university of groningen
faculty of science and engineering
mathematics and applied mathematics

Topics in Betweenness and Enriched Categories

Master's Project Mathematics
August 2023

Author: Ruben Mud
First supervisor: Prof. Dr. Oliver Lorscheid
Second assessor: Prof. Dr. Jaap Top

Abstract

By making an adjustment in the axiomatization of betweenness, we find new but very general examples of betweenness spaces. In particular it turns out that enriched categories come equipped with a betweenness relation and that betweenness spaces can be thought of as enriched categories. In this line of thought we construct functors between the category of betweenness spaces and the category of enriched categories equipped with some suitable notion of morphism. We introduce the concepts of betweenness and enriched categories formally and accompanied by various examples.

Contents

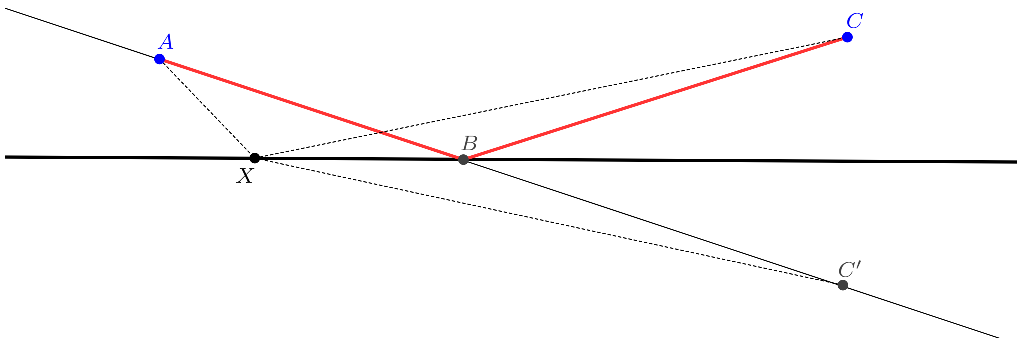
1 Prologue

1.1 Acknowledgment

My special thanks to Arthemy Kiselev for planting a seed, Alef Sterk for making it sprout, Jaap Top for saving it from early frost, Oliver Lorscheid for his many useful remarks and suggestions towards the soil. On a personal note, I want to thank Esther de Boer. Without these people I would have graduated with a very different thesis.

1.2 Introduction

(i) Betweenness relations were first formally defined by Moritz Pasch [3] in 1882. However, betweenness existed in nature before it was described in literature. The greek mathematician Heron of Alexandria solved an optimization problem by (perhaps implicitly) making use of betweenness around 60 AD. The problem is as follows. Given two points A and C on one side of a line, find a point B on the straight line that minimizes $AB + BC$. The solution is to reflect the point C in the straight line to find a point C'. Draw the line AC' and find its intersection with the straight line. This intersection point is the solution denoted B.



To see this is indeed the solution, note that for an arbitrary point X on the straight line we have $XC = XC'$. Because B is *between* A and C', we have $AB + BC' = AC' \leq AX + XC'$. This shows that $AB + BC \leq AX + XC$ for all points X on the straight line. Thus B is indeed the solution.

What is important about the work of Pasch is that he defines betweenness as a property instead of implicitly making use of it when thinking of proofs. This opened the door to reason about betweenness *per se*. Later on, various definitions were invented to capture betweenness as a property in increasingly abstract contexts. For example, a good overview of betweenness in ordered geometry is given in [4]. Moreover, many of these newly found axioms and how they interrelate have been systematically studied in [5].

Recent publications [6, 7, 8, 9] show that the topic of betweenness is still an active research topic. In fact, research is currently carried out on the modal logic of betweenness at the Institute for Logic, Language and Computation (ILLC) in Amsterdam. Furthermore, grant applications that are visible online suggest the structure of betweenness preserving maps is being investigated in the Czech Republic. Various applications of betweenness exist. For instance, betweenness relations can be used to construct a notion of line for which a generalization of the Sylvester Gallai theorem is proven in [10]. Furthermore, betweenness relations are used to model causality. There is a notion of causal betweenness introduced by Reichenbach which is characterized in [11]. Betweenness relations are even finding applications in artificial intelligence [12] and they are taken into consideration for the purpose of modelling social choice theory [13]. Because there are so many examples of betweenness, one concludes there are either many different notions, or there is a very general framework that captures the property.

(ii) In this thesis we contribute an axiomatization of betweenness that allows us to view it as a property intrinsic to the general theory of enriched categories. Explicitly, every enriched category comes with a betweenness relation, and every betweenness relation has an associated enriched category. We prove this correspondence is functorial. In order to achieve this result, we make use of the following key observation.

> The commonly used *minimality* axiom of betweenness is too restrictive. It states that "$[a, b, a]$ implies $a = b$". Instead we introduce the weaker and more symmetric version "$[a, b, a]$ and $[b, a, b]$ imply $a = b$".

This allows us to define betweenness on arbitrary enriched categories in terms of their composition morphism. The payoff is that the intuitive clarity of betweenness could aid the use of ordered geometry in enriched categories. Since enriched categories are highly complex mathematical structures they are in literature often considered with simplifications. Thus betweenness may prove itself a usefull mnemonic device. In the other direction, if we are thinking of composition morphisms as generalized triangle inequalities, then by analogy we can think of hom-objects as generalized distances. Since betweenness geometry is a geometry without reference to measurement, this thesis arguably labours to find abstract forms of measurement for arbitrary betweenness spaces by interpreting them as enriched categories. However, these two views only concern the object part of our functors. The morphism part describes how seemingly unrelated structures can have the same betweenness geometry.

Among the new contributions in this thesis we find

+ A symmetric version of the minimality axiom for betweenness in section 2.1.
+ A notion of betweenness for ideals in a commutative ring in section 2.2.
+ A topological characterization of betweenness preserving functions in section 2.5.
+ Lattice betweenness in terms of presheaves in section 3.3.
+ A notion of morphism of enriched categories for non-fixed monoidal category in section 4.6.
+ A notion of betweenness on enriched categories in section 4.5 and the functoriality of this interpretation in section 4.7.
+ The associated enriched category of a betweenness space and the functoriality of this interpretation in section 5.

(iii) The document is structured as follows. In chapter 2 we explain what betweenness relations are, how its defining axioms interact and we provide various examples. After betweenness preserving functions are defined, we show an application of the existence of a particular betweenness preserving function acting on metric lattices. We moreover show a characterization of betweenness preserving functions in terms of topology.

In chapter 3 we explain how Lawvere's observation can be used to interpret betweenness categorically. We then explain how the composition in cartesian closed categories is internalized and as a result we obtain that the familliar notion of lattice betweenness is a property of internal composition morphisms.

In chapter 4 we define enriched categories accompanied with examples. After defining the needed enriched categorical notions, we formulate a betweenness relation on arbitrary enriched categories and see that the minimality axiom is closely related to the Cantor-Schroeder-Bernstein theorem. We proceed to define morphisms between arbitrary enriched categories that preserve the betweenness structure. These morphisms can be seen as a generalization of $\mathcal{V}$-functors as described in [15]. In contrast to the notion of $\mathcal{V}$-functor, our morphisms do not require its domain and codomain to be enriched over the same monoidal category. This generalization allows us to capture familiar notions of morphism such as the usual notion of functor, Lipschitz functions and geometric morphisms.

In chapter 5 we show that betweenness spaces themselves can be seen as enriched categories, that the betweenness on these associated enriched categories is compatible with the underlying betweenness space and that these shifts in interpretation are functorial. We conclude with a suggestion for further research.

2 Betweenness Relations

2.1 Definition and Examples

Betweenness relations are a geometrically intuitive notion that is axiomatized differently by various authors. Typically, for some space X a ternary relation $B \subseteq X^3$ (the betweenness relation) is defined. Then we say that b is between a and c whenever $(a, b, c) \in B$, and write $[a, b, c]_B$ to indicate this. If there is no risk of confusion with respect to which betweenness relation b is between a and c, then we may forget about the subscript and write $[a, b, c]$ instead of $[a, b, c]_B$. We axiomatize betweenness by means of the following properties.

Definition 2.1. A ternary relation $B \subseteq X^3$ on a set X is called a betweenness relation whenever the relation B satisfies:

(B1) Symmetry: $[a, b, c]_B$ if and only if $[c, b, a]_B$,

(B2) Reflexivity: $[a, b, b]_B$ holds for all $a, b \in X$,

(B3) Minimality: if $[a, b, a]_B$ and $[b, a, b]_B$, then $a = b$,

(B4) Transitivity: if $[a, b, c]_B$ and $[a, c, d]_B$, then $[a, b, d]_B$.

If in addition the betweenness relation B satisfies:

(B5) Cancellation : if $[a, b, c]_B$ and $[a, c, d]_B$, then $[b, c, d]_B$,

then the betweenness relation is called metrizable. There is another property some betweenness relations can have, namely that of separation.

(B6) Separation: $[a, b, c]_B$ and $[a, c, b]_B$, if and only if $b = c$.

Remark 2.2. In literature, the minimality condition is often formulated to state "$[a, b, a]_B$ implies $a = b$". We will refer to this notion as *strong minimality*. Our notion of minimality is slightly weaker. We will later see an interpretation of betweenness for which we require both $[a, b, a]_B$ and $[b, a, b]_B$ to hold, before $a = b$ can be concluded. This requirement is not too exotic since the notion of strong minimality can be proven as a lemma from the axioms defined above.

Lemma 2.3. *Suppose a ternary relation* $[\cdot, \cdot, \cdot] \subseteq X^3$ *satisfies axioms* B1, B2, B3 *and* B5*, then* $[a, b, a]$ *implies* $a = b$*.*

Proof. Suppose that $[a, b, a]$ holds, then from B2 we know that $[b, a, a]$ holds and from B1 we have $[a, a, b]$. Combining this with B5, we have $[a, a, b]$ and $[a, b, a]$ imply $[b, a, b]$. This means that both $[a, b, a]$ and $[b, a, b]$ hold so that by B3 we have $a = b$.

Lemma 2.4. *There are the following dependencies*

1) B6 *implies* B2*,*

2) B1 *and* B6*, imply* B3*,*

3) B2*,* B3 *and* B5*, imply* B6*.*

Proof. If B6 holds, then automatically B2 holds. Namely $b = b$ and therefore $[a, b, b]_B$ holds for all a and b. If in addition, B1 holds, then $[b, a, a]_B$ implies $[a, a, b]_B$. This way we retrieve B3, because if $[a, b, a]_B$ holds, then in combination with $[a, a, b]_B$ one uses B6 to conclude that $a = b$. Lastly, whenever B2, B3 and B5 hold, then one can deduce B6. In particular, if $b = c$ then $[a, b, c]_B$ and $[a, c, b]_B$ hold by B2. For the converse suppose that $[a, b, c]_B$ and $[a, c, b]_B$ hold, then by B5 one obtains $[b, c, b]_B$ and $[c, b, c]_B$ so that by using B3 we find $b = c$.

From the above definition, the geometric intuition of betweenness may not be immediately clear. To clarify, we consider the following examples.

Example 2.5. This example is due to Pasch [3] and dates 1882. It is the first formal definition of betweenness in literature. Suppose there are four colinear points A, B, C and D and suppose that D is not on the segment $\overline{AB}$. Then C is between A and B whenever the segment $\overline{AB}$ goes through the segment $\overline{CD}$.

This example is already very general, as it only depends on the notions of point, line and segment. One straightforward specialization would be to consider points in the Euclidean plane with the usual lines. If one has colinear points A, B, C in the plane $\mathbb{R}^2$, one sees that C is between A and B in the sense of Pasch precisely when $|A-C|+|C-B| = |A-B|$. At the time, the idea of length of a segment was not new, however the general notion of metric only came later. In 1906, Fréchet introduced the notion of écart (semi-metric). Afterwards, in 1928, as part of a larger project on the foundation of geometry inside the theory of metric spaces, Menger introduces the notion of metric betweenness in [16]. It is stated as follows:

Example 2.6. For a metric space (X, ρ) and points $a, b, x \in X$, it is said that x is between a and b whenever $a \neq x \neq b$ and:

$$\rho(a, x) + \rho(x, b) = \rho(a, b).$$

Thus the triangle inequality is an equality for these points. In particular, when the metric space is a finite undirected graph, and the metric is the shortest path metric, then x being in between a and b means that x lies on a shortest path from a to b. The requirement that $a \neq x \neq b$ is often omitted. Metric betweenness then satisfies all the properties B1 to B6. The properties B1 to B3 are immediate and B4, B5 can be seen because $[a, x, b]$ and $[a, y, x]$ imply that

$$\begin{aligned}\rho(a,b) &= \rho(a,x) + \rho(x,b),\\ &= \rho(a,y) + \rho(y,x) + \rho(x,b),\\ &\geq \rho(a,y) + \rho(y,b),\\ &\geq \rho(a,b).\end{aligned}$$

The above inequalities must be equalities, because certainly $\rho(a,b) = \rho(a,b)$. It follows that $[a, y, b]$ and $[y, x, b]$ hold. Since B2, B3 and B5 hold, also B6 holds. We call property B5 cancellation since we used the cancellation property of the monoidal operator $+$ to prove that it holds.

Example 2.7. A lattice $(X, \wedge, \vee, \leq)$ is called a metric lattice whenever it admits a positive valuation, i.e. a function $v : X \to \mathbb{R}$ that satisfies:

$$v(a) + v(b) = v(a \wedge b) + v(a \vee b),$$

and

$$a < b \text{ implies } v(a) < v(b).$$

Such a valuation induces a metric on the lattice, namely $\rho(a,b) := v(a \vee b) - v(a \wedge b)$. Glivenko proved [17, 18] that in a metric lattice, a point x is metrically between a and b if and only if

$$(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b).$$

As this relation can be formulated without reference to any metric, Pitcher and Smiley study it as a definition for betweenness in more general lattices (possibly without valuation). They show in [5, Thm. 9.1] that lattice betweenness satisfies property B4 if and only if the lattice is modular. Smiley later establishes a criterion in [19] for when metric and lattice betweenness are the same relation in arbitrary lattices equipped with a metric.

Example 2.8. Bankston introduces in [7] a notion of road system which yields a betweenness relation. For a set X, a road system $\mathcal{R}$ is defined as a collection of subsets of X, such that 1) $\{a\} \in \mathcal{R}$ for all $a \in X$, and 2) for all $a, b \in X$ there exists at least one road $R \in \mathcal{R}$ such that $a, b \in R$. Any road system induces a betweenness relation in the following way. Given a road system $(X, \mathcal{R})$, a point x is said to be between points a and b whenever any road R containing both a and b must also contain x. Informally this means that one cannot avoid x when traveling from a to b. Formally we write

$$[a, x, b]_{\mathcal{R}} \quad \text{if and only if} \quad x \in R(a,b), \quad \text{where} \quad R(a,b) := \bigcap_{a,b \in R \in \mathcal{R}} R.$$

Bankston proves that a betweenness relation is induced by a road system if and only if it satisfies the following axioms:

(R1) Symmetry: $[a, x, b]$ if and only if $[b, x, a]$,

(R2) Reflexivity: $[a, b, b]$,

(R3) Strong minimality: $[a, b, a]$ implies $b = a$,

(R4) Strong transitivity: if $[a, x, b]$ and $[a, y, b]$ and $[x, z, y]$, then $[a, z, b]$.

Indeed, properties R1 and R2 are the same as properties B1 and B2. Furthermore, B3 is implied by R3. Transitivity B4 is implied by strong transitivity R4 combined with R1, R2. Furthermore, Bankston points out that for distributive lattices, lattice betweenness satisfies the strong transitivity axiom by [5, Thm. 9.3] and that it is consequently induced by a road system.

2.2 Ideal Betweenness

Let R be a commutative ring. An ideal A is a subgroup of the additive group of R such that $xA \subseteq A$ for all $x \in R$. Here $xA = \{xa : a \in A\}$. For ideals A, B their product is an ideal defined by $AB := \{\sum_i a_i b_i : a_i \in A, b_i \in B\}$. Their quotient ideal is defined by $(A : B) := \{x \in R : xB \subseteq A\}$.

Lemma 2.9 (triangle inequality)**.** *For all ideals A, B, C belonging to the commutative ring R, we have the inclusion $(A : B)(B : C) \subseteq (A : C)$.*

Proof. To see the inclusion holds, take a point $x \in (A : B)(B : C)$. Then we can write $x = \sum_n y_n z_n$ as a finite sum where $y_n \in (A : B)$ and $z_n \in (B : C)$. If $c \in C$, then $xc = \sum_n y_n z_n c$. Here $z_n c \in B$ and so $y_n(z_n c) \in A$. Consequently we must have that $xc \in A$. In particular, $xC \subseteq A$ and hence $x \in (A : C)$. This proves the result. ∎

Using the analogy between the triangle inequality for ideals and for metric spaces, we can define a betweenness relation on the set of ideals of R as follows.

Definition 2.10. We say that an ideal B is between A and C and write $[A, B, C]$ whenever

$$(A : B)(B : C) = (A : C) \quad \text{and} \quad (C : B)(B : A) = (C : A).$$

The thus defined ternary relation is called ideal betweenness.

Proposition 2.11. *Ideal betweenness is a betweenness relation.*

Proof. We verify that this indeed defines a betweenness relation. By construction property B1 holds. Moreover, we have $(B : B) = R$ for all ideals B, as by definition of ideal, $xB \subseteq B$ for all $x \in R$. This means that both $(B : B)(B : A) = (B : A)$ and $(A : B)(B : B) = (A : B)$ hold. In other words, $[A, B, B]$ holds for all ideals A and B and hence property B2 is satisfied. To see property B3, suppose that $[A, B, A]$ holds, we have $(A : B)(B : A) = (A : A)$. Then we certainly have $R = (A : B)(B : A) \subseteq (A : B) \cap (B : A)$. Because $1 \in R$, we have that $1A \subseteq B$ and $1B \subseteq A$ so that $A = B$. Lastly to see that property B4 holds, Take ideals A, B, C, D that satisfy both $[A, B, C]$ and $[A, C, D]$. We want to show that $[A, B, D]$ holds. There are two equalities to prove, but their proof is the same. That is, we only need to prove the inclusion $(A : D) \subseteq (A : B)(B : D)$ as the other inclusion follows from Lemma 2.9. Indeed, $(A : D) = (A : C)(C : D) = (A : B)(B : C)(C : D) \subseteq (A : B)(B : D)$ by Lemma 2.9. Conclude that ideal betweenness is indeed a betweenness relation.

Besides betweenness in terms of quotient ideals, we can also consider the betweenness relation induced by the lattice structure of the ideals of R. If we denote $\text{Ideal}(R)$ for the set of ideals in R, we see that $(\text{Ideal}(R), \cap, +, \subseteq)$ defines a lattice. Indeed, ideals are partially ordered by set inclusion. For two ideals A and B, their intersection $A \cap B$ is again an ideal. This is the largest ideal contained in both A and B with respect to the partial ordering. Furthermore, their sum $A + B = \{a + b : a \in A, b \in B\}$ is again an ideal. It is the smallest ideal containing both A and B. This lattice satisfies the modular law, meaning $A+(B\cap C) = (A+B)\cap C$ whenever $A \subseteq C$. Consequently, by [5, Thm. 9.1] lattice betweenness satisfies axiom B4. This means that lattice betweenness for the ideals in a commutative ring defines a betweenness relation in the sense of Definition 2.1. Comparing this to our notion of ideal betweenness we have the following results.

