Generating Contact Terms for Scalar Field Interactions with Polynomial Rings

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Abstract

Mandelstam polynomials in scattering amplitudes correspond to contact contributions, and emerge from taking derivative terms in the Lagrangian. Equivalence relations between polynomials can be observed by employing the equations of motion, momentum conservation, and Gram constraints, which is powerful since they can serve as substitutes to finding corresponding redundancies in the Lagrangian, which are instead found with integration by parts and field redefinitions. The objective of this work is to study the use of polynomial rings to generate possible n-point contact terms in d-dimensions up to these equivalences, along with the representations in which Mandelstam invariants live. Contact contributions live in a polynomial ring, modded out by an ideal generated by momentum conservation and Gram conditions. Without Gram conditions, this becomes a study of the representations of the symmetric group acting on a set of unordered pairs, which describes the behaviour of Mandelstam invariants. In 4-point amplitudes in particular, elementary symmetric polynomials can be used as generators instead, such that there are a different number of independent polynomials at each order in the Mandelstam variables.

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Contents

Chapter 1 Introduction

The standard model of particle physics, or simply the *standard model*, is a combination of *quantum field theories* which has been successful in describing interactions between elementary particles [3, 10]. It does so with *cross-sections*, experimentally measurable quantities which in quantum mechanics are related to particle scattering probabilities such that the convey interaction strength [24]. They are proportional to the *scattering amplitude* squared, computed with the scattering matrix $\langle f| \, S \, |i\rangle$ [10, 24]. Therefore, scattering amplitudes and the S-matrix have been major objects of study in the landscape of quantum field theory.

Traditionally, scattering amplitudes are computed perturbatively with propagators and Feynman rules derived from the Lagrangian, summing the iconic so-called Feynman diagrams [10, 24]. This process can become computationally rather tedious with increasing number of particles for instance [11]. While the Feynman approach had provided initial insight, the more modern approach is to build S-matrix elements from physical criteria instead [5]. Then the Lagrangian could be inferred back from (parts of) the amplitudes [15].

Such methods include for instance bootstrapping amplitudes, and the double-copy framework [10]. Elvang [10], Cheung [5], Carrasco [4], and Elvang and Huang [11] provide useful review on the topic for further reading, with Li, Roest, and Veldhuis [20] as an example of recent advancements in the double-copy landscape.

This thesis outlines one particular approach of generating parts of scattering amplitudes, specifically for real massless scalar fields based on constraints of momenta. By considering proper physical constraints, polynomial rings can be used to generate possible contact contributions in scattering amplitudes, which is powerful partially because it more easily reveals redundancies in both the amplitude and derivative terms in the Lagrangian [15]. (Other) examples of their use can be seen in Henning et al. [14] and Beisert et al. [2]. This is closely related to the study of operator counting with the Hilbert series [9, 14, 15].

The contents of this thesis are closely based and inspired on the works by Henning et al. [15], Top [27], Serre [25], and Diaconis [7]. The main goal of this thesis is to provide a more nuanced introduction for both generating contact terms through polynomial rings, and representation theory of the symmetric group. The latter is explored first, such that the behaviour of Mandelstam invariants is contextualised within the framework of symmetric group representations. Studying the generators of contact term polynomials is namely a study of the $[n] \oplus [n-1,1] \oplus [n-2,2]$ representations [15, 20]. Then, an introduction to rings is presented to lay the groundwork of modding out ideals from polynomial rings, which highlights equivalences between polynomials through momentum constraints. First the 4-point contributions in 4 dimensions are considered as an introductory problem, after which a more general approach to n-point contributions in d-dimensions is presented. In this work, Natural units are assumed.

1.1 ► Motivations and Research Questions

To clarify the paragraph above, the motivations for this work will be illustrated in more context and detail below, formulating two research questions related to contact contributions and one to representations of the symmetric group.

1.1.1 ► Physical Criteria of Scattering Amplitudes

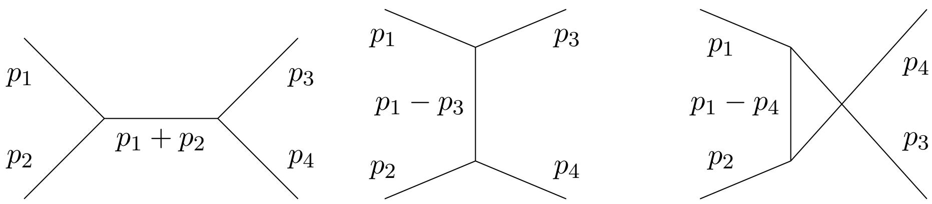
The following physical criteria have to be taken into account to build tree-level (leading order) scattering amplitudes for spin-0 systems:

- **Lorentz invariance**: Amplitudes should be Lorentz invariant (covariant under little group action for particles with spin) [5, 10, 11]. Therefore it must be built from quantities which are Lorentz invariant.
- **Locality**: Since the Lagrangian is local, interaction vertices don't give any pole terms; only the on-shell ($\Box\phi$) propagators do such that the poles are of the form $1/(\sum_i^k p_i)^2$ [11].
- **Dimensional Analysis**: The mass dimension of amplitudes must be the same as the mass dimensions of the coupling constants generating the amplitude from the Lagrangian [5, 11]

In having the constraints for the amplitudes defined, it is useful to see in action what possible terms could make up a valid amplitude. Interesting behaviour can already be observed in 4-point amplitudes for massless on-shell real indistinguishable scalar fields as an introductory problem to the topic.

1.1.2 ► Mandelstam Polynomials in 4-point Amplitudes

This section outlines the example of a 4-point scattering amplitude to clarify observations of polynomial equivalence and the number of invariant monomials in contact contributions. What possible Lorentz invariant terms could it be made of? In 4-point this results in the amplitude beings a function of Mandelstam variables,



(a) s-channel. **(b)** t-channel. **(c)** u-channel.

Figure 1.1: Feynman 4-pt interaction diagrams for massless on-shell scalar fields, with p_i 4-momenta from a ϕ^3 interaction term [24].

$$\begin{aligned} s &:= (p_1 + p_2)^2 = (p_3 + p_4)^2 \,, \\ t &:= (p_1 - p_3)^2 = (p_2 - p_4)^2 \,, \\ u &:= (p_1 - p_4)^2 = (p_2 - p_3)^2 \,, \end{aligned} \tag{1.1}$$

often quickly introduced in field theory textbooks [5, 24]. These correspond to the different propagation channels in 4-point amplitudes as well as seen in figure 1.1 for a ϕ^3 interaction term [24]. Cheung [5] describes how only simple poles and polynomial terms in Mandelstams are permitted due to locality in 4-point. Possible 4-point terms from different Lagrangian origins are listed in Cheung [5] up to and including degree-3 in Mandelstams. Henriette Elvang [16] expands on this list in her TASI lecture, and writes the possible scattering amplitude terms as

$$\begin{aligned} A_4 = {} & \text{Pole \& constant term} \\ & + c_2(s^2 + t^2 + u^2) && : \phi^2(\partial\partial\phi)^2, \quad (\partial\phi)^2(\partial\phi)^2, \quad \ldots \\ & + c_3(s^3 + t^3 + u^3) && : \phi^2(\partial\partial\partial\phi)^2, \quad (\partial\phi)^2(\partial\partial\phi)^2, \quad \ldots \\ & + c_4(s^4 + t^4 + u^4) && : \phi^2(\partial\partial\partial\partial\phi)^2, \quad (\partial\partial\phi)^2(\partial\partial\phi)^2, \quad \ldots \\ & + c_5(s^5 + t^5 + u^5) && : \phi^2(\partial\partial\partial\partial\partial\phi)^2, \quad (\partial\partial\phi)^2(\partial\partial\partial\phi)^2, \quad \ldots \\ & + c_6(s^6 + t^6 + u^6) && : \phi^2(\partial\partial\partial\partial\partial\partial\phi)^2, \quad (\partial\partial\partial\phi)^2(\partial\partial\partial\phi)^2, \quad \ldots \\ & + c_6'(stu)^2 && : (\partial\partial\partial\partial\partial\phi)^2(\partial\partial\phi)(\partial\partial\phi)(\partial\partial\phi) \\ & + \ldots \,. \end{aligned} \tag{1.2}$$

The operator counterparts from the Lagrangian for the remaining terms were either inferred from the observations in Cheung [5] and Henning et al. [15], or explained in Henriette Elvang [16], which will be elaborated on in in sections 4.1 and 5.1.4.

Of interest here are the polynomial terms corresponding to contact contributions [15]. The possible constant term comes from familiar $(\lambda/4!)\phi^4$ theory [23]. The polynomials in Mandelstams then appear when derivatives are distributed among the fields in ϕ^4, since derivatives translate to momentum contributions through the Fourier transform [5, 15, 22, 24]. Thus, the degree of Mandelstam monomials is dictated by the amount of derivatives acting on the scalar fields as seen on the right [5, 11, 15].

The polynomials above are symmetric in s, t, and u, which is a requirement for indistinguishable fields[1] [14, 15]. However, other polynomials such as stu and $st + su + tu$ are symmetric as well. A good question to ask (which the audience in Elvang her lecture asked as well) would be why those possibilities aren't listed above. They are in fact valid terms. However, they can be rewritten into the terms above through momentum conservation $s+t+u = \sum m_j^2 = 0$ [5, 11, 15, 16, 24]. For instance,

$$\begin{aligned} s^3 + t^3 + u^3 &= s^3 + t^3 + (-s-t)^3 \qquad &: (\partial\phi)^2(\partial\partial\phi)^2 \\ &= -3(s^2 t + st^2) & \\ &= 3st(-s-t) & \\ &= 3stu, &: (\partial\partial\partial\phi)(\partial\phi)(\partial\phi)(\partial\phi) \end{aligned} \tag{1.3}$$

as discussed in the lecture by Henriette Elvang [16]. Any combination of s, t, and u giving rise to a symmetric degree-3 term can be rewritten into another such that it is sufficient to write a single term at this degree, provided the coefficients are accounted for [5, 16].

A similar process can be applied to other symmetric polynomials. This is particularly powerful since it directly shows that different derivative configurations in operator terms in the Lagrangian produce equivalent polynomial terms, and therefore equivalent scattering amplitudes [11, 15, 24]. Finding these equivalences on a Lagrangian level with operators is usually much more cumbersome since it involves integration by parts, or field re-definitions (leaving the amplitude invariant), while the derivative configurations can be inferred back from the polynomials present in the amplitude [11, 15].

In her lecture, Elvang explains a peculiarity for polynomials of degree 6, where now two independent polynomial terms emerge [16]. In the above example, these are the $s^6 + t^6 + u^6$ and $(stu)^2$ terms. They can not be rewritten into each other with $u = -s - t$:

$$\begin{gathered} s^6 + t^6 + u^6 = 2(s^6 + t^6) + 6(s^5 t + st^5) + 15(s^4 t^2 + s^2 t^4) + 20 s^3 t^3, \\ (stu)^2 = s^4 t^2 + s^2 t^4 + 2 s^3 t^3. \end{gathered} \tag{1.4}$$

Other possible combinations of s, t and u can be rewritten as a linear combination of these two, for instance, $s^3t^3 + s^3u^3 + t^3u^3 = -\frac{1}{2}(s^6 + t^6 + u^6) + \frac{9}{2}(stu)^2$. According to Henriette Elvang [16], there will be 3 independent terms at degree 12, and 4 independent ones at degree 18 for this 4-point amplitude. On a Lagrangian level, this means that particular derivative configurations, with the amount of derivatives corresponding to these degrees in Mandelstam invariants, produce distinct polynomials.

[1] If the fields were distinguishable, the symmetry requirement drops [15]

1.1.3 ▸ In Summary

The previous section illustrated equivalences in contact contributions due to momentum constraints in the example of a 4-point scattering amplitude. During her treatment of this example, Henriette Elvang [16] made a remark which was the inspiration for this thesis;

"... one way you can think of this construction is as a polynomial ring ... in the Mandelstam variables. And then you basically mod out an ideal that is generated by the constraints of momentum conservation." Henriette Elvang [16].

In principle this remark seems general enough such that the machinery of polynomial rings would be applicable to n-point amplitudes in d-dimensions. One of the goals of this thesis is to essentially unpack this remark, and to present a foundation with the machinery of polynomial rings for n-point contact contributions in d-dimensions, closely following Henning et al. [15] yet presenting it in a more digestible manner. The first research question reads as follows:

> *1. How can factor rings be constructed such that they generate valid Mandelstam polynomial terms in tree-level amplitudes for real on-shell massless scalar fields, and which rings would they be?*

Another peculiarity which was outlined in the 4-point example is the amount of invariants at each degree in Mandelstams, which seems to differ. As such, specifically for 4-point, the following was asked as a second research question:

> *2. Why are there different amounts of independent terms at different orders in Mandelstams for the 4-point contact contributions?*

1.1.4 ▸ Mandelstam Invariants and S_n Representations

Typically, (parts of) amplitudes need to be symmetric under permutations of quantities such as momenta, and therefore Mandelstam invariants [15, 20]. This can be described as the symmetric group S_n acting on the indices of the Mandelstam invariants s_{ij} for instance, with the indices understood to correspond to the momenta [7, 15, 26]. Another example would be the construction of BCJ factors as described in Li, Roest, and Veldhuis [20].

Here, focus is put more on the former. Mandelstam invariants are said to live in the $[n-2, 2]$ (Mandelstam) representation, which finds itself in the composite sum of the permutation representation $[n] \oplus [n-1, 1] \oplus [n-2, 2] \leftrightarrow s_{ij} = s_{ji}, i \neq j$ [7, 15, 20, 25]. The following research question is asked to clarify why this is the case:

3. How does the "Mandelstam representation" manifest from the properties of Mandelstam variables?

In the structure of this work, this question will be answered first to provide a foundation for Mandelstam invariants before answering the other two research questions. The outline of this work is described in the section below.

1.2 ▸ Outline

Chapter 2: Representations of the symmetric Group

The symmetric group specifically is the group of all bijections of a set onto itself [8, 28]. Often the set of integers is chosen, writing S_n, since $S_\Sigma \cong S_{n=|\Sigma|}$ [8, 28]. Representations, which are homomorphisms $\rho : \mathrm{G} \to \mathrm{GL}(V)$, contain information on how a group acts on a set through basis vectors of a vector space [25].

