Bachelor Thesis

Invariants of Planar Random Geometric Graphs

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Abstract

In this paper we look at Random Geometric Graphs, these are graphs constructed from random distributed points on a space and with vertices connected by an edge if they are sufficiently close to each other. We are specifically interested in the case where the vertices are distributed on the plane.

We also give attention to the special case where the distribution is the uniform distribution on the unit-square. Furthermore, the threshold distance for connecting pairs of vertices is chosen such that it fixes the expected average degree of each vertex.

In these Random Geometric Graphs, we are interested in several graph invariants. Our main focus lies on the clique number ω, the maximum number points that are all pair-wise connected, and the chromatic number χ, the minimum number of colours needed to colour the vertices such that no two adjacent vertices are of the same colour. A large part of this paper is inspired by the paper of McDiarmid [Colin McDiarmid. "Random channel assignment in the plane". In: *Random Structures and Algorithms* 22.2 (Mar. 2003)].

Bachelor's Project Mathematics

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1 Introduction

The book from Mathew D. Penrose [7], covers many results of random geometric graphs, also on the clique number, chromatic number and maximal degree within the graphs. It also considers several definitions for random geometric graphs. It also provides applications within statistics and computer science. However The article that this thesis is mainly based on is the article from McDiarmid, *Random Channel Assignments in the Plane* [4]. This thesis aims to simplify the proofs and give insight into the structure of them. Besides giving a concise overview of the subject, the thesis integrates a graph invariant from [6] into the results of [4]. And the proof of the upper bound in the "dense case" is simplified by removing the concept of a triangular lattice from [4].

Let us start with the definition of the **planar random geometric graphs**, mostly known as *random scaled unit disk graphs.* For $n \in \mathbb{N}$ consider the following random geometric graph $G_n = G(V_n, d)$ containing n vertices. We have a common probability distribution X over the plane, where X is a continuous distribution. We have the i.i.d. points $X_1, X_2, \ldots,$ distributed according to the distribution X. The set of vertices V_n contains the first n of these points. We also have a threshold distance d for connecting vertices of the graph, where any pair of vertices is connected when the Euclidean distance between them is smaller than the threshold distance d. We let ν_{max} denote the supremum of the values that the distribution function X takes. This definition of random geometric graphs is also used in [4], for other definitions see [7].

We are only interested in the case where $d \to 0$ as $n \to \infty$. We differentiate between a sparse case and a dense case. The sparse case assumes that d^2n is $o(\ln n)$ and $n^{o(1)}$, while the dense case assumes that $\frac{d^2n}{\ln(n)} \to \infty$ as $n \to \infty$. We also give special attention to the case where the expected degree of every vertex stays constant as $n \to \infty$, more on that in section 2.1.

We need the following definitions for sets in the probability space: For a Lebesgue-set S, let $\lambda(S)$ be the area of S and let $\nu(S)$ be the probability that X is an element of S.

Consider the following definitions for graph invariants on an arbitrary graph G. These definitions are inspired by [4], except for the definition of the **stability quotient** ψ, which can be found in [6]. Despite [4] only giving results for the other five graph invariants, my contribution includes extending these results to also apply to this stability quotient ψ.

Let ω^- denote the **hitting number**, defined as the maximum number of points $X_1, \ldots, X_n$ in an open disk in the plane of diameter d centred around any point P in the plane: $\omega^- := \max_{P \in \mathbb{R}^2} |\{x \in X_1, \ldots, X_n \mid x \in B(P, \frac{d}{2})\}|$.

Let ω be the **clique number**, the maximum number of points of V all pair-wise connected by an edge.

Let ψ be the **stability quotient**, as also defined in [6], as the supremum over all subgraphs H of G of $\frac{|H|}{\alpha(H)}$, where the stability number $\alpha(H)$ denotes the order of the largest independent subset of H. So we define $\psi := \sup_{H \subset G} \frac{|H|}{\alpha(H)}$[prox].

Let χ be the **chromatic number**, the minimum number of colours in all vertex-colourings of G_n such that all pairs of neighbouring vertices have distinct colours.

Let δ^* be the **degeneracy** of the graph G, where δ^* is the smallest integer for which all subgraphs $H \subset G$ have a vertex of degree at most δ^*.

Let Δ denote the **maximal degree** of the graph G.

In Lemma 1, we have an ordering for the aforementioned graph invariants within G_n. The first inequality only applies to proximity graphs, while the other inequalities are true within any arbitrary graph G. The main focus of the thesis is the clique number ω and the chromatic number χ. However, the other graph invariants are used for finding the required upper and lower bounds.

Lemma 1. $\omega^- \leq \omega \leq \psi \leq \chi \leq \delta^* + 1 \leq \Delta + 1$

This lemma is essential for introducing the main results that are Theorem 1 and Theorem 2.Now consider the following notation. For probability functions $A(n)$ and $B(n)$ with variable $n \in \mathbb{N}$, we write $A(n) \sim B(n)$ when as $n \to \infty$ we have $\frac{A(n)}{B(n)} \to 1$ in probability. Which leads us to the main two theorems, we follow the distinction by [4] of a sparse case and a dense case. Where it is also mentioned that these terms do not hold the usual definition from graph theory. We first give the results of the sparse case:

Theorem 1. *Suppose that as $n \to \infty$ we have $d \to 0$ and we have that $d^2 n$ is $o(\ln n)$ and $n^{o(1)}$. Let*

$$\mathcal{K}_n = \frac{\ln n}{\ln\left(\frac{\ln n}{d^2 n}\right)},$$

We have $\mathcal{K}_n \to \infty$ as $n \to \infty$, and we have for the aforementioned graph invariants:

$$\omega^-(G_n) \sim \mathcal{K}_n, \omega(G_n) \sim \mathcal{K}_n, \psi(G_n) \sim \mathcal{K}_n, \chi(G_n) \sim \mathcal{K}_n, \delta^*(G_n) \sim \mathcal{K}_n \text{ and } \Delta(G_n) \sim \mathcal{K}_n.$$

The proof of Theorem 1 can be found in section 2.1, it consists of a lower bound on the hitting number $\omega^-(G_n)$ and an upper bound for the maximal degree $\Delta(G_n)$. We now continue with the results on the dense case. For the results on the degeneracy $\delta^*(G_n)$ and the maximal degree $\Delta(G_n)$ in the dense case, we refer to Theorem 2.3 of [4]. Where the corresponding proofs are seperated into lower and upper bounds. The proof on the upper bounds of $\delta^*(G_n)$ and $\Delta(G_n)$ can be found in Lemma 5.6, the proof of the lower bound of $\Delta(G_n)$ in Lemma 5.7 and Lemma 5.8 and the proof of the lower bound of $\delta^*(G_n)$ in Lemma 5.9 and Lemma 5.10, all in the aformentioned article [4].