Proposition 2.12. *Lattice betweenness is stronger than ideal betweenness.*

Proof. Suppose that an ideal B is between ideals A and C, in the sense of lattice betweenness. Then we know they satisfy the relation

$$(A \cap B) + (B \cap C) = B = (A + B) \cap (B + C).$$

To verify that B is between A and C in the sense of ideal betweenness, we have to verify that $(A : B)(B : C) = (A : C)$ and $(C : B)(B : A) = (C : A)$. By symmetry it is enough to verify only one of these equalities. It is enough to prove only the inclusion $(A : C) \subseteq (A : B)(B : C)$. Since $B = (A \cap B) + (B \cap C)$ we may rewrite

$$(A : B) = (A : (A \cap B) + (B \cap C)) = (A : A \cap B) \cap (A : B \cap C) = (A : B \cap C).$$

Thus if $x \in (A : C)$, then $x(B \cap C) \subseteq xC \subseteq A$. Consequently, it follows that $(A : C) \subseteq (A : B \cap C) = (A : B)$. Similarly, we may rewrite $B = (A+B) \cap (B+C)$ to obtain

$$(B : C) = ((A + B) \cap (B + C) : C) = (A + B : C) \cap (B + C : C) = (A + B : C).$$

Thus if $x \in (A : C)$, we know $xC \subseteq A \subseteq A + B$. This means that $x \in (A + B : C)$ so that $(A : C) \subseteq (A + B : C) = (B : C)$. Consequently, $(A : C) \subseteq (A : B)(B : C)$ which concludes the proof.

Proposition 2.13. *If lattice betweenness and ideal betweenness coincide, then two ideals A and B are coprime precisely whenever $(A : B) = A$ and $(B : A) = B$.*

Proof. Suppose that $A + B = 1$, then since $(A \cap (A + B)) + ((A + B) \cap B) = A + B = (A + (A + B)) \cap ((A + B) + B)$ we have that 1 is between A and B in the sense of lattice betweenness. Consequently, 1 is between A and B in the sense of ideal betweenness. We thus find $A = (A : 1)(1 : B) = (A : B)$. The argument to show that $B = (B : A)$ is similar. Conversely, if 1 is between A and B in the sense of ideal betweenness, then it must be between A and B in the sense of lattice betweenness. This means that $(A \cap 1) + (1 \cap B) = 1 = (A + 1) \cap (1 + B)$ so that A and B are coprime.

Corollary 2.14. *Ideal betweenness and lattice betweenness need not coincide.*

Proof. Consider a polynomial ring $k[X, Y]$. We have that $(X : Y) = X$ and $(Y : X) = Y$. However $X + Y \neq 1$. The contrapositive of Proposition 2.13 proves the two betweenness relations are distinct.

Corollary 2.15. *Ideal betweenness and lattice betweenness need not be distinct.*

Proof. Consider a commutative ring whose ideals are totally ordered by inclusion, in particular, a valuation ring. We show for such a ring that ideal betweenness implies lattice betweenness. Suppose we are given ideals A, B, C that satisfy both $(A : B)(B : C) = (A : C)$ and $(C : B)(B : A) = (C : A)$. Then because the ideals are totally ordered by inclusion, we have $A \subseteq C$ or $C \subseteq A$. By symmetry, we may assume without loss of generality that $A \subseteq C$. Consequently, $(C : A) = 1$. This means that

$$1 = (C : A) = (C : B)(B : A) \subseteq (C : B) \cap (B : A).$$

Therefore must have that $(C : B) = 1 = (B : A)$. Consequently $A \subseteq B \subseteq C$. One can now verify that the equations $(A \cap B) + (B \cap C) = B = (A + B) \cap (B + C)$ hold, so that B is also between A and C in the sense of lattice betweenness. Indeed, lattice and ideal betweenness are equivalent for these types of rings.

2.3 Betweenness Preserving Mappings

In the previous section we have defined the notion of betweenness relation. To study the structure of sets with a betweenness relation, we consider structure preserving morphisms.

Definition 2.16. Let (X, B_X) and (Y, B_Y) be betweenness spaces. A function $f : X \to Y$ is said to be betweenness preserving whenever:

$$[a, b, c]_X \quad \text{implies} \quad [f(a), f(b), f(c)]_Y.$$

In other words, for points $a, b, c \in X$, if b lies between a and c, then $f(b)$ lies between $f(a)$ and $f(c)$ in Y.

With this above notion of structure preserving morphism, there is a category denoted **Bet**. Objects of **Bet** are tuples (X, B) called betweenness spaces. Here X is a set, and B is a betweenness relation on the set X. The morphisms consist of betweenness preserving functions. Indeed, given a pair of betweenness preserving functions, $f : X \to Y$, and $g : Y \to Z$, their composition $gf : X \to Z$ is also betweenness perserving. To see this, note that $[a, b, c]_X$ implies $[f(a), f(b), f(c)]_Y$ which in turn implies $[g(f(a)), g(f(b)), g(f(c))]_Z$. Moreover, for each betweenness space (X, B) there is an identity morphism, $\mathrm{id}_X : X \to X : x \mapsto x$. This arrow identically preserves the betweenness relation.

Many examples of betweenness preserving mappings are known. Here are some examples.

Example 2.17. Let V be a vector space over the complex numbers $\mathbb{C}$. Suppose we say a point $x \in V$ is between points a and b whenever $x = \lambda a + (1 - \lambda)b$ for some $\lambda \in [0, 1]$. Then any linear map $f : V \to V$ is betweenness preserving because, $f(x) = f(\lambda a + (1 - \lambda)b) = \lambda f(a) + (1 - \lambda)f(b)$.

Example 2.18. An isometry $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) satisfies by definition that $d_X(a, b) = d_Y(f(a), f(b))$. Such a function f is then betweenness preserving. Namely, if $[a, b, c]_X$, then

$$d_Y(f(a), f(c)) = d_X(a, c) = d_X(a, b) + d_X(b, c) = d_Y(f(a), f(b)) + d_Y(f(b), f(c)).$$

Thus $[f(a), f(b), f(c)]_Y$.

Example 2.19. Consider the set X consisting of sequences of 0's and 1's. We have that $X := \{x = (x_i)_i : x_i \in \{0, 1\} \text{ for all } i \in \mathbb{N}\}$ is a metric space when equipped with the metric $\rho(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$. Then x is between a and b precisely whenever $|a_i - x_i| + |x_i - b_i| = |a_i - b_i|$ for all $i \in \mathbb{N}$. This in turn happens precisely whenever $x_i \in \{a_i, b_i\}$ for all $i \in \mathbb{N}$. Consider the shift map $S : X \to X$ acting on sequences such that

$$S : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, \dots).$$

Then S is betweenness preserving, since $[a, x, b]$ means $x_i \in \{a_i, b_i\}$ for all $i \in \mathbb{N}$. So then it follows that $S(x)_i = x_{i+1} \in \{a_{i+1}, b_{i+1}\} = \{S(a)_i, S(b)_i\}$ for all $i \in \mathbb{N}$. Thus we have that $[S(a), S(x), S(b)]$ holds. Note however that S is not an isometry since given $y, z \in X$ such that $y - z = (0, 1, 1, 1, \dots)$ we have that $\rho(y, z) = 1$ while $\rho(Sy, Sz) = 2$.

A semi-metric space is a metric space that need not satisfy the property "$d(a, b) = 0$ implies $a = b$". We can define equivalence classes $[a] := \{b \in X : d(a, b) = 0\}$. Then the set $[X] := \{[a] \subseteq X : a \in X\}$ with the metric $d'([a], [b]) := d(a, b)$ defines a metric space. Metric betweenness can be defined for semi-metric spaces and in this context there is the following result.

Theorem 2.20 (Smiley [19])**.** *If $(X, d, \wedge, \vee, \leq)$ is a semi-metric space which is also a lattice, then metric betweenness $[\cdot, \cdot, \cdot]_M$ and lattice betweenness $[\cdot, \cdot, \cdot]_L$ coincide in X if and only if:*

i) For every $a, b, c \in X$ the inequalities $a \leq b \leq c$ implies that $[a, b, c]_M$.

ii) For every $a, b \in X$, $d(a, b) = d(a \vee b, a \wedge b)$ and $d(a, a \vee b) = d(b, a \wedge b)$.

The conditions i) and ii) hold if and only if for each $a^* \in M$ the functional $m[a] := d(a \vee a^*, a^*) - d(a^*, a \wedge a^*)$ is a sharply positive modular functional and X is a metric lattice with metric $d(a, b) = m[a \vee b] - m[a \wedge b]$

Example 2.21. Theorem 2.20 gives a setting in which metric and lattice betweenness coincide. Namely, the setting of metric lattices. So the identity function $\mathrm{id}_X : (X, B_M) \to (X, B_L) : x \mapsto x$ is a betweenness preserving isomorphism. Note that apriori it is not clear if these betweenness spaces are isomorphic.

2.4 Application

One reason to study betweenness preserving mappings is to find different formulations of the same geometric structure. This can provide interesting interactions between the interpretations. In particular, suppose that $(X, d, \wedge, \vee, \leq)$ is both a complete metric space and a distributive lattice, and suppose that metric betweenness and lattice betweenness coincide. Then we can prove a best approximation theorem for closed convex sets.

Definition 2.22. Given a set X equipped with a betweenness relation B_X, we say a subset $S \subseteq X$ is convex, whenever $a, b \in S$ and $[a, x, b]_X$ imply $x \in S$.

Define the closed metric interval of points between a and b as follows,

$$[a,b] := \{x \in X : d(a,x) + d(x,b) = d(a,b)\}.$$

We can reformulate the definition of convex to the requirement that $a, b \in S$ implies $[a,b] \subseteq S$. There is the following Lemma:

Lemma 2.23. *Let (X,d) be a metric space and let $a, b, x \in X$, then we have*

$$d(x,[a,b]) \geq \frac{1}{2}(d(a,x) + d(x,b) - d(a,b)),$$

where $d(x,[a,b]) := \inf\{d(x,y) : y \in [a,b]\}$.

Proof. Suppose that $y \in [a,b]$, then we have

$$\begin{aligned} 2d(x,y) &\geq |d(a,x) - d(a,y)| + |d(x,b) - d(b,y)|, \\ &\geq |d(a,x) + d(x,b) - d(a,y) - d(y,b)|, \\ &= d(a,x) + d(x,b) - d(a,b). \end{aligned}$$

Then taking the infimum over $y \in [a,b]$ provides the result. ∎

The lower bound in the above lemma is called the Gromov product. We will denote it by $(a,b)_x := 1/2(d(a,x) + d(x,b) - d(a,b))$.

Lemma 2.24. *Suppose $(X, d, \wedge, \vee, \leq)$ is a distributive metric lattice, then for all points $a, b, x \in X$ we have $d(x,[a,b]) = (a,b)_x$.*

Proof. We will first show there is a point $y \in [a,b] \cap [b,x] \cap [a,x]$ and then its existence implies the result. This point need not exist in arbitrary metric spaces (just remove it from the set and we have a metric space in which the property fails). However, for a distributive lattice we can construct a point which is between a, b, x in terms of lattice betweenness. Let $y := (a \wedge b) \vee (b \wedge x) \vee (x \wedge a)$. We can verify that $y \in [a,b] \cap [b,x] \cap [x,a]$. To do this we will use that:

$$\begin{aligned} (a \wedge b) \vee (b \wedge x) \vee (x \wedge a) &= (a \wedge (b \vee x)) \vee (b \wedge x), \\ &= (a \vee (b \wedge x)) \wedge ((b \vee x) \vee (b \wedge x)), \\ &= (a \vee (b \wedge x)) \wedge (b \vee x), \\ &= (a \vee b) \wedge (b \vee x) \wedge (x \vee a). \end{aligned}$$

Now we see $a \wedge y = a \wedge (a \vee b) \wedge (b \vee x) \wedge (x \vee a) = a \wedge (b \vee x)$, and similarly $b \wedge y = b \wedge (a \vee x)$. Then $(a \wedge y) \vee (y \wedge b) = ((a \wedge (b \vee x)) \vee (b \wedge (a \vee x))) = (a \wedge b) \vee (b \wedge x) \vee (x \wedge a) = y$. The other equalities are dual or symmetric to this case. So indeed there exists a point $y \in [a,b] \cap [b,x] \cap [x,a]$. For this y we have $y \in [a,b]$ and hence $d(x,y) \geq d(x,[a,b])$. But notice that $d(x,y) = (a,b)_x$. ∎

Theorem 2.25 (Proximinality)**.** *Let (X, d) be a complete metric space such that for all $a, b, x \in X$ we have $d(x, [a, b]) = (a, b)_x$. Suppose that $S \subseteq X$ is closed and convex. Then for all $x \in X$ there exists a unique $s \in S$ such that $d(x, s) = d(x, S)$.*

Proof. Let us denote $d := d(x, S)$. Since S is convex, we know for all $a, b \in S$ that $[a, b] \subseteq S$. Since $d = \inf\{d(x, s) : s \in S\}$ we obtain that $d \leq d(x, [a, b]) = (a, b)_x$. Moreover we have the existence of a sequence $(s_n)_{n\in\mathbb{N}}$ in S such that $d(x, s_n) \to d$. Therefore we know that for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d(x, s_n) < d + \varepsilon/2$ and $d(x, s_m) < d + \varepsilon/2$. We must obtain

$$\begin{aligned}\frac{d(s_n, s_m)}{2} &= \frac{d(x, s_n) + d(x, s_m)}{2} - (s_n, s_m)_x,\\ &= \frac{d(x, s_n) + d(x, s_m)}{2} - d(x, [s_n, s_m]),\\ &\leq \frac{d(x, s_n) + d(x, s_m)}{2} - d,\\ &< \frac{d + d + \varepsilon}{2} - d,\\ &= \frac{\varepsilon}{2}.\end{aligned}$$

We see that (s_n) is a cauchy sequence and because X is complete this sequence converges to some $s \in X$. By the fact that S is closed we have that $s \in S$. Because $d(x, s) \leq d(x, s_n) + d(s_n, s) \to d$ we see that $d(x, s) \leq d$ but because $d = \inf\{d(x, s) : s \in S\}$ we find $d(x, s) = d(x, S)$ so that a best approximation exists.

To prove the approximation is unique, suppose that $d(x, s) = d = d(x, t)$ for some $s, t \in S$. Then we see

$$\begin{aligned}\frac{d(s, t)}{2} &= \frac{d(x, s) + d(x, t)}{2} - (s, t)_x,\\ &= \frac{d(x, s) + d(x, t)}{2} - d(x, [s, t]),\\ &\leq \frac{d(x, s) + d(x, t)}{2} - d,\\ &= \frac{d + d}{2} - d = 0.\end{aligned}$$

So that $s = t$. ∎

Definition 2.26. Let (X, d) be a metric space as above and suppose that $S \subseteq X$ is a closed convex set. We define the map $P_S : X \to S$ so that $P_S(x)$ is the unique element in S satisfying the property $d(x, P_S(x)) = d(x, S)$. We refer to this map as the projection onto S. If it is clear from the context what set we project onto, then we just write P instead of P_S.

Lemma 2.27. *Let $S \subseteq X$ be closed and convex. Then $P_S(x) \in [x, s]$ for all $s \in S$.*

Proof. We have that $(P_S(x), s)_x = d(x, [P_S(x), s]) = d(x, P_S(x))$ for all $s \in S$. Writing this out in full gives

$$d(x, P_S(x)) = \frac{1}{2}(d(x, P_S(x)) + d(x, s) - d(s, P_S(x))).$$

This is equivalent to $d(x, P_S(x)) + d(P_S(x), s) = d(x, s)$ which is equivalent to the desired result $P_S(x) \in [x, s]$. ∎

Proposition 2.28. *Let $S \subseteq X$ be closed and convex, then the projection map $P_S : X \to S$ is continuous.*

Proof. For arbitrary $x, y \in X$ we have $Py \in S$. Therefore $Px \in [x, Py]$ by Lemma 2.27. In other words we have:

$$\begin{aligned} d(Px, Py) &= d(x, Py) - d(x, Px), \\ &\leq d(x, y) + d(y, Py) - d(x, Px), \\ &= d(x, y) + d(y, S) - d(x, S). \end{aligned}$$

By symmetry we obtain $d(Px, Py) \leq d(x, y) + |d(x, S) - d(y, S)| \leq 2d(x, y)$ so that P is continuous. ∎

Proposition 2.29. *Suppose $A \subseteq X$ is a convex set. Then its closure $\overline{A}$ is convex.*

Proof. Take $a_0 \in A$ and $a \in \overline{A}$. There exits a sequence $(a_n)_{n \in \mathbb{N}}$ so that $a_n \to a$. Let us define $f_n(x) := (a_0, a_n)_x$ then f_n is continuous and converges uniformly to $f(x) := (a_0, a)_x$. Namely, for all $\varepsilon > 0$ there exists N such that $n \geq N$ implies $d(a_n, a) < \varepsilon$. But then we also have:

$$\begin{aligned} \|f_n - f\|_\infty &= \sup_{x \in X} |(a_0, a_n)_x - (a_0, a)_x|, \\ &\leq d(a_n, a) < \varepsilon. \end{aligned}$$

Because A is convex we have that $[a_0, a_n] \subseteq A$. Therefore we obtain $f_n(x) = (a_0, a_n)_x = d(x, [a_0, a_n]) \geq d(x, \overline{A})$. Consequently, we have $f(x) \geq d(x, \overline{A})$ so that $f(x) = 0$ implies that $x \in \overline{A}$, but $f(x) = 0$ if and only if $x \in [a_0, a]$ so that $[a_0, a] \subseteq \overline{A}$. Repeating the argument for $a_0 \in \overline{A}$ shows that $\overline{A}$ is convex. ∎

2.5 Topology Induced by Betweenness

Now that we have a notion of betweenness preserving mapping, we will look closer at what structure these maps actually preserve. Given a set with a betweenness relation (X, B) we find that each point $x \in X$ induces a pre-order on X given by

$$a \leq_x b \quad \text{if and only if} \quad [x, a, b]_B.$$

To see this defines a pre-order note that axiom B2 states that $[x, a, a]_B$ holds for all $x, a \in X$. Therefore we obtain $a \leq_x a$ for all $a \in X$. This proves reflexivity. Moreover, from axiom B4 we have that $[x, a, b]_B$ and $[x, b, c]_B$ imply $[x, a, c]$. This means that $a \leq_x b$ and $b \leq_x c$ imply $a \leq_x c$. This proves transitivity so that for each $x \in X$ we have a pre-order $\leq_x$. This is the reason for referring to B2 and B4 as *reflexivity* and *transitivity* respectively.

Remark 2.30. If in addition the betweenness relation satisfies the separation axiom B6, then each pre-order $\leq_x$ is in fact a partial order. Namely $a \leq_x b$ and $b \leq_x a$ both state that $[x, a, b]$ and $[x, b, a]$ so that B6 implies $a = b$. We will see why we refer to B6 as *separation*, and not as *antisymmetry*.

The family of partial orders induced by a betweenness relation that satisfies axiom B6 has been studied in [20]. There is the following Theorem:

Theorem 2.31 ([20] Thm. 5)**.** *Let $\{\leq_x : x \in X\}$ be a family of partial orders on a set X that satisfies $y \leq_x z$ if and only if $y \leq_z x$ for all $x, y, z \in X$. Then the ternary relation $B_X := \{(x, a, b) \in X^3 \mid a \leq_x b\}$ defines a betweenness relation.*

Proof. The fact that each binary relation $\leq_x$ is a partial order states that B_X satisfies axioms B2, B4 and B6. The property $y \leq_x z$ if and only if $y \leq_z x$ is precisely B1. Then Lemma 2.4 together with B1 and B6 provides B3. ∎

The above construction does not work in the weaker setting where one only considers a family of pre-orders. For example, take $X = \{a, b\}$ and let the pre-orders $\leq_a$ and $\leq_b$ be given by $\leq_a = \leq_b = \{(a, a), (b, b), (a, b), (b, a)\}$. Then B_X constructed as above, is given by:

$$B_X := \{(a, a, a), (b, b, b), (a, b, b), (b, a, a), (a, a, b), (b, b, a), (a, b, a), (b, a, b)\}.$$

This ternary relation does satisfy B1, B2 and B4, however, it does not satisfy B3, since we do have $[a, b, a]$ and $[b, a, b]$ but $a \neq b$. This means that it is not a betweenness relation in the sense of Definition 2.1. Since we consider betweenness relations that do not necessarily satisfy B6, we cannot guarantee that the induced pre-orders are partial orders. This also means that we cannot characterize betweenness the same way as was done in [20]. However, for some distinguished point $x \in X$, each such pre-order $\leq_x$ still defines a topological space (X, τ_x) when we set

$$\tau_x := \{U \subseteq X : \text{if } a \in U \text{ and } a \leq_x b \text{ then } b \in U\}.$$

In words, the topology τ_x is the collection of upward closed subsets of X.

