\tau + d \neq 0$ and that $(a, b; c, d)\tau \in \mathcal{H}$. Suppose that $c\tau + d = 0$. It follows that $c = d = 0$ as τ has nonzero imaginary part. No $(a, b; 0, 0) \in \mathrm{SL}_2(\mathbb{Z})$ exists. For $\tau \in \mathcal{H}$ and $(a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$,

$$\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau\right) = \frac{\mathrm{Im}((a\tau + b)(c\overline{\tau} + d))}{|c\tau + d|^2} = \frac{(ad - bc)\,\mathrm{Im}\,\tau}{|c\tau + d|^2} = \frac{\mathrm{Im}\,\tau}{|c\tau + d|^2}$$

which is greater than 0. This shows that $(a, b; c, d)\tau \in \mathcal{H}$. Moreover, $(1, 0; 0, 1)\tau = \tau$ and $\alpha(\beta\tau) = (\alpha\beta)\tau$ for $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$. ■

Let $N > 1$ be an integer. The homomorphism $\pi_N \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ induced by the projection homomorphism $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ has kernel

$$\ker \pi_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

This normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is denoted by $\Gamma(N)$ for integer $N > 1$ and is referred to as the *principle congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$. As a convention, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. The map π_N is surjective [8, Theorem 3.2], its image $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is a finite set. By the first homomorphism theorem we see that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = |\,\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| < \infty$$

Definition 1.21. [13, Definition 1.2.1] A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup* if there is an integer $N \geq 1$ such that $\Gamma(N) \leq \Gamma$. ▲

Let Γ be a congruence subgroup. From the inclusion $\Gamma(N) \leq \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ we obtain

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \leq [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] < \infty.$$

Any congruence group Γ inherits a group action from $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ as in Proposition 1.20. The set of orbits is denoted by

$$Y(\Gamma) = \{\Gamma\tau : \tau \in \mathcal{H}\} = \Gamma\backslash\mathcal{H}.$$

The expression $\Gamma\backslash\mathcal{H}$ is a left quotient and is interpreted as identifying Γ equivalent points. Let $\pi \colon \mathcal{H} \to Y(\Gamma)$ denote the projection map. With the subspace topology on $\mathcal{H}$ we endow $Y(\Gamma)$ with the quotient topology where we define $U \subset Y(\Gamma)$ to be open whenever $\pi^{-1}(U)$ is an open subset of $\mathcal{H}$. With this topology, π is continuous and open. To see that π is open, let $U \subset \mathcal{H}$ be an open subset, then

$$\pi^{-1}(\pi(U)) = \bigcup_{\gamma \in \Gamma} \pi^{-1}(\gamma(U)). \tag{1.5}$$

Interpret γ as a homeomorphism $\mathcal{H} \to \mathcal{H}$ with continuous inverse γ^{-1}. We see that $\gamma(U)$ is open and from (1.5) it follows that $\pi(U)$ is open.

By second countability of $\mathcal{H}$ it follows that $Y(\Gamma)$ also has this property. Moreover by continuity of π and connectedness of $\mathcal{H}$ we see that $\pi(\mathcal{H}) = Y(\Gamma)$ is connected. Next, we show that $Y(\Gamma)$ is Hausdorff. This requires some more work. However, the intermediate results we obtain are important when defining the domains of the charts on $Y(\Gamma)$. The following lemma is proven in the proof of Proposition 2.1.1 of [13]. The proof we include here fills in some of the missing details.

Lemma 1.22. [13, Proposition 2.1.1] Let Γ be a congruence subgroup. For open sets U_1, U_2 of $\mathcal{H}$ with compact closure the set

$$\{\gamma \in \Gamma : \gamma(U_1) \cap U_2 \neq \varnothing\}$$

has finite cardinality. ⬣

Proof. First note that for $(a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$ we have $ad - bc = 1$. This implies that $\gcd(c, d) = 1$. Let $(c, d) \in \mathbb{Z}^2$ be any pair such that $\gcd(c, d) = 1$. Define the real numbers

$$\begin{aligned} y_1 &= \inf\{\mathrm{Im}(\tau) \colon \tau \in U_1\} \\ Y_1 &= \sup\{\mathrm{Im}(\tau) \colon \tau \in U_1\} \\ y_2 &= \inf\{\mathrm{Im}(\tau) \colon \tau \in U_2\}. \end{aligned}$$

Existence of these is ensured by the fact that U_1 and U_2 have compact closure in $\mathcal{H}$. Let $\gamma = (a, b; c, d) \in \Gamma$ and $\tau \in U_1$, then

$$\begin{aligned}\operatorname{Im}(\gamma\tau) &= \frac{\operatorname{Im}\tau}{|c\tau+d|^2} = \frac{\operatorname{Im}\tau}{(c\operatorname{Re}\tau+d)^2+(c\operatorname{Im}\tau)^2} \leq \min\left\{\frac{\operatorname{Im}\tau}{(c\operatorname{Im}\tau)^2}, \frac{\operatorname{Im}\tau}{(c\operatorname{Re}\tau+d)^2}\right\} \\ &= \min\left\{\frac{1}{c^2\operatorname{Im}\tau}, \frac{\operatorname{Im}\tau}{(c\operatorname{Re}\tau+d)^2}\right\} \leq \min\left\{\frac{1}{c^2 y_1}, \frac{Y_1}{(c\operatorname{Re}\tau+d)^2}\right\}.\end{aligned} \tag{1.6}$$

For $|c|$ large enough we have $1/(c^2y_1) < y_2$, say, for $|c| \geq R$ with $R \in \mathbb{Z}_{>0}$. There are only finitely many c such that $|c| \leq R$. For these c we have

$$\frac{Y_1}{(c\operatorname{Re}\tau+d)^2} < y_2$$

for d large enough, say, $|d| \geq Q$ with $Q \in \mathbb{Z}_{>0}$. Combining these two results and the bound obtained above, we see that for all but finitely many pairs $(c, d) \in \mathbb{Z}^2$ with $\gcd(c, d) = 1$ we have $\operatorname{Im}(\gamma\tau) < y_2$ where γ has bottom row (c, d). By taking the supremum of (1.6) we obtain

$$\sup\{\gamma\tau \colon \gamma \in \Gamma \text{ has bottom row } (c, d), \tau \in U_1\} < y_2$$

which holds for all but finitely many pairs (c, d) such that $\gcd(c, d) = 1$. In particular, this implies that $\gamma(U_1) \cap U_2 = \varnothing$ for all but finitely many of such pairs.

Next, we show that there are only finitely many $\gamma_{c,d} \in \Gamma$ with fixed bottom row (c, d) such that $\gamma_{c,d}(U_1) \cap U_2 \neq \varnothing$. Combining this fact with the fact that $\gamma(U_1) \cap U_2 = \varnothing$ for all but finitely many pairs (c, d) proves the lemma. To find the elements $\gamma_{c,d} = (a, b; c, d) \in \Gamma$ with fixed bottom row (c, d), we note that by Bézout's identity the equation $ad - bc = 1$ is solved by $(a, b) = (a + kc, b + kd)$ for any $k \in \mathbb{Z}$. Therefore, any matrix in Γ with bottom row (c, d) has the form

$$\begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \gamma_{c,d}.$$

The action of $(1, k; 0, 1)\gamma_{c,d}$ on $\tau \in U_1$ is

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \gamma_{c,d}\tau = \gamma_{c,d}\tau + k.$$

Such a matrix translates $\gamma_{c,d}$ by integer amounts, and since both U_1 and U_2 are bounded, there are only finitely many integers $k \in \mathbb{Z}$ such $\big(\gamma_{c,d}(U_1)+k\big)\cap U_2 \neq \varnothing$. As argued above, this proves the statement. ■

Proposition 1.23. [13, Proposition 2.1.1] Let Γ be a congruence subgroup. Suppose $x, y \in \mathcal{H}$ satisfy $\pi(x) \neq \pi(y)$. Then there exists open neighborhoods U of x and V of y such that

$$\gamma(U) \cap V = \varnothing$$

for all $\gamma \in \Gamma$. ⬣

Proof. Pick open neighborhoods A of x and B of y with compact closure. By Lemma 1.22 the set

$$S = \{\gamma \in \Gamma : \gamma(A) \cap B \neq \varnothing\}$$

is finite. By assumption, $\gamma x \neq y$ for all $\gamma \in S$. We can therefore (since $\mathcal{H}$ is Hausdorff) pick disjoint open neighborhoods U_γ and V_γ of γx and y, respectively. Define

$$U = A \cap \Big(\bigcap_{\gamma\in S} \gamma^{-1}U_\gamma\Big) \qquad V = B \cap \Big(\bigcap_{\gamma\in S} V_\gamma\Big).$$

Then U and V are open since the intersections are finite. Moreover, $x \in U$ and $y \in V$. Additionally U and V satisfy $\gamma(U) \cap V = \varnothing$ for every $\gamma \in \Gamma$. To see this, suppose $\gamma \notin S$. Then $\gamma(U) \subset \gamma(A)$ and $\gamma(U) \cap V \subset \gamma(A) \cap B = \varnothing$. It follows that $\gamma(U) \cap V = \varnothing$. Conversely, suppose $\gamma \in S$ we have $\gamma(U) \subset U_\gamma$ and $\gamma(U) \cap V \subset U_\gamma \cap V_\gamma = \varnothing$. It follows that $\gamma(U) \cap V = \varnothing$ for all $\gamma \in \Gamma$ ■

Corollary 1.24. [13, Corollary 2.1.2] Let Γ be a congruence subgroup. The topological space $Y(\Gamma)$ is Hausdorff. ⬣

Proof. Let $\pi(x), \pi(y) \in Y(\Gamma)$ be distinct points. By Theorem 1.23, there exist open neighborhoods U and V of x and y such that $\gamma(U) \cap V = \varnothing$ for all $\gamma \in \Gamma$. This implies that $\pi(U) \cap \pi(V) = \varnothing$. To see this, for $x \in \pi(U) \cap \pi(V)$ we have $x = \pi(u)$ and $x = \pi(v)$ for some $u \in U$ and $v \in V$. By definition, there exists some $\gamma \in \Gamma$ such that $\gamma u = v$. This implies that $v \in \gamma(U) \cap V$, a contradiction. Since π is an open map, we have $\pi(U)$ and $\pi(V)$ open in $Y(\Gamma)$. Moreover $\pi(U)$ and $\pi(V)$ contain $\pi(x)$ and $\pi(y)$ respectively, which were taken to be arbitrary, proving that $Y(\Gamma)$ is Hausdorff. ■

Next, we define charts on $Y(\Gamma)$. As with the complex torus, around most points $\tau \in \mathcal{H}$ we can find an open neighborhood of τ such that π is locally invertible. However, there are some points for which this is not possible. For example, the point $i \in \mathcal{H}$ is fixed by $\gamma = (0, -1; 1, 0)$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} i = -\frac{1}{i} = i.$$

Since $\gamma \colon \mathcal{H} \to \mathcal{H}$ is a homeomorphism and fixes i, any neighborhood around i is mapped to a neighborhood of i. This implies that there are always two $\mathrm{SL}_2(\mathbb{Z})$ equivalent points in any neighborhood of i and π is not locally invertible.

Definition 1.25. [13, Definition 2.2.1] Let Γ be a congruence subgroup. A point $\tau \in \mathcal{H}$ is called an *elliptic point* for Γ if its stabilizer

$$\Gamma_\tau = \{\gamma \in \Gamma \colon \gamma\tau = \tau\}$$

contains any element other than I or $-I$. ▲

For example $\mathrm{SL}_2(\mathbb{Z})_i$ contains $(0, -1; 1, 0)$ and i is an elliptic point for $\mathrm{SL}_2(\mathbb{Z})$. The elliptic points are well behaved as the stabilizer groups are uncomplicated groups.

Proposition 1.26. [13, Corollary 2.3.5] For every $\tau \in \mathcal{H}$ the stabilizer Γ_τ is cyclic and finite. ⬣

This result leads to the following definition.

Definition 1.27. [13, Section 2.2] Let Γ be a congruence subgroup. The *period* of $\tau \in \mathcal{H}$ is

$$h_\tau = \begin{cases} |\Gamma_\tau|/2 & \text{if } -I \in \Gamma \\ |\Gamma_\tau| & \text{if } -I \notin \Gamma. \end{cases}$$

▲

This definition counts the number of elements $\gamma \in \Gamma$ which fix τ up to $-I$, since γ and $-\gamma$ act identically on $\mathcal{H}$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The point $\tau \in \mathcal{H}$ is an elliptic point whenever its period h_τ is greater than 1. An alternate way to calculate h_τ is as $h_\tau = |\{\pm I\}\Gamma_\tau/\{\pm I\}|$ where $\{\pm I\}\Gamma_\tau$ is a product of subgroups. For $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the period of τ with respect to Γ is equal to the period of $\alpha\tau$ with respect to $\alpha\Gamma\alpha^{-1}$. To see this, let $\gamma \in \Gamma_\tau$. Then

$$\alpha\gamma\alpha^{-1}(\alpha\tau) = \alpha\gamma\tau = \alpha\tau.$$

It follows that the isomorphism

$$\begin{aligned} \Gamma_\tau &\to (\alpha\Gamma\alpha^{-1})_{\alpha\tau} \\ \gamma &\mapsto \alpha\gamma\alpha^{-1}, \end{aligned}$$

is well-defined. In particular, when $\alpha \in \Gamma$ we have $\alpha\Gamma\alpha^{-1} = \Gamma$ and $h_\tau = h_{\alpha\tau}$. Showing that the period is well-defined on $Y(\Gamma)$.

Proposition 1.28. [13, Corollary 2.2.3] Let Γ be a congruence subgroup. For every $\tau \in \mathcal{H}$ there exists an open neighborhood $U \subset \mathcal{H}$ of τ such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\text{if} \quad \gamma(U) \cap U \neq \varnothing \quad \text{then} \quad \gamma \in \Gamma_\tau.$$

Moreover, the only possible elliptic point in U is τ. ⬣

For a point $\tau \in \mathcal{H}$ which is not an elliptic point, Theorem 1.28 provides an open neighborhood of τ such that no distinct points are equivalent. This means that π restricts to a bijection $U \to \pi(U)$ and hence a homeomorphism as π is open and continuous. The open sets $\pi(U)$ serve as the domains of the charts on $Y(\Gamma)$.

Proof of Proposition 1.28. Let $\tau \in \mathcal{H}$ and let $V \subset \mathcal{H}$ be an open neighborhood of τ with compact closure. By Lemma 1.22, the set

$$S = \{\gamma \in \Gamma \colon \gamma(V) \cap V \neq \varnothing, \gamma\tau \neq \tau\}$$

has finite cardinality. For every $\gamma \in S$ let U_γ and V_γ be disjoint open neighborhoods of τ and $\gamma\tau$ respectively. These sets exist since $\gamma\tau \neq \tau$ and $\mathcal{H}$ is Hausdorff. Define

$$U = V \cap \Big(\bigcap_{\gamma \in S} U_\gamma \cap \gamma^{-1}(V_\gamma) \Big).$$

Then U is an open neighborhood of τ as the intersection is finite. Suppose that $\gamma(U) \cap U \neq \varnothing$, then $\gamma(V) \cap V \neq \varnothing$. In addition, suppose for a contradiction that $\gamma\tau \neq \tau$ so that $\gamma \in S$. Then $U \subset U_\gamma$ and $\gamma(U) \subset V_\gamma$ and hence

$$\varnothing \neq \gamma(U) \cap U \subset U_\gamma \cap V_\gamma = \varnothing,$$

a contradiction. Therefore, $\gamma \notin S$. It follows that $\gamma\tau = \tau$ and $\gamma \in \Gamma_\tau$.

For the second assertion, suppose that τ is not an elliptic point and that there is some elliptic point $z \in U$ distinct from τ. Then there is a nontrivial $\gamma \in \Gamma_z$ and $\gamma(U) \cap U \neq \varnothing$ implying $\gamma \in \Gamma_\tau$, a contradiction. Suppose τ is an elliptic point and that there is some other elliptic point $z \in U$. The same argument as above shows that $\Gamma_z \subset \Gamma_\tau$. Therefore, for $(a, b; c, d) \in \Gamma_z$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \tau \Leftrightarrow \frac{a\tau + b}{c\tau + d} = \tau \Leftrightarrow c\tau^2 + (d - a)\tau + b = 0.$$

If $c \neq 0$ then the last equation is only satisfied for τ and $\overline{\tau}$. This is because the coefficients a, b, c and d are real. Therefore, $(a, b; c, d)$ does not fix z and $c = 0$. It is easily verified that for $c = 0$ the equation above implies $(a, b; c, d) = \pm I$. Therefore, z is not an elliptic point. We conclude that the only possible elliptic point in U is τ. ■

Proposition 1.29. [13, Section 2.2] Let Γ be a congruence subgroup. The set of orbits $Y(\Gamma)$ is a Riemann surface. ⬣

Proof. Recall that $Y(\Gamma)$ is second countable and connected. Additionally, by Corollary 1.23, $Y(\Gamma)$ is Hausdorff. What remains to be shown is that there is a chart around every point in $Y(\Gamma)$. To do this, define the *general linear group over* $\mathbb{C}$

$$\mathrm{GL}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2\times 2} : ad - bc \neq 0 \right\}.$$

The group $\mathrm{GL}_2(\mathbb{C})$ acts on the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \begin{cases} \infty & \text{if } cz + d = 0 \\ \frac{az+b}{cz+d} & \text{otherwise} \end{cases} \quad \text{for} \quad z \in \mathbb{C}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \infty = \begin{cases} \infty & \text{if } c = 0 \\ \frac{a}{c} & \text{otherwise.} \end{cases}$$

Let $\tau \in \mathcal{H}$, then $\delta_\tau = (1, -\tau; 1, -\overline{\tau}) \in \mathrm{GL}_2(\mathbb{C})$ sends $\tau \mapsto 0$ and $\overline{\tau} \to \infty$ via its action on $\mathcal{H}$. Let $U \subset \mathcal{H}$ be as in Theorem 1.28. Interpret δ_τ as the map $\delta_\tau \colon U \to \mathbb{C}$ induced by the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathcal{H}$. When τ is not an elliptic point, the map δ_τ serves as a chart $U \to \delta_\tau(U)$. More care is required when τ is an elliptic point. Define

$$\begin{aligned} \psi \colon U &\to \mathbb{C} \\ \nu &\mapsto \rho\delta_\tau(\nu) \end{aligned} \tag{1.7}$$

where $\rho(z) = z^{h_\tau}$ and h_τ is the period of τ. Points in $\mathcal{H}$ which are Γ equivalent have the same image under ψ. To see this, suppose that $\tau_1, \tau_2 \in U$ are such that $\pi(\tau_1) = \pi(\tau_2)$. By the choice of U we must have $\tau_1 \in \Gamma_\tau \tau_2$. Applying δ_τ on both sides we obtain that this is equivalent to

$$\delta_\tau(\tau_1) \in \delta_\tau \Gamma_\tau \tau_2 = \delta_\tau \Gamma_\tau \delta_\tau^{-1}(\delta_\tau(\tau_2)). \tag{1.8}$$

For $\delta_\tau \gamma \delta_\tau^{-1} \in \delta_\tau \Gamma_\tau \delta_\tau^{-1}$ we have

$$0 \xmapsto{\delta_\tau^{-1}} \tau \xmapsto{\gamma} \tau \xmapsto{\delta_\tau} 0 \qquad\qquad \infty \xrightarrow{\delta_\tau^{-1}} \overline{\tau} \xrightarrow{\gamma} \overline{\tau} \xrightarrow{\delta_\tau} \infty.$$

Elements in $\mathrm{GL}_2(\mathbb{C})$ which fix 0 and ∞ act as multiplication. Therefore, elements in $\delta_\tau \Gamma_\tau \delta_\tau^{-1}$ act as multiplication. Using this fact and (1.8) we obtain

$$\delta_\tau(\tau_1) = \delta_\tau \gamma \delta_\tau^{-1}(\delta_\tau(\tau_1)) = \zeta \delta_\tau(\tau_2) \quad \text{for some} \quad \zeta \in \mathbb{C} \quad \text{and} \quad \gamma \in \Gamma_\tau.$$

By Proposition 1.26 we know that $\delta_\tau \Gamma_\tau \delta_\tau^{-1}$ is cyclic of order h_τ or $2h_\tau$. Therefore $(\delta_\tau \alpha \delta_\tau^{-1})^{h_\tau} = \pm I$ which acts trivially on $\delta_\tau(\tau_2)$. It follows that $\zeta^{h_\tau} = 1$ and hence ζ is an $h_\tau-$th root of unity, write $\zeta = \zeta_{h_\tau}^d$ for some $d \in \mathbb{Z}$. Then

$$\delta_\tau(\tau_1) = \zeta_{h_\tau}^d \delta_\tau(\tau_2) \Leftrightarrow \big(\delta_\tau(\tau_1)\big)^{h_\tau} = \big(\delta_\tau(\tau_2)\big)^{h_\tau} \Leftrightarrow \psi(\tau_1) = \psi(\tau_2).$$

Doing this calculation in reverse we obtain that $\pi(\tau_1) = \pi(\tau_2)$ if and only if $\psi(\tau_1) = \psi(\tau_2)$. Set $V = \psi(U)$ which is open by the open mapping theorem [9, Theorem IV.7.5]. Define $\varphi\colon \pi(U) \to V$ to be the (unique) map that makes the following diagram commute.

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "\pi"] & V \\ \pi(U) \arrow[ur, "\varphi"] & \end{tikzcd} \tag{1.9}$$

By the argument above, φ is well-defined and injective. Since π and ψ are open, continuous and surjective, so is φ. The homeomorphism $\varphi : \pi(U) \to V$ satisfies Definition 1.1 and defines a chart on $Y(\Gamma)$. In [13, Section 2.2] the transition maps of these charts are calculated which turn out to be holomorphic. The charts $\varphi\colon \pi(U) \to V$ are defined around any point $\tau \in \mathcal{H}$. It follows that $Y(\Gamma)$ is a Riemann surface. ■

Next, we add a point at ∞ to $Y(\Gamma)$. To do this we compactify $\mathcal{H}$ by adding ∞. By attempting to extend the action of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathcal{H} \cup \{\infty\}$ by defining

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \begin{cases} \infty & \text{if } c = 0 \\ \frac{a}{c} & \text{otherwise} \end{cases}$$

we see that this extension of $\mathcal{H}$ also requires $\mathbb{Q}$ to be added. Denote the extended upper half plane by $\mathcal{H}_\infty = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ and for $p/q \in \mathbb{Q}$ define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{p}{q} = \begin{cases} \infty & \text{if } pc + qd = 0 \\ \frac{pa+qb}{pc+qd} & \text{otherwise.} \end{cases}$$

Any congruence subgroup Γ inherits its action on $\mathcal{H}_\infty$ from $\mathrm{SL}_2(\mathbb{Z})$.

Definition 1.30. Let Γ be a congruence subgroup. The *modular curve* $X(\Gamma)$ is the set of orbits of $\mathcal{H}_\infty$

$$X(\Gamma) = \Gamma \backslash \mathcal{H}_\infty = Y(\Gamma) \cup \Gamma \backslash (\mathbb{Q} \cup \{\infty\}).$$

A *cusp* of $X(\Gamma)$ is an orbit of the form $\Gamma s \in \Gamma \backslash (\mathbb{Q} \cup \{\infty\})$. ▲

Lemma 1.31. [13, Lemma 2.4.1] The modular curve $X(\mathrm{SL}_2(\mathbb{Z}))$ has one cusp. If Γ is a congruence subgroup then $X(\Gamma)$ has a finite number of cusps. ⬣

Proof. Let $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. By Bézout's identity there exists integers $s, t \in \mathbb{Z}$ such that $ps - tq = 1$ then $(p, t; q, s) \in \mathrm{SL}_2(\mathbb{Z})$ and

$$\begin{pmatrix} p & t \\ q & s \end{pmatrix} \infty = \frac{p}{q}.$$

Therefore, ∞ is in the same orbit as any rational number. As a consequence, $\mathrm{SL}_2(\mathbb{Z})\infty$ is the only cusp of $X(\mathrm{SL}_2(\mathbb{Z}))$.

For the second assertion, let Γ be a congruence subgroup. The subgroup Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, therefore, there exist $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\mathrm{SL}_2(\mathbb{Z}) = \bigcup_{i=1}^{n} \Gamma\gamma_i$$

where $n = [\mathrm{SL}_2(\mathbb{Z})\colon \Gamma]$. Let $s \in \mathbb{Q} \cup \{\infty\}$. By the previous argument, there is some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $s = \alpha(\infty)$ and we can write $\alpha = \delta\gamma_j$ for some $\delta \in \Gamma$ and $j \in \{1, \dots, n\}$. This implies $\Gamma s = \Gamma\gamma_j(\infty)$ and hence $|\Gamma\backslash(\mathbb{Q} \cup \{\infty\})| \leq [\mathrm{SL}_2(\mathbb{Z})\colon \Gamma] < \infty$. ■

The topology on $\mathcal{H}_\infty$ has the same basis as the topology of $\mathcal{H}$ along with the sets

$$\alpha(\mathcal{N}_M \cup \{\infty\}) \quad \text{for} \quad \alpha \in \mathrm{SL}_2(\mathbb{Z})$$

where

$$\mathcal{N}_M = \{\tau \in \mathcal{H} : \mathrm{Im}(\tau) > M\} \quad \text{for some positive} \quad M \in \mathbb{R}. \tag{1.10}$$

Let $\pi\colon \mathcal{H}_\infty \to X(\Gamma)$ be the projection $\tau \mapsto \Gamma\tau$. The modular curve $X(\Gamma)$ is endowed with the quotient topology. With respect to this topology, π is continuous and open. Additionally, $X(\Gamma)$ has the following topological properties.

Proposition 1.32. [13, Proposition 2.4.2] Let Γ be a congruence subgroup. The modular curve $X(\Gamma)$ is Hausdorff, second countable, connected and compact. ⬣

Proposition 1.33. [13, Section 2.4] Let Γ be a congruence subgroup. The modular curve $X(\Gamma)$ is a compact Riemann surface. ⬣

To prove this, we require an analogue of the period for the cusps of $X(\Gamma)$. This is required for the definition of the charts around the cusps. A swift computation shows that the stabilizer $\mathrm{SL}_2(\mathbb{Z})_\infty$ of ∞ is of the form

$$\mathrm{SL}_2(\mathbb{Z})_\infty = \{\pm I\} \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle. \tag{1.11}$$

Let $s \in \mathbb{Q} \cup \{\infty\}$, then by Lemma 1.31 there is some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha(\infty) = s$. The group $\{\pm I\}(\alpha\Gamma\alpha^{-1})_\infty$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})_\infty$ and using (1.11) has the form

$$\{\pm I\}(\alpha\Gamma\alpha^{-1})_\infty = \{\pm I\} \left\langle \begin{pmatrix} 1 & h_s \\ 0 & 1 \end{pmatrix} \right\rangle \tag{1.12}$$

for some integer $h_s \geq 1$. Such an integer h_s exists since there is some positive $N \in \mathbb{Z}$ such that

$$\Gamma(N) \leq \Gamma \quad \Leftrightarrow \quad \Gamma(N)_\infty \leq \{\pm I\}(\alpha\Gamma\alpha^{-1})_\infty$$

where we used that $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$. From the above inclusion we have that $(1, N; 0, 1) \in \{\pm I\}(\alpha\Gamma\alpha^{-1})_\infty$ which shows existence of h_s and that $h_s \leq N$. Define the *width* of s (for Γ) to be the positive integer h_s. Combining (1.11) and (1.12) we see that the set of cosets has representatives

$$\mathrm{SL}_2(\mathbb{Z})_\infty/(\{\pm I\}\alpha\Gamma\alpha^{-1})_\infty = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\{\pm I\}\alpha\Gamma\alpha^{-1}), \dots, \begin{pmatrix} 1 & h_s - 1 \\ 0 & 1 \end{pmatrix} (\{\pm I\}\alpha\Gamma\alpha^{-1}) \right\}.$$

This gives the formula $h_s = |\,\mathrm{SL}_2(\mathbb{Z})_\infty/(\{\pm I\}\alpha\Gamma\alpha^{-1})_\infty|$. From this formula, we see that h_s is independent of the choice of α as $\pm\alpha\gamma\alpha^{-1} \mapsto \pm\gamma$ is an isomorphism $\{\pm I\}(\alpha\Gamma\alpha^{-1})_\infty \to \{\pm I\}\Gamma_s$. It follows that

$$h_s = |\,\mathrm{SL}_2(\mathbb{Z})_\infty/(\{\pm I\}\alpha\Gamma\alpha^{-1})_\infty| = |\,\mathrm{SL}_2(\mathbb{Z})_\infty/(\{\pm I\}\Gamma_s)|.$$

This shows that h_s is independent of the choice of α. Let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. The width $h_{\gamma s}$ of γs for $\alpha\Gamma\alpha^{-1}$ is

$$h_{\gamma s} = |\,\mathrm{SL}_2(\mathbb{Z})_\infty/(\{\pm I\}\beta\alpha\Gamma\alpha^{-1}\beta^{-1})_\infty| = |\,\mathrm{SL}_2(\mathbb{Z})_\infty/(\{\pm I\}(\beta\alpha)\Gamma(\beta\alpha)^{-1})_\infty|$$

where $\beta \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\beta\gamma s = \infty$. The expression on the right is equal to the period h_s for Γ. In particular, if $\alpha \in \Gamma$, then $\alpha\Gamma\alpha^{-1} = \Gamma$ and $h_s = h_{\gamma s}$. This shows that the width is well-defined on $X(\Gamma)$.