Since $s_{ij} = s_{ji}$, and $s_{ii} = 0$, studying the permutation representation of S_n acting on Mandelstam variables boils down to understanding how it acts on unordered pairs of cardinality $\binom{n}{2}$ [7]. Defining ρ as $\rho(\sigma)e_{\{i,j\}} := e_{\{\sigma(i),\sigma(j)\}},\ e_{\{i,j\}} \in V$, this permutation representation becomes a composition $[n] \oplus [n-1,1] \oplus [n-2,2]$, which are 1, $n-1$, and $n(n-3)/2$ dimensional irreps (irreducible representations) respectively [7, 15, 25, 26]. $[n] \oplus [n-1,1]$ is the natural permutation representation, resulting from S_n acting on a single index of basis vectors [15, 25, 26]. This means that $X_i = \sum_j e_{\{i,j\}}$ forms a basis of this sub representation [15, 26].

Momentum conservation forces $X_i = \sum_j s_{ij} = 0$, killing off the natural rep, from which $[n-2,2]$ remains [7, 15]. Therefore, this "Mandelstam representation" encodes the behaviour of Mandelstam invariants with momentum conservation, answering the third research question.

Chapter 3: Polynomial Rings

This chapter presents relevant definitions, theorems and lemmas closely following the relevant ones in Top [27]. Rings are an algebraic structure over a set R: $(R, +, \cdot, 0, 1)$, often just written as R, with addition $+$, multiplication $\cdot$, and 0 & 1 the neutral and unit elements respectively [27]. Addition is the composition following Abelian group axioms, while multiplication satisfies associativity, distributivity, and the existence of the unit element [27].

A polynomial ring $R[X]$ is one which contains polynomials in variable X (not X_i from above) with coefficients in the ring R [27]. Polynomial rings can be modded out by an ideal $I \subset R$, forming factor rings of residue classes similarly to factor groups [18, 27]. A ring of representants of each equivalence class can then describe this construction [27]. In essence, these factor rings and representants allow for constraints to be modded out, leaving only independent polynomials. In case polynomials are required to be symmetric, the fundamental theorem of symmetric polynomials can be used to express them in terms of elementary symmetric polynomials, which can be done uniquely [6].

Chapter 4: Generating the 4-Point Contact Terms

This section builds further on the contents of section 1.1.2, following examples from Henning et al. [15]: The machinery of polynomial quotient rings is first applied to the generation of contact contributions in terms of Mandelstam invariants in the 4-point amplitude. Relevant constraints are on-shell conditions leading to $p^2 = 0$ (already taken into account by only considering $[s, t, u]$), and momentum conservation leading to $s + t + u = 0$ [11, 15, 24].

Momentum conservation is used to generate the ideal $\langle s + t + u \rangle$ used to mod out $\mathbb{C}[s, t, u]$. Contact terms in Mandelstam invariants then live in the polynomial ring of representants $\mathbb{C}[s, t] \to \mathbb{C}[s, t, u]/\langle s + t + u \rangle$. This need not necessarily be symmetric such that it generally describes terms from distinguishable fields [15].

Polynomials are required to be symmetric when the fields are indistinguishable, which through the fundamental theorem of symmetric polynomials leads to contact polynomials living in $\mathbb{C}[st + su + tu, stu]$ [6, 15]. This answers the first research question only for the 4-point case.

The second one is answered by observing that the generators are of degree 2 and 3 respectively, such that the number of independent polynomial terms is dictated by the amount of possible combinations of generators at each order in Mandelstams.

Chapter 5: Generalisations to n-Point Amplitudes

This chapter mostly follows Henning et al. [15] closely while providing more context to their work. Derivative terms in operators on Lagrangian level translate to momentum contributions in amplitudes through the Fourier transform [11, 24]. In particular, Equation of Motion (EoM) and Integration by Parts (IBP) contributions translate to $p^2 = 0$ and momentum conservation conditions in momentum space respectively such that they form equivalence relations between momentum polynomials [14, 15].

For n momenta and d dimensions, only at most d momenta can be linearly independent, which result in Gram constraints through which equivalence classes are further formed [15, 17]. These conditions are only relevant when $n > d + 1$, and are obtained through vanishing $(d + 1) \times (d + 1)$ minors [15, 17].

As a result, with $\{s_{ij}\}$ living in the $[n] \oplus [n-1, 1] \oplus [n-2, 2]$ representation, $\mathbb{C}[\{s_{ij}\}]/\langle \{X_i\}, \{\Delta\} \rangle$, with $X_i = \sum_j s_{ij} = 0$ and $\{\Delta\}$ the set of vanishing minors, generates possible n-point contact contributions in d dimensions for distinguishable massless on-shell real scalar fields [15]. For indistinguishable fields, each possible polynomial is required to be symmetric, such that they can be generated by $(\mathbb{C}[\{s_{ij}\}]/\langle \{X_i\}, \{\Delta\} \rangle)^{\mathrm{S}_n}$ [15].

Chapter 2 Representations of the symmetric Group

The aim of this chapter is to gain a better understanding of some of the representations of the symmetric group, and how they can be used to describe the properties of Mandelstam invariants. This provides a framework with which to understand Mandelstam invariants before using them as variables for polynomial rings.

2.1 ▸ Recap

2.1.1 ▸ Symmetric Group

The symmetric group is defined as the set of all bijections[1] from a non-empty set onto itself, under the composition of maps [28]:

Definition 2.1.1 (Dixon and Mortimer [8] and Top [28])**.** For Σ a non-empty set, The set S_Σ of *all* bijections $\sigma : \Sigma \to \Sigma$ under the composition of maps $\circ$ forms a group called the *symmetric group* $(\mathrm{S}_\Sigma, \circ, \mathrm{id}_\Sigma)$, with $\mathrm{id}_\Sigma : \Sigma \to \Sigma,\ a \mapsto a\ \forall\, a \in \Sigma$ the identity map. A bijection σ is called a *permutation*. This group is often referred to simply by its set of bijections S_Σ.

This indeed satisfies all group axioms [28]. For the discussions in this thesis it suffices to work with the symmetric group on n integers $\mathrm{S}_{\{1,2,\dots,n\}} = \mathrm{S}_n$, since $\mathrm{S}_\Sigma \cong \mathrm{S}_{n=|\Sigma|}$ [8, 28]. S_n has $n!$ elements, and is a permutation group: a group of permutations (note the difference between a group of permutations and a group of all permutations: a permutation group is a subgroup of the symmetric group) [8].

Recall that each permutation of S_n can explicitly be written in the matrix-like notation,

$$\begin{aligned} \tau : \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}, \\ \tau'\tau : \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma'(\sigma(1)) & \sigma'(\sigma(2)) & \dots & \sigma'(\sigma(n)) \end{pmatrix}, \end{aligned} \tag{2.1}$$

[1]A bijection is a map which is injective (one-to-one), and surjective (onto) [8].

or as products of disjoint k-cycles, $(a_1 a_2 \ldots a_k)$ with distinct $a_i \in \{1, \ldots, n\}$, $\sigma(a_{l \bmod k}) = a_{l+1 \bmod k}$ for $\sigma \in \mathrm{S}_n$ [8, 28]. For instance,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \in S_4, \tag{2.2}$$

can be expressed as disjoint cycles $(124)(3) = (124)$.

The symmetric group also finds an interesting role with group actions.

Definition 2.1.2 (Dixon and Mortimer [8], Serre [25], and Top [28])**.** Let X be a non-empty set, and G a group with neutral element e. A group action is defined as a map of $\mathrm{G} \times X \to X$, written as $(g, x) \mapsto gx$, which satisfies the following two conditions for all $x \in X$:

A1 $(e, x) = x$, also written as $ex = x$,

A2 $\forall\, g, h \in \mathrm{G}$: $(gh, x) = (g, (h, x))$, also written as $(gh)x = g(hx)$.

Since a group action is a mapping of $\mathrm{G} \times X$ into X, $g \in \mathrm{G}$ can be understood to be a map which maps X onto itself with $x \mapsto gx$ [8]. This is a bijection since it is invertible, with the inverse corresponding to the map associated with g^{-1} [8, 28]. Let m be a map such that $m(g)x = gx$.

$$\begin{aligned} m(g^{-1}g)x &= (g^{-1}g)x = g^{-1}(gx) = g^{-1}(m(g)x) = m(g^{-1})(m(g)x) \\ &= m(e)x = ex = x, \end{aligned} \tag{2.3}$$

and similarly for $m(gg^{-1})$ [28]. The map m is a bijection of X into itself. Now recall that elements of the symmetric group are bijections of a non-empty set onto itself, as per definition 2.1.1. Therefore it might not be surprising that one finds that $m : \mathrm{G} \to \mathrm{S}_X$, with $m(g)x = gx$, which is in fact a homomorphism (see Top [28] for the proof and a more rigorous description of this homomorphism). This homomorphism can then serve as another way of describing group actions [8, 28].

For further review on the symmetric group, refer to Dixon and Mortimer [8] and Top [28].

2.1.2 ▸ Representations

Representations are rather useful in revealing properties of (sub)spaces of some vector space which may be invariant under group action. They find use in quantum mechanics for instance with addition of angular momentum [12]. Here they will be used to identify basis vectors of subspaces invariant under group action, and in the end reveal how properties of Mandelstam invariants manifest themselves. The definition of a representation is as follows.

Definition 2.1.3 (Serre [25])**.** Let $\mathrm{G} = (\mathrm{G}, \circ, e)$ be a finite group[2], V a vector space over a field, and $\mathrm{GL}(V)$ the group of isomorphisms[3] $V \to V$. The group homomorphism[4] $\rho : \mathrm{G} \to \mathrm{GL}(V)$ is called a *linear representation*. Then V is called a *representation space* of G.

By the above definition, ρ is the linear representation, yet often V is referred to as the representation by abuse of language [25]. As such, if a vector space V is referred to as a representation here, what is meant is the homomorphism of G to $\mathrm{GL}(V)$. Furthermore, a linear representation will simply be referred to as a representation unless otherwise specified. Moreover, if V is of dimension n, it is said that the representation is of *dimension* n as well[5] [7], Furthermore, Recall that matrices can be used to express linear maps [19].

Definition 2.1.4 (Serre [25])**.** For a vector space V with subspace W, a representation $\rho^W : \mathrm{G} \to \mathrm{GL}(W)$ is a *sub-representation* of $\rho : \mathrm{G} \to \mathrm{GL}(V)$ if W is invariant under the action of G [25],

$$w \in W : \rho(g)w \in W \quad \forall\, g \in G. \tag{2.4}$$

As such, a representation restricted to a subspace can be defined if this subspace is invariant under the action of a group. Trivial invariant subspaces would be 0 and V itself.

Definition 2.1.5 (Serre [25])**.** Let $V \neq 0$. A representation $\rho : \mathrm{G} \to \mathrm{GL}(V)$ is *irreducible* (irrep) if it has no invariant subspaces other than 0 and V itself. It can not be written as a direct sum[6] of other representations except 0 and V.

Irreducible representation can serve as building blocks for other representations. Serre [25] will be quoted for the following theorem:

Theorem 2.1.6 ([25])**.** *Every representation is a direct sum of irreducible representations.*

The proof will not be discussed here, but is indeed explained in Serre [25]. The beauty in this theorem is that it reduces the study of any representation to studying its individual parts, which proves to be crucial in understanding the behaviour of Mandelstam invariants.

[2] finite dimensional representations with lie groups can be defined as well. See Hall [13] for further details

[3] Recall that an isomorphism between vector spaces over the same field is by definition linear [19, 25]. Consider vector spaces V and W over the same field $\mathbb{F}$, and the invertible map $m : V \to W$. If $m(\alpha a + \beta b) = \alpha m(a) + \beta m(b)\ \forall\, a, b \in V\ \&\ \forall\, \alpha, \beta \in \mathbb{F}$, it is a linear map [19]. Moreover, if m is bijective, it is an isomorphism [17, 19].

[4] Recall that for given groups $(\mathrm{G}, \circ, e_G)$ & $(\mathrm{H}, \cdot, e_H)$, a map $\rho : \mathrm{G} \to H$ for which $\forall\, a, b \in \mathrm{G}$: $\rho(a \circ b) = \rho(a) \cdot \rho(b)$ is a group homomorphism [28]

[5] Some texts might refer to it as *degree* like Serre [25]. To avoid confusion with polynomial degrees, dimension will be used in the remainder of this work.

[6] Recall that a direct sum means $V = W \oplus W'$ if $W \cap W' = 0$ and $\dim(V) = \dim(W) + \dim(W')$ [25]

2.2 ▶ Examples of Representations

Let $\rho : \mathrm{G} \to \mathrm{GL}(V)$ with G a finite group and V an undetermined finite-dimensional vector space. In the subsections below a few basic representations will be given.

2.2.1 ▶ Trivial Rep

The *trivial representation* is one for which $\rho(g) = 1$ for $g \in \mathrm{G}$ [25]. Each element g is assigned to the identity map [7]. The representation is of dimension 1 [7, 25]. Thus $V = \mathrm{Span}(e)$ has a single basis vector. This is often useful for quantities which are invariant under the actions of a group.

2.2.2 ▶ Permutation Representation

Consider a set X, which G acts on: $x \mapsto gx$ for $x \in X$, $g \in \mathrm{G}$. Let the representation $\rho : \mathrm{G} \to \mathrm{GL}(V)$ be defined such that,

$$\rho(g)e_x = e_{gx}, \tag{2.5}$$

for permutations $x \mapsto gx$, where $(e_x)_{x \in X}$ is the basis of V. This is a *permutation representation*, which is $|X|$-dimensional [25].

For instance, consider $X = \{1, 2, \dots, n\}$. Then, the representation is n dimensional, and $\rho(g)e_i = e_{gi}$: the index of a basis vector is permuted (indeed a permutation since maps associated with group actions are bijective, see section 2.1) [7]. It then suffices to come up with matrices for the representation for which the elements are either 1 or 0 which transforms a basis vector into another. For example, if $g1 = 2$ & $g2 = 1$

$$\rho(1 \leftrightarrow 2)(ae_1 + be_2 + \dots) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} = \begin{pmatrix} b \\ a \\ \vdots \end{pmatrix} = ae_2 + be_1 + \dots, \tag{2.6}$$

with I an $n-2 \times n-2$ identity matrix. Because of the nature of the definition of this representation, it is also sometimes referred to as the *natural representation* of S_n with the set $\{1, \dots, n\}$ in the context of some other representation where it appears within the composite sum (see section 2.3.2) [15, 26]. In a more abstract sense, the homomorphism associated with group action $m : \mathrm{G} \to \mathrm{S}_X$ is a permutation representation, though this wouldn't be classified as a linear representation by Serre [25] [8].