Lemma 2.32. *For a fixed $x \in X$, write $(\uparrow a)_x := \{b \in X : a \leq_x b\}$. Then the family $\{(\uparrow a)_x : a \in X\}$ defines a basis for the topology τ_x.*

Proof. Each set $(\uparrow a)_x$ is open in τ_x because if $b \in (\uparrow a)_x$ and $b \leq_x c$, then we have $[x, a, b]$ and $[x, b, c]$. It follows by axiom B4 that $[x, a, c]$ holds and consequently $c \in (\uparrow a)_x$. Furthermore, for any $U \in \tau_x$ we have $\bigcup_{a \in U} (\uparrow a)_x = U$. Indeed, $a \in (\uparrow a)_x$ because $[x, a, a]$ holds by axiom B2. It follows that $U \subseteq \bigcup_{a \in U} (\uparrow a)_x$. To see the reverse inclusion, if $b \in (\uparrow a)_x$ and $a \in U$, then we have by definition that $a \leq_x b$ so that by upward closedness of U, we find $b \in U$. Therefore we have $(\uparrow a)_x \subseteq U$ for all $a \in U$. We must conclude that $\bigcup_{a \in U} (\uparrow a)_x = U$.

Remark 2.33. Axiom B6 is satisfied precisely whenever each topological space τ_x satisfies the separation axiom T_0. To see this, take $a \neq b$, then for each x we have $\neg[x, a, b]$ or $\neg[x, b, a]$ which states $a \notin (\uparrow b)_x$ or $b \notin (\uparrow a)_x$. But we do have that $a \in (\uparrow a)_x$ and $b \in (\uparrow b)_x$ so that τ_x satisfies T_0. Conversely if axiom B6 is not satisfied, then there exist x, a, b with $a \neq b$ such that $[x, a, b]$ and $[x, b, a]$ hold. In particular $a \in (\uparrow b)_x$ and $b \in (\uparrow a)_x$. Since the upwards closed sets form a basis, τ_x does not satisfy the separation axiom T_0.

Definition 2.34. An Alexandrov topological space (X, τ) is a set X equipped with a family τ of subsets of X, that satisfies the following properties.

1) $\emptyset, X \in \tau$,

2) if $\{U_i\}_{i \in I} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$ and $\bigcap_{i \in I} U_i \in \tau$.

Proposition 2.35. *The family τ_x defines an Alexandrov topology.*

Proof. Note that $\emptyset \in \tau_x$ vacuously. Moreover, we have that $X \in \tau_x$ since if $a \in X$ and $a \leq_x b$, then $(x, a, b) \in B \subseteq X^3$ and therefore $b \in X$. Suppose that $\{U_\lambda : \lambda \in \Lambda\}$ is some family of open sets in τ_x. Then $\bigcup U_\lambda \in \tau_x$ because if $a \in \bigcup U_\lambda$ and $a \leq_x b$, then there exists some λ_0 such that $a \in U_{\lambda_0}$. Since U_{λ_0} is upward closed, we have $b \in U_{\lambda_0}$ and hence $b \in \bigcup U_\lambda$. The proof for arbitrary intersections is similar.

Proposition 2.36. *Let (X, B_X) and (Y, B_Y) be sets equipped with a betweenness relation. For a function $f : X \to Y$, the following are equivalent:*

1) The function f is betweenness preserving,

2) f is monotone with respect to $\leq_x$ and $\leq_{f(x)}$ for all $x \in X$,

3) f is continuous with respect to τ_x and $\tau_{f(x)}$ for all $x \in X$.

Proof. We first prove that *1)* is equivalent to *2)*. Suppose that f is betweenness preserving, fix some $x \in X$. Then $a \leq_x b$ holds if and only if $[x, a, b]_X$. Since f preserves betweenness, we obtain $[f(x), f(a), f(b)]_Y$ which is equivalent to $f(a) \leq_{f(x)} f(b)$. Similarly, suppose f is monotone with respect to $\leq_x$ and $\leq_{f(x)}$ for all $x \in X$. Then $[a, b, c]_X$ implies $b \leq_a c$ and hence $f(b) \leq_{f(a)} f(c)$. This implies that $[f(a), f(b), f(c)]_Y$ holds.

We prove that *2)* is equivalent to *3)*. Suppose that f is monotone, Fix some $x \in X$ and take an open set $U \in \tau_{f(x)}$. We want to see that $f^{-1}(U)$ is open in τ_x. Pick some $y \in f^{-1}(U)$, suppose we have that $y \leq_x z$. Then $f(y) \leq_{f(x)} f(z)$. Since $f(y) \in U$, we have that $f(z) \in U$ because U is open (and hence upwards closed). This means that $z \in f^{-1}(U)$ so that $f^{-1}(U)$ is open in (X, τ_x) meaning that f is $\tau_x/\tau_{f(x)}$ continuous. Conversely, suppose that f is $\tau_x/\tau_{f(x)}$ continuous. We want to see that f is monotone. Suppose we have $a, b \in X$ such that $a \leq_x b$. Since the set $(\uparrow f(a))_{f(x)} = \{y : f(a) \leq_{f(x)} y\}$ is open in $(Y, \tau_{f(x)})$, we therefore also have that $f^{-1}((\uparrow f(a))_{f(x)})$ is open in (X, τ_x). Now because $[f(x), f(a), f(a)]_Y$ holds, we know that $a \in f^{-1}((\uparrow f(a))_{f(x)})$. Because we're given that $a \leq_x b$, we know (by definition of τ_x) that $b \in f^{-1}((\uparrow f(a))_{f(x)})$. This means that $f(b) \in (\uparrow f(a))_{f(x)}$ so that $f(a) \leq_{f(x)} f(b)$. Conclude that f is monotone.

3 Betweenness in a Categorical Setting

3.1 Motivation

In section 2.2 we've seen a notion of triangle inquality for quotient ideals that can be used to define a betweenness relation on the ideals of a commutative ring. While this particular notion of ideal betweenness is not described in previous literature, the idea of a more general version of triangle inequality is not new.

In [14], Lawvere points out that the triangle inequality for metric spaces

$$d(b, c) + d(a, b) \geq d(a, c),$$

looks very similar to the composition of morphisms in a category

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C).$$

He continues to describe how metric spaces can be seen as a category enriched over a monoidal category and proves that many notions in metric spaces have meaningfull interpretations in this categorical context. It turns out that this categorical context provides a very general example of betweenness relation. In particular, we will exploit the composition law (the triangle inequality) to define a betweenness relation for enriched categories. Intuitively, given objects A, B, C in a category $\mathcal{C}$, we can think of B to lie between A and C whenever every morphism $A \longrightarrow C$ factors through B. This reminds vaguely of the notion of betweenness induced by road systems, "every road from A to C goes via B". Equivalently we can require that every morphism from A to C is the composition of a pair of morphisms $(g, f) \in \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B)$. In other words, the composition is surjective. By the axiom of choice, every surjective function has a section. In the analogy between composition and triangle inequality, we can think of the inequality $d(b, c) + d(a, b) \geq d(a, c)$ as a composition morphism. Then this morphism has a section precisely whenever $d(b, c) + d(a, b) \leq d(a, c)$ so that b is between a and c in terms of metric betweenness. We will generalize this further in a later section. Before we get into the topic of enriched categories, we first provide some categorical preliminaries.

3.2 Exponentials and Internalization

Definition 3.1. For objects X and Y in a category $\mathcal{C}$ with finite limits, an *exponential* is an object Y^X in $\mathcal{C}$ with an arrow $\mathrm{ev} : Y^X \times X \longrightarrow Y$ having the property that for every object $A \in \mathcal{C}$ and arrow $h : A \times X \longrightarrow Y$ there is an unique arrow $H : A \longrightarrow Y^X$ that makes the diagram

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\text{ev}} & Y \\ {\scriptstyle H\times 1}\uparrow & \nearrow_{h} & \\ A \times X & & \end{array}$$

commute. The arrow H and h are called each others exponential transpose.

Definition 3.2. A category $\mathcal{C}$ is said to be *Cartesian closed* whenever it has finite products, a terminal object (denoted I), and all exponentials. The latter meaning that for any pair of objects X and Y, there exists an object Y^X in $\mathcal{C}$.

Proposition 3.3. *In a Cartesian closed category $\mathcal{C}$, for each object X of $\mathcal{C}$, we have a functor $(-)^X$.*

Proof. The object part is given by $Y \mapsto Y^X$. For a morphism $f : Y \longrightarrow Z$ we have a uniquely induced arrow f^X making the diagram

$$\begin{array}{ccc} Z^X \times X & \xrightarrow{\text{ev}} & Z \\ {\scriptstyle f^X\times 1}\uparrow & & \uparrow{\scriptstyle f} \\ Y^X \times X & \xrightarrow[\text{ev}]{} & Y \end{array}$$

commute. By uniqueness, we have $(gf)^X = g^X f^X$ and $(\text{id}_Y)^X = \text{id}_{(Y^X)}$.

The previous proposition shows we have a natural transformation

$$\text{ev} : (-)^X \times X \Longrightarrow 1_{\mathcal{C}}.$$

Proposition 3.4. *The functor $(-)^X$ is a right adjoint to the functor $(-) \times X$.*

Proof. Let $m_{A,Y} : \text{Hom}_{\mathcal{C}}(A \times X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(A, Y^X)$, be the map that sends an arrow h to its exponential transpose $m(h)$. Then this map is injective, because if $m(h) = m(g)$, then

$$h = \text{ev} \circ (m(h) \times 1) = \text{ev} \circ (m(g) \times 1) = g.$$

Moreover, the map is surjective, because if $H : A \to Y^X$, then denoting $h = \text{ev} \circ (H \times 1)$, we have $H = m(h)$ because H is the unique arrow that makes the following diagram commute

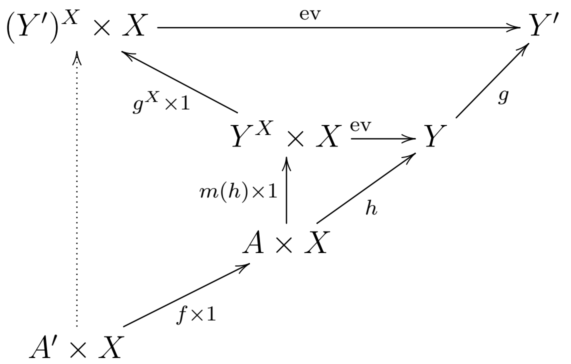
$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\text{ev}} & Y \\ {\scriptstyle H\times 1}\uparrow & \nearrow_{h} & \\ A \times X. & & \end{array}$$

Thus $m_{A,Y}$ is indeed a bijection.

Naturality means that for arrows $f : A' \to A$ and $g : Y \to Y'$, the following square commutes,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A \times X, Y) & \xrightarrow{m_{A,Y}} & \mathrm{Hom}_{\mathcal{C}}(A, Y^X) \\ \Big\downarrow{\scriptstyle \mathrm{Hom}(f\times 1, g)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(f, g^X)} \\ \mathrm{Hom}_{\mathcal{C}}(A' \times X, Y') & \xrightarrow[m_{A',Y'}]{} & \mathrm{Hom}_{\mathcal{C}}(A', (Y')^X). \end{array}$$

To see this square indeed commutes, pick an arrow $h : A \times X \to Y$. Then in the diagram



the top square commutes by naturality of $\mathrm{ev} : (-)^X \times X \Rightarrow 1_{\mathcal{C}}$ and the inner triangle commutes by the exponential property and the fact that $m(h)$ is the exponential transpose of h. Thus we find $m(g \circ h \circ (f \times 1)) = g^X \circ m(h) \circ f$ as this choice of transpose makes the outer triangle commute. This is precisely saying that m is natural in A and Y.

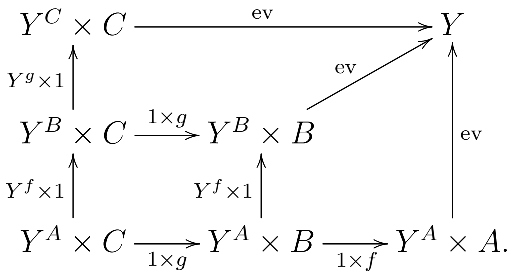
Proposition 3.5. *Let $\mathcal{C}$ be Cartesian closed, we have a functor* $Y^{(-)} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}$.

Proof. Given an arrow $f : A \longrightarrow B$ in $\mathcal{C}^{\mathrm{op}}$ there is an arrow $f : B \longrightarrow A$ in $\mathcal{C}$. Thus from the exponential property we know there is a uniquely induced arrow Y^f that makes the following diagram commute.

$$\begin{array}{ccc} Y^B \times B & \xrightarrow{\mathrm{ev}} & Y \\ \Big\uparrow{\scriptstyle Y^f \times 1} & & \Big\uparrow{\scriptstyle \mathrm{ev}} \\ Y^A \times B & \xrightarrow[1\times f]{} & Y^A \times A. \end{array}$$

From this we find that if $1_A : A \to A$ is the identity arrow, then $Y^{1_A} = 1_{Y^A}$

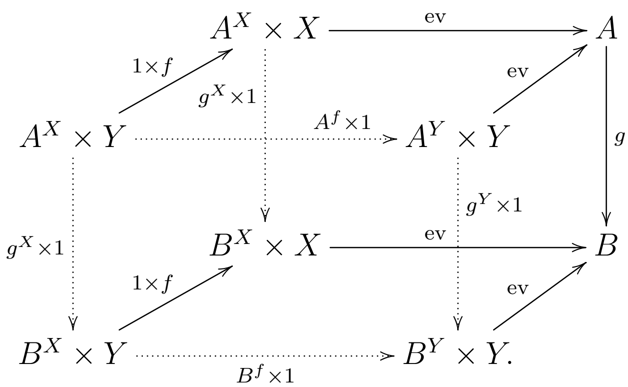
Moreover, given $g : B \to C$ in $\mathcal{C}^{\mathrm{op}}$, we have a commutative diagram



From the outside square, we see that $Y^{gf} = Y^g Y^f$. ✐

Corollary 3.6. *There is a bifunctor* $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$ *whose action on objects is given by* $(X, Y) \mapsto Y^X$.

Proof. The action on objects is well defined. To define the action on morphisms, take an arrow $(f, g) : (X, A) \to (Y, B)$ in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, and consider the following diagram



The back square of this cube commutes since g^X is the unique arrow that makes it commute. The right square commutes since g^Y is the unique arrow that makes it commute. The left square commutes trivially. The top square commutes since A^f is the unique arrow that makes it commute. The bottom square commutes since B^f is the unique arrow that makes it commute. Since the arrows $g^Y \circ A^f$ and $B^f \circ g^X$ are both the exponential transpose of the same arrow, we see that they are equal. Thus the front square commutes. For the action on morphisms, the bifunctor sends the arrow (f, g) to $g^Y \circ A^f = B^f \circ g^X$. This is indeed a bifunctor because it preserves identity and moreover, for a pair of morphisms $(f_1, g_1) : (X, A) \longrightarrow (Y, B)$ and

$(f_2, g_2) : (Y, B) \longrightarrow (Z, C)$, we have a commutative diagram

$$\begin{array}{ccccc} A^X & \xrightarrow{A^{f_1}} & A^Y & \xrightarrow{A^{f_2}} & A^Z \\ {\scriptstyle g_1^X}\downarrow & & {\scriptstyle g_1^Y}\downarrow & & {\scriptstyle g_1^Z}\downarrow \\ B^X & \xrightarrow{B^{f_1}} & B^Y & \xrightarrow{B^{f_2}} & B^Z \\ {\scriptstyle g_2^X}\downarrow & & {\scriptstyle g_2^Y}\downarrow & & {\scriptstyle g_2^Z}\downarrow \\ C^X & \xrightarrow{C^{f_1}} & C^Y & \xrightarrow{C^{f_2}} & C^Z. \end{array}$$

By commutativity we have that $(g_2 g_1)^Z \circ A^{(f_2 f_1)} = (g_2^Z \circ B^{f_2}) \circ (g_1^Y \circ A^{f_1})$. Indeed this defines a functor.

Remark 3.7. Because I is a terminal object, there is an isomorphism $l : I \times X \cong X$ (and similarly an isomorphism $r : X \times I \cong X$). So there is a natural bijection

$$\begin{array}{ccccc} \mathrm{Hom}(X, Y) & \longrightarrow & \mathrm{Hom}(I \times X, Y) & \longrightarrow & \mathrm{Hom}(I, Y^X) \\ f & \longmapsto & f \circ l & \longmapsto & m(f \circ l). \end{array}$$

Therefore, the functor $\mathrm{Hom}(I, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ can be seen as a map that sends Y^X to the set of morphisms $X \to Y$. This way it makes sense to think of exponentials as generalized hom-sets. For this reason the object Y^X is called the *internal hom* of X and Y. Going further, there is an induced arrow $M_{XYZ} : Z^Y \times Y^X \to Z^X$ which is the exponential transpose of the arrow

$$(Z^Y \times Y^X) \times X \xrightarrow{a} Z^Y \times (Y^X \times X) \xrightarrow{1 \times \mathrm{ev}} Z^Y \times Y \xrightarrow{\mathrm{ev}} Z,$$

where a denotes the associativity isomorphism. This arrow M_{XYZ} is called the internal composition. Given two arrows $f : X \to Y$ and $g : Y \to Z$, there are corresponding arrows:

$$\overline{f} : I \longrightarrow Y^X, \qquad \overline{g} : I \longrightarrow Z^Y, \qquad \langle \overline{g}, \overline{f} \rangle : I \longrightarrow Z^Y \times Y^X.$$

Thus we have an arrow

$$M_{XYZ} \langle \overline{g}, \overline{f} \rangle : I \longrightarrow Z^X.$$

To see this arrow corresponds to the composition $gf \in \mathrm{Hom}(X, Z)$, we note that it corresponds to $\mathrm{ev}(M_{XYZ}\langle \overline{g}, \overline{f}\rangle \times 1_X)$. We compute:

$$\begin{aligned}
\mathrm{ev}(M_{XYZ}\langle \overline{g}, \overline{f}\rangle \times 1) &= \mathrm{ev}(M_{XYZ} \times 1)(\langle \overline{g}, \overline{f}\rangle \times 1)\\
&= \mathrm{ev}(1 \times \mathrm{ev})a(\langle \overline{g}, \overline{f}\rangle \times 1)\\
&= \mathrm{ev}(1 \times \mathrm{ev})a((\overline{g} \times \overline{f}) \times 1)l^{-1}\\
&= \mathrm{ev}(1 \times \mathrm{ev})(\overline{g} \times (\overline{f} \times 1))l^{-1}\\
&= \mathrm{ev}(\overline{g} \times \mathrm{ev}(\overline{f} \times 1))l^{-1}\\
&= \mathrm{ev}(\overline{g} \times fl)l^{-1}\\
&= \mathrm{ev}(\overline{g} \times 1)(1 \times fl)l^{-1}\\
&= gl(1 \times fl)l^{-1}\\
&= gfl : I \times X \to Z.
\end{aligned}$$

This means that $M_{XYZ}\langle \overline{g}, \overline{f}\rangle = m(gfl) = \overline{gf}$ is the exponential transpose of gf. Indeed, the term internal composition is justified. One might ask if this composition is associative and the answer is yes. Consider the following diagram

$$\begin{array}{ccccc}
(Z^Y \times Y^X) \times X^W & & \xrightarrow{\quad a \quad} & & Z^Y \times (Y^X \times X^W)\\
\big\downarrow{\scriptstyle M_{XYZ}\times 1} & & & & \big\downarrow{\scriptstyle 1\times M_{WXY}}\\
Z^X \times X^W & & & & Z^Y \times Y^W\\
& \searrow^{M_{WXZ}} & & \swarrow^{M_{WYZ}} & \\
& & Z^W. & &
\end{array}$$

This diagram commutes because both the arrows $M(M \times 1)$ and $M(1 \times M)a$ are the exponential transpose of iterated evaluation

$$Z^Y \times Y^X \times X^W \times W \xrightarrow{1\times 1\times \mathrm{ev}} Z^Y \times Y^X \times X \xrightarrow{1\times \mathrm{ev}} Z^Y \times Y \xrightarrow{\mathrm{ev}} Z.$$

Because exponential transposes are unique, we find that $M(M \times 1) = M(1 \times M)a$. Thus we can say that the internal composition is associative.

