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An exploration of sub-Riemannian Orbifolds

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Abstract

Orbifolds and sub-Riemannian manifolds are generalizations of the concept of manifold. Orbifolds generalize manifolds by incorporating singularities, while sub-Riemannian manifolds exclude specific geodesics and restrict movement to chosen subsets. In this thesis we discuss the possibility to define a sub-Riemannian structure on an orbifold. First, we sketch a method to define sub-Riemannian structure on the regular part of an orbifold, similar to the known construction of sub-Riemannian structures on lens spaces. However, problems for the horizontal distribution occur around the singularities on the orbifold. It turns out that a sub-Riemannian distribution on an orbifold is well-defined around the singular points if it is equivariant with respect to the actions on the orbifold. As a result we define sub-Riemannian structures on orbifolds obtained by reflections, rotation and the (p, q)-Hopf action and find geodesics in these cases. We also sketch an extension of a result by Herr, to find a sub-Riemannian structure on any closed cyclic 3-orbifolds.

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Introduction

In the field of differential geometry, group actions on manifolds have been and are thoroughly studied. We know from the 'Quotient manifold theorem' that if a Lie group G acts smoothly, freely and properly on a smooth manifold M, then the quotient M/G is a smooth manifold. For example, we can create a circle by letting the group $\mathbb{Z}$ freely, properly and smoothly act on $\mathbb{R}$ by translation. Similarly, we can construct a 2-torus by letting $\mathbb{Z} \times \mathbb{Z}$ act on $\mathbb{R}^2$ by translation. But what would happen if we let go of the assumption that the action is free? This leads to the notion of a *developable orbifold*, which is the quotient of a manifold M by a discrete group Γ acting properly on M.

As an example of a developable orbifold, one can consider the $\mathbb{Z}/n\mathbb{Z}$-action on $\mathbb{R}^2$ by rotation. If we assume we have polar coordinates (r, θ) on $\mathbb{R}^2$, the action is given by $(r, \theta) \mapsto (r, \theta + \frac{2\pi}{n})$. Under the quotient by this action, all vectors in $\mathbb{R}^2$ that differ by an angle $\frac{2\pi}{n}$ are identified. After the identification what remains is a cone with angle $\frac{2\pi}{n}$. This cone has a tip, which is the place where the origin of $\mathbb{R}^2$ is mapped onto itself n times. At such a point we no longer have a smooth manifold structure, hence we call the point *singular*. In general, we can think of developable orbifolds as a generalization of the concept of manifold that contain some of these singular points. For the case $n = 3$ the construction of the cone is illustrated in figure 1, here the red lines are being 'folded' onto each other.

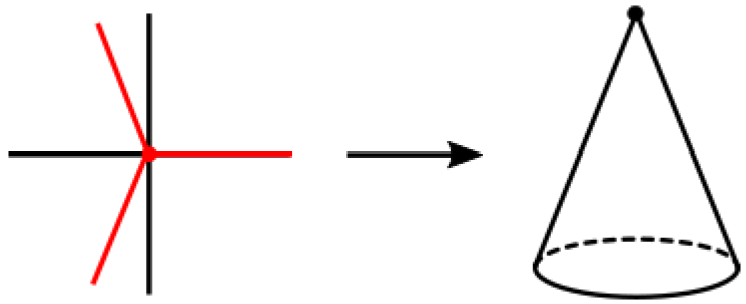

Figure 1: Cone orbifold obtained from $\mathbb{Z}/3\mathbb{Z}$-action on $\mathbb{R}^2$, the red lines are identified under the quotient, the origin in $\mathbb{R}^2$ becomes a singular point.

Instead of considering a global quotient on a manifold, we can also consider a seperate action on each chart. In general, orbifolds are then topological spaces that are locally homeomorphic to $\mathbb{R}^n/\Gamma$ for a discrete group Γ acting properly on $\mathbb{R}^n$. Orbifolds appear in various disguises in both algebraic and differential geometry. They are also of interest in Mathematical Physics, since they model how symmetries act on given spaces. Orbifolds also appear in the mathematical study of music theory (see for example [1]).

An orbifolds is also an interesting object in its own right. One question one can ask is if the constructions and results we have for manifolds generalize to orbifolds. It turns out that many theorems and constructions on manifolds, have an analogy for orbifolds. For example, one can generalize the theory for Riemannian manifolds to Riemannian orbifolds and for instance the famous Gauss-Bonnet theorem has an analogous Gauss-Bonnet theorem for orbifolds [2, Theorem 4.3.16]. One structure on manifolds that does not yet have an analogy on orbifolds is *sub-Riemannian geometry*.

Sub-Riemannian manifolds are a generalization of Riemannian manifolds in which one cannot move freely. To illustrate this, consider a car that has to parallel park as in figure 2. In a Riemannian setting, the shortest path would be for the car to shift to the right. However, we know that the wheels of a car simply cannot rotate that way. Given the constraints on the wheels, we want to find the shortest path the car can take in order to parallel park. If the car moves on a manifold M, the constraints on the wheels give us a tangent subbundle $\mathcal{D}_q \subseteq T_qM$ for each $q \in M$. The subbundle is called a *distribution*. The paths for the car that are possible are described by the curves $\gamma : [0,T] \to M$ that have a tangent vector $\gamma'(t)$ in the distribution $\mathcal{D}_{\gamma(t)}$ for all $t \in [0,T]$. These paths will be called *horizontal curves*. On the distribution we can define a metric, called the *sub-Riemannian metric*, which induces a notion of distance and length. Under this notion of length, the 'shortest' path is called a *sub-Riemannian geodesic*. With some modifications, many sub-Riemannian geodesics can be obtained from Hamilton's equations for the so-called *sub-Riemannian Hamiltonian*, this is a variation of the regular Hamiltonian found in classical mechanics.

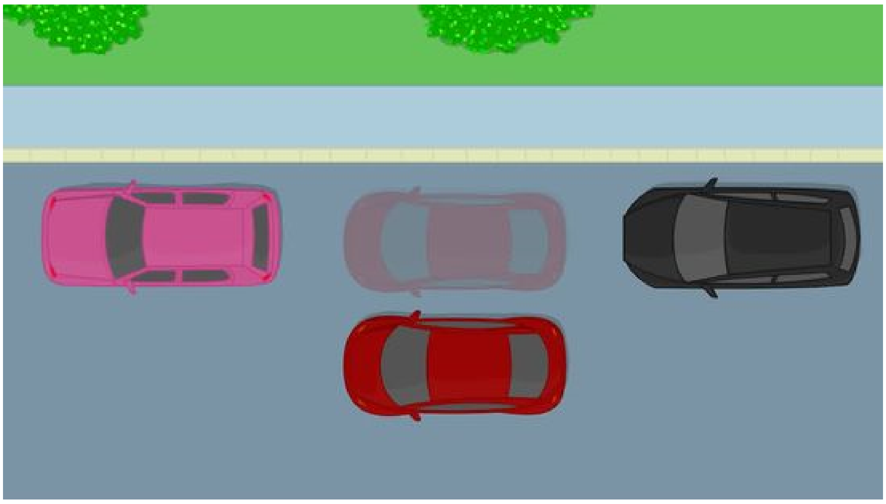

Figure 2: Parallel parking problem

One key difference between Riemannian and sub-Riemannian geometry is that there exist *abnormal geodesics* for some cases in sub-Riemannian geometry. These are the shortest paths under the sub-Riemannian metric that do not show up as solutions to Hamilton's equations. Studying abnormal minimizers is a large part of research into sub-Riemannian geometry.

In this thesis we discuss the relation between sub-Riemannian geometry and orbifold theory. In general, we do not know whether it is possible to define a sub-Riemannian structure on an orbifold. We sketch the problems you run into when defining a sub-Riemannian structure on an orbifold. However, we will see that in case of a reflection or rotation action on some manifold, we can obtain an orbifold with a sub-Riemannian structure. A similar result will be shown for the so called (p,q)-Hopf action, which is a variation of the classical Hopf action. To our knowledge these results were not shown in literature before. Moreover, we will sketch a result from [3] that shows that on a closed and cyclic 3-dimensional orbifold, there always exists a contact form ξ

with $\ker(\xi)$ the *contact distribution*. From this we show there exists a sub-Riemannian structure coming from the contact distribution can be defined, and does not admit abnormal geodesics.

Before we studied the problem of sub-Riemannian structures on orbifolds, we also studied the already studied cases of sub-Riemannian structures on Prinicpal G-bundles and homogeneous spaces in the hope of using some of these techniques for orbifolds. For context of defining sub-Riemannian structures on quotient spaces and possible future research we included a detailed explanation of sub-Riemannian structures on prinicpal G-bundles as described in [4].

The outline of the thesis is as follows: In Chapter 1 we give the basic definitions and results from sub-Riemannian geometry. We discuss sub-Riemannian geodesics and how to find them using the sub-Riemannian Hamiltonian. Next, we discuss contact distributions and prove that no abnormal minimizers can exist in these cases.

In chapter 2 we define sub-Riemannian structures on a principal G-bundle $\pi : Q \to M$. We show that if there exists a 'metric of constant bi-invariant type' on Q, then we have a closed form formula for all normal sub-Riemannian geodesics on Q. As examples we discuss sub-Riemannian structures on Lie groups, homogeneous spaces, the 'falling cat problem' and the Hopf action.

We introduce all necessary results and definition for orbifold theory in Chapter 3. For this we first define orbifolds in general, after which we specify to developable orbifolds. Moreover, we discuss how to construct a tangent bundle, differential form and Riemannian metrics on an orbifolds.

In chapter 4 we give the results of this thesis. First, we study the sub-Riemannian structure on Lens spaces. Lens spaces are in general not orbifolds, but the techniques for Lens spaces can be extended to the non-singular parts of orbifolds. Then we define sub-Riemannian structures on quotients of $\mathbb{R}^3$ under reflective actions, a cyclic action and the (p, q)-Hopf action. After this we sketch a result that shows that we can define a sub-Riemannian structure on any cyclic closed 3-orbifold.

We also added an Appendix on group actions. In this section we define all the terms and theorems we used from group theory.

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Chapter 1

Sub-Riemannian Geometry

In this chapter we discuss the basic theory of sub-Riemannian geometry. In section 1.1 we will discuss the definition of a sub-Riemannian structure and horizontal curves. Next, we will define normal and abnormal Pontryagin extremals (section 1.2) and the sub-Riemannian Hamiltonian (section 1.3), in order to compute sub-Riemannian geodesics in section 1.4. When we define geodesics, we will find that there can exist abnormal geodesics. We will prove that these abnormal geodesics cannot exist in the case of a so called contact distribution. The definitions and results will be stated in 1.5. Finally, we discuss an analogous construction of the exponential map in sub-Riemannian geometry in section 1.6. Most of the material in this chapter can be found in [5].

Notation Throughout this chapter M will be a smooth connected n-dimensional manifold unless stated otherwise. We also fix a notation for the projection $\pi : T^*M \to M$.

1.1 Sub-Riemannian geometry

In this section, we give the definition of a sub-Riemannian structure on a manifold. After the definition we define lengths and distances induced by such structures. Moreover, we state and explain the Chow-Rashevskii theorem. Before, we can give the definition of a sub-Riemannian structure we need to define distributions.

Definition 1.1. *Let M be a smooth n-dimensional manifold. A* ***distribution*** *on M is a family of vector subspaces $\mathcal{D}_q \subset T_qM$ for every point $q \in M$. We say the distribution is* ***regular of rank*** *k if the subspaces $\mathcal{D}_q$ are all of dimension k. A distribution $\mathcal{D}$ is said to be* ***smooth*** *if at every point $q \in M$ there exists a neighbourhood U such that there exist smooth vector fields $X_1, \dots, X_k : U \to TM$ such that for all $x \in U$*

$$\mathcal{D}_x = \operatorname{span}\{X_1(x), \dots, X_k(x)\}.$$

One can also think of a regular distribution of rank k as a rank k-subbundle of the tangent bundle. In this thesis we assume all distributions to be smooth and regular. Using this assumption we can at least locally write every distribution as the span of vector fields $X_1, \dots, X_k$. Moreover, notice that because our rank k distribution is smooth and regular at each point $q \in M$ there exists an open neighbourhood U on which there exist a family of differential 1-forms $\eta^1, \dots, \eta^{n-k}$ such that for all $x \in U$ we have $\mathcal{D}_x = \ker(\eta^1)|_x \cap \dots \cap \ker\left(\eta^{n-k}\right)|_x$. A proof of this fact can be found in [6, Lemma 19.5].

We also define a specific kind of distribution, called a *bracket-generating distribution*.

Definition 1.2. *Consider a distribution $\mathcal{D}_q = \operatorname{span}\{X_1(q), \dots, X_k(q)\}$ on M. The Lie algebra generated by the distribution is defined as:*

$$\operatorname{Lie}_q(\mathcal{D}_q) \coloneqq \operatorname{span}\{[X_1, \dots, [X_{j-1}, X_j]], X_i(q) \in \mathcal{D}_q, j \geq 1\}.$$

*The distribution $\mathcal{D}_q$ is called **bracket-generating** if $\operatorname{Lie}_q(\mathcal{D}) = T_qM$ for all $q \in M$.*

Remark 1.3. Let us now sketch why bracket-generating distributions are important to us. In contrast to bracket generating distributions, we have involutive distributions. A distribution $\mathcal{D}$ is called **involutive** if for any two vector fields X, Y on $\mathcal{D}$ we have $[X, Y] = 0$. By Frobenius theorem ([6, Theorem 19.12]), we know that every involutive distribution is in fact completely integrable. From this point of view a bracket-generating distribution can also be called a **completely non-integrable distribution**. In this context completely integrable means that all maximal connected integral manifolds of $\mathcal{D}$ form the leaves of a folitation. In other words, if we have an initial position on a leaf of the foliation, we cannot leave the leaf via a curve that is tangent to the distribution. This would mean that not all points on the manifold can be connected via curves that are tangent to the distribution. In a bracket-generating distribution this is not the case, and we find curves connecting any two points that are tangent to the distribution. In figure 1.1a an involutive distribution is shown. The leaves of the folitation are given by the 'layers'. If we start on a given layer, we can never move from one layer to another. In the bracket-generating distribution shown in figure 1.1c, we see that we can actually move from one layer to another while staying within the distribution.

We consider a few examples of distributions.

Example 1.4.

1. On $\mathbb{R}^3$ at the point (x, y, z) the distribution $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is of rank 2. It is involutive and not bracket-generating, since $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = [\frac{\partial}{\partial x}, \frac{\partial}{\partial x}] = [\frac{\partial}{\partial y}, \frac{\partial}{\partial y}] = 0$.

2. On $\mathbb{R}^3$ we can define the **Heisenberg distribution** [1]. This is a rank 2 distribution generated by the vector fields $X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$. We find $[X, Y] = \frac{\partial}{\partial z}$, hence it follows that that the Heisenberg distribution is bracket generating. A drawing of this distribution is shown in 1.1b.

3. On $\mathbb{R}^{2n+1}$ for $n \in \mathbb{N}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ we can define a so-called contact form [2] $\alpha = dz + \sum_{i=1}^{n} x_i dy_i \in \Omega^1(\mathbb{R}^{2n+1})$. Its kernel generates a distribution $\mathcal{D}$ that is spanned by the vector fields $\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z_i}$ for $i = 1, \dots, n$. The distribution is called the **contact distribution**. One can check that this distribution is bracket generating. For the case $\mathbb{R}^3$ this distribution is shown in figure 1.1c.

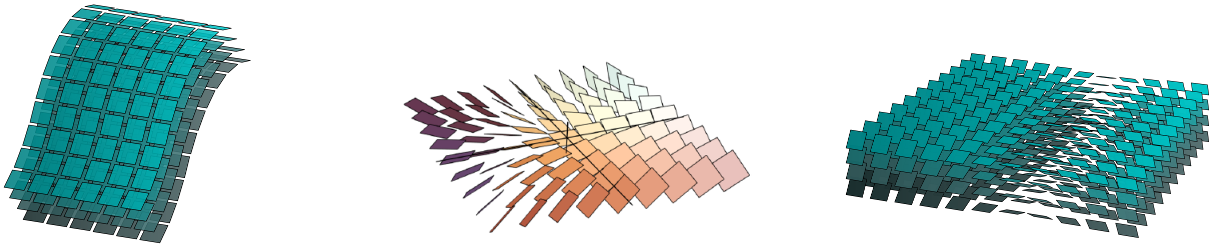
4. On $\mathbb{R}^4$ we can define the **Engel distribution**. Consider coordinates (x, y, z, w) on $\mathbb{R}^4$, then we define the rank 2 distibution $\mathcal{D} = \operatorname{span}\{X, Y\}$ with $X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial w}$.

[1] The Heisenberg distribution is closely related to the Heisenberg group and algebra and the isoperimetric problem the connection can be found in [7, Chapter 1]

[2] A *Contact form* on a $(2k+1)$-manifold M is a 1-form ξ such that $\xi \wedge (\xi)^k \neq 0$.

The Engel distribution can also be written as $\mathcal{D} = \ker(\eta_1) \cap \ker(\eta_2)$ with $\eta_1 = dz - wdx$ and $\eta_2 = dy - zdx$. One can check that this distribution is also bracket-generating.

$\triangle$



(a) Involutive distribution (b) Heisenberg distribution (c) Contact distribution

Figure 1.1: Three different distributions (pictures taken from [8], [9] and [10])

Now that we have the definition of a distribution, we can give a formal definition of sub-Riemannian geometry.

Definition 1.5. *A* ***sub-Riemannian structure*** *on a smooth connected manifold M is given by a bracket-generating distribution $\mathcal{D}$ with an inner product g on $\mathcal{D}$. A* ***sub-Riemannian manifold*** *consists of the triple $(M, \mathcal{D}, g)$.*

From now on, unless stated otherwise, we assume that any sub-Riemannian manifold is given by $(M, \mathcal{D}, g)$ for M a smooth connected n-dimensional manifold, $\mathcal{D}$ a smooth regular rank k distribution, and g a metric on $\mathcal{D}$.

In general, we want that the distribution encodes a way to restrict the directions we can move in. This leads to the notion of horizontal curves.

Definition 1.6. *Given a sub-Riemannian structure $(M, \mathcal{D}, g)$ a* ***horizontal curve*** *is a curve $\gamma : [0, T] \to M$ such that $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0, T]$.*

In other words, these are the curves that stay tangent to the distribution, and hence are the paths that are restricted by the distribution.

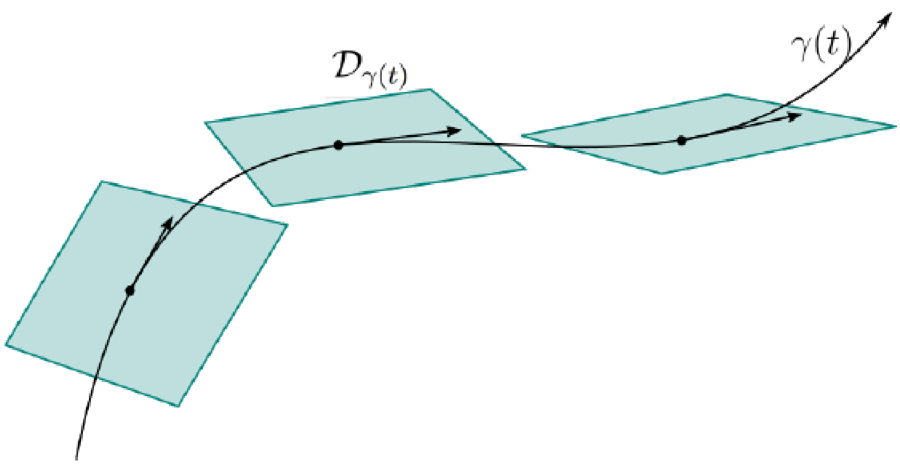


Figure 1.2: A horizontal curve with a distribution (Figure taken from [10])

Remark 1.7. Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold with a local frame for the distribution $\mathcal{D}_q = \text{span}\{X_1, \ldots, X_k\}$. We notice that for any horizontal curve $\gamma : [0, T] \to M$ we can write the tangent map as

$$\gamma'(t) = \sum_{i=1}^{k} u_i(t) X_i(\gamma(t)) \tag{1.1}$$

for some L^1-functions $u_1, \ldots, u_k$ called the **control functions**.

The notion of horizontal curves makes it possible to define lengths of a horizontal curve $\gamma : [0, T] \to M$ as follows.

$$\ell(\gamma) := \int_0^T \sqrt{g(\gamma'(t), \gamma'(t))} dt. \tag{1.2}$$

Analogously, to the Riemannian case we find distances between two points is found by considering the infimum of the lengths of horizontal curves connecting the two points.

Definition 1.8. *Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold. For points $x_1, x_2 \in M$ we define the* ***Carnot-Carathéodory distance*** *d_{SR} as*

$$d_{SR}(x_1, x_2) = \inf\{\ell(\gamma) | \gamma : [0, T] \to M \text{ is a horizontal curve}, \gamma(0) = x_1, \gamma(T) = x_1\}.$$

Using this terminology, we can formalize remark 1.3.

Theorem 1.9 ([5, Theorem 3.31])**.** *[Chow-Rashevskii's theorem] Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold, then*

1. *(M, d_{SR}) is a metric space such that for $x_1, x_2 \in M$ we have $d_{SR}(x_1, x_2) < \infty$, i.e. each two points can be connected by a horizontal curve;*
2. *the topology induced by (M, d) is equivalent to the manifold topology.*

Remark 1.10. Some authors do not assume the distribution to be bracket-generating in the definition of a sub-Riemannian manifold, in this case the bracket-generating assumption should be added in order for Theorem 1.9 to hold.

1.2 Pontryagin extremals in the Hamiltonian setting

In this section we define normal and abnormal length minimizers in a terms of Hamiltonian vector fields. Suppose we have a horizontal curve γ on a sub-Riemannian manifold $(M, \mathcal{D}, g)$. Recall that in equation 1.1, we wrote the tangent map as $\gamma'(t) = \sum_{i=1}^{k} u_i(t) X_i(\gamma(t))$ for some L^1-functions u_i. Using this we associate to γ a time-dependent vector field

$$\vec{u}_t = \sum_{i=1}^{k} u_i(t) X_i.$$

The integral curve of the vector field $\vec{u}_t$ starting at $\gamma(0)$ is the curve $\gamma(t)$. Consider the flow $\Phi(t)$ of $\vec{u}_t$. Using the pullback we define the flow on the cotangent bundle as

$$\left(\Phi(t)^{-1}\right)^* : T_q^*M \to T^*_{\Phi(t)(q)}M.$$

We first want to find a vector field in $\mathfrak{X}(T^*M)$ generating $(\Phi(t)^{-1})^*$. In order to do this we need a way of lifiting vector fields from M to T^*M. Smooth vector fields on a manifold M are in one-to-one correspondence with smooth functions in $C^\infty(T^*M)$ that are linear on the fibers of T^*M. To see this, we note each vector field Y on M can be associated with a function $f_Y : T^*M \to \mathbb{R}$, where $f_Y(\lambda) = \langle \lambda, Y(q)\rangle$ for $\lambda \in T^*M$, $q = \pi(\lambda)$ and $\langle \cdot, \cdot \rangle$ is the pairing of vectors and covectors. Using this specific lift of vector fields, we can define the Poisson bracket on vector fields as follows.

Definition 1.11. *The **Poisson bracket** is a map*

$$\{\cdot,\cdot\} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$$

defined by

$$\{f_X, f_Y\} = f_{[X,Y]}$$

for f_X, f_Y and $f_{[X,Y]}$ functions in $C^\infty(T^*M)$ associated to the vector fields $X, Y, [X,Y] \in \mathfrak{X}(M)$ as defined above.

In coordinates (q,p) on T^*M we can compute that

$$\{f,g\} = \sum_{i=1}^n \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}$$

for $f, g \in C^\infty(T^*M)$. Using the Poisson bracket we can define the Hamiltonian vector field.

Definition 1.12. *Given a function $F \in C^\infty(T^*M)$ we define the **Hamiltonian vector field** as $X_F = \{F, \cdot\}$.*

Theorem 1.13. *[5, Proposition 4.12] Let $\vec{u}_t = \sum_{i=1}^k u_i(t)X_i$ be a time-dependent vector field with flow $\Phi(t)$. The flow $(\Phi(t)^{-1})^*$ on T^*M is generated by the Hamiltonian time-dependent vector field X_h on T^*M. Here $X_h = \{h, \cdot\}$ for the smooth function*

$$h(\lambda) = \langle \lambda, \vec{u}_t(q)\rangle \in C^\infty T^*M,$$

with $q = \pi(\lambda)$.

Let us give an example of the Hamiltonian vector field for a specific distribution.

Example 1.14. Let us consider the Heisenberg distribution from 1.4 and compute X_h. At a point $q = (x,y,z) \in \mathbb{R}^3$ we first compute the time-dependent vector field $\vec{u}_t$ for the distribution to be

$$\begin{aligned}\vec{u}_t(q) &= u_1(t)\left(\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}\right) + u_2(t)\left(\frac{\partial}{\partial y}\right)\\ &= u_1(t)\frac{\partial}{\partial x} + u_2(t)\frac{\partial}{\partial y} + \left(u_2(t)\frac{x}{2} - u_1(t)\frac{y}{2}\right)\frac{\partial}{\partial z}.\end{aligned}$$

Take an arbitrary covector $\lambda = (x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3$. Then the function $h \in C^\infty(T^*M)$ is given by

$$\begin{aligned} h(\lambda) &= \langle \lambda, \vec{u}_t(q) \rangle \\ &= p_x u_1(t) + p_y u_2(t) + \left(u_2(t)\frac{x}{2} - u_1(t)\frac{y}{2} \right) p_z. \end{aligned}$$

The Hamiltonian vector field related to h is

$$\begin{aligned} X_h(\lambda) &= \{h(\lambda), \cdot\} \\ &= u_1(t)\frac{\partial}{\partial x} - \frac{1}{2}u_2(t)p_z\frac{\partial}{\partial p_x} + u_2(t)\frac{\partial}{\partial y} + \frac{1}{2}p_z\frac{\partial}{\partial p_y} + \left(\frac{1}{2}u_2(t)x - \frac{1}{2}u_1(t)y \right)\frac{\partial}{\partial z}. \end{aligned}$$

To see that the projection of the flow of this vector field indeed yields the curve $\gamma(t)$ and satisfies Equation (1.1), we make the following computation. Consider an integral curve $\lambda(t)$ of the vector field X_h and project it to a curve $\gamma(t) = \pi(\lambda(t))$ on $\mathbb{R}^3$. The tangent vector of this curve is

$$\begin{aligned} \gamma'(t) &= d\pi\left(X_h(\lambda)\right) \\ &= u_1(t)\frac{\partial}{\partial x} + u_2(t)\frac{\partial}{\partial y} + \left(u_2(t)\frac{x}{2} - u_1(t)\frac{y}{2} \right)\frac{\partial}{\partial z} \\ &= \vec{u}_t(\gamma(t)). \end{aligned}$$

△

Let us reformula the vector field X_h and the function h in Theorem 1.13 in a more convenient way. Define $h_i(\lambda) := \langle \lambda, X_i(q) \rangle$ for $q = \pi(\lambda)$. Then we write:

$$\begin{aligned} h(\lambda) &= \langle \lambda, \sum_{i=1}^k u_i(t) X_i(q) \rangle \\ &= \sum_{i=1}^k u_i(t) h_i(\lambda), \end{aligned}$$

In the same way the vector field X_h is given by

$$\begin{aligned} X_h(\lambda) &= \{h(\lambda), \cdot\} \\ &= \left\{ \sum_{i=1}^k u_i(t) h_i(\lambda), \cdot \right\} \\ &= \sum_{i=1}^k u_i(t) X_{h_i} \end{aligned}$$

where X_{h_i} the Hamiltonian vector fields associated to the functions h_i. Using the Hamiltonian vector field, we can define Pontryagin extremals.