Proof of Proposition 1.33. By Proposition 1.32, all that remains to show is that around every point in $X(\Gamma)$ there is a chart. For points $\pi(\tau) \in Y(\Gamma) \subset X(\Gamma)$ we use the charts as in Proposition 1.29. The points we have to account for are the cusps of $X(\Gamma)$.

Let $s \in \mathbb{Q} \cup \{\infty\}$. By Lemma 1.31 there is some $\delta \in \mathrm{SL}_2(\mathbb{Z})$ such that $\delta(s) = \infty$. Let $U = \delta^{-1}(\mathcal{N}_2 \cup \{\infty\})$ be an open neighborhood of s where $\mathcal{N}_2$ is as in (1.10). As before, define

$$\begin{aligned} \psi\colon U &\to \mathbb{C} \\ \nu &\mapsto \rho\delta(\nu) \end{aligned} \tag{1.13}$$

where we interpret δ as a holomorphic bijection $\mathcal{H}_\infty \to \mathcal{H}_\infty$ and

$$\begin{aligned} \rho\colon \mathcal{N}_2 \cup \{\infty\} &\to \mathbb{C} \\ z &\mapsto \begin{cases} 0 & \text{if } z = \infty \\ e^{2\pi i z/h_s} & \text{otherwise} \end{cases} \end{aligned}$$

where h_s is the width of s. Points in U which are Γ equivalent have the same image under ψ. To see this, let $\tau_1, \tau_2 \in U$ such that $\tau_1 = \gamma\tau_2$ for some $\gamma \in \Gamma$. By applying δ on both sides we obtain

$$\delta(\tau_1) = \delta(\gamma\tau_2) \Leftrightarrow \delta(\tau_1) = \delta\gamma\delta^{-1}(\delta(\tau_2)). \tag{1.14}$$

Both $\delta(\tau_1)$ and $\delta(\tau_2)$ are elements in $\mathcal{N}_2 \cup \{\infty\}$ and hence have imaginary part larger than 2 or are both equal to ∞. A computation verifies that this implies that $\delta\gamma\delta^{-1} = (1, n; 0, 1)$ for some $n \in \mathbb{Z}$. Then $\delta\gamma\delta^{-1}$ fixes ∞ and

$$\delta\gamma\delta^{-1} \in (\delta\Gamma\delta^{-1})_\infty \subset \{\pm I\}\left\langle \begin{pmatrix} 1 & h_s \\ 0 & 1 \end{pmatrix} \right\rangle.$$

The equation on the right of (1.14) is then

$$\delta(\tau_1) = \delta(\tau_2) + mh_s \quad \text{for some} \quad m \in \mathbb{Z}.$$

Apply ρ on both sides to obtain

$$\psi(\tau_1) = e^{2\pi i\delta(\tau_1)/h_s} = e^{2\pi i(\delta(\tau_2)+mh_s)/h_s} = e^{2\pi i\delta(\tau_2)/h_s} = \psi(\tau_2).$$

By reversing the argument above we obtain that $\pi(\tau_1) = \pi(\tau_2)$ if and only if $\psi(\tau_1) = \psi(\tau_2)$. Set $V = \psi(U)$, then V is open by the open mapping theorem. Let $\varphi\colon \pi(U) \to V$ be the unique map such that

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "\pi"'] & V \\ \pi(U) \arrow[ur, "\varphi"'] & \end{tikzcd} \tag{1.15}$$

commutes. The map φ is well-defined and injective by the equivalence obtained above. Since π and ψ are surjective, open and continuous, so is φ. This shows that φ satisfies Definition 1.1 and defines a chart around the cusps $\Gamma s \in \Gamma\backslash(\mathbb{Q} \cup \{\infty\})$. Section 2.4 of [13] computes the transition functions of the charts around the cusps and shows that these are compatible. It follows that $X(\Gamma)$ is a compact Riemann surface. ■

2 Meromorphic differentials and the genus

The coordinate transformation $(x, y) \mapsto x + iy$ is a smooth map from an open subset of $\mathbb{R}^2$ to an open subset of $\mathbb{C}$ when viewed as real manifolds. It follows that a Riemann surface X is a 2−dimensional real manifold. It turns out that X is also orientable as a 2−dimensional real manifold, this follows from the fact that the transition functions of X satisfy the Cauchy-Riemann equations. The compact oriented 2−dimensional smooth manifolds are classified in the sense that every such manifold is isomorphic (as real manifolds) to a sphere with $g \in \mathbb{Z}_{\geq 0}$ handles attached to it [4, Section 7]. By combining these facts, we obtain the following.

Proposition 2.1. [28, Proposition 1.23] Every Riemann surface is an orientable path-connected 2−dimensional smooth real manifold. Every compact Riemann surface is isomorphic (as real manifolds) to a sphere with g handles attached to it for some unique integer $g \geq 0$. ⬢

The first homology group is the set of loops in X with zero boundary modulo the boundaries of triangles in X (see [4, Section 8]). The homology group of a sphere with g handles is isomorphic to $\mathbb{Z}^{2g}$. Combining this fact with Proposition 2.1 we obtain the following definition.

Definition 2.2. Let X be a Riemann surface. The *genus* g of X is defined to be the positive integer such that

$$H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

where $H_1(X, \mathbb{Z})$ is the first homology group of X. ▲

In this section we introduce meromorphic functions and meromorphic differentials. In particular, we investigate the meromorphic and holomorphic differentials on $X(\Gamma)$. We relate the latter to the genus of $X(\Gamma)$ using the Riemann-Roch theorem. Studying meromorphic differentials on $X(\Gamma)$ gives rise to meromorphic functions on $\mathcal{H}$ which preserve a certain symmetry of Γ. These functions turn out to be classified by meromorphic differentials on $X(\Gamma)$. This allows us to relate them to the genus of $X(\Gamma)$. Finally, we give explicit formulas for calculating the genus of a specific type of congruence subgroup. We use [28, Section II] for most definitions relating to Riemann surfaces. In the rest of this section we follow the ideas of [13, Section 3].

2.1 Meromorphic functions

Meromorphic functions are objects of great interest when studying residue theory. On Riemann surfaces, the notion of a meromorphic function generalizes this concept using charts. In this section meromorphic functions are defined using [28, Chapter II] and are related to holomorphic functions on the Riemann sphere.

Definition 2.3. [28, Definition II.1.13] Let X be a Riemann surface. Let $U \subset X$ be open, $x \in U$ and suppose $f : U \setminus \{x\} \to \mathbb{C}$ is holomorphic. We say that

1. f has a *pole* at x if and only if there is a chart $\varphi : U \to V$ such that $f\varphi^{-1} : \varphi(U) \to V$ has a pole at $\varphi(x)$;
2. f has a *removable singularity* if and only if there is a chart $\varphi : U \to V$ such that $f\varphi^{-1} : \varphi(U) \to V$ has a removable singularity at $\varphi(x)$;
3. f has an *essential singularity* if and only if there is a chart $\varphi : U \to V$ such that $f\varphi^{-1} : \varphi(U) \to V$ has an essential singularity at $\varphi(x)$. ▲

As in Section 1.1, this definition is independent of the choice of chart.

Definition 2.4. [28, Definition II.1.15] A function $f : X \to \mathbb{C}$ is *meromorphic* at $x \in X$ if f is holomorphic at x, has a removable singularity at x, or has a pole at x. We say that f is *meromorphic* if it is meromorphic at every point $x \in X$. ▲

Example 2.5. Let Γ be a congruence subgroup and $f : X(\Gamma) \to \mathbb{C}$ be meromorphic. Let $\varphi : \pi(U) \to V$ be a chart of $X(\Gamma)$. Then $\varphi\pi = \psi$ where ψ is as in (1.7) or (1.13). By definition, $f\varphi^{-1} : V \to \mathbb{C}$ is a meromorphic function of open subsets of the complex plane. Elements in V take the form $\psi(\tau)$. Then $f\varphi^{-1}(\psi(\tau)) = f(\pi(\tau))$. This means that $f\varphi^{-1}$ is Γ invariant for every chart φ of $X(\Gamma)$. The fact that f is meromorphic in particular implies that f is meromorphic at every $\Gamma s \in \pi(\mathbb{Q} \cup \{\infty\})$. It follows that $f\alpha\varphi^{-1}$ is meromorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ (see Definition 2.20). We obtain a one-to-one correspondence between meromorphic functions on $X(\Gamma)$ and meromorphic functions $f : \mathcal{H} \to \mathbb{C}$ which are meromorphic at the cusps and satisfy $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma$. ♦

Proposition 2.6. [28, Example II.1.17] Let X be a Riemann surface. Let $f, g : X \to \mathbb{C}$ be meromorphic functions. Then $f \pm g$ and $f \cdot g$ are meromorphic on X. If g is not identically 0 on X, then f/g is meromorphic on X. ⬣

From Proposition 2.6, we see that the set of meromorphic functions on X form a field with usual pointwise multiplication and addition which we denote by $\mathbb{C}(X)$. For a meromorphic function $f\colon X \to \mathbb{C}$ and a chart $\varphi\colon U \to V$ of X, the function $f\varphi^{-1}\colon \varphi(U) \to V$ is a meromorphic function of open subsets of $\mathbb{C}$. It has the Laurent series around $z_0 \in \varphi(U)$

$$f\varphi^{-1}(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n. \tag{2.1}$$

for some $m \in \mathbb{Z}$.

Definition 2.7. [28, Definition II.1.27] Let X be a Riemann surface. Suppose $f\colon X \to \mathbb{C}$ is meromorphic. Let $\varphi : U \to V$ be a chart of X. The *order* of f at a point $x \in U$ is the smallest m such that $a_n = 0$ for all $n < m$ in (2.1). The order of f at x is denoted by $\mathrm{ord}_x(f)$. ▲

Due to compatibility, the order of a meromorphic function at a point is independent of the chart chosen [28, Section II.1].

Proposition 2.8. [28, Lemma II.1.28] Let X be a Riemann surface. Suppose $f\colon X \to \mathbb{C}$ is meromorphic at $x \in X$. Then f

1. is holomorphic at x if and only if $\mathrm{ord}_x(f) \geq 0$;
2. has a zero at x if and only if $\mathrm{ord}_x(f) > 0$;
3. has a pole at x if and only if $\mathrm{ord}_x(f) < 0$. ⬣

Proposition 2.9. [28, Lemma II.1.29] Let X be a Riemann surface and $f, g\colon X \to \mathbb{C}$ meromorphic at $x \in X$. Then

1. $\mathrm{ord}_x(fg) = \mathrm{ord}_x(f) + \mathrm{ord}_x(g)$
2. $\mathrm{ord}_x(f \pm g) \geq \min\{\mathrm{ord}_x(f), \mathrm{ord}_x(g)\}$ with equality if $\mathrm{ord}_x(f)$ and $\mathrm{ord}_x(g)$ are distinct. ⬣

Suppose $f\colon X \to \mathbb{C}$ is a meromorphic map on a Riemann surface X. At the poles, f tends to ∞. The Riemann sphere allows us to make this precise by defining the holomorphic map $\Psi\colon X \to \mathbb{C}_\infty$ as

$$\Psi(x) = \begin{cases} \infty & \text{if } x \text{ is a pole of } f \\ f(x) & \text{otherwise.} \end{cases}$$

Via this construction we obtain a bijection between the meromorphic functions $f\colon X \to \mathbb{C}$ and holomorphic maps $\Psi\colon X \to \mathbb{C}_\infty$ which are not identically ∞ [28, Proposition II.3.13].

2.2 Meromorphic differentials

Meromorphic differentials are objects defined on a Riemann surfaces. As with most objects on Riemann surfaces we first define these objects on open subsets of the complex plane.

Definition 2.10. [13, Section 3.3] Let $V \subset \mathbb{C}$ be open. A *meromorphic differential* on V is an object of the form $f(z)dz$ where $f\colon V \to \mathbb{C}$ is meromorphic and z is a coordinate on V. The set of meromorphic differentials on V is denoted by $\Omega(V)$. ▲

For all open $V \subset \mathbb{C}$ the set $\Omega(V)$ is a $\mathbb{C}-$vector space with vector addition and scalar multiplication

$$\begin{aligned} f(z)dz + g(z)dz &= (f(z) + g(z))dz \\ \alpha(f(z)dz) &= \alpha f(z)dz. \end{aligned} \tag{2.2}$$

where $f(z)dz, g(z)dz \in \Omega(V)$ and $\alpha \in \mathbb{C}$.

Definition 2.11. [13, Section 3.3] Let V_1 and V_2 be open subsets of $\mathbb{C}$ with coordinates z_1 and z_2, respectively. Let $\Psi\colon V_1 \to V_2$ be holomorphic. The *pull back* of Ψ is the map $\Psi^*\colon \Omega(V_2) \to \Omega(V_1)$ defined by

$$\Psi^*(f(z_2)dz_2) = f(\Psi(z_1))\Psi'(z_1)dz_1$$

where $\Psi'\colon V_1 \to V_2$ denotes the derivative of Ψ. ▲

When d is interpreted as the exterior derivative, the pull back is obtained by plugging in the coordinates $z_2 = \Psi(z_1)$ as

$$f(\Psi(z_1))d(\Psi(z_1)) = f(\Psi(z_1))\Psi'(z_1)dz_1.$$

The pull back is linear and satisfies the following properties.

Proposition 2.12. [13, Section 3.3] Suppose $V, V_1, V_2, V_3 \subset \mathbb{C}$ are open, $\Psi_1\colon V_1 \to V_2$ and $\Psi_2\colon V_2 \to V_3$ holomorphic maps. Then

1. the pull back $(\Psi_2\Psi_1)^*\colon \Omega(V_3) \to \Omega(V_1)$ is equal to $\Psi_1^*\Psi_2^*$;
2. the identity map $\mathrm{id}_V\colon V \to V$ has pull back $(\mathrm{id}_V)^* = \mathrm{id}_{\Omega(V)}$;
3. if $\Psi_1\colon V_1 \to V_2$ is surjective then $\Psi^*\colon \Omega(V_2) \to \Omega(V_1)$ is injective. ⬣

If $\Psi\colon V_1 \to V_2$ is an isomorphism, then by combining Proposition 2.12.1 and 2.12.2 we obtain that Ψ^* is a isomorphism of vector spaces with inverse $(\Psi^*)^{-1} = (\Psi^{-1})^*$.

Definition 2.13. [13, Section 3.3] Let X be a Riemann surface with charts $\{\varphi_j\colon U_j \to V_j : j \in J\}$ for some indexing set J. A *meromorphic differential on* X is an object of the form

$$(\omega_j)_{j\in J} \in \prod_{j\in J} \Omega(V_j)$$

such that for every transition function $\varphi_j\varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ with $i, j \in J$, we have

$$(\varphi_j\varphi_i^{-1})^*(\omega_j|_{\varphi_j(U_i\cap U_j)}) = \omega_i|_{\varphi_i(U_i\cap U_j)}. \tag{2.3}$$

A meromorphic differential is *compatible* whenever it satisfies (2.3) for every transition function. Denote the set of meromorphic differentials on X by $\Omega(X)$. ▲

Compatibility ensures that this definition is independent of the choice of chart. A meromorphic differential on a Riemann surface X is a patchwork of meromorphic differentials on the local coordinates of X. The compatibility requirement ensures that the meromorphic differentials agree on the overlapping coordinates. The set $\Omega(X)$ forms a $\mathbb{C}-$vector space with addition and scalar multiplication as in (2.2). Additionally, $\Omega(X)$ forms a $\mathbb{C}(X)-$vector space via pointwise addition and scalar multiplication as follows. Let $\varphi_j\colon U_j \to V_j$ be a chart of X. If $\omega \in \Omega(X)$ has local expression $\omega_j = g_j(z_j)dz_j$, then for $f \in \mathbb{C}(X)$, the scalar multiple $f\omega$ has local expression

$$(f\omega)_j = f(\varphi_j^{-1}(z_j))g_j(z_j)dz_j.$$

This is well-defined since $f\omega$ is compatible. To see this, let $\varphi_i\colon U_i \to V_i$ be another chart of X. Then

$$(\varphi_j\varphi_i^{-1})^*(f\omega)_j = f(\varphi_j^{-1}\varphi_j\varphi_i^{-1}(z_i))g_j(\varphi_j\varphi_i^{-1}(z_i))(\varphi_j\varphi_i^{-1})'(z_i)dz_i = f(\varphi_i^{-1}(z_i))(\varphi_i\varphi_j^{-1})^*(\omega_j) = (f\omega)_i$$

where we omitted the domain restrictions to ease notation. It follows that $f\omega$ is compatible whenever ω is compatible. It turns out that the structure of $\Omega(X)$ as a $\mathbb{C}(X)-$vector space is quite simple.

Lemma 2.14. [28, Lemma V.1.12] Let X be a Riemann surface. The $\mathbb{C}(X)-$vector space of meromorphic differentials $\Omega(X)$ has dimension one. ⬣

Proof. Let ω_1 and ω_2 be elements of $\Omega(X)$. We show that there exists a function $f \in \mathbb{C}(X)$ such that $f\omega_1 = \omega_2$. It then follows that $\Omega(X) = \mathrm{span}\{\omega_1\}$ and the statement follows. Let $\varphi_i\colon U_i \to V_i$ be a chart of X. Then, locally on V_i,

$$(\omega_1)_i = g_1(z_i)dz_i \quad \text{and} \quad (\omega_2)_i = g_2(z_i)dz_i$$

for meromorphic functions g_1 and g_2. Define f locally on U_i as

$$f(x) = \frac{g_2(\varphi_i(x))}{g_1(\varphi_i(x))} = \frac{g_2(z_i)}{g_1(z_i)}.$$

Then

$$(f\omega_1)_i = \frac{g_2(z_i)}{g_1(z_i)} g_1(z_i)dz_i = g_2(z_i)dz_i = (\omega_2)_i.$$

To see that f can be extended to all of X, let $\varphi_j \colon U_j \to V_j$ be another chart such that $U_i \cap U_j \neq \varnothing$. Write

$$(\omega_1)_j = h_1(z_j)dz_j \quad \text{and} \quad (\omega_2)_j = h_2(z_j)dz_j.$$

By compatibility of ω_1 and ω_2, we have

$$(\varphi_i\varphi_j^{-1})^*((\omega_k)_j|_{\varphi_j(U_i\cap U_j)}) = (\omega_k)_j|_{\varphi_j(U_i\cap U_j)} = h_k(z_j)dz_j|_{\varphi_i(U_i\cap U_j)} \tag{2.4}$$

for $k = 1, 2$. On the other hand,

$$(\varphi_i\varphi_j^{-1})^*((\omega_k)_j|_{\varphi_j(U_i\cap U_j)}) = g_k(\varphi_i\varphi_j^{-1}(z_j))(\varphi_i\varphi_j^{-1})'(z_j)dz_j|_{\varphi_i(U_i\cap U_j)}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$g_k(\varphi_i\varphi_j^{-1}(z_j)) = \frac{h_k(z_j)}{(\varphi_i\varphi_j^{-1})'(z_j)} \quad \text{for all} \quad z_j \in \varphi_i(U_i \cap U_j). \tag{2.6}$$

This is well-defined since $(\varphi_i\varphi_j^{-1})' \neq 0$ by [28, Lemma I.1.7]. Then for $x \in U_i \cap U_j$,

$$\frac{g_2(\varphi_i(x))}{g_1(\varphi_i(x))} = \frac{g_2(\varphi_i\varphi_j^{-1}\varphi_j(x))}{g_1(\varphi_i\varphi_j^{-1}\varphi_j(x))} = \frac{g_2(\varphi_i\varphi_j^{-1}(z_j))}{g_1(\varphi_i\varphi_j^{-1}(z_j))} = \frac{h_2(z_j)/(\varphi_i\varphi_j^{-1})'(z_j)}{h_1(z_j)/(\varphi_i\varphi_j^{-1})'(z_j)} \overset{(2.6)}{=} \frac{h_2(\varphi_j(x))}{h_1(\varphi_j(x))}$$

It follows that defining $f(x) = h(\varphi_j(x))/h(\varphi_j(x))$ on U_j yields a well-defined meromorphic function on $U_j \cup U_i$. Doing this on every chart, we obtain a meromorphic function $f \colon X \to \mathbb{C}$ such that $f\omega_1 = \omega_2$. ■

Example 2.15. Let X be a complex torus $\mathbb{C}/\Lambda$ as in Section 1.2. Denote the local inverses of the projection $\mathbb{C} \to \mathbb{C}/\Lambda$ by $\varphi_j \colon U_j \to V_j$ where j is in some indexing set J. The transition maps are of the form $\varphi_j\varphi_i^{-1}(z) = z + \omega$ for some $\omega \in \Lambda$. For $j \in J$, denote the coordinates on V_j by z_j. Then $(dz_j)_{j\in J}$ is a meromorphic differential since

$$(\varphi_j\varphi_i^{-1})^*(dz_j|_{\varphi_j(U_i\cap U_j)}) = (\varphi_j\varphi_i^{-1})'(z)dz_i = dz_i|_{\varphi_i(U_j\cap U_i)}.$$

Since dz_j pulls back to dz_i for all $i, j \in J$. We can write dz as a global expression for $(dz_j)_{j\in J}$. ♦

Definition 2.16. [13, Section 6.2] Let X and Y be Riemann surfaces and $\Psi \colon X \to Y$ a holomorphic map. The *pull back of* Ψ is the map $\Psi^* \colon \Omega(Y) \to \Omega(X)$ mapping $\omega \mapsto \Psi^*\omega$. Here, $\Psi^*\omega$ is locally defined as

$$(\Psi^*\omega)_j = (\tilde{\varphi}_j\Psi\varphi_j^{-1})^*(\omega_j) \in \Omega(V_j) \quad \text{for} \quad \omega_j \in \Omega(\tilde{V}_j)$$

where $\varphi_j \colon U_j \to V_j$ and $\tilde{\varphi}_j \colon \tilde{U}_j \to \tilde{V}_j$ are charts of X and Y, respectively, such that $\Psi(U_j) = \tilde{U}_j$. ▲

With notation as in Definition 2.16, the object ω_j is an element in $\Omega(\tilde{V}_j)$ so it has the form $\omega_j = g(\tilde{z}_j)d\tilde{z}_j$. The local pull back is then calculated as

$$(\Psi^*\omega)_j = (\tilde{\varphi}_j\Psi\varphi_j^{-1})^*(\omega_j) = g(\tilde{\varphi}_j\Psi\varphi_j^{-1}(z_j))(\tilde{\varphi}_j\Psi\varphi_j^{-1})'(z_j)dz_j.$$

Some wariness is required for Definition 2.16 as we claim that Ψ^* maps to $\Omega(X)$ which requires $\Psi^*\omega$ to be compatible for $\omega \in \Omega(Y)$.

Proposition 2.17. Let X and Y be Riemann surfaces and $\Psi \colon X \to Y$. The pull back $\Psi \colon \Omega(Y) \to \Omega(X)$ is a well-defined linear map. ⬢

Proof. Let φ_i and φ_j be charts of X, $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ be charts of Y and $\omega \in \Omega(X)$. To ease notation, define $W_i = \varphi_i(U_i \cap U_j)$, define W_j, $\tilde{W}_i$ and $\tilde{W}_j$ in a similar fashion. Then

$$\begin{aligned}
(\varphi_j\varphi_i^{-1})^*((\Psi^*\omega)_j|_{W_j}) &= (\varphi_j\varphi_i^{-1})^*(\tilde{\varphi}_j\Psi\varphi_j^{-1})^*(\omega_j|_{\tilde{W}_j}) = (\varphi_j\varphi_i^{-1})^*(\tilde{\varphi}_j\tilde{\varphi}_i^{-1}\tilde{\varphi}_i\Psi\varphi_j^{-1})^*(\omega_j|_{\tilde{W}_j}) \\
&\overset{2.12.1}{=} (\varphi_j\varphi_i^{-1})^*(\tilde{\varphi}_i\Psi\varphi_j^{-1})^*(\tilde{\varphi}_j\tilde{\varphi}_i^{-1})^*(\omega_j|_{\tilde{W}_j}) = (\tilde{\varphi}_i\Psi\varphi_j^{-1}\varphi_j\varphi_i^{-1})^*(\omega_i|_{\tilde{W}_i}) \\
&= (\tilde{\varphi}_i\Psi\varphi_i^{-1})^*(\omega_i|_{\tilde{W}_i}) = (\Psi^*\omega)_i|_{W_i}
\end{aligned}$$

where the fourth equality follows from compatibility of ω. Linearity of Ψ^* follows from linearity of $(\tilde{\varphi}_i\Psi\varphi_i^{-1})^*$ ■

Properties similar to 1 and 2 of Proposition 2.12 hold for the pull back Ψ^* in Definition 2.16.

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Denote the charts of $X(\Gamma)$ by $\varphi_i\colon \pi(U_i) \to V_i$ with $\varphi_i\pi = \psi_i$ with $i \in I$ for some index set I. Here, ψ_i is as in (1.7) or (1.13) and φ_i is as in (1.9) or (1.15). Remove the cusps from the U_i to obtain $U_i' = U_i \cap \mathcal{H}$ and define $V_i' = \psi_i(U_i')$. Then $\mathcal{H}$ is a Riemann surface with charts $\mathrm{id}_i\colon U_i' \to U_i'$. With these charts on $\mathcal{H}$, the restriction of the projection $\pi\colon \mathcal{H} \to X(\Gamma)$ is a holomorphic map of Riemann surfaces. Let $\omega \in \Omega(X(\Gamma))$, then $\varphi_i|_{\pi(U_i')}\colon \pi(U_i') \to V_i'$ is a chart and the local pull back of π is

$$(\pi^*\omega)_i|_{U_i'} = (\varphi_i\pi\mathrm{id}^{-1})^*(\omega_i|_{V_i'}) = \psi_i^*(\omega_j|_{V_j'}). \tag{2.7}$$

This construction actually gives a global meromorphic differential. To see this, we have

$$\varphi_j^{-1}\psi_j|_{U_j'\cap U_i'} = \pi|_{U_j'\cap U_i'} = \varphi_i^{-1}\psi_i|_{U_j'\cap U_i'} \quad \Leftrightarrow \quad \varphi_i\varphi_j^{-1}\psi_j|_{U_j'\cap U_i'} = \psi_i|_{U_j'\cap U_i'}.$$

Define the complex function $\sigma_{i,j}$ as

$$\sigma_{i,j} = \varphi_i\varphi_j^{-1}|_{\varphi_j(\pi(U_j')\cap\pi(U_i'))}. \tag{2.8}$$

By pulling back (2.7), we obtain $\psi_i^* = \psi_j^*\sigma_{i,j}^*$. Then

$$(\pi^*\omega)_i|_{U_i'\cap U_j'} = \psi_i^*(\omega_i|_{\psi_i(U_i'\cap U_j')}) = \psi_j^*\sigma_{i,j}^*(\omega_i|_{\psi_i(U_i'\cap U_j')}) = \psi_j^*(\omega_j|_{\psi_j(U_i'\cap U_j')}) = (\pi^*\omega)_j|_{U_i'\cap U_j'}$$

where we used compatibility of ω in the third equality. This shows that $\pi^*\omega$ agrees on the overlapping charts of $\mathcal{H}$ and $\pi^*\omega$ is of the form $\pi^*\omega = f(\tau)d\tau$ for some meromorphic $f\colon \mathcal{H} \to \mathbb{C}$. Let $\gamma = (a,b;c,d) \in \Gamma$, interpret γ as a holomorphic map $\mathcal{H} \to \mathcal{H}$. Pull back the equation $\pi\gamma = \pi$ to obtain $\gamma^*\pi^* = \pi^*$. Then

$$f(\tau)d\tau = \pi^*(\omega) = \gamma^*\pi^*(\omega) = \gamma^*(f(\tau)d\tau) = f(\gamma(\tau))\gamma'(\tau)d\tau = (c\tau+d)^{-2}f(\gamma(\tau))d\tau.$$

We see that $f\colon \mathcal{H} \to \mathbb{C}$ satisfies $f(\tau) = (c\tau+d)^{-2}f(\gamma\tau)$.