Moreover, we have that internal composition satisfies the unit axiom. There is a commutative diagram

$$\begin{array}{ccccc}
Y^Y \times Y^X & \xrightarrow{M} & Y^X & \xleftarrow{M} & Y^X \times X^X\\
\big\uparrow{\scriptstyle j_Y\times 1} & \nearrow_{l} & & \nwarrow_{r} & \big\uparrow{\scriptstyle 1\times j_X}\\
I \times Y^X & & & & Y^X \times I,
\end{array}$$

where the arrows j_X corresponds to the identity arrow id_X via the natural bijection $\mathrm{Hom}(X, X) \cong \mathrm{Hom}(I \times X, X) \cong \mathrm{Hom}(I, X^X)$. To see that the diagram commutes, note that both l and $M(j_Y \times 1)$ are the exponential transpose of the map

$$I \times Y^X \times X \xrightarrow{1\times \mathrm{ev}} I \times Y \xrightarrow{l} Y.$$

Example 3.8. For a category $\mathcal{C}$, the category of its presheaves $\widehat{\mathcal{C}}$ is Cartesian closed. If for any two presheafs X and Y there exits a presheaf $Y^X(-)$ then by the Yoneda lemma it satisfies $Y^X(C) \cong \mathrm{Hom}_{\widehat{\mathcal{C}}}(y_C, Y^X) \cong \mathrm{Hom}_{\widehat{\mathcal{C}}}(y_C \times X, Y)$. This bijection is natural in C. The latter term is a well defined presheaf, so it is taken as the *definition* of Y^X.

3.3 Lattice Betweenness Revisited

Definition 3.9. For a lattice $(L, \leq, \wedge, \vee)$ and for points $a, x, b \in L$, we say that x is between a and c whenever it satisfies

$$(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b).$$

Remark 3.10. For a distributive lattice, this definition is equivalent to the property $a \wedge b \leq x \leq a \vee b$. Note that this definition is self dual, that is, if x is between a and b in L, then that remains the case in the lattice L^{op} with reverse partial ordering.

Considering L as a category, we say there exists an arrow $a \longrightarrow b$ precisely whenever $a \leq b$. Explicitly,

$$\mathrm{Hom}(a, b) = \begin{cases} \{*\}, & \text{if } a \leq b \\ \emptyset, & \text{otherwise.} \end{cases}$$

This indeed defines a category since the fact that $a = a$ for all $a \in L$ provides existence of identity morphisms for all $a \in L$. Furthermore, there is a composition of morphisms that follows from the transitivity of the partial order $\leq$. The lattice L can be embedded in its category of presheaves $\widehat{L} = [L^{\mathrm{op}}, \mathbf{Set}]$ via the Yoneda embedding. Since presheaf categories are Cartesian closed, we can consider the betweenness relation on L defined by internal composition of the internal homs in its presheaves. Explicitly the internal hom objects are given by

$$y_b^{y_a}(-) = \mathrm{Hom}_{\widehat{L}}(y_{(-)} \times y_a, y_b).$$

Now at each component c, we have $y_b^{y_a}(c) = \mathrm{Hom}_{\widehat{L}}(y_c \times y_a, y_b)$. And because L has finite products, we see that the functor $y_c \times y_a$ is representable. In particular,

$$\mathrm{Hom}(-, c) \times \mathrm{Hom}(-, a) \cong \mathrm{Hom}(-, c \wedge a).$$

That is; $y_c \times y_a \cong y_{c \wedge a}$. The Yoneda Lemma now provides a natural isomorphism

$$\mathrm{Hom}_{\widehat{L}}(y_c \times y_a, y_b) \cong \mathrm{Hom}_{\widehat{L}}(y_{c \wedge a}, y_b) \cong \mathrm{Hom}_L(c \wedge a, b).$$

In terms of the internal composition law for $\widehat{L}$, we say that x is between a and b whenever the composition morphism $M : L(x,b) \times L(a,x) \to L(a,b)$ admits a section. Letting φ denote the right inverse, and using the above computation, we find that at each component c, we have a map;

$$\varphi_c : \mathrm{Hom}(c \wedge a, b) \to \mathrm{Hom}(c \wedge x, b) \times \mathrm{Hom}_L(c \wedge a, x).$$

Because these homsets are either singleton or empty, we need not say a lot about what this map is. We just find that its existence is equivalent to the condition

$$c \wedge a \leq b \quad \text{if and only if,} \quad c \wedge a \leq x \text{ and } c \wedge x \leq b, \quad \text{for all } c \in L.$$

If we in addition require that the composition in L^{op}, enriched over its presheaves $[L, \mathbf{Set}]$, admits a section, then we find that this $\widehat{L^{\mathrm{op}}}$-betweenness is equivalent to the property

$$c \vee a \geq b \quad \text{if and only if,} \quad c \vee a \geq x \text{ and } c \vee x \geq b, \quad \text{for all } c \in L.$$

Assuming the lattice L is distributive, we find that the above two conditions for $\widehat{L}$- and $\widehat{L^{\mathrm{op}}}$-betweenness, are equivalent to lattice betweenness.

Proposition 3.11. *For a distributive lattice L, we have that a point x is between points a and b in terms of lattice betweenness if and only if it is so in terms of $\widehat{L}$- and $\widehat{L^{\mathrm{op}}}$- betweenness.*

Proof. Suppose $[a,x,b]_L$, that is, x is between a and b in the sense of lattice betweenness. Then $(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b)$. Using distributivity, we find $x \wedge (a \vee b) = x = x \vee (a \wedge b)$. This means that $a \wedge b \leq x \leq a \vee b$. Therefore, if $c \wedge a \leq b$, then $c \wedge a = c \wedge a \wedge b \leq c \wedge x \leq x$. Moreover, $c \wedge x \leq c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b) \leq b$. This shows that x is between a and b in the sense of $\widehat{L}$-betweenness. The proof for $\widehat{L^{\mathrm{op}}}$ is dual. So we conclude that lattice betweenness implies $\widehat{L}$- and $\widehat{L^{\mathrm{op}}}$- betweenness. Conversely, if x is between a and b in the sense of $\widehat{L}$ and $\widehat{L^{\mathrm{op}}}$, then because $b \wedge a \leq b$, we find by definition that $b \wedge a \leq x$. Moreover, $b \vee a \geq b$ implies $b \vee a \geq x$. So that x is between a and b in the sense of lattice betweenness.

The main point of the above proposition is that even if we are not be able to find a real valued metric that is compatible with the betweenness relation on a distributive lattice in the sense of Smiley[19], we can still obtain the betweenness relation from a composition morphism. In fact we used intuition from betweenness in metric spaces and road systems to see the analogy with Lawvere's work. The remarkable fact that we re-obtain lattice betweenness shows the analogy is fitting. Consequently we consider the notion of betweenness in enriched category theory.

4 Enriched Categories

We will recall some notions from enriched category theory. A comprehensive textbook on enriched categories is the book by Kelly [15]. We will first define what an enriched category is and then give various examples. We furthermore prove some lemmas that we need in order to define a notion of betweenness in enriched categories.

4.1 Definition

Definition 4.1. A monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$ consists of:

i) a category $\mathcal{V}_0$,

ii) a functor $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$,

iii) an object I of $\mathcal{V}_0$,

iv) natural isomorphisms: $a_{XYZ} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$,
$l_X : I \otimes X \longrightarrow X$ and $r_X : X \otimes I \longrightarrow X$.

subject to two *coherrence axioms* expressed by commutativity of the following diagrams:

$$\begin{array}{ccccc} ((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{a} & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle 1\otimes a} & & & & \uparrow{\scriptstyle 1\otimes a} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[a]{\hspace{6em}} & & W \otimes ((X \otimes Y) \otimes Z), \end{array}$$

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{a} & X \otimes (I \otimes Y) \\ & {\scriptstyle r\otimes 1} \searrow \quad \swarrow {\scriptstyle 1\otimes l} & \\ & X \otimes Y. & \end{array}$$

Definition 4.2. A monoidal functor $\mathcal{V} \longrightarrow \mathcal{U}$ between monoidal categories $\mathcal{V} = (\mathcal{V}_0, \otimes, I_{\mathcal{V}}, a, l, r)$ and $\mathcal{U} = (\mathcal{U}_0, \otimes, I_{\mathcal{U}}, a, l, r)$ is a functor $F : \mathcal{V}_0 \longrightarrow \mathcal{U}_0$ on the underlying categories together with an isomorphism $\varphi_0 : I_{\mathcal{U}} \to F(I_{\mathcal{V}})$ and natural isomorphisms $\varphi_{2,X,Y} : F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$ such that the following diagrams

$$
\begin{array}{ccc}
(F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{a} & F(X)\otimes (F(Y)\otimes F(Z)) \\
\downarrow{\scriptstyle \varphi_2\otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi_2} \\
F(X\otimes Y)\otimes F(Z) & & F(X)\otimes F(Y\otimes Z) \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2} \\
F((X\otimes Y)\otimes Z) & \xrightarrow[Fa]{} & F(X\otimes (Y\otimes Z)),
\end{array}
$$

$$
\begin{array}{ccc}
I\otimes F(X) & \xrightarrow{l} & F(X) \\
\downarrow{\scriptstyle \varphi_0\otimes \mathrm{id}} & & \uparrow{\scriptstyle F(l)} \\
F(I)\otimes F(X) & \xrightarrow[\varphi_2]{} & F(I\otimes X),
\end{array}
\qquad
\begin{array}{ccc}
F(X)\otimes I & \xrightarrow{r} & F(X) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\varphi_0} & & \uparrow{\scriptstyle F(r)} \\
F(X)\otimes F(I) & \xrightarrow[\varphi_2]{} & F(X\otimes I),
\end{array}
$$

commute for all objects $X, Y, Z \in \mathrm{ob}(\mathcal{V})$. The monoidal functor is said to be strict if the isomorphisms φ_0 and φ_2 are identities.

Definition 4.3. A $\mathcal{V}$-category $\mathcal{A}$ consists of:

i) a set of objects, denoted $\mathrm{ob}(\mathcal{A})$,

ii) a hom-object $\mathcal{A}(A, B) \in \mathcal{V}_0$ for each pair of objects A, B in $\mathrm{ob}(\mathcal{A})$,

iii) a composition law $M_{AXB} : \mathcal{A}(X, B) \otimes \mathcal{A}(A, X) \to \mathcal{A}(A, B)$ for each triple of objects $A, X, B \in \mathrm{ob}(\mathcal{A})$,

iv) and an identity element $j_A : I \longrightarrow \mathcal{A}(A, A)$ for each object $A \in \mathrm{ob}(\mathcal{A})$,

subject to the associativity and unit axioms expressed respectively by the commutativity of the following diagrams:

$$
\begin{array}{ccccc}
(\mathcal{A}(X,B)\otimes\mathcal{A}(Y,X))\otimes\mathcal{A}(A,Y) & & \xrightarrow{a} & & \mathcal{A}(X,B)\otimes(\mathcal{A}(Y,X)\otimes\mathcal{A}(A,Y)), \\
\downarrow{\scriptstyle M_{YXB}\otimes 1} & & & & \downarrow{\scriptstyle 1\otimes M_{AYX}} \\
\mathcal{A}(Y,B)\otimes\mathcal{A}(A,Y) & & & & \mathcal{A}(X,B)\otimes\mathcal{A}(A,X) \\
& \searrow{\scriptstyle M_{AYB}} & & \swarrow{\scriptstyle M_{AXB}} & \\
& & \mathcal{A}(A,B) & &
\end{array}
$$

$$
\begin{array}{ccccc}
\mathcal{A}(B,B)\otimes\mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,B) & \xleftarrow{M} & \mathcal{A}(A,B)\otimes\mathcal{A}(A,A). \\
\uparrow{\scriptstyle j_B\otimes 1} & \nearrow{\scriptstyle l} & & \nwarrow{\scriptstyle r} & \uparrow{\scriptstyle 1\otimes j_A} \\
I\otimes\mathcal{A}(A,B) & & & & \mathcal{A}(A,B)\otimes I
\end{array}
$$

4.2 Examples

Example 4.4. A small category $\mathcal{C}$ is enriched over the category **Set**. The objects are given by the objects of the category $\mathcal{C}$. For each two pair objects A, B in $\mathcal{C}$ there is a hom-object in **Set**, namely the set of arrows $\mathrm{Hom}_{\mathcal{C}}(A, B)$. For any triple of objects, A, B, C in $\mathcal{C}$, there is a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C) : (g, f) \mapsto gf.$$

The identity element is given by a map $j_A : \{*\} \to \mathrm{Hom}_{\mathcal{C}}(A, A) : * \mapsto \mathrm{id}_A$. The associativity axiom $(hg)f = h(gf)$, and the unit axiom $f\,\mathrm{id}_A = f = \mathrm{id}_B\, f$ for all $f : A \to B$ ensures that the required diagrams commute.

Example 4.5 (Lawvere, [14])**.** A metric space (X, d) is enriched over the category $\overline{\mathbb{R}}_+$. The objects are the points in X. For two points a, b, there is a hom-object $d(a, b)$ in $\overline{\mathbb{R}}_+$. Note that $+$ is the monoidal operation in $\overline{\mathbb{R}}_+$ and that for $r, s \in \overline{\mathbb{R}}_+$ there is an arrow $r \to s$ precisely whenever $r \geq s$. Thus, for any triple of points a, b, c, there is a composition morphism $d(b, c) + d(a, b) \geq d(a, c)$. Moreover, there is an identity element since $0 \geq d(a, a)$ for all points $a \in X$. With these definitions of composition and identity, we see that the required diagrams commute.

Remark 4.6. This means the condition that a point x is between two other points a and b in a metric space can be seen as a property of the composition morphism M_{axb}. Namely, the triangle inequality becomes an equality, or the composition is an isomorphism.

Example 4.7. In section 3.2 we see that a Cartesian closed category $\mathcal{C}$ is enriched over itself when one considers the hom-objects $\mathcal{C}(X, Y)$ to be the exponentials Y^X. The composition morphism of this enriched category is then the internal composition. Its unit is given by the arrow j_X corresponding to the identity arrow id_X under the natural bijection $\mathrm{Hom}_{\mathcal{C}}(X, X) \cong \mathrm{Hom}_{\mathcal{C}}(I \times X, X) \cong \mathrm{Hom}_{\mathcal{C}}(I, X^X)$.

Example 4.8. Let R be a commutative ring and denote $\mathrm{Ideal}(R)$ for its collection of ideals. We can view $\mathrm{Ideal}(R)$ as a monoidal category. For two ideals I, J there exists precisely one arrow $I \to J$ whenever $I \subseteq J$. Recall there is a monoidal product of two ideals I and J denoted $IJ = \{\sum_i a_i b_i : a_i \in I, b_i \in J\}$, that is, the ideal consisting of all finite sums of products of elements of I and J. Moreover, the ideal quotient is defined $(J : I) := \{x \in R : xI \subseteq J\}$. In terms of categories, there is a pair of adjoint functors $I(-) : \mathrm{Ideal}(R) \longrightarrow \mathrm{Ideal}(R) : J \longmapsto IJ$ and $(- : I) : \mathrm{Ideal}(R) \longrightarrow \mathrm{Ideal}(R) : J \longmapsto (J : I)$. These form an adjunction since $IJ \subseteq K$ holds if and only if $J \subseteq (K : I)$ we see that the quotient ideal is a right adjoint to the monoidal product. Thus the category $\mathrm{Ideal}(R)$ is monoidally closed. We can therefore think of the ideal $(I : J)$ as the internal hom. In doing so we can view Ideal(R) as enriched over itself. Namely, let the set of objects be the set of ideals Ideal(R). For the hom-object belonging to two ideals I, J we consider their ideal quotient. Then there is a composition morphism:

$$M_{IJK} : (K : J)(J : I) \longrightarrow (K : I).$$

This morphism is the inclusion from lemma 2.9. Moreover, for any ideal I, we have an identity element

$$j_I : R \longrightarrow (I : I),$$

because for all $x \in R$, we have that $xI \subseteq I$. The associativity and unit diagrams commute because the category Ideal(R) is posetal, so all existing arrows are unique. Indeed, the ideals of R are enriched over themselves.

Example 4.9. This example is due to Simon Willerton. Given a group G, a G-torsor is a set T together with a group action $a : G \times T \longrightarrow T$ such that the map $G \times T \longrightarrow T \times T : (g, t) \longmapsto (a(g, t), t)$ is a bijection. We will write gt instead of $a(g, t)$. The group G can be thought of as a monoidal category by taking for the objects, the members of G. The monoidal operation is then the group multiplication. There are only identity morphisms in this category. We can then view T as a category enriched over G. The objects are given by the set T. For any pair of objects $t_1, t_2 \in T$, there is a hom object $g(t_2, t_1) \in G$ which is by definition the unique element of G that satisfies $g(t_2, t_1)t_1 = t_2$. For any triple of objects $t_3, t_2, t_1 \in T$, we have that $g(t_3, t_2)g(t_2, t_1)t_1 = g(t_3, t_2)t_2 = t_3$. By uniqueness this means that $g(t_3, t_2)g(t_2, t_1) = g(t_3, t_1)$, so there is a composition morphism. Consequently, given any object $t \in T$, we have that $g(t, t)g(t, t) = g(t, t)$. This means that $e = g(t, t)$ so that there is an identity element. From the fact that G is a group and that e is its neutral element we find that diagrams expressing the associativity and unit axioms must commute.

Example 4.10. For any monoidal category $\mathcal{V}$, there is the *unit* $\mathcal{V}$-category denoted $\mathcal{I}$ with $\mathrm{ob}(\mathcal{I}) = \{0\}$ and with $\mathcal{I}(0, 0) = I$ being the tensor unit.