Theorem 1.15 ([5, Theorem 4.20], Hamiltonian characterization of Pontryagin extremals)**.** *Consider the length-minimizing horizontal curve $\gamma(t)$ with $\gamma'(t) = \vec{u}_t(\gamma(t))$ on M. There exists a* *curve $\lambda : [0,T] \to T^*M$ with $\pi(\lambda(t)) = \gamma(t)$ and $\lambda'(t) = \sum_{i=1}^k u_i(t) X_{h_i}(\lambda(t))$ satisfying one of the following two conditions:*

(N) $h_i(\lambda(t)) = u_i(t)$;

(A) $h_i(\lambda(t)) = 0$.

Using this theorem, we can define Pontryagin extremals.

Definition 1.16. *The curve $\lambda : [0,T] \to M$ as defined in theorem 1.15 is called a **Pontryagin extremal**. If λ satisfies condition (N) then it is called a **normal extremal**, and when λ satisfies condition (A) then it is called an **abnormal extremal**. If we find a geodesic that is a projection of a normal Pontryagin extremal, we call it a **normal geodesic**. Similarly, for abnormal Pontryagin extremals.*

1.3 Sub-Riemannian Hamiltonian

Our goal is to compute geodesics in the sub-Riemannian setting. For a Riemannian manifold (M, g) we know how to find the geodesics, first one determines the Lagrangian $L : TM \to \mathbb{R}$ given by $L(q,v) = \frac{1}{2} g_q(v,v)$ for the metric g_q at the point $q = (q_1, \ldots, q_n) \in M$ and $v = (v_1, \ldots, v_n) \in T_qM$. From the Lagrangian one either solves the Euler-Lagrange equations given by $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial q}$. Otherwise, one can find the Hamiltonian $H : T^*M \to \mathbb{R}$ defined by the Legendre transform $H(q,p) = \sum_{i=1}^n p_i v_i - L(q,v)$ for $p = (p_1, \ldots, p_n) \in T^*M$ and solve Hamilton's equations given by

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

However, on a sub-Riemannian manifold $(M, \mathcal{D}, g)$ both methods no longer work. This is because the metric is not defined on the whole tangent bundle, but only on the distribution. Moreover, since we only consider curves that are tangent to the distribution, there will be an infinite number of vectors in the tangent bundle that do project to the same curve which we do not take the Legendre transform over. In figure 1.3 the vectors projecting to the same curve are shown in red.

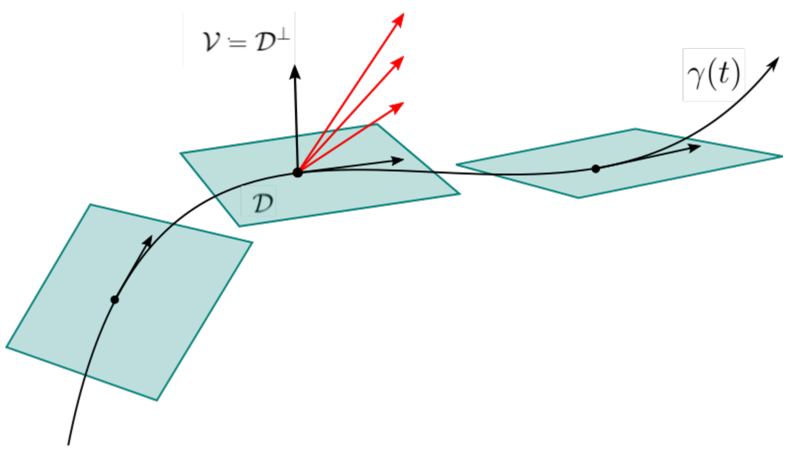


Figure 1.3: Red vectors all project to the horizontal curve $\gamma(t)$, but do not lie in the distribution $\mathcal{D}$.

To resolve these problem, we consider a Legendre-like transform, namely we consider all possible functions $u = (u_1, \dots, u_k)$ such that the vector $\vec{u}_t(q) = \sum_{i=1}^k u_i X_i(q) \in \mathcal{D}_q$ for all $q \in M$. Usin this we find the Lagrangian

$$L(q, \vec{u}_t(q)) = \frac{1}{2} g_q(\vec{u}_t(q), \vec{u}_t(q)).$$

If we assume the local frame $X_1, \dots X_k$ are orthonormal, we find $g_q(\vec{u}_t(q), \vec{u}_t(q))$ becomes $|u|^2$. If we denote $u = (u_1, \dots, u_k)$, we find $L(q, g_q(\vec{u}_t(q)) = \frac{1}{2}|u|^2$. Using a Legendre-like transform, which maximizes over u instead of v we define a sub-Riemannian Hamiltonian as follows.

Definition 1.17. *The **sub-Riemannian Hamiltonian** is the function $H : T^*M \to \mathbb{R}$ given by*

$$H(\lambda) = \sup_u \left(\langle \lambda, \vec{u}_t(q) \rangle - \frac{1}{2}|u|^2 \right)$$

where the supremum is taken over all controls u and $q = \pi(\lambda)$.

We can rewrite the sub-Riemannian Hamiltonian in such a way that we do not need the supremum.

Proposition 1.18. *[5, Prop 4.22] The sub-Riemannian Hamiltonian can be written as*

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^k \langle \lambda, X_i(q) \rangle^2$$

where $q = \pi(\lambda)$.

Remark 1.19. Notice that we can only state Definition 1.17 and Proposition 1.18 when the local frame for $\mathcal{D}_q$ is orthonormal with respect to the sub-Riemannian metric on $\mathcal{D}_q$. When the frame is not orthonormal, we can either write a sub-Riemannian Hamiltonian via the so-called cometric as in [4, Proposition 1.5.5.] or apply a Gram-Schmidt process (conform [11, Proposition 11.3]) to make the frame orthonormal. If we already have an orthogonal frame, we can rescale the sub-Riemannian metric to make the frame also orthonormal. In the rest of this thesis we assume that the local frame for the distribution is orthogonal, unless stated otherwise.

Next we relate normal Pontryagin extremals to the vector field $X_H = \{H, \cdot\}$. We show that the normal Pontryagin extremals are integral curves of the vector field generated by the sub-Riemannian Hamiltonian H.

Theorem 1.20. *[5, Theorem 4.25] A curve in the cotangent space of M given by $\lambda : [0, T] \to T^*M$ is a normal extremal if and only if it is a solution to*

$$\lambda'(t) = X_H(\lambda(t)).$$

Moreover, given a normal extremal, the corresponding normal extremal trajectory $\gamma(t) = \pi(\lambda(t))$ is smooth and has constant speed satisfying

$$\frac{1}{2}\|\gamma'(t)\|^2 = H(\lambda(t))$$

for all $t \in [0, T]$.

In order to find the sub-Riemannian geodesics, we need an analogous set of equations to the Hamilton equations for a Riemannian Hamiltonian. If we write the sub-Riemannian Hamiltonian vector field in coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on T^*M we obtain:

$$X_H = \sum_{i=1}^{k} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Comparing this to $\lambda'(t)$, which in coordinates is given by $\lambda'(t) = \sum_{i=1}^{k} \dot{q}_i(t)\frac{\partial}{\partial q_i} + \dot{p}_i(t)\frac{\partial}{\partial p_i}$,

we find that

$$\dot{q}(t) = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p}(t) = -\frac{\partial H}{\partial q} \tag{1.3}$$

which are precisely the Riemannian Hamilton's equations, but now for the sub-Riemannian Hamiltonian.

The following corollary gives us a way of parametrizing the normal trajectories by arclength.

Corollary 1.21. *[5, Corollary 4.27] A normal trajectory $\gamma(t)$ is parametrized by arclength if and only if its associated Pontryagin extremal lift $\lambda(t)$ is contained in the level set $H^{-1}(1/2)$.*

Proof. Notice that for a normal Pontryagin extremal we have $H(\lambda(t)) = \frac{1}{2}\|\gamma'(t)\|^2$ for the normal trajectory $\gamma(t) = \pi(\lambda(t))$. Notice that if $\gamma(t)$ is parametrized by arclength we find that $\|\gamma'(t)\| = 1$, which is the case if and only if $H(\lambda(t)) = \frac{1}{2}$. So we find $\lambda(t) \in H^{-1}(\frac{1}{2})$. $\square$

1.4 Sub-Riemannian geodesics

In this section we define sub-Riemannian geodesics and we state that projections of normal Pontryagin extremals are indeed always sub-Riemannian geodesics. Let us first define what we mean by a sub-Riemannian geodesics.

Definition 1.22. *A horizontal curve $\gamma : [0,T] \to M$ parametrized by arclength is a **geodesic** if for all $t \in [0,T]$ there exists some interval $[a,b] \subset [0,T]$ around t such that*

$$\ell\left(\gamma|_{[a,b]}\right) = d\left(\gamma(a), \gamma(b)\right).$$

Here ℓ and d are the length and distance of horizontal curves as defined in Section 1.1. In other words, $\gamma(t)$ locally minimizes the length for all $t \in [0,T]$.

In order to state the theorem that normal Pontryagin extremals always project to sub-Riemannian geodesics, we need some technical notions.

Fix a smooth function $f \in C^\infty(M)$ then define the smooth submanifold

$$\mathcal{L}_0 := \{df_q : q \in M\} \subseteq T^*M.$$

Note that the projection $\pi|_{\mathcal{L}_0} : T^*M \to M$ is a diffeomorphism, because for each point $q \in M$ we defined a unique covector field $df_q \in \mathcal{L}_0$. Hence, we know $\dim(\mathcal{L}_0) = \dim(M) = n$ as manifolds. Assume that the flow of X_H is complete (i.e. defined for all $t \in \mathbb{R}$)and denote the flow of the Hamiltonian vector field X_H by e^{tX_H}. Considering the image of $\mathcal{L}_0$ under the flow, we obtain

$$\mathcal{L}_t := \exp(X_H t)\mathcal{L}_0$$

for all $t \in [0, T]$. The collection of spaces $\mathcal{L}_t$ for each t defines a manifold

$$\mathcal{L} := \{(t, e^{tX_H}\lambda_0) \in \mathbb{R} \times T^*M : \lambda_0 \in \mathcal{L}_0, t \in [0, T]\}.$$

Using this terminology, we can state the main theorem saying that all normal extremals project to sub-Riemannian geodesics.

Theorem 1.23. *[5, Theorem 4.62] Let $f \in C^\infty(T^*M)$. Consider the manifold $\mathcal{L}_t$ for f such that the restriction $\pi|_{\mathcal{L}_t}$ is a diffeomorphism for all $t \in [0, T]$. If $\lambda_0 \in \mathcal{L}_0$, then the normal trajectory*

$$\gamma(t) := \pi \circ \exp(tX_H)(\lambda_0) \tag{1.4}$$

is a strict length minimizer among all horizontal curves between $\gamma(0)$ and $\gamma(T)$.

Let us present two examples of sub-Riemannian minimizers, one normal and one abnormal case.

Example 1.24 (Heisenberg distribution)**.** Consider the Heisenberg distribution $\mathcal{D}_{(x,y,z)} = \text{span}\{X, Y\} = \text{span}\{\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}\}$ on $\mathbb{R}^3$ like in example 1.4. Notice that X and Y are orthogonal under the standard inner product of $\mathbb{R}^3$. Hence, we know from 1.18 the sub-Riemannian Hamiltonian for $\lambda = (x, y, z, p_x, p_y, p_z) \in T^*M$ is given by

$$H(\lambda) = \frac{1}{2}\langle\lambda, \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}\rangle^2 + \frac{1}{2}\langle\lambda, \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}\rangle^2.$$

From this we find that the sub-Riemannian Hamilton equations are given by the set of equations:

$$\begin{cases} x'(t) = p_x - \frac{y}{2}p_z \\ y'(t) = p_y + \frac{x}{2}p_z \\ z'(t) = -\frac{y}{2}\left(p_x - \frac{y}{2}p_z\right) + \frac{x}{2}\left(p_y + \frac{x}{2}p_z\right) \\ p_x'(t) = -\frac{p_z}{2}\left(p_y + \frac{x}{2}p_z\right) \\ p_y'(t) = \frac{p_z}{2}\left(p_x - \frac{y}{2}p_z\right) \\ p_z'(t) = 0. \end{cases} \tag{1.5}$$

Solving these equations can be done using the Mathemematica code in Appendix C.1. This code finds both the analytic and the numerical solution to the system of equations in 1.5. A plot of a sub-Riemannian geodesic for the Heisenberg distribution is shown in figure 1.4.

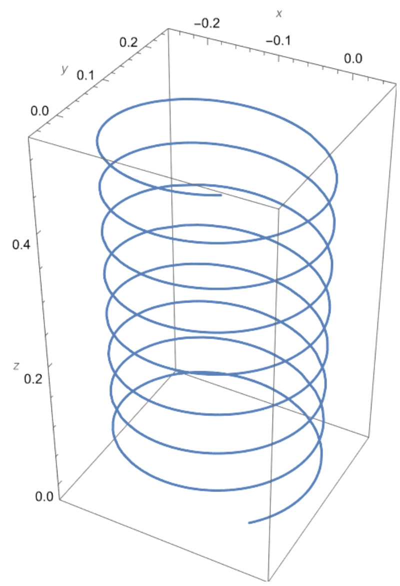


Figure 1.4: Sub-Riemannian geodesic for the Heisenberg distribution.

△

A similar computation can be made for the parallel parking problem as mentioned in the introduction. For the explicit computation see [4] or [10].

Example 1.25 (Abnormal curve)**.** Let us give an example of a length minimizer that is an abnormal geodesics. This example was taken from [12]. Consider the distribution $\mathcal{D} = \operatorname{span}\{X, Y\}$ with $X_{(x,y,z)} = \frac{\partial}{\partial x}$ and $Y_{(x,y,z)} = (1-x)\frac{\partial}{\partial y}x^2\frac{\partial}{\partial z}$ on $\mathbb{R}^3$. It turns out that

$$[X, Y] = -\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z} \qquad [X, [X, Y]] = 2\frac{\partial}{\partial z}$$

and all other bracket are zero. We note that $X, Y, [X, Y]$ are linearly independent if $x \neq 0$ and $x \neq 2$, while $X, Y, [X, [X, Y]]$ are linearly independent if $x \neq 1$. Hence, together $X, Y, [X, Y]$ and $[X, [X, Y]]$ span the tangent space for all points $(x, y, z) \in \mathbb{R}^3$. Therefore, the distribution is bracket-generating and by Theorem 1.9, we know every two points are connected by a horizontal curve. Moreover, we find that for the metric $g = dx^2 + ((1-x)^2 + x^4)^{-1}(dy^2 + dz^2)$ the frame X and Y are orthonormal. Therefore, by Theorem 1.18 we find the sub-Riemannian Hamiltonian for $\lambda = (x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3$ is of the form

$$H(\lambda) = \frac{1}{2}p_x^2 + \frac{1}{2}\left((1-x)p_y + x^2 p_z\right)^2.$$

Hamilton's equations are given by

$$\begin{cases} x'(t) = p_x \\ y'(t) = ((1-x)p_y + x^2 p_z)(1-x) \\ z'(t) = x^2((1 - xp_y + x^2 p_z) \\ p_x'(t) = ((1-x)p_y + x^2 p_z)\,(p_y - 2xp_z) \\ p_y'(t) = 0 \\ p_z'(t) = 0. \end{cases} \tag{1.6}$$

In [12, Proposition 1] it is shown that the curve $\gamma : [a, b] \to \mathbb{R}^3$ defined by $t \mapsto (0, t, 0)$ is a length minimizer when $b - a \leq \frac{2}{3}$. We want to show that in this case the system (1.6) is not satisfied for $\lambda(t) = \pi(\gamma(t))$, and hence γ is an abnormal geodesic for this distribution. Since $\gamma(t) = \pi(\lambda(t))$, we find $x(t) = z(t) = 0$ and the system (1.6) reduces to

$$\begin{cases} x'(t) = p_x \\ y'(t) = p_y \\ z'(t) = 0 \\ p_x'(t) = p_y^2 \\ p_y'(t) = 0 \\ p_z'(t) = 0. \end{cases} \tag{1.7}$$

From this, we find that since $x(t)$ is constant, we have $p_x(t) = 0$ and hence $p_y(t) = 0$. But then we find that $y'(t) = 0$, in other words $y(t)$ is constant. However, for the curve $\gamma(t) = (0, t, 0)$ we know that $y(t)$ is not constant. Therefore, $\gamma(t)$ does not satisfy the system (1.6) and hence it is not a normal sub-Riemannian geodesic of the distribution $\mathcal{D}$. △

1.5 Abnormal extremals and contact distributions

In this section we discuss abnormal sub-Riemannian geodesics. Thusfar, we have seen that we can characterize normal geodesics as the projections of normal Pontryagin extremals or as solutions of the sub-Riemannian Hamiltonian. Computing abnormal minimizers is not as easy. Moreover, we mention that not every sub-Riemannian structure admits abnormal geodesics ([4]). Therefore, it would be useful to have a result that tells us whether there exist abnormal geodesics. We will present one such result, which tells us that in the case of contact distribution in dimension three (like discussed in 1.4) there are no abnormal minimizers to be found. Before we state this, we want to characterize abnormal geodesics. For this characterization we will use *characteristic curves*. First, we will characterize normal extremals again, after which we continue with abnormal extremals. This new characterizataion will be crucial to state the non existence result of abnormal minimizers for a contact structure in dimension three.

Definition 1.26. *Let M be a smooth manifold with a 2-form σ. A smooth curve $\gamma : [0, T] \to M$ is a **characteristic curve** for the form σ if, for almost every $t \in [0, T]$, we have*

$$\gamma'(t) = \ker(\sigma_{\gamma(t)}).$$

Using the terminology of characteristic curves, we first give a new characterization for normal extremals.

Theorem 1.27 ([5, Proposition 4.30], Normal extremals via Characteristic curves)**.** *Let H be the sub-Riemannian Hamiltonian and assume that $c > 0$ is a regular value of H. Then a smooth curve on $H^{-1}(c)$ is a characteristic curve for the standard symplectic form $\omega|_{H^{-1}(c)}$ if and only if the curve is a reparametrization of a normal extremal trajectory.*

Now, we want to state a similar theorem for abnormal extremals. Recall that according to 1.15 an abnormal minimizer was defined as a non-vanishing solution to the system of equations

$$\begin{cases} \lambda'(t) & = \sum_{i=1}^k u_i(t) X_{h_i}(\lambda(t)) \\ h_i(\lambda(t)) & = 0. \end{cases} \tag{1.8}$$

In particular, we have that every abnormal extremal is contained in the zero level set of the sub-Riemannian Hamiltonian $H^{-1}(0)$. In order to define characteristic curves on $H^{-1}(0)$, we need that $H^{-1}(0)$ is a submanifold. However, 0 is never a regular value of H. In order to solve this problem, we need the regularity assumption we already made for our distributions.

Since we assumed that H is the sub-Riemannian Hamiltonian associated to a regular sub-Riemannian structure, we find by the constant-rank level set theorem ([6, Theorem 5.12]) that each level set of H is a properly embedded submanifold of codimension r in M. This assumption makes it possible to talk about characteristic curves in the submanifold $H^{-1}(0)$ of T^*M.

Theorem 1.28 ([5, Proposition 4.34], Abnormal extremals via Characteristic curves)**.** *Let H be a sub-Riemannian Hamiltonian associated to a sub-Riemannian structure. A smooth curve on $H^{-1}(0)$ is a characteristic curve for $\omega|_{H^{-1}(0)}$ if and only if it is a reparametrization of an abnormal extremal.*

We now need to relate the level sets of the Hamiltonian back to the distribution $\mathcal{D}$ we defined on our manifold. In general we can think of $H^{-1}(0)$ as the subspace of covectors that annihilate the distribution, i.e.

$$H^{-1}(0) = \{\lambda \in T^*M : \langle \lambda, v \rangle = 0 : v \in \mathcal{D}_{\pi(\lambda)}\} = \mathcal{D}^{\perp}.$$

Let us pick a basis $\omega_1, \dots, \omega_k$ for $\mathcal{D}^{\perp}$ with $\omega_i \in \Omega^1(T^*M)$. Then following remark A.3 we find $\lambda = \sum_{i=1}^n h_i \omega_i$ for $h_i : TM \to \mathbb{R}$ as in Equation 1.8. The canonical symplectic form becomes

$$\omega|_{\mathcal{D}^{\perp}} = \sum_{i=1}^k dh_i \wedge \omega_i + h_i d\omega_i. \tag{1.9}$$

1.5.1 Contact sub-Riemannian structure

In the previous section we have characterized abnormal extremals in terms of characteristic curves on the level set $H^{-1}(0)$ for a sub-Riemannian Hamiltonian H associated to a regular sub-Riemannian structure. We will now use this result to show that on a contact distribution no abnormal extremals can exist.

Definition 1.29. *Let M be a 3-dimensional manifold. A distribution $\mathcal{D} = \ker \xi$ of corank one is called a* ***contact distribution*** *if*
$xi \wedge d\xi \neq 0$.

We can then state the following theorem.

Theorem 1.30. *[5, Theorem 4.38] Let M be a 3-dimensional manifold with a distribution $\mathcal{D} = \ker \xi$ of corank one. All nontrivial abnormal extremal trajectories are contained in the* ***Martinet set***

$$\mathfrak{M} := \{q \in M : (\xi \wedge d\xi)|_q = 0\}.$$

In particular, if the distribution is a contact distribution the Martinet set is empty and no nontrivial abnormal extremal trajectories exist.

Before we can prove this result, we first need the following lemma.

Lemma 1.31. *Let M be an $2n$-dimensional manifold for $n \in \mathbb{Z}$ with a two-form $\omega \in \Omega^2(M)$. Then ω is non-degenerate on M if and only if $\wedge^n \omega \neq 0$.*

Proof. First, the equivalence is proven from right to left by contraposition. Take an arbitrary point $q \in M$. Assume ω is degenerate, then there exists $v \in TN$ such that $\omega(u, v) = 0$ for all $u \in T_qN$. Therefore, $\bigwedge^n \omega(v, u_1, \dots, u_{2n-1}) = 0$.

Conversely, assume that ω is non-degenerate and consider some local coordinates $(x_1, \dots, x_{2n})$. Then ω must be of the form:

$$\omega = \sum_{i,j=1}^{2n} c^{ij} dx_i \wedge dx_j.$$

For ω to be non-degenerate we need that $dx_i \wedge dx_j$ is included for every $1 \leq i, j \leq 2n$. Since suppose that $dx_l \wedge dx_r$ for $1 \leq l, r \leq 2n$ is not included, then $\omega(\frac{\partial}{\partial x_r}, u) = 0$ for all $u \in T_qN$, which contradicts the non-degeneracy. Moreover, $c^{ij} \neq 0$ as the form ω is nonvanishing. We can now compute that

$$\bigwedge^n \omega = C dx_1 \wedge \dots \wedge dx_{2n},$$

where $C = \prod_{i,j=1}^{2n} c^{ij} \neq 0$. Since the wedge product of all basis vectors is non-zero, this proves the implication from left to right. $\square$

Now we can prove the theorem.

Proof of theorem 1.30. From theorem 1.28 any abnormal extremal $\lambda \in T^*M$ is a characteristic curve of the symplectic form $\omega|_{\mathcal{D}^\perp}$. On $\mathcal{D}^\perp$ we have that $\lambda = h_1\omega_1 + h_2\omega_2 + h_3\omega_3$, where ω_1, ω_2 and ω_3 form a basis for $\mathcal{D}^\perp$. Here we take without loss of generality that $\omega_1 = \xi$ with $\mathcal{D} = \ker(\omega)$. We notice that $h_i(\lambda) = 0$ for $\lambda \neq 0$. On $\mathcal{D}^\perp$ we have $\langle \lambda, v \rangle = 0$ for $v \in \mathcal{D}_{\pi(\lambda)} = \ker(\xi)$. In other words, we find that using the fact that $v \in \ker(\omega_1)$

$$\begin{aligned} 0 &= \langle h_1\omega_1 + h_2\omega_2 + h_3\omega_3, v \rangle \\ &= \langle h_2\omega_2, v \rangle + \langle h_3\omega_3, v \rangle \end{aligned}$$

Hence, $h_2 = 0$ and $h_3 = 0$, since ω_2 and ω_3 are nontrivial. Moreover, we find $h_1 \neq 0$. Using this, we find that the symplectic form is given by

$$\omega|_{\mathcal{D}^\perp} = dh_1 \wedge \xi + h_1 d\xi. \tag{1.10}$$

For $\lambda(t)$ to be an abnormal extremal with extremal trajectory $\gamma(t)$, we need that $\gamma'(t) \in \ker(\omega|_{\mathcal{D}^\perp})$ for $\gamma(t)$ non-trivial. Therefore, we need that $\omega|_{\mathcal{D}^\perp}$ is degenerate. By lemma 1.31 we find that $\omega|_{\mathcal{D}^\perp}$ is degenerate if and only if $(\omega \wedge \omega)|_{\mathcal{D}^\perp} = 0$. Computing this form, we obtain:

$$(\omega \wedge \omega)|_{\mathcal{D}^\perp} = 2h_1 \wedge \xi \wedge d\xi.$$

For this form to vanish, we need that $\xi \wedge d\xi = 0$, because $h_0 \neq 0$. Therefore, we conclude that for $\lambda(t)$ to be an abnormal extremal is equivalent to requiring that $\xi \wedge d\xi = 0$ along the associated trajectory $\gamma(t)$. This is precisely equivalent to $\gamma(t)$ being in the Martinet set $\mathfrak{M}$.

We notice that in case $\mathcal{D}$ is a contact distribution, then the Martinet set is empty because $\xi \wedge d\xi \neq 0$. Hence, in this case no abnormal minimizers exist. □

Contrary to the contact case, we find that on even dimensional manifolds with a distribution of codimension one there always exist abnormal extremals.