Theorem 2. *Suppose that as $n \to \infty$ we have $d \to 0$ and also we have $\frac{d^2 n}{\ln(n)} \to \infty$ as $n \to \infty$. Then for $\mathcal{K}_n = \nu_{max} \frac{\pi}{4} d^2 n$, we have:*

$$\omega^-(G_n) \sim \mathcal{K}_n,\ \omega(G_n) \sim \mathcal{K}_n,\ \psi(G_n) \sim \frac{2\sqrt{3}}{\pi}\mathcal{K}_n,\ \chi(G_n) \sim \frac{2\sqrt{3}}{\pi}\mathcal{K}_n.$$

The proof of this theorem can also be found in section 2.1. The proof consists of lower bounds on the hitting number $\omega^-(G_n)$ and the stability quotient $\psi(G_n)$ and upper bounds on the clique number $\omega(G_n)$ and the chromatic number $\chi(G_n)$.

2 Results

Here we present the main results of the paper. Most proofs have been inspired by [4], and it can be assumed that the results are from [4], unless stated otherwise.

In section 2.1, titled "Main results" we give proofs on the theorems and lemma provided in the introduction. The bounds for the proofs of this section are given in the other three sections. Section 2.2 will provide proofs on two theorems with regard to the sparse case. Meanwhile, section 2.3 and 2.4 will provide lower bounds and, respectively, upper bounds for the graph invariants.

2.1 Main results

We first prove the lemma of the ordering of the graph invariants of G_n.

Lemma 1. *Suppose that as $n \to \infty$ we have $d \to 0$ and we have that $d^2 n$ is $o(\ln n)$ and $n^{o(1)}$. Let*

$$\mathcal{K}_n = \frac{\ln n}{\ln\left(\frac{\ln n}{d^2 n}\right)},$$

We have $\mathcal{K}_n \to \infty$ as $n \to \infty$, and we have for the aforementioned graph invariants:

$$\omega^-(G_n) \sim \mathcal{K}_n, \omega(G_n) \sim \mathcal{K}_n, \psi(G_n) \sim \mathcal{K}_n, \chi(G_n) \sim \mathcal{K}_n, \delta^*(G_n) \sim \mathcal{K}_n \text{ and } \Delta(G_n) \sim \mathcal{K}_n.$$

Proof Lemma 1. (a) All of the points of $X_1, \ldots, X_n$ that are in a disk with diameter d also form a clique in G_n. Therefore no such disk can contain more points than the clique number ω, so we find $\omega^- \leq \omega$.

(b) For a largest clique in the graph $C_{max} \subset G_n$ we have: $\frac{|C_{max}|}{\alpha(C_{max})} = \omega$, therefore using the definition of the stability quotient ψ we find $\omega \leq \psi$.

(c) By definition of χ there is a colouring that partitions G_n into χ stable sets $S_1, \ldots, S_\chi$. For any subgraph $H \subset G_n$, the induced sets $S_1', \ldots, S_\chi'$ of $S_1, \ldots, S_\chi$ in H are also stable and thus each have at most $\alpha(H)$ vertices. The induced sets $S_1', \ldots, S_\chi'$ also form a partition of H, for the number of vertices $|H|$ we find therefore $|H| \leq \chi \cdot \alpha(H)$. Since the argument holds for any subgraph $H \subset G_n$ we conclude that $\psi := \sup_{H \subset G_n} \frac{|H|}{\alpha(H)} \leq \chi$.

(d) By Proposition 1 of [3], the degeneracy $\delta * (G)$ of a graph G, with n vertices, is equivalent to the smallest integer k for which there is an ordering of the vertices $v_1, \ldots, v_n$ in which the the degree of every vertex v_i is at most k within the induced subgraph of vertex set $\{v_1, \ldots, v_i\}$. Consider such an ordering for the vertices of G, we colour the graph with $\delta^* + 1$ colours by giving every vertex, represented as v_i in the ordering, a colour different from its neighbours in the induced subgraph of the vertex set $\{v_1, \ldots, v_i\}$. From Proposition 1 in [3] we know that there are at most δ^* such neighbours, so for every vertex v_i there is a colour that has not been used before in the ordering. We conclude that $\chi(G) \leq \delta^* + 1$.

(e) Any vertex in a graph G is smaller than the maximum degree $\Delta(G)$, therefore all vertices of all subgraphs $H \subset G$ have degree at most $\Delta(G)$. Giving us $\delta^* + 1 \leq \Delta + 1$. □

The following two theorems will allow us to prove the results on the graph invariants in the sparse case from Theorem 1. The first theorem provides us with a lower bound for $\omega^-(G_n)$ and the second with an upper bound for $\Delta(G_n)$. By the graph invariant ordering of Lemma 1, the result of Theorem 3 is also true for the other graph invariants, namely for the clique number $\omega(G_n)$, the stability quotient $\psi(G_n)$, the chromatic number $\chi(G_n)$, the degeneracy $\delta^*(G_n)$ and the maximal degree $\Delta(G_n)$. Similarly, the result of Theorem 4 holds for these graph invariants and the hitting number $\omega^-(G_n)$ as well.

Theorem 3. *For any $\epsilon > 0$ there is the following lower bound on the hitting number $\omega^-(G_n)$:*

$$\mathbb{P}[\omega^-(G_n) < (1 - \epsilon)\mathcal{K}_n] = o(1).$$

Theorem 4. *For any $\epsilon > 0$ there is the following upper bound on the maximal degree $\Delta(G_n)$:*

$$\mathbb{P}[\Delta(G_n) > (1+\epsilon)\mathcal{K}_n] = o(1).$$

The proof of Theorem 3 and Theorem 4 can be found in Section 2.2. We use these theorems to prove the main result of the sparse case:

Theorem 1. *Suppose that as $n \to \infty$ we have $d \to 0$ and we have that $d^2 n$ is $o(\ln n)$ and $n^{o(1)}$. Let*

$$\mathcal{K}_n = \frac{\ln n}{\ln\left(\frac{\ln n}{d^2 n}\right)},$$

We have $\mathcal{K}_n \to \infty$ as $n \to \infty$, and we have for the aforementioned graph invariants:

$$\omega^-(G_n) \sim \mathcal{K}_n, \omega(G_n) \sim \mathcal{K}_n, \psi(G_n) \sim \mathcal{K}_n, \chi(G_n) \sim \mathcal{K}_n, \delta^*(G_n) \sim \mathcal{K}_n \text{ and } \Delta(G_n) \sim \mathcal{K}_n.$$