4.3 $\mathcal{V}$-categorical notions

For any category enriched over a monoidal category $\mathcal{V}$ there are the notions of $\mathcal{V}$-functor and $\mathcal{V}$-natural transformation. They are defined as follows:

Definition 4.11. For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ consists of a function

$$T : \mathrm{ob}(\mathcal{A}) \longrightarrow \mathrm{ob}(\mathcal{B})$$

together with for each pair $A, B \in \mathrm{ob}(\mathcal{A})$ a map $T_{AB} : \mathcal{A}(A,B) \longrightarrow \mathcal{B}(TA,TB)$, subject to the compatibility with composition and with the identities expressed by the commutativity of the diagrams

$$\begin{array}{ccc}
\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,C) \\
{\scriptstyle T\otimes T}\downarrow & & \downarrow{\scriptstyle T} \\
\mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) & \xrightarrow[M]{} & \mathcal{B}(TA,TC),
\end{array}$$

and

$$\begin{array}{ccc}
 & & \mathcal{A}(A,A) \\
 & \overset{j}{\nearrow} & \\
I & & \downarrow{\scriptstyle T} \\
 & \underset{j}{\searrow} & \\
 & & \mathcal{B}(TA,TA).
\end{array}$$

We say a $\mathcal{V}$-functor is *fully faithfull* if each T_{AB} is an isomorphism.

Example 4.12. In the case that $\mathcal{V} = \mathbf{Set}$ we refind the usual notion of functor between categories $\mathcal{A}$ and $\mathcal{B}$. Indeed, the usual notion of functor consists of a function on objects, together with a function on morphisms that sends an arrow $f \in \mathrm{Hom}_{\mathcal{A}}(A,B)$ into an arrow $Tf \in \mathrm{Hom}_{\mathcal{B}}(TA,TB)$. This moreover satisfies that first composing and then mapping to the image under T, commutes with first mapping to the image under T and then composing. Thus commutativity of the top square states that $TgTf = M(T \times T)(g,f) = TM(g,f) = T(gf)$. Commutativity of the lower triangle states that $T\,\mathrm{id}_A = \mathrm{id}_{TA}$. The terminology of calling a $\mathcal{V}$-functor fully faithfull whenever each T_{AB} is an isomorphism is consistent with the terminology for the usual notion of fully faithfull functor. This is the case because the isomorphism T_{AB} in the case of $\mathcal{V} = \mathbf{Set}$ is a bijection $\mathrm{Hom}_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{\mathcal{B}}(TA,TB)$.

Example 4.13. For metric spaces (X, d_X) and (Y, d_Y) a metric map is a function $f : X \longrightarrow Y$ such that $d_Y(f(a), f(b)) \le d_X(a,b)$ for all $a, b \in X$. This is precisely the notion of $\overline{\mathbb{R}}_+$ functor when thinking of metric spaces as categories enriched over $\overline{\mathbb{R}}_+$. Note that isometries are precisely given by the fully faithfull $\overline{\mathbb{R}}_+$ functors.

Example 4.14. For G-torsors, T and S, a G-functor is a function $f : T \longrightarrow S$ together with arrows $f_{s,t} : g(s,t) = g(fs, ft)$. In other words, a function such that the following diagram commutes, also known as equivariant map

$$\begin{array}{ccc}
G \times T & \xrightarrow{a} & T \\
{\scriptstyle 1\times f}\downarrow & & \downarrow{\scriptstyle f} \\
G \times S & \xrightarrow{a} & S.
\end{array}$$

Definition 4.15. For $\mathcal{V}$-functors $T, S : \mathcal{A} \longrightarrow \mathcal{B}$, a $\mathcal{V}$-natural transformation $\alpha : T \longrightarrow S$ is an ob($\mathcal{A}$) indexed family of *components* $\alpha_A : I \longrightarrow \mathcal{B}(TA, SA)$ that satisfy the $\mathcal{V}$-naturality condition expressed by the commutativity of

$$\begin{array}{ccccc} & I \otimes \mathcal{A}(A,B) \xrightarrow{\alpha_B \otimes T} \mathcal{B}(TB,SB) \otimes \mathcal{B}(TA,TB) & \\ {}^{l^{-1}}\nearrow & & \searrow^{M} \\ \mathcal{A}(A,B) & & \mathcal{B}(TA,SB). \\ {}_{r^{-1}}\searrow & & \nearrow_{M} \\ & \mathcal{A}(A,B) \otimes I \xrightarrow[S \otimes \alpha_A]{} \mathcal{B}(SA,SB) \otimes \mathcal{B}(TA,SA) & \end{array}$$

The *vertical composition* $\beta \cdot \alpha$ of $\alpha : T \longrightarrow S$ and $\beta : S \longrightarrow R$ has the component $(\beta \cdot \alpha)_A$ given by

$$I \cong I \otimes I \xrightarrow[\beta_A \otimes \alpha_A]{} \mathcal{B}(SA, RA) \otimes \mathcal{B}(TA, SA) \xrightarrow[M]{} \mathcal{B}(TA, RA).$$

Example 4.16. For a category enriched over $\mathcal{V} = \mathbf{Set}$, we see that $\mathcal{V}$-natural transformations coincide with the usual notion of natural transformation. Given functors $S, T : \mathcal{A} \longrightarrow \mathcal{B}$, then a **Set**-natural transformation $\alpha : T \longrightarrow S$ specifies a family of components $\alpha_A : \{*\} \longrightarrow \mathrm{Hom}_B(TA, SA)$. Since the domain of each α_A is the singleton set, this family determines a family of arrows $\{ TA \xrightarrow{\alpha_A(*)} SA : A \in \mathrm{ob}(\mathcal{A})\}$. The commutativity of the hexagon above states that this family of arrows is a natural transformation. Namely, given an arrow $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, then applying the top row of the hexagon $M(\alpha_B \otimes T)(f) = \alpha_B(*) \circ Tf$ must equal applying the bottom row of the hexagon $M(S \otimes \alpha_A)(f) = Sf \circ \alpha_A(*)$. This states precisely that $\alpha(*) : T \Longrightarrow S$ is a natural transformation

$$\begin{array}{ccc} TA & \xrightarrow{\alpha_A(*)} & SA \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle Sf} \\ TB & \xrightarrow[\alpha_B(*)]{} & SB. \end{array}$$

Example 4.17. If there is a $\overline{\mathbb{R}}_+$-natural transformation between metric maps $f, g : X \longrightarrow Y$, then there is a family of components $\{0 \geq d_Y(f(x), g(x))\}_{x \in X}$. This means that $f = g$. Thus there are no non-trivial natural transformations between metric maps.

4.4 Underlying Category

There is a functor $(-)_0 : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}$ mapping a $\mathcal{V}$-category into its underlying category. For each $\mathcal{V}$-category $\mathcal{A}$ there is a category $\mathcal{A}_0 := \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{I}, \mathcal{A})$, the $\mathcal{V}$-functor category. Here $\mathcal{I}$ denotes the unit $\mathcal{V}$-category. The objects of $\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{I}, \mathcal{A})$ are $\mathcal{V}$-functors $A : \mathcal{I} \longrightarrow \mathcal{A}$. Because $\mathrm{ob}(\mathcal{I}) = \{0\}$ is a singleton set, such a $\mathcal{V}$-functor determines precisely one object in $\mathrm{ob}(\mathcal{A})$. Thus we can think of such $\mathcal{V}$-functors as the objects of $\mathcal{A}$. The morphisms of $\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{I}, \mathcal{A})$ are $\mathcal{V}$-natural transformations $f : A \longrightarrow B$. Because such a $\mathcal{V}$-natural transformation is $\mathrm{ob}(\mathcal{I})$ indexed, there is a single component $\{ f_0 : I \longrightarrow \mathcal{A}(A, B) \}$. This means the underlying category $\mathcal{A}_0$ has the same objects as $\mathcal{A}$, while a morphism $f : A \longrightarrow B$ is just an arrow $f : I \longrightarrow \mathcal{A}(A, B)$ in $\mathcal{V}$. Composition of arrows in $\mathcal{A}_0$ is as follows. Given a pair of arrows $f \in \mathcal{A}_0(A, B)$ and $g \in \mathcal{A}_0(B, C)$ their composite is an arrow denoted $gf \in \mathcal{A}_0(A, C)$ given by

$$I \cong I \otimes I \xrightarrow{g \otimes f} \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \xrightarrow{M} \mathcal{A}(A, C).$$

Associativity of this composition follows from the commutativity of the pentagonal diagram in Definition 4.3. Explicitly,

$$\begin{aligned} h(gf) &= M(1 \otimes M)(h \otimes (g \otimes f)) \\ &= M(1 \otimes M)a((h \otimes g) \otimes f) \\ &= M(M \otimes 1)((h \otimes g) \otimes f) \\ &= (hg)f. \end{aligned}$$

The identity arrow id_A in $\mathcal{A}_0(A, A)$ is given by $j_A : I \longrightarrow \mathcal{A}(A, A)$. This also satisfies the required properties with respect to composition, since we have a commutative diagram

$$\begin{array}{ccccc} I \cong I \otimes I & \xrightarrow{j_B \otimes f} & \mathcal{A}(B, B) \otimes \mathcal{A}(A, B) & \xrightarrow{M} & \mathcal{A}(A, B) \\ & \searrow^{1 \otimes f} & \uparrow{\scriptstyle j_B \otimes 1} & \nearrow_{l} & \\ & & I \otimes \mathcal{A}(A, B). & & \end{array}$$

Again commutativity follows from Definition 4.3. This shows that $M(j_B \otimes f) = f$, and the proof for $M(f \otimes j_A) = f$ is similar. Indeed, $\mathcal{A}_0$ is a category.

For a $\mathcal{V}$-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$, the underlying functor $(T)_0 = T_0 : \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ is given as follows. For the action of $T_0(A) = TA$ for all $A \in \mathrm{ob}(\mathcal{A}_0) = \mathrm{ob}(\mathcal{A})$. Given a morphism $f \in \mathcal{A}_0(A, B)$ we have that $T_0(f)$ is given by the morphism

$I \xrightarrow{f} \mathcal{A}(A,B) \xrightarrow{T_{AB}} \mathcal{B}(TA,TB)$. This is indeed functorial, because given a pair of arrows $f \in \mathcal{A}_0(A,B)$ and $g \in \mathcal{A}_0(B,C)$ then we find

$$T_0(gf) = T_{AC}M_{ABC}(g\otimes f) = M_{TATBTC}(T_{BC}\otimes T_{AB})(g\otimes f) = T_0(g)T_0(f)$$

To see the functor T_0 respects unity, note that $T_0(\mathrm{id}_A) = T_{AA}j_A = j_{TA} = \mathrm{id}_{T_0A}$.

Definition 4.18. We say two objects A and B in a $\mathcal{V}$-category $\mathcal{A}$ are $\mathcal{V}$-isomorphic and write $A \cong_{\mathcal{V}} B$ whenever A and B are isomorphic objects in the underlying category $\mathcal{A}_0$.

Lemma 4.19. *The relation $\cong_{\mathcal{V}}$ is an equivalence relation.*

Example 4.20. Consider a locally small category $\mathcal{C}$. This is a category enriched over **Set**. If two objects A and B are **Set**-isomorphic, then by definition, there are elements $f : \{*\} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $g : \{*\} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(B,A)$ that satisfy both $M(g\times f) = j_A$ and $M(f\times g) = j_B$. In the underlying category $\mathcal{C}$, this means that $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$, so A and B are isomorphic objects in $\mathcal{C}$.

Example 4.21. Let (X,d) be a metric space. If $a,b \in X$ are $\overline{\mathbb{R}}_+$-isomorphic, then $0 \geq d(a,b)$ so that $a = b$.

Lemma 4.22. *If $A \cong_{\mathcal{V}} B$ then $\mathcal{A}(A,X) \cong \mathcal{A}(B,X)$ and $\mathcal{A}(X,A) \cong \mathcal{A}(X,B)$ for all objects X in* $\mathrm{ob}(\mathcal{A})$*.*

Proof. Since $A \cong_{\mathcal{V}} B$, there are arrows $f : I \longrightarrow \mathcal{A}(A,B)$ and $g : I \longrightarrow \mathcal{A}(B,A)$ in $\mathcal{V}$ for which $M(g\otimes f)r^{-1} = j_A$ and $M(f\otimes g)r^{-1} = j_B$. Thus there are arrows

$$\mathcal{A}(B,X) \xrightarrow{r^{-1}} \mathcal{A}(B,X)\otimes I \xrightarrow{1\otimes f} \mathcal{A}(B,X)\otimes\mathcal{A}(A,B) \xrightarrow{M} \mathcal{A}(A,X),$$
$$\mathcal{A}(A,X) \xrightarrow{r^{-1}} \mathcal{A}(A,X)\otimes I \xrightarrow{1\otimes g} \mathcal{A}(A,X)\otimes\mathcal{A}(B,A) \xrightarrow{M} \mathcal{A}(B,X).$$

Their composition satisfies

$$\begin{aligned}
M(1\otimes f)r^{-1}M(1\otimes g)r^{-1} &= M(1\otimes f)(M\otimes 1)r^{-1}(1\otimes g)r^{-1}\\
&= M(M\otimes 1)(1\otimes 1\otimes f)r^{-1}(1\otimes g)r^{-1}\\
&= M(M\otimes 1)(1\otimes 1\otimes f)(1\otimes g\otimes 1)r^{-1}r^{-1}\\
&= M(M\otimes 1)(1\otimes g\otimes f)r^{-1}r^{-1}\\
&= M(1\otimes M)a(1\otimes g\otimes f)r^{-1}r^{-1}\\
&= M(1\otimes M(g\otimes f))r^{-1}r^{-1}\\
&= M(1\otimes j_A)r^{-1}\\
&= \mathrm{id}_{\mathcal{A}(A,X)}\,.
\end{aligned}$$

Similarly we find that $M(1\otimes g)r^{-1}M(1\otimes f)r^{-1} = \mathrm{id}_{\mathcal{A}(B,X)}$. This proves that $\mathcal{A}(A,X) \cong \mathcal{A}(B,X)$ are isomorphic in $\mathcal{V}$. The proof for $\mathcal{A}(X,A) \cong \mathcal{A}(X,B)$ is similar.

4.5 Betweenness and the Minimality Axiom

In this section we define a relation on the objects of enriched categories that corresponds to equality of the triangle inequality. Namely, we consider triples (A, B, C) such that the composition morphisms M_{ABC} and M_{CBA} are split epimorphisms.

Definition 4.23. An arrow $f : A \longrightarrow B$ in a category $\mathcal{C}$ is called a split epimorphism whenever there exists a $g : B \longrightarrow A$ (called a section) such that $fg = \mathrm{id}_B$.

Theorem 4.24. *For any $\mathcal{V}$-category $\mathcal{A}$, the relation $[-,-,-]_{\mathcal{V}} \subseteq \mathrm{ob}(\mathcal{A})^3$ given by*

$$[-,-,-]_{\mathcal{V}} = \{(A,B,C) \in \mathrm{ob}(\mathcal{A})^3 : M_{ABC} \text{ and } M_{CBA} \text{ are split epimorphisms}\},$$

satisfies axioms B1, B2 and B4 of definition 2.1. We will write $[A,B,C]_{\mathcal{V}}$ to denote $(A,B,C) \in [-,-,-]_{\mathcal{V}}$.

Proof. Recall that M_{ABC} denotes the composition morphism

$$M_{ABC} : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C).$$

We point out that the axiom of symmetry, B1, holds by construction. Indeed, $[A,B,C]_{\mathcal{V}}$ is equivalent to $[C,B,A]_{\mathcal{V}}$ because M_{ABC} and M_{CBA} are split epimorphisms precisely whenever M_{CBA} and M_{ABC} are split epimorphisms. Next we prove B2, i.e. that $[A,B,B]_{\mathcal{V}}$ holds. By definition of $\mathcal{V}$-category, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(B,B) \otimes \mathcal{A}(A,B) & \xrightarrow{M_{ABB}} & \mathcal{A}(A,B) \\ {\scriptstyle j_B \otimes 1}\uparrow & \nearrow{\scriptstyle l} & \\ I \otimes \mathcal{A}(A,B). & & \end{array}$$

Because $M_{ABB}(j_B \otimes 1) = l$, and because l is an isomorphism, we obtain that M_{ABB} is a split epimorphism, because $M_{ABB}(j_B \otimes 1)l^{-1} = \mathrm{id}_{\mathcal{A}(A,B)}$. The proof that M_{BBA} is a split epimorphism, is similar. To prove property B4, we assume that $[A,B,C]_{\mathcal{V}}$ and $[A,C,D]_{\mathcal{V}}$. We want to show that $[A,B,D]_{\mathcal{V}}$ holds. Note that by definition, we have a commutative diagram:

$$\begin{array}{ccccc} (\mathcal{A}(C,D) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(A,B) & & \xrightarrow{\quad a \quad} & & \mathcal{A}(C,D) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)). \\ \downarrow{\scriptstyle M_{BCD} \otimes 1} & & & & \downarrow{\scriptstyle 1 \otimes M_{ABC}} \\ \mathcal{A}(B,D) \otimes \mathcal{A}(A,B) & & & & \mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \\ & \searrow{\scriptstyle M_{ABD}} & & \swarrow{\scriptstyle M_{ACD}} & \\ & & \mathcal{A}(A,D) & & \end{array}$$

We can write $M_{ABD}(M_{BCD} \otimes 1) = M_{ACD}(1 \otimes M_{ABC})a$. Since a is an isomorphism, and since M_{ABC} and M_{ACD} are split epimorphisms, with sections ϕ_{ABC} and ϕ_{ACD} respectively, we find

$$\begin{aligned} M_{ABD}(M_{BCD} \otimes 1)a^{-1}(1 \otimes \phi_{ABC})\phi_{ACD} &= M_{ACD}(1 \otimes M_{ABC})(1 \otimes \phi_{ABC})\phi_{ACD} \\ &= \mathrm{id}_{\mathcal{A}(A,D)} . \end{aligned}$$

Thus M_{ABD} is a split epimorphism with section $(M_{BCD} \otimes 1)a^{-1}(1 \otimes \phi_{ABC})\phi_{ACD}$. A similar argument works for showing that M_{DBA} is a split epimorphism. Thus showing that $[A, B, D]_{\mathcal{V}}$ holds.

Proposition 4.25. *The relation $[-, -, -]_{\mathcal{V}}$ is stable under $\mathcal{V}$-isomorphisms. That is, for $A \cong_{\mathcal{V}} B$ we have:*

1. *$[A, X, Y]_{\mathcal{V}}$ holds if and only if $[B, X, Y]_{\mathcal{V}}$ holds,*
2. *$[X, A, Y]_{\mathcal{V}}$ holds if and only if $[X, B, Y]_{\mathcal{V}}$ holds.*

Proof. By Lemma 4.22 we have that $\mathcal{A}(A, X) \cong \mathcal{A}(B, X)$ and $\mathcal{A}(A, Y) \cong \mathcal{A}(B, Y)$ We claim that the diagram

$$\begin{array}{ccc} \mathcal{A}(X,Y) \otimes \mathcal{A}(A,X) & \xrightarrow{M} & \mathcal{A}(A,Y) \\ {\scriptstyle 1\otimes M(1\otimes g)r^{-1}} \downarrow & & \downarrow {\scriptstyle M(1\otimes g)r^{-1}} \\ \mathcal{A}(X,Y) \otimes \mathcal{A}(B,X) & \xrightarrow[M]{} & \mathcal{A}(B,Y). \end{array}$$

commutes. Here $M(1 \otimes g)r^{-1}$ denotes the isomorphism $\mathcal{A}(A, X) \longrightarrow \mathcal{A}(B, X)$ from Lemma 4.22. Indeed this diagram commutes because

$$\begin{aligned} M(1 \otimes M(1 \otimes g)r^{-1}) &= M(1 \otimes M)(1 \otimes (1 \otimes g))(1 \otimes r^{-1}) \\ &= M(1 \otimes M)a((1 \otimes 1) \otimes g)(1 \otimes r^{-1}) \\ &= M(M \otimes 1)((1 \otimes 1) \otimes g)(1 \otimes r^{-1}) \\ &= M(1 \otimes g)r^{-1}M. \end{aligned}$$

Consequently, if M_{AXY} is a split epimorphism, then it admints a section φ_{AXY}. Since the inverse of $M(1\otimes g)r^{-1}$ is given by $M(1\otimes f)r^{-1}$ we see that M_{BXY} admits a section (the arrow going counter clockwise through the commutative square).

$$\begin{aligned} &M_{BXY}(1 \otimes M(1 \otimes g)r^{-1})\varphi_{AXY}M(1 \otimes f)r^{-1} = \dots \\ &\dots = M(1 \otimes g)r^{-1}M_{AXY}\varphi_{AXY}M(1 \otimes f)r^{-1} = \mathrm{id}_{\mathcal{A}(B,Y)} \end{aligned}$$

In other words, if $A \cong_{\mathcal{V}} B$, then M_{AXY} is a split epimorphism implies that M_{BXY} is a split epimorphism. The properties that remain to be shown can be proven similarly.