Definition 2.18. [13, Definition 1.1.1] Let Γ be a congruence subgroup and let $f\colon \mathcal{H} \to \mathbb{C}$ be meromorphic. Let $(a,b;c,d) \in \mathrm{SL}_2(\mathbb{Z})$. Let $k \in \mathbb{Z}$, the meromorphic function $f[\gamma]_k\colon \mathcal{H} \to \mathbb{C}$ is defined by

$$f[\gamma]_k(\tau) = (c\tau+d)^{-k}f(\gamma\tau).$$

The map f is *weakly modular of weight* k (with respect to Γ) if $f = f[\gamma]_k$ for all $\gamma \in \Gamma$. ▲

With this new terminology, we see that the function $f\colon \mathcal{H} \to \mathbb{C}$ such that $\pi^*\omega = f(\tau)d\tau$ is weakly modular of weight 2. Next, we study the behaviour of f around the cusps. To explore this, we require some more terminology.

Let $(1,h;0,1) \in \Gamma$ be such that $|h|$ is as small as possible and nonzero. A weakly modular meromorphic function $g\colon \mathcal{H} \to \mathbb{C}$ of weight k with respect to Γ satisfies

$$g(\tau+h) = g\left(\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\tau\right) = (0\tau+1)^k g(\tau) = g(\tau).$$

It follows that g is $h-$periodic. If there is some region $D = \{z \in \mathcal{H}\colon \operatorname{Im} z > R\}$ with $R \in \mathbb{R}_{>0}$ in which g has no poles, then g has Laurent series expression

$$g(z) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{2\pi i z n}{h}} \quad \text{where} \quad z \in D. \tag{2.9}$$

Note that such a region D does not necessarily exist. For example $1/\sin(\tau i)$ is meromorphic on $\mathcal{H}$ but a region D does not exist for this function.

Definition 2.19. [13, Section 3.2] Let Γ be a congruence subgroup. Let $g\colon \mathcal{H} \to \mathbb{C}$ be meromorphic and weakly modular of weight k with respect to Γ. Suppose that an expansion (2.9) exists for g. The map g is *meromorphic at* ∞ if there exists an $m \in \mathbb{Z}$ such that $a_n = 0$ for all $n \leq m$ with a_n as in (2.9). The *order of* g *at* ∞ is the smallest such m and is denoted by $\mathrm{ord}_\infty(g)$. If $\mathrm{ord}_\infty(g) \geq 0$ then g is *holomorphic at* ∞. ▲

Suppose that g is holomorphic at ∞. By taking the limit on both sides of (2.9) we obtain $\lim_{y\to\infty} g(iy) = a_0$. Verifying whether a meromorphic weakly modular function is holomorphic at ∞ amounts to checking if this limit exists because it does not exist for functions which are not holomorphic at ∞.

Let $\omega \in \Omega(X(\Gamma))$ and $f\colon \mathcal{H} \to \mathbb{C}$ such that $\pi^*\omega = f(\tau)d\tau$. Let $\alpha = (a,b;c,d) \in \mathrm{SL}_2(\mathbb{Z})$, then $f[\alpha]_2$ is a weakly modular and meromorphic map $\mathcal{H} \to \mathbb{C}$. Additionally, $f[\alpha]_2$ is meromorphic at ∞. To see this, set $s = \alpha(\infty) \in \mathbb{Q} \cup \{\infty\}$. Furthermore, let $\psi\colon U \to V$ be as in (1.13) write, $\varphi\pi = \psi$ for some chart φ around Γs as in (1.15). The map ψ has the form $\psi = \rho\alpha^{-1}$ where $\rho(z) = e^{2\pi i z/h}$ with h the width of s. Let τ, z and q denote coordinates on $U, \alpha^{-1}(U)$ and V respectively. On V, ω has the form $\omega|_V = g(q)dq$ for some meromorphic $g\colon \mathcal{H} \to \mathbb{C}$. By definition, ω is meromorphic at s. The point s is sent to $q = 0$ via ψ. We see that g is then meromorphic at $q = 0$ and since $q = e^{2\pi i z/h}$, g has a Laurent series expression as in (2.9) with finite starting index. It follows that g is meromorphic at ∞. From (2.7) we obtain

$$\psi^*(g(q)dq|_{V\setminus\{0\}}) = \psi^*(\omega|_{V\setminus\{0\}}) = (\pi^*\omega)|_{U\setminus\{s\}} = f(\tau)d\tau|_{U\setminus\{s\}}. \tag{2.10}$$

The left hand side of the above is

$$\psi^*(g(q)dq)) = g(\rho\alpha^{-1}(\tau))\rho'(\alpha^{-1}(\tau))(\alpha^{-1})'(\tau)d\tau = g(\rho\alpha^{-1}(\tau))\frac{2\pi i}{h}e^{2\pi i\alpha^{-1}(\tau)/h}(a-c\tau)^{-2}d\tau.$$

By (2.10) and the above, on $U \setminus \{s\}$ we have

$$f(\tau) = g(\rho\alpha^{-1}(\tau))\frac{2\pi i}{h}e^{2\pi i\alpha^{-1}(z)/h}(a-c\tau)^{-2}$$

and

$$f[\alpha]_2(z) = (cz+d)^{-2}f(\alpha(z)) = g(e^{2\pi i z/h})\frac{2\pi i}{h}e^{2\pi i z/h}.$$

The right hand side is meromorphic at ∞, therefore so is $f[\alpha]_2$. If we require g to be holomorphic on U then the right hand side of the above has Laurent series expression of the form (2.9) with starting index $n = 1$ so that $f[\alpha]_2(iy) \to 0$ as $y \to \infty$.

Definition 2.20. [13, Definition 3.2.1] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A function $f\colon \mathcal{H} \to \mathbb{C}$ is *meromorphic at* $s = \alpha(\infty) \in \mathbb{Q} \cup \{\infty\}$ when $f[\alpha]_2$ is meromorphic at ∞. We say that $f\colon \mathcal{H} \to \mathbb{C}$ is an *automorphic form of weight* 2 with respect to Γ if

1. f is meromorphic on $\mathcal{H}$;
2. f is weakly modular of weight 2 with respect to Γ;
3. f is meromorphic at every $s \in \mathbb{Q} \cup \{\infty\}$.

The set of automorphic forms of weight 2 with respect to Γ is denoted by $\mathcal{A}_2(\Gamma)$. ▲

Example 2.21. The modular $j-$invariant is the map $j\colon \mathcal{H} \to \mathbb{C}$ defined by

$$j(\tau) = 1728\frac{g_4(\tau)^3}{(2\pi)^{12}\eta(\tau)^{24}}$$

where $g_4(\tau) = 60G_4(\mathbb{Z}\tau \oplus \mathbb{Z})$ (see Section 4.2) and η is the Dedekind eta function (see Chapter 3). Both $\tau \mapsto \eta(\tau)^{24}$ and $\tau \mapsto g_4(\tau)^3$ are weakly modular of weight 12 with respect to $\mathrm{SL}_2(\mathbb{Z})$, see Section 4.2 and Chapter 3 for more details. Due to this fact, it follows that j is weakly modular of weight 0 with respect to $\mathrm{SL}_2(\mathbb{Z})$. In other words, j satisfies

$$j(\gamma\tau) = j(\tau) \quad \text{for all} \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.11}$$

The $j-$invariant is holomorphic on $\mathcal{H}$ since both η and g_4 are holomorphic on $\mathcal{H}$ and η is non-vanishing on $\mathcal{H}$ by Proposition 3.6. The $\eta-$function vanishes at ∞ and hence, j is meromorphic at ∞ and has a pole there (and hence also at every cusp by Lemma 1.31). The derivative j' also inherits these properties. By taking the derivative of (2.11), we obtain

$$j'(\gamma\tau)(c\tau+d)^{-2} = j'(\tau) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

It follows that $j' \in \mathcal{A}_2(\mathrm{SL}_2(\mathbb{Z}))$. In particular $j' \in \mathcal{A}_2(\Gamma)$ for every congruence subgroup Γ. ♦

The set $\mathcal{A}_2(\Gamma)$ is a vector space with pointwise addition and scalar multiplication. By the argument above, the functions $f\colon \mathcal{H} \to \mathbb{C}$ such that $\pi^*\omega = f(\tau)d\tau$ for some meromorphic differential $\omega \in \Omega(X(\Gamma))$ are elements of $\mathcal{A}_2(\Gamma)$. Conversely, it turns out that every automorphic form $f \in \mathcal{A}_2(\Gamma)$ comes from a meromorphic form $\omega \in \Omega(X(\Gamma))$.

Theorem 2.22. [13, Theorem 3.3.1] Let Γ be a congruence subgroup. The map $\varsigma\colon \omega \mapsto f$, where f is such that $\pi^*\omega = f(\tau)d\tau$, is an isomorphism of $\mathbb{C}-$vector spaces $\Omega(X(\Gamma)) \to \mathcal{A}_2(\Gamma)$. ⬣

We require a lemma which simplifies the proof of Theorem 2.22. The lemma states that the calculation that $\pi^*\omega$ has a global form can be turned around.

Lemma 2.23. [13, Section 3.3] Let Γ be a congruence subgroup. Let $\{\varphi_i\colon \pi(U_i) \to V_i : i \in I\}$ denote the charts of $X(\Gamma)$ for some indexing set I. Suppose we are given a collection of meromorphic differentials $(\omega_i)_{i\in I} \in \prod_{i\in I} \Omega(V_i)$. Remove the cusps of U_i to obtain U_i' and define $V_i' = \psi_i(U_i')$. If every $\omega_i|_{V_i'}$ satisfies $\psi_i^*(\omega_i|_{V_i'}) = f(\tau)d\tau|_{U_i'}$ for some $f \in \mathcal{A}_2(\Gamma)$ then $(\omega_i)_{i\in I}$ is compatible and hence $(\omega_i)_{i\in I} \in \Omega(X(\Gamma))$. ⬣

Proof. As before, we have $\psi_i^* = \psi_j^*\sigma_{i,j}^*$ with $\sigma_{i,j} = \varphi_i\varphi_j^{-1}|_{\varphi_j(\pi(U_j')\cap\pi(U_i'))}$. Then

$$\psi_j^*(\omega_j|_{\psi(U_j'\cap U_i')}) = f(\tau)d\tau|_{U_j'\cap U_i'} = \psi_i^*(\omega_i|_{\psi(U_j'\cap U_i')}) = \psi_j^*(\sigma_{i,j}^*(\omega_i|_{\psi(U_j'\cap U_i')}))$$

The map $\psi_j\colon U_j' \to V_j'$ is surjective. By Proposition 2.12.3, ψ_j^* is injective. Therefore, the above equality implies

$$\omega_j|_{\psi(U_j'\cap U_i')} = \sigma_{i,j}^*(\omega_i|_{\psi(U_j'\cap U_i')})$$

which shows compatibility. We conclude $(\omega_i)_{i\in I} \in \Omega(X(\Gamma))$. ∎

Proof of Theorem 2.22. Linearity of the map $\varsigma\colon \omega \mapsto f$ follows from linearity of π^*. It follows that ς is injective by using Proposition 2.12.3 and the fact that π is surjective.

For surjectivity of ς, let $f \in \mathcal{A}_2(\Gamma)$. Using the same notation as in Lemma 2.23, it suffices to construct a meromorphic differential $\omega_i \in \Omega(V_i')$ which pulls back under ψ_i to restrictions $f(\tau)d\tau|_{U_i'} \in \Omega(U_i')$. The maps $\psi_i\colon U_i \to V_i$ are of the form $\psi_i = \rho_i\delta_i$ where $\delta_i \in \mathrm{GL}_2(\mathbb{C})$. Denote coordinates of $U_i, \delta_i(U_i)$ and V_i by τ, z and q respectively. Define $\lambda_i \in \Omega(\delta_i(U_i))$ as

$$\lambda_i = (\delta_i^{-1})^*(f(\tau)d\tau|_{U_i'}) = f(\delta_i^{-1}(z))(\delta_i^{-1})'(z)dz =: \tilde{f}(z)dz.$$

Since $(\delta_i^{-1})^* = (\delta_i^*)^{-1}$ we have $\delta_i^*\lambda_i = f(\tau)d\tau|_{U_i'}$. The form λ_i is $\delta_i\Gamma\delta_i^{-1}$ invariant since the pull back of λ_i by $\delta_i\gamma\delta_i^{-1} \in \delta_i\Gamma\delta_i^{-1}$ is

$$(\delta_i^{-1})^*\gamma^*\delta_i^*\lambda_i = (\delta_i^{-1})^*\gamma^*(f(\tau)d\tau|_{U_i'}) = (\delta_i^{-1})^*(f(\tau)d\tau|_{U_i'}) = \lambda_i$$

where we used Γ invariance of $f(\tau)d\tau$ in the second equality.

If U_i does not contain a cusp, then δ_i takes some $\tau_i \in U_i$ to 0. By Proposition 1.26, the group $G_i := \{\pm I\}\delta_i\Gamma_{\tau_i}\delta_i^{-1}/\{\pm I\}$ is cyclic of order h_i where h_i is the period of τ_i. Then $\rho_i(z) = z^{h_i}$. An element that generates G_i, say, r acts as $r\colon z \mapsto \zeta_h z$ on $\mathcal{H}$ where ζ_h is a primitive h_i-th root of unity. Since λ_i is $\delta_i\Gamma\delta_i^{-1}$ invariant, it is also G_i invariant, hence

$$\tilde{f}(z)dz = \lambda_i = r^*\lambda_i = r^*(\tilde{f}(z)dz) = \tilde{f}(\zeta_h z)\zeta_h dz.$$

It follows that $z\tilde{f}(z) = \tilde{f}(\zeta_h z)\zeta_h z$. Therefore it is possible to write $z\tilde{f}(z) = g_i(z^{h_i})$ for some meromorphic $g_i\colon V_i \to \mathbb{C}$. Define the meromorphic differential

$$\omega_i = \frac{g_i(q)}{h_i q}dq \in \Omega(V_i).$$

Then

$$\rho_i^*\omega_i = \frac{g_i(\rho_i(z))}{h_i\rho_i(z)}\rho_i'(z)dz = \frac{g_i(z^{h_{\tau_i}})}{h_i z^{h_i}}h_i z^{h_i-1}dz = \frac{z\tilde{f}(z)}{z}dz = \lambda_i.$$

Then $\psi_i^*\omega_i = \delta_i^*\rho_i^*\omega_i = \delta_i^*\lambda_i = f(\tau)d\tau|_{U_i'}$. This shows that ω_i pulls back to $f(\tau)d\tau|_{U_i'}$ under ψ_i.

If U_i contains a cusp $s_i \in \mathbb{Q} \cup \{\infty\}$ then $\delta_i \in \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{C})$ such that δ_i sends s_i to ∞. And $\rho_i(z) = e^{(2\pi i z)/h_i}$ where h_i is the period of s_i. Similarly to the above, λ_i is $\delta_i \Gamma \delta_i^{-1}$ invariant and therefore also $\{\pm I\}(\delta_i \Gamma \delta_i^{-1})_\infty$ invariant. This subgroup has the form

$$\{\pm I\}(\delta_i \Gamma \delta_i^{-1})_\infty = \{\pm I\} \left\langle \begin{pmatrix} 1 & h_i \\ 0 & 1 \end{pmatrix} \right\rangle.$$

The generator $r = \pm(1, h_i; 0, 1)$ acts as $r \colon z \mapsto z + h_i$ on $\mathcal{H}_\infty$. By invariance of λ_i we obtain

$$\tilde{f}(z)dz = \lambda_i = r^*\lambda_i = r^*(\tilde{f}(z)dz) = \tilde{f}(z + h_i)dz.$$

It follows that $\tilde{f}(z)$ is h_i periodic. As above, we may write $\tilde{f}(z) = g_i(e^{(2\pi i z)/h_i})$ for some meromorphic $g_i \colon V_i \to \mathbb{C}$. Define the meromorphic differential

$$\omega_i = \frac{h_i g_i(q)}{2\pi i q} dq \in \Omega(V_i).$$

Then

$$\rho_i^* \omega_i = \frac{h_i g_i(\rho_i(z))}{2\pi i \rho_i(z)} \rho_i'(z) dq = \frac{h_i g_i(e^{(2\pi i z)/h_i})}{2\pi i} e^{-(2\pi i z)/h_i} \frac{2\pi i}{h_i} e^{(2\pi i z)/h_i} dz = \tilde{f}(z)dz = \lambda_i.$$

And hence ω_i pulls back to $f(\tau)d\tau|_{U_i'}$ under ψ_i. By Lemma 2.23, we conclude that ς is an isomorphism. ■

Thus far we have only discussed meromorphic differentials. The holomorphic differentials on $X(\Gamma)$ are worth studying as they give a lot of information about $X(\Gamma)$.

Definition 2.24. Let $V \subset \mathbb{C}$ be open with coordinate z. A *holomorphic differential* on V is a meromorphic differential $f(z)dz$ where $f \colon V \to \mathbb{C}$ is holomorphic. Denote the set of holomorphic differentials on V by $\Omega_{\mathrm{hol}}(V)$. Let X be a Riemann surface with charts $\{\varphi_j \colon U_j \to V_j : j \in J\}$ for some indexing set J. A *holomorphic differential on* X is an object of the form

$$(\omega_j)_{j \in J} \in \prod_{j \in J} \Omega_{\mathrm{hol}}(V_j)$$

such that $(\omega_j)_{j \in J}$ is compatible. Denote the set of holomorphic differentials on X by $\Omega_{\mathrm{hol}}(X)$. ▲

The set of holomorphic differentials $\Omega_{\mathrm{hol}}(X)$ on X is a linear subspace of the meromorphic differentials $\Omega(X)$ on X. Next, we investigate the image of the isomorphism ς in Theorem 2.22 when restricted to $\Omega_{\mathrm{hol}}(X(\Gamma))$.

Let $f \in \varsigma(\Omega_{\mathrm{hol}}(X(\Gamma)))$, and let $\tau_0 \in \mathcal{H}$. Using the same notation as in the proof of Theorem 2.22, there is some chart about τ_0, say, $\varphi_0 \colon \pi(U_0) \to V_0$ such that $\varphi_0 \pi = \psi_0 = \rho_0 \delta_0$ where $\delta_0(\tau) = (\tau - \tau_0)/(\tau - \overline{\tau}_0)$ as (1.7). By assumption, there is some $\omega_0 \in \Omega_{\mathrm{hol}}(V_0)$ such that $\psi_0^* \omega_0 = f(\tau)d\tau|_{U_0}$. In other words, $\varsigma^{-1}(f) = \omega_0$. From the proof of Theorem 2.22, the holomorphic differential ω_0 has the form

$$\omega_0 = \frac{g_0(q)}{hq} dq. \tag{2.12}$$

Where g_0 is some meromorphic and obtained as in the proof of Theorem 2.22 and h is the period of τ_0. Since ω_0 is holomorphic on V_0, it is holomorphic at $q = 0$. Therefore, by Proposition 2.8 $\mathrm{ord}_0(g_0) \geq 1$. Via the relation $g_0(z^h) = z\tilde{f}(z)$ we see that $\mathrm{ord}_0(\tilde{f}) \geq 0$. The map $\tilde{f}$ relates to f via $\delta_0^*(\tilde{f}(z)dz) = f(\tau)d\tau|_{U_0}$. Then

$$f(\tau) = \tilde{f}(\delta_0(z))\delta_0'(z) = 2\,\mathrm{Im}(\tau_0)\tilde{f}\Big(\frac{\tau - \tau_0}{\tau - \overline{\tau}_0}\Big)(\tau - \overline{\tau}_0)^{-2} \quad \text{for} \quad \tau \in U_0.$$

We see that $\mathrm{ord}_{\tau_0}(f) = \mathrm{ord}_0(\tilde{f}) \geq 0$ and by Proposition 2.8, f is holomorphic at τ_0. As τ_0 was taken to be arbitrary, f is holomorphic on $\mathcal{H}$.

Let $s \in \mathbb{Q} \cup \{\infty\}$ be a cusp and let $\varphi_0 \colon \pi(U_0) \to V_0$ be a chart around $\pi(s)$. Using the same notation as above, $\varsigma^{-1}(f) = \omega_0$ has the form

$$\omega_0 = \frac{h g_0(q)}{2\pi i q} dq \tag{2.13}$$

where g_0 is obtained as in the proof of Proposition 2.22. By assumption, ω_0 is a holomorphic differential on V_0. It follows that $\mathrm{ord}_0(g_0) \geq 1$ and since $\tilde{f}(z) = g_0(q)$ with $q = e^{2\pi i z/h}$, $\mathrm{ord}_\infty(\tilde{f}) = \mathrm{ord}_0(g_0) \geq 1$. It follows that $\tilde{f}$ is holomorphic at ∞ and vanishes there. Recall that δ_0 is such that $\delta_0(s) = \infty$, $\tilde{f}$ relates to f via $\tilde{f} = f[\delta_0^{-1}]$. As s is taken to be arbitrary we see that for $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha(\infty) = s$, the map $f[\alpha]_2$ is holomorphic at ∞ and vanishes there for every $s \in \mathbb{Q} \cup \{\infty\}$.

We see that the elements of $\varsigma(\Omega_{\mathrm{hol}}(X(\Gamma)))$ are elements of $\mathcal{A}_2(\Gamma)$ which satisfy additional properties.

Definition 2.25. [13, Definition 1.1.3] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. A function $f\colon \mathcal{H} \to \mathbb{C}$ is *holomorphic at* $s = \alpha(\infty) \in \mathbb{Q} \cup \{\infty\}$ when $f[\alpha]_2$ is holomorphic at ∞. We say that f is a *cusp form of weight 2* with respect to Γ if

1. f is holomorphic on $\mathcal{H}$;
2. f is weakly modular of weight 2 with respect to Γ;
3. f is holomorphic at every $s \in \mathbb{Q} \cup \{\infty\}$ and vanishes there.

Denote the set of cusp forms of weight 2 with respect to Γ as $\mathcal{S}_2(\Gamma)$. ▲

The set $\mathcal{S}_2(\Gamma)$ is a linear subspace of $\mathcal{A}_2(\Gamma)$.

Corollary 2.26. [13, Exercise 3.3.6] Let Γ be a congruence subgroup. The restriction map $\varsigma\colon \omega \mapsto f$, where $\pi^*\omega = f(\tau)d\tau$, is an isomorphism of vector spaces $\Omega_{\mathrm{hol}}(X(\Gamma)) \to \mathcal{S}_2(\Gamma)$. ⬢

Proof. The restriction $\varsigma|_{\Omega_{\mathrm{hol}}(X(\Gamma))}$ is injective and linear since ς is. It remains to show surjectivity. We do this by showing that $\varsigma(\Omega_{\mathrm{hol}}(X(\Gamma)) = \mathcal{S}_2(\Gamma)$. By the argument above, via the isomorphism ς of Theorem 2.22, the holomorphic differentials on $X(\Gamma)$ are mapped to a subspace of $\mathcal{S}_2(\Gamma)$. To obtain the reverse inclusion, let $f \in \mathcal{S}_2(\Gamma)$. Around $\tau_0 \in \mathcal{H}$ the meromorphic differential $\varsigma^{-1}(f)$ is locally of the form (2.12). We have that $\mathrm{ord}_{\tau_0}(f) \geq 0$ implies $\mathrm{ord}_0(g_0) \geq 1$. It follows that ω_0 is locally a holomorphic differential. Similarly, around the cusps $s \in \mathbb{Q} \cup \{\infty\}$, $\varsigma^{-1}(f)$ takes the form (2.13). The fact that f is holomorphic at every cusp and vanishes there ensures that $\mathrm{ord}_0(g_0) \geq 1$. This implies that $\varsigma^{-1}(f)$ is a holomorphic differential around the cusps. By combining these two facts, it follows that $\varsigma^{-1}(f)$ is a holomorphic differential. As f is taken to be arbitrary, we obtain the reverse inclusion and hence an equality of $\mathbb{C}-$vector spaces. ■

2.3 The Riemann-Roch theorem

The Riemann-Roch theorem relates the genus of a compact Riemann surface X to the meromorphic differentials on X. For some congruence group Γ, Riemann-Roch allows us to compute the dimension of the holomorphic differentials on $X(\Gamma)$ and, by Corollary 2.26, also the dimension of $\mathcal{S}_2(\Gamma)$. In this section we cover the preliminaries required for Riemann-Roch and we state the theorem and some consequences.

Definition 2.27. [13, Section 3.4] Let X be a compact Riemann surface. A *divisor* D on X is a formal sum

$$D = \sum_{x \in X} n_x x$$

where $n_x = 0$ for all but finite x. Denote the set of divisors on X by $\mathrm{Div}(X)$. The set of divisors is an abelian group with group law

$$\sum_{x \in X} n_x x + \sum_{x \in X} n'_x x = \sum_{x \in X} (n_x + n'_x) x.$$

If $D' = \sum_{x \in X} n'_x x \in \mathrm{Div}(X)$ then we write $D \geq D'$ if and only if $n_x \geq n'_x$ for all $x \in X$. The *degree* $\deg(D)$ of the divisor D is the sum $\deg(D) = \sum_{x \in X} n_x$. ▲

Definition 2.28. [13, Section 3.4] Let X be a compact Riemann surface and $f\colon X \to \mathbb{C}$ a nonzero meromorphic map. The *divisor of* f, denoted by $\mathrm{div}(f)$, is the element

$$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f) x \in \mathrm{Div}(X).$$

▲

Note that $\operatorname{ord}_x(f) = 0$ for all but finitely many $x \in X$ as X is compact. By Proposition 2.9.1, $\operatorname{div}\colon \mathbb{C}(X)^\times \to \operatorname{Div}(X)$ is a homomorphism of groups.

Definition 2.29. [13, Section 3.4] Let X be a compact Riemann surface and $D \in \operatorname{Div}(X)$. The *linear space* $L(D)$ *of* D is

$$L(D) = \{\tilde{f} \in \mathbb{C}(X) : \operatorname{div}(\tilde{f}) + D \geq 0\} \cup \{0\}.$$

▲

Let $f\colon X \to \mathbb{C}$ be meromorphic, $x \in X$ and $\alpha \in \mathbb{C}$. By the identity $\operatorname{ord}_x(\alpha f) = \operatorname{ord}_x(f)$ and by Proposition 2.9.2, $L(D)$ is a $\mathbb{C}-$vector space. Denote the dimension of $L(D)$ as a $\mathbb{C}-$vector space by $\ell(D)$.

Definition 2.30. [13, section 3.4] Let X be a compact Riemann surface and $\omega \in \Omega(X)$ nonzero. Let $x \in X$ and $\varphi\colon U \to V$ be a chart about x. Then $\omega|_V = f(z)dz$ for some meromorphic $f\colon U \to V$. The *order of* ω *at* x is the integer $\operatorname{ord}_{\varphi(x)}(f)$ and is denoted by $\operatorname{ord}_x(\omega)$. Define

$$\operatorname{div}(\omega) = \sum_{x \in X} \operatorname{ord}_x(\omega) x.$$

A *canonical divisor on* $\operatorname{Div}(X)$ is a divisor of the form $\operatorname{div}(\omega)$ for some nonzero $\omega \in \Omega(X)$. ▲

Compatibility of the meromorphic differentials on X ensures that the order of a meromorphic differential does not depend on the choice of chart.