Proposition 1.32. *Let M be an even dimensional manifold with a constant rank distribution of codimension one, then there always exist abnormal extremals.*

Proof. Let n be an even number. Notice that if $\dim \mathcal{D}_q = n-1$, then we can find the dimensions of the subbundles $\mathcal{D} \subseteq TM$ and $\mathcal{D}^\perp \subseteq T^*M$ to be:

$$\begin{aligned}\dim \mathcal{D} &= \dim M + \operatorname{rank} \mathcal{D} \\ &= n + n - 1 \\ &= 2n - 1\end{aligned}$$

$$\begin{aligned}\dim \mathcal{D}^\perp &= \dim M + \operatorname{rank} \mathcal{D}^\perp \\ &= n + 1.\end{aligned}$$

Since ω is a symplectic linear form, we know that it must be skew-symmetric. Any skew-symmetric map has even rank. Hence, by the rank-nullity theorem applied to the linear map $\omega|_{\mathcal{D}^\perp}$, we find

$$n + 1 = \dim \mathcal{D} = \dim \ker \omega|_{\mathcal{D}^\perp} + \dim \operatorname{im} \omega|_{\mathcal{D}^\perp}.$$

Since, $n+1$ is odd and the image of the form is even, we need that the kernel of $\omega|_{\mathcal{D}^\perp}$ is odd, and hence non-trivial. Therefore, there always exists some non-trivial characteristic curve for $\omega|_{\mathcal{D}^\perp}$, which implies there exists an abnormal extremal trajectory by theorem 1.28. □

1.6 Exponential map

In Riemannian geometry, we define the exponential map to be the map that sends each tangent vector to a corresponding geodesic. To be more precise, if on a Riemannian manifold (M, g), we consider a point $q \in M$ and a tangent vector $v \in T_qM$, then $\exp : T_qM \to M$ is defined as $\exp_q(v) = \gamma_v(1)$. Here γ_v is the unique curve through the point q with tangent vector v. Each Riemannian geodesic can be written as $\gamma_v(t) = \exp(tv)$. Analogously, we want to define a similar construction for sub-Riemannian manifolds. This is however not always possible since it might be that the vector $v \in T_qM$ does not lie in the admitted distribution. This is the same issue as we had before while defining the Legendre transform. In other words, there are in infinte number of vectors $v \in T_qM$ that project to the geodesic γ_v, so the exponential map would not be well-defined. Similarly as for the sub-Riemannian Lagrangian (See section 1.3) this problem can be solved by lifting to the cotangent bundle. Consider the set of curves $\lambda(t)$ in T_q^*M solving the Hamiltonian system

$$\lambda'(t) = X_H(\lambda(t))$$

such that the solution $\lambda(t)$ is well-defined on the interval $[0, 1]$. Each such curve has an intial covector $\lambda_0 = \lambda(0)$. Consider the set of these initial covectors λ_0 and denote is $\mathcal{A}$. Taking the set $\mathcal{A}$ as the domain, the following definition of the exponential map is well-defined.

Definition 1.33. *Let $q \in M$. The* ***sub-Riemannian exponential map*** *based at q is the map $\exp_q : \mathcal{A} \to M$ defined by $\exp(\lambda_0) = \pi \circ e^{X_H}(\lambda_0)$.*

Intuitively this definition means that we look at the image of an initial covector under the flow of the sub-Riemannian Hamiltonian vector field X_H, and project this to the base space. It remains to show this construction yields the normal extremal trajectories. To show this, we first need to prove a lemma about the flow.

Lemma 1.34 ([5, Lemma 8.35], Homogeneity of the exponential map)**.** *Let H be the sub-Riemannian Hamiltonian. Then for every covector $\lambda \in T^*M$ and any constant $\alpha \in \mathbb{R}_{>0}$ we have*

$$e^{X_H t}(\alpha\lambda) = \alpha e^{\alpha X_H t}(\lambda)$$

for all $t \in \mathbb{R}_{>0}$.

Using this lemma we can state that the exponential map, maps normal extremals to normal geodesics.

Lemma 1.35. *Let $\lambda(t)$ be a normal extremal that satisfies the initial condition $\lambda(0) = \lambda_0 \in T_{q_0}^*M$. Then the normal extremal path $\gamma(t) = \pi(\lambda(t))$ satisfies $\gamma(t) = \exp_{q_0}(t\lambda_0)$.*

Proof. By definition, we have $\exp_{q_0}(t\lambda_0) = \pi\left(e^{X_H}(t\lambda_0)\right)$. Then using the homogeneity property in 1.34, we find $\pi\left(e^{X_H}(t\lambda_0)\right) = \pi\left(e^{tX_H}(\lambda_0)\right)$. By definition of the flow, we have $e^{tX_H}(\lambda_0) = \lambda(t)$. Therefore, $\pi\left(e^{tX_H}(\lambda_0)\right) = \pi(\lambda(t)) = \gamma(t)$. □

The homogeneity property in lemma 1.34 has another nice consequence.

Corollary 1.36. *The level set $H^{-1}(\frac{1}{2})$ of the sub-Riemannian Hamiltonian H is diffeomorphic to the cylinder of normalized covectors $\Lambda_{q_0} \subseteq T_{q_0}^*M$.*

The proof of this fact can be found in [5, Remark 8.37]. But to give some intuition, we know that if we start with a normalized covector it must lay within a circle. Then by the homogeneity property we find that if we increase time the covector stays with in the circle but can move in some direction, namely vertically to the distribution (if the rank of the distribution is less than the dimension of the manifold).

Chapter 2

Sub-Riemannian structure on a principal G-bundle

In this chapter we discuss how we can find a sub-Riemannian structure on a principal G-bundle $\pi : Q \to M$. It turns out that we can give a closed form expression for the normal sub-Riemannian geodesics on Q, if there exists a Riemannian metric of so-called 'constant bi-invariant type' on the space Q. The statement and proof of this result are the content of section 2.2. This will allow us to give sub-Riemannian structures on some examples like Lie groups and homogeneous spaces in section 2.3. Moreover, it will be useful if we want to define a sub-Riemannian structure for the famous Hopf-fibration in section 2.3.1. We initially studied Sub-Riemannian principal bundles to see if we could extend techniques from this proof to sub-Riemannian structures on orbifolds, this turned out not to be possible within the scope of the thesis. We still include it for potential future research. This section is heavily inspired by [4] and [13].

2.1 Sub-Riemannian structures of bundle type

First we need some notions of differential geometry. Let $\pi : Q \to M$ be a submersion of manifolds. The fiber through $q \in Q$ is the submanifold $Q_m := \pi^{-1}(m)$ for $m = \pi(q) \in M$. By the implicit function theorem the fiber is a submanifold.

Definition 2.1. *The* ***vertical space*** $\mathcal{V}_q$ *is the tangent space to fiber through* q*, i.e.*

$$\mathcal{V}_q = T_q(Q_m) = \ker(d\pi_q).$$

The collection of all vertical spaces is called the ***vertical distribution*** $\mathcal{V} \subseteq TQ$*.*

The distribution $\mathcal{V}$ is an integrable distribution since $\mathcal{V}_q = T_q(Q_m)$ for each point $q \in Q_m$ (and $Q_m \subseteq Q$ and immersed submanifold). As we are interested in completely non-integrable distributions, we consider its complement as follows.

Definition 2.2. *An* ***Ehresmann connection*** *for* $\pi : Q \to M$ *is a distribution* $\mathcal{H} \subseteq TQ$ *which is everywhere transversal to the vertical distribution* $\mathcal{V}$*, i.e. for every* $q \in Q$

$$\mathcal{V}_q \oplus \mathcal{H}_q = T_q Q. \tag{2.1}$$

A connection on a principal G-bundle can be thought of as a choice of horizontal complement to the tangent bundle on Q. We will refer to the distribution $\mathcal{H}$ as the *horizontal distribution* or the *sub-Riemannian distribution*. On the horizontal distribution $\mathcal{H}$ we want to give a sub-Riemannian metric.

Lemma 2.3. *The map $r = d\pi_q|_{\mathcal{H}_q} : \mathcal{H}_q \to T_{\pi(q)}M$ is a linear isomorphism.*

Proof. Because π is a submersion $d\pi_q : T_qQ \to T_{\pi(q)}M$ is a surjective linear map with kernel $\mathcal{V}_q$. Hence, we find that by the first isomorphism theorem $T_qQ/\mathcal{V}_q \cong T_{\pi(q)}M$. On the other hand, we know that by the Ehresmann connection we have that $T_qQ = \mathcal{V}_q \oplus \mathcal{H}_q$. So, $T_qQ/\mathcal{V}_q \cong \mathcal{H}_q$. Therefore, we find that $\mathcal{H}_q \cong T_{\pi(q)}M$ via r. □

Using the previous lemma, we can use that given a Riemannian metric g on M, the pullback metric r^*g under $r = d\pi_q|_{\mathcal{H}_q} : \mathcal{H}_q \to T_{\pi(q)}M$ yields a metric on $\mathcal{H}_q$. In this way we obtain a sub-Riemannian metric on the underlying distribution $\mathcal{H}_q$.

Definition 2.4. *For a given submersion $\pi : Q \to M$ with a metric g on M, the triple $(Q, \mathcal{H}, r^*g)$, as constructed above, is called the induced* ***sub-Riemannian structure****. If $\pi : Q \to M$ is a principal G-bundle then we call the metric r^*g on $\mathcal{H}$ a* ***metric of bundle type****.*

Given a curve on M, we want to be able to lift it to Q, and preferably to $\mathcal{H}$. This can be done via a *horizontal lift.*

Definition 2.5. *Let $c : [0, 1] \to M$ be a path starting at m in M, then the* ***horizontal lift*** *of c through $q \in Q_m$ is defined to be the unique curve $\gamma_c : [0, 1] \to Q$ which starts at q, is tangent to $\mathcal{H}$, and projects to c (i.e. $\pi \circ \gamma_c = c$).*

Then we find that from a Riemannian metric on Q, we can make a sub-Riemannian metric via the following definition.

Definition 2.6. *Let $\pi : Q \to M$ a submersion with horizontal distribution $\mathcal{H}$. We say that a Riemannian metric on Q is* ***compatible*** *with the induced sub-Riemannian metric on Q if $TQ = \mathcal{V} \oplus \mathcal{H}$ is an orthogonal splitting, i.e. $\mathcal{H} = \mathcal{V}^\perp$ under the Riemannian metric.*

2.2 Sub-Riemannian geodesics on a principal G-bundle

For the rest of this chapter let $\pi : Q \to M$ be a Principal G-bundle and $\mathfrak{g}$ the Lie algebra of G. Using the structure of this principal bundle we want to compute sub-Riemannian geodesics on Q. In Chapter 1, we have seen that a convenient way to find sub-Riemannian geodesics is to compute a sub-Riemannian Hamiltonian. In [4] the following definition is made.

Definition 2.7. *Suppose we have a principal G-bundle $\pi : Q \to M$. Let g be a Riemannian metric on M, with Hamiltonian $H_R : T^*M \to \mathbb{R}$. Given a projection* $\mathrm{Pr} : T^*Q \to T^*M$ *we define the sub-Riemannian Hamiltonian induced by the Riemannian Hamiltonian on M as*

$$H_{SR} = H_R \circ Pr.$$

We note that the distribution related to the sub-Riemannian structure on Q is then given by the horizontal lifts of an orthogonal frame for the Riemannian structure on M. The projection $Pr : T^*Q \to T^*M$ is however not obvious. Using the Ehresmann connection, we find that $TQ = \mathcal{H} \oplus \mathcal{V}$. Dualizing, yields $T^*Q = \mathcal{H}^* \oplus \mathcal{V}^*$. Let us denote the projection onto the first factor by $\mathrm{pr}_1 : T^*Q \to \mathcal{H}^*$. Now, it remains to find a map from $\mathcal{H}^*$ to T^*M. Consider any element $f \in H_q^*$, this is a linear map $f : \mathcal{H}_q^* \to \mathbb{R}$, from this we want to construct a map $T_{\pi(q)}M \to \mathbb{R}$. Consider any vector v in $T_{\pi(q)}$. These vectors have a unique horizontal lift $h_q v$ on $\mathcal{H}_q$. Then we define the map $\mathrm{pr}_2 : \mathcal{H}_q \to T_{\pi(q)}M$ by $\mathrm{pr}_2(f)(v) = f(h_q v)$. Combining the maps pr_1 and pr_2 we obtain the map $\mathrm{Pr} : T^*Q \to T^*M$ defined by $\mathrm{pr}_2 \circ \mathrm{pr}_1$ as desired.

Let us fix a metric of bundle type on a principal G-bundle $\pi : Q \to M$. Moreover, let $\mathfrak{g}$ be the Lie algebra of G. We will state a result that relates the sub-Riemannian geodesics on Q to the Riemannian geodesics of M. For this we will first need some notions from Riemannian geometry and Lie theory. First of all let us lift the action on Q to its tangent bundle.

Definition 2.8. *The **infinitesimal generator** of the group action of G on Q is the map $\sigma_q : \mathfrak{g} \to T_qQ$ defined by*

$$\sigma_q(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} q \exp_G(\xi t)$$

for $q \in Q$ and $\xi \in \mathfrak{g}$.

Lemma 2.9. *[11, Proposition 27.8] For a principal G-bundle $\pi : Q \to M$ we have $\sigma_q : \mathfrak{g} \to V_q$ is a linear isomorphism.*

Because we can identify $\mathcal{V}_q$ with $\mathfrak{g}$, we can define a connection 1-form.

Definition 2.10. *Given a horizontal space $\mathcal{H}$ on $\pi : Q \to M$ a **connection 1-form** $A \in \Omega^1(Q;\mathfrak{g})$ is a G-equivariant $\mathfrak{g}$-valued 1-form on Q such that $A(\sigma_q(\xi)) = \xi$ and $A(\xi) = 0$ for $\xi \in \mathcal{H}$.*

Every connection $\mathcal{H}$ on a principal G-bundle comes with such a connection 1-form, since we can construct it as the projection $A_q : T_qQ \to \mathcal{V}_q$ such that $\mathcal{H}_q = \ker(A_q)$ for all $q \in Q$. Notice that the condition $A(\sigma_q(\xi)) = \xi$ is well-defined then since $A(\sigma_q(\xi)) \in V_q \cong \mathfrak{g}$ by 2.9. Conversely, given such a connection 1-form A, we can give the horizontal space $\mathcal{H}_q = \ker(A)$.

Let us define a Riemannian metric $\langle \cdot, \cdot \rangle$ on Q that is G-invariant [1], then

Definition 2.11. *The bilinear form*

$$\mathbb{I}_q(\xi, \eta) = \langle \sigma_q\xi, \sigma_q\eta \rangle,$$

*for $\xi, \eta \in \mathfrak{g}$, is called the **moment of inertia tensor** at q.*

Using the moment of inertia tensor, we can define the main condition for our principal bundles to have normal geodesics of a specific form.

Definition 2.12. *[13] The Riemannian metric $\langle \cdot, \cdot \rangle$ on Q is said to be of **constant bi-invariant type** if its moment of inertia tensor $\mathbb{I}_q$ does not depend on the point $q \in Q$.*

[1]Let $\alpha : G \times M \to M$ be a group action G-action on a manifold M. We call a Riemannian metric β G-invariant if $\beta(d\alpha(v), d\alpha(w)) = \beta(v, w)$.

Example 2.13. Let G be a connected matrix Lie group that acts on itself by right-translation $R_g(q) = qg$. For $q \in G$ and $\xi \in \mathfrak{g}$. Consider a G-invariant Riemannian metric β on G. We find the infinitesimal generator is given by

$$\begin{aligned}\sigma_q(\xi) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} q\exp(t\xi) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} q\left(I + t\xi + \frac{(t\xi)^2}{2} + \dots\right) \\ &= q\xi.\end{aligned}$$

The moment of inertia tensor for arbitrary $\xi, \eta \in \mathfrak{g}$ is given by

$$\mathbb{I}_q(\xi,\eta) = \beta_q(\sigma_q\xi, \sigma_q\eta)$$

Consider any other point $p \in G$. Then we know that $p = qg$ for some $g \in G$ we have

$$\begin{aligned}\mathbb{I}_p(\xi,\eta) &= \mathbb{I}_{qg}(\xi,\eta) \\ &= \beta_{qg}(qg\xi, qg\eta) \\ &= \beta_q(q\xi, q\eta).\end{aligned}$$

Here the last equality follows by the G-invariance of β. Hence, we find that $\mathbb{I}_q$ does not depend on the point q. △

These definitions allow us to state the following theorem which relates the Riemannian to the sub-Riemannian geodesics.

Theorem 2.14. *[4, Theorem 11.2.5] Let $\pi : Q \to M$ be a principal G-bundle with a Riemannian metric of constant bi-invariant type on Q. Let $\mathcal{H}$ be the Ehresmann connection with connection 1-form $A \in \Omega^1(Q;\mathfrak{g})$. Take $\exp_R$ to be the Riemannian exponential map, so that $\gamma_R(t) = \exp_R(tv)$ is the Riemannian geodesic through $q \in Q$ with tangent vector $v \in T_qQ$. Then any horizontal lift of the projection $\pi \circ \gamma_R$ is a normal sub-Riemannian geodesic on Q given by*

$$\gamma(t) = \exp_R(tv)\exp_G(-tA(v)).$$

Moreover, all normal geodesics on Q can be obtained in this way.

To give some intuition behind this statement, given a Riemannian geodesic γ_R on Q we can find a sub-Riemannian geodesic by first projecting to the base M, $\pi \circ \gamma_R$. Then we apply a horizontal lift to get a a curve on Q again. The way to apply the horizontal lift is to consider the inverse of the flow of the vector field $A(v)$ projected to the Lie group G.

The proof of this theorem can be found in [4, Section 11.2], we include it here for clarity and to provide some extra level of detail in some of the terse parts of the original proof.

Let us write H_R for the Riemannian Hamiltonian, H_{SR} the sub-Riemannian Hamiltonian and H_G for the vertical part of H_R with respective flows Φ_R, Φ_{SR} and Φ_G. Then we can write

$$H_R = H_{SR} + H_G. \tag{2.2}$$

For the proof we need to prove four lemma's.

Lemma 2.15. *Any bi-invariant function Poisson commutes with any right-invariant function [2] on T^*G. Similarly, for left-invariant functions.*

Proof. Let the Lie group G act on itself by right translation $R_g(h) = hg$. The infinitesimal generators for the right translation are given by the left-invariant vector fields $h \mapsto (dL_h)_e \xi$, where $\xi \in \mathfrak{g}$, L_h is the left translation action on G, and $dL_h : \mathfrak{g} \to T_gG$ is the derivative of the left translation action at the identity e. Similarly, the infinitesimal generators of the left actions are given by the right-invariant vector fields.

Given a tangent vector $\xi \in \mathfrak{g}$ we can extend it to a right-invariant vector field ξ^r by setting

$$\xi^r(g) \coloneqq dR_g(\xi).$$

The following computation confirms that this vector field is indeed right-invariant. Consider a point $s \in G$, let an element $g \in G$ act on s to get the point $q = sg \in G$, then we compute

$$\begin{aligned} dR_h\left(\xi^r\right)_q &= dR_h\left(dR_g\right)(\xi)_s \\ &= d\left(R_{gh}\right)(\xi)_s \\ &= \xi^r_{sgh} \\ &= \xi^r_{qh}. \end{aligned}$$

Hence, the vector field ξ^r is right-invariant. A similar construction can be made for the left-invariant vector field ξ^l.

Consider coordinates (q, p) on Q. We define the momentum maps J_R corresponding to L_g as follows

$$\begin{aligned} J_R^\xi(q,p) &\coloneqq p(\xi^l(g)_q) \\ &= p\left(dL_g(\xi_q)\right). \end{aligned}$$

In other words, the momentum map is given by lifting the action on G to T^*G, hence we have that $J_R(q,p) = dL_g^*(p)$. Similarly, we can write $J_L = dR_g^*(p)$.

On T^*G we want to compute the Poisson bracket on the cotangent bundle, this bracket is defined in detail in Appendix A.2. In order to do this, we need to find a general form for left-, right- and bi-invariant functions on T^*G. To find this form we will make use of the fact that the quotient T^*G/G, formed by letting G act on T^*G by the lift of the left-translation, is diffeomorphic to $\mathfrak{g}^*$. Here, we consider without loss of generality the left-action. To see the afformentioned diffeomorphism, note that the trivialization of T^*G by left-translation is given by

$$\lambda : \alpha_g \in T_g^*G \mapsto (g, T_e^*L_g(\alpha_g)) = (g, J_R(\alpha_g)) \in G \times \mathfrak{g}^*$$

such that the cotangent lift of the left-translation on G is given by the G-action on $G \times \mathfrak{g}^*$ defined by

$$g \cdot (h, \mu) = (gh, \mu) \tag{2.3}$$

[2]A function $f : T^*G \to \mathbb{R}$ is right-invariant if $R_g^* f = f$ for R_g the right multiplication by $g \in G$

for $g, h \in G$ and $\mu \in \mathfrak{g}^*$. Hence, T^*G is diffeomorphic to $(G \times \mathfrak{g}^*)/G$, which in turn is diffeomorphic to $\mathfrak{g}^*$ since G does not act on $\mathfrak{g}^*$, as can be seen in (2.3). We note that the momentum map $J_R : T^*G \to T^*G/G$ is the canonical projection map.

Now, we find a general form for left-invariant functions. Consider a basis $\{e_1, \ldots, e_n\}$ on $\mathfrak{g}$ with a corresponding dual basis $\{\varepsilon^1, \ldots, \varepsilon^n\}$. Under the projection J_R this dual basis gives rise to a basis consisting of left-invariant functions $\{\varepsilon_l^1, \ldots, \varepsilon_l^n\}$ where $\varepsilon_l^i = J_R^{e_i}$ for any $i \in \{1, \ldots, n\}$. Therefore, any left-invariant function can be written as $f = f_1\varepsilon_l^1 + \cdots + f_n\varepsilon_l^n$. Similarly, we can find such a basis for right-invariant functions. Taking a right invariant function $h : T^*G \to \mathbb{R}$, we find

$$\{\varepsilon_l^i, h\} = \{J_R^{e_i}, h\} = 0,$$

since $J_R^{e_i}$ is acting from the right on h, which was assumed to be right-invariant. If we consider h to be bi-invariant then it is also right-invariant, hence we find $\{\varepsilon_l^i, h\} = 0$. Now, let us consider any left-invariant function $\varphi : T^*G \to \mathbb{R}$, then $\varphi = \varphi_1\varepsilon_l^1 + \cdots + \varphi_n\varepsilon_l^n$. Then we find

$$\{\varphi, h\} = \sum_{i=1}^{n} \varphi_i \{\varepsilon_l^i, h\} = 0$$

by using the Leibniz identity and the bilinearity for the Poisson bracket. A similar computation shows the same result for right-invariant functions. □

Lemma 2.16. *For a bi-invariant metric β on a Lie group G, the geodesics through the identity coincide with the one-parameter subgroup of G. More explicitly stated: the Riemannian exponential map $\exp_R$ and $\exp_G : \mathfrak{g} \to G$ coincide, i.e.*

$$\exp_G(t\xi) = \exp_R(t\xi)$$

for $\xi \in \mathfrak{g}$.

Proof. Consider the Hamiltonian related to the metric β given by $H : T^*G \to \mathbb{R}$. The Hamiltonian H will be bi-invariant because the metric is bi-invariant. In terms of the basis $\{\varepsilon_r^1, \ldots, \varepsilon_r^n\}$, discussed in the proof of Lemma 2.15, we find

$$H = H_{ij}\varepsilon_r^i\varepsilon_r^j.$$

For any vector field η on G, we consider the momentum function $P_\eta : T^*G \to \mathbb{R}$ defined as

$$P_\eta(q, p) = p(\eta(q)) = J^\eta(q, p)$$

for $(q, p) \in T^*G$. Hence, $\varepsilon_r^i = P_{E_i}$ where $\{E_1, \ldots, E_n\}$ is a basis for left-invariant vector fields corresponding to the basis vectors e_i. In terms of the position variable q and the momentum variable $P_{E_i} = \varepsilon_r^i$ we find Hamilton's equations to be as follows:

$$\dot{q}^i = \sum_{j=1}^{n} H_{ij}\varepsilon_r^j E_j, \tag{2.4}$$

$$\begin{aligned} \dot{P}_{E_i} &= \dot{\varepsilon}_r^i \\ &= \{\varepsilon_r^i, H\}, \\ &= 0. \end{aligned} \tag{2.5}$$

where the last equality follows from using lemma 2.15, the fact that H is bi-invariant and the right-invariance of ε_r. Let us consider a geodesic $\gamma : [0,1] \times G \to G$ on G that starts at the identity element, i.e. $\gamma(0) = I$, and a tangent vector $\xi(q) = \sum_{j=1} H_{ij}\varepsilon_r^j E_j(q)$ for any $q \in G$. Then by Hamilton's equations, we know for a geodesic γ that it can be written as $\gamma'(t) = \gamma(t)\xi$, where ξ is constant by the momentum part (Equation (2.5)) of the Hamilton's equations. Solving this equation, we find:

$$\exp_R(t\xi) = \gamma(t) = \exp_G(t\xi).$$

Which proves the lemma. $\square$

Lemma 2.17. *The Hamiltonians H_R, H_{SR} and H_G Poisson commute pairwise.*

Proof. First, we prove that H_{SR} and H_G Poisson commute. Recall that we have a principal G-bundle $\pi : Q \to M$. Consider an open set $U \subseteq M$. It suffices to prove this statement in a locally trivialized neighbourhood $N = \pi^{-1}(U)$ with local trivialization $\pi_U : \pi^{-1}(U) \to U \times G$. The local trivialization induces the diffeomorphism

$$T^*Q|_N \cong T^*(U \times G) \cong T^*U \times T^*G.$$

Since T^*U and T^*G have canonical Poisson structures, we can give $T^*Q|_N$ the product Poisson bracket structure $\{\cdot,\cdot\}_{T^*Q|_N}$ which can be identified (locally) with

$$\{\cdot,\cdot\}_{T^*U} + \{\cdot,\cdot\}_{T^*G}$$

as discussed in Appendix A.2. Consider coordinates (q,p) on T^*U and (g,μ) on T^*G. Notice that we have T^*G is diffeomorphic to $G \times \mathfrak{g}^*$. The Hamiltonian restricted on T^*G does not depend on (q,p) and because of the G-invariance of the metric β on Q we have that the Hamiltonian is given by β^* which only depends on μ, i.e.

$$H_G(q,p,g,\mu) = \beta^*(\mu).$$

Similarly, noticing that H_{SR} is given by a function $f : T^*Q \to \mathbb{R}$ only depending on q, p and μ since the metric is G-invariant, i.e.

$$H_{SR}(q,p,g,\mu) = f(q,p,\mu).$$

If we now compute the bracket using β^* and f and the afforementioned Poisson bracket we find

$$\begin{aligned}\{H_{SR}, H_G\}_{T^*Q|_N} &= \{f,\beta^*\}_{T^*Q|_N}\\ &= \{f,\beta^*\}_{T^*U} + \{f,\beta^*\}_{T^*G}\\ &= \{f,\beta^*\}_{T^*G}.\end{aligned}$$

Here, the last equality follows because β^* is constant on T^*U, and hence the Poisson bracket will vanish. Take $n = \dim(G)$. Since f is bi-invariant (this follows from the fact it does not depend on G), we can similarly to what was done in the proof of lemma 2.15 write $f = \sum_{i=1}^n f_i(q,p)\mu_r^n$,for some right invariant basis $\{\mu_r^1, \ldots, \mu_r^n\}$ for $\mathfrak{g}^*$. Hence, we compute using the bilinearity of the Poisson bracket,

$$\begin{aligned}\{f, \beta^*\}_{T^*G} &= \{\sum_{i=1}^n f_i(q,p)\mu_r^i, \beta^*\}_{T^*G} \\ &= \sum_{i=1}^n f_i(q,p)\{\mu_r^i, \beta^*\}_{T^*G} \\ &= 0.\end{aligned}$$

Here the last equality follows from Lemma 2.15 and the fact that β^* is a bi-invariant function on T^*G. Therefore, we find that

$$\{H_{SR}, H_G\} = 0. \tag{2.6}$$

For the other two commutation relations, notice that $H_R = H_{SR} + H_G$, hence, we find

$$\begin{aligned}\{H_R, H_G\} &= \{H_{SR} + H_G, H_G\} \\ &= \{H_{SR}, H_G\} + \{H_G, H_G\} \\ &= 0.\end{aligned}$$

Similarly we find $\{H_R, H_{SR}\} = 0$. This proves the lemma for the local trivialization N. Proving this for all local trivializations and gluing together yields the global result. □

Lemma 2.18. *Any integral curve in T^*Q related to the Hamiltonian vector field X_{H_G} projects to a curve of the form $q\exp_R(t\xi)$ for some $\xi \in \mathfrak{g}$.*

Proof. Let us again consider a local trivialization of T^*Q given by $\pi^{-1}(U) \to U \times G$ for an open set $U \subseteq M$. This trivialization induces the diffeomorphism $T^*Q|_N \cong T^*U \times T^*G$. The flow Φ_G of the Hamiltonian vector field X_{H_G}, will only change over time in T^*G. Therefore, let us write

$$\Phi_G(t)(q,p,g,\mu) = (q, p, \Phi_\beta(t)(q,\mu)).$$

Here, let $\Phi_\beta(t) : T^*G \to T^*G$ denote the flow associated to β^*. Notice that if we project $\Phi_\beta(t)(q,\mu)$, we get by lemma 2.16 that

$$\pi\left(\Phi_\beta(t)(g,\mu)\right) = \exp_G(t\xi) = \exp_R(t\xi)$$

for some $\xi \in \mathfrak{g}$. So we obtain

$$\pi\left(\Phi_G(t)(q,p,g,\mu)\right) = q\exp_R(t\xi)$$

as desired. □

Using the lemma's above, the main theorem can be proven.