Theorem 4.24 together with Proposition 4.25 shows that for any enriched category there is a well behaved ternary relation, defined on the objects, satisfying axioms B1, B2 and B4. However, something is left to be said about axiom B3. Recall that axiom B3 states $[a, b, a]$ and $[b, a, b]$ imply $a = b$. This property need not hold for arbitrary enriched category. The best we can do is via the following equivalence relation.

Lemma 4.26. *Suppose a ternary relation $[\cdot, \cdot, \cdot] \subseteq X^3$ satisfies axioms* B1, B2 *and* B4*, then the following defines an equivalence relation*

$$a \sim b \quad \text{if and only if} \quad [a, b, a] \text{ and } [b, a, b].$$

Proof. By the reflexivity axiom B2 we have that $[a, a, a]$ holds for all a in X. Thus $a \sim a$ which shows the relation $\sim$ is reflexive. Suppose that $a \sim b$, then $[a, b, a]$ and $[b, a, b]$ hold. Then indeed $[b, a, b]$ and $[a, b, a]$ hold so that $b \sim a$. Thus the relation $\sim$ is symmetric. To show that $\sim$ is also transitive, suppose that $a \sim b$ and $b \sim c$. We have $[b, b, a]$ by B1 and B2. Moreover, $[b, c, b]$ holds by $b \sim c$. Consequently, using axiom B4 we find $[b, c, a]$ which is equivalent to $[a, c, b]$ under B1. Since $a \sim b$ we have $[a, b, a]$ so that in combination with $[a, c, b]$ these imply $[a, c, a]$ under B4. The proof for $[c, a, c]$ is the same but with the roles of a and c interchanged. We conclude that $a \sim c$ which proves $\sim$ is an equivalence relation.

Lemma 4.27. *A ternary relation satisfying* B1, B2 *and* B4 *is stable under the equivalence relation $\sim$. Meaning that if $a \sim b$, then*

1) $[x, y, a]$ holds if and only if $[x, y, b]$ holds,

2) $[x, a, y]$ holds if and only if $[x, b, y]$ holds.

Proof. Firstly, suppose that $[x, y, a]$ holds. From B2 we know that $[x, b, b]$ holds. From B1 also $[b, b, x]$ holds. Since $[b, a, b]$ holds, we can use axiom B4 to conclude that $[x, a, b]$ holds. Since $[x, y, a]$ holds by assumption, we find under B4 that $[x, y, b]$ holds. Interchanging the roles of a and b provides the converse. Secondly, suppose that $[x, a, y]$ holds. From axiom B2 we know that $[x, a, a]$ holds so that under B1 also $[a, a, x]$ holds. Because $a \sim b$ we have $[a, b, a]$ so that together $[a, a, x]$ and $[a, b, a]$ imply that $[a, b, x]$ holds under B4. Axiom B1 then shows we have $[x, b, a]$. By assumption we have $[x, a, y]$ so that using B4 we can conclude $[x, b, y]$. The converse follows by interchanging the roles of a and b.

Definition 4.28. For a pair of objects A, B of a $\mathcal{V}$-category $\mathcal{A}$, we say they are $\mathcal{V}$-equivalent and write $A \sim_{\mathcal{V}} B$ precisely whenever $[A, B, A]_{\mathcal{V}}$ and $[B, A, B]_{\mathcal{V}}$ hold.

Corollary 4.29. *If $A \cong_{\mathcal{V}} B$ then $A \sim_{\mathcal{V}} B$.*

Proof. By Proposition 4.25 the relation $[-,-,-]_{\mathcal{V}}$ is stable under $\mathcal{V}$-isomorphisms. Since we have that $[A,A,A]_{\mathcal{V}}$ holds, and since we assumed that $A \cong_{\mathcal{V}} B$, we find that $[A,B,A]_{\mathcal{V}}$ holds. Similarly $[B,A,B]_{\mathcal{V}}$ holds so that $A \sim_{\mathcal{V}} B$.

Remark 4.30. The ternary relation $[-,-,-]_{\mathcal{V}}$ satisfies the minimality axiom B3 up to $\mathcal{V}$-equivalence in arbitrary enriched categories. This equivalence $\sim_{\mathcal{V}}$ can be distinct from the notion of $\mathcal{V}$-isomorphism but need not be. For particular choices of enriched category (metric spaces, ideals enriched over themselves, distributive lattices), axiom B3 holds and the notions of equality, $\mathcal{V}$-isomorphism and $\mathcal{V}$-equivalence coincide. We will illustrate the issue with the minimality condition by means of examples.

Theorem 4.31 (Cantor-Schroeder-Bernstein)**.** *Let A and B be sets and suppose there are injective maps $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, then there exists a bijection between A and B.*

Example 4.32. Consider the category of sets **Set**. Suppose that $[A,B,A]_{\mathbf{Set}}$ and $[B,A,B]_{\mathbf{Set}}$ hold. Then the composition morphisms

$$\begin{aligned}\mathrm{Hom}(B,A) \times \mathrm{Hom}(A,B) &\longrightarrow \mathrm{Hom}(A,A),\\ \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,A) &\longrightarrow \mathrm{Hom}(B,B),\end{aligned}$$

are split epimorphisms. This means that there exist arrows $f,h : A \longrightarrow B$ and $g,k : B \longrightarrow A$ such that $gf = \mathrm{id}_A$ and $hk = \mathrm{id}_B$. Consequently, f and k are injective functions. Since there exist injections from A into B and from B into A, by the Cantor-Schroeder-Bernstein Theorem, there is a bijection $A \cong B$. So A and B are **Set**-isomorphic. Thus, in this particular example of enriched category the notion of $\mathcal{V}$-equivalence and $\mathcal{V}$-isomorphism coincide.

The above example shows that strong minimality ($[a,b,a]$ implies $a = b$) is too strong for the notion of betweenness in **Set**. Indeed, if $[A,B,A]_{\mathbf{Set}}$ holds but not $[B,A,B]_{\mathbf{Set}}$, then A is a retract of B, but they are not isomorphic.

Furthermore, there are examples of enriched categories for which the notions of $\mathcal{V}$-equivalent and $\mathcal{V}$-isomorphic do not coincide.

Example 4.33. Given a G-torsor T, the composition morphisms $g(a,b)g(b,a) = g(a,a)$ and $g(b,a)g(a,b) = g(b,b)$ are split epimorphisms for all a and b. However, if $a \neq b$, then $[a,b,a]_G$ and $[b,a,b]_G$ hold and $(b,a) \neq (a,a)$. Since the map $(g(t_2,t_1),t_1) \mapsto (t_2,t_1)$ is a bijection, we must conclude that $g(b,a) \neq g(a,a) = e$. So there do not exist arrows between a and b in the underlying category T_0. This means that a and b are not isomorphic in the underlying category T_0 so that a and b are not G-isomorphic. In particular, while $a \sim_G b$ for all $a,b \in T$, we have $a \cong_G b$ if and only if $a = b$. In other words, the betweenness relation on a G-torsor is not compatible with the underlying category.

4.6 A Category of Enriched Categories

Since we have a notion of betweenness in enriched categories, we may ask what a betweenness preserving morphism of enriched categories is. For that matter, we need a notion of morphism of enriched categories. We will define a category of enriched categories which we will denote by **EnCat**. The author is aware of issues with self referential statements and Russels paradox. We are also aware of strategies around these issues such as the usage of Grothendieck universes and conglomerates. These topics are beyond the scope of this thesis.

Definition 4.34. Denoted $(\mathcal{A}, \mathcal{V})$ for the $\mathcal{V}$-category $\mathcal{A}$, and $(\mathcal{B}, \mathcal{U})$ for the $\mathcal{U}$-category $\mathcal{B}$. A morphism $(\mathcal{A}, \mathcal{V}) \longrightarrow (\mathcal{B}, \mathcal{U})$ of enriched categories consist of

$$\begin{aligned} f : \mathrm{ob}(\mathcal{A}) &\longrightarrow \mathrm{ob}(\mathcal{B}), && \text{a function,} \\ g : \mathcal{U} &\longrightarrow \mathcal{V}, && \text{a monoidal functor,} \end{aligned}$$

such that f is a $\mathcal{V}$-functor when viewing $\mathcal{B}$ as enriched over $\mathcal{V}$ via g. This means that there is an arrow $\mathcal{A}(A, B) \longrightarrow g\mathcal{B}(fA, fB)$ for each pair of objects A, B in $\mathrm{ob}(\mathcal{A})$ such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,C) \\ \downarrow{\scriptstyle f \otimes f} & & \downarrow{\scriptstyle f} \\ g\mathcal{B}(fB,fC) \otimes g\mathcal{B}(fA,fB) & \xrightarrow[g(M)]{} & g\mathcal{B}(fA,fC), \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{j_A} & \mathcal{A}(A,A) \\ & {\scriptstyle g(j_{fA})\varphi_0} \searrow & \downarrow{\scriptstyle f} \\ & & g\mathcal{B}(fA,fA). \end{array}$$

Here the arrow $\varphi_0 : I_{\mathcal{V}} \longrightarrow g(I_{\mathcal{U}})$ denotes the isomorphism that comes with the datum of the monoidal functor g.

Example 4.35. A morphism of enriched categories $(f, \mathrm{id}_{\mathcal{V}}) : (\mathcal{A}, \mathcal{V}) \longrightarrow (\mathcal{B}, \mathcal{V})$ where the monoidal functor is given by the identity functor on $\mathcal{V}$, coincides with the usual notion of $\mathcal{V}$-functor. Thus morphisms of enriched categories are a generalization of $\mathcal{V}$-functors.

Example 4.36. The shiftmap from Example 2.19 can be seen as a morphism of enriched categories. Recall that we have a set $X = \{(x_i)_{i \in \mathbb{N}} : x_i \in \{0, 1\}\}$ of sequences of 0's and 1's equipped with a metric $\rho(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$. This is a category enriched over $\overline{\mathbb{R}}_+$. The shiftmap is given by $S(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$

thus we obtain $\frac{1}{2}\rho(Sx, Sy) = \frac{1}{2}\sum_{i=0}^{\infty}\frac{|x_{i+1}-y_{i+1}|}{2^i} = \sum_{i=1}^{\infty}\frac{|x_i - y_i|}{2^i} \leq \rho(x, y)$. This means that for each pair $x, y \in X$ we have an arrow $\rho(x,y) \longrightarrow \frac{1}{2}\rho(Sx, Sy)$. Since multiplication by $\frac{1}{2}$ is a monoidal functor $\overline{\mathbb{R}}_+ \longrightarrow \overline{\mathbb{R}}_+$, we see that the arrow $(S, \frac{1}{2}\cdot) : (X, \overline{\mathbb{R}}_+) \longrightarrow (X, \overline{\mathbb{R}}_+)$ defines a morphism in **EnCat** because the required diagrams commute. The same reasoning shows that any arbitrary (non constant) lipschitz continuous function $f : A \longrightarrow B$ between metric spaces is a morphism in **EnCat**. These functions satisfy $d_B(fx, fy) \leq K d_A(x, y)$ for some constant $K > 0$. Thus $(f, \frac{1}{K})$ defines such a morphism.

Example 4.37. A geometric morphism $\mathcal{F} \longrightarrow \mathcal{E}$ between toposes is an adjoint pair of functors $f^* \dashv f_*$ with $f^* : \mathcal{E} \longrightarrow \mathcal{F}$ and $f_* : \mathcal{F} \longrightarrow \mathcal{E}$ where f^* preserves finite limits. This can be seen as a morphism of enriched categories $(f^*, f_*) : (\mathcal{E}, \mathcal{E}) \longrightarrow (\mathcal{F}, \mathcal{F})$. Because toposes are Cartesian closed we can think of $\mathcal{F}$ and $\mathcal{E}$ as enriched over themselves via the internal hom functor written out in section 3.2. Note that because f^* preserves finite limits, we have an arrow

$$f^*(B^A) \times f^*(A) \cong f^*(B^A \times A) \xrightarrow{f^*(\mathrm{ev})} f^*(B).$$

This arrow then corresponds to an exponential transpose

$$\overline{f^*_{A,B}} : f^*(B^A) \longrightarrow f^*(B)^{f^*(A)}.$$

Under the adjunction $f^* \dashv f_*$, this in turn corresponds to the arrow

$$f^*_{A,B} : B^A \longrightarrow f_*\left(f^*(B)^{f^*(A)}\right).$$

By the unicity of all involved arrows we find, for each triple of objects $A, B, C \in \mathrm{ob}(\mathcal{E})$, a commutative diagram

$$\begin{array}{ccc} C^B \times B^A & \xrightarrow{\quad M \quad} & C^A \\ \big\downarrow{\scriptstyle f^* \times f^*} & & \big\downarrow{\scriptstyle f^*} \\ f_*\left(f^*(C)^{f^*(B)}\right) \times f_*\left(f^*(B)^{f^*(A)}\right) & \xrightarrow[f_*(M)]{} & f_*\left(f^*(C)^{f^*(A)}\right). \end{array}$$

To see that $f^*M = f_*(M)(f^* \times f^*)$, note that under the adjunction $f^* \dashv f_*$, the arrow f^*M corresponds to the arrow

$$f^*(C^B) \times f^*(B^A) \xrightarrow{f^*(M)} f^*(C^A) \xrightarrow{\overline{f^*}} f^*(C)^{f^*(A)}.$$

Its exponential transpose is given by:

$$\begin{aligned} \mathrm{ev}(\overline{f^*} f^*(M) \times 1) &= \mathrm{ev}(\overline{f^*} \times 1)(f^*(M) \times 1) \\ &= f^*(\mathrm{ev})(f^*(M) \times 1) \\ &= f^*(\mathrm{ev}(M \times 1)) \\ &= f^*(\mathrm{ev}(1 \times \mathrm{ev})a). \end{aligned}$$

Furthermore, the arrow $f_*(M)(f^* \times f^*)$ corresponds under the adjunction $f^* \dashv f_*$ to the arrow

$$f^*(C^B) \times f^*(B^A) \xrightarrow{\overline{f^*} \times \overline{f^*}} f^*(C)^{f^*(B)} \times f^*(B)^{f^*(A)} \xrightarrow{M} f^*(C)^{f^*(A)}.$$

Its exponential transpose is given by

$$\begin{aligned} \mathrm{ev}(M(\overline{f^*} \times \overline{f^*}) \times 1) &= \mathrm{ev}(M \times 1)((\overline{f^*} \times \overline{f^*}) \times 1) \\ &= \mathrm{ev}(1 \times \mathrm{ev})a((\overline{f^*} \times \overline{f^*}) \times 1) \\ &= \mathrm{ev}(\overline{f^*} \times 1)(1 \times \mathrm{ev}(\overline{f^*} \times 1)) \\ &= f^*(\mathrm{ev})(1 \times f^*(\mathrm{ev})) \\ &= f^*(\mathrm{ev}(1 \times \mathrm{ev})a). \end{aligned}$$

Because these exponential transposes are equal, we find that that they correspond to the same arrow, thus $\overline{f^*} f^*(M) = M(\overline{f^*} \times \overline{f^*})$. Consequently, these arrows correspond to the same arrow under the adjunction $f^* \dashv f_*$. This means that $f^*M = f_*(M)(f^* \times f^*)$ so that the square commutes. Similarly, there is a commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{j_A} & A^A \\ & \searrow^{f_*(j_{f^*(A)})\varphi_0} & \downarrow f^* \\ & & f_*(f^*(A)^{f^*(A)}). \end{array}$$

To see that $f^* j_A = f_*(j_{f^*(A)})\varphi_0$ note that under the adjunction $f^* \dashv f_*$ the arrow $f^* j_A$ corresponds to

$$f^*(I) \xrightarrow{f^*(j_A)} f^*(A^A) \xrightarrow{\overline{f^*}} f^*(A)^{f^*(A)}.$$

Its exponential transpose is given by

$$\mathrm{ev}(\overline{f^*} f^*(j_A) \times 1) = \mathrm{ev}(\overline{f^*} \times 1)(f^*(j_A) \times 1) = f^*(\mathrm{ev}(j_A \times 1)) = f^*(\mathrm{id}_A\, l) = f^*(l).$$

The arrow $f_*(j_{f^*(A)})\varphi_0$ corresponds under the adjunction $f^* \dashv f_*$ to the arrow

$$f^*(I) \xrightarrow{\varphi_0^{-1}} I \xrightarrow{j_{f^*(A)}} f^*(A)^{f^*(A)}.$$

Its exponential transpose is given by

$$\mathrm{ev}(j_{f^*(A)}\varphi_0^{-1}\times 1) = \mathrm{ev}(j_{f^*(A)}\times 1)(\varphi_0^{-1}\times 1) = \mathrm{id}_{f^*(A)}\, l(\varphi_0^{-1}\times 1) = f^*(l).$$

Here the last equality follows from Definition 4.2. In particular, we use that the diagram

$$\begin{array}{ccc} I\times f^*(A) & \xrightarrow{\quad l \quad} & f^*(A) \\ {\scriptstyle \varphi_0\times 1}\downarrow & & \uparrow{\scriptstyle f^*(l)} \\ f^*(I)\times f^*(A) & \xrightarrow[\varphi_2]{} & f^*(I\times A) \end{array}$$

commutes. Again, because the exponential transposes are equal, we find by unicity that $\overline{f^*}f^*(j_A) = j_{f^*(A)}\varphi_0^{-1}$. Consequently they correspond to the same arrow under the adjunction $f^* \dashv f_*$. This means that $f^*j_A = f_*(j_{f^*(A)\varphi_0})$. Because all the required diagrams commute, a geometric morphism indeed fits the definition of a morphism in **EnCat**.