There is a subspace of V which is invariant under the action of G, namely $\mathrm{Span}(\sum_i^n e_i)$. In other words, $\forall\, g \in \mathrm{G},\ \rho(g)w = w$ for $w \in \mathrm{Span}(\sum_i^n e_i)$:

$$\rho(g) \begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix} \tag{2.7}$$

The sub-representation associated with this invariant subspace is the trivial representation $\rho_1 : \mathrm{G} \to \mathrm{GL}(\mathrm{Span}(\sum_i^n e_i))$. It is said that $\sum_i^n e_i$ transforms according to the trivial representation.

Therefore, ρ is not an irrep, but consists of sub representations, one of which being the trivial representation $\rho_1 : \mathrm{G} \to \mathrm{GL}(\mathrm{Span}(\sum_i^n e_i))$, and the other being the complement [7, 25]. The representation can then be written as a block diagonal matrix acting on a vector of which one of the new basis vectors $e'_1 = \sum_i^n e_i$.

$$\begin{pmatrix} 1 & \dots \\ \vdots & O \end{pmatrix} \tag{2.8}$$

2.2.3 ▶ The $(n-1)$-Dimensional Representation of the Symmetric Group

The $(n-1)$-dimensional sub-representation of the permutation representation for S_n on a set $\{1, \dots, n\}$ which is complement to the trivial sub-representation is the *standard representation*, which is defined as [7],

$$\rho_{n-1} : \mathrm{S}_n \to \mathrm{GL}\left(W = \left\{x \in V : \sum_i^n x_i = 0\right\}\right). \tag{2.9}$$

In other words; the subspace for which all vectors in it have their components summing to 0 is an invariant subspace under group action [7]. Indeed; the indices of the base vectors get permuted $\rho(g)e_i = e_{gi}$, which doesn't change the sum of components $\sum_i^n x_i = 0$. For example, for S_3,

$$\rho(1 \leftrightarrow 3)(e_1 + e_2 - 2e_3) = (e_3 + e_2 - 2e_1), \tag{2.10}$$

where the sum of components didn't change, and equates to 0.

This is a $n-1$-dimensional irrep [7]. Suppose $w \in W$ with components w_i such that $\sum_{i=1}^n w_i = 0$. Then choose to write $w_n = -\sum_{i-1}^{n-1} w_i$ (taking inspiration from the 3-dimensional example in Diaconis [7]),

$$\begin{aligned} w &= w_1e_1 + w_2e_2 + \dots + w_{n-1}e_{n-1} + w_ne_n \\ &= w_1e_1 + w_2e_2 + \dots + w_{n-1}e_{n-1} - \left(\sum_{i=1}^{n-1} w_i\right) e_n \\ &= w_1(e_1 - e_n) + w_2(e_2 - e_n) + \dots + w_{n-1}(e_{n-1} - e_n) \\ &= w_1f_1 + w_2f_2 + \dots + w_{n-1}f_{n-1}. \end{aligned} \tag{2.11}$$

It is easy to verify that cycles of (ij) for which $i, j \neq n$ acting on this vector indeed keeps the sum of components in the (f_i) basis,

$$w_if_i + w_jf_j = w_i(e_i - e_n) + w_j(e_j - e_n) \mapsto w_i(e_j - e_n) + w_j(e_i - e_n) = w_if_j + w_jf_1, \tag{2.12}$$

leaving the sum of components in the (e_i) basis unaltered. What about an (in) cycle? Then,

$$\begin{aligned}
w_i f_i = w_i(e_i - e_n) &\mapsto w_i(e_n - e_i) = -w_i f_i, \\
w_j f_j = w_j(e_j - e_n) &\mapsto w_j(e_j - e_i + (e_n - e_n)) = w_j(f_j - f_i), \\
\sum_{j=1}^{n-1} w_j f_j &\mapsto -\left(\sum_{j-1}^{n-1} w_j\right) f_i + \sum_{j\neq i} w_j f_j \\
&= -\left(\sum_{j-1}^{n-1} w_j\right)(e_i - e_n) + \sum_{j\neq i} w_j(e_j - e_n) \\
&= \sum_{j\neq i} w_j e_j - \left(\sum_{j-1}^{n-1} w_j\right) e_i + \left(\sum_{j-1}^{n-1} w_j - \sum_{j\neq i} w_j\right) e_n \\
&= \sum_{j\neq i} w_j e_j - w_n e_i + w_i e_n,
\end{aligned} \tag{2.13}$$

which is a vector of components w_i, summing to 0, since $\sum_{i=1}^{n} w_i = 0$. As such, this sub-representation is $(n-1)$-dimensional, with $(f_i)_{i\in\{1,\dots,n-1\}}$ a basis. Diaconis [7] discusses an example of S_3, where they show a table of transformations and matrices for the basis $f_1 = e_1 - e_2$, $f_2 = e_2 - e_3$, and show that this sub-representation is an irreducible one (will not be shown here) [7, 25]. As such, the permutation representation for S_n acting on the set $\{1,\dots,n\}$ can be decomposed into,

$$\rho = \rho_1 \oplus \rho_{n-1}, \tag{2.14}$$

with ρ_1 the 1-dimensional trivial representation and ρ_{n-1} the complementary $(n-1)$-dimensional representation [25].

2.3 ▸ Describing Mandelstam Properties with Representations

2.3.1 ▸ Mandelstam Invariants

Recall that scattering amplitudes are required to be Lorentz invariant as discussed in section 1.1.1, and as such will be composed of Lorentz invariant quantities such as the inner products of momenta $p_{i\mu}p_j^{\mu}$ [5, 10, 15, 24]. These inner products are described by *Mandelstam variables*, or *invariants*, which are defined as,

$$s_{ij\dots} := -(p_i + p_j + \dots)^2, \tag{2.15}$$

with d-vectors p_i such that it is symmetric in indices [11]. For the discussion in this chapter, and thesis for that matter, the scalar fields associated with these momenta are massless, which means that $p_i^2 = 0$, such that $(p_i + p_j + \ldots)^2 = 2(p_{i\mu}p_j^\mu + \ldots)$. It therefore suffices to use

$$s_{ij} = p_{i\mu}p_j^\mu, \tag{2.16}$$

as a definition of Mandelstam invariants in two indices like in Henning et al. [15], omitting the factors of -2. Then $s_{ijk\ldots}$ would be proportional to $s_{ij}+s_{ik}+s_{jk}+\ldots$. Mandelstams in more indices than two can be generated by s_{ij} in the massless case, and therefore it suffices to only consider s_{ij} here.

With this definition of the Mandelstam invariants, the following properties are evident and relevant for the purposes of this chapter:

C1 $s_{ij} = s_{ji}$ which comes from index symmetry in equation 2.16 [11].

C2 $i \neq j$ which comes from $p_{i\mu}p_i^\mu = 0$ if a particle is massless [11].

For n momenta these conditions lead to $n(n-1)/2$ independent Mandelstam invariants. One way to find this is in observing that for each index i, there are $n - i$ available indices j to find elements with due to C1 and C2. Summing this for each i gives,

$$\begin{aligned}
(n-1)+(n-2)+...+1 &= \sum_{m=1}^{n-1} m \\
&= -n + \sum_{m=1}^{n} m \\
&= -n + n + (n-1) + \cdots + (n-(n-1)) \qquad (2.17) \\
&= -n + nn - \sum_{m=1}^{n-1} m, \\
\Rightarrow \sum_{m=1}^{n-1} m &= \tfrac{1}{2}n(n-1).
\end{aligned}$$

Another way of understanding it is through combinatorial theory. Recall that out of n objects, k amount can be picked in $\binom{n}{k}$ different ways, where [1],

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{2.18}$$

Such that for $k = 2$, $\binom{n}{k} = \frac{1}{2}n(n-1)$.

Another way of describing the construction of criteria C1 and C2 is through a set of unordered pairs $\{i, j\}$ of cardinality $\binom{n}{2}$ (which is $n(n-1)/2$) with $i \neq j$ [7]. This framework is rather useful when describing representations of S_n, since a basis $e_{\{i,j\}}$ can be formed to span an $\binom{n}{2}$-dimensional vector space; a pair of indices for each basis vector [7].

2.3.2 ► Acting on Unordered Pairs

Let the symmetric group S_n act on the set of unordered pairs of cardinality $n(n-1)/2$ as,

$$\sigma\{i,j\} := \{\sigma(i), \sigma(j)\}, \tag{2.19}$$

for $\sigma \in \mathrm{S}_n$. Then, consider the representation $\rho : \mathrm{S}_n \to \mathrm{GL}(V)$ with vector space $V = \mathrm{Span}(e_{\{i,j\}})$, such that,

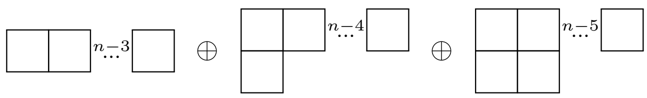
$$\rho(\sigma)e_{\{i,j\}} := e_{\sigma\{i,j\}} = e_{\{\sigma(i),\sigma(j)\}}. \tag{2.20}$$

This representation is $n(n-1)/2$ dimensional, and is a permutation representation [7, 25].

It was already understood before that the permutation representation is not an irreducible one. In this case there are three subspaces of V stable under group action, such that ρ is decomposed into three irreducible representations,

$$\rho = \rho_1 \oplus \rho_{(n-1)} \oplus \rho_{n(n-3)/2}, \tag{2.21}$$

where the subscript denotes the dimension of the sub-representation [7, 15, 26]. This decomposition is often written in terms of partitions, such that one writes $[n] \oplus [n-1, 1] \oplus [n-2, 2]$, referring to the size of each row of a Young tableaux [7, 15, 26]:



(2.22)

The 1-dimensional invariant subspace is spanned by [7],

$$\mathcal{S}_0 = \sum_{\{i,j\}} e_{\{i,j\}} = e_{\{1,2\}} + e_{\{1,3\}} + \dots\ , \tag{2.23}$$

said to transform according to the trivial representation. The $(n-1)$-dimensional invariant subspace is spanned by [7],

$$\mathcal{S}_i = \sum_{j \neq i} e_{\{i,j\}} - c \sum_{\{i,j\}} e_{\{i,j\}} = \sum_{j \neq i} e_{\{i,j\}} - c\mathcal{S}_0, \tag{2.24}$$

with c such that each $\mathcal{S}_i$ is orthogonal to $\mathcal{S}_0$ [7]. One might notice that this is in fact n-dimensional, but also observe that only one index survives the sum; the basis has a single index. This suggests that $\rho_{(n-1)}(\sigma)\mathcal{S}_i = \mathcal{S}_{\sigma(i)}$.

Take for instance $n = 4$. Then (e_{ij} is shorthand for $e_{\{i,j\}}$),

$$\mathcal{S}_0 = e_{12} + e_{13} + e_{14} + e_{23} + e_{24} + e_{34}, \tag{2.25}$$

is invariant under group action. Now, take $\mathcal{S}_2$:

$$\mathcal{S}_2 = e_{12} + e_{23} + e_{24} - c(e_{12} + e_{13} + e_{14} + e_{23} + e_{24} + e_{34}) \tag{2.26}$$

A permutation like (kl) where $k \neq 2$ & $l \neq 2$ indeed leaves $\mathcal{S}_2$ invariant, while $(2k)$ maps $\mathcal{S}_2 \mapsto \mathcal{S}_k$. Then c can be chosen such that $\mathcal{S}_i \cdot \mathcal{S}_0 = 0$ with both subspaces orthogonal [7].

The sub-representations $\rho_1 \oplus \rho_{n-1} = \rho_n$ together form the *natural representation*, which is n dimensional [15, 26]. The subspace associated with this representation has a (much more useful) basis,

$$X_i = \sum_{j \neq i} e_{\{i,j\}}, \tag{2.27}$$

which acts in a way such that simply the indices are permuted; $\rho_n(\sigma)X_i = X_{\sigma(i)}$ (left as an exercise to verify) [15, 26]. As such, this sub-representation ρ_n can be treated similarly to an n-dimensional permutation representation as discussed in the previous section, constructing bases for the invariant subspaces ρ_1 & ρ_{n-1} out of X_i. The addition of the $\rho_{n(n-3)/2}$ sub-representation in the composite sum of equation 2.21 is a consequence of S_n acting on a set of unordered pairs of cardinality $\binom{n}{2}$ instead of acting on a single integer-numbered index.

2.3.3 ► The Mandelstam Representation

Previously only the conditions C1 and C2 (See description 2.3.1) were considered, which meant there were $n(n-1)/2$ independent Mandelstam variables for n momenta. The aim of previous observations on unordered pairs is to apply it to Mandelstam variables, which for external on-shell fields also abide by momentum conservation, and therefore integration by parts (IBP) relations [15]. Momentum conservation is stated as,

$$\sum_{j=1}^{n} p_j^{\mu} = 0, \tag{2.28}$$

taking all momenta as incoming (or phrased another way with the help of Yang Li from Li, Roest, and Veldhuis [20]; outgoing momenta are redefined by absorbing the negative sign into the vector itself, such that there is no sign distinction between incoming and outgoing momenta) [11]. Multiplying this by $p_{i\mu}$ yields (here using Mandelstam invariants as a basis),

$$p_{i\mu} \sum_{j=1}^{n} p_j^{\mu} = \sum_{i \neq j} s_{ij} = 0, \tag{2.29}$$

on the assumption that the momenta correspond to massless on-shell scalar fields ($p_{i\mu} p_i^{\mu} = 0$) [15]. This reduces the amount of independent Mandelstam variables by n, since each X_i allows for an s_{ij} to be expressed in terms of others ($n \leq n(n-1)/2$

for $n \geq 3$). Left are $\frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$ independent Mandelstam variables [15]. This number corresponds to the dimension of the $\rho_{n(n-3)/2}$ representation.

Indeed: With momentum conservation introducing $X_i = 0$, The representations ρ_1 and ρ_{n-1} in $\rho = \rho_1 \oplus \rho_{n-1} \oplus \rho_{n(n-3)/2}$ from the previous section are killed off [15]. What is left is the subspace complement to the subspace corresponding to $\rho_1 \oplus \rho_{n-1}$, which is the one corresponding to $\rho_{n(n-3)/2}$ [7, 15]. Therefore, it might be said that *Mandelstam invariants transform according to the $\rho_{n(n-3)/2}$ representation* [20]. In speaking, this representation has sometimes been referred to as the *Mandelstam representation*, since it encodes the behaviour of Mandelstam invariants for massless on-shell scalar fields.

An example of $n = 4$ is first treated to get a grasp of determining a basis for this representation.