Proof of theorem 2.14. Consider the Riemannian exponential map $\exp_R : T_qQ \to Q$ for some $q \in Q$ as in definition 1.33. Then a Riemannian geodesic can be written as $\gamma_R(t) = \exp_R(tv)$

for $v \in T_qQ$. Using equation (2.2) we find that $H_{SR} = H_R - H_G$. Using 2.17 we know that the flows related to the Hamiltonian vector fields for the Hamiltonians H_{SR}, H_G and H_R commute. Hence, we find

$$\Phi_{SR}(t) = \Phi_R(t) \circ \Phi_G(-t).$$

Consider the projection $\mathrm{pr}_Q : T^*Q \to Q$. Then we find the normal geodesic $\gamma_{SR}(t)$ can be written as $\mathrm{pr_Q}\left(\Phi_{SR}(t)\right)$. In other words we can compute as follows,

$$\begin{aligned}\gamma_{SR}(t) &= \mathrm{pr}_Q\left(\Phi_{SR}(t)\right)\\ &= \mathrm{pr}_Q\left(\Phi_R(t) \circ \Phi_G(-t)\right)\\ &= \gamma_R(t) \exp_G(-t\xi),\end{aligned}$$

where the last equality folllows from lemma 2.18. Since the sub-Riemannian geodesics γ_{SR} must be horizontal, we find that $\xi = A(v)$ for some the connection 1-form $A \in \Omega^1(Q, \mathfrak{g})$ and some $v \in \mathfrak{g}$. Expressing the Riemannian geodesic in exponetial form this shows the formula

$$\gamma_{SR}(t) = \exp_R(tv) \exp_G(-tA(v)).$$

Since any normal sub-Riemannian geodesic can be expressed as the projection of the sub-Riemannian Hamiltonian flow (Theorem 1.20), we find that all normal geodesics on Q can be obtained as above. □

2.3 Examples of sub-Riemannian principal bundles

Using the techniques from the previous section we give a few larger classes of examples.

Example 2.19 (Lie groups)**.** Consider a compact Lie group G with a closed subgroup $K \subset G$. Let K act on G from the right, that is $R_g : K \to G$ such that $R_g(k) = gk$. This action yields a principal K-bundle $G \to G/K$. Define a K-invariant Riemannian metric β on G of constant bi-invariant type. Consider the Lie algebra $\mathfrak{k}$ of K. Let $(R_g)_*(\mathfrak{k})$ be the vertical space $\mathcal{V}_g$ at the point $g \in G$. The horizontal space is defined as the orthognal with respect to β given by

$$\mathcal{H}_g = \mathcal{V}_g^\perp = \{v \in T_gG : \beta(v, k) = 0 \text{ for all } k \in \mathcal{V}_g\}.$$

Since we have a Riemannian metric of bi-invariant type, Theorem 2.14 yields a sub-Riemannian structure on G with distribution $\mathcal{H}_g$. In order to find a closed form for the normal sub-Riemannian structure we need to find the connection 1-form. We want to find a projection $A : T_gG \to \mathcal{V}_g$ such that $A_g(\sigma_q(\xi)) = \xi$ for all $\xi \in T_gG$ and $A_g(\xi) = 0$ for $\xi \in \mathcal{H}_g$. We note that any $\xi \in T_gG$ can be written as a vector $\xi_v + \xi_h$, with $\xi_h \in \mathcal{H}_g$ and $\xi_v \in \mathcal{V}_g$. Define the map A to be the orthogonal projection: $A_g(\xi) = \xi_v$. Then for $\xi \in \mathfrak{g}$ we have $A(\xi) = 0$ since $\xi = \xi_h$. Using this result, for any $\xi = \xi_v \in \mathcal{V}_g$:

$$A_g(\xi) = A_g(\xi_v + \xi_h) = A_g(\xi_v) + A_g(\xi_h) = A_g(\xi_v).$$

Because $\sigma_g(\xi) \in T_gG$, we have $\sigma_g(\xi) = \sigma_g(\xi)_v + \sigma_g(\xi)_h$, hence $A(\sigma_g(\xi)) = \sigma_g(\xi)_v \in \mathcal{V}_q$. Using the isomorphism from Lemma 2.9, we find that $\sigma_g(\xi)_v = \xi_v = \xi$ as desired. So A is indeed a connection 1-form. Using this form, and the exponential maps $\exp : \mathfrak{g} \to G$ and $\exp_K : \mathfrak{k} \to K$ we find the normal sub-Riemannian geodesics through $g \in G$ are given by

$$\gamma(t) = \exp_G(t\xi)\exp_K(tA(\xi)). \tag{2.7}$$

for any $\xi \in \mathcal{H}_g$. △

Example 2.20 (Homogeneous spaces)**.** Consider the bundle $\pi : G \to G/K$ from example 2.19. If we consider a closed subgroup $H \subset G$ that commutes with K, i.e. $HK = KH$, then we can quotient by H on both sides of π. We obtain a new Principal K-bundle $G/H \to G/(H \times K)$. Notice that the projection $G \to G/H$ is a Riemannian submersion. So the metric β on G gives rise to a metric $\widetilde{\beta}$ on G/H. Since we still consider the K-action this metric will be of constant bi-invariant type and the connection 1-form will not change. However, we do need to assume that ξ is orthogonal to the Lie algebra of H: $\mathfrak{h}$ in order for the flow to be defined on G/H instead of G. The normal sub-Riemannian geodesics on G/H will be of the form (2.7). △

Both the Lie group and the homogeneous space cases give rise to many examples which can be found in [4, Section 11.3]. We discuss two additional exampls, first the falling cat problem.

Example 2.21 (Falling cat problem)**.** Let us consider a falling cat. As is commonly known a cat will always land on its feet independent of its starting position and orientation. From a physics point of view this is strange however, since a cat cannot simply rotate itself in midair as this this would violate the conservation of angular momentum. Instead, it follows from biological research that a cat can rotate its front-half and its back-half separately. Hence, first rotating its front-half and then its back-half rotates the cat while adhering to the conservation of angular momentum. This process is illustrated in figure 2.1.

To find the most efficient way for the cat to fall and 'change its shape' we can find the geodesics in the configuration space of this model. Mathematically, the very simplified model will be as follows. The configuration space is given by $Q = \mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathbb{R}^3$. The two copies of $\mathrm{SO}(3)$ give the orientation of the front- and back-halves of the cat and $\mathbb{R}^3$ the positon of the cat. The group of rigid-body rotations and translation is defined by $G = \mathrm{SE}(3) \cong \mathrm{SO}(3) \rtimes \mathbb{R}^3$, we let this act on Q. We note that $\mathrm{SE}(3)$ acts freely and properly on Q hence we get a principal $\mathrm{SE}(3)$-bundle given by

$$\mathrm{SE}(3) \to Q \to Q/\,\mathrm{SE}(3).$$

Note that $S = Q/\,\mathrm{SE}(3)$ is isomorphic to $\mathrm{SO}(3)$. The space S will be called the shape space of the cat, and it describes the one half of the cat relative to the other half of the cat. On Q let us define a bi-invariant (pseudo-)Riemannian metric β, for the falling cat problem we can for example choose to minimize the length of paths with respect to the metric kinetic energy. Using theorem 2.14 we know the form of the normal sub-Riemannian geodesics on Q if the metric is of constant bi-invariant type. We note that $\mathrm{SE}(3)$ is a matrix Lie group, with matrix Lie algebra $\mathfrak{se}(3)$. Hence, using Example 2.13 the infinitesimal generator is given given by $\sigma_q(\xi)$ for $\xi \in \mathfrak{se}(3)$

The moment of inertia tensor for arbitrary $\xi, \eta \in \mathfrak{se}(3)$ is given by

$$\begin{aligned}\mathbb{I}_q(\xi,\eta) &= \beta(\sigma_q\xi, \sigma_q\eta)\\ &= \beta(\xi q, \eta q)\end{aligned}$$

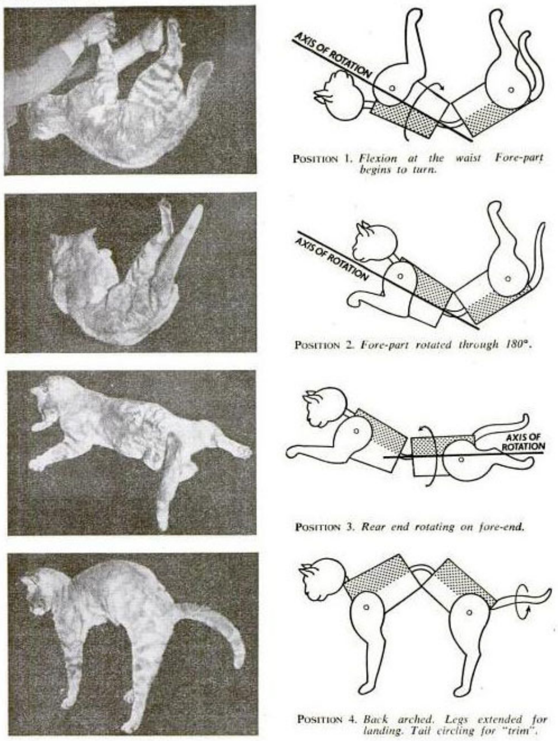


Figure 2.1: Falling cat [15]

Since ξq and ηq are both in $T_q\,\mathrm{SE}(3)$, they are invariant under rotations and translation. Hence, changing the point q does not change $\mathbb{I}(\xi, \eta)$. Therefore, we find that the metric β must be of constant bi-invariant type. Hence the normal sub-Riemannian geodesics through $q \in Q$ are of the form

$$\gamma(t) = \exp(t\xi)\exp(-tA(\xi))$$

where $\xi \in T_qQ$ and A is the connection 1-form related to the horizontal distribution one would choose. This would, dependent on the chosen horizontal distribution (which would need to incorporate the fact that the angular momentum needs to be preserved), indeed give the observed rotating motion. A more detailed explanation and result can be found in [16]. $\triangle$

2.3.1 Sub-Riemannian Hopf action

Consider the Hopf action

$$S^1 \to S^3 \xrightarrow{\pi} S^2$$

Notice that $S^3 := \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$. The map π is given by

$$\pi(z_0, z_1) = \left(2z_0\overline{z}_1, |z_0|^2 - |z_1|^2\right).$$

One can check that $\pi(z_0, z_1) \in S^2 := \{(z, x) \in \mathbb{C}\times\mathbb{R} : z\overline{z}+x^2 = 1\}$ by a computation. Using that the Lie group $U(1) = \{z \in \mathbb{C} : z\overline{z} = 1\}$ is diffeomorphic to S^1 we can consider the Hopf action as a principal $U(1)$-bundle. Using the results in section 2.2 we will find the sub-Riemannian geodesics on S^3 from the Hopf action, by checking that the Riemannian structure on S^3 is of constant bi-invariant type. This can be done using a similar construction as for Lie groups in

Example 2.19. Then a horizontal distribution can be found using the Riemannian metric on S^3. However, for this example we want to find the distribution using a slightly other way. The result is borrowed from [17].

As we have seen in section 2.2 in order to find a horizontal distribution $\mathcal{H}$ we first need to find the vertical distribution and consider its orthogonal complement. In this case, we consider the vertical space to be the tangent space of the action that is obtained by the right multiplication of an element in S^1 on S^3. To find this vertical space note that the space S^3 can be realized as the set of unit quaternions, i.e. $S^3 = \{q \in \mathbb{H} : \|q\| = 1\}$ where $\|q\| = q\bar{q}$. The multiplicative structure of $\mathbb{H}$ yields a right S^1-action on S^3 defined by $R_p(x) := x \cdot p$. In coordinates the right action is defined by

$$\begin{aligned} R_p(x) = {} & (x_0p_0 - x_1p_1 - x_2p_2 - x_3p_3) + (x_1p_0 + x_0p_1 - x_3p_2 + x_2p_3)\, i + \\ & + (x_2p_0 + x_3p_1 + x_0p_2 - x_1p_3)\, j + (x_3p_0 - x_2p_1 + x_1p_2 + x_0p_3)\, k \end{aligned}$$

with $x = x_0 + x_1 i + x_2 j + x_3 k, p = p_0 + p_1 i + p_2 j + p_3 k \in S^3$. The tangent map of $R_p(x)$ is given by $(dR_p(x))^T$, which as a matrix is defined as

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ -p_1 & p_0 & -p_3 & p_2 \\ -p_2 & p_3 & p_0 & -p_1 \\ -p_3 & -p_2 & p_1 & p_0 \end{pmatrix}.$$

If we calculate the action on the standard basis of $\mathbb{R}^4$ we obtain the following four vector fields

$$\begin{aligned} N(p) &= p_0\partial_{p_0} + p_1\partial_{p_1} + p_2\partial_{p_2} + p_3\partial_{p_3} \\ V(p) &= -p_1\partial_{p_0} + p_0\partial_{p_1} - p_3\partial_{p_2} + p_2\partial_{p_3} \\ X(p) &= -p_2\partial_{p_0} + p_3\partial_{p_1} + p_0\partial_{p_2} - p_1\partial_{p_3} \\ Y(p) &= -p_3\partial_{p_0} - p_2\partial_{p_1} + p_1\partial_{p_2} + p_0\partial_{p_3} \end{aligned}$$

Notice $N(p)$ is the normal vector to S^3 at $p \in S^3$ with respect to the standard inner product on $T\mathbb{R}^4$. The vector $V(p)$ will be our vertical space $\mathcal{V}_p$ in the language of 2.2.

Lemma 2.22. *The set $\{X(p), Y(p), V(p)\}$ form an orthonormal basis for T_pS^3.*

Proof. Note that the vectors $X(p), Y(p)$ and $V(p)$ lie in T_pS^3 because they are orthogonal to the normal vector $N(p)$ at p. Since $\dim T_pS^3 = 3$ and we have three linearly independent vectors we find that $\{X(p), Y(p), V(p)\}$ form an basis for T_pS^3. We note that by computing their inner products we can see that the vector $\{X(p), Y(p), V(p)\}$ are orthogonal and all have norm 1. Therefore, $\{X(p), Y(p), V(p)\}$ form an orthonormal basis for T_pS^3. □

We notice that we can split $T_pS^3 = V(q) \oplus V(q)^\perp = V(q) \oplus (\mathrm{span}\{X(p), Y(p)\})$. So let us define the horizontal distribution $\mathcal{H} = \mathrm{span}\{X, Y\}$ on S^3. The distribution $\mathcal{H}$ is bracket generating since

$$[X, Y] = 2V.$$

Notice that equivalently we can also get a distribution if we take X or Y the vertical space, and we consider the orthogonal complement of these.

We can also define $\mathcal{H} = \ker(\omega)$ where ω is the 1-form corresponding to the vector field V. We find

$$\omega = -p_1 dp_0 + p_0 dp_1 - p_3 dp_2 + p_2 dp_3.$$

Lemma 2.23. *$\mathcal{H}$ is a contact distribution.*

Proof. We want to show that $\omega \wedge d\omega \neq 0$. Note

$$d\omega = 2dp_0 \wedge dp_1 + 2dp_2 \wedge dp_3.$$

Then we compute

$$\omega \wedge d\omega = -2p_3 dp_0 \wedge dp_1 \wedge dp_2 + 2p_2 dp_0 \wedge dp_1 \wedge dp_3 - 2p_1 dp_2 \wedge dp_3 \wedge dp_0 + 2p_0 dp_2 \wedge dp_3 \wedge dp_1 \neq 0.$$

Therefore, ω is a contact form and $\mathcal{H}$ is a contact distribution. □

Since $\mathcal{H}$ is a contact distribution, theorem 1.30 gives us that there are no singular sub-Riemannian geodesics on $\mathcal{H}$. We will now use the construction for sub-Riemannian structures on principal bundles to give explicit sub-Riemannian geodesics on S^3. First, let us show that the metric on S^3 is of constant bi-invariant type. The Lie algebra of $U(1)$ is given by $\mathfrak{u}(1) = \{z \in \mathbb{C} : z = -\overline{z}\}$. Since $z = -\overline{z}$ implies that z has no real part we can write $\xi = i\alpha$ for $\xi \in \mathfrak{u}(1)$ and $\alpha \in \mathbb{R}$. Consider $q \in S^3$ then the infinitesimal generator for the action $\sigma : \mathfrak{u}(1) \to T_q S^3$ is given by

$$\sigma_q(\xi) = iq\alpha,$$

conform the computation inExample 2.13.

The moment of inertia tensor is then given by

$$\begin{aligned} \mathbb{I}_q(\xi, \eta) &= \langle \sigma_q(\xi), \sigma_q(\eta) \rangle \\ &= \langle q\xi, q\eta \rangle \\ &= \langle iq\alpha, iq\beta \rangle \\ &= -\alpha\beta \langle q, q \rangle \\ &= -\alpha\beta \end{aligned}$$

for $\xi = i\alpha$ and $\eta = i\beta$ in $\mathfrak{u}(1)$, $\alpha, \beta \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ the Riemannian metric on S^3 induced by the standard inner product on $\mathbb{R}^4$. The last equality follows because $q \in S^3$. Hence, $\mathbb{I}_q$ does not depend on q. Therefore, the Riemannian metric on S^3 is of constant bi-invariant type. This means that we have an explicit form of the sub-Riemannian geodesics in terms of the Riemannian geodesics of S^3.

To find the sub-Riemannian geodesics on S^3 we need the connection 1-form $A \in \Omega^1(S^3, \mathfrak{u}(1))$. The form A is given by $\sigma^{-1}|_{V_q} : V_q \to \mathfrak{u}(1)$ and defined by $A(v) = i\langle v, V(q) \rangle$ for $q \in S^3$ and $v \in T_q S^3$ (i.e. the projection of v onto the space $V(q)$).

We recall that the Riemannian geodesics on S^3 starting at p are given by the great circles $\gamma_R(t) = (z_0(t), z_1(t), z_2(t))$ with $\gamma_R(0) = p$ and $\gamma_R'(0) = v$. Using theorem 2.14 the fact that the metric on S^3 is of constant bi-invariant type yields that the sub-Riemannian geodesics are given by

$$\gamma(t) = \left(z_0 e^{-it\langle v, V(p)\rangle}, z_1 e^{-it\langle v, V(p)\rangle} z_2 e^{-it\langle v, V(p)\rangle}\right)$$

where $\gamma(0) = p$ and $\gamma'(0) = v$.

Chapter 3

Orbifolds

In this chapter we introduce the second important topic of this thesis: orbifolds. Orbifolds, then called V-manifolds, were first described in 1956 by Satake [18]. In 1980, Thurston gave orbifolds their name in [19]. As an intuition one can think about orbifolds as manifolds with isolated singularities. They are the quotient manifolds of discrete groups acting properly on each chart of the manifold. In this section, we will first describe orbifolds structures in general (section 3.1). Then in section 3.2 we will present 'developable' orbifolds, which roughly speaking are manifolds that are quotiented by a discrete group. Finally, in section 3.3, we describe orbifold tangent spaces, differential forms and Riemannian metrics. In this chapter we use some theory about group actions, for convenience this is summarized in appendix B. The material in this section was mostly found in [20], [19] and [21].

3.1 Orbifold structure

Let Q be a paracompact [1] Hausdorff topological space with an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ which is closed under finite intersections. Let us fix a positive integer n.

Definition 3.1. *An* ***n-dimensional smooth orbifold chart*** *associated to an open set* $U_i \in \mathcal{U}$ *is given by a triple* $\left(\widehat{U}_i, \Gamma_i, \varphi_i\right)$ *where*

- $\widehat{U}_i$ *is a connected open subset of* $\mathbb{R}^n$*;*
- Γ_i *is a finite group acting effectively and smoothly by diffeomorphism on* $\widehat{U}_i$*;*
- $\varphi_i : \widehat{U}_i \to U_i$ *is a continuous surjective map that induces a homeomorphism fom* $\widehat{U}_i/\Gamma_i$ *onto* U_i*.*

Definition 3.2. *If* Γ_i *acts effectively on* $\widehat{U}_i$*, then the orbifold chart is said to be* ***reduced*** *or* ***effective****.*

In general not every orbifold chart is reduced, however for this thesis we assume that our charts are always effective.

[1] Paracompact means that every open cover has a locally finite refinement, i.e. every point has a neighbourhood that intersects only finitely many sets in the cover.

Similar to the manifold case, we want to give a compatibility condition on the orbifold charts to create atlasses. For orbifolds, this is done in a similar way to the manifold case, but we need to keep track of the group actions. Consider two orbifold charts $\left(\widehat{U}_i, \Gamma_i, \varphi_i\right)$ and $\left(\widehat{U}_j, \Gamma_j, \varphi_j\right)$ corresponding to two subsets U_i and U_j such that $U_i \subseteq U_j$.

Definition 3.3. *An* ***embedding*** *between orbifold charts is a pair*

$$\left(\widehat{\varphi}_{ij}, \widehat{\lambda}_{ij}\right) : \left(\widehat{U}_i, \Gamma_i, \varphi_i\right) \to \left(\widehat{U}_j, \Gamma_j, \varphi_j\right)$$

consisting of a smooth embedding $\widehat{\varphi}_{ij} : \widehat{U}_i \hookrightarrow \widehat{U}_j$ and an injective group homomorphism $\lambda_{ij} : \Gamma_i \to \Gamma_j$ such that $\widehat{\varphi}_{ij}$ is λ_{ij}-equivariant [2].

The definition of an embedding allows us to give the following definition of an atlas.

Definition 3.4. *An* ***orbifold atlas*** *$\mathcal{A}$ on Q associated to $\mathcal{U}$ is a collection of orbifold charts $\left\{\left(\widehat{U}_i, \Gamma_i, \varphi_i\right)\right\}_{i\in I}$ which satisfy the following local compatibility condition. Given two charts $\left(\widehat{U}_i, \Gamma_i, \varphi_i\right)$ and $\left(\widehat{U}_j, \Gamma_j, \varphi_j\right)$ there exists an open set $U_k \subseteq U_i \cap U_j$ with an associated orbifold chart $\left(\widehat{U}_k, \Gamma_k, \varphi_k\right)$ that embeds in $\left(\widehat{U}_i, \Gamma_i, \varphi_i\right)$ and $\left(\widehat{U}_j, \Gamma_j, \varphi_j\right)$.*

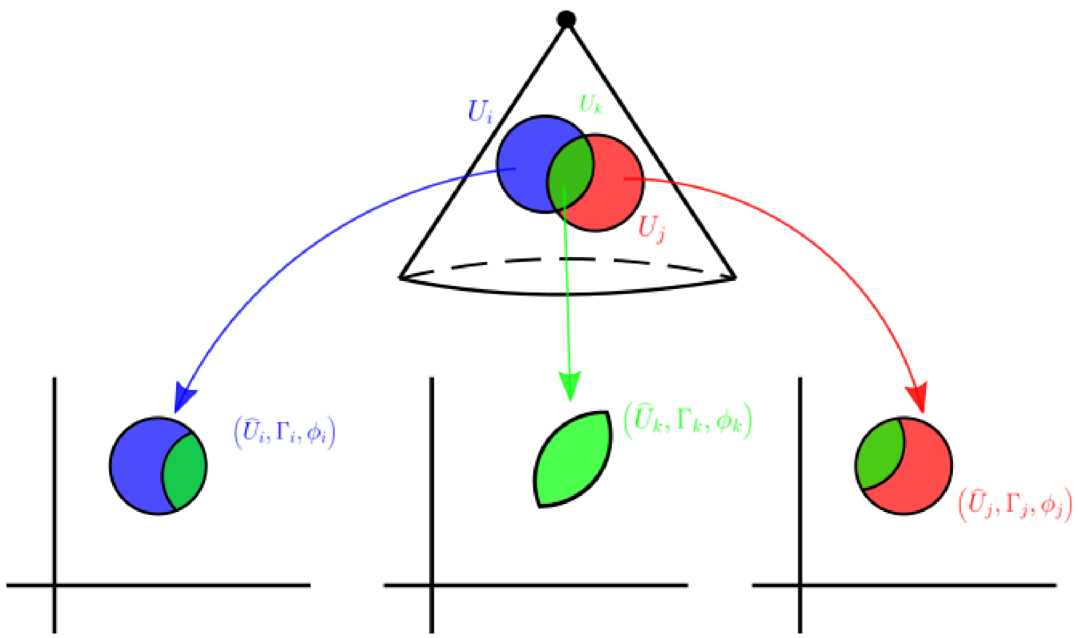


Figure 3.1: Pictorial explanation of an orbifold atlas.