Remark 4.38. A pair of morphisms $(\mathcal{A},\mathcal{W}) \xrightarrow{(f_1,g_1)} (\mathcal{B},\mathcal{V}) \xrightarrow{(f_2,g_2)} (\mathcal{C},\mathcal{U})$ in **EnCat** can be composed to give a morphism $(f_2f_1, g_1g_2) : (\mathcal{A},\mathcal{W}) \longrightarrow (\mathcal{C},\mathcal{U})$ consisting of the function f_2f_1 and the monoidal functor g_1g_2. To see the required diagrams commute, consider the square

$$\begin{array}{ccc} \mathcal{A}(B,C)\otimes\mathcal{A}(A,B) & \xrightarrow{\qquad M \qquad} & \mathcal{A}(A,C) \\ {\scriptstyle f_1\otimes f_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ g_1\mathcal{B}(f_1(B)f_1(C))\otimes g_1\mathcal{B}(f_1(A),f_1(B)) & \xrightarrow{\quad g_1(M) \quad} & g_1\mathcal{B}(f_1(A),f_1(C)) \\ {\scriptstyle g_1(f_2\otimes f_2)}\downarrow & & \downarrow{\scriptstyle g_1(f_2)} \\ g_1g_2\mathcal{C}(f_2f_1(B),f_2f_1(C))\otimes g_1g_2\mathcal{C}(f_2f_1(A),f_2f_1(B)) & \xrightarrow[g_1g_2(M)]{} & g_1g_2\mathcal{C}(f_2f_1(A),f_2f_1(C)). \end{array}$$

The upper square commutes since (f_1,g_1) is a morphism in **EnCat**. The lower square commutes since it is the g_1 image of a commutative square. Namely (f_2,g_2) is a morphism in **EnCat**. Similarly, for the unit axiom we have a commutative diagram

$$\begin{array}{ccc} I_{\mathcal{W}} & \xrightarrow{\qquad j_A \qquad} & \mathcal{A}(A,A) \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ g_1(I_{\mathcal{V}}) & \xrightarrow[g_1(j_{f_1(A)})]{} & g_1\mathcal{B}(f_1(A),f_1(A)) \\ {\scriptstyle g_1(\varphi_0)}\downarrow & & \downarrow{\scriptstyle g_1(f_2)} \\ g_1g_2(I_{\mathcal{U}}) & \xrightarrow[g_1g_2(j_{f_2f_1(A)})]{} & g_1g_2\mathcal{C}(f_2f_1(A),f_2f_1(A)). \end{array}$$

The upper square commutes since (f_1, g_1) is a morphism in **EnCat** and the lower square commutes because it is the g_1 image of a commutative square. Thus we have a well defined composition for this definition of morphism in **EnCat**. The fact that this composition is associative follows from the fact that composition on functions and monoidal functors is associative. Explicitly

$$(f_3, g_3)((f_2, g_2)(f_1, g_1)) = (f_3, g_3)(f_2 f_1, g_1 g_2) = (f_3(f_2 f_1), (g_1 g_2) g_3) = \dots$$
$$\dots = ((f_3 f_2) f_1, g_1(g_2 g_3)) = (f_3 f_2, g_2 g_3)(f_1, g_1) = ((f_3, g_3)(f_2, g_2))(f_1, g_1).$$

The identity arrow on an enriched category $(\mathcal{A}, \mathcal{V})$ is given by the pair $(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{V}})$. Thus we have a well defined category **EnCat**.

4.7 Betweenness Preserving Morphisms

Definition 4.39. A morphism of enriched categories $(f, g) : (\mathcal{A}, \mathcal{V}) \longrightarrow (\mathcal{B}, \mathcal{U})$ is called betweenness preserving whenever $[A, B, C]_{\mathcal{V}}$ implies $[fA, fB, fC]_{\mathcal{U}}$.

We denote the category of enriched categories equipped with betweenness preserving morphisms by $\mathbf{EnCat}'$.

Theorem 4.40. *There is a functor* $L : \mathbf{EnCat}' \longrightarrow \mathbf{Bet}$ *of which the action on objects is given by* $L(\mathcal{A}, \mathcal{V}) := (\mathrm{ob}(\mathcal{A})/{\sim_{\mathcal{V}}}, [-,-,-]_{\mathcal{V}})$. *That is, it sends a* $\mathcal{V}$*-category* $\mathcal{A}$ *to the set of* $\mathcal{V}$*-equivalence classes on* $\mathrm{ob}(\mathcal{A})$ *equipped with the betweenness relation* $[-,-,-]_{\mathcal{V}}$.

Proof. It is understood, when we denote $[A]_{\mathcal{V}} := \{B \in \mathrm{ob}(A) : A \sim_{\mathcal{V}} B\}$, that we have $([A], [B], [C]) \in [-,-,-]_{\mathcal{V}}$ if and only if $[A, B, C]_V$. This is a well defined betweenness relation by Theorem 4.24 and Lemma 4.27. It satisfies axiom B3 by construction of the equivalence relation $\sim_{\mathcal{V}}$. Thus, $L(\mathcal{A}, \mathcal{V}) = (\mathrm{ob}(\mathcal{A})/{\sim_{\mathcal{V}}}, [-,-,-]_{\mathcal{V}})$ indeed defines a betweenness space. For the morphism part, Given a betweenness preserving morphism of enriched categories $(f, g) : (\mathcal{A}, \mathcal{V}) \longrightarrow (\mathcal{B}, \mathcal{U})$, we set $L(f, g) := f$. This is well defined, since if $A \sim_{\mathcal{V}} B$, then $[A, B, A]_{\mathcal{V}}$ and $[B, A, B]_{\mathcal{V}}$ hold. By the fact that the morphism (f, g) is in $\mathbf{EnCat}'$ we find that $[fA, fB, fA]_{\mathcal{U}}$ and $[fB, fA, fB]_{\mathcal{U}}$ hold so that $fA \sim_{\mathcal{U}} fB$. In other words $[A] = [B]$ implies $f[A] = [fA] = [fB] = f[B]$. Hence we have a function $f : \mathrm{ob}(\mathcal{A})/{\sim_{\mathcal{V}}} \longrightarrow \mathrm{ob}(\mathcal{B})/{\sim_{\mathcal{U}}}$. This function is a morphism in **Bet** because if $[[A], [B], [C]]_{\mathcal{V}}$ holds, then equivalently $[A, B, C]_{\mathcal{V}}$ holds so that $[fA, fB, fC]_{\mathcal{U}}$ holds. This means $[[fA], [fB], [fC]]_{\mathcal{U}}$ which shows that $L(f, g)$ is a morphism in **Bet**. To see that L is functorial, note that it respects composition and unity

$$L((f_2, g_2)(f_1, g_1)) = L(f_2 f_1, g_1, g_2)) = f_2 f_1 = L(f_2, g_2) L(f_1, g_1),$$

$$L(\mathrm{id}_A, \mathrm{id}_{\mathcal{V}}) = \mathrm{id}_A : \mathrm{ob}(\mathcal{A})/{\sim_{\mathcal{V}}} \longrightarrow \mathrm{ob}(\mathcal{A})/{\sim_{\mathcal{V}}} = \mathrm{id}_{(\mathrm{ob}(\mathcal{A})/\sim_{\mathcal{V}}, [-,-,-]_{\mathcal{V}})}.$$

Example 4.41. We have seen in Example 2.19 and Example 4.36 that the shiftmap is a betweenness preserving morphism of enriched categories.

Example 4.42. Every fully faithfull $\mathcal{V}$-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is betweenness preserving. Recall that a $\mathcal{V}$-functor T is fully faithfull whenever for each pair of objects $A, B \in \mathrm{ob}(\mathcal{A})$ the arrow $\mathcal{A}(A,B) \longrightarrow \mathcal{B}(TA,TB)$ is an isomorphism. By commutativity of the diagram

$$\begin{array}{ccc} \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) & \xrightarrow{M_{ABC}} & \mathcal{A}(A,C) \\ {\scriptstyle T\otimes T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) & \xrightarrow[M_{TATBTC}]{} & \mathcal{B}(TA,TC) \end{array}$$

and by the fact that T and $T \otimes T$ are isomorphisms, we see that M_{ABC} is a split epimorphism if and only if M_{TATBTC} is a split epimorphism. This means that $(T, \mathrm{id}_{\mathcal{V}}) : (\mathcal{A}, \mathcal{V}) \longrightarrow (\mathcal{B}, \mathcal{V})$ is a betweenness preserving morphism of enriched categories.

Example 4.43. Similarly, continuing on Example 4.37, if a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ is also a logical functor, meaning it preserves finite limits, exponentials and subobject identifiers, then it is also betweenness preserving. To see this, recall we have a commutative diagram

$$\begin{array}{ccc} C^B \times B^A & \xrightarrow{M} & C^A \\ {\scriptstyle f^*\times f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ f_*\left(f^*(C)^{f^*(B)}\right) \times f_*\left(f^*(B)^{f^*(A)}\right) & \xrightarrow[f_*(M)]{} & f_*\left(f^*(C)^{f^*(A)}\right). \end{array}$$

and that under the adjunction $f^* \dashv f_*$ this corresponds to a commutative diagram

$$\begin{array}{ccc} f^*(C^B) \times f^*(B^A) & \xrightarrow{f^*(M)} & f^*(C^A) \\ {\scriptstyle \overline{f^*}\times \overline{f^*}}\downarrow & & \downarrow{\scriptstyle \overline{f^*}} \\ f^*(C)^{f^*(B)} \times f^*(B)^{f^*(A)} & \xrightarrow[M]{} & f^*(C)^{f^*(A)}. \end{array}$$

If M_{ABC} is a split epimorphism with section φ_{ABC}, then $f^*(M_{ABC})$ is also a split epimorphism. Since f is a logical functor, we have by definition that the arrows $\overline{f^*}$ and $\overline{f^*} \times \overline{f^*}$ are isomorphisms. Thus it follows that also $M_{f^*(A)f^*(B)f^*(C)}$ is a split epimorphism whose section is given by $(\overline{f^*} \times \overline{f^*}) f^*(\varphi_{ABC}) \overline{f^*}^{-1}$. Indeed logical functors can be seen as betweenness preserving morphisms.

Different examples exist. We will continue to prove a sufficient condition for enriched morphisms to be betweenness preserving.

Definition 4.44. An arrow $B \xrightarrow{f} C$ is said to be a regular epimorphism whenever it fits in a coequalizer diagram

$$A \rightrightarrows B \xrightarrow{f} C.$$

Lemma 4.45. *A split epimorphism is a regular epimorphism.*

Proof. Say $B \xrightarrow{f} C$ is a split epimorphism, let g be its right inverse, then we have a coequalizer diagram

$$B \xrightarrow{f} C \xrightarrow{g} B \xrightarrow{f} C, \qquad \mathrm{id}_B : B \longrightarrow B.$$

Indeed, for any $k : B \longrightarrow X$ that coequalizes id_B and gf, we have $kgf = k$. So k factors through f. To see this factorization is unique, if $hf = k$ is an other factorization, then $hf = kgf$ implies $h = kg$ by the fact that f is an epimorphism.

Remark 4.46. The converse of Lemma 4.45 need not hold. The statement that the two notions coincide is referred to as the external regular axiom of choice.

Definition 4.47. A category $\mathcal{C}$ is called regular if:

i) it has all finite limits,

ii) the kernel pair

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_0} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

of any morphism $f : X \longrightarrow Y$ admits a coequalizer $X \times_Y X \overset{p_0}{\underset{p_1}{\rightrightarrows}} X \longrightarrow E$,

iii) the pullback of a regular epimorphism along any morphism is again a regular epimorphism.

Definition 4.48. In a regular category an object P is called (regular) projective if for every regular epimorphism $f : X \longrightarrow Y$, any arrow $P \longrightarrow Y$ factors through f.

Lemma 4.49. *In a regular category, an object P is projective if and only if every regular epimorphism with codomain P is a split epimorphism.*

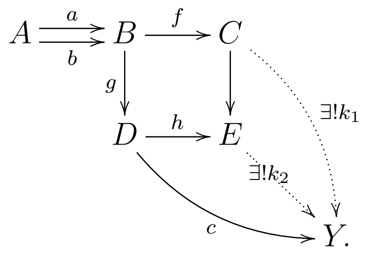
Proof. Suppose that P is projective and let $f : X \longrightarrow P$ be a regular epimorphism. Then the identity arrow $\mathrm{id}_P : P \longrightarrow P$ factors through f. In other words, there exists some $g : P \longrightarrow X$ such that $fg = \mathrm{id}_P$, so f is split. Conversely, suppose every regular epimorphism with codomain P is split. Pick a regular epimorphism $f : X \longrightarrow Y$. For any arrow $g : P \longrightarrow Y$ we have a pullback diagram

$$\begin{array}{ccc} P \times_Y X & \xrightarrow{g^*(f)} & X \\ {\scriptstyle f^*(g)}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow[g]{} & Y. \end{array}$$

Because in a regular category, regular epimorphisms are preserved under pullbacks, we find that the arrow $f^*(g)$ is a regular epimorphism with codomain P. By hypothesis, this means it is a split epimorphism. If k is its section, then $g = gf^*(g)k = fg^*(f)k$. This shows that g factors through f. We conclude that P is projective.

Lemma 4.50. *Regular epimorphisms are preserved under pushouts.*

Proof. Consider the following diagram



Suppose that $A \overset{a}{\underset{b}{\rightrightarrows}} B \xrightarrow{f} C$ is a coequalizer diagram and that the square is a pushout. Given an arrow $D \xrightarrow{c} Y$ such that $c(ga) = c(gb)$, we then have an arrow $B \xrightarrow{cg} Y$ that factors through f by its coequalizer property, i.e. $k_1 f = cg$. This means the outer square is commutative. Since the inner square is a pushout, there is a unique arrow k_2 such that the two triangles commute. In other words, $c = k_2 h$. But this must mean that $A \overset{ga}{\underset{gb}{\rightrightarrows}} D \xrightarrow{h} E$ is a coequalizer diagram. Conclude that h is a regular epimorphism.

Corollary 4.51. *Suppose the external regular axiom of choice holds in the category $\mathcal{V}$. If a $\mathcal{V}$-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ between $\mathcal{V}$-categories satisfies that for all objects $A, B, C \in \mathrm{ob}(\mathcal{A})$, the diagram*

$$\begin{array}{ccc} \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,C) \\ \downarrow{\scriptstyle T\otimes T} & & \downarrow{\scriptstyle T} \\ \mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) & \xrightarrow[M]{} & \mathcal{B}(TA,TC), \end{array}$$

is a pushout diagram. Then T is betweenness preserving.

Proof. If M_{ABC} is a split epimorphism, then it is a regular epimorphism by Lemma 4.45. Because regular epimorphisms are preserved under pushout diagrams, we find that the composition M_{TATBTC} is a regular epimorphism, c.f. Lemma 4.50. Because the notion of regular epimorphism and split epimorphism coincide in $\mathcal{V}$, we find that T is betweenness preserving. ∎

Remark 4.52. From Lemma 4.49 we see that even if $\mathcal{V}$ does not satisfy the regular axiom of choice, but that the hom objects of the $\mathcal{V}$-category $\mathcal{B}$ are projective, then each composition morphism M has as codomain a projective object in $\mathcal{V}$. From the property that the above square is a pushout we find that $\mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) \xrightarrow{M} \mathcal{B}(TA,TC)$ is a regular epimorphism. Then because $\mathcal{B}(TA,TC)$ was assumed to be projective, M is a split epimorphism so that T is betweenness preserving. More generally, the same result holds if in the above Corollary we replace $\mathcal{V}$-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ with morphism of enriched category $(f,g) : (\mathcal{A},\mathcal{V}) \longrightarrow (\mathcal{B},\mathcal{U})$ where g is a monoidal functor that reflects split epimorphisms. Simply because if $g : \mathcal{U} \longrightarrow \mathcal{V}$ is such a monoidal functor, then we can think of $\mathcal{B}$ as a $\mathcal{V}$-category by taking the hom object of $X, Y \in \mathrm{ob}(\mathcal{B})$ to be given by $g\mathcal{B}(X,Y)$. Direct application of Corollary 4.51 provides the result.

5 Betweenness Space as an Enriched Category

We have seen that enriched categories can be seen as a set equipped with a betweenness relation. Conversely, we can think of sets with a betweenness relation, as enriched categories.

Suppose we have a set X with a betweenness relation $B \subseteq X^3$. We can then view X as a category enriched over the powerset of X denoted 2^X, or V if it is clear with respect to which set X and betweenness relation B it is defined. The arrows in V are given by set inclusion, we say there is an arrow $S \to T$ precisely whenever $S \subseteq T$. The tensor product in V is given by intersection and the tensor unit is given by X. The objects of this V-category X are given by the points in X. Given a pair of points $x, y \in X$, there is a hom object $X(x, y)$ in V given by $X(x, y) := \{a \in X : [a, y, x]_B\}$. Furthermore, we have a composition law

$$X(x, y) \cap X(y, z) \subseteq X(x, z).$$

Namely, if $a \in X(x, y) \cap X(y, z)$ then both $[a, y, x]_B$ and $[a, z, y]_B$ hold. By the transitivity axiom B4, this means that $[a, z, x]_B$ holds so that $a \in X(x, z)$, which proves the inclusion holds. Moreover, for each $x \in X$ there is an identity arrow

$$X \subseteq X(x, x).$$

Because $a \in X$ implies that $[a, x, x]_B$ by reflexivity B2. Since V is posetal, the unit and associativity diagrams commute.

Definition 5.1. The enriched category $(X, 2^X)$ constructed above, is called the associated enriched category to the betweenness space (X, B).

Proposition 5.2. *In the associated enriched category $(X, 2^X)$, the notions of 2^X-isomorphic and 2^X-equivalence and equality coincide.*

Proof. From Corollary 4.29 we know that $a \cong_V b$ implies $a \sim_V b$. For the other direction, suppose $a \sim_V b$, then $[a, b, a]_V$ holds. This means that we have

$$X(a, b) \cap X(b, a) = X(a, a).$$

Since $X(a, a) = X$, the set inclusions $X \subseteq X(a, b)$ and $X \subseteq X(b, a)$ hold. This gives arrows $f : a \longrightarrow b$ and $g : b \longrightarrow a$ in the underlying category X_0. Their composition gf is given by the arrow

$$X \subseteq X(a, b) \cap X(b, a) = X(a, a),$$

which is the identity arrow id_a. Similarly $fg = \mathrm{id}_b$. This shows that $a \cong_V b$. Notice that since $X = X(a, b) = X(b, a)$ we have that $a \in X(a, b)$ and $b \in X(b, a)$ so that $[a, b, a]_B$ and $[b, a, b]_B$ hold, which implies $a = b$ by axiom B3. ∎

5.1 Compatibility

Since any associated enriched category induces a betweenness relation on its own, we can ask if V-betweenness coincides with the relation B. With regards to this question, we have the following results.

Proposition 5.3. *$[a, x, b]_V$ implies $[a, x, b]_B$.*

Proof. Suppose that $[a, x, b]_V$, then $X(x, b) \cap X(a, x) = X(a, b)$. Hence, for all $y \in X$ we have that $[y, a, b]_B$ implies $[y, a, x]_B$ and $[y, x, b]_B$. Since $[a, a, b]_B$ holds we find by substituting $y = a$ that $[a, a, x]_B$ and $[a, x, b]_B$ hold. ∎

Proposition 5.4. *$[a, x, b]_V$ is equivalent to $[a, x, b]_B$ if and only if B (and then also V) satisfies the cancelation axiom B5.*

Proof. Suppose $[a, x, b]_V$ and $[a, x, b]_B$ are equivalent. We want to prove they satisfy the cancelation property. Suppose that $[a, x, b]_B$ and $[a, y, x]_B$ hold, we want to conclude $[y, x, b]_B$. By the equivalence of betweenness relations, we find that $[a, y, x]_V$ holds. This means that $X(a, y) \cap X(y, x) = X(a, x)$. Now since $[a, x, b]_B$ is assumed to hold, we have by symmetry that $[b, x, a]_B$ holds, and therefore we know that $b \in X(a, x)$. It follows from $X(a, y) \cap X(y, x) = X(a, x)$ that $b \in X(y, x)$. This means that $[b, x, y]_B$ holds and by symmetry we find $[y, x, b]_B$ so that the betweenness relation satisfies the cancelation property.