Example 2.3.1.

$$\begin{aligned} X_1 &= s_{12} + s_{13} + s_{14} = 0, \quad X_2 = s_{12} + s_{23} + s_{24} = 0, \\ X_3 &= s_{13} + s_{23} + s_{34} = 0, \quad X_4 = s_{14} + s_{24} + s_{34} = 0. \end{aligned} \tag{2.30}$$

This for instance leads to;

$$s_{12} = -s_{13} - s_{14} = s_{23} + s_{34} + s_{24} + s_{34} = -s_{12} + 2s_{34}. \tag{2.31}$$

Similarly for s_{13} and s_{14}, this leads to,

$$\begin{aligned} s_{12} &= s_{34}, \\ s_{13} &= s_{24}, \\ s_{14} &= s_{23}, \\ s_{14} &= -s_{12} - s_{13}. \end{aligned} \tag{2.32}$$

Left are the two invariants s_{12} and s_{13} (others could have been chosen), which corresponds to the amount of invariants of $n(n-3)/2 = 2$.

Let s_{12} and s_{13} form a basis of the invariant subspace. To understand its behaviour, consider a simple permutation (12) (this is an example discussed with Tonnis ter Veldhuis). Then s_{12} is left untouched, but $s_{13} \mapsto s_{23} = s_{14} = -s_{12} - s_{13}$. The matrix corresponding to $\rho_2((12))$ is then found to be,

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \tag{2.33}$$

such that for the basis vectors $s_{12} := (1,0)^\mathsf{T}$, $s_{13} := (0,1)^\mathsf{T}$;

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \tag{2.34}$$

The resulting vector is still within the space spanned by (s_{12}, s_{13}) indeed.

For a slightly more general approach, the reader will be referred to Cheung [5] for determining a basis for Mandelstam invariants.

Chapter 3 Polynomial Rings

This chapter is mostly a recap of some of the essential definitions and features of (polynomial) Rings as required for the topic of this paper. It mostly follows definitions and corollaries from Top [27]. This with the aim to build the groundwork for generating polynomials in Mandelstam invariants, which in the previous chapter have been described with representations of unordered pairs.

3.1 ► Ring Algebraic Structure

3.1.1 ► Definitions

Groups in particular are algebraic structures which find lots of applications in physics. Anything related to symmetry is often tackled with the wisdom from group and/or representation theory. For some examples, see Jones [18]. On the contrary, rings don't find as many applications in physics. However, polynomial rings in particular provide for a useful framework for generating polynomials under various constraints, which is exactly what is explored in this thesis, and other works such as Henning et al. [14, 15]. Even though Groups are very likely treated in most physics programmes, rings likely are not.

With this in mind, this section lists a few relevant fundamental definitions taken from Top [27] , starting with the most fundamental one:

Definition 3.1.1 (Top [27])**.** A *unitary ring* is a 5-tuple $(R, +, \cdot, 0, 1)$, with R a set, maps $+$ & $\cdot$ (*addition* and *multiplication*) defined as ($\cdot$ is often not written)

$$+ : R \times R \to R, \qquad (a, b) \mapsto a + b \quad \forall\, a, b \in R, \tag{3.1}$$

$$\cdot\ : R \times R \to R, \qquad (a, b) \mapsto ab \qquad \forall\, a, b \in R, \tag{3.2}$$

and $0, 1 \in R$ (zero and unit element), such that the following properties hold:

R1 $(R, +, 0)$ is an *abelian group* over R, with composition $+$ and neutral element 0. As such it follows group axioms [18, 28]:

G0 $a + b \in R$, (closure)

G1 $a + (b + c) = (a + b) + c \quad \forall\, a, b, c \in R$, (associativity)

G2 $0 + a = a + 0 = a \quad \forall\, a \in R$, (zero / neutral element)

G3 $\forall\, a \in R,\ \exists\,(-a) \in R$ s.t. $a + (-a) = (-a) + a = 0,$ (inverse)

Since it's abelian:

A $a + b = b + a \quad \forall\, a, b \in R,$ (abelian / commutative)

R2 $a(bc) = (ab)c \quad \forall\, a, b, c \in R,$ (associativity)

R3 $a(b + c) = (ab) + (bc) \quad \forall\, a, b, c \in R,$ (distributivity)

R4 $1a = a1 = a \quad \forall\, a \in R.$ (unit element)

A ring $(R, +, \cdot, 0, 1)$ is often referred to in notation by its set R. For instance; "Let R be a ring...". Similarly, the abelian group $(R, +, 0)$ embedded within the ring could be referred to with R as well, for instance; "...subgroup of R...". Relations with inverse elements of the subgroup of a ring R are often written as for instance $a - a$ instead of $a + (-a)$ for $a \in R$. A ring is *non-unitary* if (R4) from definition 3.1.1 is not satisfied; the ring does not have 1 [27].

For certain theorems or lemmas, it may be necessary that a ring is a *field*. This is not to be confused with the fields (scalar or vector fields for instance) in physics. A ring is a field if it is both *commutative* (if $\forall\, a, b \in R : ab = ba$), and is a *division ring* (if $\forall\, a \in R\backslash\{0\}, \exists\, a^{-1} \in R : aa^{-1} = a^{-1}a = 1$) [27]. Examples of fields are real $\mathbb{R}$ and complex $\mathbb{C}$ numbers [27].

Another fairly fundamental object is the *ideal*, which will be important when considering factor rings in section 3.1.2. The definition of an ideal is as follows:

Definition 3.1.2 (Top [27])**.** An *ideal* (also said to be a two sided ideal) $I \subset R$ of a ring R is a subset which satisfies:

I1 I is a subgroup of R (not a subring):

H0 $0 \in I,$

H1 $a - b \in I \quad \forall\, a, b \in I,$

I2 $\forall\, r \in R \quad \& \quad a \in I; \quad ra, ar \in I.$

Condition I2 in definition 3.1.2 could be replaced by a weaker one, namely either [27];

- $\forall\, r \in R \quad \& \quad a \in I; \quad ra \in I,$ (*left ideal*)

or;

- $\forall\, r \in R \quad \& \quad a \in I; \quad ar \in I.$ (*right ideal*)

An example of an ideal would be $3\mathbb{Z}$ for the ring $\mathbb{Z}$, since $0 \in 3\mathbb{Z}$, and multiplying an integer multiple of 3 with any integer is another multiple of 3: $a \in \mathbb{Z} : a3\mathbb{Z} \in 3\mathbb{Z}$. In fact, $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ for any $n \in \mathbb{Z}$ [27]. Moreover, this is an example of a generated ideal, which is defined as follows:

Definition 3.1.3 (Top [27]). Consider R a unitary ring. $a_1, \dots, a_n \in R$ *generate* a left ideal as follows:

$$Ra_1 + \cdots + Ra_n = \left\{ r_1, \dots, r_n \in R : \sum_{i>o}^{n} r_i a_i \right\}, \tag{3.3}$$

with $\sum$ denoting a sequence of $+$ additions. A right ideal is generated in a similar way. With R commutative, one speaks of just an ideal being generated, and the notation $\langle a_1, \dots, a_n \rangle := Ra_1 + \cdots + Ra_n$ is often used. An ideal is *principal* if it is generated by a single $a : \langle a \rangle \subset R$.

In this chapter, the $\langle\ \rangle$ brackets will exclusively be reserved as notation for generating ideals unless otherwise specified.

Even though ideals are not subrings, they are subgroups. In fact, they are normal:

Corollary 3.1.4 (Top [28]). *For R a ring and $I \subset R$ an ideal, I is a normal subgroup, that is:*

$$a + I - a = I \quad \forall\, a \in R. \tag{3.4}$$

Proof (Top [27]). This follows from R, therefore I, being abelian;

$$a + b - a = a - a + b = 0 + b = b \quad \forall\, a \in R \quad \& \quad \forall\, b \in I.$$

■

3.1.2 ► Factor Rings

From group theory it is well known that a factor group G/H can be defined for H a *normal subgroup* of G [28]. By defining multiplication well on R/I for R a ring and I an ideal, R/I (R modulo I) is a ring called the *factor ring* [27].

Definition 3.1.5 (Top [27]). Let R be a ring and $I \subset R$ an ideal. The set of *residue classes*;

$$R/I = \{a \in R : \ \overline{a} := a + I \subset R\}, \tag{3.5}$$

forms a ring, called the *factor ring* with addition;

$$(a + I) + (b + I) = (a + b) + I, \qquad \overline{a} + \overline{b} = \overline{a + b}, \tag{3.6}$$

and multiplication;

$$(a + I) \cdot (b + I) := ab + I, \qquad \overline{a} \cdot \overline{b} := \overline{ab}, \tag{3.7}$$

with notation $\overline{a} := a + I$.

See Top [27] on the validity of multiplication as defined in definition 3.1.5. It could be said that the factor ring is a ring of residue classes modulo the ideal. The elements of R/I are simply the unique residue classes $\overline{a} = a + I$. This is indeed a ring: the axioms can be verified to hold with multiplication as defined in definition 3.1.5 [27]. This will not be done here. To be able to determine whether two elements in a ring belong to the same residue class, the following corollary is a rather powerful tool:

Corollary 3.1.6 (Top [28])**.** *For R a ring and $I \subset R$ an ideal, the elements $\overline{a}, \overline{b} \in R/I$ are equal if and only if $a - b \in I$, or in other words, when a and b are 'separated' by a 'multiple' of an element in the ideal.*

Proof (Top [27]).

- $\Rightarrow$: $\quad \overline{a} = \overline{b} \Rightarrow a + I = b + I \Rightarrow a = b + i,\ \forall\, i \in I, \Rightarrow a - b = i \in I.$

- $\Leftarrow$: $\quad a - b \in I \Rightarrow i = a - b \in I \Rightarrow a + I = (b + i) + I = b + I$ since $i \in I$ (subgroup closure).

■

continuing with the example of $3\mathbb{Z}$ being an ideal of $\mathbb{Z}$, what would the factor ring $\mathbb{Z}/3\mathbb{Z}$ look like? This is illustrated in the example below.

Example 3.1.7. Consider the ring $\mathbb{Z}$ (integers), and the ideal $3\mathbb{Z} = \{3 \cdot a : \ a \in Z\} \subset \mathbb{Z}$ (integer multiples of 3). Then $\overline{a} = a + 3\mathbb{Z} = \{\dots, a - 3, 0, a + 3, \dots\}$ denotes a residue class for $a \in Z$. According to definition 3.1.5, $\mathbb{Z}/3\mathbb{Z}$ is a ring of residue classes. As such, by corollary 3.1.6;

$$\mathbb{Z}/3\mathbb{Z} = \left\{\overline{0}, \overline{1}, \overline{2}\right\}, \tag{3.8}$$

with

$$\begin{aligned} \overline{0} &= \{\dots, -6, -3, 0, 3, 6, \dots\}, \\ \overline{1} &= \{\dots, -5, -2, 1, 4, 7, \dots\}, \\ \overline{2} &= \{\dots, -4, -1, 2, 5, 8, \dots\}. \end{aligned} \tag{3.9}$$

Some $a \in \mathbb{Z}$ then lives in a partition corresponding to one of the three residue classes of $\mathbb{Z}/3\mathbb{Z}$. To illustrate multiplication:

$$\overline{2} \cdot \overline{5} = \overline{2 \cdot 5} = \overline{1}.$$

This example illustrates how the notion of factor rings for integers is related to modular arithmetic, and can also be thought of as such in more abstract cases. Because factor rings are a set of residue classes, they are rather useful for classifying polynomials based on equivalence constraints when the ring in question is a polynomial ring, which will be described in the section below.

3.2 ▸ Polynomial Rings

3.2.1 ▸ Single & Multivariable Polynomials

Definition 3.2.1. Let R be a ring. A polynomial ring $R[X]$ is a ring of polynomials in a single *variable* X with *coefficient* a_i in R [27]:

$$R[X] := \left\{ a_i \in R : \sum_{i=0}^{<\infty} a_i X^i \right\}. \tag{3.10}$$

This is indeed a ring as it can be shown that the axioms of definition 3.1.1 hold (See Top [27]). Moreover, $R[X]$ will be commutative if R is as well [27]. A polynomial can be evaluated through the evaluation homomorphism, which will not be of relevance here [27]. The *degree* of a polynomial, written as $\deg f$ for f a polynomial in $R[X]$, is the largest i for which the coefficient $a_i \neq 0$ [27]. Then, the *leading coefficient* is $a_{\deg f}$ [27].

This is a nice starting point to generate polynomials. However, aiming to describe scattering amplitudes at a later stage, it will be necessary to be able to define what a polynomial ring in multiple variables looks like. In the 4-point case from section 1.1.2 there are already 3 variables (s, t, u). Fear not! There is a rather elegant way of defining a polynomial ring in multiple variables. Key insight is that polynomial rings are rings themselves:

Definition 3.2.2. Similarly to a polynomial ring in a single variable, since $R[X]$ is a ring, a polynomial ring in multiple variables is inductively defined as [27]:

$$R[X_1, \ldots, X_n] := (R[X_1, \ldots, X_{n-1}])\,[X_n]. \tag{3.11}$$

Therefore [27];

$$R[X_1, \ldots, X_n] := \left\{ a_{i_1,\ldots,i_n} \in R : \sum_{i_1,\ldots,i_n=0}^{<\infty} a_{i_1,\ldots,i_n} X_1^{i_1} \cdots X_n^{i_n} \right\}. \tag{3.12}$$

3.2.2 ▸ Division With Remainder

As alluded to at the end of section 3.1.2, factor rings will be composed of polynomial rings as well with ideals generated by polynomials. In this section, the equivalence classes will first be classified before constructing the factor ring with polynomials. The following theorem provides a method to dissect polynomials as a first step, and find the remainder with which to define equivalence classes.

Theorem 3.2.3 (Top [27])**.** *Consider R a unitary ring. Let $f, g \in R$ with $g \neq 0$ and $a_{\deg g}$ is invertible (if $\exists b \in R : ab = ba = 1$). Then;*

$$f = qg + r, \tag{3.13}$$

where $q, r \in R[X]$, the quotient *and* remainder *respectively upon division by g, are unique with* $\deg r < \deg g$.

The proof for this can be found in Top [27], and is not of importance for understanding the train of thought of this section. The idea now is to perform division with remainder on polynomials and construct the equivalence classes, similarly to how this was achieved with integers modulo an integer. To help illustrate division by g, the *long division* procedure with polynomials can be used to perform division with remainder [27].