Proof Theorem 1. By applying the ordering of the graph invariants in Lemma 1 to Theorem 3, we also find for any $\epsilon > 0$ that the lower bound $(1-\epsilon)\mathcal{K}_n$ also holds for the maximal degree $\Delta(G_n)$ with probability $1 - o(1)$. By also applying Lemma 1 to Theorem 4 we similarly find that the upper bound $(1+\epsilon)\mathcal{K}_n$ also holds for the hitting number ω^- with probability $1 - o(1)$, again for any $\epsilon > 0$. We thus find that for any $\epsilon > 0$ we have $\mathbb{P}[\ |\frac{\omega^-(G_n)}{\mathcal{K}_n} - 1| \geq \epsilon] = o(1)$, as well as $\mathbb{P}[\ |\frac{\Delta(G_n)}{\mathcal{K}_n} - 1| \geq \epsilon] = o(1)$. We have the same results for the other invariants in Lemma 1 that are between $\omega^-(G_n)$ and $\Delta(G_n)$. We conclude that the convergence $\frac{\omega^-(G_n)}{\mathcal{K}_n} \to 1$ and $\frac{\Delta(G_n)}{\mathcal{K}_n} \to 1$ hold in probability. For the clique number ω, the stability quotient ψ, the chromatic number χ and the degeneracy δ^* of G_n we find the same results. This gives us the desired result of asymptotic equivalence to $\mathcal{K}_n$ for these graph invariants of G_n. □

Now that we have established all the necessary theorems for the sparse case we are ready to answer the question raised in the introduction about the special case. The proof of the corollary is part of my contribution.

Corollary 1. *If we have $G_n = G(V_n, d)$, where $d(n)$ is such that $d^2 n = C$ is constant, then the expected degree of every vertex $X_1, \ldots, X_n$ is constant and the following graph invariants $\omega^-(G_n), \omega(G_n), \psi(G_n), \chi(G_n), \delta^*(G_n), \Delta(G_n)$ are all asymptotically equivalent to $\mathcal{K}_n$ as defined in Theorem 1.*

Proof Corollary 1. For any point X_i we have that, almost surely, for sufficiently large n, the disk D centred around X_i with radius d is contained in the unit square, because the event that X_i lies on the boundary of the unit square has probability 0. The probability of any vertex being inside the disk D is proportional to the area $\lambda(D)$ of the disk: $\nu(D) = \lambda(D)$. For the area of the disk we find $\lambda(D) = \frac{\pi}{4}d^2$. So for any vertex X_i we find that for sufficiently large n the expected order of X_i is given by: $\mathbb{E}[d(X_i)] = (n-1)\cdot\nu(D) = \frac{\pi}{4}d^2(n-1) \to \frac{\pi}{4}C$ as $n \to \infty$. This shows the first part of Corollary 1. The second part follows from applying Theorem 1, since $d^2 n$ is both $o(\ln n)$ and $n^{o(1)}$. To see this, notice that for any constant C we have $\frac{C}{\ln n} \to 0$ and also $C = n^{\frac{\ln C}{\ln n}} = n^{o(1)}$. □

We prove the theorem 2 of the dense case using the following two theorems:

Theorem 5. *Suppose that $d \to 0$ as $n \to \infty$. For any $0 < \sigma < \nu_{max}$, we have that lower bounds on the clique number $\omega^-(G_n)$ and the stability quotient $\psi(G_n)$ hold with probability $1 - e^{-\Omega(d^2 n)}$:*

$$\omega^-(G_n) \geq \sigma \frac{\pi}{4} d^2 n$$

and

$$\psi(G_n) \geq \sigma \frac{\sqrt{3}}{2} d^2 n$$

Theorem 6. *Suppose that as $n \to \infty$ we both have $d \to 0$ and $\frac{d^2 n}{\ln n} \to \infty$. For any $\nu_{max} < \sigma$, the following upper bounds on $\omega(G_n)$ and $\chi(G_n)$ hold with probability $1 - e^{-\Omega(d^2 n)}$:*

$$\omega(G_n) \leq \frac{\pi}{4} \sigma d^2 n$$

and

$$\chi(G_n) \leq \frac{\sqrt{3}}{2} \sigma d^2 n$$

The proof of Theorem 5 can be found in Section 2.3 and the proof of Theorem 6 in section 2.4. From these theorem, we prove the main results of the dense case:

Theorem 2. *Suppose that as $n \to \infty$ we have $d \to 0$ and also we have $\frac{d^2 n}{\ln(n)} \to \infty$ as $n \to \infty$. Then for $\mathcal{K}_n = \nu_{max} \frac{\pi}{4} d^2 n$, we have:*

$$\omega^-(G_n) \sim \mathcal{K}_n,\ \omega(G_n) \sim \mathcal{K}_n,\ \psi(G_n) \sim \frac{2\sqrt{3}}{\pi} \mathcal{K}_n,\ \chi(G_n) \sim \frac{2\sqrt{3}}{\pi} \mathcal{K}_n.$$

Proof Theorem 2. Consider arbitrary σ, σ' satisfying $0 < \sigma < \nu_{max} < \sigma'$. We apply the graph invariant ordering of Lemma 1 to Theorem 5 and Theorem 6 to find that the upper bounds of ω and χ also hold with probability $1 - e^{-\Omega(d^2 n)}$ for ω^- and ψ respectively. Similarly, the lower bounds of ω^- and ψ also hold with probability $1 - e^{-\Omega(n)}$ for ω and χ respectively. This means that:

$$\mathbb{P}\left[\frac{\sigma}{\nu_{max}} \mathcal{K}_n \leq \omega^- \leq \frac{\sigma'}{\nu_{max}} \mathcal{K}_n\right] = 1 - e^{\Omega(d^2 n)} - e^{-\Omega(n)},$$

and we find the same for ω, $\frac{\pi}{2\sqrt{3}} \cdot \psi$ and $\frac{\pi}{2\sqrt{3}} \cdot \chi$. We thus find for arbitrary $\epsilon > 0$ that:

$$\mathbb{P}\left[\left|\frac{\omega^-}{\mathcal{K}_n} - 1\right| > \epsilon\right] = 1 - e^{\Omega(d^2 n)},$$

and so $\omega^- \sim \mathcal{K}_n$. With the same reasoning we find $\omega \sim \mathcal{K}_n$, $\psi \sim \frac{2\sqrt{3}}{\pi} \mathcal{K}_n$ and $\chi \sim \frac{2\sqrt{3}}{\pi} \mathcal{K}_n$ as well. □

2.2 Bounds in the sparse case

Within this section we give proofs on the sparse case, both on the lower bound in Theorem 3 and on the upper bound in 4, for which we assume that $d^2 n$ is both $o(\ln n)$ and $n^{o(1)}$ as $n \to \infty$. We start by stating a lemma that is used in the lower bounds, both in the sparse case of Theorem 3 and in the dense case of Theorem 5.