Definition 3.5. *If $\mathcal{U}'$ is another open cover of Q that refines $\mathcal{U}$, we say that the associated atlas $\mathcal{A}'$* ***refines*** *$\mathcal{A}$ if every orbifold chart in $\mathcal{A}$ can be embedded in some orbifold chart of $\mathcal{A}$. We call two atlasses* ***compatible*** *if they have a common refinement.*

The compatibility gives rise to an equivalence relation on orbifold atlasses. Similar to the manifold case, every orbifold atlas is contained in in a unique maximal orbifold atlas. We can

[2]The statement $\widehat{\varphi}_{ij}$ is λ_{ij}-equivariant means that for $\gamma \in \Gamma_i$ we have $\widehat{\varphi}_{ij}(\gamma x) = \lambda_{ij}(\gamma)\widehat{\varphi}_{ij}(x)$ for $x \in \widehat{U}_i$.

then redefine the compatibility condition by saying two orbifold atlasses are compatible if and only if they are contained in the same maximal orbifold atlas. As for manifolds, we mostly work with the maximal orbifold atlas.

Definition 3.6. *A smooth n-dimensional* ***orbifold*** $\mathcal{Q}$ *is a paracompact Hausdorff space* Q *with an equivalence class of orbifold atlasses.*

Remark 3.7. It is possible to have more than one (non-equivalent) orbifold structures on the same topological space Q. Orbifolds are not smooth manifold, and also not in general topological manifolds. Except for dimension two the underlying topological space Q need not have the structure of a topological manifold [20, Example 2.38].

Let us state a few technical results that will be used later.

Theorem 3.8. *[21] Given two embeddings of orbifold charts* $\lambda, \mu : \left(\widehat{U}, \Gamma, \varphi\right) \hookrightarrow \left(\widehat{V}, H, \psi\right)$, *there exists a unique* $h \in H$ *such that* $\mu = h \cdot \lambda$.

Proof. For a proof of this theorem see the appendix of [22]. □

This implies the following corollaries. Both proofs can also be found in the Appendix of [22].

Corollary 3.9. *An embedding of orbifold charts* $\lambda : \left(\widehat{U}, \Gamma, \varphi\right) \hookrightarrow \left(\widehat{V}, H, \psi\right)$ *induces an injective group homomorphism* $\widetilde{\lambda} : \Gamma \hookrightarrow H$.

Corollary 3.10. *Consider an embedding* $\lambda : \left(\widehat{U}, \Gamma, \varphi\right) \hookrightarrow \left(\widehat{V}, H, \psi\right)$. *If there exists* $h \in H$ *such that* $\lambda(\widehat{U}) \cap h \circ \lambda(\widehat{U}) \neq \emptyset$, *then* h *lies in the image of* λ. *In other words,* $\lambda(\widehat{U}) = h \circ \lambda(\widehat{U})$.

Orbifolds could be thought of as manifolds with singularities. But how do we obtain these singularities? In order to answer this question we need to talk about isotropy groups of an orbifold.

Definition 3.11. *For an orbifold* $\mathcal{Q} = (Q, \mathcal{U})$, *pick* $x \in Q$. *If* $\left(\widehat{U}, \Gamma, \varphi\right)$ *is any orbifold chart around* x *and* $y \in \widehat{U}$ *such that* $\varphi(y) = x$, *the* ***local group/isotropy group*** *at* x *is defined as*

$$\Gamma_x \coloneqq \{\gamma \in \Gamma : \gamma \cdot y = y\}.$$

This means the isotropy group of a point on an orbifold, is given by the isotropy group on the chart in $\mathbb{R}^n$. We need to check the isotropy group is indeed well-defined.

Lemma 3.12. *The isotropy group is well-defined. In other words, it does not depend on the chosen chart and it is independent of the chosen lift.*

Proof. First we show that the istoropy group does not depend on the chosen chart. Suppose we use a different chart $\left(\widehat{V}, H, \psi\right)$ around x. By the compatibility of charts we know there must exist a chart $\left(\widehat{W}, K, \mu\right)$ together with embeddings $\lambda_1 : \left(\widehat{W}, K, \mu\right) \hookrightarrow \left(\widehat{U}, \Gamma, \varphi\right)$ and $\lambda_2 : \left(\widehat{V}, H, \psi\right)$ such that the inclusions $\widehat{W} \hookrightarrow \widehat{U}$ and $\widehat{W} \hookrightarrow \widehat{V}$ are equivariant. Using lemma 3.9 we get injective group homomorphisms $\widetilde{\lambda}_1 : K_y \hookrightarrow \Gamma_y$ and $\widetilde{\lambda}_2 : K_y \hookrightarrow H_y$. To see these homomorphisms are also surjective, we note that by the definition of the isotropy group there exists $\gamma \in \Gamma_y$ such that $\lambda_1(\widehat{W}) \cap \gamma \circ \lambda_1(\widehat{W}) \neq \emptyset$. By 3.10 we find that γ must lie in the image of λ, hence $\widetilde{\lambda}_1$ is a surjection. Therefore, we find K_y is isomorphic to Γ_y. Similarly, one can show that $\widetilde{\lambda}_2$ is an isomorphism of groups, so K_y is isomorphic to H_y. Consequently, H_y is isomorphic to Γ_y, so the isotropy group does not depend on the chosen chart.

Secondly, we need to show the isotropy group for a point $x \in Q$ is independent of the chosen lift $y \in \widehat{U}$ for $\left(\widehat{U}, \Gamma, \varphi\right)$ a local chart around x. Suppose we pick $y' \in \widehat{U}$ such that $x = \varphi(y')$. Since y and y' both lie in the same chart there exists $\gamma \in \Gamma$ such that $y' = \gamma \cdot y$. Therefore, $\Gamma_y = \Gamma_{\gamma \cdot y'} = \gamma \cdot \Gamma_{y'}$. Hence, Γ_y is conjugate to $\Gamma_{y'}$. So the definition of the isotropy group is up to conjugation.

Together, the above show that the isotropy group is well-defined. □

Definition 3.13. *A point $x \in Q$ is called **singular** if its isotropy group is non-trivial. Non-singular points will be called **regular**. Let us denote the set of singular points in Q by Σ and the set of regular points by Q_{reg}.*

In the next section we will see examples of these singularities. An orbifold $\mathcal{Q}$ with empty singular set is a smooth manifold. For some group actions we find that an orbifold becomes a smooth manifold. If for example the group acts freely or trivially then the isotropy group is trivial, and hence the orbifold chart is a smooth manifold chart.

Proposition 3.14. *Let $\mathcal{Q} = (Q, \mathcal{U})$ be a smooth orbifold. If for each orbifold chart $(widehatU_i, \Gamma_i, \varphi_i)$ the groups Γ_i act trivially or freely on U_i, then the orbifold Q is a smooth manifold.*

The singular points on an orbifold are isolated. In other words, there exists no region where we cannot contain a singular set in its own open set.

Proposition 3.15. *[20, Proposition 2.8] The singular set of an orbifold is closed and nowhere dense.*

From the previous proposition we note that $Q_{\text{reg}} = Q \setminus \Sigma$ must be a dense open set of Q, and therefore can be endowed with a manifold structure.

3.2 Developable orbifolds and examples

In general orbifolds are spaces that locally look like $\mathbb{R}^n/\Gamma$, however they can also appear as global quotient of a manifold. In this section we prove that if we take a manifold and let a discrete group act on it properly, then the quotient space will have an orbifold structure. After we have seen the proof, we give a few examples of orbifolds.

Theorem 3.16. *Let Γ be a discrete group acting properly on a manifold M. The quotient space M/Γ has a natural orbifold structure.*

Proof. Define the space $Q \coloneqq M/\Gamma$. Note that because the action is proper, by proposition B.13 Q is a Hausdorff space. Let us construct an orbifold atlas for Q. Consider the quotient map

$q : M \to Q$. Take a class $[x] \in Q$ and a lift $\widehat{x} \in M$ such that $q(\widehat{x}) = [x]$. Moreover, consider the isotropy group $\Gamma_{\widehat{x}} := \{\gamma \in \Gamma : \gamma \cdot \widehat{x} = \widehat{x}\}$. Using Proposition B.13(iii) each $\widehat{x} \in M$ admits an open neighbourhood $\widehat{U}_{\widehat{x}}$ which is invariant under $\Gamma_{\widehat{x}}$ and such that

$$\{\gamma \in \Gamma : \left(\gamma \cdot \widehat{U}_{\widehat{x}}\right) \cap \widehat{U}_{\widehat{x}} \neq \emptyset\} = \Gamma_{\widehat{x}}.$$

In other words the open set $\widehat{U}_{\widehat{x}}$ is disjoint from all translations by elements of Γ that are not in $\Gamma_{\widehat{x}}$. The first open sets in the open cover of Q are defined as $U_x = q\left(\widehat{U}_{\widehat{x}}\right)$. The quotient map $q|_{\widehat{U}_{\widehat{x}}} : \widehat{U}_{\widehat{x}} \to U_x$ is a homeomorphism.

Since M is a smooth manifold, we can pick an maximal atlas $\widehat{\mathcal{U}}$ for M. Using the method above, we can associate an open set U to each $\widehat{U} \in \widehat{\mathcal{U}}$. Together, these open sets U form an open cover $\mathcal{U}$ for Q. The orbifold charts in this atlas $\mathcal{U}$ will be of the form

$$\left(\widehat{U}_{\widehat{x}}, \Gamma_{\widehat{x}}, q|_{\widehat{U}_{\widehat{x}}}\right).$$

To make this an orbifold covering we need that $\mathcal{U}$ is closed under intersections. To do this let us add all finite intersections of U_x's to $\mathcal{U}$ and show that these intersections indeed form valid orbifold charts. Take arbitrary points $x_1, \dots, x_k \in Q$, where $k \in \mathbb{Z}_{>0}$, such that the finite intersection of associated orbifold charts is nonempty, i.e.

$$I = U_{x_1} \cap \dots \cap U_{x_k} \neq \emptyset.$$

We need to show that I is indeed a valid orbifold chart. Let $\widehat{U}_{\widehat{x}_i}$ be the open set in M associated to U_{x_i}, for $i = 1, \dots, k$. Consider the set $q^{-1}\left(U_{x_1} \cap \dots \cap U_{x_k}\right) \subseteq M$. Since Γ is a discrete group, it acts on $q^{-1}\left(U_{x_1} \cap \dots \cap U_{x_k}\right)$ by permuting its connected components, so there must be some $\gamma_1, \dots, \gamma_k \in \Gamma$ such that

$$\widehat{I} := \gamma_1 \cdot \widehat{U}_{\widehat{x_1}} \cap \dots \gamma_k \cdot \widehat{U}_{\widehat{x_k}} \neq \emptyset.$$

So we can take $\widehat{I}$ as the chart for I. The group acting on it is given by:

$$\gamma_1 \Gamma_{\widehat{x_1}} \gamma_1^{-1} \cap \dots \cap \gamma_k \Gamma_{\widehat{x_k}} \gamma_k^{-1}.$$

The chart map is given by $q|_{\widehat{I}}$. Together, the above make I into a valid orbifold chart.

Next, we need to show $\mathcal{U}$ is indeed an atlas, or in other words that the charts are indeed compatible. Take two element U and V in $\mathcal{U}$ such that $V \subseteq U$. We want to show these charts embed nicely into each other. Pick a point $x \in V$ and let $\widehat{x} \in q^{-1}(x)$. Using the construction from above, we construct an open set $\widehat{U}_{\widehat{x}}$ associated to U and the isotropy group $\Gamma_{\widehat{x}}$. Moreover, construct a set $\widehat{V}_{\widehat{x}}$ such that it contains $\widehat{x}$. To show the embedding, it suffices to show $\widehat{V}_{\widehat{x}} \subseteq \widehat{U}_{\widehat{x}}$, as in both charts the chart maps and the isotropy groups are the same at $\widehat{x}$. To show the inclusion, assume by contradiction there exists $\widehat{p} \in \widehat{V}_{\widehat{x}} \setminus \widehat{U}_{\widehat{x}}$. Notice that $q(\widehat{p}) = p \in V \subset U$. By construction, we know that there exists an element $\gamma \in \Gamma_{\widehat{x}}$ such that $\gamma \cdot \widehat{p} \in \widehat{U} \cap \widehat{V}$. Notice that both $\widehat{U}_{\widehat{x}}$ and $\widehat{V}_{\widehat{x}}$ are invariant under the action of $\Gamma_{\widehat{x}}$. Hence, the intersection $\widehat{U}_{\widehat{x}} \cap \widehat{V}_{\widehat{x}}$ is invariant under the $\Gamma_{\widehat{x}}$-action. Therefore, $\gamma \cdot \widehat{p} = \widehat{p}$ so $\widehat{p} \in \widehat{U}_{\widehat{x}} \cap \widehat{V}_{\widehat{x}}$. Which is a contradiction, hence $\widehat{V}_{\widehat{x}} \subseteq \widehat{U}_{\widehat{x}}$. So the chart V embeds into the chart U. Since U and V where chosen arbitrarily we find that each two charts in $\mathcal{U}$ are compatible, proving that $\mathcal{U}$ forms an orbifold atlas for Q. □

We can now summarize the content of Theorem 3.16 in the following definition.

Definition 3.17. *An orbifold is called* ***developable*** *if it is the quotient of a proper action of a discrete group* Γ *on a manifold* M.

To see what is happening for developable orbifolds, let us work out a simple example in detail.

Example 3.18 (Reflection orbifold)**.** Consider the action $\alpha : \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(n, (x, y)) \mapsto ((-1)^n x, y)$. The space $Q = \mathbb{R}^2/(\mathbb{Z}/2\mathbb{Z})$ found by this action is the half plane $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. Let us make an inconvenient choice of open cover on Q (otherwise the example would be trivial) given by

$$U_1 \coloneqq [0, 1) \times \mathbb{R} \text{ and } U_2 \coloneqq (0, \infty) \times \mathbb{R}.$$

These sets are open in $\mathbb{R}^2$ under the subspace topology. Let us construct orbifold charts associated to this open cover.

$$\widehat{U}_1 \coloneqq \{(x, y) \in \mathbb{R}^2 : x < 1\} \quad \text{and} \quad \widehat{U}_2 \coloneqq \{(x, y) \in \mathbb{R}^2 : x > 0\}$$

.

Consider the map $\widehat{\varphi}_1 : \widehat{U}_1 \to \widehat{U}_1$ defined by

$$\varphi_1(x, y) = \begin{cases} (x, y) & \text{if } x \geq 0 \\ (0, y) & \text{if } x \leq 0 \end{cases}.$$

This is a continuous and surjective map. We see that $\widehat{U}_1/(\mathbb{Z}/2\mathbb{Z}) = [0, 1) \times \mathbb{R}$ which is homeomorphic (identical) to U_1. Similarly, we can define $\varphi_2 : \widehat{U}_2 \to U_2$ by $\varphi_2(x, y) = (x, y)$, which is continuous and surjective. Moreover, $\widehat{U}_2/(\mathbb{Z}/2\mathbb{Z}) = (0, \infty) \times \mathbb{R}$ is homeomorphic (identical) with U_2. To make this open cover into an atlas we need to make sure it is closed under finite intersections. That means we need to construct an orbifold chart for $U_1 \cap U_2$. We know $U_1 \cap U_2 = (0, 1) \times \mathbb{R}$. Notice that the lift of the open set is given by

$$\widehat{U_1 \cap U_2} = \widehat{U}_1 \cap \widehat{U}_2 = (0, 1) \times \mathbb{R}.$$

The orbifold chart for $U_1 \cap U_2$ will be given by $((0, 1) \times \mathbb{R}, \mathbb{Z}/2\mathbb{Z}, \mathrm{Id})$. This means we can construct an atlas

$$\mathcal{U} \coloneqq \{\left(\widehat{U}_1, \mathbb{Z}/2\mathbb{Z}, \varphi_1\right), \left(\widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \varphi_2\right), \left(\widehat{U}_1 \cap \widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \mathrm{Id}\right)\}.$$

To check this is indeed an atlas, we need to show that all charts V are compatible. To show this, we need that for $\left(\widehat{U}_1, \mathbb{Z}/2\mathbb{Z}, \varphi_1\right)$ and $\left(\widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \varphi_2\right)$ there exists an open set $U_3 \subseteq U_1 \cap U_2$ such that its orbifold chart $\left(\widehat{U}_3, \mathbb{Z}/2\mathbb{Z}, \varphi_3\right)$ embeds into $\left(\widehat{U}_1, \mathbb{Z}/2\mathbb{Z}, \varphi_1\right)$ and into $\left(\widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \varphi_2\right)$. Take $U_3 = U_1 \cap U_2$, as was just shown this open set has an orbifold chart $\left(\widehat{U}_1 \cap \widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \mathrm{Id}\right)$. The inclusions $\iota_1 : U_1 \cap U_2 \hookrightarrow U_1$ and $\iota_2 : U_1 \cap U_2 \hookrightarrow U_2$, give us two embedings:

$$(\iota_1, \mathrm{Id}) : \left(\widehat{U}_1 \cap \widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \mathrm{Id}\right) \hookrightarrow \left(\widehat{U}_1, \mathbb{Z}/2\mathbb{Z}, \varphi_1\right)$$
$$(\iota_2, \mathrm{Id}) : \left(\widehat{U}_1 \cap \widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \mathrm{Id}\right) \hookrightarrow \left(\widehat{U}_2, \mathbb{Z}/2\mathbb{Z}, \varphi_2\right)$$

where both ι_1 and ι_2 are trivially equivariant with respect to Id. Therefore, $\mathcal{U}$ is an orbifold atlas for Q, so we have an orbifold $\mathcal{Q} = (Q, \mathcal{U})$. We note that the line $\{(0, y) \in \mathbb{R}^2\}$ is the singular set of this orbifold, since for each point $(0, y)$ the isotropy group is given by

$$\begin{aligned} \Gamma_{(0,y)} &= \{\gamma \in \mathbb{Z}/2\mathbb{Z} : \gamma \cdot (0, y) = (0, y)\} \\ &= \{\overline{0}, \overline{1} \in \mathbb{Z}/2\mathbb{Z}\}, \end{aligned}$$

which is non-trivial. △

We can think of many more developable orbifolds. We list two more examples.

Example 3.19. Consider the plane $\mathbb{R}^2$ with polar coordinates (r, θ) and the discrete group $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Let the Γ-action on $\mathbb{R}^2$ be defined as

$$\gamma \cdot (r, \theta) = (r, \theta + \frac{2\pi\gamma}{n}).$$

In other words, the Γ-action rotates each vector in $\mathbb{R}^2$ by $2\pi/n$ around the origin. The quotient $\mathbb{R}^2/\left(\mathbb{Z}/n\mathbb{Z}\right)$ is a developable orbifold by Theorem 3.16. We notice that the isotropy group $\Gamma_{(r,\theta)}$ is trivial for all points $(r, \theta) \in \mathbb{R}^2$, except when $r = 0$. This yields one singular point. The orbifolds $\mathbb{R}^2/\left(\mathbb{Z}/n\mathbb{Z}\right)$ will have the form of a cone in which the tip has angle $\frac{2\pi}{n}$. This orbifold was already shown in figure 1 in the introduction. Let us note this example can be extended to $\mathbb{R}^3$. The orbifold $\mathbb{R}^3/\left(\mathbb{Z}/n\mathbb{Z}\right)$ will be a cone over the projective space $\mathbb{RP}^2$. △

Example 3.20. Consider a 2-torus $\mathbb{T}^2$ in $\mathbb{R}^3$ centered at the origin. Let $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and let it act on $\mathbb{T}^2$ by rotating by an angle π around an axis of $\mathbb{R}^3$. There are four points of intersection between the torus and the axis of rotation: $\widetilde{p}, \widetilde{q}, \widetilde{r}$ and $\widetilde{s}$. These points will be mapped onto themselves by the action, and hence have isotropy group $\mathbb{Z}/2\mathbb{Z}$. The resulting orbifold $\mathbb{T}^2/\left(\mathbb{Z}/2\mathbb{Z}\right)$ will have the four singular points (corresponding to the intersections with the axis of rotation) and will have the shap of a 'pillow case'. The orbifold is shown in figure 3.2.

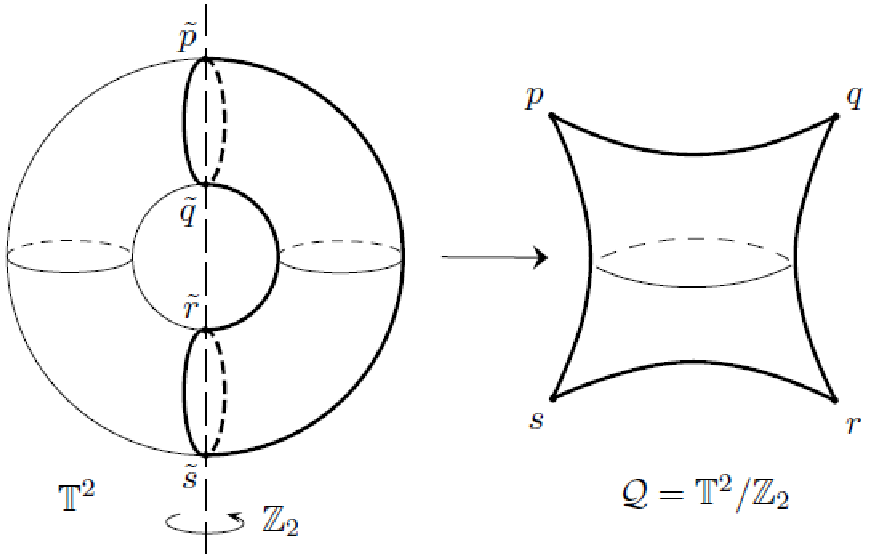


Figure 3.2: Pillow case orbifold obtained from rotation action on $\mathbb{T}^2$ (figure is taken from [20]).

△

Next we give an example of a non-developable orbifold.

Example 3.21 ($\mathbb{Z}_n$-Teardrop)**.** Let n be a positive integer. Consider an atlas for the sphere S^2

$$\mathcal{U} := \{(U_1, \varphi_1), (U_2, \varphi_2)\} = \{(\{S\}, \varphi_1), (S^2 \setminus \{S\}, \varphi_2)\}$$

where $\{S\}$ denotes the southpole of S^2. We can then consider an orbifold with charts $\{\widehat{U}_1, \Gamma_1, \varphi_1\} = \{\mathbb{R}^2, \mathbb{Z}/n\mathbb{Z}, \varphi_1\}$ and $\{\widehat{U}_2, \Gamma_2, \varphi_2\} = \{\mathbb{R}^2, \{e\}, \varphi_2\}$. Here $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{R}^2$ by rotation, i.e. for polar coordinates on $\mathbb{R}^2$ we have the action $(n, (r, \theta)) \mapsto (r, \theta + \frac{2\pi}{n})$. Moreover, we consider $\varphi_1(0) = O$. In this case we do not have to worry about intersections, since $U \cap V = \emptyset$. The orbifold obtained from this will look as in figure 3.3.

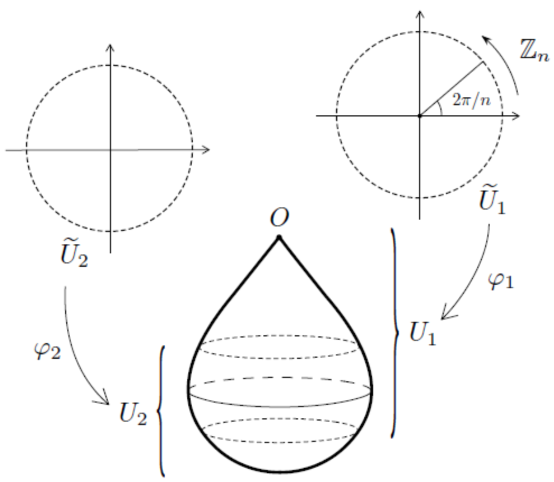


Figure 3.3: $\mathbb{Z}/n\mathbb{Z}$-Teardrop [20]

The only singular point is given by O since the isotropy group at this point is

$$\begin{aligned}\Gamma_O &= \Gamma_0 \\ &= \{\gamma \in \mathbb{Z}/n\mathbb{Z} : \gamma \cdot 0 = 0\} \\ &= \mathbb{Z}/n\mathbb{Z}\end{aligned}$$

is non-trivial. We remark that this orbifold is non-developable i.e. it cannot be represented as a manifold quotiented by a proper action of a discrete group. To show this we need more algebraic topolopgy on orbifold as introduced in [20, Example 2.36]. △

3.3 Tangent bundles, differential forms and Riemannian orbifolds

As is the case for manifolds, we can define tangent spaces and tangent bundles on orbifolds. Generalizing the tangent bundle on an orbifold to a so called orbibundle yields a large number of geometric structures that can be defined on an orbifolds. In this section we sketch the construction of a tangent bundle for a developable orbifold with an underlying differentiable manifold.

First, we define the tangent bundle as described in [20]. Consider a developable orbifold $\mathcal{Q} = M/\Gamma$, where M is a differentiable manifold. By definition the action $\alpha : \Gamma \times M \to M$ is smooth. The smoothness of the action makes it possible to lift the action from M to TM. Take a point $(\widehat{x}, v) \in TM$, then for all $\gamma \in \Gamma$ we define the action

$$\gamma \cdot (\widehat{x}, v) = (\alpha(\gamma, \widehat{x}), d\alpha_{\widehat{x}}(\widehat{x}, \gamma)(v))\,.$$

Using this action, we can define the **tangent bundle** of $\mathcal{Q}$ to be the quotient $T\mathcal{Q} = TM/\Gamma$. Because the lifted action is also proper, Theorem 3.16 implies the space $T\mathcal{Q}$ has an orbifold structure.

Remark 3.22. The definition as stated here is only valid for developable orbifolds. However, in the non-developable case this construction can be done on each orbifold chart seperately and we can 'glue' the charts together to obtain the tangent bundle. For this construction we refer the reader to [20, Section 2.6].

Pick a point $x \in \mathcal{Q}$ with a lift $\widehat{x} \in M$. The fiber of $T\mathcal{Q}$ above a point $x \in \mathcal{Q}$ turns out to be isomorphic to $T_{\widehat{x}}M/\Gamma_x$. We define the **tangent cone** at $x \in Q$ denoted by T_xQ to be $T_{\widehat{x}}M/\Gamma_x$. From this definition, we infer that a **tangent vector** of $\mathcal{Q}$ is an equivalence class in $T_x\mathcal{Q}$.

Lemma 3.23. *For a developable orbifold $\mathcal{Q}$ the tangent cone $T_x\mathcal{Q}$ is a vector space if and only if x is non-singular.*

Proof. We note $T_x\mathcal{Q}$ is isomorphic to $T_{\widehat{x}}M/\Gamma_x$, this space is a vector space if and only if the isotropy group Γ_x is trivial. By definition Γ_x is trivial if and only if x is not a singular point. □

Analogously to the manifold case, we can, for example, define vector fields on orbifolds.

Definition 3.24. *For an orbifold $T\mathcal{Q}$ with projection $\pi : T\mathcal{Q} \to \mathcal{Q}$, we define a* ***vector field*** *on $\mathcal{Q}$ to be a section of $T\mathcal{Q}$, i.e. a smooth orbifold map $X : \mathcal{Q} \to T\mathcal{Q}$ such that $\pi \circ X = \mathrm{id}$.*

Using similar constructions as for the tangent bundle, one can for example also define the cotangent bundle $T^*\mathcal{Q}$, the k-th exterior bundle $\bigwedge^k \mathcal{Q}$ and the (j,i)-tensor bundle $\bigotimes_j^i \mathcal{Q}$, which all have an orbifold structure. Two interesting constructions that can be made from these are the following.