Example 3.2.4 (Long Division)**.** Let $f = X^3+6X-4 \in \mathbb{R}[X]$. Consider division by $g = X^2 + X - 1 \in \mathbb{R}[X]$ (notice unit leading coefficient) such that $f = qg + r, \quad q, r \in \mathbb{R}[X]$. The procedure of long division can be used to find q and r. Repeatedly subtract a multiple h_i of g from f until the degree of the remaining polynomial is less than $\deg g$. This becomes the remainder r while summing the multiples h_i becomes q.

$$\begin{array}{rl} & \qquad\qquad q = \quad X - 1 \\ \cline{2-2} X^2 + X - 1 & \big)\, X^3 \qquad\quad + 6X - 4 \\ & \underline{X^3 + X^2 - \ X \qquad} \ - \\ & \qquad -X^2 + 7X - 4 \\ & \qquad \underline{-X^2 - \ X + 1} \ - \\ & \qquad\quad r = \quad 8X - 5 \end{array} \tag{3.14}$$

In the above example, $h_1 g = X^3 + X^2 - X$ is first subtracted from f, after which $h_2 g = -X^2 - X + 1$ is subtracted. Therefore $q = h_1 + h_2 = X - 1$ and $r = 8X - 5$. Then f can be reconstructed:

$$f = qg + r = (X - 1)(X^2 + X - 1) + (8X - 5) = X^3 + 6X - 4. \tag{3.15}$$

Multiple different polynomials $f \in R[X]$ can have the same remainder $r \in R[X]$ upon division by $g \in R[X]$ up to a quotient $q \in R[X]$. These can be classified into residue classes modulo the principal ideal $\langle g \rangle \subset R[X]$ [27]. This is a similar idea to the example of factor rings with integers. Observe that $qg \in \langle g \rangle$ when looking at $f = qg + r$: $f + \langle g \rangle = r + \langle g \rangle$ [27]:

$$r = f \mod g. \tag{3.16}$$

As such;

$$\overline{f} = f + \langle g \rangle, \tag{3.17}$$

form residue classes. Two polynomials $f_1, f_2 \in R[X]$ belong to the same residue class (have the same remainder) if $f_1 - f_2 \in \langle g \rangle$: $f_2 = f_1 + qg$ for $q, g \in R[X]$ and $qg \in \langle g \rangle$ (see corollary 3.1.6). Notice that elements of the principal ideal g have remainder $r = 0$.

In order to quantify the different equivalence classes, a polynomial ring of representants can be constructed. These representants are ones for each residue class.

Theorem 3.2.5 (Top [27])**.** *Consider R a unitary ring and $g \in R[X]$ with* $\deg g > 0$ *& $a_{\deg g}$ invertible. Then*

$$\begin{aligned}\{h \in R[X] : \deg h < \deg g\} &\to R[X]/\langle g\rangle, \\ h &\mapsto h + \langle g\rangle.\end{aligned} \tag{3.18}$$

is a bijection.

The proof will once again not be given but can be found in Top [27], but intuitively it makes sense since polynomials of lower degree than g can not be divided out by g, but adding multiples of g now allows for division while the remainders of both instances were the same.

The essence of this theorem is indeed that unique representants of the elements of $R[X]/\langle g\rangle$ can be found for each residue class modulo the principal ideal [27]. These representants should not be thought of as the elements of the factor ring, since the factor ring is a set equivalence classes, not polynomials themselves. They are a way to quantify the different equivalence classes in a digestible manner. As such, when it is mentioned that a factor ring generates polynomials in this thesis, what is meant is that a ring of representants can be made which generates the polynomials. To be clear, this does not mean that scattering amplitudes can not include a polynomial which is not an element of this ring of representants. In the end it is a result of the field theory. Rather, representants are a useful way of taking into account the residue classes when constructing amplitudes from physical criteria.

The procedure to find representants is to find polynomials $h \in R[X]$ for which $\deg h < \deg g$. This is illustrated in the two examples below.

Example 3.2.6. consider again $f = X^3 + 6X - 4 \in \mathbb{R}[X]$ and $g = X^2 + X - 1 \in \mathbb{R}[X]$ from the previous example. Then, using theorem 3.2.5, the representants are:

$$\{a, b \in \mathbb{R} : aX + b\} \to \mathbb{R}[X]/\langle X^2 + X - 1\rangle. \tag{3.19}$$

As such:

$$\mathbb{R}[X]/\langle X^2 + X - 1\rangle = \left\{a, b \in \mathbb{R} : \overline{a}\overline{X} + \overline{b}\right\}. \tag{3.20}$$

Example 3.2.7 (A funky example)**.**
The inspiration to look into this case was from a brief discussion with Jimin Park. Consider $\mathbb{R}[X]/(X^2 + 1)$. Using theorem 3.2.5:

$$\mathbb{R}[X]/\langle X^2 + 1\rangle = \left\{a, b \in \mathbb{R} : \overline{a}\overline{X} + \overline{(}b)\right\}, \tag{3.21}$$

(Note that it is not the same as the above example since $\overline{X}$ is different). This ring exhibits interesting yet familiar behaviour. With $g = X^2 + 1$:

$$\begin{aligned}\overline{X}^2 &= X^2 + \langle g\rangle = X^2 + (-X^2 - 1) + \langle g\rangle = -1 + \langle g\rangle \\ &= \overline{-1},\end{aligned} \tag{3.22}$$

where multiplication as defined in definition 3.1.5 was used for the first equality and subgroup closure for the second. The same result could have been obtained by corollary 3.1.6:

$$X^2 + 1 \in \langle X^2 + 1 \rangle \quad \Rightarrow \quad \overline{XX} = \overline{X}^2 = \overline{-1}. \tag{3.23}$$

It can also be seen through long division, dividing X^2 by $X^2 + 1$. Then, upon multiplication of two elements $f = \overline{a}\overline{X} + \overline{b}, f' = \overline{a'}\overline{X} + \overline{b'} \in \mathbb{R}[X]/(X^2 + 1)$:

$$\begin{aligned} f_1 f_2 &= (\overline{a}\overline{X} + \overline{b})(\overline{a'}\overline{X} + \overline{b'}) \\ &= \overline{a}\overline{a'}\overline{XX} + \overline{bb'} + \overline{a}\overline{b'}\overline{X} + \overline{a'bX} \\ &= (\overline{a}\overline{a'}\ \overline{-1} + \overline{bb'}) + (\overline{ab'} + \overline{a'b})\overline{X}. \end{aligned} \tag{3.24}$$

As one might suspect, it turns out that $\mathbb{R}[X]/\langle X^2 + 1 \rangle \cong \mathbb{C}$ as mentioned in Top [27], the proof of which is given there.

In the case of multiple variables, definition 3.2.2 is exploited to understand what polynomial factor rings for multiple variables look like:

$$R[X_1, \ldots, X_n]/\langle g \rangle = R[X_1, \ldots, X_{n-1}][X]/\langle g \rangle, \quad g \in R[X_1, \ldots, X_n], \tag{3.25}$$

since $R[X_1, \ldots, X_{n-1}]$ is a ring itself. Theorem 3.2.5 holds with $R[X_1, \ldots, X_{n-1}]$ as ring and the variable being X_n, provided of course that the ring is unitary and that the leading coefficient $a_{\deg g}$ is invertible in $R[X_1, \ldots, X_{n-1}]$. Specifically the case of multiple variables is of importance since the n-point amplitudes depend on Mandelstam variables. As such it is useful to treat an example with two variables.

Example 3.2.8. The following example is inspired by an example in Top [27]. Consider $\mathbb{R}[X, Y]$. Let $g = X^2 + Y^2 - 1$ generate a principal ideal:

$$\langle g \rangle = \mathbb{R}[X][Y]g = \left\{ q \in \mathbb{R}[X][X] : q(X^2 + Y^2 + 1) \right\}. \tag{3.26}$$

By theorem 3.2.3:

$$f = q(X^2 + Y^2 + 1) + r, \quad q, r \in \mathbb{R}[X][Y],\ \deg_Y r < \deg_Y g. \tag{3.27}$$

One way this can be understood is by observing that $X^2 + 1 \in \mathbb{R}[X]$, such that g in $f = qg + r$ is a polynomial in the variable Y and coefficients in $\mathbb{R}[X]$. With this observation, using theorem 3.2.5 [27]:

$$\mathbb{R}[X, Y]/\langle X^2 + Y^2 + 1 \rangle = \left\{ a, b \in \mathbb{R}[X] : \overline{a}\overline{Y} + \overline{b} \right\}. \tag{3.28}$$

Here;

$$\overline{Y}^2 = Y^2 + (-X^2 - Y^2 - 1) + \langle g \rangle = -X^2 - 1 + \langle g \rangle = -\overline{X}^2 - \overline{1}. \tag{3.29}$$

Or simply $\overline{Y}^2 = (\overline{Y}^2 + \overline{X}^2 + \overline{1}) - \overline{X}^2 - \overline{1} = -\overline{X}^2 - \overline{1}$.

The reader is invited to use long division for the above examples to see the polynomial division in action for these cases.

3.2.3 ► Symmetric Polynomials

In some cases it may be necessary that each polynomial from a polynomial ring is symmetric under permutations of its variables. An example is when scalar fields are indistinguishable such that the scattering amplitude is symmetric [14, 15].

Definition 3.2.9. Consider the polynomial ring $R[X_1, \ldots, X_n]$ with $\{X_i\}$ independent variables and R a field. let $f \in R[X_1, \ldots, X_n]$. For all permutations the Symmetric group $\sigma \in S_n$ [6]:

$$f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = f(X_1, \ldots, X_n) \quad \Rightarrow \quad f \text{ is symmetric.} \tag{3.30}$$

Symmetric polynomials in $R[X_1, \ldots, X_n]$ form a subring [21]

$$R[X_1, \ldots, X_n]^{S_n} \subset R[X_1, \ldots, X_n]. \tag{3.31}$$

There is a rather particular set of symmetric polynomials which fill a specific role in generating symmetric polynomials with the fundamental theorem of symmetric polynomials. Before showing this theorem, the definition of elementary symmetric polynomials is presented below.

Definition 3.2.10 (Cox [6] and Macdonald [21])**.** Consider n amount of distinct variables X_i. *Elementary Symmetric Polynomials* are defined as:

$$\begin{aligned} e_0 &= 1, \\ e_r &= \sum_{1 \leq i_1 < \cdots < i_r \leq n} X_{i_1} \cdots X_{i_r}, \end{aligned} \tag{3.32}$$

To grasp what this means, consider for instance 3 distinct variables (X_1, X_2, X_3). Then,

$$\begin{aligned} e_1 &= \sum_{1 \leq i_1 \leq 3} X_i &&= X_1 + X_2 + X_3, \\ e_2 &= \sum_{1 \leq i_1 < i_2 \leq 3} X_{i_1} X_{i_2} &&= X_1 X_2 + X_1 X_3 + X_2 X_3, \\ e_3 &= \sum_{1 \leq i_1 < i_2 < i_3 \leq 3} X_{i_1} X_{i_2} X_{i_3} &&= X_1 X_2 X_3, \end{aligned} \tag{3.33}$$

are the 3 elementary symmetric polynomials. These are then used in the fundamental theorem of symmetric polynomials.

Theorem 3.2.11 (Fundamental Theorem of Symmetric Polynomials [6])**.** *Let R be a field. Any symmetric polynomial in $R[X_1, \ldots, X_n]$ can be expressed as a polynomial in the elementary symmetric polynomials e_i with the coefficients in R.*

Theorem 3.2.12 ([6])**.** *Let R be a field. A symmetric polynomial in $R[X_1, \ldots, X_n]$ can uniquely be expressed as a polynomial in elementary symmetric polynomials e_i.*

The proofs for both of these theorems will not be elaborated on here and can be found in Cox [6], though importantly the proof of the latter theorem discovers the following ring isomorphism,

$$\varphi : R[U_1, \dots, U_n] \to R[e_1, \dots, e_n], \qquad\qquad U_i \mapsto e_i, \tag{3.34}$$

showing uniqueness [6]. This becomes relevant when treating contact terms in scattering amplitudes for indistinguishable fields, since they can be generated using elementary symmetric polynomials instead, which will be applied in the latter half of the next chapter [15].

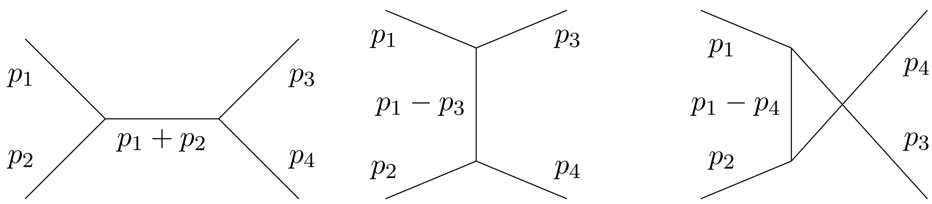
Chapter 4 Generating the 4-Point Contact Terms

The material from the previous chapter will be applied to the example of 4-point amplitudes in 4 dimensions for massless scalar theories. The aim here is to generate contact terms from the 4-point amplitude for massless on-shell real scalar fields. Generating them forn-point amplitudes in d-dimensions is further explored in chapter 5.

4.1 ▸ Invariants of 4-Point Interactions

When treating the 4-pt amplitude, any quantum field theory text book likely quickly introduces the Mandelstam variables as s, t, and u, seen before in section 1.1.2, equation 1.1 [20, 24]. They appear either through propagators or as a result of two derivatives in the corresponding operator [24].

Recall that Mandelstam invariants were defined in equation 2.16 as $s_{ij} = p_{i\mu}p_j^\mu$ for massless scalar fields, from which the properties $s_{ij} = s_{ji}$ and $s_{ii} = 0 \Rightarrow i \neq j$ emerge from C1 & C2 in description 2.3.1 [11]. Then momentum conservation is taken into account as $\sum_i p_i^j = 0$ with all momenta treated as incoming as seen in equation 2.28, due to which it suffices to consider,



(a) s-channel. **(b)** t-channel. **(c)** u-channel.

Figure 4.1: Feynman 4-pt interaction diagrams for massless on-shell scalar fields, with p_i 4-momenta [24].

$$\begin{aligned} s &:= s_{12} = s_{34}, \\ t &:= s_{13} = s_{24}, \\ u &:= s_{14} = s_{23}, \end{aligned} \tag{4.1}$$

where the second equality in each line follows from momentum conservation as seen in section 2.3.3 [11]. Each invariant finds correspondence in the s-, t-, and u-channels as seen in section 1.1.2, diagrams of which are illustrated in figure 4.1. Momentum conservation then manifests itself as [15, 24],

$$p_{1\mu} \sum_{j \neq 1} p_j^{\mu} = s + t + u = 0. \tag{4.2}$$

This result is crucial for the discussion, since it is the source of equivalence relations for polynomials in Mandelstam Variables. For instance,

$$\begin{aligned} s^2 + t^2 + u^2 &= s^2 + t^2 + (-t - s)^2 \\ &= 2(s^2 + t^2 + st) \\ &= 2(-s(-s - t) - t(-s - t) - st) \\ &= -2(st + su + tu), \\ &\Rightarrow s^2 + t^2 + u^2 \sim -2(st + su + tu), \end{aligned} \tag{4.3}$$

is one such equivalence.