Theorem 2.31. [13, Theorem 3.4.1](Riemann-Roch) Let X be a compact Riemann surface with genus g. Let $\operatorname{div}(\omega)$ be a canonical divisor of X. Then

$$\ell(D) = \deg(D) - g + 1 + \ell(\operatorname{div}(\omega) - D)$$

for every divisor $D \in \operatorname{Div}(X)$. ⬣

The interested reader is referred to [36] for a proof of the Riemann-Roch theorem. We are mostly interested in the following consequence of the Riemann-Roch theorem.

Corollary 2.32. [13, Exercise 3.4.3 and 3.3.6] Let Γ be a congruence subgroup and g the genus of $X(\Gamma)$. Then $\dim \Omega_{\text{hol}}(X(\Gamma)) = g$ and $\dim \mathcal{S}_2(\Gamma) = g$. ⬣

It holds more generally that the dimension of the holomorphic differentials of a compact Riemann surface is equal to the genus. Because of this, the genus of a compact Riemann surface X is often defined as the dimension of $\Omega_{\text{hol}}(X)$.

Proof. The strategy of proving Corollary 2.32 is by relating the meromorphic differentials on $X(\Gamma)$ to the meromorphic functions on $X(\Gamma)$. This relation is used to find a correspondence between the holomorphic differentials on $X(\Gamma)$ and the linear space of a canonical divisor ω. With these relations in place, the result follows from the Riemann-Roch Theorem 2.31.

By Lemma 2.14, we can write $\Omega(X(\Gamma)) = \mathbb{C}(X(\Gamma))\omega$ as $\mathbb{C}(X(\Gamma))-$vector spaces for some nonzero $\omega \in \Omega(X(\Gamma))$. The map $\mu\colon \Omega(X(\Gamma)) \to \mathbb{C}(X(\Gamma))$ sending $f\omega \mapsto f$ is a $\mathbb{C}-$vector space isomorphism with inverse $f \mapsto f\omega$.

Via μ we have $\Omega_{\text{hol}}(X(\Gamma)) \cong L(\operatorname{div}(\omega))$. To see this, let $\eta \in \Omega_{\text{hol}}(X(\Gamma))$ and let $f_\eta = \mu(\eta) \in \mathbb{C}(X(\Gamma))$ so that $f\eta\omega = \eta$. The differential η has a global expression $g(\tau)d\tau$ where g is holomorphic, so that $\operatorname{div}(\eta) = \operatorname{div}(g) \geq 0$. We obtain

$$\operatorname{div}(\eta) = \operatorname{div}(f_\eta) + \operatorname{div}(\omega) \geq 0$$

and $f_\eta \in L(\operatorname{div}(\omega))$. Conversely, let $f \in L(\operatorname{div}(\omega)) \subset \mathbb{C}(X(\Gamma))$, then $\mu^{-1}(f) = f\omega$ which is a holomorphic differential by definition of $L(\operatorname{div}(\omega))$. We see that f lies in the image $\mu(\Omega_{\text{hol}}(X(\Gamma)))$ and hence $\Omega_{\text{hol}}(X(\Gamma)) \cong L(\operatorname{div}(\omega))$ for any nonzero $\omega \in \Omega(X(\Gamma))$.

Invoke the Riemann-Roch Theorem 2.31 for $0 \in \operatorname{Div}(X(\Gamma))$ to obtain

$$\ell(0) = \deg(0) - g + 1 + \ell(\operatorname{div}(\omega)) = -g + 1 + \dim(\Omega_{\text{hol}}(X(\Gamma))).$$

Then $\dim(\Omega_{\text{hol}}(X(\Gamma))) = g$ since $\ell(0) = 1$. To see this, let $f \in L(0)$ be nonzero so that $\operatorname{div}(f) \geq 0$. Then f is holomorphic on $X(\Gamma)$. Suppose for a contradiction that f is not constant. By Proposition 1.10, f is surjective and its image, which is $\mathbb{C}$, is compact, a contradiction. We conclude f is constant. It follows that $L(0) \cong \mathbb{C}$ and $\ell(0) = 1$. The second assertion of Corollary 2.32 follows directly from Corollary 2.26. ■

2.4 Genus of $X_0(N)$

The genus of $X(\Gamma)$ is fully determined by its elliptic points of period 2 and 3, its cusps and a positive integer associated to the projection map $X(\Gamma) \to X(\mathrm{SL}_2(\mathbb{Z}))$. In this section we introduce this associated integer and give an explicit formula for the genus of $X(\Gamma)$. For a positive integer N, we then specialise to the congruence subgroup,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

and compute the genus of its associated modular curve $X_0(N) := X(\Gamma_0(N))$.

Locally, holomorphic maps between Riemann surfaces are very simple. That is, in coordinates they are power maps.

Proposition 2.33. [28, Proposition II.4.1](Local Normal Form) Let X and Y be Riemann surfaces and $\Psi\colon X \to Y$ a non-constant holomorphic mapping defined at $x \in X$. Then there is a unique integer $m \geq 1$ which satisfies the following property: for every chart $\varphi_2\colon U_2 \to V_2$ centered at $\Psi(x)$, there exists a chart $\varphi_1\colon U_1 \to V_1$ centered at x such that $\varphi_2\Psi\varphi_1^{-1}(z) = z^m$. ⬣

Definition 2.34. [28, Definition II.4.2] With notation as in Proposition 2.33, the *multiplicity* of Ψ at $x \in X$, denoted $\mathrm{mult}_x(\Psi)$, is the unique integer m such that, locally, Ψ has the form $z \mapsto z^m$. ▲

Consider the open unit disc $D = \{z \in \mathbb{Z} : |z| < 1\}$ and the holomorphic and onto map $f\colon D \to D$ mapping $z \mapsto z^m$ for some positive integer m. The preimage of $w \neq 0$ contains the m $m-$th roots of w, each having multiplicity 1. The preimage of $w = 0$ under f is the singleton set containing 0 where 0 has multiplicity m. We see that the preimages of f all have the same number of elements when counting multiplicity. A similar property holds for compact Riemann surfaces.

Proposition 2.35. [28, Proposition II.4.8] Let X and Y be compact Riemann surfaces and $\Psi\colon X \to Y$ a holomorphic map. For $y \in Y$ define

$$d_y(\Psi) = \sum_{x \in \Psi^{-1}(y)} \mathrm{mult}_x(\Psi).$$

Then $d_y(\Psi)$ is independent of y. ⬣

Note that $d_y(\Psi)$ above is well-defined for all $y \in Y$ by Proposition 1.11.

Definition 2.36. [28, Definition II.4.9] Let X and Y be compact Riemann surfaces and let $\Psi\colon X \to Y$ be holomorphic. The *degree* $\deg(\Psi)$ of Ψ is the positive integer $d_y(\Psi)$ for any $y \in Y$. ▲

For a holomorphic map $\Psi\colon X \to Y$ between compact Riemann surfaces, the Riemann–Hurwitz formula relates the genus of X and Y to the degree of Ψ. In [13, Section 3.1] this relation is used to compute an explicit formula for the genus of a modular curve.

Theorem 2.37. [13, Theorem 3.1.1] Let Γ be a congruence subgroup and $\Psi_\Gamma\colon X(\Gamma) \to X(\mathrm{SL}_2(\mathbb{Z}))$ the projection. Let ε_2 and ε_3 denote the number of elliptic points of order 2 and 3 in $X(\Gamma)$, and ε_∞ the number of cusps of $X(\Gamma)$. Then the genus g of $X(\Gamma)$ is

$$g = 1 + \frac{\deg(\Psi_\Gamma)}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}.$$

⬣

Limiting our attention to the case where $\Gamma = \Gamma_0(N)$ for some positive integer N, in [13, Section 3.7], Diamond and Shurman reduce the problem of finding ε_2 and ε_3 of $X_0(N)$ to counting the ideals J of $\mathbb{Z}[i]$ and $\mathbb{Z}[\zeta_3]$ such that $\mathbb{Z}[i]/J \cong \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}[\zeta_3]/J \cong \mathbb{Z}/N\mathbb{Z}$. From this, we obtain explicit formulas for ε_2 and ε_3.

Proposition 2.38. [13, Corollary 3.7.2] Let $N > 1$ be an integer. Let $\varepsilon_2(\Gamma_0(N))$ and $\varepsilon_3(\Gamma_0(N))$ denote the number of elliptic points of period 2 and 3 for $\Gamma_0(N)$. Then

$$\varepsilon_2(\Gamma_0(N)) = \begin{cases} \prod_{p \mid N}\left(1 + \left(\frac{-1}{p}\right)\right) & \text{if } 4 \nmid N \\ 0 & \text{if } 4 \mid N \end{cases}$$

where $(-1/p)$ is ± 1 if $p \equiv 1 \bmod 4$ and 0 if $p = 2$, and

$$\varepsilon_3(\Gamma_0(N)) = \begin{cases} \prod_{p|N}(1 + \left(-\frac{3}{p}\right)) & \text{if } 9 \nmid N \\ 0 & \text{if } 9 \mid N \end{cases}$$

where $(-3/p)$ is ± 1 if $p \equiv \pm 1 \bmod 3$ and 0 if $p = 3$. ⬣

In [13, Section 3.8] an explicit formula for the number of cusps of $X_0(N)$ is given. The proof is mainly a calculation where the definition of $\Gamma_0(N)$ is central.

Proposition 2.39. [13, Section 3.8] Let $N > 1$ be an integer. Let $\varepsilon_\infty(\Gamma_0(N))$ denote the number of cusps of $X_0(N)$. Then

$$\varepsilon_\infty(\Gamma_0(N)) = \sum_{d|N} \phi(\gcd(d, N/d))$$

where ϕ is the Euler totient function. ⬣

Finally, in [13, Section 3.8 and 3.9], the degree $\deg(\Psi_{\Gamma_0(N)})$ of the projection holomorphism $\Psi_{\Gamma_0(N)} \colon X_0(N) \to X(\mathrm{SL}_2(\mathbb{Z}))$ is computed.

Proposition 2.40. [13, Section 3.9] Let $N > 1$ be an integer. The degree $\deg(\Psi_{\Gamma_0(N)})$ of the projection $\Psi \colon X_0(N) \to X(\mathrm{SL}_2(\mathbb{Z}))$ is

$$\deg(\Psi_{\Gamma_0(N)}) = \frac{N^2}{\phi(N)} \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

⬣

The four formulas above allow us to calculate the genus of $X_0(N)$ for any $N > 1$. Incidentally, by Corollary 2.32, Theorem 2.37 allows us to compute the dimension of $\mathcal{S}_2(\Gamma_0(N))$ for any $N > 1$. The first few values are

Genus $g(X_0(N))$ of $X_0(N)$

N	2	...	10	11	12	13	14	15	16	17	18	19	20
$g(X_0(N))$	0	...	0	1	0	0	1	1	0	1	0	1	1

The sequence $g(X_0(N))$ is sequence A001617 in the OEIS [30].

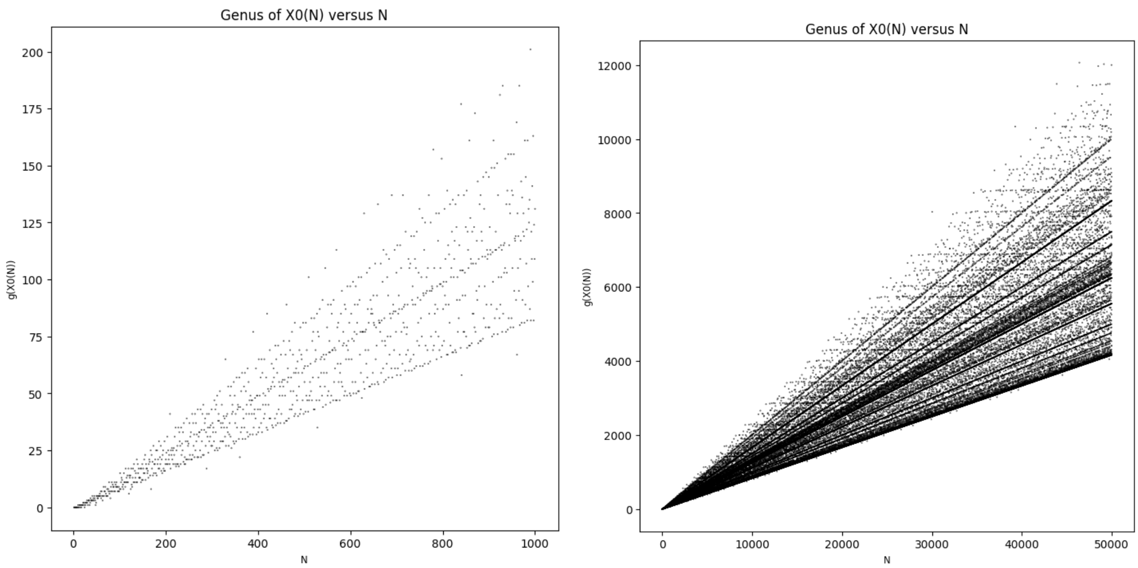

Figure 1: The genus of $X_0(N)$ versus N for $N \leq 1000$ (left) and $N \leq 50000$ (right)

In Figure 1 two plots of the genus of $X_0(N)$ are provided for two maximum values of N. From the image it seems like $g(X_0(N))$ is bounded above and below by N. The paper [11] gives such bounds explicitly, the lower bound is

$$g(X_0(N)) \geq (N - 5\sqrt{N} - 8)/12. \tag{2.14}$$

We are mostly interested in the values of N for which $g(X_0(N)) = 1$. By the bound (2.14), we only have to check the values of N smaller than 59. The values of N such that the genus of $X_0(N)$ is 1 are

$$N = 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49. \tag{2.15}$$

Precisely for these values of N the vector space $\mathcal{S}_2(\Gamma_0(N))$ is generated by a single element. In Chapter 3 we write down explicit formulas for these elements for some of the values in (2.15).

3 The $\eta-$function and eta products

In the previous section we established a relation between the holomorphic differentials on $X(\Gamma)$ and $\mathcal{S}_2(\Gamma)$. The elements in $\mathcal{S}_2(\Gamma)$ are weakly modular holomorphic functions $\mathcal{H} \to \mathbb{C}$ which vanish at the cusps. The purpose of this section is to write down generators of $\mathcal{S}_2(\Gamma_0(N))$ for some N among (2.15). It is in general a difficult task to write down closed form expressions for these generators. However, in some cases it is possible to do this with an eta product. This is a product of the $\eta-$function $\eta\colon \mathcal{H} \to \mathbb{C}$. In this section we investigate how η transforms under $\mathrm{SL}_2(\mathbb{Z})$. With this transformation property, we determine how eta products transform under $\mathrm{SL}_2(\mathbb{Z})$. We do this following the ideas of [21].

Let Γ be a congruence subgroup and let $f \in \mathcal{S}_2(\Gamma)$. Let h be the smallest positive integer such that $(1, h; 0, 1) \in \Gamma$. Then, by Definition 2.25.2, $f(z + h) = f(z)$. This means that f is determined on the region

$$\mathcal{H}_h = \{\tau \in \mathcal{H} : |\operatorname{Re}\tau| \leq h/2\}.$$

The map $z \mapsto e^{2\pi i\tau/h}$ is a homeomorphism from the region $\mathcal{H}_h$ to the punctured unit disc D. By changing coordinates, f is defined on D. To make this precise, define the holomorphic map

$$\begin{aligned}\tilde{f}\colon D &\to \mathbb{C}\\ q &\mapsto f\left(\frac{h\log q}{2\pi i}\right).\end{aligned}$$

This map is well-defined due to $h-$periodicity of f. We have that f is holomorphic at ∞ and vanishes there by Definition 2.25.3. Since $e^{2\pi i\tau/h} \to 0$ if and only if $\operatorname{Im}\tau \to \infty$, we extend $\tilde{f}$ to $D \cup \{0\}$ by setting $\tilde{f}(0) = 0$. From this we obtain a holomorphic function on the unit disc which then has Fourier expansion

$$\tilde{f}(q) = \sum_{n=1} a_n q^n \quad \text{with} \quad q \in D.$$

Definition 3.1. Let Γ, f and h be as above and let $q = e^{2\pi i\tau/h}$. The *$q-$expansion of $f \in \mathcal{S}_2(\Gamma)$* is the Fourier series

$$f(\tau) = \sum_{n=1} a_n q^n$$

for some $a_n \in \mathbb{C}$. ▲

Some of the cusp forms that are important to us are built from a function $\eta\colon \mathcal{H} \to \mathbb{C}$. This function is defined as a $q-$expansion.

Definition 3.2. The *Dedekind eta function* $\eta\colon \mathcal{H} \to \mathbb{C}$ is the infinite product

$$\eta(\tau) = e^{2\pi i\tau/24}\prod_{n=1}^{\infty}(1 - q^n) \tag{3.1}$$

where $q = e^{2\pi i\tau}$. ▲

The $\eta-$function is widely studied and has applications in algebraic geometry and number theory, among other fields (see [21]). To determine how $\mathrm{SL}_2(\mathbb{Z})$ acts on η we require the following fact.

Proposition 3.3. [8, Theorem 1.1] The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

⬣

The action $(\gamma, f) \mapsto f[\gamma]_k$ is a group action of $\mathrm{SL}_2(\mathbb{Z})$ on holomorphic functions on $\mathcal{H}$. From this fact and Proposition 3.3 it is sufficient to deduce the actions of S and T on a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ to determine the action of $\mathrm{SL}_2(\mathbb{Z})$ on f. The generators S and T act as

$$f(T\tau) = f(\tau + 1) \quad \text{and} \quad f(S\tau) = f\left(-\frac{1}{\tau}\right).$$

The $\eta-$function satisfies

$$\eta(\tau + 1) = e^{2\pi i/24}\eta(\tau) \quad \text{for all} \quad \tau \in \mathcal{H}. \tag{3.2}$$

The action of S on η requires some more work, we will define a function G_2 which transforms a certain way under S. This function then appears when we transform η under S.

Proposition 3.4. [7, Proposition 2.7] For every $\tau \in \mathcal{H}$ we have

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$$

where the branch of $\sqrt{-i\tau}$ is taken to have positive real part. ⬣

In Figure 2, η is plotted when considered as a function of the unit disc D. On the right, the identity map is depicted as a reference, the color of $q \in D$ represents the argument of q. On the left, the colour represents the argument of $\eta(q) \in \mathbb{C}$.

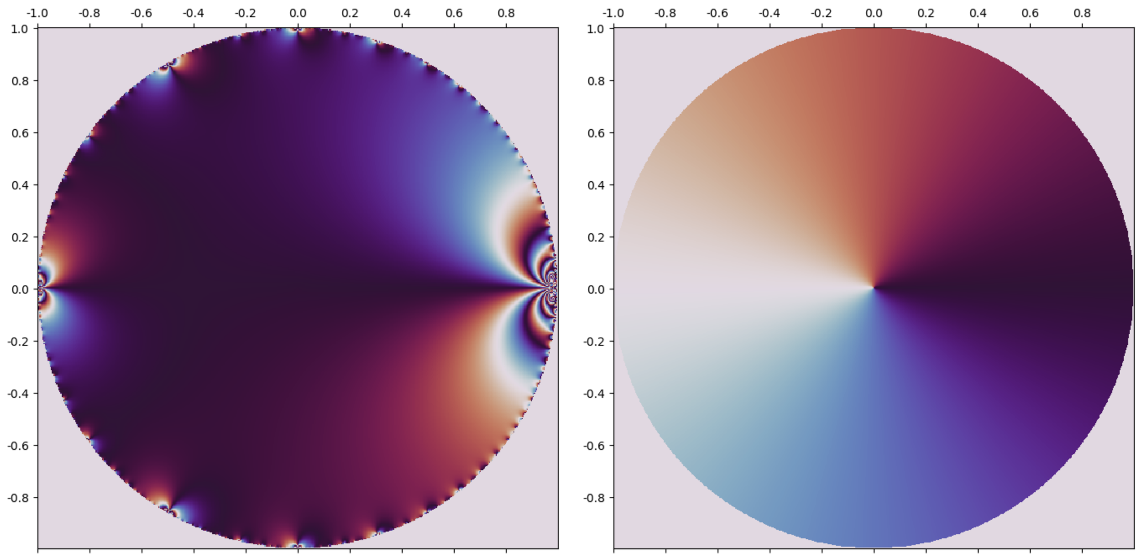

Figure 2: Colour plot of η when considered as a function of D (left) and the identity map $q \mapsto q$ (right)

First, we show that η is a holomorphic function on $\mathcal{H}$. This amounts to showing that the infinite product defining η is holomorphic on the unit disc. To do this, we require the following result in complex analysis.

Lemma 3.5. [9, Section IIV.5] Let $U \subset \mathbb{C}$ be open and $\{f_n : U \to \mathbb{C} : n \in \mathbb{Z}_{n\geq 1}\}$ a family of holomorphic functions such that

$$\sum_{n=1} |f_n|$$

converges uniformly on U. Then the product

$$F = \prod_{n=1}^{\infty} (1 + f_n)$$

converges uniformly on compact subsets of U and is holomorphic. Additionally, for $z \in U$ such that $F(z) \neq 0$, we have

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{1 + f_n(z)}.$$

⬣

Proposition 3.6. The Dedekind eta function $\eta \colon \mathcal{H} \to \mathbb{C}$ is holomorphic and non-vanishing on $\mathcal{H}$. ⬣

Proof. For q in the open unit disc D, the series

$$\sum_{n=1}^{\infty} |-q^n| = \sum_{n=1}^{\infty} |q|^n$$

is a geometric series and converges uniformly on D. By Lemma 3.5, the map defined by the product

$$\prod_{n=1}^{\infty} (1 - q^n)$$

is holomorphic on D. Since the maps $\tau \mapsto q = e^{2\pi i\tau}$ and $\tau \mapsto e^{2\pi i\tau/24}$ are holomorphic, so is η. The fact that $\eta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$ follows from the fact that $q = 0$ if and only if $\mathrm{Im}(\tau) \to \infty$ and $q = 1$ if and only if τ is an integer. ■

Proving Proposition 3.4 requires introducing a second function on the upper half complex plane. Much like η, this function has properties that come close to being weakly modular of weight 2.

Definition 3.7. The *Eisenstein series of weight 2* is the function $G_2\colon \mathcal{H} \to \mathbb{C}$ defined by

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau+n)^2}. \tag{3.3}$$

▲

The function $G_2(z)$ converges but fails to converge absolutely. This means that rearranging the terms changes the value of the sum. This is why G_2 is not weakly modular. It is close however.

Proposition 3.8. [7, Proposition 2.4] The Eisenstein series of weight 2 satisfies

$$G_2\left(-\frac{1}{\tau}\right) = \tau^2 G_2(\tau) - 2\pi i\tau \tag{3.4}$$

for all $\tau \in \mathcal{H}$. ⬣

A colour plot of G_2 is provided in Figure 3. The colour of a point $\tau \in \mathcal{H}$ represents the argument of $G_2(\tau) \in \mathbb{C}$.

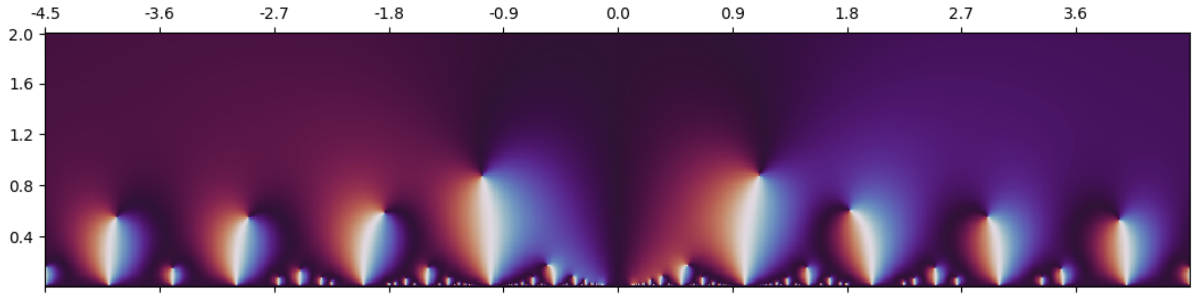

Figure 3: Colour plot of G_2 on a segment of $\mathcal{H}$.

The proof of [33, Theorem VII.4.6] contains a proof for Proposition 3.8 and incidentally shows that G_2 converges. The proof comes down to advanced manipulation of infinite series. Additionally, in his proof, Serre shows that G_2 has $q-$expansion

$$G_2(\tau) = \frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} \sigma_1(k) q^k, \tag{3.5}$$

where $\sigma_1(n)$ is the sum of positive divisors of $n \in \mathbb{Z}$. These are all the tools needed for the proof of Proposition 3.4.

Proof of Proposition 3.4. By Lemma 3.5 and Proposition 3.6, the logarithmic derivative of η is

$$\begin{aligned}\frac{d}{d\tau}\log\eta(\tau) &= \frac{\eta'(\tau)}{\eta(\tau)} = \frac{2\pi i}{24} + \sum_{n=1}^{\infty} \frac{\frac{d}{d\tau}(-q^n)}{1-q^n} = \frac{2\pi i}{24} - 2\pi i \sum_{n=1}^{\infty} nq^n \frac{1}{1-q^n} \\ &= \frac{2\pi i}{24} - 2\pi i \sum_{n=1}^{\infty} nq^n \sum_{m=0}^{\infty} q^{nm} = \frac{2\pi i}{24} - 2\pi i \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nq^{nm}\end{aligned}$$

In the last expression, for $k \in \mathbb{Z}_{>0}$, the coefficient in front of q^k is the sum of integers $n \in \mathbb{Z}_{>0}$ such that there exists an integer $m \in \mathbb{Z}_{>0}$ with $mn = k$. In other words, the coefficient in front of q^k is $\sigma_1(k)$. We obtain

$$\frac{d}{dz}\log\eta(\tau) = \frac{2\pi i}{24} - 2\pi i \sum_{k=1}^{\infty}\sigma_1(k)q^k = \frac{i}{4\pi}\left(\frac{\pi^2}{3} - 8\pi^2\sum_{k=1}^{\infty}\sigma_1(k)q^k\right) \overset{(3.5)}{=} \frac{i}{4\pi}G_2(\tau). \tag{3.6}$$

Additionally,

$$\frac{d}{d\tau}\log\left(\eta\left(-\frac{1}{\tau}\right)\right) = \frac{1}{\tau^2}\left[\frac{d}{d\mu}\log(\eta(\mu))\right]_{\mu=-\frac{1}{\tau}} \overset{(3.6)}{=} \frac{i}{4\pi\tau^2}G_2\left(-\frac{1}{\tau}\right)$$
$$\overset{(3.4)}{=} \frac{i}{4\pi}G_2(\tau) + \frac{1}{2\tau} = \frac{d}{d\tau}\log(\sqrt{-i\tau}\eta(\tau)).$$

By taking an anti-derivative with respect to τ of the above we obtain

$$\log\left(\eta\left(-\frac{1}{\tau}\right)\right) = \log(\sqrt{-i\tau}\eta(\tau)) + C \quad \text{for some} \quad C \in \mathbb{C}$$
$$\Leftrightarrow \quad \eta\left(-\frac{1}{\tau}\right) = B\sqrt{-i\tau}\eta(\tau) \quad \text{where} \quad B = e^C.$$

The formula above also holds when $\tau = i$, from which we obtain $\eta(i) = B\eta(i)$. Since $\eta(i) \neq 0$ by Proposition 3.6, it follows that $B = 1$. This completes the proof. ■

Proposition 3.4 and (3.2) show that the generators T and S of $\mathrm{SL}_2(\mathbb{Z})$ act on η as

$$\eta(T\tau) = \zeta_{24}\eta(\tau) \quad \text{and} \quad \eta(S\tau) = \zeta_{24}^3\sqrt{\tau}\eta(\tau) \quad \text{for all} \quad \tau \in \mathcal{H}$$

where $\zeta_{24} = e^{2\pi i/24}$. Since these elements generate $\mathrm{SL}_2(\mathbb{Z})$ by Proposition 3.3, a similar fact holds for general elements of $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.9. [21, Section 1.3, (1.16)] Let $\alpha = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\eta(\alpha\tau) = v_\eta(\alpha)\sqrt{c\tau + d}\eta(\tau)$$

where $v_\eta(\alpha)$ is a 24−th root of unity depending on α. ⬣

In [19, Section 4.1] an explicit formula for $v_\eta(\alpha)$ is computed. To write this down, we require an extension of the Jacobi symbol (see [32, Section 11.1]).