Definition 3.25. *A* ***differential k-form*** *on an orbifold $\mathcal{Q}$ is a section of the bundle $\bigwedge^k T^*\mathcal{Q}$. The set of all k-forms on $\mathcal{Q}$ is denoted $\Omega^k(\mathcal{Q})$.*

One could also think of a differential k-form on an orbifold as a family of Γ-invariant differential k-forms on each orbifold chart that are equal on the intersection of the charts. Similarly, to the manifold case one can define the *exterior derivative*, which is a map $d : \Omega^k \to \Omega^{k+1}$. The precise construction can be found in [2, Section 3.4]. From these construction we can define symplectic and contact orbifolds analogously to the manifold case. Having these geometric structures on orbifolds will be relevant in section 4.4.

Similarly as for differential froms, the (j,i)-tensor bundle $\bigotimes_j^i \mathcal{Q}$ yields a way to construct Riemannian metrics on an orbifold. For this we follow [2, Section 4]. A section of $\bigotimes_j^i \mathcal{Q}$ will be an **(i,j)-tensor field** on the orbifold $\mathcal{Q}$. Hence we can define Riemannian metrics.

Definition 3.26. *On an orbifold $\mathcal{Q}$, a* ***Riemannian metric*** *is a positive definite, symmetric tensor field $g \in \bigotimes_0^2(\mathcal{Q})$. We define a* ***Riemannian orbifold*** *as the pair $(\mathcal{Q}, g)$.*

As is the case for differential forms, we can think of Riemannian metrics on orbifolds as a family of Γ-invariant Riemannian metrics on each chart. If we have a Riemannian metric on an orbifold $\mathcal{Q}$, then for a chart $\left(\widehat{U}, \Gamma, \varphi\right)$ we have a Riemannian metric on $\widehat{U}$. The metric induces an inner product $\langle \cdot, \cdot \rangle_x \coloneqq g_x(\cdot, \cdot)$ on $T_x\mathcal{Q}$ for all $x \in \mathcal{Q}$. This means that on a Riemannian orbifold $(\mathcal{Q}, g)$ we can define the length of a piecewise smooth curve $\gamma : [a,b] \to \mathcal{Q}$ to be

$$\ell(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle}.$$

Analogously to the Riemannian manifold case, we can define connections and in particular the Levi-Civita connection ∇. Using this construction, we can define Riemannian geodesics on an orbifold.

Definition 3.27. *A smooth curve $\gamma : [a,b] \to \mathcal{Q}$ is a* ***geodesic*** *if it locally lifts to chart where the curve satisfies the geodesic equation $\nabla_{\gamma'}\gamma' = 0$.*

As in the Riemannian case we find any locally length minimizing curve is a geodesic. For a more detailed exposition of Riemannian orbifolds we refer the reader to [20, Section 2.6], [2, Section 4] or [23].

Chapter 4

Sub-Riemannian Orbifolds

Now that we have seen sub-Riemannian geometry and orbifolds, one might wonder whether we can define a sub-Riemannian structure on an orbifold. This is in general not possible, however in some cases we can give a sub-Riemannian structure on an orbifold. The main problem will be to define a distribution on an orbifold that is well-defined around singular points. In this chapter we sketch the problems one runs into when defining sub-Riemannian orbifolds and give one way of resolving problem, in the specific case where the distribution is equivariant with respect to the action. We will also give examples of cases in which we can define a sub-Riemannian structure on an orbifold. We will also sketch on general result which allows us to find sub-Riemannian structures on cyclic closed 3-orbifolds.

Before, we can dive into the sub-Riemannian orbifolds, we first consider sub-Riemannian structures on 'lens spaces'. We will show that lens spaces are in general not orbifolds, but they are a good model for what happens on orbifolds at points that are not singular. On Lens spaces we define a so-called Cartan decomposition, which is a way of generating a distribution on an orbifold with some interesting properties.

4.1 Cartan decomposition

The first goal is to define sub-Riemannian structures on lens spaces. Before we can do this, we need to first define sub-Riemannian structures on three-dimensional matrix Lie groups, specifically $\mathrm{SU}(2)$. These turn out to be example of $k \oplus p$-manifolds, which are simple Lie groups admitting a Cartan decomposition. Most results in this section can be found in [24].

Let us assume all Lie groups and algebras mentioned consist of matrices, moreover consider all Lie algebras to be finite dimensional over a field $\mathbb{F}$ of characteristic zero, in our case $\mathbb{C}$. First we need some definitions from Lie theory.

Definition 4.1. *A Lie algebra is called* ***simple*** *if it is a non-Abelian Lie algebra without nontrivial proper ideals* [1]. *A Lie group is called* ***simple*** *if it is a connected non-Abelian Lie group without nontrivial connected normal subgroups, in other words if its Lie algebra is simple. A* ***semisimple*** *Lie algebra is a direct sum of simple Lie algebras.*

Remark 4.2. We note that any simple Lie algebra is semisimple.

[1]An ideal is a subalgebra $\mathfrak{i} \subseteq \mathfrak{g}$ such taht $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$.

Definition 4.3. *Consider a Lie algebra* $\mathfrak{g}$ *over a field* $\mathbb{F}$*, then the symmetric bilinear form* $\mathrm{Kil} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ *defined by*

$$\mathrm{Kil}(x, y) = \mathrm{Tr}\,(\mathrm{ad}(x)\,\mathrm{ad}(y))$$

is called the ***Killing form****.*

Proposition 4.4. *[25, Theorem 5.1] A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if its Killing form is nondegenerate.*

Definition 4.5. *A Lie algebra* $\mathfrak{g}$ *is called* ***compact*** *if there exists a compact Lie group* G *such that* T_eG *is isomorphic to* $\mathfrak{g}$*.*

Proposition 4.6. *[26, Proposition 6.6] Let* $\mathfrak{g}$ *be a semisimple Lie algebra, then* $\mathfrak{g}$ *is compact if and only its Killing form is negative definite.*

In order to define a distribution on a simple Lie group we consider a decomposition of semisimple Lie algebras.

Definition 4.7. *A decomposition of a semisimple Lie algebra* $\mathfrak{g}$ *of the form* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is called a* ***Cartan decomposition*** *if it satisfies*

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \tag{4.1}$$

where $\mathfrak{k}$ *and* $\mathfrak{p}$ *are subspace of* $\mathfrak{g}$*.*

Proposition 4.8. *[26, Chapter 7]. Any semisimple real Lie algebra admits a Cartan decomposition.*

For the purpose of this section it is enough to consider simple Lie algebras and groups. Using the Cartan decomposition we can endow each simple Lie group with a specific sub-Riemannian structure.

Definition 4.9. *Let* G *be a compact simple Lie group with Lie algebra* $\mathfrak{g}$*. Consider the Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$*. Consider the distibution* $\mathcal{D}_q = q\mathfrak{p}$ *at the point* $q \in G$ *endowed with a Riemannian metric* β *given by*

$$\beta_q(v_1, v_2) = \langle g^{-1}v_1, g^{-1}v_2 \rangle$$

where $\langle \cdot, \cdot \rangle := \alpha K(\cdot, \cdot)|_{\mathfrak{p}}$ *for* K *the Killing form restricted to* $\mathfrak{p}$ *and* $\alpha \in \mathbb{R}_{<0}$*.The triple* $(G, \mathcal{D}, g)$ *is called a* $\mathfrak{k} \oplus \mathfrak{p}$***-manifold****.*

Here the choice for a negative α is made to make sure that the metric is positive definite following 4.6. The distribution $\mathcal{D}_q$ is defined by considering $\mathfrak{p}$, which is a subspace of T_eG. We can translate $\mathfrak{p}$ to any other tangent space on the manifold by considering the derivative of the left-translation $L_g(x) = gx$ for $g, x \in G$. Therefore, we find $\mathcal{D}_q = (L_q)_*(\mathfrak{p})$.

Remark 4.10. In case G is not compact, we need to assume $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{g}$. We define a maximal compact subalgebra of $\mathfrak{g}$ as a compact subalgebra of $\mathfrak{g}$ that is not properly contained in any other compact subalgebra. The existence of the maximal compact subalgebra in the Cartan decomposition of a (possibly) non-compact Lie group G is discussed in [26, Prop 7.4]. The choice of the maximal compact subalgebra is made in order to define a distribution that is non-compact. In this case following 4.6, we know that the Killing form is positive definite. Therefore, α must be a positive scaling instead.

It turns out that $\mathfrak{k} \oplus \mathfrak{p}$-manifolds have some nice properties. First of all, we can show that the Pontryagin maximum principle, as stated in Theorem 1.15 on a right-invariant $\mathfrak{k} \oplus \mathfrak{p}$-manifold G gives rise to a completely integrable [2] Hamiltonian system. This integrability gives rise to a closed form formula for normal geodesic. Given a normal geodesic $\gamma : [0, T] \to G$, at the the point $\gamma(0)$, there is a vector $v \in T_{\gamma(0)}G$ by the right-invariance of G, we know we can write $v \in \mathfrak{g}$. So, we can write $v = A_k + A_p$ for $A_k \in \mathfrak{k}$ and $A_p \in \mathfrak{p}$. The closed form of the normal geodesics becomes:

$$\gamma(t) = e^{-A_k t} e^{(A_k + A_p)t} \gamma(0). \tag{4.2}$$

A derivation of this formula and a proof of the complete integrability of the of the Hamiltonian system can be found in [27, Appendix B]. But let us mention that it is a reformulation of the formula for normal sub-Riemannian geodesics on Lie groups we have seen in Example 2.19.

Secondly, we can show that on a $\mathfrak{k} \oplus \mathfrak{p}$-manifold abnormal minimizers can exist, however they will never be global length-minimizers (i.e. the geodesic will never be optimal). The proof of this fact is presented in [27, Appendix C].

4.2 Lens spaces

Using the Cartan decomposition described in the previous section, we can give a few interesting examples of sub-Riemannian structures. First, we construct an example on the Lie group SU(2). Defining a specifc action on SU(2) we construct Lens spaces. We will transfer the sub-Riemannian structure on SU(2) to Lens spaces, which yields a more general technique manifolds with a free action.

4.2.1 Special unitary matrices

Consider the Lie group SU(2), which is defined as

$$\mathrm{SU}(2) := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}) : |a|^2 + |b|^2 = 1 \right\}.$$

The Lie algebra of SU(2) can be computed as

$$\mathfrak{su}(2) := \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \in GL(2, \mathbb{C}) : a \in \mathbb{R}, b \in \mathbb{C} \right\}.$$

This Lie algebra has a basis of the form

$$p_1 := \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad p_2 := \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad k := \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

[2] In this context completely integrable mean that for each degree of freedom there is a constant of motion.

with commutation relations

$$[p_1, p_2] = k, \qquad [p_2, k] = p_1, \qquad [k, p_1] = p_2. \tag{4.3}$$

We notice that SU(2) is a simple Lie group. Let us define a sub-Riemannian structure on SU(2). In general we can find that the Killing form on $\mathfrak{su}(n)$ is given by

$$\mathrm{Kil}(x, y) = 2n\,\mathrm{Tr}(xy)$$

for $x, y \in \mathfrak{su}(n)$. Hence, on $\mathfrak{su}(2)$ we find that $\mathrm{Kil}(x, y) = 4\,\mathrm{Tr}(xy)$ for $x, y \in \mathfrak{su}(2)$. Using the commutation relations (4.3) we have a Cartan decomposition for $\mathfrak{su}(2)$ as follows,

$$\mathfrak{k} = \mathrm{span}\{k\} \qquad \text{and} \qquad \mathfrak{p} = \mathrm{span}\{p_1, p_2\}.$$

We notice that $\mathrm{Kil}(p_i, p_j) = -2\delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Let us define the metric as

$$\langle \cdot, \cdot \rangle := -\frac{1}{2}\,\mathrm{Kil}(\cdot, \cdot)|_{\mathfrak{p}}.$$

The set $\{p_1, p_2\}$ form an orthonormal frame with respect to this metric. We can define a distribution at $g \in \mathrm{SU}(2)$ using the formula

$$\Delta(g) = g\mathfrak{p}$$

and endow it with a sub-Riemannian metric

$$\beta_g(v_1, v_2) = \langle g^{-1}v_1, g^{-1}v_2 \rangle.$$

The triple $(\mathrm{SU}(2), \Delta, \beta)$ forms a sub-Riemannian $\mathfrak{k} \oplus \mathfrak{p}$-manifold.

Using the fact that SU(2) is a $\mathfrak{k} \oplus \mathfrak{p}$-manifold, we know from the general form of its normal sub-Riemannian geodesics from equation (4.2). In 1.36 we find that the initial covector of a normal geodesic through a point $q_0 \in \mathrm{SU}(2)$ lies in the cylinder Δ_{q_0}. In coordinates we take the intitial covector $\lambda(\theta, c) = \cos(\theta)p_1 + \sin(\theta)p_2 + ck$. The coefficients in (4.2) are given by $A_k = ck$ and $A_p = \cos(\theta)p_1 + \sin(\theta)p_2$. Then the normal geodesics are given by

$$\gamma(t) = e^{(\cos(\theta)p_1 + \sin(\theta)p_2 + ck)t}e^{-ckt}.$$

A more concrete form for the geodesics can be obtained by noticing that SU(2) is diffeomorphic to the sphere S^3 which is embedded in $\mathbb{C}^2$. Hence, we write $\gamma(t)$ as a vector in $\mathbb{C}^2$. From [24] we find the expression $\gamma(t) = \begin{pmatrix} a \\ b \end{pmatrix}$ with

$$
\begin{aligned}
a =& \frac{c\sin\left(\frac{ct}{2}\right)\sin\left(\sqrt{1+c^2}\frac{t}{2}\right)}{\sqrt{1+c^2}} + \cos\left(\frac{ct}{2}\right)\cos\left(\sqrt{1+c^2}\frac{t}{2}\right) \\
&+ i\left(\frac{c\cos\left(\frac{ct}{2}\right)\sin\left(\sqrt{1+c^2}\frac{t}{2}\right)}{\sqrt{1+c^2}} - \sin\left(\frac{ct}{2}\right)\cos\left(\sqrt{1+c^2}\frac{t}{2}\right)\right), \\
b =& \frac{\sin\left(\sqrt{1+c^2}\frac{t}{2}\right)}{\sqrt{1+c^2}}\left(\cos\left(\frac{ct}{2}+\theta\right) + i\sin\left(\frac{ct}{2}+\theta\right)\right).
\end{aligned}
\tag{4.4}
$$

Let us remark that for other three-dimensional Lie groups such as SO(3) and SL(2) we can find a general form for normal geodesics in a very similar way. The full computations can be found in [24, Section 3].

4.2.2 Sub-Riemannian Lens spaces

Using the sub-Riemannian structure for SU(2) we can construct a sub-Riemannian structure on three-dimensional Lens spaces. Lens spaces will give us a way of defining a sub-Riemannian structure on regular parts of orbifolds. First, we define Lens spaces.

Definition 4.11. *Consider coprime integers $p, q \in \mathbb{Z}$ and the unit three-sphere $S^3 \subseteq \mathbb{C}^2$. We define the $\mathbb{Z}/p\mathbb{Z}$-action on S^3 given by*

$$(z_1, z_2) \mapsto \left(e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2\right). \tag{4.5}$$

The quotient space formed by this action is called a ***Lens space*** *denoted by $L(p,q)$.*

In this definition lens spaces are compact three-manifold [3], but are not in general homogeneous spaces or Lie goups. Therefore, the techniques described in 2.3 are not applicable. However, since the $\mathbb{Z}/p\mathbb{Z}$-action is free the lens space will have no orbifold singularities. To see that the action is free, notice that the only way (z_1, z_2) is fixed by the action, is when $e^{2\pi i/p} = 1$ and $e^{2\pi i q/p} = 1$. So $1/p$ and q/p should be integers. The only way in which this can be is if $p = 1$, but then we act by the identity element of $\mathbb{Z}/p\mathbb{Z}$. Hence, the action is free.

One convenient way to think about lens spaces, is to consider the action on SU(2) instead of S^3. In order to see this, we notice that SU(2) and S^3 are diffeomorphic via the map

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

The $\mathbb{Z}/p\mathbb{Z}$-action can then be described as

$$\begin{pmatrix} z_1 & z_2 \\ -\overline{z_1} & \overline{z_2} \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi i q/p} \end{pmatrix}.$$

[3]This definition can be generalized to higher dimensions by considering the action for more coprime integers on higher dimensional spheres.

Let us now define a sub-Riemannian structure on $L(p,q)$. Using Theorem B.16 the canonical quotient map $\pi : \mathrm{SU}(2) \to L(p,q)$ is a local diffeomorphism. Therefore, using lemma B.15, we know the tangent map π_* is a local linear isomorphism. Using this construction, we can make a correspondence between the sub-Riemannian structure $(\mathrm{SU}(2), \Delta, \beta)$ defined in section 4.2.1 and a sub-Riemannian structure on $L(p,q)$.

Theorem 4.12. *[24, Proposition 9] The sub-Riemannian structure on* $\mathrm{SU}(2)$ *induces a sub-Riemannian structure* $\left(L(p,q), \widetilde{\Delta}, \widetilde{\beta}\right)$ *via the quotient map* $q : \mathrm{SU}(2) \to L(p,q)$. *Here we have*

(i) For $[g] \in L(p,q)$, *the distribution is given by the two-dimensional subspace of* $T_{[g]}L(p,q)$*:* $\widetilde{\Delta}_{[g]} := \pi_*\left(\Delta_h\right))$ *with a representative* $h \in [g]$.

(ii) The sub-Riemannian metric is given by the smooth positive definite inner product defined as

$$\widetilde{\beta}_{[g]}\left(\pi_*(v), \pi_*(w)\right) := \beta_h(v,w)$$

for $h \in [g]$ *and* $v, w \in T_h\,\mathrm{SU}(2)$.

In order to prove this lemma, we need the following lemma.

Lemma 4.13. *[24, Proposition 10] For two representatives* $h_1, h_2 \in [g]$ *for* $g \in L(p,q)$, *the map* $\varphi : \mathfrak{p} \to \mathfrak{p}$ *defined by*

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\frac{2\pi(q-1)}{p} & \sin\frac{2\pi(q-1)}{p} \\ -\sin\frac{2\pi(q-1)}{p} & \cos\frac{2\pi(q-1)}{p} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = A \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

is a bijection, an isometry with respect to the inner product on $\widetilde{\Delta}_{[g]}$ *and the identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} h_1 e^{t\eta} = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} h_2 e^{t\varphi(\eta)} \tag{4.6}$$

holds for all $\eta \in \mathfrak{p}$.

Proof. The fact that φ is a bijection follows because the matrix A is invertible. In order to check φ is an isometry, it suffices to check that $A^*A = I$ for A^* the Hermitian of A. Lastly, we show equation (4.6) holds. Consider $\eta = np_1 + mp_2 \in \mathfrak{p}$, rewriting yields

$$t\eta = \frac{t}{2}\begin{pmatrix} 0 & n+im \\ im-n & 0 \end{pmatrix}.$$

We note that $(t\eta)^2 = -x^2 I$ for $x = \frac{t}{2}\sqrt{n^2+m^2}$ and I the identity matrix. Now, we can compute the exponential by splitting it in even and odd terms as follows:

$$
\begin{aligned}
e^{t\eta} &= I + t\eta + \frac{t^2\eta^2}{2!} + \frac{t^3\eta^3}{3!} + \frac{t^4\eta^4}{4!} + \dots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) I + \left(1 - \frac{x^2}{3!} + \frac{x^5}{5!} - \dots\right) t\eta \\
&= \cos(x) I + \frac{\sin(x)}{x} t\eta \\
&= \cos\left(\frac{t}{2}\sqrt{n^2+m^2}\right) I + \frac{\sin\left(\sqrt{n^2+m^2}\right)}{x} \begin{pmatrix} 0 & n+im \\ im-n & 0 \end{pmatrix}.
\end{aligned}
$$

If we now pick some $h = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ in SU(2), we find

$$
he^{t\eta} = \begin{pmatrix} a\cos\left(\sqrt{n^2+m^2}\frac{t}{2}\right) - b\sin\left(\sqrt{n^2+m^2}\frac{t}{2}\right)\frac{n-im}{\sqrt{n^2+m^2}} \\ b\cos\left(\sqrt{n^2+m^2}\frac{t}{2}\right) - a\sin\left(\sqrt{n^2+m^2}\frac{t}{2}\right)\frac{n+im}{\sqrt{n^2+m^2}} \end{pmatrix}.
$$

Here, we use the identification of SU(2) with S^3. Consider $\eta_i = n_i p_1 + m_i p_2$ for $i = 1, 2$. Since h_1, h_2 are both representatives in the equivalence class $[g]$, we have $h_2 = \begin{pmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi iq/p} h_1 \end{pmatrix}$. Take $h_1 = \begin{pmatrix} a_1 & b_1 \\ -\overline{b_1} & \overline{a_1} \end{pmatrix}$ Using this, we find

$$
h_1 e^{t\eta_1} = \begin{pmatrix} a_1\cos\left(\sqrt{n_1^2+m_1^2}\frac{t}{2}\right) - b_1\sin\left(\sqrt{n_1^2+m_1^2}\frac{t}{2}\right)\frac{n_1-im_1}{\sqrt{n_1^2+m_1^2}} \\ b_1\cos\left(\sqrt{n_1^2+m_1^2}\frac{t}{2}\right) - a_1\sin\left(\sqrt{n_1^2+m_1^2}\frac{t}{2}\right)\frac{n_1+im_1}{\sqrt{n_1^2+m_1^2}} \end{pmatrix}
$$

and

$$
\begin{aligned}
h_2 e^{t\eta_1} &= \begin{pmatrix} e^{2\pi i/p} a_1\cos\left(\sqrt{n_2^2+m_2^2}\frac{t}{2}\right) - e^{2\pi iq/p} h_1 b_1\sin\left(\sqrt{n_2^2+m_2^2}\frac{t}{2}\right)\frac{n_2-im_2}{\sqrt{n_2^2+m_2^2}} \\ e^{2\pi iq/p} h_1 b_1\cos\left(\sqrt{n_2^2+m_2^2}\frac{t}{2}\right) - e^{2\pi i/p} a_1\sin\left(\sqrt{n_2^2+m_1^2}\frac{t}{2}\right)\frac{n_2+im_2}{\sqrt{n_2^2+m_2^2}} \end{pmatrix} \\
&= \begin{pmatrix} a_1\cos\left(\sqrt{n_2^2+m_2^2}\frac{t}{2}\right) - e^{2\pi i(q-1)/p} h_1 b_1\sin\left(\sqrt{n_2^2+m_2^2}\frac{t}{2}\right)\frac{n_2-im_2}{\sqrt{n_2^2+m_2^2}} \\ h_1 b_1\cos\left(\sqrt{n_2^2+m_2^2}\frac{t}{2}\right) - e^{2\pi i(1-q)/p} a_1\sin\left(\sqrt{n_2^2+m_1^2}\frac{t}{2}\right)\frac{n_2+im_2}{\sqrt{n_2^2+m_2^2}} \end{pmatrix}.
\end{aligned}
$$

Taking the derivative and setting $t = 0$, we find that equation (4.6) holds if and only if $n_1^2 + m_1^2 = n_2^2 + m_2^2$, $n_1 - im_1 = e^{2\pi i(q-1)/p}(n_2 - im_2)$ and $n_1 + im_1 = e^{2\pi i(1-q)/p}(n_2 - im_2)$. These three equations are equivalent to requiring that $e^{2\pi i(q-1)/p}(n_1 + im_1) = n_2 - im_2$, which is the case if and only if $\eta_2 = \varphi(\eta_1)$ which was true by assumption. □

Proof of Theorem 4.12. Since, $\pi : \mathrm{SU}(2) \to L(p,q)$ is a local diffeomorphism, the tangent map at a point $g \in \mathrm{SU}(2)$ given by $\pi_* : T_g\,\mathrm{SU}(2) \to T_{\pi(g)}L(p,q)$ is a local linear isomorphism by lemma B.15. Given the distribution $\Delta_g = g\mathfrak{p}$ on SU(2) is a 2-dimensional subspace, the space $\widetilde{\Delta}_{[g]} = \pi_*|_g(\Delta_g)$ will be a 2-dimensional subspace in $T_{\pi(g)}L(p,q)$. We notice that Δ_g is bracket generating, since $[p_1, p_2] = k$. Hence, $\widetilde{\Delta}_{[}g]$ is also bracket generating, since

$$[\pi_*(p_1), \pi_*(p_2)] = \pi_* ([p_1, p_2]) = \pi_* k$$

by the naturality of the Lie bracket. The fact that $\widetilde{\beta}_{[g]} (\pi_*(v), \pi_*(w))$ is a smooth positive-definite inner product, follows because $\beta_h(v, w)$ is a smooth positive-definite inner product on SU(2). It remains to show that the sub-Riemannian structure is well-defined. This will be shown in the following two claims.

Claim 1. *The distribution $\Delta_{[g]}$ is well-defined: For every representatives $h_1, h_2 \in [g]$ we have*

$$\pi_*|_{h_1} (\Delta_{h_1}) = \pi_*|_{h_2} (\Delta_{h_2}).$$

Claim 2. *The inner product $\langle \pi_*(v), \pi_*(w)\rangle_{[g]}$ is well-defined: For every $h_1, h_2 \in [g]$ and $v_1, w_1 \in T_{h_1}\,\mathrm{SU}(2)$, $v_2, w_2 \in T_{h_2}\,\mathrm{SU}(2)$ such that $\pi_*|_{h_1}(v_1) = \pi_*|_{h_1}(w_1)$ and $\pi_*|_{h_2}(v_2) = \pi_*|_{h_2}(w_2)$, then we have $\langle v_1, w_1\rangle_{h_1} = \langle v_2, w_2\rangle_{h_2}$.*

Lemma 4.13 tells us that if we have some paths $h_1 e^{t\eta_1}$ and $h_2 e^{t\eta_2}$ that are equivalent on $L(p, q)$ under the equivalence relation, then under the quotient their tangent vectors are projected onto the same vector. In particular, we find that starting in $\mathfrak{p}$, we do not escape this distribution. This implies claim 1. The second claim follows from a similar argument and using the fact that φ is an isometry. Therefore, we obtain a well-defined sub-Riemannian structure on $L(p, q)$ as desired. □

Using the construction above, we find that SU(2) and $L(p, q)$ are locally isometric. Therefore, if we project the normal sub-Riemannian geodesics on SU(2) given in equation (4.4), we find the normal sub-Riemannian geodesics on $L(p, q)$. An interesting question one can ask, is what potential abnormal minimizers would look like on $L(p, q)$. The abnormal minimizers on SU(2) are not optimal, would the abnormal minimizers on $L(p, q)$ also be non-optimal? Currently, we do not know, but this might be interesting for future research.