With the machinery of polynomial rings, let $s+t+u$ generate an ideal $\langle s + t + u \rangle$ of the polynomial ring $\mathbb{C}[s, t, u]$, using complex numbers as a field without loss of generality [15]. Modding out the ideal from $\mathbb{C}[s, t, u]$ gives,

$$\mathbb{C}[s, t] \to \mathbb{C}[s, t, u]/\langle s + t + u \rangle, \tag{4.4}$$

with $\mathbb{C}[s, t]$ understood as the representants as per theorem3.2.5 [15, 27]. Indeed, left are $n(n - 3)/2 = 2$ Mandelstam invariants of the $\rho_{n(n-3)/2}$ representation, as seen in section 2.3.3, to generate the contact term contributions in the 4-pt amplitude for distinguishable on-shell massless scalar fields with $\mathbb{C}[s, t]$ (with the polynomials not necessarily being symmetric) [15].

Each term of a polynomial in $\mathbb{C}[s, t, u]/\langle s + t + u \rangle$ can be traced back to operators in the Lagrangian. Using the representants $\mathbb{C}[s, t]$ avoids finding equivalent operator terms in the Lagrangian. For instance, the term $s_{12}^n s_{13}^m = s^n t^m \in \mathbb{C}[s, t]$ is associated with a term

$$\left(\partial_{\mu_1} \cdots \partial_{\mu_n} \partial_{\nu_1} \cdots \partial_{\nu_n} \phi_1\right) \left(\partial^{\mu_1} \cdots \partial^{\mu_n} \phi_2\right) \left(\partial^{\nu_1} \cdots \partial^{\nu_n} \phi_3\right) \phi_4, \tag{4.5}$$

on Lagrangian level, observing that each derivative acting on field ϕ_i produces momentum p_i^{μ} [15].

4.2 ► Indistinguishable Fields

The polynomial ring from equation 4.4 holds for distinguishable fields ϕ_i, assigning a flavor index to each field to distinguish one field from another [15]. In this case the operator term with flavor indices can be traced back from the momentum indices. In case the fields are in fact indistinguishable, the fields are not labeled, such that the contact term contributions should be symmetric under permutations [14, 15]. Each term needs to be complemented by others such that the resulting combination is symmetric under interchanging momenta, effectively resulting in the contact term polynomials needing to be symmetric under S_4 for the 4-pt contact contributions,

$$\tilde{A}_4\left(p^\mu_{\sigma(1)}, p^\mu_{2\sigma(2)}, p^\mu_{\sigma(3)}, p^\mu_{\sigma(4)}\right) = \tilde{A}_4\left(p^\mu_1, p^\mu_2, p^\mu_3, p^\mu_4\right), \tag{4.6}$$

as per definition 3.2.9 [6, 15].

For polynomials in Mandelstam variables (s, t, u), symmetry under S_3 is imposed instead [15]. an example of how this relates to permutation of momenta is given below.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} s_{12} & s_{13} & s_{14} \\ \downarrow & \downarrow & \downarrow \\ s_{23} & s_{24} & s_{21} \end{pmatrix} = \begin{pmatrix} s & t & u \\ \downarrow & \downarrow & \downarrow \\ u & t & s \end{pmatrix}. \tag{4.7}$$

This is by no means a formal proof. However, the polynomials will be treated as if they exhibit S_3 symmetry in Mandelstams [15].

Key insight in integrating S_3 symmetry into contact term polynomials is that the variables of the ring don't necessarily have to be the Mandelstams themselves, but could be elementary symmetric polynomials instead if every polynomial in that ring is symmetric. Recall that any symmetric polynomial in a polynomial ring over a field can be written in terms of elementary symmetric polynomials by the fundamental theorem of polynomials (theorem 3.2.11) [6]. Considering the variables (s, t, u), the 4 elementary symmetric polynomials are given by (see definition 3.2.10),

$$\begin{aligned} e_0 &= 1, \\ e_1 &= s + t + u, \\ e_2 &= st + su + tu, \\ e_3 &= stu. \end{aligned} \tag{4.8}$$

With these elementary symmetric polynomials, Henning et al. [15] makes the following proposition,

$$\left[\mathbb{C}[s, t, u]/\left\langle s + t + u\right\rangle\right]^{S_3} = \mathbb{C}[e_1, e_2, e_3]/\left\langle e_1\right\rangle, \tag{4.9}$$

Such that instead the elementary symmetric polynomials are used as variables, and the ideal is generated by e_1 instead, since $e_1 = s + t + u = 0$. Therefore, by theorem 3.2.5 [15],

$\mathcal{O}(s_{ij})$	$\mathcal{O}(p_i^\mu)$	m	Monomials
1	2	1	$e_1 = 0$
2	4	1	e_2
3	6	1	e_3
4	8	1	e_2e_2
5	10	1	e_2e_3
6	12	2	$e_2e_2e_2$ e_3e_3
7	14	1	$e_2e_2e_3$
8	16	2	$e_2e_2e_2\ e_2$ $e_3e_3\ e_2$
9	18	2	$e_2e_2e_2\ e_3$ $e_3e_3\ e_3$
10	20	2	$e_2e_2e_2\ e_2e_2$ $e_3e_3\ e_2e_2$
11	22	2	$e_2e_2e_2\ e_2e_3$ $e_3e_3\ e_2e_3$
12	24	3	$e_2e_2e_2\ e_2e_2e_2$ $e_2e_2e_2\ e_3e_3$ $e_3e_3\ e_3e_3$
13	26	2	$e_2e_2e_2\ e_2e_2e_3$ $e_3e_3\ e_2e_2e_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4.1: This table shows the possible independent monomial terms of $\mathbb{C}[e_2, e_3]$. m denotes the amount of independent monomials at each order in Mandelstams.

$$\mathbb{C}[e_2, e_3] \to \mathbb{C}[e_1, e_2, e_3]/\left\langle e_1 \right\rangle, \tag{4.10}$$

with $\mathbb{C}[e_2, e_3]$ the representants of the residue classes of $\mathbb{C}[e_1, e_2, e_3]/\left\langle e_1 \right\rangle$. Therefore, contact term polynomials for indistinguishable massless on-shell real scalar fields live in $\mathbb{C}[e_2, e_3]$ [15].

Monomial terms as shown in table 4.1 can appear in the 4-pt amplitude depending on the presence of certain operators. It can be seen that 2 independent combinations of e_2 and e_3 pop up at Mandelstam degree 6, 3 at 12, and so on. It can also be seen that at orders 7 and 13 the amount of independent monomials is reduced by one momentarily. The hypothesis is that this behaviour continues up to higher order, however this was not explicitly computed. This coincides with results of Dujava [9], where operators were counted with the Hilbert series.

Generating symmetric polynomials for contact terms like this is powerful since it is rather simple to relate back to operator terms in the Lagrangian as seen in equation 4.5. Particularly, performing integration by parts or redefining fields to find equivalent Lagrangians can be much more cumbersome than finding polynomials for contact terms which satisfy the necessary physical criteria [11, 15]. Incorporating momentum symmetry can be achieved through the use of the fundamental theorem of symmetric polynomials, which allows for immediate generation of symmetric contact terms without having to refer back to the Lagrangian.

Limitations of polynomial ring generators are that they don't contain information on how many invariant terms there are at each degree directly. Instead, a different object, the Hilbert series (or Molien series), provides this information through its Taylor coefficients. This is not explored here, but finds use in Henning et al. [14] and Li, Roest, and Veldhuis [20].

Chapter 5 Generalisations to n-Point Amplitudes

This chapter mostly follows some of chapter 5 of Henning et al. [15]. However, the goal is to treat the story in a more nuanced way such that certain concepts and statements are elaborated on in more detail, in the hopes of making the subject easier to grasp. Having found generators for contact terms in 4-point amplitudes, the procedure is approached in a more general way here for n-point amplitudes in d-dimensions. Here, spacetime is assumed to be Euclidean, with complex momenta [15].

5.1 ▸ Constraints of Momenta

5.1.1 ▸ Polynomials in Momentum Space

Momentum polynomials in momentum space show up due to derivatives in operators on a Lagrangian level due to the Fourier transform [10, 15, 24]. Write the Fourier transform of real scalar fields as,

$$\begin{aligned}\mathcal{F}[\phi(x)] = \hat{\phi}(p) &= \frac{1}{(2\pi)^{d/2}}\int_{-\infty}^{\infty}\phi(x)e^{-ip\cdot x}d^dx,\\ \mathcal{F}^{-1}[\hat{\phi}(p)] = \phi(x) &= \frac{1}{(2\pi)^{d/2}}\int_{-\infty}^{\infty}\hat{\phi}(p)e^{ip\cdot x}d^dp.\end{aligned} \tag{5.1}$$

with $\phi(x) = \mathcal{F}_d^{-1}[\mathcal{F}_d[\phi(x)]]$, provided that $\phi(x)$ goes to 0 as $|x| \to \infty$ [22]. Indeed, recall that taking the derivative of a function is equivalent to multiplying its Fourier transform by momentum in momentum space [22]. For instance, for two derivatives:

$$\partial_\mu\partial_\nu\phi(x) = \partial_\mu\left(\mathcal{F}^{-1}[ip_\nu\hat{\phi}(p)]\right) = \mathcal{F}^{-1}\left[-p_\mu p_\nu\hat{\phi}(p)\right], \tag{5.2}$$

for which μ and ν can be contracted by other terms in the same operator, such as $(\partial_\mu\partial_\nu\phi_1)(\partial^\mu\phi_2)(\partial^\nu\phi_3)\phi_4$, leading to a monomial $p_{1\mu}p_2^\mu p_{1\nu}p_3^\nu = (p_1\cdot p_2)(p_1\cdot p_3)$ [15]. Therefore, for a given operator $\mathcal{O}^{n,k}$ consisting of n fields and k derivatives,

$$\mathcal{O}^{n,k} \propto \int \left(\prod_{j=1}^{n} d^d p_j \hat{\phi}_j(p_j) \right) F^{n,k}(\{p_i\}) e^{i \sum_{l=1}^{n} p_l x}, \tag{5.3}$$

where $F^{n,k}(\{p_i\})$ is a polynomial in n different momenta corresponding to the n different fields, of degree k, which depends on the configuration of the derivatives distributed among the different fields [15]. This polynomial is the study of this chapter, and of chapter 5 in Henning et al. [15].

5.1.2 ▸ Translating Operator Conditions to Momentum Space

On Lagrangian level with massless real scalar fields, operators are considered equivalent if they are related by either the equation of motion $\Box\phi(x)$ (EoM), or by a total derivative term $\partial_\mu \mathcal{O}^\mu$. Operators could be rewritten into each other through either field redefinitions or integration by parts (IBP) leaving the scattering amplitude invariant [11, 15]. Henning et al. [15] refers to this as EoM and IBP redundancies.

First the *EoM* condition: Two operators are considered equivalent if they are related by the EoM ($\Box\phi$ in this case) [14, 15].

$$\Box\phi(x) = \Box\left(\mathcal{F}^{-1}\left[\hat{\phi}(p)\right]\right) = \mathcal{F}^{-1}\left[-p^2\hat{\phi}(p)\right] = \mathcal{F}^{-1}\left[\mathcal{F}\left[\Box\phi(x)\right]\right]. \tag{5.4}$$

For a massless on-shell scalar field, $p^2 = -m^2 = 0$ [11]. As such, the EoM equivalence relation translates to an equivalence relation in $p^2 = 0$ for momentum polynomials, as stated in Henning et al. [15].

Second is the *IBP condition*; equivalence up to total derivative terms for which $\partial_\mu \mathcal{O}^\mu = 0$, which could occur from integration by parts, translates to equivalence up to momentum conservation $\sum_i p_i^\mu = 0$, where the sign of the momenta is defined such that this sum indeed equals 0 [11, 14]. On the operator level, if two operators are related by a total derivative term, they are considered equivalent since the integral of the total derivative term is assumed to vanish [14]. $\mathcal{O}^\mu$ can be rewritten as a product $\mathcal{O}^\mu = \mathcal{O}_0^\mu \prod_i \mathcal{O}_i$ where $\mathcal{O}_j$ is a local operator associated with a single field, taking into account $\mathcal{O}_0^\mu$ can be separated by product rule in case it involves a $\partial^\mu \cdots$ term. Then (omitting the factors of $1/(2\pi)^{d/2}$);

$$\begin{aligned}
\partial_\mu \mathcal{O}^\mu &= \partial_\mu \left(\mathcal{O}_0^\mu \prod_j \mathcal{O}_j \right) \\
&= \partial_\mu \mathcal{F}^{-1}\left[\mathcal{O}_0^\mu\right] \prod_j \mathcal{F}^{-1}\left[\mathcal{O}_j\right] \\
&= \partial_\mu \left(\int_{-\infty}^{\infty} \hat{\mathcal{O}}_0^\mu e^{ip_0 x} d^d p_0 \prod_j \int_{-\infty}^{\infty} \hat{\mathcal{O}}_j e^{ip_j x} d^d p_j \right)
\end{aligned} \tag{5.5}$$

$$= \int_{-\infty}^{\infty} \left(\hat{\mathcal{O}}_0^\mu \prod_j \hat{\mathcal{O}}_j \right) \partial_\mu e^{i \sum_k p_k x} d^d p_0 \prod_l d^d p_l$$
$$= \int_{-\infty}^{\infty} \left(\hat{\mathcal{O}}_0^\mu \prod_j \hat{\mathcal{O}}_j \right) \left(\sum_m p_{m\mu} \right) e^{i \sum_k p_k x} d^d p_0 \prod_l d^d p_l,$$

where the conservation of momentum $\sum_i p_{i\mu} = 0$ is found, as stated in Henning et al. [14] and Henning et al. [15]. As such the statement of operators being equivalent up to total derivative terms translates to polynomials in momenta being equivalent if they are related by momentum conservation.