Lemma 2. *Consider a distribution in the plane ν with finite ν_{max}. For any $0 < \sigma < \nu_{max}$ there exist $\rho > 0$ and $\eta > 0$, where for all $0 < r < \rho$ the following holds: there are ηr^{-2} pairwise disjoint open disk where each disk D has radius r and satisfies $\nu(D) > \sigma\lambda(D)$.*

Proof Lemma 2. For the proof we refer to the proof of Lemma 4.3 in [4]. We include a proof of the special case of the uniform distribution on the unit square here. We denote this distribution ν, for which we have $\nu_{max} = 1$. Here we have for any $\sigma < \nu_{max}$ that any ball B that is contained in the unit square satisfies $\nu(B) > \sigma\lambda(B)$. Any square with sides of length d contains a ball of diameter $d = 2r$. So the example can be proven by showing that for some $\rho, \eta > 0$ there exist for any $r < \rho$ at least ηr^{-2} pairwise distinct squares all contained within the unit square. For squares with sides of length d, $\lfloor \frac{1}{d} \rfloor$ such squares fit length-wise in the unit square. We have the following algebraic property for $C > 4$: $\lceil \frac{C}{2} \rceil \leq \frac{C}{2} + 1 < C - 1 \leq \lfloor C \rfloor$. By applying this property we deduce that we can fit at least $\lceil \frac{1}{2d} \rceil^2$ pairwise distinct squares of side d in the unit square, when $d < \frac{1}{4}$. So for $r < \frac{1}{8}$ we have at least $\frac{1}{16} r^{-2}$ pairwise distinct disks where for each disk D we have $\nu(D) > \sigma\lambda(D)$, proving the example for $\rho = \frac{1}{8}$ and $\eta = \frac{1}{16}$. □

The next lemma gives bounds on the tails of the binomial distribution, the lower bound will be used in Theorem 3 and the upper bound in Theorem 4.

Lemma 3. *For a binomial distribution $B(n,p)$, where $n \in \mathbb{N}$ and $0 \leq p \leq 1$, we have the following bounds on the tails of the distribution: For each integer k with $\mu := np \leq k \leq n$, for a random variable $X \sim B(n,p)$, we have,*

$$\left(\frac{\mu}{ek}\right)^k \leq \mathbb{P}[X \geq k] \leq 2\left(\frac{e\mu}{k}\right)^k.$$

Proof Lemma 3. For this proof we refer to Lemma 4.4 of [4].

□

We now state a lemma that simplifies the proof of Theorem 3 and Theorem 4.

Lemma 4. *For $\mathcal{K}_n$ defined as:*

$$\mathcal{K}_n := \frac{\ln n}{\ln\left(\frac{\ln n}{d^2 n}\right)},$$

as $n \to \infty$ we have $\mathcal{K}_n \to \infty$. Also, for any function $f(n)$ that is $\Theta(d^2 n)$, we have the following asymptotic equivalence:

$$\mathcal{K}_n \ln\left(\frac{\mathcal{K}_n}{f(n)}\right) \sim \ln n.$$

Proof Lemma 4. For the proof that $\mathcal{K}_n \to \infty$ as $n \to \infty$, we refer to Lemma 5.3 in [4].
For the asymptotic equivalence, we first prove that: $\ln\left(\frac{\mathcal{K}_n}{f(n)}\right) \sim \ln\left(\frac{\ln n}{d^2 n}\right)$. For the first we find by using $f(n) = \Theta(d^2 n)$:

$$\begin{aligned}
\ln\left(\frac{\mathcal{K}_n}{f(n)}\right) &= \ln\left(\frac{\mathcal{K}_n}{\Theta(d^2 n)}\right) \\
&= \ln\left(\frac{\mathcal{K}_n}{d^2 n}\right) - \ln(\Theta(1)) \\
&= \ln\left(\frac{\ln n}{d^2 n}\right) - \ln\left(\ln\left(\frac{\ln n}{d^2 n}\right)\right) - \Theta(1),
\end{aligned}$$

where the latter follows from applying the definition of $\mathcal{K}_n$ and subsequently rewriting the logarithm. Substituting the numerator in the following fraction we find for the limit:

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\ln\left(\frac{\mathcal{K}_n}{f(n)}\right)}{\ln\left(\frac{\ln n}{d^2 n}\right)}\right) &= \lim_{n\to\infty}\left(\frac{\ln\left(\frac{\ln n}{d^2 n}\right)-\ln\left(\ln\left(\frac{\ln n}{d^2 n}\right)\right)-\Theta(1)}{\ln\left(\frac{\ln n}{d^2 n}\right)}\right)\\ &= \lim_{n\to\infty}\left(1-\frac{\ln\left(\ln\left(\frac{\ln n}{d^2 n}\right)\right)}{\ln\left(\frac{\ln n}{d^2 n}\right)}-0\right)\\ &= \lim_{\ln\left(\frac{\ln n}{d^2 n}\right)\to\infty}\left(1-\frac{\ln\left(\ln\left(\frac{\ln n}{d^2 n}\right)\right)}{\ln\left(\frac{\ln n}{d^2 n}\right)}\right)\\ &= 1,\end{aligned}$$

because as $n \to \infty$ we also have $\ln\left(\frac{\ln n}{d^2 n}\right) \to \infty$. This shows the asymptotic equivalence $\ln\left(\frac{\mathcal{K}_n}{f(n)}\right) \sim \ln\left(\frac{\ln n}{d^2 n}\right)$. The Lemma follows by multiplication of $\mathcal{K}_n$ on both sides. □

By using Lemma 4, we find the following result which appears both in Theorem 3 and Theorem 4:

Corollary 2. *For a function $f(n)$ that is $\Theta(d^2 n)$ we have that:*

$$\left(\frac{\mathcal{K}_n}{f(n)}\right)^{\mathcal{K}_n} = n^{1+o(1)}.$$

Proof Corollary 2. Since $f(n)$ is $\Theta(d^2 n)$, we use Lemma 4 to conclude that:

$$\left(\frac{\mathcal{K}_n}{f(n)}\right)^{\mathcal{K}_n} = n^{\mathcal{K}_n \frac{\ln(\mathcal{K}_n/f(n))}{\ln n}} = n^{1+o(1)}.$$

□

Using the previous lemmas and Corollary 4, we prove Theorem 3:

Theorem 3. *For any $\epsilon > 0$ there is the following lower bound on the hitting number $\omega^-(G_n)$ of the graph G_n:*

$$\mathbb{P}[\omega^- < (1-\epsilon)\mathcal{K}_n] = o(1).$$

Proof Theorem 3. Consider $p := (1-\epsilon)\nu_{max} d^2 \frac{\pi}{4}$, for which we have the distribution $Z \sim B(n,p)$ with $\mu = np = \Theta(d^2 n)$. We have for sufficiently large n that $(1-\epsilon)\mathcal{K}_n > \mu$, because if $(1-\epsilon)\mathcal{K}$ where $O(d^2 n)$, then using Lemma 4 we would find for the function $f(n) := \mu = \Theta(d^2 n)$ the following function $g(n) := \mathcal{K}_n \ln\left(\frac{\mathcal{K}_n}{f(n)}\right) = O(d^2 n)$, with the property that $g(n) \sim \ln n$, this contradicts our assumption that $d^2 n = o(\ln n)$. This means that we can apply Lemma 3 for the lower bound on the tail of the binomial distribution:

$$\mathbb{P}[Z > (1-\epsilon)\mathcal{K}_n] \geq \left(\frac{\mu}{e(1-\epsilon)\mathcal{K}_n}\right)^{(1-\epsilon)\mathcal{K}_n} = \left(\frac{e(1-\epsilon)\cdot\mathcal{K}_n}{\mu}\right)^{\mathcal{K}_n\cdot -(1-\epsilon)},$$

for which we apply Corollary 2 with $f(n) = \frac{\mu}{e(1-\epsilon)} = \Theta(d^2 n)$. Whereby we find:

$$\mathbb{P}[Z > (1-\epsilon)\mathcal{K}_n] \geq n^{-(1+o(1))(1-\epsilon)} = n^{-(1-\epsilon+o(1))}.$$

We are interested in a family of pairwise disjoint disks, for which every disk D has radius d and with $\nu(D) > (1-\epsilon)\nu_{max}\lambda$. By Lemma 2 we can find ηd^{-2} such disks for some factor η and for sufficiently large n, such that $d < \rho$ for some $\rho > 0$. For any specific disk D from the family we thus have $\nu(D) > p$ and therefore the probability that the disk D contains more than $(1-\epsilon)\mathcal{K}_n$ vertices is at least $\mathbb{P}[Z > (1-\epsilon)\mathcal{K}_n]$. For the hitting number ω^- to be less than $(1-\epsilon)\mathcal{K}_n$, all the disks in the family need to contain less than $(1-\epsilon)\mathcal{K}_n$ disks. This gives us:

$$\mathbb{P}[\omega^- < (1-\epsilon)\mathcal{K}_n] \leq (1 - \mathbb{P}[Z > (1-\epsilon)\mathcal{K}_n])^{\eta d^{-2}} \leq e^{-\mathbb{P}[Z>(1-\epsilon)\mathcal{K}_n]\eta d^{-2}}.$$

We conclude the proof with the notion that $d^2 n$ is $n^{o(1)}$, giving us:

$$\mathbb{P}[\omega^- < (1-\epsilon)\mathcal{K}_n] \leq e^{-n^{-(1-\epsilon+o(1))}\eta d^{-2}} = e^{-\frac{n^{\epsilon+o(1)}}{d^2 n}} = e^{-n^{\epsilon+o(1)}} = o(1),$$

thus showing that for every $\epsilon > 0$, we have $\mathbb{P}[\omega^- < (1-\epsilon)\mathcal{K}_n] = o(1)$. □

Using Lemma 3 and Corollary 2, we can proof Theorem 4:

Theorem 4. *Suppose that for $n \to \infty$, we have $d^2 n = o(\ln n)$. For any $\epsilon > 0$ we have the following upper bound on the maximal degree $\Delta(G_n)$:*

$$\mathbb{P}[\Delta(G_n) > (1+\epsilon)\mathcal{K}_n] = o(1).$$

Proof Theorem 4. Let $\epsilon > 0$. Let $p = \nu_{max}\pi d^2$, meaning that for any disk B of radius d we have $\nu(B) \leq p$. We have distribution $Z = B(n,p)$ with $\mu = np$, this means that $\mu = \Theta(d^2 n)$. We have $(1+\epsilon)\mathcal{K}_n > \mu$ for sufficiently large n, as shown in Theorem 3, so we can apply Lemma 3 to find the following upper bound on the tail of distribution Z:

$$\mathbb{P}[Z \geq (1+\epsilon)\mathcal{K}_n] \leq 2\left(\frac{\mathcal{K}_n}{e\mu}\right)^{-(1+\epsilon)\mathcal{K}_n} = 2\left(\frac{\mathcal{K}_n}{f(n)}\right)^{-(1+\epsilon)\mathcal{K}_n},$$

where we let $f(n) = e\mu = \Theta(d^2 n)$. We now apply Corollary 2, so that we find:

$$\mathbb{P}[Z \geq (1+\epsilon)\mathcal{K}_n] \leq n^{o(1)} \cdot n^{-(1+o(1))(1+\epsilon)} = n^{o(1)-(1+\epsilon)}.$$

Now, let the disks $D_1, \ldots, D_n$ be of radius d and centred around the vertices $X_1, \ldots, X_n$ of G_n respectively. In order for the maximum degree to satisfy $\Delta(G_n) \geq (1+\epsilon)\mathcal{K}_n$, at least one of the disks $D_1, \ldots, D_n$ needs to contain at least $(1+\epsilon)\mathcal{K}_n$ vertices. We find therefore for the upper bound of $\Delta(G_n)$:

$$\begin{aligned}\mathbb{P}[\Delta(G_n) \geq (1+\epsilon)\mathcal{K}_n] &\leq \sum_{D_1,\ldots,D_n} \mathbb{P}[D_i \text{ contains at least } (1+\epsilon)\mathcal{K}_n \text{ vertices}] \\ &= n\mathbb{P}[Z \geq (1+\epsilon)\mathcal{K}_n] \\ &= n^{o(1)-\epsilon} \\ &= o(1),\end{aligned}$$

meaning that $\mathbb{P}[\Delta(G_n) \geq (1+\epsilon)\mathcal{K}_n] \to 0$ as $n \to \infty$. We conclude that for any $\epsilon > 0$, the upper bound $\Delta(G_n) \leq (1+\epsilon)\mathcal{K}_n$ holds in probability as $n \to \infty$. □

2.3 Lower bounds in the dense case

In this section we discuss the proofs of the lower bounds of the hitting number $\omega^-(G_n)$ and the stability quotient $\psi(G_n)$ from Theorem 5. The upper bounds for the clique number $\omega(G_n)$ and the chromatic number $\chi(G_n)$ will be discussed in the next section. In both sections we assume that as $n \to \infty$ we have $d \to 0$ and $\frac{d^2 n}{\ln n} \to \infty$. We first give a preliminary result, which is a simplification of the "Chernoff Bound" in [4]. This lemma will be used in this section and the next.