Definition 3.10. [21, Section 1.3] Let c and d be integers such that $\gcd(c, d) = 1$, d odd and $c \neq 0$. Then

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right) \quad \text{and} \quad \left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right) \cdot (-1)^{(\mathrm{sgn}(c)-1)(\mathrm{sgn}(d)-1)/4}$$

where $(\frac{c}{d})$ is the Jacobi symbol. Define

$$\left(\frac{0}{1}\right)^* = \left(\frac{0}{-1}\right)^* = 1, \quad \left(\frac{0}{1}\right)_* = 1 \quad \text{and} \quad \left(\frac{0}{-1}\right)_* = -1.$$

▲

Proposition 3.11. [21, Theorem 1.7] For $\alpha = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$v_\eta(\alpha) = \left(\frac{c}{d}\right)_* \zeta_{24}^{(a+d)c-bd(c^2-1)+3d-3-3cd} \quad \text{if } c \text{ is even.}$$
$$v_\eta(\alpha) = \left(\frac{d}{c}\right)^* \zeta_{24}^{(a+d)c-bd(c^2-1)-3c} \quad \text{if } c \text{ is odd}$$

⬣

As mentioned, for some values of N, the η−function serves as the building block for the generators of $\mathcal{S}_2(\Gamma_0(N))$. These elements have the following form.

Definition 3.12. [21, Section 2.1] An *eta product of level* N is a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ defined by the product

$$f(\tau) = \prod_{m|N} \eta(m\tau)^{a_m} \tag{3.7}$$

where the product runs through the divisors of N and the a_m are positive integers. ▲

Let f be an eta product as in (3.7) and $\gamma = (a, b; c, d) \in \Gamma_0(N)$. Then f transforms as

$$f(\gamma\tau) = v_f(\gamma)(c\tau + d)^{\frac{1}{2}\sum_m a_m} f(\tau) \tag{3.8}$$

(see [21, Section 2.1]). Here, $v_f(\gamma)$ is a 24−th root of unity associated to f. It is equal to

$$v_f(\gamma) = v_f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \prod_{m|N} v_\eta\left(\begin{pmatrix} a & mb \\ c/m & d \end{pmatrix}\right)^{a_m}. \tag{3.9}$$

The following proposition writes down explicit eta products of level N. These are generators for $\mathcal{S}_2(\Gamma_0(N))$ for some N among (2.15). The proof provided in [26] uses Hecke operators and Dirichlet characters.

Proposition 3.13. [26, Theorem 1] The following list of eta products are generators for the one dimensional vector space $\mathcal{S}_2(\Gamma_0(N))$ for the indicated N.

$$\begin{array}{ll}
N = 11 & \eta(\tau)^2\,\eta(11\tau)^2 \\
N = 14 & \eta(\tau)\,\eta(2\tau)\,\eta(7\tau)\,\eta(14\tau) \\
N = 15 & \eta(\tau)\,\eta(3\tau)\,\eta(5\tau)\,\eta(15\tau) \\
N = 20 & \eta(2\tau)^2\,\eta(10\tau)^2 \\
N = 24 & \eta(2\tau)\,\eta(4\tau)\,\eta(6\tau)\,\eta(12\tau) \\
N = 27 & \eta(3\tau)^2\,\eta(9\tau)^2 \\
N = 32 & \eta(4\tau)^2\,\eta(8\tau)^2 \\
N = 36 & \eta(6\tau)^4.
\end{array}$$

⬣

The map $\tau \mapsto f(\tau)$ where f is an eta product can be interpreted as a function of the unit disc D via $q = e^{2\pi i\tau}$. For example, using (3.1), the $N = 14$ product in Proposition 3.13, dubbed f_{14}, has $q-$expansion

$$f_{14}(\tau) = q\prod_{n=1}^{\infty}(1 - q^n)(1 - q^{2n})(1 - q^{7n})(1 - q^{14n})$$

Let f_{24} be the eta product corresponding to $N = 24$. In Figure 4 colour plots of f_{14} and f_{24} are depicted.

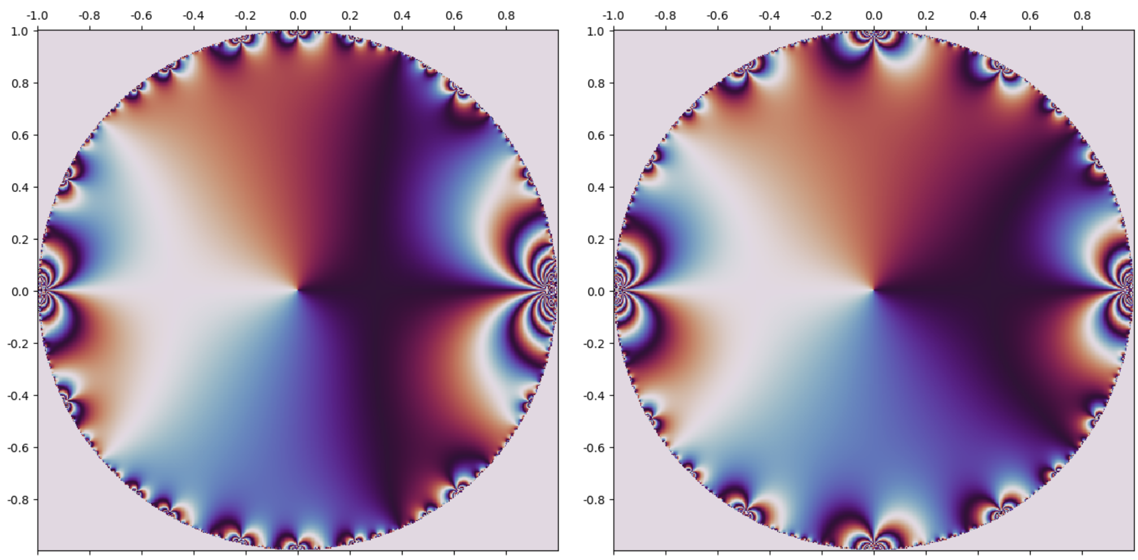

Figure 4: Colour plot of f_{14} (left) and of f_{24} (right) when considered as a function of D

An alternative proof of Proposition 3.13 is done as follows. First, find generators of $\Gamma_0(N)$, this can be done using the computer algebra system Sage [37] or by using the algorithm described in [20]. Then, using these generators and the transformation (3.8) show that the eta products are weakly modular of weight 2 with respect to these generators. Property 1 and 3 of Definition 2.25 follow since η has these properties. For example, for $N = 14$, by running the command `Gamma0(14).generators()` in Sage, we obtain that $\Gamma_0(14)$ is generated by the matrices

$$\begin{pmatrix} 9 & -2 \\ 14 & -3 \end{pmatrix}, \quad \begin{pmatrix} 41 & -11 \\ 56 & -15 \end{pmatrix}, \quad \begin{pmatrix} -29 & 9 \\ -42 & 13 \end{pmatrix}, \quad \begin{pmatrix} -11 & 4 \\ -14 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.10}$$

For the first matrix in (3.10), by (3.8) we obtain

$$f_{14}\left(\begin{pmatrix} 9 & -2 \\ 14 & -3 \end{pmatrix}\tau\right) = v_{f_{14}}\left(\begin{pmatrix} 9 & -2 \\ 14 & -3 \end{pmatrix}\right)(14\tau - 3)^2 f(\tau). \tag{3.11}$$

Using Proposition 3.11 and (3.9), we compute the 24−th root of unity $v_{f_{14}}$ for this matrix to be equal to

$$v_{f_{14}}\left(\begin{pmatrix} 9 & -2 \\ 14 & -3 \end{pmatrix}\right) = \prod_{m|14} v_\eta\left(\begin{pmatrix} 9 & -2m \\ 14/m & -3 \end{pmatrix}\right) = \zeta_{24}^{-960} \cdot \zeta_{24}^{-555} \cdot \zeta_{24}^{-96} \cdot \zeta_{24}^{3} = \zeta_{24}^{-1608} = \left(\zeta_{24}^{24}\right)^{67} = 1.$$

By substituting this into (3.11), we see that f_{14} respects the modularity property with respect to this matrix. Similarly f_{14} respects the modularity condition for the other matrices in (3.10). Since these are generators of $\Gamma_0(14)$, it follows that f_{14} is weakly modular of weight 2 with respect to $\Gamma_0(14)$.

For the remaining values of N, that is, the values for which $X_0(N)$ has genus one but is not among those in Proposition 3.13, the space $\mathcal{S}_2(\Gamma_0(N))$ is generated by a single element. However, this element may or may not have a closed form expression. In [10, Section 2.9], a method to compute the q−expansions of these generators is explained. This method uses Hecke operators and exploits the duality between $\mathcal{S}_2(\Gamma_0(N))$ and $H_1(X_0(N), \mathbb{Z})$. To do this, Cremona introduces modular symbols, these symbols give a description of $H_1(X_0(N), \mathbb{Q})$. In Section 5.2 we introduce these symbols and use them to calculate the period lattice. These, and many more, q−expansions are found in the LMFDB [24].

4 Elliptic curves as Riemann surfaces

In this section we encounter the third major example of a Riemann surface, elliptic curves. First we introduce elliptic curves and state some results which are important for our purposes. In the second subsection it is made apparent that elliptic curves are actually complex tori.

4.1 Elliptic curves over $\mathbb{C}$

In this subsection we briefly discuss definitions and results relating to elliptic curves which are of particular interest to us. We mainly use [34] and [18].

A *Weierstrass equation over* $\mathbb{C}$ is an equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \tag{4.1}$$

where $a_1, \dots, a_6 \in \mathbb{C}$. Define the associated polynomial $F \in \mathbb{C}[X, Y, Z]$ as

$$F(X, Y, Z) = Y^2Z + a_1XYZ + a_3YZ^2 - X^3 + a_2X^2Z - a_4XZ^2 - a_6Z^3. \tag{4.2}$$

Then F is a *homogeneous polynomial of degree 3*, that is, for $\lambda \in \mathbb{C}$, F satisfies $F(\lambda X, \lambda Y, \lambda Z) = \lambda^3 F(X, Y, Z)$. The fact that F is homogeneous ensures that the *zero set of* F in $\mathbb{P}^2(\mathbb{C})$

$$\{(X : Y : Z) \in \mathbb{P}^2(\mathbb{C}) : F(X, Y, Z) = 0\}$$

is well-defined.

Definition 4.1. [28, Definition I.3.2] A homogeneous polynomial $F(X, Y, Z)$ of degree 3 is *nonsingular* if

$$F = \frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0$$

has no solutions in $\mathbb{C}^3$. ▲

Definition 4.2. [34, Section III.3] An *elliptic curve* E *over* $\mathbb{C}$ is a curve defined by the equation (4.1) where $a_1, \dots, a_6 \in \mathbb{C}$ are such that its associated polynomial (4.2) is nonsingular. We say that E is *defined over* $\mathbb{Q}$ if $a_1, \dots, a_6 \in \mathbb{Q}$. ▲

Proposition 4.3. [28, Proposition I.3.6] Let $F \in \mathbb{C}[X, Y, Z]$ be a homogeneous nonsingular polynomial of degree 3. Then the zero set of F in $\mathbb{P}^2(\mathbb{C})$ is a compact Riemann surface. ⬣

The charts on such a zero set are local projections. The fact that F is nonsingular ensures that these projections are homeomorphisms. In particular, by Proposition 4.3, the points satisfying the equation of an elliptic curve E over $\mathbb{C}$ is a Riemann surface. Set $Z = 0$ in (4.1), then $(0 : 1 : 0)$ is the only solution to the equation. This point is referred to as the *point at infinity*. With this point in mind, assume $Z \neq 0$ and transform coordinates as $x = X/Z$ and $y = Y/Z$ to obtain

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{4.3}$$

An elliptic curve E is then the curve defined by (4.3) along with the point at infinity which we denote by $\mathcal{O}$.

It is possible to simplify the equation (4.3) by making suitable changes of coordinates. Define

$$\begin{aligned}
b_2 &= a_1^2 + 4a_4 \\
b_4 &= 2a_4 + a_1a_3 \\
b_6 &= a_3^2 + 4a_6 \\
b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\
c_4 &= b_2^2 - 24b_4 \\
c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 \\
\Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6
\end{aligned} \tag{4.4}$$

Changing coordinates as $y \mapsto \frac{1}{2}(y - a_1x - a_3)$ changes (4.3) to

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6. \tag{4.5}$$

Further replacing (x, y) with

$$\left(\frac{x - 3b_2}{36}, \frac{y}{108}\right)$$

simplifies the equation (4.5) to

$$y^2 = x^3 - 27c_4x - 54c_6. \tag{4.6}$$

The *discriminant* Δ is of particular interest to us, due to the following result.

Proposition 4.4. [18, Theorem 3.2] The polynomial (4.2) is nonsingular if and only if $\Delta \neq 0$. ⬣

In [18, Section III.2] Knapp shows that the discriminant Δ coincides with the discriminant of the polynomials on the right hand side of (4.5) and (4.6). The discriminant is a (very) rough measure of how complicated a polynomial is. A fact that promotes this interpretation is that a polynomial has a repeated root if and only if its discriminant vanishes (see [22, Chapter IV.6]). Combining these facts, we obtain the following result.

Proposition 4.5. [18, Proposition 3.5] A curve of the form

$$y^2 = x^3 - \alpha x^2 + \beta x - \gamma \quad \text{with} \quad \alpha, \beta, \gamma \in \mathbb{C}$$

is an elliptic curve if and only if $f(x) = x^3 - \alpha x^2 + \beta x - \gamma$ has distinct roots in $\mathbb{C}$. ⬣

Using projective geometry, a binary operation can be constructed on E as follows. Let $E \subset \mathbb{P}^2(\mathbb{C})$ be an elliptic curve over $\mathbb{C}$ (considered as a set of points satisfying a Weierstrass equation) and let $A, B \in E$ be two points. Let $L \subset \mathbb{P}^2(\mathbb{C})$ be the (unique) line connecting A and B, if $A = B$ then L is the tangent line at A. By [17, Corollary I.7.8], L intersects E in a third point (counting multiplicity), say, $C \in E$. Let L' be the line connecting C with the point at infinity $\mathcal{O}$. Again, L' intersects E in a third point (counting multiplicity) which is denoted by $A \oplus B$. Then $\oplus \colon E \times E \to E$ is a binary operation on E.

Proposition 4.6. [34, Proposition 2.2] Let E be an elliptic curve. The binary operation $\oplus$ on E turns E into an abelian group with identity element $\mathcal{O}$. ⬣

Two elliptic curves E_1 and E_2 over $\mathbb{C}$ can be isomorphic as groups and as Riemann surfaces even though they have two different Weierstrass equations describing them. Silverman describes the coordinate transformations which yield isomorphic elliptic curves.

Proposition 4.7. [34, Proposition III.3.1.b] Let E_1 and E_2 be elliptic curves over $\mathbb{C}$ with Weierstrass equations (4.3). Then E_1 and E_2 are isomorphic as Riemann surfaces and as groups if and only if the Weierstrass equations of E_1 and E_2 are related via a coordinate transformation of the form

$$\begin{aligned} x &= u^2x' + r \\ y &= u^3y' + su^2x' + t \end{aligned} \tag{4.7}$$

where $u \in \mathbb{C}^\times$ and $r, s, t \in \mathbb{C}$. ⬣

Definition 4.8. Let E_1 and E_2 be elliptic curves over $\mathbb{Q}$ with Weierstrass equations (4.3). We say that E_1 and E_2 are *isomorphic over* $\mathbb{Q}$ if they are related via a coordinate transformation of the form (4.7) where $u \in \mathbb{Q}^\times$ and $r, s, t \in \mathbb{Q}$. ▲

The formulation of Proposition 4.7 is different from [34, Proposition III.3.1.b]. However, these are equivalent statements. This will become apparent when we delve deeper into the Riemann surface structure of E, see Proposition 4.25. The change of coordinates that simplifies (4.3) to (4.6) is of the form (4.7). Changing coordinates as in (4.7) affects the c_i and Δ in (4.4) as

$$\begin{aligned} u^4c_4' &= c_4 \\ u^6c_6' &= c_6 \\ u^{12}\Delta' &= \Delta. \end{aligned} \tag{4.8}$$

For an elliptic E with Weierstrass equation (4.3) we introduce the $j-$*invariant* of E as the quantity

$$j = \frac{c_4^3}{\Delta}. \tag{4.9}$$

By (4.8), via a change of variables of the form (4.7) the $j-$invariant remains unchanged. The converse is also true.

Proposition 4.9. [34, Proposition III.1.4.b] Two elliptic curves over $\mathbb{C}$ are isomorphic if and only if they have the same $j-$invariant. ⬣

In its full generality, Proposition 4.9, states that two elliptic curves over algebraically closed fields are isomorphic. In particular, if two elliptic curve defined over $\mathbb{Q}$ have the same $j-$invariant, then they may or may not be isomorphic. The $j-$invariant encodes information about the field of definition of an elliptic curve E.

Proposition 4.10. [18, Proposition 3.7.b] If $j_0 \in \mathbb{Q}$ then there exists an elliptic curve over $\mathbb{Q}$ with $j-$invariant equal to j_0. ⬣

Suppose that E is an elliptic curve over $\mathbb{Q}$ with Weierstrass equation (4.3). By the above propositions, there are a multitude of curves with different Weierstrass equations that are isomorphic to E (over $\mathbb{Q}$). We wish to have an equation for E which is as 'uncomplicated' as possible. This is formalized as follows.

Definition 4.11. [10, Chapter 3.1] Let E be an elliptic curve over $\mathbb{Q}$. A *minimal Weierstrass equation* for E is a Weierstrass equation of the form (4.3) with $a_1 \dots, a_6 \in \mathbb{Z}$ minimizing $|\Delta|$ such that the elliptic curve given by this equation is isomorphic to E over $\mathbb{Q}$. ▲

As mentioned before, the discriminant roughly measures how complicated an equation is. An equation for E which minimizes the discriminant then gives the least complicated Weierstrass equation for E, roughly speaking.

Proposition 4.12. [34, Proposition VII.1.3 (a) and (b)] Let E be an elliptic curve defined over $\mathbb{Q}$. Then E has a minimal Weierstrass equation. This minimal Weierstrass equation is unique up to change of coordinates of the form (4.7) where $u = \pm 1$ and $r, s, t \in \mathbb{Z}$. ⬣

4.2 Elliptic functions

In this section we introduce the notion of an elliptic function. These are essentially meromorphic functions on complex tori. Studying these functions leads us to the correspondence between complex tori and elliptic curves. The main ideas in this section are from [34, Section VI] and [27, Section III].

Let $\Lambda \subset \mathbb{C}$ be a lattice. There is a bijection between meromorphic functions $f\colon \mathbb{C}/\Lambda \to \mathbb{C}$ and meromorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $f(z+\omega) = f(z)$ for all $\omega \in \Lambda$ and $z \in \mathbb{C}$. If $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ then this property is equivalent to

$$f(z+\omega_1) = f(z) \quad \text{and} \quad f(z+\omega_2) = f(z) \quad \text{for all} \quad z \in \mathbb{C}. \tag{4.10}$$

Definition 4.13. [34, Section VI.2] Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice. An *elliptic function (for Λ)* is a meromorphic function $f\colon \mathbb{C} \to \mathbb{C}$ satisfying (4.10). ▲

An elliptic function f for Λ is fully determined on the closure of a fundamental domain $\overline{D}_z$, where $\overline{D}_z$ is as in (1.2). Such a set is compact, therefore, a holomorphic elliptic function is bounded on $\overline{D}_z$. Since f satisfies (4.10) it follows that f is bounded on $\mathbb{C}$. Liouville's theorem implies that such a function is constant. We obtain the following.

Proposition 4.14. [34, Proposition IV.2.1] A holomorphic function which is elliptic is constant. ⬣

Proposition 4.15. [27, Proposition III.2.1] Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonzero elliptic function and $\Lambda \subset \mathbb{C}$ a lattice. Let D_z be a fundamental domain for Λ such that f has no zeroes or poles on the boundary. Then

1. $\sum_{x \in D_z} \mathrm{Res}_x(f) = 0$;
2. $\sum_{x \in D_z} \mathrm{ord}_x(f) = 0$,

where $\mathrm{Res}_x(f)$ is the residue of f at x (see [9, Definition V.2.1]). ⬣

Proof. By the residue theorem [9, Theorem V.2.2],

$$\int_{\partial D_z} f(z)dz = \sum_{x \in D_z} \mathrm{Res}_x(f).$$

Since f is elliptic, the opposite boundaries of the integral on the left hand side cancel out. It follows that this integral is equal to 0, this proves 1. For 2, note that f'/f is an elliptic function and that $\mathrm{Res}_x(f'/f) = \mathrm{ord}_x(f)$. Applying the residue theorem to f'/f proves 2. ■

From Proposition 4.15 and 4.14 it follows that non-constant elliptic functions have at least two poles in $\overline{D}_z$. To see this, let $f \colon \mathbb{C} \to \mathbb{C}$ be elliptic with exactly one pole in $\overline{D}_z$. By Proposition 4.15 the residue at this pole is 0 and f is actually holomorphic and hence constant by Proposition 4.14, a contradiction. With this in mind, we look for an elliptic function with a pole of order 2 at every $z \in \Lambda$ so that every fundamental domain $\overline{D}_z$ contains at least 2 poles. We obtain the following.

Definition 4.16. [34, Section VI.3] Let $\Lambda \subset \mathbb{C}$ be a lattice. The *Weierstrass $\wp$−function (for Λ)* is the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

▲

Proposition 4.17. [34, Theorem VI.3.1 (b) and (c)] Let $\Lambda \subset \mathbb{C}$ be a lattice. The series defining the Weierstrass $\wp$−function is absolutely and uniformly convergent. Furthermore, $\wp$ is an elliptic function which is holomorphic on $\mathbb{C} \setminus \Lambda$. ⬣

In the proof of this theorem, Silverman uses the following intermediate result to find a uniform bound on the series defining $\wp$.

Lemma 4.18. Let $\Lambda \subset \mathbb{C}$ be a lattice and $k > 1$ an integer. Then the series

$$\sum_{\substack{\omega \in \Lambda \\ |\omega| \geq 1}} \frac{1}{|\omega|^{2k}}$$

converges. ⬣

As $\wp$ is uniformly convergent, it is justified to differentiate term by term in order to determine its derivative

$$\wp'(z) = -\sum_{\omega \in \Lambda} \frac{2}{(z-\omega)^3}.$$

The derivative $\wp'$ is again an elliptic function. As $\wp$ and $\wp'$ are meromorphic on $\mathbb{C}$, they have Laurent series expansions around every point. We derive these series around 0. Incidentally we find that $\wp$ satisfies a differential equation which is of particular interest to us. To do all this, we first study the following series.

Definition 4.19. [34, Section VI.3] Let $\Lambda \subset \mathbb{C}$ be a lattice and $k > 1$ an integer. The *Eisenstein series of weight $2k$ (for Λ)* is the series

$$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^{2k}}.$$

▲

By Lemma 4.18, $G_{2k}(\Lambda)$ converges absolutely and hence converges. For $\alpha \in \mathbb{C}$ nonzero we have $G_{2k}(\alpha\Lambda) = \alpha^{-2k} G_{2k}(\Lambda)$ so that for generators $\omega_1, \omega_2 \in \mathbb{C}$ of Λ,

$$G_{2k}(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) = \frac{1}{\omega_2^{2k}} G_{2k}(\mathbb{Z}(\omega_1/\omega_2) \oplus \mathbb{Z}). \tag{4.11}$$

The function $G_{2k}(\tau) = G_{2k}(\mathbb{Z}\tau \oplus \mathbb{Z})$ is holomorphic and is weakly modular of weight $2k$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ ([13, Section 1.1]). Additionally, G_{2k} is holomorphic at the cusps but does not vanish there. Such a function is called a *modular form* with respect to $\mathrm{SL}_2(\mathbb{Z})$, for more details on modular forms see [13, Section 1.1]. The Eisenstein series of weight 2 has an analogous definition to that of G_{2k}. However, G_2 is not weakly modular of weight 2 with respect to $\mathrm{SL}_2(\mathbb{Z})$ as discussed in (3.2) and Proposition 3.4.

Proposition 4.20. [34, Theorem VI.3.5.a] Let $\Lambda \subset \mathbb{C}$ be a lattice. The Laurent series of $\wp$ and $\wp'$ for Λ around $z = 0$ are

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2}(\Lambda) z^{2k-1} \tag{4.12}$$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty} 2k(2k+1) G_{2k+2}(\Lambda) z^{2k-1}. \tag{4.13}$$

⬣

Proof. Recall the Laurent expression

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \quad \text{for} \quad |z| < 1.$$

For $|z| < \min\{|\omega| : \omega \in \Lambda\}$,

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{(1-z/\omega)^2} - 1\right) = \sum_{n=1}^{\infty} (n+1) \frac{z^n}{\omega^{n+2}}.$$

Plugging this into the definition of $\wp$ we obtain (4.12). Differentiating term by term yields (4.13). ∎

Using (4.12) and (4.13), we obtain the expansions

$$\begin{aligned}
\wp'(z)^2 &= \frac{4}{z^6} - \frac{24 G_4(\Lambda)}{z^2} - 80 G_6(\Lambda) - \dots \\
\wp(z)^3 &= \frac{1}{z^6} + \frac{9 G_4(\Lambda)}{z^2} + 15 G_6(\Lambda) + \dots \\
\wp(z) &= \frac{1}{z^2} + 3 G_4(\Lambda) z^2 + 5 G_6(\Lambda) z^4 + \dots .
\end{aligned}$$

Using these we see that the elliptic function $f(z) = \wp'(z)^2 - 4\wp(z)^3 + 60 G_4(\Lambda)\wp(z) + 160 G_6(\Lambda)$ has no pole at $z = 0$. It follows that f is holomorphic in a fundamental domain D_z containing 0. Therefore f is holomorphic and elliptic and hence constant by Proposition 4.14. Since $f(0) = 0$ we obtain the following.

Proposition 4.21. [34, Theorem VI.3.5.a] Let $\Lambda \subset \mathbb{C}$ be a lattice. The Weierstrass $\wp$−function satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - 60 G_4(\Lambda)\wp(z) - 140 G_6(\Lambda).$$

⬣

It is customary to set $g_4 = g_4(\Lambda) = 60 G_4(\Lambda)$ and $g_6 = g_6(\Lambda) = 140 G_6(\Lambda)$. Then $(\wp(z), \wp'(z)) \in \mathbb{C}^2$ satisfies (4.5) where $b_2 = 0, b_4 = -g_4/2$ and $b_6 = -g_6$.

Proposition 4.22. [34, Proposition VI.3.6.a] Let $\Lambda \subset \mathbb{C}$ be a lattice with associated quantities g_4 and g_6. Then $f(z) = 4z^3 - g_4 z - g_6$ has distinct roots. ⬣

Proof. Let ω_1 and ω_2 be generators of Λ. Using the fact that the elliptic function $\wp'$ for Λ is odd we obtain
$$\wp'(\omega_1/2) = \wp'(\omega_1/2 - \omega_1) = \wp'(-\omega_1/2) = -\wp'(\omega_1/2).$$
It follows that $\omega_1/2$ is a zero of $\wp'$. By Proposition 4.21, $\wp(\omega_1/2)$ is a root of f. Similarly $\wp(\omega_2/2)$ and $\wp((\omega_1+\omega_2)/2)$ are roots of f. By [22, Theorem IV.1.4], f does not have more than 3 roots (counting multiplicity) in $\mathbb{C}$. Therefore, it remains to be shown that $\wp(\omega_1/2), \wp(\omega_2/2)$ and $\wp((\omega_1+\omega_2)/2)$ are distinct.