4.3 Singular examples

Now that we know what sub-Riemannian structures on Lens spaces look like, we wonder how to extend this to orbifolds. The following remark is crucial for the construction. Suppose we have a sub-Riemannian manifold $(M, \mathcal{D}, g)$. If let a discrete group Γ act on M freely and properly, we can consider the space M/Γ. This is in general not an orbifold, but we can give a sub-Riemannian structure on it in the same way as we did for lens spaces. In other words, we can lift the distibution $\mathcal{D}_x$ for $x \in M$ to a distribution $q_*(\mathcal{D}_x)$ on $T_{q(x)}M/\Gamma$ and find the the sub-Riemannian geodesics by projecting the sub-Riemannian geodesics on M to $M\Gamma$.

However, a developable orbifold is not obtained by a free action. If the action is not free, we have seen singularities occur. In this section we discuss a method to develop a sub-Riemannian structure in these singular cases.

We note that any orbifold is 'locally free'. In other words, at all points in our orbifold the isotropy group is trivial, except for some isolated singularities 3.15. This means that at the non-singular points we can use the construction above to lift the sub-Riemannian structure of the underlying manifold to the developable orbifold. At the singular points however, there occur some problems that need to be solved.

Consider a singular point $y \in M/\Gamma$. The first problem that occurs is that for a discrete group Γ a non-free Γ-action on a manifold M, the quotient map $\pi : M \to M/\Gamma$ is not a local diffeomorphism around y. This implies that, for a point in $q^{-1}(y) \subset M$, the lifted quotient $\pi_* : T_{q^{-1}(y)}M \to T_y M/\Gamma$ is no longer a local is linear isomorphism. Therefore, the tangent space $T_y M/\Gamma$ is not a vector space.

A second problem that occurs is that given a path γ on M, the induced path $\pi(\gamma)$ on M/Γ is not unique. Since for two different representatives μ_1, μ_2 of M/Γ, we can have $\mu_1 = \pi(\gamma) = \mu_2$. This means that we cannot lift our sub-Riemannian structure on M to a unique sub-Riemannian structure on M/Γ. As an example, in the cone described in Example 3.19 we find that for $n = 3$, there are 3 paths on $\mathbb{R}^2$ that describe a path over the tip of the cone on $\mathbb{R}^2/\left(\mathbb{Z}/3\mathbb{Z}\right)$. These paths are represented by three pairs of colours.

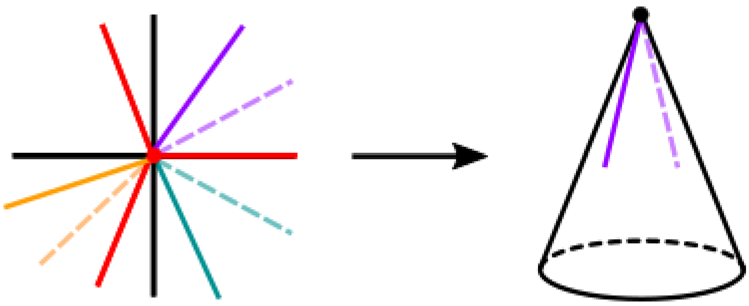


Figure 4.1: A sketch of three curves on $\mathbb{R}^2$ that all lift to the same curve on the cone.

In special cases we can solve this problem. These special cases are the cases when the distribution is equivariant under the action. Consider an action $\alpha : G \times M \to M$, with a lift $\alpha_*(x) : T_e\Gamma \to T_x M$ for $x \in M$ and e the unit element in Γ. This induces a quotient map $\pi : M \to M/\Gamma$, with a lift $q_* : TM \to T(M\Gamma)$. Moreover, take a distribution $\mathcal{D}_x \subseteq T_x M$ for $x \in M$. Then the distribution is called **equivariant** if

$$\pi_* \left(\alpha_* \left(\mathcal{D}_x\right)(g)\right) = \pi_* \left(\mathcal{D}_{\alpha(x,g)}\right) \tag{4.7}$$

for every $x \in M$ and $g \in \Gamma$.

If a distribution is equivariant, we can define the sub-Riemannian structure on M/Γ by lifting the sub-Riemannian structure on M via the quotient on the non-singular points. Because of the equivariance, the distribution will stay consistent around the singularity. We try to give some intuition for this. Let us fix an arbitrary element $g \in \gamma$. On a point $x \in M$ we have a distribution $\mathcal{D}_x$. If we let g act on the point x we find a new point $\alpha(g, x) \in M$ with a distribution $\mathcal{D}_{\alpha(g,x)}$. The equivariance requirement means that if we act on a point in the base M, then the distribution on the tangent bundle follows along via the lifted action. If we now consider the projection $q : M \to M/\Gamma$, then we know that $q(\alpha(g, x)) = q(x)$. By the equivariance requirement, we find that the distribution at $q(\alpha(g, x))$ and $q(x)$ must also be the same when projected to $T_{q(x)}M/\Gamma$. Therefore, the distribution will remain well-defined around a singular point.

We will now give a few examples in which the distribution is equivariant around the singularity.

4.3.1 Reflections

Consider the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{R}^3$, $\alpha : \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$(x, y, z) \mapsto (x, -y, -z).$$

This corresponds to a reflection in the x-axis. The quotient space $\mathbb{R}^3/(\mathbb{Z}/2\mathbb{Z})$ is a developable orbifold, in which the singular stratum is the x-axis because at the point $(x, 0, 0)$ the isotropy group is non-trivial for all $x \in \mathbb{R}$. We denote the canonical quotient map $\pi : \mathbb{R}^3/(\mathbb{Z}/2\mathbb{Z}) \to \mathbb{R}^3$.

Consider the distribution $\mathcal{D} = \ker(\xi)$ for $\xi = dz + xdy$ the standard contact form on $\mathbb{R}^3$. This distribution is generated by the vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y}.$$

We can check the equivariance as follows, $\alpha_*(v_x, v_y, v_z) = (v_x, -v_y, -v_z)$, hence

$$\alpha_*(X_{(x,y,z)})) = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z} \qquad \alpha_*(Y_{(x,y,z)})) = -\frac{\partial}{\partial y}.$$

On the right-hand side we find

$$X_{\alpha(x,y,z)} = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z} \qquad Y_{\alpha(x,y,z)} = \frac{\partial}{\partial y}.$$

Noticing that projecting all equations by π_*, we identify the axes $\frac{\partial}{\partial y}$ and $-\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ with $-\frac{\partial}{\partial z}$, i.e. $\pi_*(-\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}$ and $\pi_*(-\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}$. Therefore,

$$\pi_* \left(\alpha_*(X_{(x,y,z)}))\right) = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} = \pi_* \left(X_{\alpha(x,y,z)}\right)$$

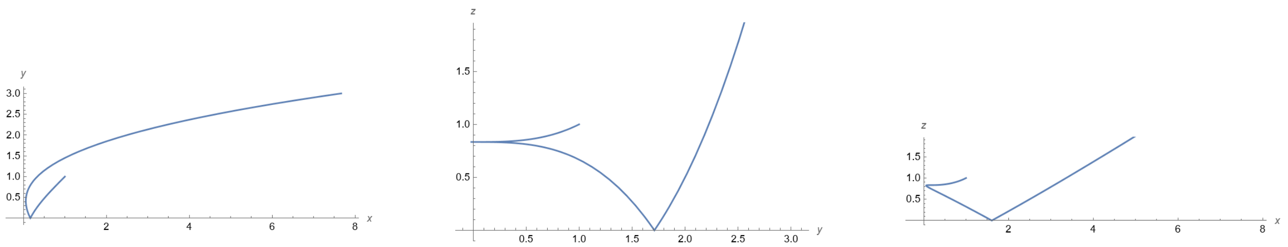
and similarly,

$$\pi_* \left(\alpha_*(Y_{(x,y,z)}))\right) = \frac{\partial}{\partial y} = \pi_* \left(Y_{\alpha(x,y,z)}\right).$$

Therefore, in this case the distribution is equivariant with respect to the reflective action α. This means, we can project the sub-Riemannian structure of $\mathbb{R}^3$ onto $\mathbb{R}^3/(\mathbb{Z}/2\mathbb{Z})$. We notice the frame $\{X, Y\}$ for $\mathcal{D}$ is orthogonal with respect to the standard inner product on $\mathbb{R}^3$. Therefore, using proposition 1.18, we can write the sub-Riemannian Hamilonian for $\lambda = (x, y, z, p_x, p_y, p_z) \in T^*M$ as

$$\begin{aligned} H(\lambda) &= \frac{1}{2}\langle \lambda, X\rangle^2 + \frac{1}{2}\langle \lambda, Y\rangle^2 \\ &= \frac{1}{2}(p_x + yp_z)^2 + \frac{1}{2}p_y^2 \end{aligned}$$

Notice that the distribution is a contact distribution. Therefore, by Theorem 1.30 we do not have to look for abnormal minimizers. The sub-Riemannian geodesics for this structure will



(a) Reflection on xy-plane (b) Reflection on yz-plane (c) Reflection on xz-plane

Figure 4.2: Geodesics from all perspectives on $\mathbb{R}^3/(\mathbb{Z}/2\mathbb{Z})$.

be the normal sub-Riemannian geodesics for $\mathbb{R}^3$, but if they hit the y- or z-axis then they will be reflected. Using Mathematica we can compute the geodesic flow and plot it. The code is included in Appendix **??** In a figure this looks as follows.

We now reflected in the x-axis, but many more reflections are possible in Euclidean spaces. For example, one could show in a very similar way that the vector fields X, Y are also equivariant under the $\mathbb{Z}/2\mathbb{Z}$-action $(x, y, z) \mapsto (-x, y, z)$. The sub-Riemannian Hamiltonian will be the same as before, however the geodesics will be reflected in a different way. Let me also note that not all reflections are allowed, for example if we consider the Heisenberg distribution which is generated by

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}.$$

If we now consider the action $(x, y, z) \mapsto (-x, y, z)$, we find

$$\alpha_*\left(Y_{(x,y,z)}\right) = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \neq \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z} = Y_{\alpha(x,y,z)}.$$

In this case we find that for the canonical quotient map π, $\pi_*(-\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}$, however this is not the case for $\frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$, hence the equivariance condition is not satisfied.

Sub-Riemannian structures on Euclidean spaces quotiented by a reflection can also be defined for higher dimensions, given we have a distribution. For example, consider $\mathbb{R}^4$ with coordinates (x, y, z, w). On $\mathbb{R}^4$ we can define the Engel distribution which is generated by the vector fields

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial w}.$$

This distribution is spanned by orthogonal vector fields and is bracket generating. If we consider the antipodal action $(x, y, z, w) \mapsto (-x, -y, -z, -w)$ on $\mathbb{R}^4$, then we find

$$\alpha_*(X_{x,y,z,w}) = -\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} \qquad \alpha_*(Y_{x,y,z,w}) = -\frac{\partial}{\partial w}.$$

And,

$$X_{\alpha(x,y,z,w)} = \frac{\partial}{\partial x} - z\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} \qquad Y_{\alpha(x,y,z,w)} = \frac{\partial}{\partial w}.$$

Under the quotient map $\pi : \mathbb{R}^4 \to \mathbb{R}^4/(\mathbb{Z}/2\mathbb{Z})$ we find that $\pi_*(\pm\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}$, $\pi_*(\pm\frac{\partial}{\partial y}) = \frac{\partial}{\partial y}$, $\pi_*(\pm\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}$, $\pi_*(\pm\frac{\partial}{\partial w}) = \frac{\partial}{\partial w}$. Therefore, we find that the Engel distribution on $\mathbb{R}^4$ is equivariant with respect to the antipodal action. In this case the sub-Riemannian Hamiltonian will be given by

$$H(\lambda) = \frac{1}{2}(p_x + zp_y + wp_z)^2 + \frac{1}{2}p_w^2$$

for $\lambda = (x, y, z, w, p_x, p_y, p_z, p_w)$.

4.3.2 Rotation

In example 3.19, we have seen that the cyclic $\mathbb{Z}/n\mathbb{Z}$-action $(r, \theta) \mapsto (r, \theta + \frac{2\pi}{n})$ on $\mathbb{R}^2$ generates a conic orbifold. If we apply the same action to $\mathbb{R}^3$, we get the action $\alpha : \mathbb{Z}/n\mathbb{Z} \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by $(r, \theta, z) \mapsto (r, \theta + \frac{2\pi}{n}, z)$. Taking the quotient of $\mathbb{R}^3$ with respect to this action yields an orbifold Q that is a cone over $\mathbb{RP}^2$, i.e. we get a series of cones of which the tops form a line. If we want to define a sub-Riemannian structure on the orbifold Q, we need a good distribution. Let us consider the following contact form $\xi = dz + r^2 d\theta$ on $\mathbb{R}^3$ in cylindrical coordinates. To check this is a contact form we notice that $\xi \wedge d\xi = 2r dz \wedge dr \wedge d\theta$ is non-zero for $r \neq 0$. Then the distribution $\ker(\xi)$ is spanned by the vector fields

$$X = \frac{\partial}{\partial z} - \frac{1}{r^2}\frac{\partial}{\partial \theta} \qquad Y = \frac{\partial}{\partial r}.$$

Now, we check this distribution is equivariant under the cyclic action α. We notice that $d\alpha(v_r, v_\theta, v_z) = (v_r, v_\theta, v_z)$. Hence,

$$\alpha_*(X_{(r,\theta,z)}) = \frac{\partial}{\partial z} - \frac{1}{r^2}\frac{\partial}{\partial \theta} \quad \text{and} \quad \alpha_*\left(X_{r,\theta,z}\right) = \frac{\partial}{\partial r}.$$

On the other hand, the action on the base yields,

$$X_{\alpha(r,\theta,z)} = \frac{\partial}{\partial z} - \frac{1}{r^2}\frac{\partial}{\partial \theta} \quad \text{and} \quad Y_{\alpha(r,\theta,z)} = \frac{\partial}{\partial r}.$$

Hence, the equivariance condition is satisfied.

The reason the equivariance is satisfied is because our distibution is orthogonal to the tangent vector of the action, hence acting on the distribution means that we move it to another point. However, since the distribution does not depend on the coordinate θ it is invariant with respect to the action. So we find the equivariance as desired.

If we want to find the sub-Riemannian geodesics on Q, we need to find the Hamiltonian. The frame $\{X, Y\}$ for the distribution is orthogonal, hence using proposition 1.18, we find that for $\lambda = (r, \theta, z, p_r, p_\theta, p_z) \in T^*\mathbb{R}^3$ we have

$$\begin{aligned}H(\lambda) &= \frac{1}{2}\langle \lambda, X\rangle^2 + \frac{1}{2}\langle \lambda, Y\rangle^2 \\ &= \frac{1}{2}\left(p_z - \frac{p_\theta}{r^2}\right)^2 + \frac{1}{2}p_r^2\end{aligned}$$

The geodesics on for example $\mathbb{R}^3/(\mathbb{Z}/4\mathbb{Z})$ will be the sub-Riemannian geodesics of $\mathbb{R}^3$ for the distribution $\mathcal{D}$, but when the geodesic hits either the x- or the y-axis, it will not continue but come out at the other axis. The following figure shows an example of this.

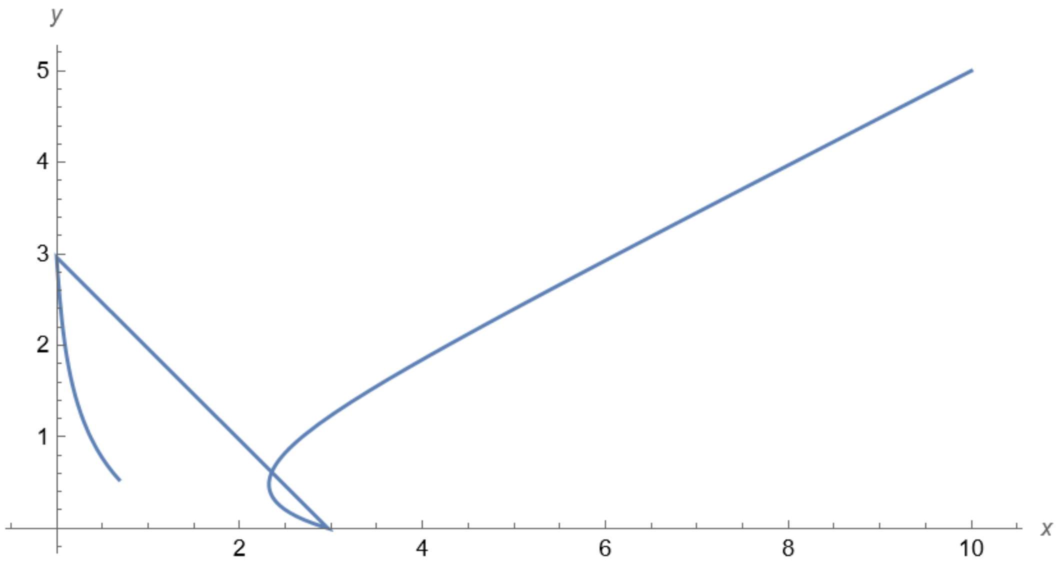


Figure 4.3: Sub-Riemannian geodesics on the (x, y) plane of $\mathbb{R}^3/(\mathbb{Z}/4\mathbb{Z})$

We see that in this figure the sub-Riemannian geodesic bounces between positive x and y axes. The Mathematica code to generate this plot is included in Appendix C.3.

4.3.3 (p,q)-Hopf action

This example was heavily inspired by [28]. Consider the (p, q)-Hopf action $\alpha : S^1 \times S^3 \to S^3$ defined by

$$e^{it} \cdot (u, v) = \left(e^{itp}u, e^{iqt}v\right).$$

In order to find a distribution on S^3 we consider the vector field along the action. The span of the vector field we obtain will be our vertical space, while its orthogonal the distribution. The vector field Z along this action is given by

$$\begin{aligned}Z &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} e^{it} \cdot (u, v) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\left(e^{ipt}u, e^{iqt}v\right) \\ &= (ipu, iqv).\end{aligned}$$

Consider $u = x_1 + ix_2$ and $v = x_3 + ix_4$, then (x_1, x_2, x_3, x_4) are coordinates on $\mathbb{R}^4$. We can write the normal vector as $N = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4}$. In a similar way, we can write

$$Z = -px_2\partial_{x_1} + px_1\partial_{x_2} - qx_4\partial_{x_4} + qx_3\partial_{x_4}.$$

The vertical space will be given by $\mathcal{V} = \operatorname{span}\{Z\}$, the horizontal space and distribution is given by $\mathcal{D} = \mathcal{V}^{\perp}$. In order to find two vector fields that span $\mathcal{D}$, we search for vector fields X, Y such that

$$\langle X, Z\rangle = \langle X, Z\rangle = \langle Y, Z\rangle = \langle Y, Z\rangle = 0$$

with respect to the standard inner product on $\mathbb{R}^4$. Finding X and Y is an underdetermined problem, so there is no unique solution we can find. One solution we can find is the following:

$$X = \begin{pmatrix} -x_1(x_3^2 + x_4^2) \\ -x_2(x_3^2 + x_4^2) \\ x_3(x_1^2 + x_2^2) \\ x_4(x_1^2 + x_2^2) \end{pmatrix} \qquad Y = \begin{pmatrix} qx_2(x_3^2 + x_4^2) \\ -qx_1(x_3^2 + x_4^2) \\ -px_4(x_1^2 + x_2^2) \\ px_3(x_1^2 + x_2^2) \end{pmatrix}. \tag{4.8}$$

For X and Y we find that $\langle X, Y\rangle = 0$, hence the frame is orthogonal. Let us now show that $\mathcal{D} = \operatorname{span}\{X, Y\}$ satisfies the equivariance condition (4.7) with respect to the (p, q)-Hopf action.

First, we compute the derivative of α in suitable coordinates. The action α at $x = (x_1, x_2, x_3, x_4) \in S^3$ is given by

$$\alpha(e^{it}, (x_1, x_2, x_3, x_4)) = \begin{pmatrix} \cos(pt)x_1 - \sin(pt)x_2 \\ \sin(pt)x_1 + \cos(pt)x_2 \\ \cos(qt)x_3 - \sin(qt)x_4 \\ \sin(qt)x_3 + \cos(qt)x_4 \end{pmatrix}$$

Consider the tangent vector $v = (v_{x_1}, v_{x_2}, v_{x_3}, v_{x_4})$ in T_xS^3 then the derivative is given by

$$\alpha_*(v_{x_1}, v_{x_2}, v_{x_3}, v_{x_4}) = \begin{pmatrix} \cos(pt)v_{x_1} - \sin(pt)v_{x_2} \\ \sin(pt)v_{x_1} + \cos(pt)v_{x_2} \\ \cos(qt)v_{x_3} - \sin(qt)v_{x_4} \\ \sin(qt)v_{x_3} + \cos(qt)v_{x_4} \end{pmatrix}.$$

Therefore, we find

$$\alpha_*\left(X_{((x_1,x_2,x_3,x_4))}\right) = \begin{pmatrix} -\cos(pt)x_1(x_3^2 + x_4^2) + \sin(pt)x_2(x_3^2 + x_4^2) \\ -\sin(pt)x_1(x_3^2 + x_4^2) - \cos(pt)x_2(x_3^2 + x_4^2) \\ \cos(qt)x_3(x_1^2 + x_2^2) - \sin(qt)x_4(x_1^2 + x_2^2) \\ \sin(qt)x_3(x_1^2 + x_2^2) + \cos(qt)x_4(x_1^2 + x_2^2) \end{pmatrix}$$

and,

$$X_{\alpha(x_1,x_2,x_3,x_4)} = \begin{pmatrix} -(\cos(pt)x_1 - \sin(pt)x_2)((\cos(qt)x_3 - \sin(qt)x_4)^2 + (\sin(qt)x_3 + \cos(qt)x_4)^2) \\ -(\sin(pt)x_1 + \cos(pt)x_2)((\cos(qt)x_3 - \sin(qt)x_4)^2 + (\sin(qt)x_3 + \cos(qt)x_4)^2) \\ (\cos(qt)x_3 - \sin(qt)x_4)((\cos(pt)x_1 - \sin(pt)x_2)^2 + (\sin(pt)x_1 + \cos(pt)x_2)^2) \\ (\sin(qt)x_3 + \cos(qt)x_4)((\cos(pt)x_1 - \sin(pt)x_2)^2 + (\sin(pt)x_1 + \cos(pt)x_2)^2) \end{pmatrix}$$
$$= \begin{pmatrix} -\cos(pt)x_1(x_3^2+x_4^2) + \sin(pt)x_2(x_3^2+x_4^2) \\ -\sin(pt)x_1(x_3^2+x_4^2) - \cos(pt)x_2(x_3^2+x_4^2) \\ \cos(qt)x_3(x_1^2+x_2^2) - \sin(qt)x_4(x_1^2+x_2^2) \\ \sin(qt)x_3(x_1^2+x_2^2) + \cos(qt)x_4(x_1^2+x_2^2) \end{pmatrix}$$

Since, $X_{\alpha(x)} = \alpha_*(X_x)$, we find that $\pi_*\left(X_{\alpha(x)}\right) = \pi_*(\alpha_*(X_x))$ as desired. Similarly, the result holds for Y_x. Therefore, we find that the distribution $\mathcal{D}_x$ is equivariant under the (p,q)-Hopf action, so we can define a sub-Riemannian structure on S^3/S^1.

Given the orthogonal frame $\{X, Y\}$ for $\mathcal{D}$ we want to find the sub-Riemannian geodesics. First, we show that no abnormal geodesics exist. Let us consider the 1-form $\xi = -px_2dx_1 + px_1dx_2 - qx_4dx_3 + qx_3dx_4$ on $\mathbb{R}^4$. If we consider the inclusion $i : S^3 \hookrightarrow \mathbb{R}^4$, then $i^*\xi$ defines a 1-form on S^3. We notice that

$$i^*\xi \wedge di^*\xi = i^*(\xi \wedge d\xi).$$

Since, $\xi \wedge d\xi \neq 0$, we find $i^*\xi$ is a contact form on S^3. We find that $\mathcal{D} = \ker(i^*\xi)$, therefore $\mathcal{D}$ is a contact distribution so by theorem 1.30 there are no abnormal geodesics for this sub-Riemannian structure.

Since the frame is orthogonal, we know from 1.18 that the Hamiltonian for

$$\lambda = (x_1, x_2, x_3, x_4, p_{x_1}, p_{x_2}, p_{x_3}, p_{x_4})$$

is given by

$$\begin{aligned} H(\lambda) &= \frac{1}{2}\langle\lambda, X\rangle^2 + \frac{1}{2}\langle\lambda, Y\rangle^2 \\ &= \frac{1}{2}(-x_1(x_3^2+x_4^2)p_{x_1} - x_2(x_3^2+x_4^2)p_{x_2} + x_3(x_1^2+x_2^2)p_{x_3} + x_4(x_1^2+x_2)p_{x_4})^2 \\ &\quad + \frac{1}{2}(qx_2(x_3^2+x_4^2)p_{x_1} - qx_1(x_3^2+x_4^2)p_{x_2} - px_4(x_1^2+x_2^2)p_{x_3} + px_3(x_1^2+x_2^2)p_{x_4})^2 \end{aligned}$$

The equations of motion from this Hamiltonian can be obtained from Hamilton's equations. Using these equations of motion we can find sub-Riemannian geodesics on S^3/S^1 under the (p,q)-Hopf action.

Let me also remark on an example that does not work. We know that a lens space $L(p,q)$ is not in an orbifold, but if we pick p and q not coprime, then the lens space action 4.5 is no longer free. Hence, we find an orbifold with singularities. We note that the distribution on $L(p,q)$ coming from the $\mathfrak{k} \oplus \mathfrak{p}$-structure on SU(2) is not equivariant with respect to the lens space action. In general, we have not been able to find a distribution on such a non-coprime lens space. But this is an interesting question for future research.

Now that we have some examples of sub-Riemannian structures on orbifolds, we can look at more general cases. In general, to our knowledge, it is not clear how to give each orbifold a sub-Riemannian structure. However, in some specific cases we can give construct sub-Riemannian structures on a larger class of examples at once. In the two forthcoming secitons, we will give constructions to define sub-Riemannian structures on cyclic 3-orbifolds and on compact orbifolds that admit a closed 2-form.

4.4 Contact orbifolds

In this section, we will sketch a result from [3] which states that on every closed cyclic developable 3-orbifold there exists a contact structure. This contact structure, as we have seen in 1.5, gives rise to a distribution in the tangent bundle of the orbifold. In this section we first need to study what contact structures on an orbifold would look like. In order to define a contact structure on each cyclic 3-orbifold, we will need a construction similar to the Giroux correspondendence for manifolds [29]. The Giroux correspondendence tells us that on each compact oriented 3-manifold we can define a so called open book decomposition, and that every open book decomposition admits a contact structure. We will first define open book decomposition and then give an analogous result for orbifolds. We do not prove the results in general, but we summarize and explain some of the steps as a pointer for future research.

In this section we work specifically on cyclic 3-orbifolds, which we define as follows.

Definition 4.14. *An orbifold Q with an atlas $\{\left(\widehat{U}_i, \Gamma_i, \varphi_i\right)\}$ is of* ***cyclic type*** *if the groups Γ_i are either trivial or cyclic* [4].