Lemma 5. *Let the random variable Y with mean μ have either a binomial distribution or a Poisson distribution. Let $\epsilon > 0$, we have the following upper bound on the tails of the distribution:*

$$\mathbb{P}[\ |Y - \mu| \geq \epsilon\mu] \leq e^{-\Omega(\mu)}$$

Proof Lemma 5. The proof follows from the Chernoff bound, which can be found in Lemma 4.5 of [4]. □

The next lemma is used in both lower bounds of the hitting number $\omega^-(G_n)$ and the stability quotient $\psi(G_n)$ in Theorem 5. It shows for arbitrary $K > 0$ the existence of a disk D with diameter Kd containing at least $\lambda(D)\tau n$ vertices. This gives us lower bounds for the graph invariants with probability $e^{-\Omega(d^2 n)}$. For the sake of simplicity, this proof somewhat differs from Lemma 5.5 of [4], where the existence of ηd^{-2} many disks is used, for some $\eta > 0$. From this the same lower bound is concluded, holding with probability $e^{-\Omega(n)}$ instead.

Lemma 6. *For the graph $G_n = G(V_n, d)$ with $\tau < \nu_{max}$ and $K > 0$, consider A_n to be the event that there exists no disk D of diameter Kd which contains more than $\lambda(D)\tau n$ of the vertices $X_1, \ldots, X_n$. The probability of this event occurring is $\mathbb{P}[A_n] = e^{-\Omega(d^2 n)}$.*

Proof Lemma 6. Let $\tau < \tau' < \nu_{max}$. By Lemma 2, we have for sufficiently large n that there exists a disk D of diameter Kd, for which the probability that a vertex is contained in it is $\nu(D) \geq \lambda(D)\tau'$. Consider the distribution $Z \sim B(n, \lambda(D)\tau')$, from the existence of the disk D, we find that the probability of the event A_n occurring is $\mathbb{P}[A_n] \leq \mathbb{P}[Z < \lambda(D)\tau n]$. From applying the Chernoff bound in Lemma 5, we find that the probability that there is no disk that contains less than $\lambda(D)\tau n$ vertices is:

$$\mathbb{P}[A_n] \leq \mathbb{P}[Z < \lambda(D)\tau n] \leq e^{-\Omega(\lambda(D)n)} = e^{-\Omega(d^2 n)}.$$

□

Together with the last lemma, Lemma 7 will support the proof of the lower bound of $\psi(G_n)$ in Theorem 5. This lemma contains a deterministic result for the lower bound of the stability quotient $\psi(G)$ for some deterministic graph $G = G(V, d)$. Lemma 7 can also be found in Lemma 5.2 of [4], with the addition here that the lower bound applies to the stability quotient as well, meanwhile [4] uses this bound for the chromatic number $\chi(G_n)$.

Lemma 7. *For any $\epsilon > 0$, there is a K for which the following is true: if for a proximity graph $G(V, d)$, with set of points V in the plane and $d > 0$, there exists a disk D of diameter at least Kd with at least $\lambda(D)\tau n$ of the points of V, then we have the following lower bound on the stability quotient: $\psi(G) \geq (1 - \epsilon)(\frac{\sqrt{3}}{2})\tau d^2 n$*

Proof Lemma 7. Let ϵ be such that $0 < \epsilon < 1$. By Lemma 5.2 of [4], there is some K for which the number of disjoint open disks of diameter d that meet a disk D of diameter at least Kd is

at most $\frac{\lambda(D)}{(1-\epsilon)(\sqrt{3}/2)d^2}$.
Consider the induced subgraph H of $G(V,d)$, such that H consists of all points on the disk D. The stability number of the subgraph $\alpha(H)$ is the largest independent set of H, meaning that it is the largest such set that has pairwise distinct disks in the disk representation. Thus $\alpha(H)$ is at most the maximum number of disjoint open disks of diameter d meeting D. Recall that D contains at least $\lambda(D)\tau n$ points, so we conclude:

$$\psi(G(V,d)) \geq \frac{|V(H)|}{\alpha(H)} \geq (1-\epsilon)\frac{\sqrt{3}}{2}\tau d^2 n$$

□

Lemma 6 and Lemma 7 provide the basis for the proof of the lower bounds of the hitting number $\omega^-(G_n)$ and the stability quotient $\psi(G_n)$ in Theorem 5.

Theorem 5. *Suppose that $d \to 0$ as $n \to \infty$. For any $0 < \sigma < \nu_{max}$, we have that lower bounds on the clique number $\omega^-(G_n)$ and the stability quotient $\psi(G_n)$ hold with probability $1 - e^{-\Omega(d^2 n)}$:*

$$\omega^-(G_n) \geq \sigma\frac{\pi}{4}d^2 n$$

and

$$\psi(G_n) \geq \sigma\frac{\sqrt{3}}{2}d^2 n$$

Proof Theorem 5. Consider $\sigma < \tau < \nu_{max}$. For arbitrary $K > 0$ we have a disk D of radius Kd. This gives $\lambda(D) = \pi(\frac{Kd}{2})^2$. Let A_n be the event that all such disk contain less than $\lambda(D)\tau n$ points. Lemma 6 shows that this event has probability $\mathbb{P}[A_n] \leq e^{-\Omega(d^2 n)}$. If we let $K = 1$, then we find that the lower bound on the clique number $\omega^-(G_n) \geq \sigma\frac{\pi}{4}d^2 n$ holds with probability $e^{-\Omega(d^2 n)}$.

What remains is to show the lower bound on the stability quotient $\psi(G_n)$. We use a weaker variant of Lemma 7, namely that for any $\epsilon > 0$ there is a K such that if there exists a disk D of diameter Kd that contains at least $\lambda(D)\tau n$ points, then the lower bound $\psi(G_n) \geq (1-\epsilon)\tau\frac{\sqrt{3}}{2}d^2 n$ holds. Note that there is also such a K when ϵ is such that $(1-\epsilon)\tau = \sigma$. Lemma 6 shows that for this K the event A_n has probability $\mathbb{P}[A_n] \leq e^{-\Omega(d^2 n)}$. In the event of A_n occurring, the conditions for applying Lemma 7 are satisfied. We therefore conclude, by Lemma 6 together with Lemma 7, that the lower bound on the stability quotient $\psi \geq (1-\epsilon)\frac{\sqrt{3}}{2}\tau d^2 n$, holds with probability $e^{-\Omega(d^2 n)}$ as well.