Define the elliptic function $g\colon z \mapsto \wp(z) - \wp(\omega_1/2)$. The elliptic function g has two poles (with multiplicity) in a fundamental domain D_z containing 0 (namely a double root at 0). By Proposition 4.15.2, g has two zeroes in D_z. This is a double zero since both g and g' have $\omega_1/2$ as a root ([22, Proposition IV.1.11]). It follows that $\wp(z) \neq \wp(\omega_1/2)$ for all $z \neq \omega_1/2$. A similar argument shows an analogous result for $\wp(\omega_2/2)$ and $\wp((\omega_1+\omega_2)/2)$. In particular $\wp(\omega_1/2), \wp(\omega_2/2)$ and $\wp((\omega_1+\omega_2)/2)$ are distinct. This completes the proof. ∎

Let $\Lambda \subset \mathbb{C}$ be a lattice with associated quantities g_4 and g_6. The curve given by the equation
$$y^2 = 4x^3 - g_4 x - g_6 \tag{4.14}$$
is of the form (4.5). Replacing (x, y) by $(x/36, y/108)$ we obtain the curve
$$y^2 = x^3 - 27c_4 x - 54c_6 \tag{4.15}$$
where $c_4 = 12g_4$ and $c_6 = 216g_6$. Suppose that Λ is generated by ω_1 and ω_2. The polynomial in x on the right hand side of (4.15) has the roots $36\wp(\omega_1/2), 36\wp(\omega_2/2)$ and $36\wp((\omega_1+\omega_2)/2)$ which are distinct by Proposition 4.22. By Proposition 4.5 it follows that (4.15) defines an elliptic curve over $\mathbb{C}$.

Proposition 4.23. [34, Proposition VI.3.6.b] Let Λ be a lattice with associated quantities g_4 and g_6. Let E be the elliptic curve with equation (4.15). Then the map
$$\begin{aligned} \phi\colon \mathbb{C}/\Lambda &\to E \subset \mathbb{P}^2(\mathbb{C}) \\ z + \Lambda &\mapsto \begin{cases} (\wp(z)/36 : \wp'(z)/108 : 1) & \text{if } z \notin \Lambda \\ (0:1:0) & \text{otherwise} \end{cases} \end{aligned}$$
is an isomorphism of groups and Riemann surfaces. ⬣

The mapping in Proposition 4.23 is well-defined by Proposition 4.21. Silverman uses techniques from complex analysis to show that ϕ is bijective. He then determines how the pull back ϕ^* acts on the invariant differentials to show that ϕ is holomorphic. A proof by more elementary means that shows that ϕ is a homomorphism is found in the lecture notes written by Sutherland [35, Theorem 15.1].

Using (4.4), the curve with equation (4.14) has discriminant $\Delta = g_4 - 27g_6 \neq 0$. Though this is not needed in this thesis, it turns out that every elliptic curve E over $\mathbb{C}$ is isomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. This fact follows from the following proposition.

Proposition 4.24. [2, Theorem 2.9] Given two complex numbers A and B such that $A - 27B \neq 0$, there exist complex numbers ω_1 and ω_2 such that $\mathrm{Im}(\omega_1/\omega_2) > 0$ and
$$g_4(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) = A \quad \text{and} \quad g_6(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) = B.$$
⬣

Proposition 4.25. [34, Corollary III.3.9] Let E_1 and E_2 be isomorphic elliptic curves over $\mathbb{C}$ with associated lattices Λ_1 and Λ_2. Then there is some $\alpha \in \mathbb{C}$ such that $\alpha\Lambda_1 = \Lambda_2$. ⬣

Proof. Via the isomorphism in Proposition 4.23 we obtain that $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$. By Proposition 1.18, there exists some α such that $\alpha\Lambda_1 = \Lambda_2$. ∎

5 The period lattice and its associated elliptic curve

The sections leading up to this one culminate in determining the structure of the following object.

Definition 5.1. Let Γ be a congruence subgroup such that $X(\Gamma)$ has genus 1 and let $f \in \mathcal{S}_2(\Gamma_0(N))$. The *period lattice of* f is the set

$$\Lambda_f = \left\{ \int_x^{\gamma x} 2\pi i f(\zeta) d\zeta : \gamma \in \Gamma \right\}.$$

▲

It is not immediate that Λ_f is a lattice, this will become apparent however when we develop the theory in this section. We start by introducing the standard notions of integrating along paths on Riemann surfaces. In the second section we introduce modular symbols. These symbols describe the homology group of $X(\Gamma)$ and allow us to determine Λ_f via this homology group. From this, we obtain an isomorphism $X_0(N) \to E_f$, where E_f is the elliptic curve isomorphic to the complex torus $\mathbb{C}/\Lambda_f$. This means that we obtain an algebraic expression for $X_0(N)$.

5.1 Integration along paths on Riemann surfaces

We start this section by defining path integrals on Riemann surfaces as in [13, Section 6.1] and [28, Section IV.3]. In particular, we look at path integrals on $X(\Gamma)$ and summarize some results.

Definition 5.2. [28, Definition 3.1] Let X be a Riemann surface. A *path* on X is a continuous map $\nu\colon [a,b] \to X$ for some $a, b \in \mathbb{R}$. The points $\nu(a)$ and $\nu(b)$ are the *endpoints* of ν. A path $\sigma\colon [a,b] \to X$ is a *loop* if $\sigma(a) = \sigma(b)$. ▲

Definition 5.3. [13, Section 6.1] and [28, Definition IV.3.8] Let X be a Riemann surface. Let $\varphi\colon U \to V$ be a chart for X and $\nu\colon [a,b] \to X$ a path such that $\nu([a,b]) \subset U$. The *integral of* $\omega \in \Omega_{\mathrm{hol}}(X)$ *along* ν is the complex number

$$\int_\nu \omega = \int_{\varphi\nu} \omega|_V = \int_{\varphi\nu} f(\zeta) d\zeta$$

where $f(\zeta)d\zeta \in \Omega_{\mathrm{hol}}(V)$ is the restriction of ω to V. ▲

Let $\varphi_1\colon U_1 \to V_1$ and $\varphi_2\colon U_2 \to V_2$ be charts. If the image of a path $\nu\colon [a,b] \to X$ lies in $U_1 \cap U_2$ then the value of the integral in Definition 5.3 may depend on the coordinates we choose on this intersection. Luckily, these values coincide as for $\omega \in \Omega_{\mathrm{hol}}(X)$, we have

$$\int_{\varphi_2\nu} \omega|_{V_2} = \int_{\varphi_2\varphi_1^{-1}\varphi_1\nu} \omega|_{V_2} = \int_{\varphi_1\nu} (\varphi_2\varphi_1^{-1})^* \omega|_{V_2} = \int_{\varphi_1\nu} \omega|_{V_1} \tag{5.1}$$

where the second equality follows from compatibility of ω. This motivates the following definition.

Definition 5.4. [28, Section IV.3] Let $\nu\colon [a,b] \to X$ be a path. Let $a_0, \ldots, a_n \in [a,b]$ be such that $a = a_0 < a_1 < \ldots < a_n = b$. A *partition* of ν is the set $\{\nu_i\}$ of n paths $\nu_i\colon [a_{i-1}, a_i] \to X$ such that $\nu_i(a_i) = \nu_{i+1}(a_i)$ for all $i = 1, \ldots n-1$. ▲

To define an integral along a general path, we would like to pick a partition of a path such that each part lies inside the domain of a chart. Lemma IV.3.7 of [28] ensures that this is possible.

Definition 5.5. [28, Definition IV.3.8] Let X be a Riemann surface. Let $\nu\colon [a,b] \to X$ be a path and let $\{\nu_i\}$ be a partition of ν such that the image of ν_i is contained in the domain of the chart $\varphi_i\colon U_i \to V_i$. The *integral of* $\omega \in \Omega_{\mathrm{hol}}(X)$ *along* ν is the complex number

$$\int_\nu \omega = \sum_i \int_{\varphi_i\nu} \omega|_{V_i}.$$

▲

The computation in (5.1) ensures that Definition 5.5 is well-defined.

Proposition 5.6. [28, Lemma IV.3.9.f] Let X and Y be Riemann surfaces and $\nu\colon [a,b] \to X$ a path. Furthermore, suppose that $\Psi\colon X \to Y$ is holomorphic. Then $\Psi\nu\colon [a,b] \to Y$ is a path and for $\omega \in \Omega_{\text{hol}}(X)$,

$$\int_{\Psi\nu} \omega = \int_{\nu} \Psi^*\omega.$$

⬣

Let Γ be a congruence subgroup and let $\tilde{\nu}\colon [a,b] \to X(\Gamma)$ be a path. Then $\tilde{\nu} = \pi\nu$ for some path $\nu\colon [a,b] \to \mathcal{H}_\infty$. Let $\omega \in \Omega_{\text{hol}}(X(\Gamma))$ and let $f \in \mathcal{S}_2(\Gamma)$ be nonzero such that $\pi^*\omega = f(\zeta)d\zeta$. Then

$$\int_{\tilde{\nu}} \omega = \int_{\pi\nu} \omega \overset{5.6}{=} \int_{\nu} \pi^*\omega = \int_{\nu} f(\zeta)d\zeta. \tag{5.2}$$

The integral on the right of (5.2) is an integral in $\mathbb{C}$ and is easily computed. The following results say something about integrals over holomorphic differentials on $X(\Gamma)$ along loops which are representatives of elements in the first homology group of $X(\Gamma)$. Integrating along these representatives is well-defined as it is independent of the choice of representative [28, Section VIII.1].

Proposition 5.7. [16, Proposition 3.6] Let Γ be a congruence subgroup. Let $\sigma_1, \dots, \sigma_{2g}$ be representatives of a basis for $H_1(X(\Gamma), \mathbb{Z})$ and $f_1, \dots, f_g$ a basis for $\mathcal{S}_2(\Gamma)$. Then the $2g$ vectors

$$\begin{pmatrix} \int_{\sigma_i} f_1(\zeta)d\zeta \\ \vdots \\ \int_{\sigma_i} f_g(\zeta)d\zeta \end{pmatrix} \qquad i = 1, \dots, 2g$$

are linearly independent over $\mathbb{R}$. ⬣

A more general version of Proposition 5.7 holds where $X(\Gamma)$ is replaced by a Riemann surface X and $\mathcal{S}_2(\Gamma)$ is replaced by $\Omega_{\text{hol}}(X)$. The proof of this general version uses de Rham cohomology which defines the genus of X as the dimension of the exact holomorphic differentials modulo the closed differentials, see [28, Definition 4.9] for more details. Additionally, in [28, Section VIII.4] this theorem is proven using Riemann's Bilinear relations. In [34, Proposition VI.5.1.a], Silverman uses the pull-back of the isomorphism in Proposition 4.23 to show that the integrals of the invariant differentials over homology generators of an elliptic curve are $\mathbb{R}-$linearly independent. This shows Proposition 5.5 in the special case where $X(\Gamma)$ is of genus one assuming the fact that if $X(\Gamma)$ is of genus one, then it is isomorphic to an elliptic curve.

Definition 5.8. Let Γ be a congruence subgroup. Let $\sigma_1, \dots, \sigma_{2g}$ be a basis for $H_1(X(\Gamma), \mathbb{Z})$ and $f_1, \dots, f_g$ a basis for $\mathcal{S}_2(\Gamma)$. The $g \times 2g$ *period matrix* Ω is defined as

$$\Omega = \begin{pmatrix} \int_{\sigma_1} f_1(\zeta)d\zeta & \cdots & \int_{\sigma_{2g}} f_1(\zeta)d\zeta \\ \vdots & \ddots & \vdots \\ \int_{\sigma_1} f_g(\zeta)d\zeta & \cdots & \int_{\sigma_{2g}} f_g(\zeta)d\zeta \end{pmatrix}$$

▲

From Proposition 5.7 we immediately see that Ω has rank g. Suppose that Γ is a congruence subgroup such that $X(\Gamma)$ has genus 1. Let $x, x_0 \in X(\Gamma)$ and $\omega \in \Omega_{\text{hol}}(X(\Gamma))$, the path integral

$$\int_{x_0}^{x} \omega$$

is dependent of the path from x_0 to x. To see this, let ν and ν' be two paths connecting x_0 and x. Let σ be the loop which traverses forward along ν and backwards along ν'. Then

$$\int_{\sigma} \omega = \int_{\nu} \omega - \int_{\nu'} \omega \quad \Leftrightarrow \quad \int_{\nu} \omega = \int_{\nu'} \omega + \int_{\sigma} \omega.$$

It follows that integration from x to x_0 is well-defined modulo integrating along loops in $X(\Gamma)$. To make this precise, let σ_1 and σ_2 be a basis of $H_1(X(\Gamma), \mathbb{Z})$ and let Λ be the lattice

$$\Lambda = \mathbb{Z}\left(\int_{\sigma_1} \omega\right) \oplus \mathbb{Z}\left(\int_{\sigma_2} \omega\right). \tag{5.3}$$

Then, the integral

$$\int_{x_0}^{x} \omega + \Lambda$$

is independent of the path between x_0 and x and hence well-defined.

Definition 5.9. [28, Section VIII.2] Let Γ be a congruence subgroup such that $X(\Gamma)$ has genus 1. Let $x_0 \in X(\Gamma)$ and let σ_1 and σ_2 be representatives of a basis of $H_1(X(\Gamma), \mathbb{Z})$. Furthermore, let Λ be the lattice generated by $\int_{\sigma_1} \omega$ and $\int_{\sigma_2} \omega$ for some nonzero $\omega \in \Omega_{\text{hol}}(X(\Gamma))$. The *Abel-Jacobi Map on* $X(\Gamma)$ is the map

$$\begin{aligned} X(\Gamma) &\to \mathbb{C}/\Lambda \\ x &\mapsto \int_{x_0}^{x} \omega + \Lambda \end{aligned}$$

▲

By the argument above, the Abel-Jacobi map is independent of the choice of path between x and x_0. Furthermore, Λ is indeed a lattice by Proposition 5.7.

Proposition 5.10. [3, Section 4] Let Γ be a congruence subgroup such that $X(\Gamma)$ has genus one. The Abel-Jacobi map is independent of the basepoint $x_0 \in X(\Gamma)$ and an isomorphism $X(\Gamma) \to \mathbb{C}/\Lambda$ of Riemann surfaces. ⬣

5.2 Modular symbols

Modular symbols describe elements of the homology group of $X(\Gamma)$. This is useful when determining the homology group explicitly; this is done in Section 6.1 using $M-$symbols. Modular symbols have more uses than this, however; in [10, Chapter II], Cremona uses the duality between $\mathcal{S}_2(\Gamma_0(N))$ and $H_1(X_0(N), \mathbb{C})$ to calculate the $q-$expansions of the so-called newforms of $\mathcal{S}_2(\Gamma_0(N))$. In this section we define modular symbols and explore this duality. We then use this duality to connect the period lattice to the homology group of $X(\Gamma)$.

Let X be a Riemann surface. By Proposition 2.1, $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ where g is the genus of X. For a given ring R, the homology group as an $R-$module is defined by

$$H_1(X, R) = H_1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} R.$$

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let $x, y \in \mathcal{H}_\infty$ be such that $x = \gamma y$ for some $\gamma \in \Gamma$. Then the path in $\mathcal{H}_\infty$ connecting x and y is a loop in $X(\Gamma)$ when passing to the quotient via $\pi \colon \mathcal{H}_\infty \to X(\Gamma)$ and therefore defines an element in $H_1(X(\Gamma), \mathbb{Z})$.

Definition 5.11. [10, Section 2.1.2] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $x, y \in \mathcal{H}_\infty$ such that $x = \gamma y$ for some $\gamma \in \Gamma$. The image of the path connecting x and y in $\mathcal{H}_\infty$ under π is denoted by the *modular symbol* $\{x, y\} \in H_1(X(\Gamma), \mathbb{Z})$. ▲

It turns out that every path $\sigma \in H_1(X(\Gamma), \mathbb{Z})$ is of the form $\{x, y\}$ for some $x, y \in \mathcal{H}_\infty$ equivalent under the action of Γ. This is proven by Manin in [25, Proposition 1.4]. He does this by explicitly constructing an element $\{x, \gamma x\}$ for a given representative of $H_1(X(\Gamma), \mathbb{Z})$. To do this, he uses the fact that the abelianization of the fundamental group is equal to the homology group. Let $f \in \mathcal{S}_2(\Gamma)$ and $\sigma \in H_1(X(\Gamma), \mathbb{Z})$ corresponding to $\{x, y\}$ we define

$$\langle \sigma, f \rangle = \int_\sigma 2\pi i f(\zeta) d\zeta = \int_x^y 2\pi i f(\zeta) d\zeta. \tag{5.4}$$

We see that σ acts as a linear functional on $\mathcal{S}_2(\Gamma)$ via $f \mapsto \langle \sigma, f \rangle$. Next, we wish to extend modular symbols to $H_1(X(\Gamma)), \mathbb{R}))$. We do this using the following result.

Proposition 5.12. [10, Section 2.1.2] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. There is a bijection between $H_1(X(\Gamma), \mathbb{R})$ and the $\mathbb{C}-$linear functionals on $\mathcal{S}_2(\Gamma)$. ⬣

Proof. Let $\sigma_1, \dots, \sigma_{2g}$ be representatives of a basis for $H_1(X(\Gamma), \mathbb{Z})$ and let $\sigma \in H_1(X(\Gamma), \mathbb{R})$. Write

$$\sigma = \sum_{i=1}^{2g} c_i \sigma_i \quad \text{with} \quad c_i \in \mathbb{R}.$$

Then, for $f \in \mathcal{S}_2(\Gamma)$ define

$$\langle \sigma, f \rangle = \sum_{i=1}^{2g} c_i \langle \sigma_i, f \rangle$$

where $\langle \sigma_i, f \rangle$ are as (5.4). Then $f \mapsto \langle \sigma, f \rangle$ is a $\mathbb{C}-$linear functional on $\mathcal{S}_2(\Gamma)$.

Conversely, suppose $\omega \colon \mathcal{S}_2(\Gamma) \to \mathbb{C}$ is linear. Let $f_1, \dots, f_g$ be a basis for $\mathcal{S}_2(\Gamma)$. The vector

$$\begin{pmatrix} \omega(f_1) \\ \vdots \\ \omega(f_g) \end{pmatrix}$$

can be expressed as a $\mathbb{R}-$linear combination of column vectors of the period matrix Ω. This expression gives simultaneous $c_1 \dots, c_{2g} \in \mathbb{R}$ such that

$$\omega(f_i) = \sum_{j=1}^{2g} c_j \langle \sigma_j, f_i \rangle \quad \text{for all} \quad i = 1, \dots, g.$$

For $f = \sum_{i=1}^{g} \alpha_i f_i \in \mathcal{S}_2(\Gamma)$ we have

$$\omega(f) = \sum_{i=1}^{g} \alpha_i \omega(f_i) = \sum_{i=1}^{g} \alpha_i \sum_{j=1}^{2g} c_j \langle \sigma_j, f_i \rangle = \sum_{j=1}^{2g} c_j \langle \sigma_j, \sum_{i=1}^{g} \alpha_i f_i \rangle = \sum_{j=1}^{2g} c_j \langle \sigma, f \rangle = \langle \sigma, f \rangle$$

where $\sigma = \sum_{j=1}^{2g} c_j \sigma_j \in H_1(X(\Gamma), \mathbb{R})$. Then $\omega = (f \mapsto \langle \sigma, f \rangle)$ as desired. ■

Let $x, y \in \mathcal{H}_\infty$. The map $f \mapsto \int_x^y f(\zeta) d\zeta$ is a linear functional on $\mathcal{S}_2(\Gamma)$ and by Proposition 5.12 it corresponds to some $\sigma \in H_1(X(\Gamma), \mathbb{R})$. Define $\{x, y\} \in H_1(X(\Gamma), \mathbb{R})$ to be this element. This agrees with the definition above in the case that x and y are equivalent under Γ. These extended modular symbols are elements of an $\mathbb{R}-$vector space and we therefore have a notion of adding them. We discuss some algebraic properties which follow from the definition. For example, for $x, y, z \in \mathcal{H}_\infty$, the element $\{x, y\} + \{y, z\}$ corresponds to

$$f \mapsto \int_x^y 2\pi i f(\zeta) d\zeta + \int_y^z 2\pi i f(\zeta) d\zeta = \int_x^z 2\pi i f(\zeta) d\zeta$$

which corresponds to $\{x, z\}$. We obtain the identity $\{x, y\} + \{y, z\} = \{x, z\}$. More identities are obtained in similar fashion summarized in the following proposition.

Proposition 5.13. [10, Proposition 2.1.1] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $x, y, z \in \mathcal{H}_\infty$, and let $\gamma, \gamma_1, \gamma_2 \in \Gamma$. Then

1. $\{x, x\} = 0$;
2. $\{x, y\} + \{y, x\} = 0$;
3. $\{x, y\} + \{y, z\} + \{z, x\} = 0$;
4. $\{\gamma x, \gamma y\} = \{x, y\}$;
5. $\{x, \gamma x\} = \{y, \gamma y\}$;
6. $\{x, \gamma_1 \gamma_2 x\} = \{x, \gamma_1 x\} + \{x, \gamma_2 x\}$;
7. $\{x, \gamma x\} \in H_1(X(\Gamma), \mathbb{Z})$. ⬣

Corollary 5.14. [25, Proposition 1.4] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $x \in \mathcal{H}_\infty$. The map $\gamma \mapsto \{x, \gamma x\}$ is a surjective group homomorphism $\Gamma \to H_1(X(\Gamma), \mathbb{Z})$ independent of x. ⬣

Corollary 5.14 in particular implies that if $X(\Gamma)$ has genus 1, then the period lattice of a nonzero $f \in \mathcal{S}_2(\Gamma)$ is equal to

$$\Lambda_f = \left\{ \int_x^{\gamma x} 2\pi i f(\zeta) d\zeta : \gamma \in \Gamma \right\} = \left\{ \int_\sigma 2\pi i f(\zeta) d\zeta : \sigma \in H_1(X(\Gamma), \mathbb{Z}) \right\} = \mathbb{Z}\langle \sigma_1, f \rangle \oplus \mathbb{Z}\langle \sigma_2, f \rangle \tag{5.5}$$

where σ_1 and σ_2 form a $\mathbb{Z}$ basis for $H_1(X(\Gamma), \mathbb{Z})$. Proposition 5.7 ensures that Λ_f is indeed a lattice. Note that Λ_f is precisely the lattice (5.3) in the definition of the Abel-Jacobi map. This map is an isomorphism by Proposition 5.10. Let E_f denote the elliptic curve which is isomorphic to $\mathbb{C}/\Lambda_f$ via the isomorphism in Proposition 4.23. The above proves the following.

Theorem 5.15. Let N be one of $11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49$. Let $f \in \mathcal{S}_2(\Gamma_0(N))$ and E_f the elliptic curve defined above. The modular curve $X_0(N)$ is isomorphic to E_f via the composition of the maps

$$X_0(N) \to \mathbb{C}/\Lambda_f \to E_f \tag{5.6}$$

where the first map is the Abel-Jacobi map. ⬣

It turns out that the modular curve $X_0(N)$ is an algebraic curve defined over $\mathbb{Q}$ and that the maps in Theorem 5.15 are rational maps defined over $\mathbb{Q}$, see [13, Section 7.7]. From this, it follows that $X_0(N)$ is isomorphic to an elliptic curve which is defined over $\mathbb{Q}$. When the genus of $X_0(N)$ is greater than one, a similar holomorphic mapping $X_0(N) \to E$ exists, where E is an elliptic curve defined over $\mathbb{Q}$. Such a map is not an isomorphism however, merely a surjection. Such an elliptic curve over $\mathbb{Q}$ comes from a so-called newform $f \in \mathcal{S}_2(\Gamma_0(N))$, see [18, Theorem 11.74]. The existence of such a map is precisely what it means for an elliptic curve over $\mathbb{Q}$ to be modular.

6 Putting the theory into numbers

Let N be such that $X_0(N)$ is of genus one. Our goal is to calculate a formula for the elliptic curve E_f over $\mathbb{Q}$ which is isomorphic to $X_0(N)$ via (5.6). To do this, the period lattice has to be calculated, we do this by using the right hand side of (5.5). This means that we need to calculate a basis for $H_1(X_0(N), \mathbb{Z})$. In this section, by following the ideas in [10, Sections 2.2-2.5], we develop the theory of $M-$symbols. This allows us to find an explicit basis of $H_1(X_0(N), \mathbb{Z})$. Furthermore, we determine efficient ways to calculate g_4 and g_6 and find error bounds for these calculations. Finally, we go through the process of calculating the elliptic curve formula's of $X_0(14)$ and $X_0(24)$. The methods of computing E_f are implemented in Python and can be found in [29].

The methods we use are the methods in Cremona's book [10] simplified to the case where the genus of $X_0(N)$ is one. In the general case, one has to work with newforms in $\mathcal{S}_2(\Gamma_0(N))$. In [10, Section 2.9], a method to calculate the $q-$expansions of these objects for a given N is described. Sections 2.10-2.14 of [10] are dedicated to calculate the equations for the elliptic curves corresponding to these newforms. The LMFDB [24] is a database of elliptic curves, modular forms, and other related objects. This database contains a little less than 300,000 newforms (as of June 2023) up to level $N = 10,000$ including those which we care about. Furthermore, the LMFDB contains a little less than 3 million elliptic curves over $\mathbb{Q}$. We shall use this database to check whether we obtained the correct formula for E_f and to obtain the $q-$expansions of the generators of $\mathcal{S}(\Gamma_0(N))$ with N in (2.15).

6.1 $M-$symbols

When calculating the period lattice, we require an explicit basis for the first homology group of $X_0(N)$. In this section we summarize results from [10, Sections 2.1-2.3] which allow us to compute $H_1(X_0(N), \mathbb{Z})$ using $M-$symbols. These sections in Cremona's book are adapted from Manin's paper [25, Chapter 1] where he introduced $M-$symbols as 'distinguished classes'.

The duality over $\mathbb{R}$ between $H_1(X(\Gamma), \mathbb{R})$ and $\mathcal{S}_2(\Gamma)$ obtained in the previous section can be extended to a duality over $\mathbb{C}$. Let $\sigma \in H_1(X(\Gamma), \mathbb{R})$ and $c \in \mathbb{C}$. Define $c\sigma \in H_1(X(\Gamma), \mathbb{C})$ to be the element corresponding to the functional $f \mapsto c\langle \sigma, f\rangle$. This construction gives a duality over $\mathbb{C}$. Next, we split $H_1(X(\Gamma), \mathbb{R})$ into two eigenspaces.

Definition 6.1. [10, Section 2.1.3] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We say that Γ is of *real type* if for every $\gamma = (a, b; c, d) \in \Gamma$ we have that

$$\gamma^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

is an element in Γ. ▲

For $z \in \mathcal{H}_\infty$ define $z^* = -\overline{z}$. It is readily shown that $(\gamma z)^* = \gamma^* z^*$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}_\infty$ meaning that the map $\pi(z) \mapsto \pi(z^*)$ is a well-defined map $X(\Gamma) \to X(\Gamma)$ if and only if Γ is of real type. If Γ is of real type we also have that $\{x, y\} \mapsto \{x^*, y^*\}$ is a well-defined linear mapping $H_1(X(\Gamma), \mathbb{R}) \to H_1(X(\Gamma), \mathbb{R})$. The map $*$ has order 2, consequently, its eigenvalues are ± 1. We can therefore decompose $H_1(X(\Gamma), \mathbb{R})$ into eigenspaces as

$$H_1(X(\Gamma), \mathbb{R}) = H_1^-(X(\Gamma), \mathbb{R}) \oplus H_1^+(X(\Gamma), \mathbb{R}). \tag{6.1}$$

We aim to use the duality between $H_1(X(\Gamma), \mathbb{R})$ and $\mathcal{S}_2(\Gamma)$ to obtain properties of $H_1(X(\Gamma), \mathbb{R})$. It turns out to be useful to define the map $f \mapsto f^*$ where $f^*(z) = \overline{f(z^*)}$. For a holomorphic function $f \colon \mathcal{H} \to \mathbb{C}$ the map $f^* \colon \mathcal{H} \to \mathbb{C}$ is again holomorphic on $\mathcal{H}$.