Total derivative terms (IBP), and terms proportional to $\Box\phi$ (EoM) vanish [15]. As such operators up to these terms are equivalent. In momentum space this translates to equivalences up to momentum conservation and $p^2 = 0$. As such, equivalence relations can be proposed between operators $\mathcal{O}$, and polynomials in momentum space $F(\{p_i\})$ [15]:

$$\begin{aligned} \mathcal{O}_1 \sim \mathcal{O}_2 + \Box\phi\mathcal{O}_3 \quad &\Rightarrow \quad F_1(\{p_i\}) \sim F_2(\{p_i\}) + p_i^2 F_3(\{p_i\}), \\ \mathcal{O}_1 \sim \mathcal{O}_2 + \partial_\mu \mathcal{O}_3^\mu \quad &\Rightarrow \quad F_1(\{p_i\}) \sim F_2(\{p_i\}) + \left(\sum_i p_i \right) F_3(\{p_i\}). \end{aligned} \tag{5.6}$$

For further discussion the equivalence relations in momentum space are of interest. Let $\mathbb{C}[p_1^\mu, \ldots, p_n^\mu]$ be a polynomial ring in momenta p_i. Invoking EoM and IBP conditions result in the polynomials being equivalent up to the generated ideal $\left\langle p_1^2, \ldots, p_n^2, \sum_i^n p_i \right\rangle$ [15]. Therefore;

$$\mathbb{C}\left[p_1, \ldots, p_n\right] / \left\langle p_1^2, \ldots, p_n^2, \sum_i^n p_i \right\rangle, \tag{5.7}$$

generates the polynomials in momentum space for massless scalar fields with the equivalence relations as stated in equation 5.6.

5.1.3 ▸ Gram Determinants

Before applying momentum conservation and the equation of motion, there is another set of constraints which are put on momenta. Namely that in d-dimensions, only at most d momenta are linearly independent, which means that for n momenta, there exist scalars a_i such that:

$$\sum_{i=1}^n a_i p_i^\mu = 0, \tag{5.8}$$

is non trivial (not all a_i are 0) [15, 17]. After all, in d-dimensions, there are at most only so many momenta which can be independently expressed in terms of each other. Once there is an additional momentum $d+1$, it is guaranteed that this momenta (or any of the momenta) has to be expressed in terms of the other ones (another way of seeing it is by understanding there are at most d basis vectors). This imposes additional constraints on momenta, and therefore on Mandelstam variables as well.

The way this manifests itself with Mandelstam variables is through the Gram matrix [15].

Definition 5.1.1 (Horn and Johnson [17])**.** Let (V, f) be an inner product space with $\langle\cdot,\cdot\rangle$ as the inner product. For vectors $v_1, \ldots, v_n \in (V, f)$, the *Gram matrix* is defined as,

$$G = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \ldots & \langle v_1, v_n\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \ldots & \langle v_2, v_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \ldots & \langle v_n, v_n\rangle \end{pmatrix}. \tag{5.9}$$

With Mandelstam variables $s_i j = p_{i\mu} p_j^\mu$, it takes the form,

$$G_s = \begin{pmatrix} p_{1\mu}p_1^\mu & \cdots & p_{1\mu}p_n^\mu \\ \vdots & \ddots & \vdots \\ p_{n\mu}p_1^\mu & \cdots & p_{n\mu}p_n^\mu \end{pmatrix} = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix} \tag{5.10}$$

The following theorem from Horn and Johnson [17] lists useful properties of this matrix:

Theorem 5.1.2 (Horn and Johnson [17])**.** *Consider the gram matrix G for vectors $v_1, \ldots, v_n \in (V, f)$ with inner product $\langle\cdot,\cdot\rangle$. Then,*

H1 *G is Hermitian and positive semidefinite ($x^*Gx \geq 0$ for $x \in \mathbb{C}^n$),*

H2 *G is positive definite ($x^*Gx > 0$ for $x \in \mathbb{C}^n$) $\Leftrightarrow$ $v_1, \ldots, v_n$ are linearly independent,*

H3 $\operatorname{rank} G = \dim(\operatorname{Span}(v_1, \ldots, v_n))$.

The proof can be found in Horn and Johnson [17]. Since G_s is Hermitian, it is *rank principal*, which means that it has a nonsingular (nonvanishing determinant) $r \times r$ *principal* submatrix for $\operatorname{rank} G = r$ (a submatrix is principal if it is made from a row index set and a column index set which are the same: $G[\alpha, \alpha]$) [17]. Since $\operatorname{rank} G_s = \dim(\operatorname{Span}(p_1^\mu, \ldots, p_n^\mu))$ by theorem 5.1.2, the rank of G_s is at most d (recall there are only at most d linearly independent momenta), and any $d \times d$ principal submatrix has a nonzero determinant [15, 17].

Theorem 5.1.3 (Horn and Johnson [17])**.** *A positive semidefinite matrix A is positive definite $\Leftrightarrow$ A is nonsingular ($\det(A) \neq 0$).*

As such, H2 in theorem 5.1.3 can be described as $\det(G) = 0 \Leftrightarrow v_1, \ldots, v_n$ are linearly independent. Therefore, for the Gram matrix of Mandelstam variables, if $n \leq d$, G_s is positive definite by theorem 5.1.2 if the momenta are all linearly independent. When $n > d$, G_s is positive semidefinite by theorem 5.1.2, and by theorem 5.1.3, it is also singular.

Observe that submatrices of G_s are also Gram matrices. This means that any $(d+1)$-square submatrix of G_s has vanishing determinant for $n > d$, since those momenta are already linearly dependent [15]. Therefore, when $n > d$, extra conditions on Mandelstam variables are imposed, and can be found by computing the determinants of $(d+1) \times (d+1)$ submatrices of G_s, and equating them to 0 [15]. These conditions will be referred to with $\{\Delta\}$ the set of these vanishing determinants [15]. Then;

$$\mathbb{C}[\{p_i^\mu\}]^{O(d)} \cong \mathbb{C}[\{s_{ij}\}]/\langle\{\Delta\}\rangle, \tag{5.11}$$

is an isomorphism by the fundamental theorems of invariant theory, which will not be explored in this thesis [15]. Here the spacetime is assumed to be Euclidean such that $O(d)$ invariance (parity) invariance is imposed [15].

Note however, that the Gram determinant does not impose any new conditions when $n = d+1$. Recall that momentum conservation implies $X_i = \sum_j s_{ij} = 0$. It is known that the Gram determinant is 0 due to the momenta being linearly dependent. However, the conditions obtained are equivalent to momentum conservation [15]. This can be seen through operations which leave the determinant invariant. The determinant of a matrix doesn't change when a multiple of a row is added to another [19]. Because of this,

$$\begin{aligned}
\det(G_s) &= \begin{vmatrix} 0 & s_{12} & \ldots & s_{1n} \\ s_{12} & 0 & \ldots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & \ldots & 0 \end{vmatrix} \\
&= \begin{vmatrix} X_1 & X_2 & \ldots & X_n \\ s_{12} & 0 & \ldots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & \ldots & 0 \end{vmatrix} \\
&= X_1 M_{11} + X_2 M_{12} + \cdots + X_n M_{1n} \\
&= 0,
\end{aligned} \tag{5.12}$$

where M_{1n} are the minors of the elements g_{1n} of G_s [19]. Momentum conservation already ensures that each $X_i = 0$, and therefore each minor M_{1k} in principle does not have to be 0, not imposing any additional conditions on Mandelstams in general. As such, only when $n > d+1$ will there be extra conditions put on the Mandelstam variables due to the Gram determinant if momentum conservation is considered [15].

5.1.4 ► Relating to Amplitudes & Lagrangians

The study of possible Mandelstam polynomials in momentum space is applicable to building amplitudes. It won't be explained rigorously here how momentum pops up in the amplitude from derivatives but a hand-wavy argument is presented instead. In deriving the propagator between a scalar field and the derivative of the scalar field, the momentum term which is a result of the Fourier transform is kept in the numerator (follow the derivation of the Feynman propagator in Schwartz [24]). This propagator then pops up in Wick contractions involving a derivative term when calculating the LSZ-Reduction formula (resulting in the scattering amplitude). Where the LSZ-formula forces external particles to be on-shell, propagators for external particles get cancelled, with the piece of momentum surviving [24]. This results in a momentum space Feynman rule of *derivative coupling* (See a follow-up example in Schwartz [24] on derivative coupling) [15].

For a general operator $\mathcal{O}^{n,k}$ of n fields as seen in equation 5.3, the degree k polynomial $F^{n,k}(\{p_i\})$, the form of which being a result of the derivative configuration, becomes the momentum space Feynman rule associated with this configuration [15]. This is a contact-term contribution to the tree level n-point amplitude [15]. In general for the amplitude itself it means that for some n-point, the contact term contributions consist of $F^{n,k}$ differing in k [15].

Because of the constraints of momenta, some polynomial terms are equivalent as discussed previously. Using l to denote only independent Feynman rules $F_l^{n,k}$, the n-point amplitude for real massless scalar fields can in general be made up of,

$$A_n(\{p_i\}) = \sum_{k,l} c_l F_l^{n,k} + \text{other non-contact terms,} \tag{5.13}$$

with c_l the Wilson coefficients [15]. These are the possible terms an amplitude in general can consist of. The final form of a particular amplitude depends on the operators $\mathcal{O}^{n,k}$ in the theory and whether the fields are indistinguishable. This picture of building contact terms for amplitudes from momentum constraints is particularly powerful since determining equivalence classes for Mandelstam polynomials is much less cumbersome than for operators, and forcing amplitudes to be symmetric under permutation of momenta is much simpler than taking all possible Wick contractions when the fields are indistinguishable [15]. A contact contribution in the amplitude can then be traced back to an operator. For instance an $s_{12}^a s_{13}^b$ contribution corresponds to an,

$$(\partial_{\mu_1} \dots \partial_{\mu_a} \partial_{\nu_1} \dots \partial_{\nu_b} \phi_1)(\partial^{\mu_1} \dots \partial^{\mu_a} \phi_2)(\partial^{\nu_1} \dots \partial^{\nu_b} \phi_3)\phi_4 \dots \tag{5.14}$$

operator term [15].

The use of Mandelstam invariants ensures Lorentz invariance [11, 15, 24]. However, another way to ensure this is through antisymmetric means with the ϵ-tensor: $\epsilon_{\mu_1 \dots \mu_n} p_{i_d}^{\mu_1} \dots p_{i_d}^{\mu_d}$. These terms have been omitted from the discussion by imposing parity invariance, since the focus was put on polynomial terms in Mandelstam invariants, but more information on them can be found in Henning et al. [15].

As such, polynomial rings are useful machinery since they can be used as generators for polynomials with the desired equivalence classes. With the equivalence classes determined through the constraints of momenta, the contact term contributions for the amplitude can be generated by such a polynomial ring.

5.2 ▸ Generating Polynomials in Mandelstam Variables

5.2.1 ▸ Distinguishable Fields

The criteria of Mandelstam invariants with momentum conservation was discussed in sections 2.3.1 and 2.3.3, and are summarised as,

M1 $s_{ij} = s_{ji}$ (Symmetric),

M2 $s_{ii} = 0$ (EoM),

M3 $X_i = \sum_j s_{ij} = 0$ (IBP / Momentum Conservation),

M4 $\{\Delta\}$ for $n > d + 1$ (Vanishing Gram determinants).

M1 and M2 ensure the indices $\{i, j\}$ are elements of a set of unordered pairs of cardinality $\binom{n}{2}$. These are generally seen as more trivial properties of Mandelstam variables such that $\mathbb{C}[\rho_1 \oplus \rho_{n-1} \oplus \rho_{n(n-3)/2}] = \mathbb{C}[\{s_{ij}\}]$ are generated by Mandelstams satisfying M1 and M2 [15]. M3 and M4 are used to define equivalences between polynomials.

Contact term contributions in Mandelstam invariants then live in a polynomial ring modded out by the ideal generated by the momentum and Gram constraints $\langle\{X_k\}, \{\Delta\}\rangle$. The procedure is similar to the discussion in chapter 3, such that

$$\mathbb{C}[\{s_{ij}\}] / \langle\{X_k\}, \{\Delta\}\rangle\,, \tag{5.15}$$

generates polynomials up to equivalence in M3 and M4 with representants as per theorem 3.2.5, and M1 and M2 embedded in choosing variables $\{s_{ij}\}$ [15]. As such it is only necessary to find the representants of each equivalence class to build unique polynomials,

$$f \in \mathbb{C}[\{s_{ij}\}] : \; \bar{f} = f + \langle\{X_k\}, \{\Delta\}\rangle\,. \tag{5.16}$$

This polynomial quotient ring holds for distinguishable fields, meaning that these polynomials need not be symmetric in momentum, and thus under exchange of indices of Mandelstam variables, for instance $s_{ij} \leftrightarrow s_{ik}$ [14, 15].

Without Gram conditions M4 ($n \leq d+1$), the problem translates to finding the basis of the $\rho_{n(n-3)/2}$ representation such that,

$$\mathbb{C}[\rho_{n(n-3)/2}] \to \mathbb{C}[\{s_{ij}\}] / \langle\{X_k\}\rangle\,, \tag{5.17}$$

generates contact contributions [15].

5.2.2 ▸ Indistinguishable Fields

If the fields are indistinguishable, the Mandelstam polynomials should be symmetric under index exchange [15]. In effect this means that a polynomial in n-momenta should be invariant under group action of S_n on the indices of Mandelstam variables (unordered pairs): $\sigma \in \mathrm{S}_n : \; \sigma\{i,j\} := \{\sigma(1), \sigma(2)\}$ [15]. The representants of,

$$(\mathbb{C}[\{s_{ij}\}]/\left\langle \{X_k\}, \{\Delta\}\right\rangle)^{\mathrm{S}_n}\,, \tag{5.18}$$

generate the contact term contributions for an n-point amplitude with indistinguishable real massless scalar fields [15]. This has the added effect that it eliminates any first-order terms in Mandelstams, since $\sum_i X_i = \sum_i \sum_{j\neq i} s_{ij}$ is invariant under action of S_n yet it vanishes (trivial representation). Without Gram conditions $n \leq d+1$, $\mathbb{C}[\rho_{n(n-3)/2}]^{\mathrm{S}_n} \to (\mathbb{C}[\{s_{ij}\}]/\left\langle \{X_k\}\right\rangle)^{\mathrm{S}_n}$, similarly to above.

These rings, and the ones for the distinguishable case, are Cohen-Macaulay, for which the result is that any polynomial in this ring can be generated uniquely by a finite set of generators [15]. This property, and its proof, will not further be explored in this work but can be studied in Henning et al. [15].

5.2.3 ▸ In Practice with n=4 Once More

The specific example of the 4-point amplitude was already studied in chapter 4 from an understanding of the Mandelstam variables s, t, & u, without discussing Gram constraints. Here the aim is to retrieve the same results from the more general context of understanding the momentum constraints and Feynman rule contributions.