□

2.4 Upper bounds in the dense case

For the dense case we assume that as $n \to \infty$ both $d \to 0$ and $\frac{d^2 n}{\ln n} \to \infty$ hold. In order to find the upper bounds in Theorem 6 for the clique number $\omega(G_n)$ and the chromatic number $\chi(G_n)$, we use partitions of the plane, which we denote here as a **tiling** consisting of **tiles** all of the same area $\mathcal{A}$. In general these tiles are not all required to be of the same shape. But the results here do depend both on properties on the shape of these tiles as well as the maximum number of vertices in a tile. The results in Lemma 8 on the clique number ω within tile structures is independent on the shape of the tiles. For the other results we specifically assume a **standardised hexagonal tiling** of the plane. For the standardised hexagonal tiling we differentiate between the distance between the centres of adjacent hexagons being 1 and this distance being scaled by some factor $s > 0$. We call the latter the scaled hexagonal tiling. We have the following precise definitions:

Definition of standardised hexagonal tiling: A standardised hexagonal tiling $\mathcal{T}$ is a tiling consisting of tiles which are all regular hexagons. These tiles are all of the same shape, in fact they are of the size such that the distance between centres of neighbouring tiles is 1. An intuitive choice for the coordinates of the centres is: $k(1, 0) + l(\frac{1}{2}, \frac{\sqrt{3}}{2})$ for $k, l \in \mathbb{Z}$.

Definition of scaled hexagonal tiling: For a factor $s > 0$, the scaled hexagonal tiling $s\mathcal{T}$ is a hexagonal tiling which is scaled such that the distance between adjacent hexagons is s instead.

In our results we are interested in specific properties of the tiles in the tile structures. We let a **diameter** τ of a tile structure be the maximum distance between two points in the same tile. We let **radius** r of a tile structure be the smallest number such that any tile has a point P for which the tile is contained in a disk $D_r(P)$ with radius r centred around P. For the standardised (and the scaled) hexagonal tiling, the area of the tiles are $\mathcal{A} = \frac{\sqrt{3}}{2}$ (and $\mathcal{A}_s = \frac{\sqrt{3}}{2}s^2$). We have a diameter of $\tau = \frac{2}{\sqrt{3}}$ (and $\tau = \frac{2}{\sqrt{3}}s$) and a radius of $r = \frac{1}{\sqrt{3}}$ (and $r = \frac{1}{\sqrt{3}}s$).

We continue by giving upper bounds on the clique number $\omega(G)$ and the chromatic number $\chi(G)$ for the proximity graph $G(V, d)$ of a deterministic set of points V. These upper bounds depend on the maximum number of points in a single tile of a tile structure. The results will form a basis for the proof on the upper bounds of the clique number $\omega(G_n)$ and the chromatic number $\chi(G_n)$ in graphs with random points.

Lemma 8. *For a tile structure in the plane with tiles each of area $\mathcal{A}$, and the diameter of the structure τ, where each tile contains at most one of the points of a set of points in the plane V, then the clique number $\omega(G)$ of the proximity graph $G = G(V, d)$ with threshold distance d has upper bound:*

$$\omega(G) \leq \frac{\pi}{4}(d + 2\tau)^2 \mathcal{A}^{-1}$$

Proof Lemma 8. Let C be the union of the tiles associated with the points in largest clique of $G(V, d)$. The diameter of C is at most $d + 2\tau$, (for τ the maximum diameter of the tiles) since for any pair of points in C, the points both have a distance of at most τ to points in their tiles that are in the clique of $G(V, d)$, therefore the total distance between the points of the pair is at most $d+2\tau$. The area of C is therefore (by Lemma 9 of [6]) at most $\frac{\pi}{4}(d+2\tau)^2$. We conclude:

$$\mathcal{A} \cdot \omega(G) \leq \frac{\pi}{4}(d + 2\tau)^2$$

□

Lemma 9. *For a (deterministic) set of points V in the plane, with threshold distance d, we have the proximity graph $G = G(V, d)$. For arbitrary $\delta > 0$, we let $s = \delta d$ be the distance between centres of adjacent hexagons of a scaled hexagonal tiling $s\mathcal{T}$. For a finite maximum number of points in each tile $\mathcal{M}$, we have the following upper bounds on $\omega(G)$ and $\chi(G)$:*

$$\omega(G) < \frac{\pi}{2\sqrt{3}} \cdot \mathcal{M}(\delta^{-1} + 3)^2$$
$$\chi(G) < \mathcal{M}(\delta^{-1} + 3)^2$$

Proof Lemma 9. We have the scaled hexagonal tiling, with each of its tiles having an area of $\mathcal{A}_s = \frac{\sqrt{3}}{2}s^2$ and a diameter of $\tau = \frac{2}{\sqrt{3}}s$. This satisfies the conditions in Lemma 8 for applying the upper bound on the clique number $\omega(G)$. We adjust Lemma 8 since each tile can contain at most $\mathcal{M}$ vertices, doing this we find:

$$\omega(G) \leq \mathcal{M}\frac{\pi}{4}\left(d + 2 \cdot \frac{2}{\sqrt{3}}s\right)^2 \left(\frac{\sqrt{3}}{2}s^2\right)^{-1} = \frac{\pi}{2\sqrt{3}} \cdot \mathcal{M}\left(\delta^{-1} + \frac{4}{\sqrt{3}}\right)^2$$

the weaker upper bound on $\omega(G)$ follows.

For the upper bound on $\chi(G)$ we build clusters of the standardised hexagons as described in [1]. These identical clusters will form a repeating pattern partitioning the plane. In fact, here we can build such clusters with the centres of their outer hexagons all lying on a rhombus.

Choose an arbitrary hexagon H to be at the centre with label A. We consider hexagons of other clusters to have the same label if they have the same relative position in the cluster. For H we now look for another closest hexagon H' with the same label A. There will be six such hexagons, all within different clusters. From our choice on the shape of the clusters, there is a straight chain of hexagons between H and H'. We want the centres of H and H' to be at least $d + 2r$ apart with r being the radius of the scaled regular hexagon. This way, no two points anywhere in the hexagons H and H' will have a distance less than d between them. Doing this means that the colours of any hexagon can be reused in any other cluster, namely for the hexagons that have the same label. Since the maximum number of vertices in any hexagon is $\mathcal{M}$, we need at most $\mathcal{M}$ different colours per hexagon of a cluster. What is left is to find the number of hexagons per cluster. Since the distance between the centres of adjacent hexagons is s, the chain from H to H' needs to consist of at least $\lceil\frac{d+2r}{s}\rceil$ hexagons. By formula (1) of section IV of [1], the number of hexagons in each cluster is $\lceil\frac{d+2r}{s}\rceil^2$, with the radius of the hexagon $r = \frac{1}{\sqrt{3}}s$ we find:

$$\chi(G) \leq \mathcal{M}\lceil\frac{d+2r}{s}\rceil^2 \leq \mathcal{M}\left(\delta^{-1} + \frac{2}{\sqrt{3}} + 1\right)^2,$$

from which the weaker upper bound on $\chi(G)$ follows as well.