Proposition 6.2. [10, Section 2.1.3] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let f be holomorphic on $\mathcal{H}$, $\gamma \in \Gamma$ and $\sigma \in H_1(X(\Gamma), \mathbb{R})$. Then

1. If f has $q-$expansion $f(z) = \sum_{n\geq 0} a_n q^n$, then f^* has $q-$expansion $f^*(z) = \sum_{n\geq 0} \overline{a_n} q^n$;
2. for $\gamma \in \Gamma$, we have $f^*[\gamma] = (f[\gamma^*])^*$;
3. if $f \in \mathcal{S}_2(\Gamma)$ then $\langle \sigma^*, f^*\rangle = \overline{\langle \sigma, f\rangle}$. ⬣

From Proposition 6.2.2 we see that if Γ is of real type, then the map $f \mapsto f^*$ is a well-defined $\mathbb{R}$ linear mapping $\mathcal{S}_2(\Gamma) \to \mathcal{S}_2(\Gamma)$ which equals the identity when composed with itself. Let $\mathcal{S}_2(\Gamma)_{\mathbb{R}}$ be the subspace fixed by $f \mapsto f^*$. From Proposition 6.2.3 we have that if $\langle \sigma, f \rangle \in \mathbb{R}$ for some $\sigma \in H_1(X(\Gamma), \mathbb{R})$ and for all $f \in \mathcal{S}_2(\Gamma)_{\mathbb{R}}$, then $\langle \sigma, f \rangle = \langle \sigma^*, f^* \rangle = \langle \sigma^*, f \rangle$ and hence $\sigma \in H_1^+(X(\Gamma), \mathbb{R})$. Similarly, if $\langle \sigma, f \rangle \in i\mathbb{R}$ for all $f \in \mathcal{S}_2(\Gamma)_{\mathbb{R}}$, then $\sigma \in H_1^-(X(\Gamma), \mathbb{R})$. Moreover, for any $f \in \mathcal{S}_2(\Gamma)_{\mathbb{R}}$,

$$\sigma \in H_1^+(X(\Gamma), \mathbb{R}) \Leftrightarrow \langle \sigma, f \rangle \in \mathbb{R} \Leftrightarrow \langle i\sigma, f \rangle \in i\mathbb{R} \Leftrightarrow i\sigma \in H_1^-(X(\Gamma), \mathbb{R}).$$

This shows that $\sigma \mapsto i\sigma$ is an isomorphism of vector spaces $H_1^+(X(\Gamma), \mathbb{R}) \to H_1^-(X(\Gamma), \mathbb{R})$. Consequently, we obtain $\dim H_1^+(X(\Gamma), \mathbb{R}) = \dim H_1^-(X(\Gamma), \mathbb{R})$ which must then be equal to g by the decomposition (6.1).

Next, we investigate the $\mathbb{Q}-$vector space $H_1(X(\Gamma), \mathbb{Q})$. This restriction to $\mathbb{Q}$ is of interested to us due to the following result.

Theorem 6.3. (Manin, Drinfeld) [10, Theorem 2.1.3] Let Γ be a congruence subgroup. The vector space $H_1(X(\Gamma), \mathbb{Q})$ is generated by the modular symbols of the form $\{x, y\}$ where $x, y \in \mathbb{Q} \cup \{\infty\} \subset \mathcal{H}_\infty$. ⬣

This theorem is proven using Hecke operators, which are linear functions $\mathcal{S}_2(\Gamma) \to \mathcal{S}_2(\Gamma)$ with eigenvalues that satisfy some useful properties. Section 2.9 of [10] gives a sketch of the argument for the special case where $\Gamma = \Gamma_0(N)$. When computing the period lattice, $H_1(X(\Gamma), \mathbb{Q})$ has to be computed explicitly. This is done using $M-$symbols. To introduce these, we need to find an explicit triangulation of $X(\Gamma)$.

For $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, define (α) to be the path in $\mathcal{H}_\infty$ connecting $\alpha(0)$ and $\alpha(\infty)$. Let $(\alpha)_\Gamma$ denote the image of this path under $\pi : \mathcal{H}_\infty \to X(\Gamma)$. Define $C(\Gamma)$ as the vector space spanned by the symbols $(\alpha)_\Gamma$. For $\gamma \in \Gamma$ we have that $(\gamma\alpha)_\Gamma = (\alpha)_\Gamma$, therefore $C(\Gamma)$ has at most dimension $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Let $B(\Gamma) \subset C(\Gamma)$ be the space spanned by

$$(\alpha S)_\Gamma + (\alpha)_\Gamma \tag{6.2}$$
$$\text{and} \quad (\alpha)_\Gamma + (\alpha TS)_\Gamma + (\alpha(TS)^2)_\Gamma. \tag{6.3}$$

Here, S and T are as in Proposition 3.3. The idea here is that (6.3) represents the image under α of the triangle with vertices $0, \infty$ and 1 which is represented by the element $(I)_\Gamma + (TS)_\Gamma + ((TS)^2)_\Gamma$. The paths $(\alpha)_\Gamma$ and $(S\alpha)_\Gamma$ are the same, but have different orientation, we therefore want $(\alpha)_\Gamma = -(S\alpha)_\Gamma$. We obtain this from (6.2) when we quotient out $B(\Gamma)$ from $C(\Gamma)$. Let $C_0(\Gamma)$ be the space spanned by elements of the form $\pi(s)$ with $s \in \mathbb{Q} \cup \{\infty\}$. In other words $C_0(\Gamma)$ is spanned by the cusps of $X(\Gamma)$. Define the boundary map

$$\begin{aligned} \delta \colon C(\Gamma) &\to C_0(\Gamma) \\ (\alpha)_\Gamma &\mapsto \pi(\alpha\infty) - \pi(\alpha 0). \end{aligned}$$

A simple computation shows that $B(\Gamma) \subset \ker \delta$. Define $H(\Gamma) = \ker \delta / B(\Gamma)$. The set $H(\Gamma)$ can be interpreted as the set of closed loops in $X(\Gamma)$ which are not of the form $(\alpha)_\Gamma + (\alpha TS)_\Gamma + (\alpha(TS)^2)_\Gamma$. This construction actually gives the rational homology on $X(\Gamma)$.

Theorem 6.4. [25, Theorem 1.9] Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then $H(\Gamma)$ is isomorphic to $H_1(X(\Gamma), \mathbb{Q})$. The isomorphism is induced by

$$(\alpha)_\Gamma \mapsto \{\alpha 0, \alpha\infty\}.$$

⬣

We interchangeably write elements of $H(\Gamma)$ as $(\alpha)_\Gamma$ with $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ or as $\{\alpha 0, \alpha\infty\}$. We can write an arbitrary $\{x, y\} \in H_1(X(\Gamma), \mathbb{Q})$ as a sum of elements in $H(\Gamma)$ as follows. First write $\{x, y\} = \{0, y\} - \{0, x\}$. Then find the continued fraction expansion $x = [a_0; a_1, \ldots, a_k]$ for some k, then $x = p_k/q_k$ where

$$\begin{aligned} p_{-2} &= 0 & q_{-2} &= 1 \\ p_{-1} &= 1 & q_{-1} &= 0 \\ p_n &= a_n p_{n-1} + p_{n-2} & q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

for $n = 0, \ldots, k$. See [32, Section 12.2] for more details. The integers p_j and q_j satisfy the relation

$$(-1)^{j-1}p_jq_{j-1} - (-1)^{j-1}p_{j-1}q_j = 1, \quad j = -1, 0, \ldots, k. \tag{6.4}$$

Then

$$\{0, x\} = \sum_{j=-1}^{k} \left\{ \frac{p_{j-1}}{q_{j-1}}, \frac{p_j}{q_j} \right\} = \sum_{j=-1}^{k} \{\alpha_j 0, \alpha_j \infty\} = \sum_{j=-1}^{k} (\alpha_j)_\Gamma \tag{6.5}$$

where we set $1/0 = \infty$ and

$$\alpha_j = \begin{pmatrix} (-1)^{j-1}p_j & p_{j-1} \\ (-1)^{j-1}q_j & q_{j-1} \end{pmatrix}$$

which is an element of $\mathrm{SL}_2(\mathbb{Z})$ by (6.4).

We now specialise to the case where $\Gamma = \Gamma_0(N)$. In the construction of $H(\Gamma)$ it is crucial to know when two elements of $\mathrm{SL}_2(\mathbb{Z})$ are in the same coset. The following result helps us determine this for $\Gamma_0(N)$.

Proposition 6.5. [10, Proposition 2.2.1.] Let $N > 1$ be an integer. For $j = 1, 2$, let $\alpha_j = (a_j, b_j; c_j, d_j) \in \mathrm{SL}_2(\mathbb{Z})$. The following are equivalent.

1. The right cosets $\Gamma_0(N)\alpha_1$ and $\Gamma_0(N)\alpha_2$ are equal;
2. $c_1d_2 \equiv c_2d_1 \mod N$;
3. there exists u coprime to N such that $c_1 \equiv uc_2 \mod N$ and $d_1 \equiv ud_2 \mod N$. ⬣

We define an equivalence relation on the pairs $(c, d) \in \mathbb{Z}^2$ with $\gcd(c, d, N) = 1$ by setting

$$(c_1, d_1) \sim (c_2, d_2) \quad \text{if and only if} \quad c_1d_2 \equiv c_2d_1 \mod N. \tag{6.6}$$

The equivalence of 2 and 3 of Proposition 6.5 ensures that this is an equivalence relation, while the equivalence of 1 and 2 gives an alternate way to write down the right cosets of $\Gamma_0(N)$.

Definition 6.6. Let $N > 1$ be an integer. The equivalence class of (c, d) for the equivalence (6.6) is denoted by $(c : d)$ and is referred to as an $M-$symbol. The set of $M-$symbols is denoted by $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. ▲

Every representative $(c : d)$ is determined modulo N and can be chosen such that $\gcd(c, d) = 1$. Proposition 6.4 and the equivalence of 1 and 2 in Proposition (6.5) gives us the following result.

Proposition 6.7. [10, Proposition 2.2.2.] Let $N > 1$ be an integer. The following maps are bijections.

$$\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \longleftrightarrow \mathrm{SL}_2(\mathbb{Z})/\Gamma_0(N) \longleftrightarrow C(\Gamma_0(N))$$

$$(c : d) \longleftarrow\!\!\!\!\!\mapsto \Gamma_0(N) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longleftarrow\!\!\!\!\!\mapsto \{b/d, a/c\}$$

where a and b are integers chosen such that $ad - bc = 1$. ⬣

The map on the left is independent of the choice of a and b since the solutions of the equation $xd - yc = 1$ have the form $(x, y) = (a + kc, b + kd)$ for some $k \in \mathbb{Z}$ so that the right coset is given by

$$\Gamma_0(N)T^k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Gamma_0(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here, we used that $T \in \Gamma_0(N)$ with T as in Proposition 3.3. Determining $H(\Gamma_0(N))$ is much less of a perilous exercise when using $M-$symbols rather than the definition of $C(\Gamma_0(N))$. To do this computation we need to carry over a few things that we defined on symbols of the form $(\alpha)_\Gamma$ to symbols of the form $(c : d)$. The bijections in Proposition 6.7 allow us to view $C(\Gamma_0(N))$ as a vector space spanned by the symbols $(c : d)$. The boundary map then becomes

$$\delta \colon (c : d) \mapsto \pi(a/c) - \pi(b/d).$$

Moreover, the right action of $\mathrm{SL}_2(\mathbb{Z})$ on $C(\Gamma_0(N))$ defined by $(\alpha)_\Gamma A := (\alpha A)_\Gamma$ induces an action on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ defined as

$$(c:d)\begin{pmatrix} p & q \\ r & s \end{pmatrix} = (cp+dr : cq+ds).$$

Then $B(\Gamma_0(N))$ is spanned by elements of the form

$$(c:d)+(c:d)S = (c:d)+(-d:c) \quad \text{and}$$
$$(c:d)+(c:d)TS+(c:d)(TS)^2 = (c:d)+(d+c:-c)+(d:-c-d).$$

The map $*\colon H_1(X_0(N),\mathbb{R}) \to H_1(X_0(N),\mathbb{R})$ restricts to $H_1(X_0(N),\mathbb{Q}) = H(\Gamma_0(N))$ and acts on the $M-$symbols as

$$*(c:d) = (-c:d).$$

Denote the eigenspaces of this restriction as $H^+(\Gamma_0(N))$ and $H^-(\Gamma_0(N))$. Finally, since we want to explicitly calculate $H(\Gamma_0(N))$ we need to determine $\ker\delta$. To simplify this process we use a result that states that cusp equivalence is equivalent to a condition that is easily verifiable.

Proposition 6.8. [10, Proposition 2.2.3.] For $j = 1,2$ let $s_j = p_j/q_j$ with p_j and q_j coprime. The following are equivalent.

1. $s_2 = \alpha(s_1)$ for some $\alpha \in \Gamma_0(N)$;
2. there is some u coprime to N such that $q_2 \equiv uq_1 \bmod N$ and $up_2 \equiv p_1 \bmod \gcd(q_1,N)$. ⬣

6.2 Calculating the period lattice

With an explicit basis for the homology group of $X_0(N)$ we can calculate the period lattice. In this section we describe the process of explicitly calculating the period lattice. We do this using a simplified version of the approach described in [10, Section 2.10]. Furthermore, we find some numerical error bounds that will allow us to make our method provably correct to any desired precision.

In this section, we assume that $N > 1$ is such that $X_0(N)$ has genus one. In other words, N is among (2.15). Define the *period* of $\gamma \in \Gamma_0(N)$ with respect to $f \in \mathcal{S}_2(\Gamma_0(N))$ as the complex number

$$\Phi_f(\gamma) = \int_x^{\gamma x} 2\pi i f(\zeta)d\zeta.$$

By Proposition 5.13, Φ_f is a homomorphism $\Gamma_0(N) \to \Lambda_f$ independent of the basepoint x. By Corollary 5.14 the image of Φ_f is the period lattice Λ_f. The first step to calculating the period lattice is to determine a basis for $H_1(X_0(N),\mathbb{Q}) = H(\Gamma_0(N))$ (and hence a set of generators for $H_1(X_0(N),\mathbb{Z})$). This can be explicitly done thanks to the theory of $M-$symbols developed in the previous section. The integral homology $H_1(X_0(N),\mathbb{Z})$ is contained in $H_1(X_0(N),\mathbb{Q})$ and is given by the elements $\{x,y\}$ where x and y are equivalent cusps in $X_0(N)$.

To calculate $H(\Gamma_0(N))$ we need to find a basis for $\ker\delta/B(\Gamma_0(N))$. This can be done without too much labour when we possess a basis for $C(\Gamma_0(N))/B(\Gamma_0(N))$ beforehand. To see that this works, we invoke the universal property for quotient spaces to obtain a unique linear map $\overline{\delta}\colon C(\Gamma_0(N))/B(\Gamma_0(N)) \to C_0(\Gamma_0(N))$ such that

$$\begin{tikzcd} C(\Gamma_0(N)) \arrow[r, "\delta"'] \arrow[d, two heads] & C_0(\Gamma_0(N)) \\ C(\Gamma_0(N))/B(\Gamma_0(N)) \arrow[ur, "\overline{\delta}"'] & \end{tikzcd}$$

commutes. A swift computation verifies that $\ker\overline{\delta} = \ker\delta/B(\Gamma_0(N))$. From now on we write δ in place of $\overline{\delta}$. By taking the quotient of $B(\Gamma_0(N))$ we are setting the expressions of the form

$$\begin{gathered}(c:d)+(-d:c)\\ (c:d)+(d+c:-c)+(d:-c-d)\end{gathered} \tag{6.7}$$

equal to 0. So $C(\Gamma_0(N))/B(\Gamma_0(N))$ can be seen as the kernel of the linear map mapping the $C(\Gamma_0(N))$ to the relations (6.7). The matrix representing this linear map is quite large, even for small values of N. The function `modulo_relations_matrix(N)` in [29] creates the matrix representation of the linear map (6.7) with respect to an ordered basis of $C(\Gamma_0(N))/B(\Gamma_0(N))$ and returns this matrix in its reduced row echelon form. From this matrix we can read off a basis of $C(\Gamma_0(N))/B(\Gamma_0(N))$ and how every $M-$symbol in $C(\Gamma_0(N))/B(\Gamma_0(N))$ is expressed in terms of this basis. Next, we use the basis of $C(\Gamma_0(N))/B(\Gamma_0(N))$ to calculate $H(\Gamma_0(N)) = \ker\delta/B(\Gamma_0(N))$. Since N is chosen such that the genus of $X_0(N)$ is 1 we expect to find two linearly independent elements that vanish under δ. To calculate $\ker\delta$ it is useful to know which cusps are equivalent. The function `equiv_test(p1,q1,p2,q2,N)` implements Proposition 6.8 and is the main tool we use to determine a complete set of cusp equivalence classes for $X_0(N)$.

Let σ_1 and σ_2 be generators of $H_1(X(\Gamma_0(N),\mathbb{Z})$ and therefore also a basis for $H(\Gamma_0(N))$. Every element $\sigma \in H(\Gamma_0(N))$ can be written as $\sigma = a_1\sigma_1 + a_2\sigma_2$ for $a_1, a_2 \in \mathbb{Q}$. We represent σ as the vector $(a_1, a_2)^T \in \mathbb{Q}^2$. The next step is to split off $H(\Gamma_0(N))$ into the one dimensional eigenspaces $H^+(\Gamma_0(N))$ and $H^-(\Gamma_0(N))$ which is possible as $\Gamma_0(N)$ is of real type. Let σ^+ and σ^- be the generators of these spaces. Since

$$H(\Gamma_0(N)) = H^+(\Gamma_0(N)) \oplus H^-(\Gamma_0(N))$$

we can write every element $\sigma \in H(\Gamma_0(N))$ uniquely as

$$\sigma = c_1\sigma^+ + c_2\sigma^- \quad \text{with} \quad c_1, c_2 \in \mathbb{Q}.$$

Write σ^+ and σ^- as vectors with respect to the ordered basis (σ_1, σ_2) and construct the invertible matrix

$$\begin{pmatrix}\sigma^+ & \sigma^-\end{pmatrix} \in \mathbb{Q}^{2\times 2}.$$

Define the row vectors v^+ and v^- as

$$\begin{pmatrix}v^+ \\ v^-\end{pmatrix} = \begin{pmatrix}\sigma^+ & \sigma^-\end{pmatrix}^{-1}.$$

Then

$$\sigma = c_1\sigma^+ + c_2\sigma^- = (v^+\sigma)\sigma^+ + (v^-\sigma)\sigma^-. \tag{6.8}$$

Let $\sigma \in H_1(X_0(N),\mathbb{Z})$. By (6.8), for a generator f of $\mathcal{S}_2(\Gamma_0(N))$ we have

$$\langle\sigma, f\rangle = (v^+\sigma)\langle\sigma^+, f\rangle + (v^-\sigma)\langle\sigma^-, f\rangle. \tag{6.9}$$

Define $x = \langle\sigma^+, f\rangle$ and $y = -i\langle\sigma^-, f\rangle$. The cusp form f has real Fourier coefficients by [10, Chapter II]. From Proposition 6.2.1 and 6.2.3 it follows that both x and y are real numbers. Then

$$\langle\sigma, f\rangle = (v^+\sigma)x + i(v^-\sigma)y. \tag{6.10}$$

Since σ_1 and σ_2 generate $H_1(X_0(N),\mathbb{Z})$, by (5.5) and (6.10) we have

$$\Lambda_f = \mathbb{Z}\langle\sigma_1, f\rangle \oplus \mathbb{Z}\langle\sigma_2, f\rangle = \mathbb{Z}(a_1x + ib_1y) \oplus \mathbb{Z}(a_2x + ib_2y).$$

where $v^+ = \begin{pmatrix}a_1 & a_2\end{pmatrix}$ and $v^- = \begin{pmatrix}b_1 & b_2\end{pmatrix}$. To get the full period lattice, all that remains is to find x and y. By (6.9) it suffices to calculate $\langle\sigma, f\rangle$ for some integral cycle $\sigma \in H_1(X_0(N),\mathbb{Z})$ and setting

$$x = \frac{\mathrm{Re}\langle\sigma, f\rangle}{(v^+\sigma)} \quad \text{and} \quad y = \frac{\mathrm{Im}\langle\sigma, f\rangle}{(v^-\sigma)}.$$

This is essentially one calculation of $\Phi_f(\gamma)$ where $\gamma \in \Gamma_0(N)$ satisfies $\sigma = \{\tau, \gamma\tau\}$ for any basepoint $\tau \in \mathcal{H}_\infty$. Note that all we need to determine is $\langle\sigma, f\rangle, v^+, v^-$ and a vector representation of σ. In particular, we do not have to determine σ^+, or σ^-. The vectors v^+ and v^- come from the right eigenvectors σ^+ and σ^-. If we represent $*$ as a matrix with respect to the ordered basis (σ_1, σ_2) then v^+ and v^- are left eigenvectors of this matrix. They may be scaled versions of the original $v^\pm$. However, scaling v^+ or v^- will inversely scale x and y, leaving Λ_f unchanged.

As noted multiple times, $\Phi_f(\gamma)$ is independent of the basepoint $\tau \in \mathcal{H}_\infty$. A general element $\gamma \in \Gamma_0(N)$ is of the form

$$\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}. \tag{6.11}$$

In [10], Cremona claims that choosing the basepoint

$$\tau = \frac{-d+i}{cN} \quad \text{so that} \quad \gamma\tau = \frac{a+i}{cN} \tag{6.12}$$

results in the fastest convergence of $\Phi_f(\gamma)$. We may assume that $c \neq 0$ as in the case where $c = 0$, we have $\Phi_f(\gamma) = 0$ by [18, Proposition 11.1]. We also require $c > 0$ so that $\tau \in \mathcal{H}_\infty$ which is achieved by replacing γ by $-\gamma$ if necessary.

Proposition 6.9. [10, Proposition 2.10.2.] Let $f \in \mathcal{S}_2(\Gamma_0(N))$ with $q-$expansion

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i\tau n}.$$

Then for any $\gamma = (a, b; cN, d) \in \Gamma_0(N)$, the period of γ is equal to

$$\Phi_f(\gamma) = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-2\pi n/cN}(e^{2\pi ina/cN} - e^{2\pi ind/cN}). \tag{6.13}$$

⬣

Proof. Without loss of generality we assume that $c > 0$, otherwise replace γ by $-\gamma$ which has the same action on $\mathcal{H}_\infty$. Setting the basepoint as in (6.12), we obtain

$$\Phi_f(\gamma) = \int_{(-d+i)/cN}^{(a+i)/cN} 2\pi i \sum_{n=1}^{\infty} a_n e^{2\pi i\zeta n} d\zeta.$$

The Fourier coefficients are bounded as $|a_n| \leq n$. This follows from a more general result by Deligne which bounds Fourier coefficients for general cusp forms, see [12, Théorème 8.2]. It also follows more easily from the fact that the Fourier coefficients a_p of f for a prime p of good reduction come from the amount of solutions of E_f reduced modulo p and the other coefficients via the fact that $a_{mn} = a_m a_n$ when $\gcd(m, n) = 1$. Combining this fact with the Hasse-Weil bound [34, Theorem V.1.1], we obtain the desired bound on $|a_n|$. Since a_n does not grow disproportionately large, this integral can be integrated term by term. Doing so yields (6.13). ■

We only have to calculate one period, we are therefore free to choose $\gamma = (a, b; c, d) \in \Gamma_0(N)$ with $c > 0$. To make the sum in Proposition 6.9 converge as fast as possible we want c to be as small as possible so that $e^{2\pi n/cN} \to 0$ as fast as possible, so preferably $c = 1$. When explicitly calculating a period we can only sum the first M terms of (6.13). Let $e(M)$ denote the absolute error between (6.13) and the first M terms of the sum.

Proposition 6.10. Let $f \in \mathcal{S}_2(\Gamma_0(N))$ and $(a, b; cN, d) \in \Gamma_0(N)$. Then for $M > 1$,

$$e(M) \leq \frac{e^{-2\pi(M+1)/cN}}{1 - e^{-2\pi/cN}}$$

⬣

Proof. By the triangle inequality,

$$e(M) = \left| \sum_{n=M+1}^{\infty} \frac{a_n}{n} e^{-2\pi n/cN}(e^{2\pi ina/cN} - e^{2\pi ind/cN}) \right| \leq \sum_{n=M+1}^{\infty} \left|\frac{a_n}{n}\right| e^{-2\pi n/cN} \left|e^{2\pi ina/cN} - e^{2\pi ind/cN}\right|.$$

We can bound this expression further by using the bound $|a_n| \leq n$ as in the proof of Proposition 6.9. The exponents in the expression $|e^{2\pi ina/cN} - e^{2\pi ina/cN}|$ are strictly complex. Therefore, by using the triangle inequality this expression is bounded by 1. We obtain

$$\begin{aligned} e(M) \leq \sum_{n=M+1}^{\infty} e^{-2\pi n/cN} &= \sum_{n=0}^{\infty} e^{-2\pi n/cN} - \sum_{n=0}^{M} e^{-2\pi n/cN} \\ &= \frac{1}{1 - e^{-2\pi/cN}} - \frac{1 - e^{-2\pi(M+1)/cN}}{1 - e^{-2\pi/cN}} = \frac{e^{-2\pi(M+1)/cN}}{1 - e^{-2\pi/cN}} \end{aligned} \tag{6.14}$$

■

Denote the expression on the right of (6.14) by $\varepsilon(M)$. All that remains is finding $\sigma \in H_1(X_0(\Gamma), \mathbb{Z})$ such that $\sigma = \{\tau, \gamma\tau\}$ for some $\tau \in \mathcal{H}_\infty$, we set this τ to 0 for convenience. Pick b/d such that $\pi(b/d) = \pi(0)$, so that $\{0, b/d\} \in H_1(X_0(N), \mathbb{Z})$. Picking d coprime to N will do the trick by Proposition 6.8. Solve $ad - bcN = 1$ to obtain $\gamma = (a, b; cN, d)$, this γ is then used to calculate the period $\Phi_f(\gamma)$. We have $\sigma = \{0, \gamma 0\} = \{0, b/d\}$. To obtain a vector expression of σ we express σ as in (6.5) and use the bijection in Proposition 6.7 to obtain

$$\{0, b/d\} \overset{(6.5)}{=} \sum_{j=-1}^{k} (\alpha_j)_{\Gamma_0(N)} \overset{6.7}{=} \sum_{j=-1}^{k} ((-1)^{j-1} q_j : q_{j-1}).$$

The first two terms of the sum on the right hand side are 0 in $C(\Gamma_0(N))/B(\Gamma_0(N))$ since

$$(q_{-1} : q_{-2}) + (-q_{-1} : q_{-1}) = (0 : 1) + (-1 : 0) = 0.$$

Hence

$$\{0, b/d\} = \sum_{j=1}^{k} ((-1)^{j-1} q_j : q_{j-1}). \tag{6.15}$$

We should be able to express this as a $\mathbb{Z}-$linear combination of σ_1 and σ_2 since $\{0, b/d\} \in H_1(X_0(N), \mathbb{Z})$.

By Proposition 4.23, the complex torus $\mathbb{C}/\Lambda_f$ is isomorphic to an elliptic curve E_f over $\mathbb{C}$ which is defined by the equation (4.15). We determine c_4 and c_6 via calculating g_4 and g_6. To calculate these coefficients g_4 and g_6, the sums that defines G_4 and G_6 are not suitable, because they converge very slowly. To obtain accurate values in a reasonable amount of time, we use the following formula's for g_4 and g_6.