For $n = 2$, $\{s_{ij}\} = \{s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}\}$ (the indices form a set of unordered pairs of cardinality $n(n-1)/2$), such that

$$\mathbb{C}[s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}]/\left\langle X_1, X_2, X_3, X_4, \{\Delta\}\right\rangle, \tag{5.19}$$

$$\begin{aligned} X_1 &= s_{12} + s_{13} + s_{14}, \\ X_2 &= s_{12} + s_{23} + s_{24}, \\ X_3 &= s_{13} + s_{23} + s_{34}, \\ X_4 &= s_{14} + s_{24} + s_{34}, \end{aligned} \tag{5.20}$$

generates the contact contributions for distinguishable fields [15]. For indistinguishable particles each polynomial has to be symmetric under index exchange, or equivalently, they should be invariant under S_n acting on the unordered pairs, which is usually denoted as a superscript. Using the definition of the generated ideal, definition 3.1.3,

$$f \in \mathbb{C}[\{s_{ij}\}] : \bar{f} = f + (\mathbb{C}X_1 + \mathbb{C}X_2 + \mathbb{C}X_3 + \mathbb{C}X_4 + \mathbb{C}\{\Delta\})\,. \tag{5.21}$$

What ends up happening is that the ideal kills 4 Mandelstams from the X_i contributions since they are of degree 1, such that the resulting representants form the ring;

$$\mathbb{C}[s_{12}, s_{13}]/\left\langle \{\Delta\} \right\rangle = \mathbb{C}[s,t]/\left\langle \{\Delta\} \right\rangle, \tag{5.22}$$

where the familiar notation $s_{12} = s$ & $s_{13} = t$ is brought back with the choice to not kill those two Mandelstams. In bringing back familiarity, $s_{14} = u$ will be written. In chapter 2 the equivalences of equation 2.32 were established,

$$\begin{aligned} s &= s_{12} = s_{34}, \\ t &= s_{13} = s_{24}, \\ u &= s_{14} = s_{23}, \\ u &= -s - t, \end{aligned} \tag{5.23}$$

revealing another reason why only two independent Mandelstams survive for 4-point. In general, $n(n-3)/2$ independent Mandelstams survive the momentum constraints, living in the $\rho_{n(n-3)/2}$ representation as a result, from which relation 5.22 could have been constructed.

As for the $\{\Delta\}$ contribution, this only becomes relevant at $d = 2$. At $d = 3$, the Gram conditions are equivalent to imposing momentum conservation, and at $d \geq 4$ all momenta could in principle be linearly independent, not imposing any general Gram constraints as understood in section 5.1.3. If $d = 2$, any minor of the Gram matrix,

$$\begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} \\ s_{12} & 0 & s_{23} & s_{24} \\ s_{13} & s_{23} & 0 & s_{34} \\ s_{14} & s_{24} & s_{24} & 0 \end{pmatrix}, \tag{5.24}$$

has to vanish. This turns out to result in a single condition $s^2 t + st^2 = 0$ [15]. As such, for indistinguishable fields:

$$\begin{aligned} \left(\mathbb{C}[s,t]\right)^{\mathrm{S}_n}, \quad & d \geq 3, \\ \left(\mathbb{C}[s,t]/\left\langle s^2 t + t^2 s \right\rangle\right)^{\mathrm{S}_n}, \quad & d = 2, \end{aligned} \tag{5.25}$$

generate possible contact terms in the amplitude. The first case has been studied in chapter 4, which led to a polynomial ring $\mathbb{C}[a_1, a_2]$ in two symmetric invariants a_1 & a_2. These two invariants then generate each element uniquely as per the ring being Cohen-Macaulay [15]. For distinguishable fields, the symmetry requirement is dropped.

Chapter 6 Discussion and Conclusions

6.1 ▸ Discussion

6.1.1 ▸ The $n(n-3)/2$ Representation

First the following research question is answered:

How does the "Mandelstam representation" manifest from the properties of Mandelstam variables?

In chapter 2 it was discussed how the $n(n-3)/2$-dimensional (Mandelstam) representation emerges in the decomposition of the permutation representation for the symmetric group S_n acting on the set of unordered pairs [7]. When the symmetric group S_n acts on a set of integers $\{1, \ldots, n\}$, the permutation representation is a decomposition into a 1-dimensional and an $(n-1)$-dimensional irreducible representations [25].

When instead acting on a set of unordered pairs of cardinality $n(n-1)/2$, those two together emerge as the natural permutation representation in the decomposition $\rho_1 \oplus \rho_{n-1} \oplus \rho_{n(n-3)/2}$ of this particular permutation representation [7, 15]. The basis of $\rho_1 \oplus \rho_{n-1}$ in this decomposition can be written with a single index such that $X_i = \sum_{j \neq i} e_{\{i,j\}}$, mimicking the behaviour of S_n acting on a set of n integers [15, 26]. Therefore, the inclusion of the $\rho_{n(n-3)/2}$ representation in the decomposition is a result of acting on a set of unordered pairs.

The indices of Mandelstam variables can be described as a set of unordered pairs due to their properties $s_{ij} = s_{ji}$ and $s_{ii} = 0$, and thus transform according to the permutation representation $\rho_1 \oplus \rho_{n-1} \oplus \rho_{n(n-3)/2}$ [15]. Momentum conservation forces the basis of the natural permutation representation to vanish, $X_i = \sum_{j \neq i} s_{ij} = 0$, such that it is killed off with the $\rho_{n(n-3)/2}$ representation remaining [15]. This is how the $n(n-3)/2$-dimensional representation emerges from the properties of Mandelstam variables, answering the question.

That being said, it would be more appropriate to instead state that Mandelstam variables of on-shell external particles, which abide by momentum conservation, live in the $n(n-3)/2$-dimensional representation [20].

This work however, has failed to define a general basis for the $n(n-3)/2$-dimensional representation, as was done for the natural permutation representation.

Cheung [5] however, presents a basis for $n(n-3)/2$ invariants using the general expression for Mandelstam invariants $s_{ij\ldots} := -(p_i + p_j + \ldots)^2$. This could be investigated in the future to concretely determine a general basis for the $n(n-3)/2$ representation with unordered pairs.

Another area where this work falls short is in presenting the tensor product between representations, and using it to find the decomposition of the product between two $n(n-3)/2$-dimensional representations. Therefore, a follow up to this work could be to describe the decompositions of $\bigotimes_k^p \rho_{n(n-3)/2}^{(k)}$, which would reveal the amount of invariants at order p in Mandelstams, similar to the classification of BCJ factors as presented in Li, Roest, and Veldhuis [20]. The tensor product could also be used to arrive at the $\rho_{n(n-3)/2}$ representation by taking the tensor product between two ρ_{n-1} representations, which describe conserved momenta [20]. The reader is invited to consult Serre [25] and Li, Roest, and Veldhuis [20] on the tensor product between representations.

6.1.2 ► Contact Term Generation

Here, the following question will be answered:

How can factor rings be constructed such that they generate valid Mandelstam polynomial terms in tree-level amplitudes for real on-shell massless scalar fields, and which rings would they be?

The generation of contact terms in scattering amplitudes for massless on-shell real scalar fields with polynomial rings was first introduced by treating 4-point interactions in 4-dimensions in chapter 4, and then more generally for n-point interactions in d-dimensions in chapter 5, with the machinery from chapter 3.

When taking the Fourier transform of a Lagrangian operator with n fields and k derivatives, a polynomial in n momenta of degree k comes out in momentum space. These polynomial terms form the Feynman rules associated with the derivative configuration, and become contact contributions in scattering amplitudes [15] Therefore, redundancies in the Lagrangian in the form of integration by parts and field redefinitions manifest themselves as equivalence relations in momentum conservation and $p^2 = 0$ for contact term contributions [15]. Considered here are polynomials in Mandelstam invariants [15].

Contact term polynomials then live in a polynomial ring in Mandelstam variables with coefficients in $\mathbb{C}$ modded out by an ideal generated by the constraints of momenta, as remarked by Henriette Elvang [16]. This construction is a factor ring which has equivalence classes defined by the momentum constraints as its elements [15, 27]. $s_{ij} = s_{ji}$ and $s_{ii} = 0$ are assumed in choosing variables such that $\mathbb{C}[\{s_{ij}\}] = \mathbb{C}[\rho_1 \oplus \rho_{n-1} \oplus \rho_{n(n-3)/2}]$ [15]. Then momentum conservation conditions $\{X_k\}$ and Gram conditions $\{\Delta\}$ (Gram conditions are only relevant when $n > d+1$) generate an ideal $\langle \{X_i\}, \{\Delta\} \rangle$ such that contact contributions live in either $\mathbb{C}[\{s_{ij}\}]/\langle \{X_k\}, \{\Delta\} \rangle$ or $(\mathbb{C}[\{s_{ij}\}]/\langle \{X_k\}, \{\Delta\} \rangle)^{S_n}$ depending on whether the fields are distinguishable [15, 17]. Without Gram conditions, finding the representants of these rings becomes a problem of finding the basis for the $n(n-3)$-dimensional representation to generate polynomials $\mathbb{C}[\rho_{n(n-3)/2}]$ with [15]. This answers the research question.

In the translation from a ring in momenta to Lorentz invariant quantities, only Mandelstam invariants were considered. This is a shortcoming since antisymmetric terms $\epsilon_{p_{i_1},\ldots,p_{i_d}} := \epsilon_{\mu_{i_1},\ldots,\mu_{i_d}} p_{i_1}^{\mu_1} \ldots p_{i_d}^{\mu_d}$ are Lorentz invariant as well [15]. These terms show up specifically when the polynomials are not required to be invariant under parity [15]. It could be studied in future research how the antisymmetric terms manifest themselves in the generation of contact contributions in scattering amplitude.

In deriving these conclusions, Euclidean spacetime was considered, taking momenta to be complex as in Henning et al. [15]. This seems to mostly be of relevance for equation 5.11, a result of invariant theory with $O(d)$ invariance [15]. It was not studied whether this result holds when considering the Poincaré group, and whether the story remains coherent [24]. This could be researched in the future to help solidify these methods for other spacetime geometries.

This work comes short on studying the Cohen-Macaulay property of rings. When a polynomial ring is Cohen-Macaulay, any element of the ring can uniquely be generated by a finite set of generators, which guarantees that there is not some other unknown polynomial term at some higher order [15]. Exploring this property is particularly important when considering nonzero spin systems, since it is not guaranteed that rings are Cohen-Macaulay in this case according to Henning et al. [15]. As such, this work can be expanded upon by studying this property.

Lastly, for future research, the *Gröbner basis* can be studied, which is a way of generating the same ideal with a different set of polynomial basis with the premise of being less computationally intensive [14]. Reduced Gröbner basis' can be generated using Buchberger's algorithm, which was applied in the work of Beisert et al. [2] for instance to compute matrix elements.

6.1.3 ► Counting Independent Polynomial Terms

Lastly, the following research question was specifically formulated from the discussion of 4-point scattering amplitudes for indistinguishable fields:

Why are there different amounts of independent terms at different orders in Mandelstams for the 4-point contact contributions?

In chapter 4, it was found that the fundamental theorem of symmetric polynomials can be used to generate any polynomial symmetric in Mandelstam variables (s, t, u) from elementary symmetric polynomials $e_1 = s+t+u$, $e_2 = st+su+tu$, and $e_3 = stu$ [6, 15]. Since contact contributions are required to be symmetric for indistinguishable fields, they live in $\mathbb{C}[e_1, e_2, e_3]/\langle e_1 \rangle$, with representants $\mathbb{C}[e_2, e_3]$ [15]. Monomial terms are then combinations of e_2 and e_3, which at certain orders can be arranged in multiple different ways as outlined in table 4.1. This answers the research question.

A shortcoming of this work is that no method of calculating the number of independent terms at each Mandelstam order was shown. It was mentioned that the Taylor coefficients of the Hilbert and Molien series count independent monomials [14, 20]. Studying the Hilbert series would lift the restriction to 4-point as well, making for a more general way of computing the number of independent monomials, since no description of n-point monomial terms generated by symmetric generators was presented in this thesis. Combining the machinery of polynomial rings with the Hilbert series would be a more complete description in the study of generating and counting contact polynomial terms in scattering amplitudes like in Henning et al. [14, 15]

6.2 ▸ Conclusion

In conclusion, contact polynomial terms in scattering amplitudes for massless on-shell real scalar fields were studied, along with invariants and properties of the S_n representations $[n] \oplus [n-1,1] \oplus [n-2,2]$. In particular, the research question,

How does the "Mandelstam representation" manifest from the properties of Mandelstam variables?

was answered by concluding that the inclusion of $[n-2,2]$ in the decomposition above is a result of the symmetric group acting on unordered pairs, with momentum conservation killing off $[n] \oplus [n-1,1]$ such that Mandelstam variables under momentum conservation inevitably live in the $[n-2,2]$ representation.

The question,

How can factor rings be constructed such that they generate valid Mandelstam polynomial terms in tree-level amplitudes for real on-shell massless scalar fields, and which rings would they be?

was answered by generating an ideal with momentum conservation and Gram conditions, and having it mod out a polynomial ring generated by Mandelstams without momentum conservation living in $[n] \oplus [n-1,1] \oplus [n-2,2]$ to describe n-point interactions in d-dimensions. This would look like $\mathbb{C}[\{s_{ij}\}]/\langle\{X_k\},\{\Delta\}\rangle$ with $\{X_k\}$ and $\{\Delta\}$ momentum conservation and Gram constraints respectively, with the added requirement of symmetry if fields are indistinguishable.

Lastly,

Why are there different amounts of independent terms at different orders in Mandelstams for the 4-point contact contributions?

was answered by finding different ways to combine elementary symmetric polynomials in Mandelstam variables (s, t, u), which through the fundamental theorem of symmetric polynomials generate the symmetric polynomials of $\mathbb{C}[s, t]^{\mathrm{S}_n}$

The general basis for the $[n-2, 2]$ representation when acting on unordered pairs could be further investigated, and studies on using the tensor product between $[n-2, 2]$ representations to reveal the symmetric terms in higher order Mandelstams.

Furthermore, the machinery studied in this work should be tested for non-euclidean spacetime, and antisymmetric terms should also be investigated. Moreover, the Cohen-Macaulay property of rings would be worth studying to show that any element of the ring can be generated by a finite set of generators. The Gröbner basis would be worth investigating for computational applications.

Lastly, the material presented in this thesis should ideally be studied alongside the Hilbert series to gain a more complete picture of generating contact term polynomials and the number of independent monomials at each other for n-point amplitudes in d-dimensions instead.

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