Conversely, if B satisfies the cancelation property, then we want to see that $[a, x, b]_B$ implies $[a, x, b]_V$, since the reverse implication is given for free. Suppose that $[a, x, b]_B$ holds, then for all $y \in X(a, b)$ we have $[a, b, y]_B$. It follows that $[a, x, y]_B$ and $[x, b, y]_B$ hold by transitivity and cancelation. In particular, $y \in X(a, x) \cap X(x, b)$. This means that $X(a, b) = X(a, x) \cap X(x, b)$ so that $[a, x, b]_V$ holds. ∎

5.2 Functoriality

From the above discussion, we know each betweenness space has an associated enriched category. It turns out that this defines the object part of a functor $R : \mathbf{Bet} \longrightarrow \mathbf{EnCat}$. For the morphism part, consider two sets X and Y equipped with betweenness relations $[-, -, -]_X$ and $[-, -, -]_Y$ respectively. Given a betweenness preserving function $f : X \longrightarrow Y$, we know by definition that $[a, x, b]_X$ implies $[f(a), f(x), f(b)]_Y$. We have the following proposition.

Proposition 5.5. *A function $f : X \to Y$ is betweenness preserving if and only if*

$$f^{-1}(Y(f(a), f(b))) = \bigcup_{[f(a), f(b), f(z)]_Y} X(a, z).$$

Proof. Suppose that f is betweenness preserving. Let $x \in \bigcup_{[f(a),f(b),f(z)]_Y} X(a,z)$. We have that there exists some $z \in X$ for which $[f(a), f(b), f(z)]_Y$ holds, such that $x \in X(a,z)$. This means that $[a,z,x]_X$ holds, and since f preserves betweenness, we also have $[f(a), f(z), f(x)]_Y$. Then by transitivity it follows that $[f(a), f(b), f(x)]_Y$. Consequently we must find that $x \in f^{-1}(Y(f(a), f(b)))$. To show the reverse inclusion, take any $x \in f^{-1}(Y(f(a), f(b)))$. Since $[f(a), f(b), f(x)]_Y$ holds, we must have that $x \in X(a,x) \subseteq \bigcup_{[f(a),f(b),f(z)]_Y} X(a,z)$. Equality is demonstrated.

Conversely, suppose that the equality of sets holds, we have to show that f preserves betweenness. Thus, for $[a,b,c]_X$ we have to show that $[f(a), f(b), f(c)]_Y$. Indeed, if $[a,b,c]_X$, then $c \in X(a,b)$. Because $[f(a), f(b), f(b)]_Y$ holds, we have that

$$c \in X(a,b) \subseteq \bigcup_{[f(a),f(b),f(z)]_Y} X(a,z) = f^{-1}(Y((f(a), f(b))).$$

This means that $f(c) \in Y(f(a), f(b))$, or in other words, $[f(a), f(b), f(c)]_Y$. This completes the proof.

Remark 5.6. The preimage $f^{-1} : 2^Y \longrightarrow 2^X$ is a monoidal functor since it preserves intersections and inclusions. Via this functor, we can view Y as a set enriched over 2^X if we take the hom-objects to be given by $f^{-1}(Y(x,y)) \subseteq 2^X$. For any triple of objects $a, b, c \in Y$, the composition morphism then becomes

$$f^{-1}(Y(a,b)) \cap f^{-1}(Y(b,c)) \subseteq f^{-1}(Y(a,c)).$$

For any object $a \in Y$, the unit element is given by $X \subseteq f^{-1}(Y) \subseteq f^{-1}(Y(a,a))$.

Proposition 5.7. *If* $f : X \longrightarrow Y$ *is a betweenness preserving function, then* (f, f^{-1}) *is a morphism of enriched categories* $(X, 2^X) \longrightarrow (Y, 2^Y)$.

Proof. Note that we have a commutative diagram:

$$\begin{array}{ccc} X(a,x) \cap X(x,b) & \longrightarrow & X(a,b) \\ \downarrow & & \downarrow \\ f^{-1}(Y(fa,fx)) \cap f^{-1}(Y(fx,fb)) & \longrightarrow & f^{-1}(Y(fa,fb)). \end{array}$$

Here the arrows are set inclusions. To see that $X(a,b) \subseteq f^{-1}(Y(fa,fb))$, take $y \in X(a,b)$ so that $[a,b,y]_X$ holds. Then since f is betweenness preserving, $[fa, fb, fy]_Y$ holds. This means that $y \in f^{-1}(Y(fa,fb))$.

Similarly, we have a diagram

$$\begin{array}{ccc} X & \longrightarrow & X(x,x) \\ & \searrow & \downarrow \\ & & f^{-1}(Y(fx,fx)). \end{array}$$

Thus the function $f : X \longrightarrow Y$ together with the collection of set inclusions $f_{a,b} : X(a,b) \longrightarrow f^{-1}(Y(fa, fb))$ can be thought of as a 2^X-functor between the 2^X-categories X and Y where we view Y as enriched over 2^X via f^{-1}.

Theorem 5.8. *There is a faithful functor* $R : \mathbf{Bet} \longrightarrow \mathbf{EnCat}$.

Proof. We have already defined the action of this functor on the objects and morphisms. Betweenness spaces are mapped to their associated enriched category, and betweenness preserving functions are mapped to their associated morphism of enriched categories. What is left to be shown is that R respects composition and unity. Given betweenness spaces X, Y and Z along with betweenness preserving morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we want to verify that $R(gf) = R(g)R(f)$. Indeed, $R(gf)$ consists of a function $gf : X \longrightarrow Z$ and a monoidal functor $(gf)^{-1} : 2^Z \longrightarrow 2^X$. But $(gf)^{-1} = f^{-1}g^{-1}$. We find commutative diagrams

$$\begin{array}{ccc}
X(a,b) \cap X(b,c) & \longrightarrow & X(a,c) \\
\downarrow & & \downarrow \\
f^{-1}Y(f(a), f(b)) \cap f^{-1}(f(b), f(c)) & \longrightarrow & f^{-1}Y(f(a), f(c)) \\
\downarrow & & \downarrow \\
f^{-1}g^{-1}Z(gf(a), gf(b)) \cap f^{-1}g^{-1}Z(gf(b), gf(c)) & \longrightarrow & f^{-1}g^{-1}Z(gf(a), gf(c))
\end{array}$$

and

$$\begin{array}{ccc}
 & \nearrow & X(a,a) \\
 & & \downarrow \\
X & \longrightarrow & f^{-1}Y(f(a), f(a)) \\
 & & \downarrow \\
 & \searrow & f^{-1}g^{-1}Z(gf(a), gf(a)).
\end{array}$$

Commutativity of the top square and top triangle has been shown in Proposition 5.7. Commutativity of the lower square and lower triangle is seen by noting that it is the image under the monoidal functor f^{-1} of a commutative diagram. Thus the outside square and triangle also commute. But this states that $R(gf) = R(g)R(f)$. To see that R respects unity, consider the betweenness preserving function $\mathrm{id}_X : X \longrightarrow X$. We have that $R(\mathrm{id}_X)$ consists of a function id_X and a monoidal functor $\mathrm{id}_X^{-1} : 2^X \longrightarrow 2^X$. such that the diagrams

$$\begin{array}{ccc} X(a,b) \cap X(b,c) & \longrightarrow & X(a,c) \\ \downarrow & & \downarrow \\ \mathrm{id}_X^{-1} X(\mathrm{id}_X(a), \mathrm{id}_X(b)) \cap \mathrm{id}_X^{-1} X(\mathrm{id}_X(b), \mathrm{id}_X(c)) & \longrightarrow & \mathrm{id}_X^{-1} X(\mathrm{id}_X(a), \mathrm{id}_X(c)) \end{array} ,$$

and

$$\begin{array}{ccc} X & \longrightarrow & X(a,a) \\ & \searrow & \downarrow \\ & & \mathrm{id}_X^{-1} X(\mathrm{id}_X(a), \mathrm{id}_X(a)) \end{array}$$

commute. But this is the identity arrow of $R(X)$, thus $R(\mathrm{id}_X) = \mathrm{id}_{R(X)}$. To see that R is faithful, Suppose that $R(g) = R(f)$ for a pair of betweenness preserving morphisms f and g. Since $(f, f^{-1}) = R(f) = R(g) = (g, g^{-1})$ we have $f = g$.

Let us denote $\mathbf{Bet}_{(B5)}$ for the category of betweenness spaces that satisfy the cancellation axiom B5 with betweenness preserving functions as the morphisms. If we restrict the functor R to $\mathbf{Bet}_{(B5)}$ we land in $\mathbf{EnCat}'$.

Corollary 5.9. *There is a faithful functor* $R : \mathbf{Bet}_{(B5)} \longrightarrow \mathbf{EnCat}'$.

Proof. From Proposition 5.4 we know that under axiom B5 we have for a betweenness space (X, B) that the betweenness relation $[-,-,-]_{V_X}$ in the associated enriched category is equivalent to the betweenness relation $[-,-,-]_X$ of the original betweenness space (X, B). Thus, given a betweenness preserving function $f : X \longrightarrow Y$, we find that $R(f) = (f, f^{-1}) : (X, 2^X) \longrightarrow (Y, 2^Y)$ is a betweenness preserving morphism of enriched categories. Writing this out explicitly, if $[a,b,c]_{V_X}$ holds, then equivalently $[a,b,c]_X$ holds. Since f is in $\mathbf{Bet}_{(B5)}$ we have $[f(a), f(b), f(c)]_Y$ which is equivalent to $[f(a), f(b), f(c)]_{V_Y}$. Conclude the morphism $R(f)$ is betweenness preserving.

Remark 5.10. Denoting for ι the inclusion functor, we find that the following diagram commutes

$$\begin{array}{ccc} \mathbf{Bet}_{(B5)} & \xrightarrow{\iota} & \mathbf{Bet} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ \mathbf{EnCat}' & \xrightarrow[\iota]{} & \mathbf{EnCat}. \end{array}$$

Corollary 5.11. *There is a commutative diagram*

$$\begin{array}{ccc} \mathbf{Bet}_{(B5)} & \xrightarrow{\iota} & \mathbf{Bet} \\ {\scriptstyle R}\downarrow & \nearrow_{L} & \\ \mathbf{EnCat}'. & & \end{array}$$

Proof. For a betweenness space (X, B) we have

$$LR(X, B) = L(X, 2^X) = (X/\sim_V, [-, -, -]_V) = (X, B)$$

where the last equality follows directly from Propositions 5.2 and 5.4. Moreover, for an arrow $f : X \longrightarrow Y$ in $\mathbf{Bet}_{(B5)}$ we have $LR(f) = L(f, f^{-1}) = f$. ∎

Remark 5.12. Let us denote $\mathbf{EnCat}'_{(B5)}$ for the full subcategory of $\mathbf{EnCat}'$ whose objects consist of those enriched categories $(\mathcal{A}, \mathcal{V})$ for which the relation $[-, -, -]_{\mathcal{V}}$ satisfies axiom B5. Then we can think of L as a functor $L : \mathbf{EnCat}'_{(B5)} \longrightarrow \mathbf{Bet}_{(B5)}$

Corollary 5.13. *We can restrict* L *to find a pair of functors*

$$\mathbf{Bet}_{(B5)} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{EnCat}'_{(B5)}$$

satisfying $LR = \mathrm{id}_{\mathbf{Bet}_{(B5)}}$. ∎

Remark 5.14. Since $\mathrm{id}_{\mathbf{Bet}_{(B5)}}$ is an epimorphism, also L is an epimorphism. This shows that we are justified in referring to betweenness relations that satisfy axiom B5, as being *metrizable*. There is some enriched category that induces the betweenness relation through its composition morphism.

6 Concluding Remarks

If there were an adjuction $L \dashv R$ then by definition we would have that for every object (X, B_X) in $\mathbf{Bet}_{(B5)}$, there exists an object $R(X, B_X)$ in $\mathbf{EnCat}'_{(B5)}$ and a morphism $\epsilon_{(X,B_X)} : LR(X, B_X) \longrightarrow (X, B_X)$, such that for every object $(\mathcal{A}, \mathcal{V})$ in $\mathbf{EnCat}'_{(B5)}$ and every morphism $f : L(\mathcal{A}, \mathcal{V}) \longrightarrow (X, B_X)$ there exists a unique morphism $g : (\mathcal{A}, \mathcal{V}) \longrightarrow R(X, B_X)$ that makes the following diagram commute.

$$\begin{array}{ccc} L(\mathcal{A}, \mathcal{V}) & & \\ {\scriptstyle L(g)}\downarrow & \searrow^{f} & \\ LR(X, B_X) & \xrightarrow[\epsilon]{} & (X, B_X). \end{array}$$

Since we know that $LR = \mathrm{id}_{\mathbf{Bet}_{(B5)}}$, commutativity boils down to the requirement that $f = L(g)$. Note that $g : (\mathcal{A}, \mathcal{V}) \longrightarrow R(X, B_X) = (X, 2^X)$ is given by a tuple (g_1, g_2) where $g_1 : \mathrm{ob}(\mathcal{A}) \longrightarrow X$ is a function and $g_2 : 2^X \longrightarrow \mathcal{V}$ is a monoidal functor. Thus we would obtain $f = L(g) = L(g_1, g_2) = g_1$. In other words, $g = (f, g_2)$. This means that if there is an adjunction $L \dashv R$, then every betweenness preserving function $f : L(\mathcal{A}, \mathcal{V}) = (\mathrm{ob}(\mathcal{A})/\!\sim_{\mathcal{V}}, [-,-,-]_{\mathcal{V}}) \longrightarrow (X, B_X)$ uniquely determines a monoidal functor $g_2 : 2^X \longrightarrow \mathcal{V}$ that makes the tuple $(f, g_2) : (\mathcal{A}, \mathcal{V}) \longrightarrow R(X, B_X) = (X, 2^X)$ a betweenness preserving morphism of enriched categories. Thus for each pair of objects $A, B \in \mathrm{ob}(\mathcal{A})$ there is an arrow $f_{A,B} : \mathcal{A}(A, B) \longrightarrow g_2(X(fA, fB))$ making the required diagrams commute. However, we have the following proposition.

Proposition 6.1. *Given a pair of adjoint functors*

$$\mathcal{C} \underset{U}{\overset{F}{\leftrightarrows}} \mathcal{D} \quad (\bot)$$

the following are equivalent:

i) The right adjoint U is fully faithful.

ii) The counit $\varepsilon : FU \longrightarrow 1_{\mathcal{C}}$ of the adjunction is a natural isomorphism of functors.

Now note that while $R : \mathbf{Bet}_{(B5)} \longrightarrow \mathbf{EnCat}'_{(B5)}$ is faithful, it is not full. For example consider $\mathbb{Z}$ and $\mathbb{Q}$ both equipped with the betweenness relation $[a, x, b]$

if and only if $a \leq x \leq b$ or $b \leq x \leq a$. This betweenness relation satisfies axiom B5. Furthermore, the function $f : \mathbb{Z} \longrightarrow \mathbb{Q} : x \longmapsto \frac{x}{1}$ is betweenness preserving. Then there is a morphism of enriched categories $(f, g) : R(\mathbb{Z}) \longrightarrow R(\mathbb{Q})$ given by

$$f : \mathbb{Z} \longrightarrow \mathbb{Q} : x \longmapsto \frac{x}{1}$$
$$g : 2^{\mathbb{Q}} \longrightarrow 2^{\mathbb{Z}} : S \longmapsto \mathbb{Z}$$

where f is the function defined above and g is the constant monoidal functor that sends everything to $\mathbb{Z}$. Then we indeed find commutative diagrams

$$\begin{array}{ccc} \mathbb{Z}(a,b) \cap \mathbb{Z}(b,c) & \longrightarrow & \mathbb{Z}(a,c) \\ \downarrow & & \downarrow \\ g\mathbb{Q}(fa,fb) \cap g\mathbb{Q}(fb,fc) & \longrightarrow & g\mathbb{Q}(fa,fc), \end{array}$$

and

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}(a,a) \\ & \searrow & \downarrow \\ & & g\mathbb{Q}(fa,fa). \end{array}$$

Moreover, since $[-,-,-]_V$ is equivalent to $[-,-,-]$, we see that this morphism of enriched categories is betweenness preserving. However, g is not the preimage of a function. To see this take $x \neq y$ in $\mathbb{Q}$, then $g\{x\} \cap g\{y\} = \mathbb{Z}$. For any preimage h^{-1} of a function h we have $h^{-1}\{x\} \cap h^{-1}\{y\} = \emptyset$. This means that (f, g) is not of the form $R(h)$ for any betweenness preserving function h and consequently we find that the functor R does not act fully on the morphisms in $\mathbf{Bet}_{(B5)}$. Because $LR = \mathrm{id}_{\mathbf{Bet}_{(B5)}}$, we see that $L \not\dashv R$ do not form an adjunction for otherwise, by the fact the counit would be a natural isomorphism, Proposition 6.1 would show R is fully faithful, A contradiction.

Thus with the current definition of betweenness preserving morphism of enriched categories there can not be an adjunction $L \dashv R$. As a sugestion for further research we propose to formulate a restriction on the morphisms in $\mathbf{EnCat}'_{(B5)}$ so that R is a fully faithful functor. It would be interesting to see if for this restricted version of morphism there can be an adjunction between L and R. In that case the category $\mathbf{Bet}_{(B5)}$ would be a reflective subcategory of some category of enriched categories.

References

[1] Jaap van Oosten. *Basic Category Theory.* 2002.

[2] Jaap van Oosten. *Topos Theory.* 2021.

[3] M. Pasch. *Vorlesungen über neuere Geometrie.* B.G. Teubner, Leipzig,, 1882.

[4] Victor Pambuccian. The axiomatics of ordered geometry: I. ordered incidence spaces. *Expositiones Mathematicae*, 29, issue 1:24–66, 2011.

[5] Everett Pitcher and M.F. Smiley. Transitivities of betweenness. *Transactions of the American Mathematical Society*, 52, No.1:95–114, jul., 1941.

[6] Robert Mendris and Pavol Zlatos. Axiomatization and undecidability results for metrizable betweenness relations. *Proceedings of the American Mathematical Society*, 123, No.3:873–882 (10 pages), 1995.

[7] Paul Bankston. Road systems and betweenness. *Bulletin of Mathematical Sciences*, 3, No.3:389–408 (19 pages), 2013.

[8] Kubiś et al. Monotone mappings and lines. *arXiv:2004.14301v2*, 2022.

[9] Aisling McCluskey Daron Anderson, Paul Bankston. Convexity in topological betweenness structures. *Topology and its Applications*, 304:1 – 20, 2021.

[10] Xiaomin Chen. The sylvester-chvátal theorem. *Discrete & Computational Geometry*, 35:193–199, 2006.

[11] Vasek Chvátal. On reichenbach's causal betweenness. *Erkenntnis*, pages 41–48, 2011.

[12] Cardin. Betweenness spaces: Morphism and aggregation functions. *Modeling Decisions for Artificial Intelligence*, 16th International Conference:92–97, 2019.

[13] Baets Pérez-Fernández. The role of betweenness relations, monometrics and penalty functions in data aggregation. *17th World Congress of International Fuzzy Systems Association and 9th International Conference on Soft Computing and Intelligent Systems (IFSA-SCIS)*, 2017.

[14] F.William Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 43:135 – 166, 1973.

[15] G.M. Kelly. *Basic Concepts of Enriched Category Theory.* Cambridge University Press, 1 edition, 2005.

[16] K. Menger. Untersuchungen über algemeine metrik. *Mathematische Annalen*, 100:75–163, 1928.

[17] V. Glivenko. Geometrie des systemes de choses normees. *American Journal of Mathematics*, 58, No.4:799 – 828 (30 pages), 1936.

[18] V. Glivenko. Contribution à l'Étude des systèmes de choses normées. *American Journal of Mathematics*, 59, No.4:941–956 (16 pages), 1937.

[19] M.F. Smiley. A comparison of algebraic, metric, and lattice betweenness. *Bulletin of the American Mathematical Society*, 49:264–252, 1942.

[20] H.P. Zhang B. de Baets, R. Pérez-Fernández. Topologies induced by the representation of a betweenness relation as a family of order relations. *Topology and its Applications*, 258:100–114, 2019.