Proposition 6.11. [27, Proposition 8.1] Let $\Lambda \subset \mathbb{C}$ be a lattice with generators ω_1 and ω_2. The values $g_4 = 60G_4(\Lambda)$ and $g_6 = 140G_6(\Lambda)$ have expressions

$$\begin{aligned} g_4 &= \frac{4\pi^4}{3\omega_2^4}\left(1 + 240\sum_{d=1}^{\infty} \frac{d^3}{e^{-2\pi i d\omega_1/\omega_2} - 1}\right) \\ g_6 &= \frac{8\pi^6}{27\omega_2^2}\left(1 - 504\sum_{d=1}^{\infty} \frac{d^5}{e^{-2\pi i d\omega_1/\omega_2} - 1}\right) \end{aligned} \tag{6.16}$$

⬣

Proof. Set $\tau = \omega_1/\omega_2 \in \mathcal{H}$. We find an expression for $G_{2k}(\tau) := G_{2k}(\mathbb{Z}\tau \oplus \mathbb{Z})$ and use (4.11) to obtain an expression for g_4 and g_6. Start with the known identity

$$\pi \cot \pi\tau = \frac{1}{\tau} + \sum_{m=1}^{\infty}\left(\frac{1}{\tau + m} + \frac{1}{\tau - m}\right)$$

which converges uniformly on compact subsets of $\mathcal{H}$. On the other hand,

$$\pi \cot \pi\tau = \pi\frac{\cos \pi\tau}{\sin \pi\tau} = i\pi - \frac{2\pi i}{1 - e^{2\pi i\tau}} = i\pi - 2\pi i\sum_{d=0}^{\infty} e^{2\pi i\tau d}.$$

We obtain

$$\frac{1}{\tau} + \sum_{m=1}^{\infty}\left(\frac{1}{\tau + m} + \frac{1}{\tau - m}\right) = i\pi - 2\pi i\sum_{d=0}^{\infty} e^{2\pi i\tau d}. \tag{6.17}$$

By taking the derivative $k-1$ times on both sides of (6.17) we obtain

$$-(2\pi i)^k \sum_{d=1}^{\infty} d^{k-1} e^{2\pi i\tau d} = (k-1)!(-1)^{k-1}\sum_{m\in\mathbb{Z}} \frac{1}{(\tau + m)^k}.$$

Rearranging this yields

$$\sum_{m\in\mathbb{Z}} \frac{1}{(\tau + m)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{d=1}^{\infty} d^{k-1} e^{2\pi i\tau d}. \tag{6.18}$$

Then

$$G_{2k}(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^{2k}} = 2\sum_{n=1}^{\infty}\frac{1}{n^{2k}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^{2k}}$$

$$\overset{(6.18)}{=} 2\sum_{n=1}^{\infty}\frac{1}{n^{2k}} + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{2k-1}e^{2\pi i m\tau d} = 2\sum_{n=1}^{\infty}\frac{1}{n^{2k}} + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{d=1}^{\infty}\frac{d^{2k-1}e^{2\pi i\tau d}}{1-e^{2\pi i\tau d}}$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{n^{2k}} + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{d=1}^{\infty}\frac{d^{2k-1}}{e^{-2\pi i\tau d}-1}$$

Using equation [1, 23.2.16] for the values of the Riemann zeta function at positive even integers and (4.11) we obtain

$$G_4(\Lambda) = \frac{1}{\omega_2^4}\left(\frac{\pi^2}{45} + \frac{16\pi^4}{3}\sum_{d=1}^{\infty}\frac{d^3}{e^{-2\pi i d\omega_1/\omega_2}-1}\right)$$

$$G_6(\Lambda) = \frac{1}{\omega_2^6}\left(\frac{2\pi^6}{945} - \frac{16\pi^6}{15}\sum_{d=1}^{\infty}\frac{d^5}{e^{-2\pi i d\omega_1/\omega_2}-1}\right).$$

Multiplying $G_4(\Lambda)$ and $G_6(\Lambda)$ by the appropriate quantities yields (6.16). ■

Via (4.4) and (4.9), the $j-$invariant of E_f is

$$j = \frac{1728g_4^3}{g_4^3 - 27g_6^2}.$$

We want to calculate the sums in (6.16), we can only sum the first M terms however. Let $e_4(M)$ and $e_6(M)$ denote the errors.

Proposition 6.12. Let ω_1 and ω_2 be generators of a lattice such that $e^{2\pi\operatorname{Im}\tau}-1>1$ where $\tau=\omega_1/\omega_2$. Then

$$\begin{aligned} e_4(M) &\le 320\frac{\pi^4}{|\omega_2|^4}\left(\frac{a(a^2+4a+1)}{(a-1)^4} - \sum_{d=1}^{M} d^3a^d\right)\\ e_6(M) &\le \frac{430}{3\pi^6|\omega_2|^6}\left(\frac{a(a^4+26a^3+66a^2+26a+1)}{(a-1)^6} - \sum_{d=1}^{M} d^5a^d\right)\end{aligned} \tag{6.19}$$

where $a = 1/(e^{2\pi\operatorname{Im}\tau}-1)$ ⬣

Proof. First, we can pick ω_1 and ω_2 such that $e^{2\pi\operatorname{Im}\tau}-1>1$. If this is not satisfied replace $(\omega_1,\omega_2)^T$ by $A(\omega_1,\omega_2)^T$ for $A\in\mathrm{SL}_2(\mathbb{Z})$ such that $\operatorname{Im}\tau\ge\sqrt{3}/2$. This leaves the lattice unchanged by Proposition 1.14 and is possible by [7, Theorem 1.5]. Let us first find a bound for the denominator of the sums in (6.16). For $d\in\mathbb{Z}_{>0}$, we have

$$|e^{-2\pi i d\tau}-1| \ge \big||e^{-2\pi i d\tau}|-1\big| = e^{2\pi d\operatorname{Im}\tau}-1 \ge (e^{2\pi\operatorname{Im}\tau}-1)^d. \tag{6.20}$$

The last inequality is proven by, for example, induction. Then, for $k=3$ or 5,

$$\left|\sum_{d=M+1}^{\infty}\frac{d^k}{e^{-2\pi i d\tau}-1}\right| \le \sum_{d=M+1}^{\infty}\frac{d^k}{|e^{-2\pi i d\tau}-1|} \overset{(6.20)}{\le} \sum_{d=M+1}^{\infty}\frac{d^k}{(e^{2\pi\operatorname{Im}\tau}-1)^d}$$

$$= \sum_{d=M+1}^{\infty} d^ka^d = \sum_{d=1}^{\infty} d^ka^d - \sum_{d=1}^{M} d^ka^d$$

The infinite series in the last expression has a closed form which is found in [5, Equation 3.4]. Filling this in for $k=3$ and $k=5$ we obtain

$$\left|\sum_{d=M+1}^{\infty}\frac{d^3}{e^{2\pi i d\tau}-1}\right| \le \frac{a(a^2+4a+1)}{(a-1)^4} - \sum_{d=1}^{M} d^3a^d$$

$$\left|\sum_{d=M+1}^{\infty}\frac{d^5}{e^{2\pi i d\tau}-1}\right| \le \frac{a(a^4+26a^3+66a^2+26a+1)}{(a-1)^6} - \sum_{d=1}^{M} d^5a^d.$$

Substituting these inequalities in $e_4(M)$ and $e_6(M)$ yields (6.19). ■

6.3 Explicit calculations for $N = 14$ and $N = 24$

Most calculations of the period lattice of the generator of $\mathcal{S}_2(\Gamma_0(N))$ are very similar. In this section we cover explicit calculations for the two cases where $N = 14$ and $N = 24$ to highlight this process. This is done using the code in [29]. The function names below refer to this code.

The more straightforward computation of the two is $N = 14$. We first determine a basis for $C(\Gamma_0(14))/B(\Gamma_0(14))$. Running the function `modulo_relations_matrix(14)` gives us that

$$C(\Gamma_0(14))/B(\Gamma_0(14)) = \operatorname{span}\{(13:2), (11:2), (1:7), (2:7), (1:0)\}.$$

This function uses a set of representatives for $C(\Gamma_0(14))/B(\Gamma_0(14))$ which is given by `generate_M_symbols(14)`. Next, we find $\ker\delta = H(\Gamma_0(14))$, where we view δ as a map $C(\Gamma_0(14))/B(\Gamma_0(14)) \to C_0(14)$. Using the function `equiv_test(p1,q1,p2,q2,14)` and the fact that $X_0(14)$ has 4 cusps by Proposition 2.39, we find that

$$C_0(\Gamma_0(14)) = \operatorname{span}\{\pi(0), \pi(\infty), \pi(1/2), \pi(1/7)\}.$$

To see how δ acts on its domain we solve $ad - bc = 1$ for $(c:d)$ and then $\delta((c:d)) = \pi(a/c) - \pi(b/d)$. The $M-$symbols $(p:1)$ where p is coprime to N have the property that

$$\delta((p:1)) = \pi(0) - \pi(1/p).$$

Then Proposition 6.8 ensures that $\pi(1/p) = \pi(0)$ so that $\delta((p:1)) = 0$. In the case where $N = 14$ we have $\delta((3:1)) = 0$ and $\delta((5:1)) = 0$. The function `modulo_relations_matrix(14)` also gives expressions for these in terms of the basis

$$(3:1) = (11:2) - (13:2) \quad \text{and} \quad (5:1) = (11:2) - (1:7) + (2:7).$$

We see that $(3:1)$ and $(5:1)$ are linearly independent elements of $H(\Gamma_0(14))$. Since $X_0(14)$ has genus 1, by Proposition 2.37, these two elements are a basis for $H(\Gamma_0(14))$. The function `representative(representatives,element,14)` gives a representative of any $M-$symbol which is useful to determine how $*$ acts on $H(\Gamma_0(14))$

$$\begin{aligned} *(5:1) &= (-5:1) = (9:1) = -(3:1) \\ *(3:1) &= (-3:1) = (11:1) = -(5:1). \end{aligned}$$

Then the matrix representation of $*$ with respect to the chosen ordered basis $((5:1),(3:1))$ is

$$*_{((5:1),(3:1))} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

We find that this matrix has the left eigenvectors $v^+ = \begin{pmatrix} -1 & 1 \end{pmatrix}$ and $v^- = \begin{pmatrix} 1 & 1 \end{pmatrix}$. By Proposition 3.13, $\mathcal{S}_2(\Gamma_0(14))$ is spanned by

$$\begin{aligned} f(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) &= q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 \\ &+ q^9 - 2q^{12} - 4q^{13} - q^{14} + q^{16} + 6q^{17} - q^{18} + 2q^{19} - 2q^{21} + O(q^{22}). \end{aligned}$$

This is the cusp form with label 14.2.a.a in the LMFDB. The period lattice Λ_f is then is generated by

$$\omega_1 = -x + iy \quad \text{and} \quad \omega_2 = x + iy$$

where

$$x = \frac{\operatorname{Re}\Phi_f(\gamma)}{v^+\sigma} \quad \text{and} \quad y = \frac{\operatorname{Im}\Phi_f(\gamma)}{v^-\sigma}$$

for some $\gamma \in \Gamma_0(14)$ such that $\sigma = \{\tau, \gamma\tau\} \in H_1(X_0(14), \mathbb{Z})$. Pick $\sigma = \{0, 1/5\}$. Via the bijection in Proposition 6.7, we find that $\sigma = (5:1)$ which corresponds to the vector $(1,0)^T \in H_1(X_0(14), \mathbb{Z})$ with respect to the ordered basis of $H(\Gamma_0(14))$. Moreover, $\sigma = \{0, \gamma 0\}$ with $\gamma = (3, 1; 14, 5)$ (so that $c = 1$ in (6.11)). Running the function `Phi_f(M, f)` gives the numerical approximation

$$\Phi_f(\gamma) = \langle \sigma, f \rangle \approx 0.9906709780334416 + 1.3254912396824865i.$$

We use the first 100 Fourier coefficients of f to calculate $\Phi_f(\gamma)$. The error is then at most

$$\varepsilon(100) = 1.3982506587477533 \cdot 10^{-19}.$$

Carrying out the other computations, we obtain

$$\omega_1 \approx -0.9906709780334416 + 1.3254912396824865i$$
$$\omega_2 \approx 0.9906709780334416 + 1.3254912396824865i.$$

The complex torus $\mathbb{C}/\Lambda_f$ is then isomorphic to the elliptic curve E_f over $\mathbb{C}$ with equation (4.15) where

$$g_4 \approx -17.91666666666667$$
$$g_6 \approx 24.495370370370367 + 7.105427357601002 \cdot 10^{-15} i.$$

We calculated the first 50 terms of (6.16). Using Proposition 6.19, we obtain the error bounds

$$e_4(50) \leq 1.80276830312386 \cdot 10^{-15}$$
$$e_6(50) \leq 3.032190829109276 \cdot 10^{-15}$$

The elliptic curve E_f approximately has $j-$invariant

$$j \approx 452.73209730320735 - 1.9383632428589129 \cdot 10^{-13} i.$$

Neglecting the imaginary part of j, using the function `rational_approximation(alpha, denom)` we find that j is a close approximation of

$$\frac{9938375}{21952} = 452.732097303207\ldots. \tag{6.21}$$

The approximate coefficients

$$c_4 = 12g_4 \approx -215.00000000000006$$
$$c_6 = 216g_6 \approx 5290.999999999999$$

of E_f are very close to integers. The actual values of c_4 and c_6 are actually integers. This follows from the fact that E_f is defined over $\mathbb{Q}$ and the so-called Manin constant is an integer. The latter fact is proven by Edixhoven in [14, Proposition 2]. Due to the small size of the error bound $\varepsilon(100)$, along with the bounds on the error $e_4(50)$ and $e_6(50)$, we recognize that the actual values of E_f are $c_4 = -215$ and $c_6 = 5291$. With these values, the $j-$invariant of E_f is equal to (6.21)

The Kraus–Laska–Connell algorithm [23, Section 2] is an algorithm which takes integers c_4 and c_6 of an elliptic curve over $\mathbb{Q}$ and outputs $a_1, \ldots, a_6$ such that (4.3) is in minimal form. This algorithm is implemented in `Kraus-Laska-Connell_algorithm.py` in [29]. Using this algorithm for the values $c_4 = -215$ and $c_6 = 5291$, we obtain the curve

$$E_f: \quad y^2 + xy + y = x^3 + x^2 + 4x - 6.$$

By Theorem 5.15, this is an algebraic description of $X_0(14)$ as this modular curve is isomorphic to E_f. The elliptic curve E_f has label 14.a6 in the LMFDB.

The case where $N = 24$ is different from $N = 14$ in the sense that the elements of the form $(p : 1) \in H(\Gamma_0(24))$ with p coprime to 24 are equal to 0 in $C(\Gamma_0(24))/B(\Gamma_0(24))$. This means that we have to put some more work into calculating $\ker \delta$. Using `modulo_relations_matrix(24)` we obtain a basis

$$C(\Gamma_0(24))/B(\Gamma_0(24)) = \operatorname{span}\{(14:3), (3:4), (5:4), (1:8), (5:8), (7:6), (3:8), (1:12), (1:0)\}.$$

Using `equiv_test(p1,q1,p2,q2,24)` and Proposition 2.39 we find that

$$C_0(\Gamma_0(24)) = \operatorname{span}\{\pi(0), \pi(1/2), \pi(1/3), \pi(1/4), \pi(1/6), \pi(1/8), \pi(1/12), \pi(\infty)\}.$$

Then

$$\delta((14:3)) = \pi(1/2) - \pi(1/3) \qquad \delta((3:4)) = \pi(1/3) - \pi(1/4)$$
$$\delta((5:4)) = \pi(0) - \pi(1/4) \qquad \delta((7:6))) = \pi(0) - \pi(1/6)$$
$$\delta((3:8)) = \pi(1/3) - \pi(1/8) \qquad \delta((1:12)) = \pi(0) - \pi(1/12)$$
$$\delta((1:8)) = \pi(0) - \pi(1/8) \qquad \delta((5:8)) = \pi(0) - \pi(1/8)$$
$$\delta((1:24)) = \pi(0) - \pi(\infty).$$

We immediately see that $\delta((1:8)) = \delta((5:8))$ so that $(1:8)-(5:8) \in \ker\delta$. We also find that,

$$\delta((5:4)+(3:8)) = \delta((3:4)+(5:8)).$$

It follows that $\ker\delta = H(\Gamma_0(24)) = \mathrm{span}\{A, B\}$ where

$$A = (1:8)-(5:8) \quad \text{and} \quad B = (3:4)-(3:8)+(5:8)-(5:4).$$

Using `representative(representative,element,24)` and expressions for $M-$symbols provided by `modulo_relations_matrix(24)`, we find that $*$ acts on A and B as

$$*A = (-1:8)-(-5:8) = (5:8)-(1:8) = -A$$
$$*B = (-3:4)-(-3:8)+(-5:8)-(-5:4) = (3:4)-(3:8)+(5:8)-(5:4) = B.$$

We choose the ordered basis $(B, A+B)$ for $\ker\delta$ so that the calculations later on work out better. The matrix representation of $*$ with respect to this basis is then

$$*_{(B,A+B)} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix},$$

which has left eigenvectors $v^+ = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $v^- = \begin{pmatrix} 0 & 1 \end{pmatrix}$. By Proposition 3.13 the vector space $\mathcal{S}_2(\Gamma_0(24))$ is spanned by

$$f(\tau) = \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau) = q - q^3 - 2q^5 + q^9 + 4q^{11} - 2q^{13} + 2q^{15}$$
$$+ 2q^{17} - 4q^{19} - 8q^{23} - q^{25} - q^{27} + 6q^{29} + 8q^{31} - 4q^{33} + 6q^{37} + \dots$$

This is the cusp form with label 24.2.a.a in the LMFDB. Trying a few elements of the form $\{0, b/d\}$, we find that $\sigma = \{0, 22/35\}$ is such that $v^+\sigma, v^-\sigma \neq 0$. The quantity $22/35$ has continued fraction expression $22/35 = [0; 1, 1, 1, 2, 4]$. By (6.15) we obtain

$$\{0, 22/35\} = (1:1)+(-2:1)+(3:2)+(-8:3)+(35:8)$$
$$= (3:4)-(5:4)-(1:8)-(3:8)+2(5:8) = -(A+B)+2B,$$

so that $\{0, 22/35\} = (2, -1)^T$ with respect to $(B, A+B)$. We can write $\sigma = \{0, 22/35\} = \{0, \gamma 0\}$ for $\gamma = (181, -22; 288, -35)$. Then

$$\Phi_f(\gamma) = \langle \sigma, f \rangle \approx -2.156515647476195 - 1.68575035485386 44i. \tag{6.22}$$

As the lower left entry of γ is quite large, we require more terms of (6.13) to accurately compute $\Phi_f(\gamma)$. To calculate (6.22), the first 1000 Fourier coefficients of f are used. The error is then at most

$$\varepsilon(1000) = 1.519255433010195 \cdot 10^{-8}.$$

Approximate values of the generators of Λ_f are calculated similarly to the above

$$\omega_1 \approx -2.156515647476195$$
$$\omega_2 \approx -2.156515647476195 + 1.6857503548538644i.$$

The coefficients g_4 and g_6 are then approximately

$$g_4 \approx 17.333333331992574 - 3.071498574454836 \cdot 10^{-16} i$$
$$g_6 \approx -10.370370368260845 - 5.055248595759895 \cdot 10^{-16} i.$$

These values are calculated with an error of at most

$$e_4(50) \leq -1.5412311129356958 \cdot 10^{-14}$$
$$e_6(50) \leq -1.894908830110192 \cdot 10^{-14}.$$

The $j-$invariant of E_f is approximately

$$j \approx 3905.7777769174277 + 7.415831332785713 \cdot 10^{-13} i.$$

Running `rational_approximation(alpha, denom)`, we find that the $j-$invariant closely approximates the rational number

$$\frac{35152}{9} = 3905.77777777777 \ldots.$$

Again, the coefficients

$$c_4 = 12g_4 \approx 207.9999999839083$$
$$c_6 = 216g_6 \approx -2239.99999954429$$

are close to integers. Running the Kraus-Laska-Conell algorithm for $c_4 = 208$ and $c_6 = -2240$, we obtain the curve

$$E_f : \quad y^2 = x^3 - x^2 - 4x + 4$$

This curve has label 24.a4 in the LMFDB.

These calculations have been carried out for the values of N in (2.15). The findings of these calculations are found in the tables in Section 7.

7 Tables

The following tables contain the values which are calculated as in Section 6.3 for all N in (2.15).

$N = 11$

f	$\tau \mapsto \eta(\tau)^2\eta(11\tau)^2$	
ω_1	$-1.269209304279553 + 0.00000000000000000i$	
ω_2	$-0.6346046521397765 + 1.458816616938495 7i$	
g_4	$41.33333333333347 + 1.509903313490213 \cdot 10^{-14}i$	
g_6	$92.62962962962979 + 7.105427357601002 \cdot 10^{-15}i$	
j	$-757.6726378600639 - 1.0271993225395667 \cdot 10^{-12}i$	$\approx -\dfrac{122023936}{161051}$

Coefficients of associated minimal curve E_f

a_1	0	a_6	-20
a_3	1	Δ	-11^5
a_2	-1	j	-757.6726378600567
a_4	-10		

$N = 14$

f	$\tau \mapsto \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$	
ω_1	$-0.9906709780334416 + 1.3254912396824865i$	
ω_2	$0.9906709780334416 + 1.3254912396824865i$	
g_4	$-17.91666666666667 + 0.00000000000000000i$	
g_6	$24.495370370370367 + 7.105427357601002 \cdot 10^{-15}i$	
j	$452.73209730320735 - 1.9383632428589129 \cdot 10^{-13}i$	$\approx \dfrac{9938375}{21952}$

Coefficients of associated minimal curve E_f

a_1	1	a_6	-6
a_3	1	Δ	$-2^6 7^3$
a_2	1	j	$452.73209730320\ldots$
a_4	4		

$N = 15$

f	$\tau \mapsto \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$	
ω_1	$0.00000000000000000 - 1.5962422222156807j$	
ω_2	$-1.4006030425344482 - 0.00000000000000000i$	
g_4	$40.083333314407994 + 0.00000000000000000i$	
g_6	$22.58796293325066 + 0.00000000000000000i$	
j	$2198.215130137807 + 0.00000000000000000i$	$\approx \dfrac{111284641}{50625}$

Coefficients of associated minimal curve E_f

a_1	1	a_6	-10
a_3	1	Δ	$3^4 5^4$
a_2	1	j	$2198.215130864197\ldots$
a_4	-10		

$N = 17$

ω_1 ω_2	$-1.5470797535511192 + 0.0000000000000000i$ $-0.7735398767755596 + 1.3728695590448772i$
g_4 g_6	$2.750000000000115 + 6.661338147750939 \cdot 10^{-15}i$ $55.62500000000008 - 9.769962616701378 \cdot 10^{-16}i$

j	$-0.43027502065354106 - 3.2787347441463774 \cdot 10^{-15}i$	$\approx -\dfrac{35937}{83521}$

Coefficients of associated minimal curve E_f

a_1	1	a_6	-14
a_3	1	Δ	-17^4
a_2	-1	j	$-0.43027502065348\ldots$
a_4	-1		

$N = 19$

ω_1 ω_2	$-1.359759733488311 + 0.0000000000000000i$ $0.6798798667441555 + 2.063546195858619i$
g_4 g_6	$37.333333333333336 - 1.2434497875801753 \cdot 10^{-14}i$ $46.70370370370374 - 3.552713678800501 \cdot 10^{-15}i$

j	$-13109.110949117812 + 9.534418332904698 \cdot 10^{-11}i$	$\approx -\dfrac{89915392}{6859}$

Coefficients of associated minimal curve E_f

a_1	0	a_6	-15
a_3	1	Δ	-19^3
a_2	1	j	$-13109.110949117947\ldots$
a_4	-9		

$N = 20$

f	$\tau \mapsto \eta(2\tau)^2\eta(10\tau)^2$
ω_1 ω_2	$-1.4121875709795586 - 1.1370825995205394i$ $0.0000000000000000 + 2.2741651990410787i$
g_4 g_6	$-14.666666666666593 - 1.9272415413648787 \cdot 10^{-15}i$ $-10.962962962963045 + 1.72732011129823 \cdot 10^{-16}i$

j	$851.839999999987 + 1.838747915423377 \cdot 10^{-13}i$	$\approx \dfrac{21296}{25}$

Coefficients of associated minimal curve E_f

a_1	0	a_6	4
a_3	0	Δ	$-2^8 5^2$
a_2	1	j	851.84
a_4	4		

$N = 21$

ω_1 ω_2	$1.8044616215539697 + 0.00000000000000000$ $0.00000000000000000 - 1.9109897807518297i$
g_4 g_6	$16.083333333333293 + 0.00000000000000000i$ $2.662037037037007 + 0.0000000000000000i$
j	$1811.301839254219 + 0.00000000000000000i$ $\approx \frac{7189057}{3969}$

Coefficients of associated minimal curve E_f

a_1	1	a_6	-1
a_3	0	Δ	$3^4 7^2$
a_2	0	j	$1811.30183925422\ldots$
a_4	-4		

$N = 24$

f	$\tau \mapsto \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$
ω_1 ω_2	$-2.156515647476195 + 0.00000000000000000i$ $-2.156515647476195 + 1.6857503548538644i$
g_4 g_6	$17.333333331992574 - 3.071498574454836 \cdot 10^{-16}i$ $-10.370370368260845 - 5.055248595759895 \cdot 10^{-16}i$
j	$3905.7777769174277 + 7.415831332785713 \cdot 10^{-13}i$ $\approx \frac{35152}{9}$

Coefficients of associated minimal curve E_f

a_1	0	a_6	4
a_3	0	Δ	$2^8 3^2$
a_2	-1	j	$3905.777\ldots$
a_4	-4		

$N = 27$

f	$\tau \mapsto \eta(3\tau)^2\eta(9\tau)^2$
ω_1 ω_2	$-0.8833172835427812 + 1.5299521924023414i$ $0.8833172835427812 + 1.5299521924023414i$
g_4 g_6	$8.099227083519532 \cdot 10^{-05} + 5.7852689110647615 \cdot 10^{-15}i$ $27.00028946174126 + 2.34980492095499 \cdot 10^{-15}i$
j	$-4.664164520028151 \cdot 10^{-14} - 9.994814492097526 \cdot 10^{-24}i$ ≈ 0

Coefficients of associated minimal curve E_f

a_1	0	a_6	-7
a_3	1	Δ	-3^9
a_2	0	j	0
a_4	0		

$N = 32$

f	$\tau \mapsto \eta(4\tau)^2\eta(8\tau)^2$	
ω_1 ω_2	$-1.3110318329942086 + 1.311024152052975i$ $1.3110318329942086 + 1.311024152052975i$	
g_4 g_6	$-16.000038300486118 + 2.4649336391457943 \cdot 10^{-15}i$ $-0.00027352518736865195 - 2.5243998287445214 \cdot 10^{-15}i$	
j	$1727.9999991478046 - 1.5730436397994827 \cdot 10^{-17}i$	≈ 1728

Coefficients of associated minimal curve E_f

a_1	0	a_6	0
a_3	0	Δ	-2^{12}
a_2	0	j	1728
a_4	4		

$N = 36$

f	$\tau \mapsto \eta(6\tau)^4$	
ω_1 ω_2	$-2.1028644299587596 - 1.2146261322417762$ $0.00000000000000000 + 2.4292522644835524i$	
g_4 g_6	$-0.00862385623628428 + 4.350252796880838 \cdot 10^{-15}i$ $-3.999359414494687 - 3.481590016285452 \cdot 10^{-15}i$	
j	$2.5662775647986147 \cdot 10^{-6} - 3.888098266200391 \cdot 10^{-18}i$	≈ 0

Coefficients of associated minimal curve E_f

a_1	0	a_6	1
a_3	0	Δ	$-2^4 3^3$
a_2	0	j	0
a_4	0		

$N = 49$

ω_1 ω_2	$-1.9333117056168114 + 0.00000000000000000$ $-0.9666558528084057 + 2.5575309899160983i$	
g_4 g_6	$8.749999999999995 - 1.1102230246251565 \cdot 10^{-15}i$ $6.12500000000000000 - 1.3322676295501878 \cdot 10^{-15}i$	
j	$-3374.999999999982 - 5.419771774230322 \cdot 10^{-13}i$	≈ -3375

Coefficients of associated minimal curve E_f

a_1	1	a_6	-1
a_3	0	Δ	-7^3
a_2	-1	j	-3375
a_4	-2		

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