In what follows it will be useful to know the topological structure of the singular locus of a three-dimensional cyclic orbifold. Consider a 3-orbifold $\mathcal{Q}$ with underlying manifold Q.

Corollary 4.15. *[3, Chapter 2] The singular locus Σ of a cyclic orbifold $\mathcal{Q}$ is a link.*

In order to give an analogous construction to the Giroux correspondendence for manifolds, we need to define contact structures on orbifolds.

Definition 4.16. *[3, Definition 2.3.1] Let $\mathcal{Q}$ be an orbifold with orbifold atlas $\{\widehat{U}_i, \Gamma_i, \varphi_i\}_{i\in I}$. An* ***orbifold contact structure*** *ξ on $\mathcal{Q}$ consists of a family of Γ_i-invariant*[5] *contact forms on each chart, such that they define the same contact form on the overlap of charts. An orbifold with such a contact structure is called a* ***contact orbifold****.*

Example 4.17. As an example, we can consider the developable orbifold $\mathbb{R}^3/(\mathbb{Z}/n\mathbb{Z})$. This orbifold is obtained from the cyclic action α defined in section 4.3.2. The standard contact form on $\mathbb{R}^3$ is given by $\xi = dz + r^2 d\theta$. We can compute for cyclic coordinates on $\mathbb{R}^3$, for a vector field X on $\mathbb{R}^3$ that

$$(\alpha^*\xi)_{(r,\theta,z)}(X) = \xi_{\alpha(x)}(d\alpha_{(r,\theta,z)}X) = \xi_{(r,\theta,z)}(X).$$

Hence, ξ defines a contact structure on the cone $\mathbb{R}^3/(\mathbb{Z}/n\mathbb{Z})$. △

[4] A group is cyclic if it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some integer n.

[5] For a group Γ and Γ-action α on a manifold M, a differential form $\omega \in \Omega(M)$ is called Γ-invariant if $\alpha^*\omega = \omega$.

4.4.1 Open book decompositions

The next step towards a Giroux correspondence for orbifolds is to define open books. In this section, let us first consider open books on a smooth manifold M. For this we first need the definition of a fibration and the homotopy lifting property.

Definition 4.18. *[30, Definition 8.1] Definition 8.1. Let X, E and B be topological spaces $p : E \to B$ be a continuous map and let $A \subseteq X$ be a topological subspace. We say that p has the* ***homotopy lifting property*** *for (X, A) if for any commutative square of continuous maps*

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r] \arrow[d, "incl"'] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

there exists a continuous map $h : X \times [0, 1] \to E$ making both triangles commutative.

Definition 4.19. *[30, Definition 8.2] Let E and B be topological spaces, the map $p : E \to B$ is a* ***fibration*** *if it has the Homotopy lifting property with respect to all spaces X.*

As an example covering maps are fibrations. Moreover, the Hopf-fibration defined in section 2.3.1 is one of the most famous examples of a fibration. Using fibrations an open book decomposition of a smooth manifold can be defined as follows.

Definition 4.20. *Let M be a closed 3-manifold. An* ***open book decomposition*** *on M is a fibration $f : M/L \to S^1$, where $L \subseteq M$ is a link in M such that L has a tubular neighbourhood $L \times D^2$ with $f|_{L\times D^2 \setminus \{0\}}$ given by $f(\theta, r, \varphi) = \varphi$ with θ the coordiantes along L and (r, φ) the coordinates on D^2. The link is called the* ***binding*** *and the closure of the fibers $P_\varphi = \overline{f^{-1}(\varphi)}$ are called the* ***pages*** *with as boundery the binding, i.e. $\partial P_\varphi = L$.*

Examples of open book decompositions can be found in [29].

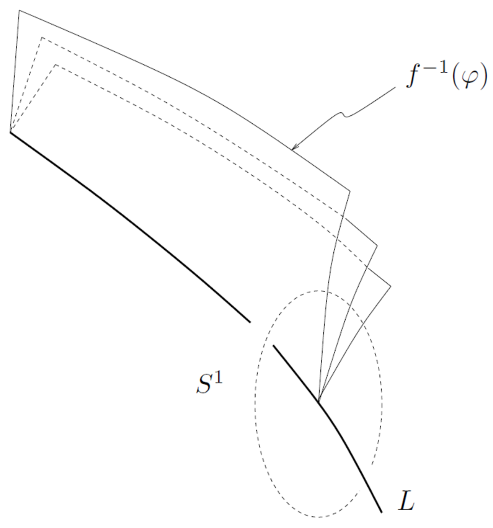


Figure 4.4: A sketch of an open book decomposition (picture taken from [31])

The notion of an open book decomposition can be extended to an orbifold as follows.

Definition 4.21. *[3, Definition 3.4.1.] An* ***open book decomposition*** *(B, f) of the cyclic 3-orbifold $\mathcal{Q}$ with singular link L, and a fibration $f : (Q \setminus B) \to S^1$ whose fibers $f^{-1}(\theta)$ are interior of compact two-orbifolds Σ_θ with boundary B.*

One of the key ingredients for defining a sub-Riemannian structure on a cyclic 3-orbifold is the following result.

Theorem 4.22. *[3, Theorem 3.4.1.] Every cyclic 3-orbifold has an open book decomposition*

These open book decompositions on an orbifold admit a contact structure, the definition is a slight variation on the definition for manifolds.

Definition 4.23. *[3, Definition 3.2.1.] An open book (B, f) on a contact 3-orbifold $(\mathcal{Q}, \xi)$ supports the contact structure if, after some isotopy of ξ through contact structures, there is a contact form α on $\mathcal{Q}$ such that α is positive on B and $d\alpha$ is the area form on every page.*

The second key ingredient for contact structures on cyclic 3-orbifolds is given by the next result.

Theorem 4.24. *[3, Theorem 4.1.2.] Every open book decomposition on a closed cyclic 3-orbifold supports a contact structure.*

Together, Theorem 4.22 and Theorem 4.24 imply the result summarized in the following corrolary.

Corollary 4.25. *[3, Corollary 4.1.5.] Every closed cyclic 3-orbifold admits a contact structure.*

Given a contact structure ξ on a cyclic closed 3-orbifold $\mathcal{Q}$, we can define the contact distribution $\mathcal{D} = \ker(\xi)$ on $\mathcal{Q}$. This yields a sub-Riemannian structure in general. Hence, we can formulate the following corollary.

Corollary 4.26. *Every cyclic closed developable 3-orbifold admits a contact sub-Riemannian distribution.*

Let us note that this is only an existence result, so not every contact form on $\mathcal{Q}$ will yield a distribution. This point can be illustrated by the following example.

Example 4.27. Consider the cyclic action $\alpha(r, \theta, z) = (r, \theta + \frac{2\pi}{n}, z)$ on $\mathbb{R}^3$ and the standard contact form $\xi = dz - \frac{1}{2}(xdy - ydx) = dz - r^2 d\theta$ on $\mathbb{R}^3$. In example 4.17 we have shown that $(\mathbb{R}^3/(\mathbb{Z}/n\mathbb{Z}), \xi)$ defines a contact orbifold. If we consider the distribution $\mathcal{D} = \ker(\xi)$, then we can find the frame

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}.$$

Rewriting this frame in cylindrical coordinates, we obtain

$$X = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta} - \frac{r\sin(\theta)}{2} \qquad \frac{\partial}{\partial z} Y = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta} - \frac{r\cos(\theta)}{2}\frac{\partial}{\partial z}.$$

However, checking the equivariance condition (4.7), we find

$$d\alpha\left(X_{(r,\theta,z)}\right) = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta)} - \frac{r\sin(\theta)}{2}$$

and

$$X_{\alpha(r,\theta,z)} = \cos\left(\theta + \frac{2\pi}{n}\right)\frac{\partial}{\partial r} - \frac{\sin\left(\theta + \frac{2\pi}{n}\right)}{r}\frac{\partial}{\partial\theta} - \frac{r\sin\left(\theta + \frac{2\pi}{n}\right)}{2}\frac{\partial}{\partial z}$$

Since the canonical quotient $\pi : \mathbb{R}^3 \to \mathbb{R}^3/(\mathbb{Z}/n\mathbb{Z})$ is only applied at the tangent level, we do not apply the quotient on the base space. In other words, we find $\pi_*\left(X_{\alpha(r,\theta,z)}\right) \neq \pi_*\left(d\alpha\left(X_{(r,\theta,z)}\right)\right)$. Therefore, we conclude that the distribution generated by X and Y is not well-defined on the orbifold $\mathbb{R}^3/(\mathbb{Z}/n\mathbb{Z})$ eventhough the distribution satisfied the relation at the contact form level. $\triangle$

Conclusion and outlook on further research

In this thesis we have studied sub-Riemannian structures on orbifolds. First of all we have given all relevant definitions and theorems related to sub-Riemannian geometry and orbifold theory. Next, we have defined what a sub-Riemannian structure on the regular part of a developable orbifold looks like. We found that if the quotient map $q : M \to M/\Gamma$ is a local diffeomorphism, then we can lift a distribution to M to a unique distribution on M/Γ. At the singular points this gave problems, since there the tangent space was no longer a vector space and the lift of curves from M to M/Γ was no longer well-defined. In order to fix this, we defined a sub-Riemannian structure around a singular point to be an equivariant distribution with respect to the action on M. We gave examples of sub-Riemannian structures on orbifolds obtained by rotation, reflections and the (p, q)-hopf action on $\mathbb{R}^3$. Moreover, we sketched a general existence result for sub-Riemannian structures on closed cyclic 3-orbifolds.

One problem that we run into with these results is that the construction of an equivariant distribution was done by hand. This means that we do not have a method to construct an equivariant distribution for a given orbifold. Especially, when working on higher dimensional orbifolds it would be essential to have a concrete way of finding an equivariant distribution. One proposed method is to use a method similar to how one finds a horizontal distribution on a principal G-bundle. Finding such a method and generating more examples would be good topic for future research. Using higher dimensional examples, one could also study the dimensions of the singular strata better and consider for what dimensions one could or could not define a sub-Riemannian distribution.

A second way we left unexplored, was to define sub-Riemannian geometry on groupoids. A groupoid is a generalization of a group in a more categorical language. It turns out that orbifolds are equivalent to 'proper étale groupoids'. More on this topic is for example explained in [32]. It turns out that sub-Riemannian structures are defined on Lie groupoids (as we found in [33]). An interesting direction of study could be to see if we can use the sub-Riemannian structure on Lie groupoids, to get sub-Riemannian structures on proper étale Lie groupoids, and hence on orbifolds. In some cases this has already been studied. One could for example define an Engel structure on a contact 3-orbifold in this way (see for example [34]).

Shortly said, we could easily have spend another year studying this topic, and we hope to do so in the future!

Appendix A

Differential Geometry

A.1 Symplectic structure on the cotangent bundle

If we want to consider T^*M as the phase space for some (sub-Riemannian) Hamiltonian dynamics, it is convenient to do this in the language of symplectic geometry. The goal of this section is to define a symplectic structure on T^*M. This construction is relevant if we talk about contact distributions and characteristic curves in section 1.5. The material discussed in this section is borrowed from [5, Section 4.2]. First, we define the tautological 1-form, which makes a correspondence between position and momentum variables.

Definition A.1. *For any covector $\lambda \in T^*M$ and $\omega \in T_\lambda(T^*M)$ the **tautological 1-form** $s \in \Omega^1(T^*M)$ given by $s : T^*M \to T_\lambda(T^*M)$ is defined by $s(\lambda) = s_\lambda$ such that*

$$\langle s_\lambda, \omega \rangle = \langle \lambda, \pi_* \omega \rangle.$$

Here, $\pi : T^*M \to M$ is the canonical projection.

From this 1-form we can make a closed 2-form by taking the exterior derivative.

Definition A.2. *The **canonical symplectic form** on T^*M is defined by*

$$\sigma = ds \in \Omega^2(T^*M).$$

Let us find an expression for this form in canonical coordinates (q, p) on T^*M. In these coordinates we have that $\lambda = \sum_{i=1}^n p_i dq_i$ and $v \in T_\lambda(T^*M)$ can be expressed as a linear combination of basis vectors of the tangent space, i.e.

$$v = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial p_i} + \beta_i \frac{\partial}{\partial q_i}.$$

Hence, we find $\pi_* v = \sum_{i=1}^n \beta_i \frac{\partial}{\partial q_i}$. We get

$$
\begin{aligned}
\langle s_\lambda, v\rangle &= \langle \lambda, \pi_* v\rangle \\
&= \langle \sum_{i=1}^n p_i dq_i, \sum_{i=1}^n \beta_i dq_i\rangle \\
&= \sum_{i=1}^n p_i \beta_i \\
&= \sum_{i=1}^n p_i \langle dq_i, v\rangle \\
&= \langle \sum_{i=1}^n p_i dq_i, v\rangle.
\end{aligned}
$$

Comparing the inner product, we find that $s_\lambda = \sum_{i=1}^n p_i dq_i$. So in coordinates we find that $s_\lambda = \lambda$, which is why we speak of the tautological 1-form. The canonical symplectic form is then given by

$$\sigma_\lambda := ds_\lambda = \sum_{i=1}^n dp_i \wedge dq_i. \tag{A.1}$$

It remains to show that σ is indeed a symplectic form. By construction σ is a closed 2-form ($d^2 = 0$). In coordinates we can see that σ is non-degenerate because it is equal to the standard symplectic form on $\mathbb{R}^{2n}$.

Remark A.3. In some cases it is convenient to start with only a basis $\eta_1, \ldots, \eta_n$ on T^*M. In this case we find $\lambda = s_\lambda = \sum_{i=1}^n \lambda_i \eta_i$. Then the canonical symplectic form becomes:

$$\sigma := ds = \sum_{i=1}^n d\lambda_i \wedge \eta_i + \lambda_i d\eta_i. \tag{A.2}$$

A.2 Poisson geometry

In this section we give the basic definition of a Poisson structure and Poisson structures on symplectic manifolds, coadjoint orbits and on products of Poisson manifolds.This material was found in [14] and [35], here one can also find the omitted proofs.

Definition A.4. *Let M be a smooth manifold. A **Poisson structure** on M is a $\mathbb{R}$-bilinear bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that for $f, g, h \in C^\infty(M)$ the following properties hold:*

- *Antisymmetry:* $\{f, g\} = -\{g, f\}$;
- *Jacobi identity:* $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$;
- *Leibniz rule:* $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

*The pair $(M, \{\cdot,\cdot\})$ is called a **Poisson manifold**.*

Using this definition, we can show that the Poisson bracket as defined in 1.11 is indeed a Poisson bracket on the contangent bundle. We give three constructions of Poisson structures on spaces we need throughout the thesis.

Example A.5 (Symplectic manifolds)**.** Let (M, ω) be a symplectic manifold. We can define a Poisson bracket $\{\cdot,\cdot\} : C^\infty \times C^\infty \to C^\infty$ on (M, ω) by

$$\{f, g\} = \omega(X_f, X_g).$$

Here the map $f \in C^\infty(M)$ induces a vector field $X_f \in \mathcal{X}(M)$ as the unique vector field X_f that satisfies $\iota_{X_f}\omega = -df$, for ι the interior multiplication. The uniqueness of this vector field is ensured by the nondegeneracy of the symplectic form.

Since, we have already seen that for any smooth manifold M the cotangent bundle T^*M is a symplectic manifold, we find that any cotangent bundle of a smooth manifold admits a Poisson structure. $\triangle$

Example A.6 (Coadjoint orbits)**.** Consider a Lie group G with a Lie algebra $\mathfrak{g}$ Its dual $\mathfrak{g}^*$ has a natural Poisson structure given by

$$\{f, g\}(\mu) = \langle \mu, [d_\mu f, d_\mu g] \rangle$$

for $\mu in \mathfrak{g}^*$ and $\langle \mu, X \rangle$ the pairing of the linear functional μ with the vector field X. This structure is called the **Lie-Poisson bracket**. $\triangle$

Example A.7 (Products of Poisson structures)**.** Consider two Poisson manifolds $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$. The manifold $M_1 \times M_2$ has a unique Poisson structure $\{\cdot,\cdot\}$. Consider $f, g \in C^\infty(M_1 \times M_2)$. For any $m_1 \in M_1$ and $m_2 \in M_2$ consider the inclusions $i_{m_1} : M_2 \hookrightarrow M_1 \times M_2$ defined by $i_{m_1}(x) = (m_1, x)$ and likewise the inclusion $i_{m_2} : M_1 \hookrightarrow M_1 \times M_2$ given by $i_{m_2}(x) = (x, m_2)$. Then the **product Poisson structure** is defined by

$$\{f, g\}(m_1, m_2) = \{f \circ i_{m_2}, g \circ i_{m_2}\}_1(m_1) + \{f \circ i_{m_1}, g \circ i_{m_1}\}_2(m_2).$$

$\triangle$

Appendix B

Group actions

In this appendix we define group actions and list a few important properties for studying orbifolds. We start by stating some definitions, for these we assume that G is a group (with unit e) and X is a space.

Definition B.1. *An **action** of G on X is a map*

$$\alpha : G \times X \to X$$

defined by $(g, x) \mapsto \gamma \cdot x$ such that, for all $x \in X$

1. *$\alpha(gh, x) = g \cdot (h \cdot x)$ for all $g, h \in G$*
2. *$\alpha(e, x) = x$*

*For such an action, X is called a **G-space**.*

If we impose some structure on X we can define several types of actions of G.

Definition B.2. *G acts on X **by homeomorphism (resp. diffeomorphism)** if for each $g \in G$ the map $x \mapsto g \cdot x$ is a homeomorphism (resp. diffeomorphism) of X.*

Definition B.3. *Let X and Y be G-spaces. A map $f : X \to Y$ is called **G-equivariant** if for all $x \in X$ and all $g \in G$, we have*

$$f(g \cdot x) = g \cdot f(x).$$

Definition B.4. *The **orbit** of a point $x \in X$ is the set*

$$G \cdot = \{g \cdot x : g \in G\} \subset X.$$

Using the action of G on X we can define an equivalence relation on X given by $x \sim y$ if and only if x and y belong to the same orbit, i.e. $y = hx$ for some $h \in G$. We can then consider the space of equivalence classes X/G.

We now list a few properties an action $\alpha : G \times X \to X$ can have.

Definition B.5. *The elements of G which leave an element $x \in X$ fixed form a subgroup called the **isotropy group** at x, and is denoted*

$$G_x := \{g \in G : gx = x\}.$$

Definition B.6. *A point $x \in X$ is called a **fixed point of the action** if the isotropy group at x is the whole group, i.e. $G_x = G$. Let us denote the set of all fixed points of the aciton by X^G.*

Definition B.7. *A subset $Y \subset X$ is called **G-invariant** if $g \cdot Y = Y$ for every $g \in G$.*

Definition B.8. *The action of G on X is **effective** if no element of the group, except the identity element, fixes all the elements of the space, i.e. $g \cdot x = x$ implies $g = e$.*

Definition B.9. *The action is called **free** if no point of X is fixed by an element of G other than the identity. In other words, $G_x = \{e\}$ for all $x \in X$.*

Definition B.10. *For an effective action $\alpha : G \times X \to X$ a point $x \in X$ is called **singular** if the the isotropy group is non-trivial. The collection of all singular points in X is denoted Σ_G and will be called the singular set.*

Notice that an effective action that is also free, hence cannnot have any singular points.

Definition B.11. *We call the action α **proper** if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (x, g \cdot x)$ is proper (i.e. premimage of every compact set is compact).*

One way to check an action is proper is via the following lemma.

Lemma B.12. *If G is a topological group endowed with the discrete topology, the action G on X is proper if and only if for any compact sets $K_1, K_2 \subseteq X$, the set*

$$\{g \in G : g \cdot K_1 \cap K_2 \neq \emptyset\}$$

is finite.

This allows us to state the following theorem.

Proposition B.13. *[20, Proposition 1.1] The action of a discrete group Γ on a locally compact topological space X is proper if and only if the following conditions hold*

i the quotient X/Γ is Haussdorf under the quotient topology

ii each $x \in X$ has finite isotropy group

iii each $x \in X$ has a Γ_x-invariant neighbourhood U such that

$$\{\gamma \in \Gamma : \gamma \cdot U \cap U \neq \emptyset\} = \Gamma_x.$$

Now that we have defined proper group actions, we will use this to prove an important fact about proper and free group actions yielding a local diffeomorphism as quotient map for the action. Quoting verbatim from [6] we define local diffeomorphisms as follows.

Definition B.14. *If M and N are smooth manifolds, a map $F : M \to N$ is a* ***local diffeomorphism*** *if every point $p \in M$ has a neighbourhood U such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.*

Local diffeomorphisms are useful for us because of the following result.

Lemma B.15. *[6, Proposition 3.6(d)] Let M and N be smooth manifolds and $F : M \to N$ a (local) diffeomorphism, then the tangent map $dF_p : T_pM \to T_{F(p)}N$ is a (local) isomorphism.*

This result is of great importance to define a sub-Riemannian structure on the regular part of an orbifold, in combination with the following result.

Theorem B.16. *Given a group Γ that acts freely on a manifold M, we know that the quotient map $q : M \to M/\Gamma$ is a local diffeomorphism.*

Appendix C

Mathematica Code

C.1 Heisenberg geodesics

Using NDSolve we find and plot the solutions to the system of Ordinary Differential Equations 1.5.

```
(*Numerically solving the system of ODEs for the Heisenberg distribution*)
s = NDSolve[{px'[t] == 1/2 (-py[t] - x[t]/2), py'[t] == 1/2 (px[t] - y[t]/2), x'[t] == px[t] - y[t]/2, y'[t] == py[t] + x[t]/2,
    z'[t] == 1/2 (py[t] + x[t]/2) x[t] - 1/2 (px[t] - y[t]/2) y[t], x[0] == 0, y[0] == 0, z[0] == 0, px[0] == 0.1, py[0] == 0.1},
   {px[t], py[t], x[t], y[t], z[t]}, {t, 0, 50}];
ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. s], {t, 0, 50}, AxesLabel → {x, y, z}, Axes → True]
```

Secondly, using DSolve, we find the analytic solutions.

```
In[1]:= sol = DSolve[{px'[t] == 1/2 (-py[t] - x[t]/2), py'[t] == 1/2 (px[t] - y[t]/2), x'[t] == px[t] - y[t]/2, y'[t] == py[t] + x[t]/2,
          z'[t] == 1/2 (py[t] + x[t]/2) x[t] - 1/2 (px[t] - y[t]/2) y[t]}, {px[t], py[t], x[t], y[t], z[t]}, t]
Out[1]= {{z[t] → t c1^2/2 + t c2^2/2 + t c2 c3/2 + t c3^2/8 - t c1 c4/2 + t c4^2/8 + c5 - 1/2 c1 c3 Cos[t] - 1/2 c2 c4 Cos[t] - 1/2 c1^2 Sin[t] - 1/2 c2^2 Sin[t] + 1/8 c3^2 Sin[t] + 1/8 c4^2 Sin[t],
         px[t] → 1/4 c4 (1 - Cos[t]) + 1/2 c1 (1 + Cos[t]) - 1/2 c2 Sin[t] - 1/4 c3 Sin[t], py[t] → 1/4 c3 (-1 + Cos[t]) + 1/2 c2 (1 + Cos[t]) + 1/2 c1 Sin[t] - 1/4 c4 Sin[t],
         x[t] → c2 (-1 + Cos[t]) + 1/2 c3 (1 + Cos[t]) + c1 Sin[t] - 1/2 c4 Sin[t], y[t] → c1 (1 - Cos[t]) + 1/2 c4 (1 + Cos[t]) + c2 Sin[t] + 1/2 c3 Sin[t]}}
```

C.2 Reflection geodesics

We use NDSolve to solve and plot the solution to the Hamilton Equations for an orbifold generated by reflection $(x, y, z) \mapsto (x, -y, -z)$ on $\mathbb{R}^3$. This code was used to plot Figure 4.2. The WhenEvent command is used to plot the sub-Riemannian geodesics on the quotient $\mathbb{R}^3/(\mathbb{Z}/2\mathbb{Z})$.

```
(*Numerically finding Geodesic flow of Reflection on R^3*)
s = NDSolve[{px'[t] == 0, py'[t] == -px[t] × py[t] - (pz[t]^2) * y[t], pz'[t] == 0, x'[t] == px[t] + py[t] * y[t], y'[t] == px[t] * y[t] + py[t],
   z'[t] == pz[t] * y[t]^2, x[0] == 1, y[0] == z[0] == -1, px[0] == -0.2, py[0] == pz[0] == 0.2}, {px[t], py[t], x[t], y[t], z[t]}, {t, 10}];
ParametricPlot3D[Evaluate[{x[t], Abs[y[t]], Abs[z[t]]} /. s], {t, 0, 10}, AxesLabel → {x, y, z}]
ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. s], {t, 0, 10}, AxesLabel → {x, y, z}]
ParametricPlot[Evaluate[{x[t], Abs[y[t]]} /. s], {t, 0, 10}, AxesLabel → {x, y}]
ParametricPlot[Evaluate[{Abs[y[t]], Abs[z[t]]} /. s], {t, 0, 10}, AxesLabel → {y, z}]
ParametricPlot[Evaluate[{x[t], Abs[z[t]]} /. s], {t, 0, 10}, AxesLabel → {x, z}]
```

C.3 Rotation geodesics

We use NDSolve to solve and plot the solution to the Hamilton Equations for an orbifold generated by reflection $(r, \theta, z) \mapsto (r, \theta + \frac{2\pi}{4}, z)$ on $\mathbb{R}^3$. This code was used to plot Figure 4.3. The WhenEvent command is used to plot the sub-Riemannian geodesics on the quotient $\mathbb{R}^3/(\mathbb{Z}/4\mathbb{Z})$.

```
(*Finding sub-Riemannian geodesics for Rotation action on R^3*)
In[2]:= H = (1/2) * (pz[t] - (1/(x[t]^2 + y[t]^2)) * (-y[t] * px[t] + x[t] * py[t]))^2 +
   (1/2) * ((x[t] * px[t]) / (Sqrt[x[t]^2 + y[t]^2]) + (y[t] * py[t]) / (Sqrt[x[t]^2 + y[t]^2]))^2

s = NDSolve[{px'[t] == -D[H, x[t]], py'[t] == -D[H, y[t]], pz'[t] == -D[H, z[t]], x'[t] == D[H, px[t]], y'[t] == D[H, py[t]],
    z'[t] == D[H, pz[t]], x[0] == 10, y[0] == 5, z[0] == 0.1, px[0] == -8, py[0] == -10, pz[0] == 1,
    WhenEvent[{x[t] == 0, y[t] == 0}, {x[t], y[t]} → {y[t], x[t]}]}, {px[t], py[t], pz[t], x[t], y[t], z[t]}, {t, 1}];
ParametricPlot[Evaluate[{Abs[x[t]], Abs[y[t]]} /. s], {t, 0, 1.1}, AxesLabel → {x, y}]
ParametricPlot3D[Evaluate[{Abs[x[t]], Abs[y[t]], z[t]} /. s], {t, 0, 1}, AxesLabel → {x, y, z}]
```

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