□

Using the previous Lemmas we can prove Theorem 6 on the upper bounds of the clique number $\omega(G_n)$ and the chromatic number $\chi(G_n)$ in the dense case.

Theorem 6. *Suppose that as $n \to \infty$ we both have $d \to 0$ and $\frac{d^2 n}{\ln n} \to \infty$. For any $\nu_{max} < \sigma$, the following upper bounds on $\omega(G_n)$ and $\chi(G_n)$ hold with probability $1 - e^{-\Omega(d^2 n)}$:*

$$\omega(G_n) \leq \frac{\pi}{4}\sigma d^2 n$$

and

$$\chi(G_n) \leq \frac{\sqrt{3}}{2}\sigma d^2 n$$

Proof Theorem 6. Choose a τ such that $\nu_{max} < \tau < \sigma$. We let $\delta > 0$ be arbitrary and we let $s = \delta d$ be the factor by which the standardised hexagonal tile structure is scaled. For the area of each tile we thus have $\mathcal{A}_s = \frac{\sqrt{3}}{2}s^2$. Let Y_n denote the maximal number of points $X_1, \ldots, X_n$

in a tile. Our aim is to show that the upper bound $\tau\mathcal{A}_s n$ for Y_n holds with probability $1-e^{-\Omega(d^2n)}$.

Let $p(n) = \mathcal{A}_s\nu_{max}$, so we have for any tile H that $\nu(H) \leq p$. We partition the plane into at most $\frac{2}{p}$ sets, where for each set S in the partition we have $\frac{p}{2} < \nu(S) \leq p$. We denote this partition as $\mathcal{P}$. We obtain $\mathcal{P}$ by starting with the partition of the hexagonal tile structure and by repeatedly unionising sets S for which $\nu(S) \leq \frac{p}{2}$. The maximal number of points in a hexagonal tile is therefore at most the maximal number of points in a set of $\mathcal{P}$.

Consider the distribution $Z_n \sim B(n,p)$. Since $\tau\mathcal{A}_s n = \frac{\tau}{\nu_{max}}np > np$ we can apply the Chernoff bound of Lemma 5:

$$\mathbb{P}[Z_n > \tau\mathcal{A}_s n] = e^{-\Omega(d^2n)},$$

giving $\tau\mathcal{A}_s n$ as the upper bound for Z_n with probability $1 - e^{-\Omega(d^2n)}$. For any set S in the partition $\mathcal{P}$ we have $\nu(S) \leq p$. Therefore, for any such S, the distribution Z_n also gives the upper bound $\tau\mathcal{A}_s n$ for the number of vertices $X_1, \ldots, X_n$ contained in S, holding with probability $1 - e^{-\Omega(d^2n)}$.

We return to our aim of finding for Y_n the upper bound $\tau\mathcal{A}_s n$ to hold with probability $1 - e^{-\Omega(d^2n)}$. Recall that the maximal number of vertices in a tile is less then the maximal number of vertices in sets of the partition $\mathcal{P}$. Therefore:

$$\mathbb{P}[Y_n > \tau\mathcal{A}_s n] \leq \sum_{S\in\mathcal{P}} \mathbb{P}[S \text{ contains more than } \tau\mathcal{A}_s n \text{ vertices of } G_n] \leq \left(\frac{2}{p}\right)\mathbb{P}[Z_n > \tau\mathcal{A}_s n],$$

from which we find that the upper bound for Y_n, holds with probability $e^{-\Omega(d^2n)}$:

$$\mathbb{P}[Y_n > \tau\mathcal{A}_s n] \leq e^{\ln\left(\frac{2}{p}\right)-\Omega(d^2n)} = e^{O(1)-2\ln(d)-\Omega(d^2n)} = e^{-\Omega(d^2n)},$$

which shows that for Y_n, the maximal number of points $X_1, \ldots, X_n$ in a tile, the upper bound $\tau\mathcal{A}_s n$ holds with probability $1 - e^{-\Omega(d^2n)}$. Let δ be specifically such that $(1+3\delta)^2\tau = \sigma$. Also, notice that when the upper bound for Y_n holds, we can apply Lemma 9 to find upper bounds on the clique number $\omega(G_n)$ and the chromatic number $\chi(G_n)$. So for the clique number $\omega(G_n)$ we find the upper bound:

$$\omega(G_n) < \frac{\pi}{2\sqrt{3}} \cdot \tau\mathcal{A}_s n \cdot (\delta^{-1}+3)^2 = \frac{\pi}{2\sqrt{3}}\left(\frac{\sqrt{3}}{2}s^2\right) \cdot (\delta^{-1}+3)^2\tau n,$$

which we simplify to be:

$$\omega(G_n) < \frac{\pi}{4}d^2(1+3\delta)^2\tau n = \frac{\pi}{4}\sigma d^2 n,$$

holds with probability $1 - e^{-\Omega(d^2n)}$. Similarly, for the chromatic number $\chi(G_n)$, we find the upper bound:

$$\chi(G_n) < \tau\mathcal{A}_s n \cdot (\delta^{-1}+3)^2 = \left(\frac{\sqrt{3}}{2}s^2\right) \cdot (\delta^{-1}+3)^2\tau n,$$

which we also simplify:

$$\chi(G_n) < \frac{\sqrt{3}}{2}d^2(1+3\delta)^2\tau n = \frac{\sqrt{3}}{2}\sigma d^2 n.$$

also holds with probability $1 - e^{-\Omega(d^2n)}$, concluding the theorem. □

The upper bound of the dense case differs from the upper bound provided in [4], since the concept of a triangular lattice is not mentioned here, instead we have results for vertices anywhere within the hexagonal tiles.

3 Generalisations

If not for time constraints, the first generalisation to have been integrated into this thesis would be the following: we have a probability variable p, where pairs of vertices that are sufficiently close to each other, are only connected with a probability p. However, there is no guarantee that the bounds presented within this thesis would provide precise results for the different graph invariants.

Another generalisation of these results for random geometric graphs in the plane, is by considering more than two dimension, as is often already done. This may already be applied to proofs within this thesis, for example by considering n-dimensional balls instead of disks, as they are used in most proofs here.

The results can also be extended to different distance norms, as is for example done in [7]. However, with the application of channel assignments in the plane [4], the chromatic number may mostly have applications in two dimensions.

In the article [4], McDiarmid shows an interest in results between the sparse case and the dense case. For this results on the ratio of the chromatic number and the clique number $\frac{\chi}{\omega}$ have been found in [5].

One might also look for results on other graph invariants in random geometric graphs. An example of this is the k-improper chromatic number χ^k, studied in [2].

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